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Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions

Cetraro, Italy 2016

Matteo Bonforte · Gabriele Grillo
Editors

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Preface

We are honoured to have been the scientific organizers of the 2016 CIME Course "Nonlocal and nonlinear diffusions and interactions. New methods and directions", which took place in Cetraro (Cosenza, Italy) on July 4–8, 2016, with the following sets of lectures:

- José Antonio Carrillo, Imperial College London, United Kingdom:
  Swarming Models with Repulsive-Attractive Effects:
  Derivation, Model Hierarchies and Pattern Stability
- Manuel del Pino, Universidad de Chile, Chile:
  Bubbling blow-up in critical parabolic problems
- Alessio Figalli, ETH Zürich, Switzerland:
  Regularity results for local and nonlocal energy interactions
- Giuseppe Mingione, Università Parma, Italy:
  Recent progresses in nonlinear potential theory
- Juan Luis Vázquez, Universidad Autónoma de Madrid, Spain:
  Nonlinear Diffusion Equations with Fractional Laplacian Operators

During the school, there has been a special dinner to celebrate the 70th birthday of Juan Luis Vázquez. Juan Luis has always had a big involvement in Italian mathematics.

This CIME course has brought together some of the leading scholars in nonlinear partial differential equations (PDEs), with a special emphasis on rapidly developing topics which are of great and increasing interest both from the theoretical point of view and as concerns applications. This summer course has been a success even beyond our expectation. We are indebted to the distinguished speakers for giving such high-level lectures and also for the effort of writing this excellent quality set of lecture notes. We counted numerous attendance; there were more than 60 participants, mostly doctoral or postdoctoral students coming from more than 10 different countries. We would like also to thank them; without their active participation, this event would not have been so meaningful.

Throughout the whole week, the atmosphere has been especially nice, friendly and scientifically stimulating, mostly because of the active involvement of both the speakers and the participants.

We believe that the scientific level achieved in this course has been outstanding, but also quite adequate and accessible to the heterogeneous audience. We have received an extremely good feedback from the participants regarding both the practical organization and especially the high level of the courses and the very nice and stimulating atmosphere. It is worth mentioning that the speakers were not only amongst the top-level mathematicians in the respective fields, but also they were quite open to discussions with the students, friendly discussing with them after the lessons as well as in other moments.

We now briefly describe the single contributions included in this set of lecture notes.

José Antonio Carrillo contributed with a paper entitled "The geometry of diffusing and self-attracting particles in a one-dimensional fair-competition regime". The authors consider aggregation-diffusion equations modelling particle interactions with nonlinear diffusion and nonlocal attraction, leading to variants of the Keller-Segel model of chemotaxis. Their analysis deals with the one-dimensional case, providing an almost complete classification. Amongst the topics dealt with, we stress the uniqueness of stationary states via suitable functional inequalities, asymptotic behaviour of solutions, convergence to equilibrium in Wasserstein distance in the critical singular kernel case and convergence to self-similarity for subcritical interaction strength. Interesting numerical simulations naturally complement the analytical part.

Manuel del Pino contributed with a paper entitled "Bubbling blow-up in critical parabolic problems". The paper is devoted to the analysis of the blowup of solutions for some parabolic equations that involve *bubbling phenomena*. The term *bubbling* refers to the presence of families of solutions which at main order look like scalings of a single stationary solution which in the limit become singular but at the same time have an approximately constant energy level. This arise in various problems where critical loss of compactness for the underlying energy appears. Three main equations are studied, namely, the Sobolev critical semilinear heat equation in $\mathbb{R}^n$, the harmonic map flow from $\mathbb{R}^2$ into S^2 and the Patlak-Keller-Segel system in $\mathbb{R}^2$.

Alessio Figalli contributed with the paper "Regularity theory for local and nonlocal minimal surfaces: an overview". This review paper begins with a very useful general introduction to the classical theory of local minimal boundaries, explaining the main ideas behind the existence and regularity theory. This is extremely useful as it provides, in the second part of the note, a natural guideline in order to address the same kind of problems in the nonlocal context. Amongst the important results discussed in this note are (1) the existence of minimal surfaces, (2) the regularity theory of minimal graphs, (3) the regularity of flat minimal surfaces and (4) the analysis of blowups and minimal cones.

Giuseppe Mingione contributed with a paper entitled "Short Tales from Nonlinear Calderón-Zygmund Theory". The paper is a very useful review of nonlinear Calderón-Zygmund theory, which aims at reproducing, in the nonlinear setting, the classical linear theory. This topic has large intersections with nonlinear potential

theory. A central theme of the paper relies on the idea that linear potential theory tools like Riesz potential can still be used to study fine properties of solutions to nonlinear equations.

Juan Luis Vázquez contributed with a paper entitled "The mathematical theories of diffusion. Nonlinear and fractional diffusion". The paper begins with an historical survey on linear and nonlinear diffusions whose prototypes are the heat equation and the porous medium equation. Then the attention turns to new nonlinear and nonlocal diffusion models, coming from anomalous diffusions in physics that take into account long-range interactions and have also further applications in areas like biology, image processing or finance. The main points of the theory have been outlined: existence, uniqueness, regularity and asymptotic behaviour, focusing attention on the different techniques used in the proofs. The occurrence of stable diffusive patterns was stressed.

We had in Cetraro an interesting, rich and friendly atmosphere, created by the speakers, the participants and the CIME Organizers, in particular Elvira Mascolo (CIME Director) and Paolo Salani (CIME Secretary). We thank all of them warmly.

Madrid, Spain — Matteo Bonforte
Milano, Italy — Gabriele Grillo

Contents

Chapter 1
The Geometry of Diffusing and Self-Attracting Particles in a One-Dimensional Fair-Competition Regime

Vincent Calvez, José Antonio Carrillo, and Franca Hoffmann

Abstract We consider an aggregation-diffusion equation modelling particle interaction with non-linear diffusion and non-local attractive interaction using a homogeneous kernel (singular and non-singular) leading to variants of the Keller-Segel model of chemotaxis. We analyse the *fair-competition regime* in which both homogeneities scale the same with respect to dilations. Our analysis here deals with the one-dimensional case, building on the work in Calvez et al. (Equilibria of homogeneous functionals in the fair-competition regime), and provides an almost complete classification. In the singular kernel case and for critical interaction strength, we prove uniqueness of stationary states via a variant of the Hardy-Littlewood-Sobolev inequality. Using the same methods, we show uniqueness of self-similar profiles in the sub-critical case by proving a new type of functional inequality. Surprisingly, the same results hold true for any interaction strength in the non-singular kernel case. Further, we investigate the asymptotic behaviour of solutions, proving convergence to equilibrium in Wasserstein distance in the critical singular kernel case, and convergence to self-similarity for sub-critical interaction strength, both under a uniform stability condition. Moreover, solutions converge to a unique self-similar profile in the non-singular kernel case. Finally, we provide a

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numerical overview for the asymptotic behaviour of solutions in the full parameter space demonstrating the above results. We also discuss a number of phenomena appearing in the numerical explorations for the diffusion-dominated and attraction-dominated regimes.

1.1 Introduction

Mean field macroscopic models for interacting particle systems have been derived in the literature [66, 69] with the objective of explaining the large time behaviour, the qualitative properties and the stabilisation of systems composed by a large number of particles with competing effects such as repulsion and attraction between particles. They find natural applications in mathematical biology, gravitational collapse, granular media and self-assembly of nanoparticles, see [33, 39, 56, 58, 60, 78] and the references therein. These basic models start from particle dynamics in which their interaction is modelled via pairwise potentials. By assuming the right scaling between the typical interaction length and the number of particles per unit area one can obtain different mean field equations, see for instance [14]. In the mean-field scaling they lead to non-local equations with velocity fields obtained as an average force from a macroscopic density encoding both repulsion and attraction, see [2, 12] and the references therein. However, if the repulsion strength is very large at the origin, one can model repulsive effects by (non-linear) diffusion while attraction is considered via non-local long-range forces [66, 78].

In this work, we concentrate on this last approximation: repulsion is modelled by diffusion and attraction by non-local forces. We will make a survey of the main results in this topic exemplifying them in the one dimensional setting while at the same time we will provide new material in one dimension with alternative proofs and information about long time asymptotics which are not known yet in higher dimensions. In order to understand the interplay between repulsion via non-linear diffusion and attraction via non-local forces, we concentrate on the simplest possible situation in which both the diffusion and the non-local attractive potential are homogeneous functions. We will focus on models with a variational structure that dissipate the free energy of the system. This free energy is a natural quantity that is already dissipated at the underlying particle systems.

The plan for this work is twofold. In a first part we shall investigate some properties of the following class of homogeneous functionals, defined for centered probability densities $\rho(x)$, belonging to suitable weighted L^p-spaces, and some interaction strength coefficient $\chi > 0$ and diffusion power $m > 0$:

$$\mathcal{F}_{m,k}[\rho] = \int_{\mathbb{R}} U_m\left(\rho(x)\right)\, dx + \chi \iint_{\mathbb{R}\times\mathbb{R}} \rho(x) W_k(x-y)\rho(y)\, dxdy := \mathcal{U}_m[\rho] + \chi \mathcal{W}_k[\rho]\,, \tag{1.1}$$

$$\rho(x) \geq 0\,, \quad \int_{\mathbb{R}} \rho(x)\, dx = 1\,, \quad \int_{\mathbb{R}} x\rho(x)\, dx = 0\,,$$

with

$$U_m(\rho) = \begin{cases} \dfrac{1}{m-1}\,\rho^m\,, & \text{if} \quad m \neq 1 \\ \rho \log \rho\,, & \text{if} \quad m = 1 \end{cases},$$

and

$$W_k(x) = \begin{cases} \dfrac{|x|^k}{k}\,, & \text{if} \quad k \in (-1,1) \setminus \{0\} \\ \log|x|\,, & \text{if} \quad k = 0 \end{cases}. \tag{1.2}$$

The center of mass of the density ρ is assumed to be zero since the free energy functional is invariant by translation. Taking mass preserving dilations, one can see that $\mathcal{U}_m[\cdot]$ scales with a power $m-1$, whilst $\mathcal{W}_k[\cdot]$ scales with power $-k$, indicating that the relation between the parameters k and m plays a crucial role here. And indeed, one observes different types of behaviour depending on which of the two forces dominates, non-linear diffusion or non-local attraction. This motivates the definition of three different regimes: the *diffusion-dominated regime* $m-1 > -k$, the *fair-competition regime* $m-1 = -k$, and the *attraction-dominated regime* $m-1 < -k$. We will here concentrate mostly on the fair-competition regime.

This work can be viewed as a continuation of the seminal paper by McCann [65] in a non-convex setting. Indeed McCann used the very powerful toolbox of Euclidean optimal transportation to analyse functionals like (1.1) in the case $m \geq 0$ and for a convex interaction kernel W_k. He discovered that such functionals are equipped with an underlying convexity structure, for which the interpolant $[\rho_0, \rho_1]_t$ follows the line of optimal transportation [81]. This provides many interesting features among which a natural framework to show uniqueness of the ground state as soon as it exists. In this paper we deal with concave homogeneous interaction kernels W_k given by (1.2) for which McCann's results [65] do not apply. Actually, the conditions on k imply that the interaction kernel W_k is locally integrable on $\mathbb{R}$ and concave on $\mathbb{R}_+$, which means that $\mathcal{W}_k[\cdot]$ is displacement concave as shown in [35]. We explain in this paper how some ideas from [65] can be extended to some convex-concave competing effects. Our main statement is that the functional (1.1)—the sum of a convex and a concave functional—behaves almost like a convex functional in some good cases detailed below. In particular, existence of a critical point implies uniqueness (up to translations and dilations). The bad functional contribution is somehow absorbed by the convex part for certain homogeneity relations and parameters χ.

The analysis of these free energy functionals and their respective gradient flows is closely related to some functional inequalities of Hardy-Littlewood-Sobolev (HLS) type [12, 29, 53, 62]. To give a flavour, we highlight the case $(m = 1, k = 0)$, called the *logarithmic case*. It is known from [9, 50] using [5, 28] that the functional $\mathcal{F}_{1,0}$ is bounded from below if and only if $0 < \chi \leq 1$. Moreover, $\mathcal{F}_{1,0}$ achieves

its minimum if and only if $\chi = 1$ and the extremal functions are mass-preserving dilations of Cauchy's density:

$$\bar{\rho}_0(x) = \frac{1}{\pi}\left(\frac{1}{1+|x|^2}\right). \tag{1.3}$$

In [28] authors have proved the uniqueness (up to dilations and translations) of this logarithmic HLS inequality based on a competing-symmetries argument. We develop in the present paper an alternative argument based on some accurate use of the Jensen's inequality to get similar results in the porous medium case $-1 < k < 0$. This goal will be achieved for some variant of the HLS inequality as in [12], indeed being a combination of the HLS inequality and interpolation estimates, see Theorem 1.3.1. The case $0 < k < 1$ has been a lot less studied, and we will show here that no critical interaction strength exists as there is no $\chi > 0$ for which $\mathcal{F}_{m,k}$ admits global minimisers. On the other hand, we observe certain similarities with the behaviour of the fast diffusion equation ($0 < m < 1$, $\chi = 0$) [79]. The mass-preserving dilation homogeneity of the functional $\mathcal{F}_{m,k}$ is shared by the range of parameters (m, k) with $N(m-1)+k = 0$ for all dimensions, $m > 0$ and $k \in (-N, N)$. This general fair-competition regime, has recently been studied in [24].

In a second stage, here we also tackle the behaviour of the following family of partial differential equations modelling self-attracting diffusive particles at the macroscopic scale,

$$\begin{cases} \partial_t \rho = \partial_{xx}\left(\rho^m\right) + 2\chi \partial_x\left(\rho\, \partial_x S_k\right), \quad t > 0, \quad x \in \mathbb{R}, \\ \rho(t=0,x) = \rho_0(x). \end{cases} \tag{1.4}$$

where we define the mean-field potential $S_k(x) := W_k(x) * \rho(x)$. For $k > 0$, the gradient $\partial_x S_k := \partial_x\left(W_k * \rho\right)$ is well defined. For $k < 0$ however, it becomes a singular integral, and we thus define it via a Cauchy principal value. Hence, the mean-field potential gradient in Eq. (1.4) is given by

$$\partial_x S_k(x) := \begin{cases} \partial_x W_k * \rho, & \text{if } 0 < k < 1, \\ \displaystyle\int_{\mathbb{R}} \partial_x W_k(x-y)\left(\rho(y) - \rho(x)\right) dy, & \text{if } -1 < k < 0. \end{cases} \tag{1.5}$$

Further, it is straightforward to check that Eq. (1.4) formally preserves positivity, mass and centre of mass, and so we can choose to impose

$$\rho_0(x) \geq 0, \quad \int \rho_0(x)\, dx = 1, \quad \int x \rho_0(x)\, dx = 0.$$

This class of PDEs are one of the prime examples for competition between the diffusion (possibly non-linear), and the non-local, quadratic non-linearity which is due to the self-attraction of the particles through the mean-field potential $S_k(x)$.

The parameter $\chi > 0$ measures the strength of the interaction. We would like to point out that we are here not concerned with the regularity of solutions or existence/uniqueness results for Eq. (1.4), allowing ourselves to assume solutions are 'nice' enough in space and time for our analysis to hold (for more details on regularity assumptions, see Sect. 1.4).

There exists a strong link between the PDE (1.4) and the functional (1.1). Not only is $\mathcal{F}_{m,k}$ decreasing along the trajectories of the system, but more importantly, system (1.4) is the formal gradient flow of the free energy functional (1.1) when the space of probability measures is endowed with the Euclidean Wasserstein metric $\mathbf{W}$:

$$\partial_t \rho(t) = -\nabla_{\mathbf{W}} \mathcal{F}_{m,k}[\rho(t)]\,. \tag{1.6}$$

This illuminating statement has been clarified in the seminal paper by Otto [70]. We also refer to the books by Villani [81] and Ambrosio et al. [1] for a comprehensive presentation of this theory of gradient flows in Wasserstein metric spaces, particularly in the convex case. Performing gradient flows of a convex functional is a natural task, and suitable estimates from below on the Hessian of $\mathcal{F}_{m,k}$ in (1.1) translate into a rate of convergence towards equilibrium for the PDE [33, 34, 81]. However, performing gradient flow of functionals with convex and concave contributions is more delicate, and one has to seek compensations. Such compensations do exist in our case, and one can prove convergence in Wasserstein distance towards some stationary state under suitable assumptions, in some cases with an explicit rate of convergence. It is of course extremely important to understand how the convex and the concave contributions are entangled.

The results obtained in the fully convex case generally consider each contribution separately, resp. internal energy, potential confinement energy or interaction energy, see [1, 33, 34, 81]. It happens however that adding two contributions provides better convexity estimates. In [33] for instance the authors prove exponential speed of convergence towards equilibrium when a degenerate convex potential W_k is coupled with strong enough diffusion, see [15] for improvements.

The family of non-local PDEs (1.4) has been intensively studied in various contexts arising in physics and biology. The two-dimensional logarithmic case ($m = 1, k = 0$) is the so-called Keller-Segel system in its simplest formulation [9, 57–59, 68, 71]. It has been proposed as a model for chemotaxis in cell populations. The three-dimensional configuration ($m = 1, k = -1$) is the so-called Smoluchowski-Poisson system arising in gravitational physics [39–41]. It describes macroscopically a density of particles subject to a self-sustained gravitational field.

Let us describe in more details the two-dimensional Keller-Segel system, as the analysis of its peculiar structure will serve as a guideline to understand other cases. The corresponding gradient flow is subject to a remarkable dichotomy, see [9, 44, 50, 52, 57, 67] . The density exists globally in time if $\chi < 1$ (diffusion overcomes self-attraction), whereas blow-up occurs in finite time when $\chi > 1$ (self-attraction overwhelms diffusion). In the sub-critical case, it has been proved that solutions decay to self-similarity solutions exponentially fast in suitable rescaled variables

[25, 26, 51]. In the super-critical case, solutions blow-up in finite time with by now well studied blow-up profiles for close enough to critical cases, see [55, 72].

Substituting linear diffusion by non-linear diffusion with $m > 1$ in two dimensions and higher is a way of regularising the Keller-Segel model as proved in [18, 76] where it is shown that solutions exist globally in time regardless of the value of the parameter $\chi > 0$. It corresponds to the diffusion-dominated case in two dimensions for which the existence of compactly supported stationary states and global minimisers of the free energy has only been obtained quite recently in [38]. The fair-competition case for Newtonian interaction $k = 2 - N$ was first clarified in [12], see also [75], where the authors find that there is a similar dichotomy to the two-dimensional classical Keller-Segel case ($N = 2, m = 1, k = 0$), choosing the non-local term as the Newtonian potential, ($N \geq 3, m = 2 - 2/N, k = 2 - N$). The main difference is that the stationary states found for the critical case are compactly supported. We will see that such dichotomy also happens for $k < 0$ in our case while for $k > 0$ the system behaves totally differently. In fact, exponential convergence towards equilibrium seems to be the generic behaviour in rescaled variables as observed in Fig. 1.1.

The paper is structured as follows: in Sect. 1.2, we give an analytic framework with all necessary definitions and assumptions. In cases where no stationary states exist for the aggregation-diffusion equation (1.4), we look for self-similar profiles instead. Self-similar profiles can be studied by changing variables in (1.4) so that stationary states of the rescaled equation correspond to self-similar profiles of the original system. Further, we give some main results of optimal transportation needed for the analysis of Sects. 1.3 and 1.4. In Sect. 1.3, we establish several functional

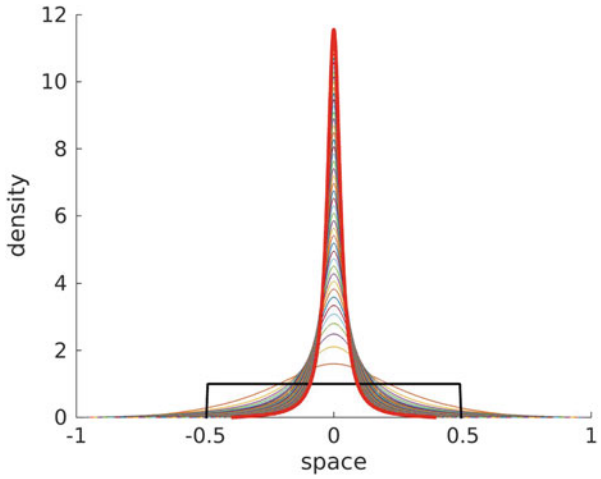

Fig. 1.1 Density evolution for parameter choices $\chi = 0.7$, $k = -0.2$, $m = 1.2$ following the PDE (1.4) in rescaled variables from a characteristic supported on $B(0, 1/2)$ (*black*) converging to a unique stationary state (*red*). For more details, see Fig. 1.6 and the explanations in Sect. 1.5

inequalities of HLS type that allow us to make a connection between minimisers of $\mathcal{F}_{m,k}$ and stationary states of (1.4), with similar results for the rescaled system. Section 1.4 investigates the long-time asymptotics where we demonstrate convergence to equilibrium in Wasserstein distance under certain conditions, in some cases with an explicit rate. Finally, in Sect. 1.5, we provide numerical simulations of system (1.4) to illustrate the properties of equilibria and self-similar profiles in the different parameter regimes for the fair-competition regime. In Sect. 1.6, we use the numerical scheme to explore the asymptotic behaviour of solutions in the diffusion- and attraction-dominated regimes.

1.2 Preliminaries

1.2.1 Stationary States: Definition and Basic Properties

Let us define precisely the notion of stationary states to the aggregation-diffusion equation (1.4).

Definition 1.2.1 Given $\bar{\rho} \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $||\bar{\rho}||_1 = 1$, it is a **stationary state** for the evolution equation (1.4) if $\bar{\rho}^m \in \mathcal{W}^{1,2}_{loc}(\mathbb{R})$, $\partial_x \bar{S}_k \in L^1_{loc}(\mathbb{R})$, and it satisfies

$$\partial_x \bar{\rho}^m = -2\chi\, \bar{\rho} \partial_x \bar{S}_k$$

in the sense of distributions in $\mathbb{R}$. If $k \in (-1, 0)$, we further require $\bar{\rho} \in C^{0,\alpha}(\mathbb{R})$ with $\alpha \in (-k, 1)$.

In fact, the function S_k and its gradient defined in (1.5) satisfy even more than the regularity $\partial_x S_k \in L^1_{loc}(\mathbb{R})$ required in Definition 1.2.1. We have from [24]:

Lemma 1.2.2 *Let $\rho \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $||\rho||_1 = 1$. If $k \in (0, 1)$, we additionally assume $|x|^k \rho \in L^1(\mathbb{R})$. Then the following regularity properties hold:*

(i) $S_k \in L^\infty_{loc}(\mathbb{R})$ for $0 < k < 1$ and $S_k \in L^\infty(\mathbb{R})$ for $-1 < k < 0$.
(ii) $\partial_x S_k \in L^\infty(\mathbb{R})$ for $k \in (-1, 1)\backslash\{0\}$, assuming additionally $\rho \in C^{0,\alpha}(\mathbb{R})$ with $\alpha \in (-k, 1)$ in the range $-1 < k < 0$.

Furthermore, for certain cases, see [24], there are no stationary states to (1.4) in the sense of Definition 1.2.1 (for a dynamical proof of this fact, see Remark 1.4.1 in Sect. 1.4.1.2), and so the scale invariance of (1.4) motivates us to look for self-similar solutions instead. To this end, we rescale Eq. (1.4) to a non-linear Fokker-Planck type equation as in [32]. Let us define

$$u(t, x) := \alpha(t)\rho\left(\beta(t), \alpha(t)x\right),$$

where $\rho(t,x)$ solves (1.4) and the functions $\alpha(t)$, $\beta(t)$ are to be determined. If we assume $u(0,x) = \rho(0,x)$, then $u(t,x)$ satisfies the rescaled drift-diffusion equation

$$\begin{cases} \partial_t u = \partial_{xx} u^m + 2\chi \partial_x \left(u\, \partial_x S_k\right) + \partial_x \left(xu\right) , \quad t > 0 , \quad x \in \mathbb{R} , \\ u(t=0,x) = \rho_0(x) \geq 0 , \quad \displaystyle\int_{-\infty}^{\infty} \rho_0(x)\, dx = 1 , \quad \int_{-\infty}^{\infty} x \rho_0(x)\, dx = 0 , \end{cases} \tag{1.7}$$

for the choices

$$\alpha(t) = e^t, \quad \beta(t) = \begin{cases} \frac{1}{2-k} \left(e^{(2-k)t} - 1\right) , & \text{if } k \neq 2 , \\ t, & \text{if } k = 2 , \end{cases} \tag{1.8}$$

and with $\partial_x S_k$ given by (1.5) with u instead of ρ. By differentiating the centre of mass of u, we see easily that

$$\int_{\mathbb{R}} xu(t,x)\, dx = e^{-t} \int_{\mathbb{R}} x\rho_0(x)\, dx = 0 , \qquad \forall t > 0 ,$$

and so the initial zero centre of mass is preserved for all times. Self-similar solutions to (1.4) now correspond to stationary solutions of (1.7). Similar to Definition 1.2.1, we state what we exactly mean by stationary states to the aggregation-diffusion equation (1.7).

Definition 1.2.3 Given $\bar{u} \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $||\bar{u}||_1 = 1$, it is a **stationary state** for the evolution equation (1.7) if $\bar{u}^m \in \mathcal{W}^{1,2}_{loc}(\mathbb{R})$, $\partial_x \bar{S}_k \in L^1_{loc}(\mathbb{R})$, and it satisfies

$$\partial_x \bar{u}^m = -2\chi\, \bar{u} \partial_x \bar{S}_k - x\, \bar{u}$$

in the sense of distributions in $\mathbb{R}$. If $-1 < k < 0$, we further require $\bar{u} \in C^{0,\alpha}(\mathbb{R})$ with $\alpha \in (-k, 1)$.

From now on, we switch notation from u to ρ for simplicity, it should be clear from the context if we are in original or rescaled variables. In fact, stationary states as defined above have even more regularity:

Lemma 1.2.4 *Let $k \in (-1,1)\backslash\{0\}$ and $\chi > 0$.*

(i) If $\bar{\rho}$ is a stationary state of Eq. (1.4) with $|x|^k \bar{\rho} \in L^1(\mathbb{R})$ in the case $0 < k < 1$, then $\bar{\rho}$ is continuous on $\mathbb{R}$.

(ii) If $\bar{\rho}_{\mathbf{resc}}$ is a stationary state of Eq. (1.7) with $|x|^k \bar{\rho}_{\mathbf{resc}} \in L^1(\mathbb{R})$ in the case $0 < k < 1$, then $\bar{\rho}_{\mathbf{resc}}$ is continuous on $\mathbb{R}$.

In the case $k < 0$, we furthermore have a non-linear algebraic equation for stationary states [24]:

Corollary 1.2.5 (Necessary Condition for Stationary States) *Let $k \in (-1, 0)$ and $\chi > 0$.*

(i) If $\bar{\rho}$ is a stationary state of Eq. (1.4), then $\bar{\rho} \in \mathcal{W}^{1,\infty}(\mathbb{R})$ and it satisfies

$$\bar{\rho}(x)^{m-1} = \frac{(m-1)}{m}\left(C_k[\bar{\rho}](x) - 2\chi\,\bar{S}_k(x)\right)_+ , \qquad \forall\, x \in \mathbb{R}\,,$$

where $C_k[\bar{\rho}](x)$ is constant on each connected component of $\mathrm{supp}(\bar{\rho})$.

(ii) If $\bar{\rho}_{\mathbf{resc}}$ is a stationary state of Eq. (1.7), then $\bar{\rho}_{\mathbf{resc}} \in \mathcal{W}^{1,\infty}_{loc}(\mathbb{R})$ and it satisfies

$$\bar{\rho}_{\mathbf{resc}}(x)^{m-1} = \frac{(m-1)}{m}\left(C_{k,\mathbf{resc}}[\bar{\rho}](x) - 2\chi\,\bar{S}_k(x) - \frac{|x|^2}{2}\right)_+ , \qquad \forall\, x \in \mathbb{R}\,,$$

where $C_{k,\mathbf{resc}}[\bar{\rho}](x)$ is constant on each connected component of $\mathrm{supp}(\bar{\rho}_{\mathbf{resc}})$.

1.2.2 Definition of the Different Regimes

It is worth noting that the functional $\mathcal{F}_{m,k}[\rho]$ possesses remarkable homogeneity properties. Indeed, the mass-preserving dilation $\rho_\lambda(x) = \lambda\rho(\lambda x)$ transforms the functionals as follows:

$$\mathcal{U}_m[\rho_\lambda] = \begin{cases} \lambda^{(m-1)}\mathcal{U}_m[\rho]\,, & \text{if} \quad m \neq 1\,, \\ \mathcal{U}_m[\rho] + \log\lambda\,, & \text{if} \quad m = 1\,, \end{cases}$$

and,

$$\mathcal{W}_k[\rho_\lambda] = \begin{cases} \lambda^{-k}\mathcal{W}_k[\rho]\,, & \text{if} \quad k \neq 0\,, \\ \mathcal{W}_k[\rho] - \log\lambda\,, & \text{if} \quad k = 0\,. \end{cases}$$

This motivates the following classification:

Definition 1.2.6 (Three Different Regimes)

$\mathbf{m + k = 1}$ This is the **fair-competition** regime, where homogeneities of the two competing contributions exactly balance. If $k < 0$, or equivalently $m > 1$, then we will have a dichotomy according to χ (see Definition 1.2.7 below). Some variants of the HLS inequalities are very related to this dichotomy. This was already proven in [9, 26, 50, 51] for the Keller-Segel case in $N = 2$, and in [12] for the Keller-Segel case in $N \geq 3$. If $k > 0$, that is $m < 1$, no critical χ exists as we will prove in Sect. 1.3.2.

$m + k > 1$ This is the **diffusion-dominated** regime. Diffusion is strong, and is expected to overcome aggregation, whatever $\chi > 0$ is. This domination effect means that solutions exist globally in time and are bounded uniformly in time [18, 75, 76]. Stationary states were found by minimisation of the free energy functional in two and three dimensions [31, 37, 73] in the case of attractive Newtonian potentials. Stationary states are radially symmetric if $2 - N \leq k < 0$ as proven in [38]. Moreover, in the particular case of $N = 2, k = 0$, and $m > 1$ it has been proved in [38] that the asymptotic behaviour is given by compactly supported stationary solutions independently of χ.

$m + k < 1$ This is the **attraction-dominated** regime. This regime is less understood. Self-attraction is strong, and can overcome the regularising effect of diffusion whatever $\chi > 0$ is, but there also exist global in time regular solutions under some smallness assumptions, see [6, 20, 42, 43, 48, 63, 74, 77]. However, there is no complete criteria in the literature up to date distinguishing between the two behaviours.

We will here only concentrate on the fair-competition regime, and denote the corresponding energy functional by $\mathcal{F}_k[\rho] = \mathcal{F}_{1-k,k}[\rho]$. From now on, we assume $m + k = 1$. Notice that the functional $\mathcal{F}_k$ is homogeneous in this regime, i.e.,

$$\mathcal{F}_k[\rho_\lambda] = \lambda^{-k}\mathcal{F}_k[\rho]\,.$$

In this work, we will first do a review of the main results known in one dimension about the stationary states and minimisers of the aggregation-diffusion equation in the fair-competition case. The novelties will be showing the functional inequalities independently of the flow and studying the long-time asymptotics of Eqs. (1.4) and (1.7) by exploiting the one dimensional setting. The analysis in the fair-competition regime depends on the sign of k:

Definition 1.2.7 (Three Different Cases in the Fair-Competition Regime)

$k < 0$ This is the **porous medium case** with $m \in (1, 2)$, where diffusion is small in regions of small densities. The classical porous medium equation, i.e. $\chi = 0$, is very well studied, see [80] and the references therein. For $\chi > 0$, we have a dichotomy for existence of stationary states and global minimisers of the energy functional $\mathcal{F}_k$ depending on a critical parameter χ_c which will be defined in (1.18), and hence separate the sub-critical, the critical and the super-critical case, according to $\chi \lessgtr \chi_c$. These are the one dimensional counterparts to the case studied in [12] where minimisers for the free energy functional were clarified. The case $k < 0$ is discussed in Sect. 1.3.1.

$k = 0$ This is the **logarithmic case**. There exists an explicit extremal density $\bar{\rho}_0$ which realises the minimum of the functional $\mathcal{F}_0$ when $\chi = 1$. Moreover, the functional $\mathcal{F}_0$ is bounded below but does not achieve its infimum for $0 < \chi < 1$ while it is not bounded below for $\chi > 1$. Hence, $\chi_c = 1$ is the critical parameter in the logarithmic case whose asymptotic behaviour was

analysed in [19] in one dimension and radial initial data in two dimensions. We refer to the results in [26, 51] for the two dimensional case.

k > 0 This is the **fast diffusion case** with $m \in (0,1)$, where diffusion is strong in regions of small densities. For any $\chi > 0$, no radially symmetric non-increasing stationary states with bounded kth moment exist, and $\mathcal{F}_k$ has no radially symmetric non-increasing minimisers. However, we have existence of self-similar profiles independently of $\chi > 0$. The fast diffusion case is discussed in Sect. 1.3.2.

When dealing with the energy functional $\mathcal{F}_k$, we work in the set of non-negative normalised densities,

$$\mathcal{Y} := \left\{ \rho \in L^1_+(\mathbb{R}) \cap L^m(\mathbb{R}) : ||\rho||_1 = 1\,,\ \int x\rho(x)\,dx = 0 \right\}.$$

In rescaled variables, Eq. (1.7) is the formal gradient flow of the rescaled free energy functional $\mathcal{F}_{k,\mathbf{resc}}$, which is complemented with an additional quadratic confinement potential,

$$\mathcal{F}_{k,\mathbf{resc}}[\rho] = \mathcal{F}_k[\rho] + \frac{1}{2}\mathcal{V}[\rho]\,, \qquad \mathcal{V}[\rho] = \int_{\mathbb{R}} |x|^2 \rho(x)\,dx\,.$$

Defining the set $\mathcal{Y}_2 := \{\rho \in \mathcal{Y} : \mathcal{V}[\rho] < \infty\}$, we see that $\mathcal{F}_{k,\mathbf{resc}}$ is well-defined and finite on $\mathcal{Y}_2$. Thanks to the formal gradient flow structure in the Euclidean Wasserstein metric $\mathbf{W}$, we can write the rescaled equation (1.7) as

$$\partial_t \rho = -\nabla_{\mathbf{W}} \mathcal{F}_{k,\mathbf{resc}}[\rho]\,.$$

In what follows, we will make use of a different characterisation of stationary states based on some integral reformulation of the necessary condition stated in Corollary 1.2.5. This characterisation was also the key idea in [19] to improve on the knowledge of the asymptotic stability of steady states and the functional inequalities behind.

Lemma 1.2.8 (Characterisation of Stationary States) *Let $k \in (-1,1)\backslash\{0\}$, $m = 1-k$ and $\chi > 0$.*

(i) Any stationary state $\bar\rho_k \in \mathcal{Y}$ of system (1.4) can be written in the form

$$\bar\rho_k(p)^m = \chi \int_{\mathbb{R}} \int_0^1 |q|^{1-m} \bar\rho_k(p - sq)\bar\rho_k(p - sq + q)\,dsdq\,. \tag{1.9}$$

Moreover, if such a stationary state exists, it satisfies $\mathcal{F}_k[\bar\rho_k] = 0$.

(ii) Any stationary state $\bar\rho_{k,\mathbf{resc}} \in \mathcal{Y}_2$ of system (1.7) can be written in the form

$$\bar\rho_{k,\mathbf{resc}}(p)^m = \int_{\mathbb{R}}\int_0^1 \left(\chi|q|^{1-m} + \frac{|q|^2}{2}\right)\bar\rho_{k,\mathbf{resc}}(p-sq)\bar\rho_{k,\mathbf{resc}}(p-sq+q)\,dsdq\,. \tag{1.10}$$

Moreover, it satisfies

$$\mathcal{F}_{k,\mathbf{resc}}[\bar\rho_{k,\mathbf{resc}}] = \frac{m+1}{2(m-1)}\,\mathcal{V}[\bar\rho_{k,\mathbf{resc}}] = \left(\frac12 - \frac1k\right)\mathcal{V}[\bar\rho_{k,\mathbf{resc}}]\,. \tag{1.11}$$

Proof We can apply the same methodology as for the logarithmic case (Lemma 2.3, [19]). We will only prove (1.9), identity (1.10) can be deduced in a similar manner. We can see directly from the equation that all stationary states of (1.4) in $\mathcal{Y}$ satisfy

$$\partial_x\left(\bar\rho_k^m\right) + 2\chi\bar\rho_k\partial_x\bar S_k = 0\,.$$

Hence, if $k \in (0,1)$, we can write for any test function $\varphi \in C_c^\infty(\mathbb{R})$

$$\begin{aligned} 0 &= -\int_{\mathbb{R}}\varphi'(p)\bar\rho_k^m(p)\,dp + 2\chi\iint_{\mathbb{R}\times\mathbb{R}}\varphi(x)|x-y|^{k-2}(x-y)\bar\rho_k(x)\bar\rho_k(y)\,dxdy \\ &= -\int_{\mathbb{R}}\varphi'(p)\bar\rho_k^m(p)\,dp + \chi\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)|x-y|^k\bar\rho_k(x)\bar\rho_k(y)\,dxdy\,. \end{aligned}$$

For $k \in (-1,0)$, the term $\partial_x\bar S_k$ is a singular integral, and thus writes

$$\begin{aligned} \partial_x\bar S_k(x) &= \lim_{\varepsilon\to0}\int_{B^c(x,\varepsilon)}|x-y|^{k-2}(x-y)\bar\rho_k(y)\,dy \\ &= \int_{\mathbb{R}}|x-y|^{k-2}(x-y)\left(\bar\rho_k(y)-\bar\rho_k(x)\right)\,dy\,. \end{aligned}$$

The singularity disappears when integrating against a test function $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}}\varphi(x)\partial_x\bar S_k(x)\,dx = \frac12\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)|x-y|^k\bar\rho_k(x)\bar\rho_k(y)\,dxdy\,. \tag{1.12}$$

In order to prove (1.12), let us define

$$f_\varepsilon(x) := \varphi(x)\int_{B^c(x,\varepsilon)}\partial_x W_k(x-y)\bar\rho_k(y)\,dy.$$

Then by definition of the Cauchy principle value, $f_\varepsilon(x)$ converges to $\varphi(x)\partial_x\bar S_k(x)$ pointwise for almost every $x \in \mathbb{R}$ as $\varepsilon \to 0$. Further, we use the fact that

$\bar\rho_k \in C^{0,\alpha}(\mathbb{R})$ for some $\alpha \in (-k, 1)$ to obtain the uniform in ε estimate

$$|f_\varepsilon(x)| \le \left(\frac{2+k+\alpha}{k+\alpha}\right)|\varphi(x)|\,, \qquad \forall\ 0<\varepsilon<1\,,$$

and therefore by Lebesgue's dominated convergence theorem,

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x)\partial_x \bar S_k(x)\,dx &= \int_{\mathbb{R}} \lim_{\varepsilon\to 0} f_\varepsilon(x)\bar\rho_k(x)\,dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f_\varepsilon(x)\bar\rho_k(x)\,dx\\ &= \lim_{\varepsilon\to 0}\iint_{|x-y|\ge\varepsilon} \varphi(x)|x-y|^{k-2}(x-y)\bar\rho_k(x)\bar\rho_k(y)\,dxdy\\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\iint_{|x-y|\ge\varepsilon}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)|x-y|^k\bar\rho_k(x)\bar\rho_k(y)\,dxdy\\ &= \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)|x-y|^k\bar\rho_k(x)\bar\rho_k(y)\,dxdy\,.\end{aligned}$$

This concludes the proof of (1.12). Hence, we obtain for any $k \in (-1,1)\backslash\{0\}$,

$$\begin{aligned}0 &= -\int_{\mathbb{R}}\varphi'(p)\bar\rho_k^m(p)\,dp + \chi\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)|x-y|^k\bar\rho_k(x)\bar\rho_k(y)\,dxdy\\ &= -\int_{\mathbb{R}}\varphi'(p)\bar\rho_k^m(p)\,dp + \chi\iint_{\mathbb{R}\times\mathbb{R}}\int_0^1\varphi'\left((1-s)x+sy\right)|x-y|^k\bar\rho_k(x)\bar\rho_k(y)\,dsdxdy\\ &= -\int_{\mathbb{R}}\varphi'(p)\bar\rho_k^m(p)\,dp + \chi\int_{\mathbb{R}}\varphi'(p)\left\{\int_{\mathbb{R}}\int_0^1|q|^k\bar\rho_k(p-sq)\bar\rho_k(p-sq+q)\,dsdq\right\}\,dp\end{aligned}$$

and so (1.9) follows up to a constant. Since both sides of (1.9) have mass one, the constant is zero. To see that $\mathcal{F}_k[\bar\rho_k] = 0$, we substitute (1.9) into (1.1) and use the same change of variables as above.

Finally, identity (1.11) is a consequence of various homogeneities. For every stationary state $\bar\rho_{k,\mathbf{resc}}$ of (1.7), the first variation $\frac{\delta\mathcal{F}_{k,\mathbf{resc}}}{\delta\rho}[\bar\rho_{k,\mathbf{resc}}] = m/(m-1)\bar\rho_{k,\mathbf{resc}}^{m-1} + 2\chi W_k * \bar\rho_{k,\mathbf{resc}} + |x|^2/2$ vanishes on the support of $\bar\rho_{k,\mathbf{resc}}$ and hence it follows that for dilations $\bar\rho_\lambda(x) := \lambda\bar\rho_{k,\mathbf{resc}}(\lambda x)$ of the stationary state $\bar\rho_{k,\mathbf{resc}}$:

$$\begin{aligned}-k\mathcal{F}_{k,\mathbf{resc}}[\bar\rho_{k,\mathbf{resc}}] + \left(\frac{k}{2}-1\right)\mathcal{V}[\bar\rho_{k,\mathbf{resc}}] &= \frac{d}{d\lambda}\mathcal{F}_{k,\mathbf{resc}}[\bar\rho_\lambda]\Big|_{\lambda=1}\\ &= \int_{\mathbb{R}}\left(\frac{\delta\mathcal{F}_{k,\mathbf{resc}}}{\delta\rho}[\bar\rho_\lambda](x)\,\frac{d\bar\rho_\lambda}{d\lambda}(x)\right)dx\Big|_{\lambda=1} = 0.\end{aligned}$$

In the fair-competition regime, attractive and repulsive forces are in balance $m+k = 1$, and so (1.11) follows. □

Recall that stationary states in rescaled variables are self-similar solutions in original variables. Tables 1.1, 1.2 and 1.3 provide an overview of results proved in this paper and in [24] in one dimension.

1.2.3 Optimal Transport Tools

This sub-section summarises the main results of optimal transportation we will need. They were already used for the case of logarithmic HLS inequalities and the classical Keller-Segel model in 1D and radial 2D, see [19], where we refer for detailed proofs.

Let $\tilde\rho$ and ρ be two density probabilities. According to [17, 64], there exists a convex function ψ whose gradient pushes forward the measure $\tilde\rho(a)da$ onto $\rho(x)dx$: $\psi'\#\left(\tilde\rho(a)da\right) = \rho(x)dx$. This convex function satisfies the Monge-Ampère equation in the weak sense: for any test function $\varphi \in C_b(\mathbb{R})$, the following identity holds true

$$\int_{\mathbb{R}} \varphi(\psi'(a))\tilde\rho(a)\,da = \int_{\mathbb{R}} \varphi(x)\rho(x)\,dx\,. \tag{1.13}$$

The convex map is unique a.e. with respect to ρ and it gives a way of interpolating measures. In fact, the interpolating curve ρ_s, $s \in [0,1]$, with $\rho_0 = \rho$ and $\rho_1 = \tilde\rho$ can be defined as $\rho_s(x)\,dx = (s\psi' + (1-s)\mathrm{id})(x)\#\rho(x)\,dx$ where id stands for the identity map in $\mathbb{R}$. This interpolating curve is actually the minimal geodesic joining the measures $\rho(x)dx$ and $\tilde\rho(x)dx$. The notion of convexity associated to these interpolating curves is nothing else than convexity along geodesics, introduced and called displacement convexity in [65]. In one dimension the displacement convexity/concavity of functionals is easier to check as seen in [30, 35]. The convexity of the functionals involved can be summarised as follows [35, 65]:

Theorem 1.2.9 *The functional $\mathcal{U}_m[\rho]$ is displacement-convex provided that $m \geq 0$. The functional $\mathcal{W}_k[\rho]$ is displacement-concave if $k \in (-1,1)$.*

This means we have to deal with convex-concave compensations. On the other hand, regularity of the transport map is a complicated matter. Here, as it was already done in [19], we will only use the fact that the Hessian measure $\det_H D^2\psi(a)da$ can be decomposed in an absolute continuous part $\det_A D^2\psi(a)da$ and a positive singular measure (Chap. 4, [81]). Moreover, it is known that a convex function ψ has Aleksandrov second derivative $D^2_A\psi(a)$ almost everywhere and that $\det_A D^2\psi(a) = \det D^2_A\psi(a)$. In particular we have $\det_H D^2\psi(a) \geq \det_A D^2\psi(a)$. The formula for the change of variables will be important when dealing with the internal energy contribution. For any measurable function U, bounded below such that $U(0) = 0$ we have [65]

$$\int_{\mathbb{R}} U(\tilde\rho(x))\,dx = \int_{\mathbb{R}} U\left(\frac{\rho(a)}{\det_A D^2\psi(a)}\right)\det_A D^2\psi(a)\,da\,. \tag{1.14}$$

Table 1.1 Overview of results in one dimension for $-1 < k < 0$

$\chi < \chi_c(k)$	$\chi = \chi_c(k)$	$\chi > \chi_c(k)$
Functional inequalities: • There are no stationary states in original variables, there are no minimisers for $\mathcal{F}_k$ [24, Theorem 2.9].. • In rescaled variables, all stationary states are continuous and compactly supported [24, Theorem 2.9]. • There exists a minimiser of $\mathcal{F}_{k,\mathbf{resc}}$. Minimisers are symmetric non-increasing and uniformly bounded. Minimisers are stationary states in rescaled variables [24, Theorem 2.9]. • If $\bar{\rho}_{\mathbf{resc}}$ is a stationary state in rescaled variables, then all solutions of the rescaled equation satisfy $\mathcal{F}_{k,\mathbf{resc}}[\rho] \geq \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_{\mathbf{resc}}]$ (Theorem 1.3.4). • Stationary states in rescaled variables and minimisers of $\mathcal{F}_{k,\mathbf{resc}}$ are unique (Corollary 1.3.6).	**Functional inequalities:** • There exists a minimiser of $\mathcal{F}_k$. Minimisers are symmetric non-increasing, compactly supported and uniformly bounded. Minimisers are stationary states in original variables [24, Theorem 2.8]. • There are no stationary states in rescaled variables in $\mathcal{Y}_2$, and there are no minimisers of $\mathcal{F}_{k,\mathbf{resc}}$ in $\mathcal{Y}_2$ (Corollary 1.3.8(ii)). • If $\bar{\rho}$ is a stationary state in original variables, then all solutions satisfy $\mathcal{F}_k[\rho] \geq \mathcal{F}_k[\bar{\rho}] = 0$, which corresponds to a variation of the HLS inequality (Theorem 1.3.2). • Stationary states in original variables and minimisers of $\mathcal{F}_k$ are unique up to dilations (Corollary 1.3.3), and they coincide with the equality cases of $\mathcal{F}_k[\rho] \geq 0$.	**Functional inequalities:** • There are no stationary states in original variables in $\mathcal{Y}$, and there are no minimisers of $\mathcal{F}_k$ in $\mathcal{Y}$ (Corollary 1.3.8(i)). • There are no stationary states in rescaled variables in $\mathcal{Y}_2$, and there are no minimisers of $\mathcal{F}_{k,\mathbf{resc}}$ in $\mathcal{Y}_2$ (Corollary 1.3.8(ii)).
Asymptotics: • Under a stability condition solutions converge exponentially fast in Wasserstein distance towards the unique stationary state in rescaled variables with rate 1 (Proposition 1.4.4).	**Asymptotics:** • Under a stability condition and for solutions with second moment bounded in time, we have convergence in Wasserstein distance (without explicit rate) to a unique (up to dilation) stationary state (Proposition 1.4.2).	**Asymptotics:** Asymptotics are not well understood yet. • If there exists a time $t_0 \geq 0$ such that $\mathcal{F}_k[\rho(t_0)] < 0$, then ρ blows up in finite time [12, 74]. • Numerics suggest that the energy of any solution becomes negative in finite time, but no analytical proof is known.

Table 1.2 Overview of results in one dimension for $k = 0$

$\chi < 1$	$\chi = 1$	$\chi > 1$
Functional inequalities: • There are no stationary states in original variables, but self-similar profiles [9, 25, 26, 50, 51].	**Functional inequalities:** • If $\bar\rho$ is a stationary state in original variables, then all solutions satisfy $\mathcal{F}_k[\rho] \geq \mathcal{F}_k[\bar\rho]$, which corresponds to the logarithmic HLS inequality [9, 19, 50]. • Stationary states are given by dilations of Cauchy's density, $\bar\rho(x) = 1/(\pi(1 + |x|^2))$, which coincide with the equality cases of the logarithmic HLS inequality. They all have infinite second moment [9, 19, 50].	**Functional inequalities:** • Smooth fast-decaying solutions do not exist globally in time [7, 9, 23, 67]. • There are no stationary states in original variables and there are no minimisers of $\mathcal{F}_0$ in $\mathcal{Y}$ (Remark 1.3.2).
Asymptotics: • Solutions converge exponentially fast in Wasserstein distance towards the unique stationary state in rescaled variables [19].	**Asymptotics:** • Solutions converge in Wasserstein distance to a dilation of Cauchy's density (without explicit rate) if the initial second moment is infinite, and to a Dirac mass otherwise [8, 11, 13, 19, 27].	**Asymptotics:** • All solutions blow up in finite time provided the second moment is initially finite [55, 72].

Luckily, the complexity of Brenier's transport problem dramatically reduces in one dimension. More precisely, the transport map ψ' is a non-decreasing function, therefore it is differentiable a.e. and it has a countable number of jump singularities. The singular part of the positive measure $\psi''(x)\,dx$ corresponds to having holes in the support of the density ρ. Also, the Aleksandrov second derivative of ψ coincides with the absolutely continuous part of the positive measure $\psi''(x)\,dx$ that will be denoted by $\psi''_{\rm ac}(x)\,dx$. Moreover, the a.e. representative ψ' can be chosen to be the distribution function of the measure $\psi''(x)\,dx$ and it is of bounded variation locally, with lateral derivatives existing at all points and therefore, we can always write for all $a < b$

$$\psi'(b) - \psi'(a) = \int_{(a,b]} \psi''(x)\,dx \geq \int_a^b \psi''_{\rm ac}(x)\,dx$$

for a well chosen representative of ψ'.

Table 1.3 Overview of results in one dimension for $0 < k < 1$

No criticality for χ
Functional inequalities: • There are no stationary states in original variables (Remark 1.4.2). In rescaled variables, there exists a continuous symmetric non-increasing stationary state [24, Theorem 2.11]. • There are no symmetric non-increasing global minimisers of $\mathcal{F}_k$. Global minimisers of $\mathcal{F}_{k,\mathbf{resc}}$ can only exist in the range $0 < k < \frac{2}{3}$ [24, Theorem 2.11]. • If $\bar{\rho}_{\mathbf{resc}}$ is a stationary state in rescaled variables, then all solutions of the rescaled equation satisfy $\mathcal{F}_{k,\mathbf{resc}}[\rho] \geq \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_{\mathbf{resc}}]$ (Theorem 1.3.9). Hence, for $0 < k < \frac{2}{3}$, there exists a global minimiser for $\mathcal{F}_{k,\mathbf{resc}}$. • For $0 < k < \frac{2}{3}$, stationary states in rescaled variables and global minimisers of $\mathcal{F}_{k,\mathbf{resc}}$ are unique (Corollary 1.3.11).
Asymptotics: • Solutions converge exponentially fast in Wasserstein distance to the unique stationary state in rescaled variables with rate 1 (Proposition 1.4.6).

The following Lemma proved in [19] will be used to estimate the interaction contribution in the free energy, and in the evolution of the Wasserstein distance.

Lemma 1.2.10 *Let $\mathcal{K} : (0, \infty) \to \mathbb{R}$ be an increasing and strictly concave function. Then, for any (a, b)*

$$\mathcal{K}\left(\frac{\psi'(b) - \psi'(a)}{b - a}\right) \geq \int_0^1 \mathcal{K}\left(\psi''_{\mathrm{ac}}([a, b]_s)\right) ds\,, \tag{1.15}$$

where the convex combination of a and b is given by $[a, b]_s = (1-s)a + sb$. Equality is achieved in (1.15) if and only if the distributional derivative of the transport map ψ'' is a constant function.

Optimal transport is a powerful tool for reducing functional inequalities onto pointwise inequalities (e.g. matrix inequalities). In other words, to pass from microscopic inequalities between particle locations to macroscopic inequalities involving densities. We highlight for example the seminal paper by McCann [65] where the displacement convexity issue for some energy functional is reduced to the concavity of the determinant. We also refer to the works of Barthe [3, 4] and Cordero-Erausquin et al. [47]. The previous lemma will allow us to connect microscopic to macroscopic inequalities by simple variations of the classical Jensen inequality.

1.3 Functional Inequalities

The first part of analysing the aggregation-diffusion equations (1.4) and (1.7) is devoted to the derivation of functional inequalities which are all variants of the Hardy-Littlewood-Sobolev (HLS) inequality also known as the weak Young's inequality [62, Theorem 4.3]:

$$\iint_{\mathbb{R}\times\mathbb{R}} f(x)|x-y|^k f(y)\,dxdy \le C_{HLS}(p,q,\lambda)\|f\|_{L^p}\|f\|_{L^q}\,, \tag{1.16}$$
$$\frac{1}{p}+\frac{1}{q}=2+k\,,\quad p,q>1\,,\quad k\in(-1,0)\,.$$

Theorem 1.3.1 (Variation of HLS) *Let $k\in(-1,0)$ and $m=1-k$. For $f\in L^1(\mathbb{R})\cap L^m(\mathbb{R})$, we have*

$$\left|\iint_{\mathbb{R}\times\mathbb{R}} f(x)|x-y|^k f(y)dxdy\right| \le C_*||f||_1^{1+k}||f||_m^m, \tag{1.17}$$

where $C_*=C_*(k)$ is the best constant.

Proof The inequality is a direct consequence of the standard HLS inequality (1.16) by choosing $p=q=\frac{2}{2+k}$, and of Hölder's inequality. For $k\in(-1,0)$ and for any $f\in L^1(\mathbb{R})\cap L^m(\mathbb{R})$, we have

$$\left|\iint_{\mathbb{R}\times\mathbb{R}} f(x)|x-y|^k f(y)dxdy\right| \le C_{HLS}||f||_p^2 \le C_{HLS}||f||_1^{1+k}||f||_m^m.$$

Consequently, C_* is finite and bounded from above by C_{HLS}. □

For instance inequality (1.17) is a consequence of interpolation between L^1 and L^m. We develop in this section another strategy which enables to recover inequality (1.17), as well as further variations which contain an additional quadratic confinement potential. This method involves two main ingredients:

- First it is required to know a priori that the inequality possesses some extremal function denoted e.g. by $\bar\rho(x)$ (characterised as a critical point of the energy functional). This is not an obvious task due to the intricacy of the equation satisfied by $\bar\rho(x)$. Without this a priori knowledge, the proof of the inequality remains incomplete. The situation is in fact similar to the case of convex functionals, where the existence of a critical point ensures that it is a global minimiser of the functional. The existence of optimisers was shown in [24].
- Second we invoke some simple lemma at the microscopic level. It is nothing but the Jensen's inequality for the case of inequality (1.17) (which is somehow degenerated). It is a variation of Jensen's inequality in the rescaled case.

1.3.1 Porous Medium Case $k < 0$

In the porous medium case, we have $k \in (-1, 0)$ and hence $m \in (1, 2)$. For $\chi = 0$, this corresponds to the well-studied porous medium equation (see [80] and references therein). It follows directly from Theorem 1.3.1, that for all $\rho \in \mathcal{Y}$ and for any $\chi > 0$,

$$\mathcal{F}_k[\rho] \geq \frac{1 - \chi C_*}{m - 1} ||\rho||_m^m ,$$

where $C_* = C_*(k)$ is the optimal constant defined in (1.17). Since global minimisers have always smaller or equal energy than stationary states, and stationary states have zero energy by Lemma 1.2.8, it follows that $\chi \geq 1/C_*$. We define the *critical interaction strength* by

$$\chi_c(k) := \frac{1}{C_*(k)} , \tag{1.18}$$

and so for $\chi = \chi_c(k)$, all stationary states of Eq. (1.4) are global minimisers of $\mathcal{F}_k$. From [24, Theorem 2.8], we further know that there exist global minimisers of $\mathcal{F}_k$ only for critical interaction strength $\chi = \chi_c(k)$ and they are radially symmetric non-increasing, compactly supported and uniformly bounded. Further, all minimisers of $\mathcal{F}_k$ are stationary states of Eq. (1.4).

From the above, we can also directly see that for $0 < \chi < \chi_c(k)$, no stationary states exist for Eq. (1.4). Further, there are no minimisers of $\mathcal{F}_k$. However, there exist global minimisers of the rescaled free energy $\mathcal{F}_{k,\mathbf{resc}}$ and they are radially symmetric non-increasing and uniformly bounded stationary states of the rescaled equation (1.7) [24, Theorem 2.9].

Theorem 1.3.2 *Let $k \in (-1, 0)$ and $m = 1 - k$. If (1.4) admits a stationary density $\bar{\rho}_k$ in $\mathcal{Y}$, then for any $\chi > 0$*

$$\mathcal{F}_k[\rho] \geq 0, \quad \forall \rho \in \mathcal{Y}$$

with the equality cases given dilations of $\bar{\rho}_k$. In other words, for critical interaction strength $\chi = \chi_c(k)$, inequality (1.17) holds true for all $f \in L^1(\mathbb{R}) \cap L^m(\mathbb{R})$.

Proof For a given stationary state $\bar{\rho}_k \in \mathcal{Y}$ and solution $\rho \in \mathcal{Y}$ of (1.4), we denote by ψ the convex function whose gradient pushes forward the measure $\bar{\rho}_k(a)da$ onto $\rho(x)dx$: $\psi' \# (\bar{\rho}_k(a)da) = \rho(x)dx$. Using (1.14), the functional $\mathcal{F}_k[\rho]$ rewrites as follows:

$$\mathcal{F}_k[\rho] = \frac{1}{m-1} \int_{\mathbb{R}} \left(\frac{\bar{\rho}_k(a)}{\psi''_{ac}(a)} \right)^{m-1} \bar{\rho}_k(a)\, da + \frac{\chi}{k} \iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\psi'(a) - \psi'(b)}{a - b} \right)^k |a - b|^k \bar{\rho}_k(a)\bar{\rho}_k(b)\, dadb$$

$$= \frac{1}{m-1}\int_{\mathbb{R}} \left(\psi''_{ac}(a)\right)^{1-m} \bar\rho_k(a)^m\, da$$
$$+ \frac{\chi}{1-m}\iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\psi'(a)-\psi'(b)}{a-b}\right)^{1-m} |a-b|^{1-m}\bar\rho_k(a)\bar\rho_k(b)\, dadb\,,$$

where ψ' non-decreasing. By Lemma 1.2.8(i), we can write for any $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}} (\psi''_{ac}(a))^{-\gamma}\bar\rho_k(a)^m\, da = \chi \iint_{\mathbb{R}\times\mathbb{R}} \left\langle \psi''_{ac}([a,b])^{-\gamma}\right\rangle |a-b|^{1-m}\bar\rho_k(a)\bar\rho_k(b)\, dadb\,,$$

where

$$\left\langle u([a,b])\right\rangle = \int_0^1 u([a,b]_s)\, ds$$

and $[a,b]_s = (1-s)a + sb$ for any $a, b \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}_+$. Hence, choosing $\gamma = m-1$,

$$\mathcal{F}_k[\rho] = \frac{\chi}{m-1}\iint_{\mathbb{R}\times\mathbb{R}} \left\{ \left\langle \psi''_{ac}([a,b])^{1-m}\right\rangle - \left(\frac{\psi'(a)-\psi'(b)}{a-b}\right)^{1-m}\right\}$$
$$\times |a-b|^{1-m}\bar\rho_k(a)\bar\rho_k(b)\, dadb\,.$$

Using the strict concavity and increasing character of the power function $-(\cdot)^{1-m}$ and Lemma 1.2.10, we deduce $\mathcal{F}_k[\rho] \geq 0$. Equality arises if and only if the derivative of the transport map ψ'' is a constant function, i.e. when ρ is a dilation of $\bar\rho_k$.

We conclude that if (1.4) admits a stationary state $\bar\rho_k \in \mathcal{Y}$, then $\mathcal{F}_k(\rho) \geq 0$ for any $\rho \in \mathcal{Y}$. This functional inequality is equivalent to (1.17) if we choose $\chi = \chi_c(k)$. □

Remark 1.3.1 (Comments on the Inequality Proof) In the case of critical interaction strength $\chi = \chi_c(k)$, Theorem 1.3.2 provides an alternative proof for the variant of the HLS inequality Theorem 1.3.1 assuming the existence of a stationary density for (1.4). More precisely, the inequalities $\mathcal{F}_k[\rho] \geq 0$ and (1.17) are equivalent if $\chi = \chi_c(k)$. However, the existence proof [24, Proposition 3.4] crucially uses the HLS type inequality (1.17). If we were able to show the existence of a stationary density by alternative methods, e.g. fixed point arguments, we would obtain a full alternative proof of inequality (1.17).

Remark 1.3.2 (Logarithmic Case) There are no global minimisers of $\mathcal{F}_0$ in the logarithmic case $k = 0$, $m = 1$ except for critical interaction strength $\chi = 1$. To see this, note that the characterisation of stationary states [19, Lemma 2.3] which corresponds to Lemma 1.2.8(i) for the case $k \neq 0$, holds true for any $\chi > 0$. Similarly, the result that the existence of a stationary state $\bar\rho$ implies the inequality $\mathcal{F}_0[\rho] > \mathcal{F}_0[\bar\rho]$ [19, Theorem 1.1] holds true for any $\chi > 0$, and corresponds

to Theorem 1.3.2 in the case $k \neq 0$. Taking dilations of Cauchy's density (1.3), $\rho_\lambda(x) = \lambda\bar{\rho}_0(\lambda x)$, we have $\mathcal{F}_0[\rho_\lambda] = (1-\chi)\log\lambda + \mathcal{F}_0[\bar{\rho}_0]$, and letting $\lambda \to \infty$ for super-critical interaction strengths $\chi > 1$, we see that $\mathcal{F}_0$ is not bounded below. Similarly, for sub-critical interaction strengths $0 < \chi < 1$, we take the limit $\lambda \to 0$ to see that $\mathcal{F}_0$ is not bounded below. Hence, there are no global minimisers of $\mathcal{F}_0$ and also no stationary states (by equivalence of the two) except if $\chi = 1$.

Further, we obtain the following uniqueness result:

Corollary 1.3.3 (Uniqueness in the Critical Case) *Let $k \in (-1,0)$ and $m = 1-k$. If $\chi = \chi_c(k)$, then there exists a unique stationary state (up to dilations) to Eq. (1.4), with second moment bounded, and a unique minimiser (up to dilations) for $\mathcal{F}_k$ in $\mathcal{Y}$.*

Proof By Calvez et al. [24, Theorem 2.8], there exists a minimiser of $\mathcal{F}_k$ in $\mathcal{Y}$, which is a stationary state of Eq. (1.4). Assume (1.4) admits two stationary states $\bar{\rho}_1$ and $\bar{\rho}_2$. By Lemma 1.2.8, $\mathcal{F}_k[\bar{\rho}_1] = \mathcal{F}_k[\bar{\rho}_2] = 0$. It follows from Theorem 1.3.2 that $\bar{\rho}_1$ and $\bar{\rho}_2$ are dilations of each other. □

A functional inequality similar to (1.17) holds true for sub-critical interaction strengths in rescaled variables:

Theorem 1.3.4 (Rescaled Variation of HLS) *For any $\chi > 0$, let $k \in (-1,0)$ and $m = 1-k$. If $\bar{\rho}_{k,\mathbf{resc}} \in \mathcal{Y}_2$ is a stationary state of (1.7), then we have for any solution $\rho \in \mathcal{Y}_2$,*

$$\mathcal{F}_{k,\mathbf{resc}}[\rho] \geq \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_{k,\mathbf{resc}}]$$

with the equality cases given by $\rho = \bar{\rho}_{k,\mathbf{resc}}$.

The proof is based on two lemmatas: the characterisation of steady states Lemma 1.2.8 and a microscopic inequality. The difference with the critical case lies in the nature of this microscopic inequality: Jensen's inequality needs to be replaced here as homogeneity has been broken. To simplify the notation, we denote by $u_{ac}(s) := \psi''_{ac}([a,b]_s)$ as above with $[a,b]_s := (1-s)a + sb$ for any $a, b \in \mathbb{R}$. We also introduce the notation

$$\langle u \rangle := \frac{\psi'(a) - \psi'(b)}{a-b} = \int_0^1 \psi''([a,b]_s)\, ds$$

with $u(s) := \psi''([a,b]_s)$. Both notations coincide when ψ'' has no singular part. Note there is a little abuse of notation since ψ'' is a measure and not a function, but this notation allows us for simpler computations below.

Lemma 1.3.5 *Let $\alpha, \beta > 0$ and $m > 1$. For any $a, b \in \mathbb{R}$ and any convex function $\psi : \mathbb{R} \to \mathbb{R}$:*

$$\alpha\langle\psi''([a,b])\rangle^{1-m} + \beta(1-m)\langle\psi''([a,b])\rangle^2 \leq (\alpha+2\beta)\langle\left(\psi''_{ac}([a,b])\right)^{1-m}\rangle - \beta(m+1), \tag{1.19}$$

where equality arises if and only if $\psi'' \equiv 1$ a.e.

Proof We have again by Lemma 1.2.10,

$$(\alpha + 2\beta)\langle u\rangle^{1-m} \leq (\alpha + 2\beta)\langle u_{ac}^{1-m}\rangle,$$

thus

$$\alpha\langle u\rangle^{1-m} + \beta(1-m)\langle u\rangle^2 \leq (\alpha + 2\beta)\langle u_{ac}^{1-m}\rangle - \beta\left[2\langle u\rangle^{1-m} + (m-1)\langle u\rangle^2\right].$$

We conclude since the quantity in square brackets verifies

$$\forall X > 0 : \ 2X^{1-m} + (m-1)X^2 \geq m + 1\,.$$

Equality arises if and only if u is almost everywhere constant and $\langle u\rangle = 1$. □

Proof (Proof of Theorem 1.3.4) We denote by $\bar\rho = \bar\rho_{k,\mathbf{resc}} \in \mathcal{Y}_2$ a stationary state of (1.7) for the sake of clarity. Then for any solution $\rho \in \mathcal{Y}_2$ of (1.7), there exists a convex function ψ whose gradient pushes forward the measure $\bar\rho(a)da$ onto $\rho(x)dx$,

$$\psi'\#\left(\bar\rho(a)da\right) = \rho(x)dx.$$

Similarly to the proof of Theorem 1.3.2, the functional $\mathcal{F}_{k,\mathbf{resc}}[\rho]$ rewrites as follows:

$$\begin{aligned}\mathcal{F}_{k,\mathbf{resc}}[\rho] = {} & \frac{1}{m-1}\int_{\mathbb{R}} (\psi''_{ac}(a))^{1-m}\bar\rho(a)^m\,da \\ & + \frac{\chi}{k}\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\psi'(a)-\psi'(b)}{a-b}\right)^k |a-b|^k\bar\rho(a)\bar\rho(b)\,dadb \\ & + \frac{1}{4}\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\psi'(a)-\psi'(b)}{a-b}\right)^2 |a-b|^2\bar\rho(a)\bar\rho(b)\,dadb\,.\end{aligned}$$

From the characterisation of steady states Lemma 1.2.8(ii), we know that for all $\gamma \in \mathbb{R}$:

$$\begin{aligned}\int_{\mathbb{R}} (\psi''_{ac}(a))^{-\gamma}\bar\rho(a)^m\,da = {} & \iint_{\mathbb{R}\times\mathbb{R}}\left\langle\psi''_{ac}([a,b])^{-\gamma}\right\rangle\left(\chi|a-b|^{1-m} + \frac{|a-b|^2}{2}\right) \\ & \times\bar\rho(a)\bar\rho(b)\,dadb\,.\end{aligned}$$

Choosing $\gamma = m-1$, we can rewrite the energy functional as

$$\begin{aligned}(m-1)\mathcal{F}_{k,\mathbf{resc}}[\rho] = {} & \iint_{\mathbb{R}\times\mathbb{R}}\left\langle\psi''_{ac}([a,b])^{1-m}\right\rangle\left(\chi|a-b|^{1-m} + \frac{|a-b|^2}{2}\right)\bar\rho(a)\bar\rho(b)\,dadb \\ & - \iint_{\mathbb{R}\times\mathbb{R}}\left(\langle\psi''([a,b])\rangle^{1-m}\chi|a-b|^{1-m}\right.\end{aligned}$$

$$+\big(\psi''([a,b])\big)^2(1-m)\frac{|a-b|^2}{4}\Big)\,\bar\rho(a)\bar\rho(b)\,dadb$$
$$\geq (m+1)\iint_{\mathbb{R}\times\mathbb{R}}\frac{|a-b|^2}{4}\bar\rho(a)\bar\rho(b)\,dadb$$
$$=\frac{m+1}{2}\int_{\mathbb{R}}|a|^2\bar\rho(a)\,da=(m-1)\mathcal{F}_{k,\mathbf{resc}}[\bar\rho]\,.$$

Here, we use the variant of Jensen's inequality (1.19) and for the final step, identity (1.11). Again equality holds true if and only if ψ'' is identically one. □

Remark 1.3.3 (New Inequality) Up to our knowledge, the functional inequality in Theorem 1.3.2 is not known in the literature. Theorem 1.3.4 makes a connection between Eq. (1.7) and this new general functional inequality by showing that stationary states of the rescaled equation (1.7) correspond to global minimisers of the free energy functional $\mathcal{F}_{k,\mathbf{resc}}$. The converse was shown in [24, Theorem 2.9].
As a direct consequence of Theorem 1.3.4 and the scaling given by (1.8), we obtain the following corollaries:

Corollary 1.3.6 (Uniqueness in the Sub-Critical Case) *Let $k\in(-1,0)$ and $m=1-k$. If $0<\chi<\chi_c(k)$, then there exists a unique stationary state with second moment bounded to the rescaled equation (1.7), and a unique minimiser for $\mathcal{F}_{k,\mathbf{resc}}$ in $\mathcal{Y}_2$.*

Proof By Calvez et al. [24, Theorem 2.9], there exists a minimiser of $\mathcal{F}_{k,\mathbf{resc}}$ in $\mathcal{Y}_2$ for sub-critical interaction strengths $0<\chi<\chi_c(k)$, which is a stationary state of Eq. (1.7). Assume (1.7) admits two stationary states $\bar\rho_1$ and $\bar\rho_2$. By Theorem 1.3.4, $\mathcal{F}_{k,\mathbf{resc}}[\bar\rho_1]=\mathcal{F}_{k,\mathbf{resc}}[\bar\rho_2]$ and it follows that $\bar\rho_1$ and $\bar\rho_2$ are dilations of each other. □

Corollary 1.3.7 (Self-Similar Profiles) *For $0<\chi<\chi_c(k)$, let $k\in(-1,0)$ and $m=1-k$. There exists a unique (up to dilations) self-similar solution ρ to (1.4) given by*

$$\rho(t,x)=((2-k)t+1)^{\frac{1}{k-2}}\;u\Big(((2-k)t+1)^{\frac{1}{k-2}}\,x\Big)\,,$$

where u is the unique minimiser of $\mathcal{F}_{k,\mathbf{resc}}$ in $\mathcal{Y}_2$.

Corollary 1.3.8 (Non-existence Super-Critical and Critical Case)

(i) If $\chi>\chi_c(k)$, there are no stationary states of Eq. (1.4) in $\mathcal{Y}$, and the free energy functional $\mathcal{F}_k$ does not admit minimisers in $\mathcal{Y}$.
(ii) If $\chi\geq\chi_c(k)$, there are no stationary states of the rescaled equation (1.7) in $\mathcal{Y}_2$, and the rescaled free energy functional $\mathcal{F}_{k,\mathbf{resc}}$ does not admit minimisers in $\mathcal{Y}_2$.

Proof For critical $\chi_c(k)$, there exists a minimiser $\bar{\rho} \in \mathcal{Y}$ of $\mathcal{F}_k$ by Calvez et al. [24, Theorem 2.8], which is a stationary state of Eq. (1.4) by Calvez et al. [24, Theorem 3.14]. For $\chi > \chi_c(k)$, we have

$$\mathcal{F}_k[\bar{\rho}] = \mathcal{U}_m[\bar{\rho}] + \chi \mathcal{W}_k[\bar{\rho}] < \mathcal{U}_m[\bar{\rho}] + \chi_c(k)\mathcal{W}_k[\bar{\rho}] = 0$$

since stationary states have zero energy by Lemma 1.2.8(i). However, by Theorem 1.3.2, if there exists a stationary state for $\chi > \chi_c(k)$, then all $\rho \in \mathcal{Y}$ satisfy $\mathcal{F}_k[\rho] \geq 0$, which contradicts the above. Therefore, the assumptions of the theorem cannot hold and so there are no stationary states in original variables. Further, taking dilations $\rho_\lambda(x) = \lambda \bar{\rho}(\lambda x)$, we have $\mathcal{F}_k[\rho_\lambda] = \lambda^{-k}\mathcal{F}_k[\bar{\rho}] < 0$, and letting $\lambda \to \infty$, we see that $\inf_{\rho \in \mathcal{Y}} \mathcal{F}_k[\rho] = -\infty$, and so (i) follows.

In order to prove (ii), observe that the minimiser $\bar{\rho}$ for critical $\chi = \chi_c(k)$ is in $\mathcal{Y}_2$ as it is compactly supported [24, Corollary 3.9]. We obtain for the rescaled free energy of its dilations

$$\mathcal{F}_{k,\mathbf{resc}}[\rho_\lambda] = \lambda^{-k}\mathcal{F}_k[\bar{\rho}] + \frac{\lambda^{-2}}{2}\mathcal{V}[\bar{\rho}] \to -\infty\,, \qquad \text{as } \lambda \to \infty\,.$$

Hence, $\mathcal{F}_{k,\mathbf{resc}}$ is not bounded below in $\mathcal{Y}_2$. Similarly, for $\chi = \chi_c(k)$,

$$\mathcal{F}_{k,\mathbf{resc}}[\rho_\lambda] = \frac{\lambda^{-2}}{2}\mathcal{V}[\bar{\rho}] \to 0\,, \qquad \text{as } \lambda \to \infty\,,$$

and so for a minimiser $\tilde{\rho} \in \mathcal{Y}_2$ to exist, it should satisfy $\mathcal{F}_{k,\mathbf{resc}}[\tilde{\rho}] \leq 0$. However, it follows from Theorem 1.3.1 that $\mathcal{F}_{k,\mathbf{resc}}[\rho] \geq \frac{1}{2}\mathcal{V}[\rho] > 0$ for any $\rho \in \mathcal{Y}_2$, and therefore, $\mathcal{F}_{k,\mathbf{resc}}$ does not admit minimisers in $\mathcal{Y}_2$ for $\chi = \chi_c(k)$.
Further, if Eq. (1.7) admitted stationary states in $\mathcal{Y}_2$ for any $\chi \geq \chi_c(k)$, then they would be minimisers of $\mathcal{F}_{k,\mathbf{resc}}$ by Theorem 1.3.4, which contradicts the non-existence of minimisers. □

Remark 1.3.4 (Linearisation Around the Stationary Density) We linearise the functional $\mathcal{F}_k$ around the stationary distribution $\bar{\rho}_k$ of Eq. (1.4). For the perturbed measure $\mu_\varepsilon = (\mathrm{id} + \varepsilon\eta')\#\bar{\mu}_k$, with $d\bar{\mu}_k(x) = \bar{\rho}_k(x)\,dx$ and $d\mu_\varepsilon(x) = \rho_\varepsilon(x)\,dx$, we have

$$\begin{aligned}
\mathcal{F}_k[\rho_\varepsilon] = \frac{\varepsilon^2}{2} m \Bigg[& \int_{\mathbb{R}} \eta''(a)^2 \bar{\rho}_k(a)^m \, da \\
& - \chi_c(k) \iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\eta'(a) - \eta'(b)}{a-b} \right)^2 |a-b|^{1-m} \bar{\rho}_k(a)\bar{\rho}_k(b)\, dadb \Bigg] + o(\varepsilon^2) \\
= \frac{\varepsilon^2}{2} m \chi_c(k) & \iint_{\mathbb{R}\times\mathbb{R}} \left\{ \left\langle \eta''([a,b])^2 \right\rangle - \left\langle \eta''([a,b]) \right\rangle^2 \right\} |a-b|^{1-m} \bar{\rho}_k(a)\bar{\rho}_k(b)\, dadb \\
& + o(\varepsilon^2)\,.
\end{aligned}$$

We define the local oscillations (in L^2) of functions over intervals as

$$\mathrm{osc}_{(a,b)}(v) := \int_{t=0}^{1} \left\{ v([a,b]_t) - \left\langle v([a,b]) \right\rangle \right\}^2 \, dt \geq 0 \,.$$

The Hessian of the functional $\mathcal{F}_k$ evaluated at the stationary density $\bar{\rho}_k$ then reads

$$D^2\mathcal{F}_k[\bar{\rho}_k](\eta,\eta) = m\chi_c(k) \iint_{\mathbb{R}\times\mathbb{R}} \mathrm{osc}_{(a,b)}(\eta'')|a-b|^{1-m}\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \geq 0.$$

Similarly, we obtain for the rescaled free energy

$$\begin{aligned}
\mathcal{F}_{k,\mathbf{resc}}[\rho_\varepsilon] &= \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_k] + \frac{\varepsilon^2}{2} m \int_{\mathbb{R}} \eta''(a)^2 \bar{\rho}_k(a)^m \, da \\
&\quad - \frac{\varepsilon^2}{2} m\chi \iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\eta'(a)-\eta'(b)}{a-b} \right)^2 |a-b|^{1-m} \bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \\
&\quad + \frac{\varepsilon^2}{4} \iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\eta'(a)-\eta'(b)}{a-b} \right)^2 |a-b|^2 \bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb + o(\varepsilon^2) \\
&= \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_k] \\
&\quad + \frac{\varepsilon^2}{2} \left[m\chi \iint_{\mathbb{R}\times\mathbb{R}} \left\{ \left\langle \eta''([a,b])^2 \right\rangle - \left\langle \eta''([a,b]) \right\rangle^2 \right\} |a-b|^{1-m} \bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \right. \\
&\quad \left. + \iint_{\mathbb{R}\times\mathbb{R}} \left\{ \frac{m}{2} \left\langle \eta''([a,b])^2 \right\rangle + \frac{1}{2} \left\langle \eta''([a,b]) \right\rangle^2 \right\} |a-b|^2 \bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \right] \\
&\quad + o(\varepsilon^2)
\end{aligned}$$

to finally conclude

$$\begin{aligned}
\mathcal{F}_{k,\mathbf{resc}}[\rho_\varepsilon] &= \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_k] \\
&\quad + \frac{\varepsilon^2}{2} \left[\iint_{\mathbb{R}\times\mathbb{R}} \mathrm{osc}_{(a,b)}(\eta'') \left(m\chi|a-b|^{1-m} + \frac{m}{2}|a-b|^2 \right) \bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \right. \\
&\quad \left. + \frac{m+1}{2} \iint_{\mathbb{R}\times\mathbb{R}} \left(\eta'(a)-\eta'(b) \right)^2 \bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \right] + o(\varepsilon^2) \,,
\end{aligned}$$

and hence, the Hessian evaluated at the stationary state $\bar{\rho}_k$ of (1.7) is given by the expression

$$\begin{aligned}
D^2\mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_k](\eta,\eta) &= \iint_{\mathbb{R}\times\mathbb{R}} \mathrm{osc}_{(a,b)}(\eta'') \left(m\chi|a-b|^{1-m} + \frac{m}{2}|a-b|^2 \right) \bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \\
&\quad + (m+1) \int_{\mathbb{R}} \eta'(a)^2 \bar{\rho}_k(a)\,da \geq 0 \,.
\end{aligned}$$

We have naturally that the functional $\mathcal{F}_{k,\mathbf{resc}}$ is locally uniformly convex, with the coercivity constant $m+1$. However, the local variations of $\mathcal{F}_{k,\mathbf{resc}}$ can be large in the directions where the Brenier's map η is large in the $\mathcal{C}^3$ norm. Interestingly enough the coercivity constant does not depend on χ, even in the limit $\chi \nearrow \chi_c(k)$.

1.3.2 Fast Diffusion Case $k > 0$

Not very much is known about the fast diffusion case where $k \in (0,1)$ and hence $m = 1-k \in (0,1)$, that is diffusion is fast in regions where the density of particles is low. In [24], we showed that Eq. (1.4) has no radially symmetric non-increasing stationary states with kth moment bounded, and there are no radially symmetric non-increasing global minimisers for the energy functional $\mathcal{F}_k$ for any $\chi > 0$. By Calvez et al. [24, Theorem 2.11], there exists a continuous radially symmetric non-increasing stationary state of the rescaled equation (1.7) for all $\chi > 0$. In this sense, there is no criticality for the parameter χ. We provide here a full proof of non-criticality by optimal transport techniques involving the analysis of the minimisation problem in rescaled variables, showing that global minimisers exist in the right functional spaces for all values of the critical parameter and that they are indeed stationary states—as long as diffusion is not too fast. More precisely, global minimisers with finite energy $\mathcal{F}_{k,\mathbf{resc}}$ can only exist in the range $0 < k < \frac{2}{3}$, that is $\frac{1}{3} < m < 1$ [24]. This restriction is exactly what we would expect looking at the behaviour of the fast diffusion equation ($\chi = 0$) [79]. In particular, for $k \in (0,1)$ and $m = 1-k \in (0,1)$, radially symmetric non-increasing stationary states, if they exist, are integrable and have bounded kth moment [24, Remarks 4.6 and 4.9]. By Calvez et al. [24, Remark 4.11] however, their second moment is bounded and $\rho^m \in L^1(\mathbb{R})$ if and only if $k < 2/3$, in which case they belong to $\mathcal{Y}_2$ and their rescaled free energy is finite. This restriction corresponds to $\frac{1}{3} < m < 1$ and coincides with the regime of the one-dimensional fast diffusion equation ($\chi = 0$) where the Barenblatt profile has second moment bounded and its mth power is integrable [16]. Intuitively, adding attractive interaction to the dynamics helps to counteract the escape of mass to infinity. However, the quadratic confinement due to the rescaling of the fast-diffusion equation is already stronger than the additional attractive force since $k < 2$ and hence, we expect that the behaviour of the tails is dominated by the non-linear diffusion effects even for $\chi > 0$ as for the classical fast-diffusion equation.

Using completely different methods, the non-criticality of χ has also been observed in [45, 46] for the limiting case in one dimension taking $m = 0$, corresponding to logarithmic diffusion, and $k = 1$. The authors showed that solutions to (1.4) with $(m = 0, k = 1)$ are globally defined in time for all values of the parameter $\chi > 0$.

In order to establish equivalence between global minimisers and stationary states in one dimension, we prove a type of reversed HLS inequality providing a bound

on $\int \rho^m$ in terms of the interaction term $\int (W_k * \rho)\rho$. The inequality gives a lower bound on the rescaled energy $\mathcal{F}_{k,\mathbf{resc}}$:

Theorem 1.3.9 *Let $k \in (0,1)$, $m = 1-k$ and $\chi > 0$. Then $\bar{\rho} \in \mathcal{Y}_{2,k}$ is a stationary state of (1.7) if and only if for any solution $\rho \in \mathcal{Y}_{2,k}$ we have the inequality*

$$\mathcal{F}_{k,\mathbf{resc}}[\rho] \geq \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}]$$

with the equality cases given by $\rho = \bar{\rho}$.

The above theorem implies that stationary states in $\mathcal{Y}_{2,k}$ of the rescaled equation (1.7) are minimisers of the rescaled free energy $\mathcal{F}_{k,\mathbf{resc}}$. Since the converse is true by Calvez et al. [24, Theorem 2.11], it allows us to establish equivalence between stationary states of (1.7) and minimisers of $\mathcal{F}_{k,\mathbf{resc}}$. To prove Theorem 1.3.9, we need a result similar to Lemma 1.3.5:

Lemma 1.3.10 *Let $\alpha, \beta > 0$ and $m \in (0,1)$. For any $a, b \in \mathbb{R}$ and any convex function $\psi : \mathbb{R} \to \mathbb{R}$:*

$$(\alpha+\beta)\left\langle \left(\psi''_{ac}([a,b])\right)^{1-m}\right\rangle \leq \alpha\left\langle \psi''([a,b])\right\rangle^{1-m} + \frac{\beta(1-m)}{2}\left\langle \psi''([a,b])\right\rangle^2 + \frac{\beta(m+1)}{2}, \tag{1.20}$$

where equality arises if and only if $\psi'' \equiv 1$ a.e.

Proof Denote $u(s) := \psi''([a,b]_s)$ with $[a,b]_s := (1-s)a + sb$ and we write u_{ac} for the absolutely continuous part of u. We have by Lemma 1.2.10,

$$(\alpha+\beta)\left\langle u_{ac}^{1-m}\right\rangle \leq (\alpha+\beta)\left\langle u\right\rangle^{1-m}.$$

Further by direct inspection,

$$\forall X > 0: \ \frac{1}{m-1}X^{1-m} + \frac{1}{2}X^2 \geq \frac{m+1}{2(m-1)},$$

thus

$$(\alpha+\beta)\left\langle u_{ac}\right\rangle^{1-m} \leq \alpha\left\langle u\right\rangle^{1-m} + \frac{\beta(1-m)}{2}\left\langle u\right\rangle^2 + \frac{\beta(m+1)}{2}$$

and equality arises if and only if u is almost everywhere constant and $\left\langle u\right\rangle = 1$. □

Proof (Proof of Theorem 1.3.9) For a stationary state $\bar{\rho} \in \mathcal{Y}_{2,k}$ and any solution $\rho \in \mathcal{Y}_{2,k}$ of (1.7), there exists a convex function ψ whose gradient pushes forward the measure $\bar{\rho}(a)da$ onto $\rho(x)dx$

$$\psi' \# (\bar{\rho}(a)da) = \rho(x)dx.$$

From characterisation (1.10) we have for any $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}} \left(\psi''_{ac}(t,a)\right)^{-\gamma} \bar\rho_k(a)^m \, da = \iint_{\mathbb{R}\times\mathbb{R}} \left(\chi|a-b|^{1-m} + \frac{|a-b|^2}{2}\right) \times \left\langle \psi''_{ac}(t,(a,b))^{-\gamma}\right\rangle \bar\rho_k(a)\bar\rho_k(b)\, dadb\,.$$

Choosing $\gamma = m-1$, the functional $\mathcal{F}_{k,\mathbf{resc}}[\rho]$ rewrites similarly to the proof of Theorem 1.3.4:

$$\begin{aligned}
\mathcal{F}_{k,\mathbf{resc}}[\rho] &= \frac{1}{m-1}\int_{\mathbb{R}} (\psi''_{ac}(a))^{1-m}\bar\rho(a)^m\, da \\
&\quad + \frac{\chi}{1-m}\iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\psi'(a)-\psi'(b)}{a-b}\right)^{1-m} |a-b|^{1-m}\bar\rho(a)\bar\rho(b)\, dadb \\
&\quad + \frac{1}{4}\iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\psi'(a)-\psi'(b)}{a-b}\right)^{2} |a-b|^{2}\bar\rho(a)\bar\rho(b)\, dadb \\
&= \frac{1}{m-1}\iint_{\mathbb{R}\times\mathbb{R}} \left\langle \psi''_{ac}([a,b])^{1-m}\right\rangle \left(\chi|a-b|^{1-m} + \frac{|a-b|^2}{2}\right)\bar\rho(a)\bar\rho(b)\, dadb \\
&\quad - \frac{1}{m-1}\iint_{\mathbb{R}\times\mathbb{R}} \Big(\left\langle \psi''([a,b])\right\rangle^{1-m}\chi|a-b|^{1-m} \\
&\quad + \left\langle \psi''([a,b])\right\rangle^{2}(1-m)\frac{|a-b|^2}{4}\Big)\bar\rho(a)\bar\rho(b)\, dadb
\end{aligned}$$

Now, using the variant of Jensen's inequality (1.20) of Lemma 1.3.10, this simplifies to

$$\begin{aligned}
\mathcal{F}_{k,\mathbf{resc}}[\rho] &\geq \frac{m+1}{m-1}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|a-b|^2}{4}\bar\rho(a)\bar\rho(b)\, dadb \\
&= \frac{m+1}{2(m-1)}\int_{\mathbb{R}} |a|^2\bar\rho(a)\, da = \mathcal{F}_{k,\mathbf{resc}}[\bar\rho]\,.
\end{aligned}$$

Here, we used identity (1.11) for the final step. Again equality holds true if and only if ψ'' is identically one. □

Remark 1.3.5 (Sign of the Rescaled Free Energy) In fact, $\mathcal{F}_{k,\mathbf{resc}}[\bar\rho] \leq 0$. Choosing $\rho_\lambda(x) = \lambda\bar\rho(\lambda x)$ a dilation of the stationary state, we obtain thanks to the homogeneity properties of the energy functional,

$$\lambda^{-k}\mathcal{U}_m[\bar\rho] + \lambda^{-k}\mathcal{W}_k[\bar\rho] + \lambda^{-2}\mathcal{V}[\bar\rho] = \mathcal{F}_{k,\mathbf{resc}}[\rho_\lambda] \geq \mathcal{F}_{k,\mathbf{resc}}[\bar\rho],$$

and so we conclude that $\mathcal{F}_{k,\mathbf{resc}}[\bar\rho]$ must be non-positive for any stationary state $\bar\rho \in \mathcal{Y}_2$ by taking the limit $\lambda \to \infty$.

Corollary 1.3.11 (Uniqueness) *Let $k \in \left(0, \frac{2}{3}\right)$ and $m = 1-k$. For any $\chi > 0$, there exists a unique stationary state with second and kth moment bounded to Eq. (1.7), and a unique minimiser for $\mathcal{F}_{k,\mathbf{resc}}$ in $\mathcal{Y}_{2,k}$.*

Proof By Calvez et al. [24, Theorem 2.11], there exists a minimiser of $\mathcal{F}_{k,\mathbf{resc}}$ in $\mathcal{Y}_{2,k}$, which is a stationary state of Eq. (1.7). Assume (1.7) admits two stationary states $\bar{\rho}_1$ and $\bar{\rho}_2$ in $\mathcal{Y}_{2,k}$. By Theorem 1.3.9, $\mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_1] = \mathcal{F}_{k,\mathbf{resc}}[\bar{\rho}_2]$ and so $\bar{\rho}_1 = \bar{\rho}_2$. □

Corollary 1.3.12 (Self-Similar Profiles) *Let $k \in (0,1)$ and $m = 1-k$. For any $\chi > 0$, if u is a symmetric stationary state of the rescaled equation (1.7), then there exists a self-similar solution to (1.4) given by*

$$\rho(t,x) = ((2-k)t+1)^{\frac{1}{k-2}}\, u\left(((2-k)t+1)^{\frac{1}{k-2}}\, x\right).$$

1.4 Long-Time Asymptotics

This part is devoted to the asymptotic behaviour of solutions, adapting the above computations, ensuring e.g. uniqueness of the functional ground state, at the level of the gradient flow dynamics. We will demonstrate convergence towards these ground states in Wasserstein distance under certain conditions, in some cases with an explicit rate. Our results rely on the fact that there is a simple expression for the Wasserstein distance in one dimension. Therefore, our methodology cannot be extended to dimension two or more so far except possibly under radial symmetry assumptions, which we would like to explore in future work.

We assume here that solutions are smooth enough so that the operations in this section are well-defined. Firstly, we require the mean-field potential gradient $\partial_x S_k(t,x)$ to be well-defined for all $t > 0$ which is guaranteed if $\rho(t,x)$ has at least the same regularity at each time $t > 0$ as provided by Definition 1.2.1 for stationary states. From now on, we assume that solutions of (1.4) satisfy $\rho(t,x) \in C\left([0,T], C^{0,\alpha}_{loc}(\mathbb{R}) \cap \mathcal{Y} \cap L^\infty(\mathbb{R})\right)$ with $\alpha \in (-k, 1)$.

Secondly, certain computations in this section remain formal unless the convex Brenier map ψ satisfying $\rho(t,x)dx = \partial_x \psi(t,x) \# \bar{\rho}_k(x)dx$ is regular enough. In the fast diffusion regime $k > 0$, stationary states are everywhere positive [24], and thus ψ'' is absolutely continuous. However, in the porous medium regime $k < 0$, stationary states are compactly supported [24], and therefore, the following computations remain formal depending on the regularity and properties of the solutions of the evolution problem. From now on, we assume that ψ'' is absolutely continuous whenever we talk about solutions of the evolution problems (1.4) or (1.7).

1.4.1 Porous Medium Asymptotics

1.4.1.1 The Critical Case $\chi = \chi_c(k)$

In the critical case, the set of global minimisers coincides with the set of stationary states of Eq. (1.4) [24, Theorem 2.8], but as we will see, it is not clear whether this set is a global attractor in the Wasserstein sense or not. We will prove here a convergence result under some conditions, which provides a dynamical proof of uniqueness up to dilations. Recall that in the fair-competition regime, we have $\mathcal{F}_k[\rho_\lambda] = \lambda^{-k}\mathcal{F}_k[\rho]$ for any dilation $\rho_\lambda(x) = \lambda\rho(\lambda x)$, $\lambda \in \mathbb{R}$ of a density $\rho \in \mathcal{Y}$, and so every stationary state provides in fact a family of stationary states by scale invariance. Given a density $\rho \in \mathcal{Y}$, $|x|^2\rho(x) \in L^1_+(\mathbb{R})$, we define the rescaling ρ_1 by

$$\rho_1(x) := \sigma\rho(\sigma x)\,, \quad \sigma^2 = \mathcal{V}[\rho] = \int_{\mathbb{R}} |x|^2\rho(x)\,dx\,, \tag{1.21}$$

and so any stationary state $\bar\rho_k$ with finite second moment has a dilation $\bar\rho_{k,1}$ with normalised second moment $\mathcal{V}[\bar\rho_{k,1}] = 1$. In particular, $\bar\rho_{k,1}$ provides a convenient representative for the family of stationary states formed by dilations of $\bar\rho_k$. Our aim here is to show that although uniqueness is degenerate due to homogeneity, we have a unique representative $\bar\rho_{k,1}$ with second moment equal to one. We will present here a discussion of partial results and open questions around the long-time behaviour of solutions in the critical case.

We first recall the logarithmic case ($m = 1, k = 0$), where the ground state is explicitly given by Cauchy's density $\bar\rho_0$ (1.3). The second momentum is thus infinite, and the Wasserstein distance to some ground state cannot be finite if the initial datum has finite second momentum. For a $\rho(t)$ satisfying (1.4), we have the estimate [19]

$$\frac{d}{dt}\mathbf{W}(\rho(t), \bar\rho_0)^2 \le 0\,,$$

where equality holds if and only if $\rho(t)$ is a dilation of $\bar\rho_0$. This makes sense only if $\rho(0)$ has infinite second momentum, and is at finite distance from one of the equilibrium configurations. Notice that possible ground states (dilations of Cauchy's density) are all infinitely far from each other with respect to the Wasserstein distance,

$$\mathbf{W}\left(\rho_{\lambda_1}, \rho_{\lambda_2}\right)^2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1\lambda_2}\mathcal{V}[\bar\rho_0] = \infty.$$

Dynamics have been described in [11] when the initial datum has finite second momentum: the solution converges to a Dirac mass as time goes to $+\infty$. However, this does not hold true in the porous medium case $k \in (-1, 0)$, $m = 1 - k$, since stationary states are compactly supported by Calvez et al. [24, Corollary 3.9]. The

case where the initial data is at a finite distance from some dilation of a thick-tail stationary state has been investigated in [13] in two dimensions.

Proposition 1.4.1 *For $\chi = \chi_c(k)$, let $\rho(t)$ satisfy (1.4) in the porous medium case $k \in (-1, 0)$ and $m = 1 - k$. If $\bar\rho_k$ is a stationary state of (1.4), then the evolution of the Wasserstein distance to equilibrium can be estimated by*

$$\frac{d}{dt}\mathbf{W}(\rho(t), \bar\rho_k)^2 \le (m-1)\mathcal{F}_k[\rho(t)]\,, \tag{1.22}$$

where equality holds if and only if $\rho(t)$ is a dilation of $\bar\rho_k$.

Proof Let ϕ be the convex Brenier map such that $\bar\rho_k(x)dx = \partial_x\phi(t,x)\#\rho(t,x)dx$ and denote by $\partial_x\psi(t,x)$ the reverse transport map, $\partial_x\phi(t,\partial_x\psi(t,a)) = a$. Following [19, 81] and using the regularity of $\rho(t,x)$ together with the argument as in the proof of Lemma 1.2.8 that allows for the singularity of the mean-field potential gradient to disappear, we have

$$\begin{aligned}
\frac12\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 &\le \int_{\mathbb{R}} (\phi'(t,x)-x)\left(\frac{\partial}{\partial x}\left(\frac{m}{m-1}\rho(t,x)^{m-1}\right)\right.\\
&\qquad \left. +2\chi_c(k)\partial_x S_k(t,x)\right)\rho(t,x)\,dx\\
&= -\int_{\mathbb{R}} \phi''(t,x)\rho(t,x)^m\,dx\\
&\qquad + \chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\phi'(t,x)-\phi'(t,y)}{x-y}\right)|x-y|^k\rho(t,x)\rho(t,y)\,dxdy\\
&\qquad + (m-1)\mathcal{F}_k[\rho(t)]\\
&= -\int_{\mathbb{R}} \left(\psi''(t,a)\right)^{-1}\left(\psi''(t,a)\right)^{1-m}\bar\rho_k(a)^m\,da\\
&\qquad + \chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\psi'(t,a)-\psi'(t,b)}{a-b}\right)^{k-1}|a-b|^k\bar\rho_k(a)\bar\rho_k(b)\,dadb\\
&\qquad + (m-1)\mathcal{F}_k[\rho(t)]
\end{aligned}$$

to finally conclude that

$$\begin{aligned}
\frac12\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 &\le -\int_{\mathbb{R}} \left(\psi''(t,a)\right)^{-m}\bar\rho_k(a)^m\,da\\
&\qquad + \chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}\int_{s=0}^1\left(\psi''(t,[a,b]_s)\right)^{-m}|a-b|^k\bar\rho_k(a)\bar\rho_k(b)\,dsdadb\\
&\qquad + (m-1)\mathcal{F}_k[\rho(t)]\,,
\end{aligned}$$

where we have crucially used the convexity of $(\cdot)^{-m}$ in the last step. We conclude as for the proof of Theorem 1.3.2 thanks to the characterisation (1.9). □

By definition of the critical value $\chi_c(k)$, the functional $\mathcal{F}_k$ is everywhere non-negative. It vanishes if and only if ρ is a dilation of some critical density. Therefore we cannot deduce from (1.22) that the density $\rho(t)$ converges to some dilation of $\bar\rho_k$. However, we can show convergence in Wasserstein distance if we assume a rather restrictive uniform $W^{2,\infty}(\mathbb{R})$-stability estimate on the Brenier map ψ connecting the solution density to the stationary state:

$$\psi''(t,x) \in L^\infty\left(\mathbb{R}_+, L^\infty(\mathbb{R})\right) \quad \text{such that} \quad ||\psi''||_{L^\infty\left(\mathbb{R}_+, L^\infty(\mathbb{R})\right)} \le 1 + \frac{1}{m}\,. \tag{1.23}$$

This condition is equivalent to

$$\forall t > 0 \quad \left\langle \psi''(t,(x,y)) \right\rangle := \int_0^1 \psi''(t,[x,y]_s)\,ds \in \left(0, 1+\frac{1}{m}\right], \quad \text{for a.e. } x,y \in \mathbb{R}\,, \quad \forall t>0\,. \tag{1.24}$$

where $[x,y]_s := (1-s)x + sy$. If we want to show convergence of a solution $\rho(t)$ to a stationary state $\bar\rho_k$ in Wasserstein distance, we need to investigate quantities that are comparable.

Proposition 1.4.2 *For $\chi = \chi_c(k)$, let $\bar\rho_k$ be a stationary state of (1.4) in the porous medium case $k \in (-1,0)$, $m = 1-k$. Let $\rho(t)$ be a solution such that*

$$\mathcal{V}_\infty := \lim_{t\to\infty} \mathcal{V}[\rho(t)] < \infty\,,$$

and we denote by ψ the transport map from $\bar\rho_k$ onto the solution,

$$\rho(t,x)dx = \partial_x\psi(t,x)\#\bar\rho_k(x)dx\,.$$

If ψ satisfies the uniform stability estimate (1.23), then

$$\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 \le 0\,,$$

where equality holds if and only if $\rho(t)$ is a dilation of $\bar\rho_k$.

Proof Note that $\mathcal{V}[\bar\rho_k] < \infty$ since $\bar\rho_k$ is compactly supported [24, Corollary 3.9]. We compute the evolution of the Wasserstein distance along the gradient flow, denoting by ϕ the inverse transport map, $\partial_x\phi(t,x) = \partial_x\psi(t,x)^{-1}$, we proceed as

in Proposition 1.4.1:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 \le &-\int_{\mathbb{R}}\phi''(t,x)\rho(t,x)^m\,dx\\ &+\chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\phi'(t,x)-\phi'(t,y)}{x-y}\right)|x-y|^k\rho(t,x)\rho(t,y)\,dxdy\\ &+\int_{\mathbb{R}}\rho(t,x)^m\,dx-\chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}|x-y|^k\rho(t,x)\rho(t,y)\,dxdy\,,\end{aligned}$$

which we can rewrite in terms of the transport map ψ' as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 \le &-\int_{\mathbb{R}}\big(\psi''(t,a)\big)^{-m}\,\bar\rho_k(a)^m\,da\\ &+\chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}\big\langle\psi''(t,(a,b))\big\rangle^{-m}|a-b|^{1-m}\bar\rho_k(a)\bar\rho_k(b)\,dadb\\ &+\int_{\mathbb{R}}\big(\psi''(t,a)\big)^{1-m}\,\bar\rho_k(a)^m\,da\\ &-\chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}\big\langle\psi''(t,(a,b))\big\rangle^{1-m}|a-b|^{1-m}\bar\rho_k(a)\bar\rho_k(b)\,dadb\,.\end{aligned}$$

Using the characterisation (1.9), we obtain for any $\gamma\in\mathbb{R}$,

$$\int_{\mathbb{R}}\big(\psi''(t,a)\big)^{-\gamma}\,\bar\rho_k(a)^m\,da=\chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}\big\langle\psi''(t,(a,b))^{-\gamma}\big\rangle|a-b|^{1-m}\bar\rho_k(a)\bar\rho_k(b)\,dadb\,.$$

Hence, the dissipation of the distance to equilibrium can be written as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 \le \chi_c(k)\iint_{\mathbb{R}\times\mathbb{R}}|a-b|^k\Big(&-\big\langle\psi''(t,(a,b))^{-m}\big\rangle+\big\langle\psi''(t,(a,b))^{1-m}\big\rangle\\ &+\big\langle\psi''(t,(a,b))\big\rangle^{-m}-\big\langle\psi''(t,(a,b))\big\rangle^{1-m}\Big)\bar\rho_k(a)\bar\rho_k(b)\,dadb\,.\end{aligned}$$

We now examinate the sign of the microscopic functional $J_m[u]$ defined for non-negative functions $u:(0,1)\to\mathbb{R}_+$ by

$$J_m[u]:=-\langle u^{-m}\rangle+\langle u^{1-m}\rangle+\langle u\rangle^{-m}-\langle u\rangle^{1-m}\,.$$

The first two terms can be written as

$$-\langle u^{-m}\rangle+\langle u^{1-m}\rangle=-\alpha\langle u\rangle^{-m}+\beta\langle u\rangle^{1-m}\,,$$

where $\alpha = \langle u \rangle^m \langle u^{-m} \rangle$ and $\beta = \langle u \rangle^{m-1} \langle u^{1-m} \rangle$. By Jensen's inequality we have $\alpha \geq 1$, $\beta \geq 1$, and by interpolation we have $\beta \leq \alpha^{m/(m+1)}$. Therefore,

$$J_m[u] \leq j_m(\langle u \rangle) = \max_{\alpha \geq 1} \left\{ -\alpha \langle u \rangle^{-m} + \alpha^{m/(m+1)} \langle u \rangle^{1-m} \right\} + \langle u \rangle^{-m} - \langle u \rangle^{1-m}.$$

We can compute explicitly the maximal value in the above expression. The first order condition gives

$$\alpha_{max} := \left(\frac{m}{m+1} \langle u \rangle \right)^{m+1}.$$

Since the function

$$g(\alpha) := -\alpha \langle u \rangle^{-m} + \alpha^{m/(m+1)} \langle u \rangle^{1-m}$$

achieves its maximum at $\alpha_{max} \leq 1$ for $\langle u \rangle \leq 1 + 1/m$ and is strictly decreasing for $\alpha > \alpha_{max}$, we have

$$\max_{\alpha \geq 1} g(\alpha) = g(1), \qquad \text{for } \langle u \rangle \leq 1 + 1/m$$

and so we conclude $j_m(\langle u \rangle) = 0$ for $\langle u \rangle \leq 1 + 1/m$. Therefore

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathbf{W}(\rho(t), \bar{\rho}_k)^2 &\leq \chi_c(k) \iint_{\mathbb{R} \times \mathbb{R}} |a-b|^k J_m[\psi''(t,(a,b))] \bar{\rho}_k(a) \bar{\rho}_k(b) \, da db \\ &\leq \chi_c(k) \iint_{\mathbb{R} \times \mathbb{R}} |a-b|^k j_m[\langle \psi''(t,(a,b)) \rangle] \bar{\rho}_k(a) \bar{\rho}_k(b) \, da db = 0 \end{aligned}$$

thanks to the stability estimate (1.24). To investigate the equality cases, note that $\beta = \alpha^{m/(m+1)}$ if and only if $u \equiv 1$ (looking at the equality cases in Hölder's inequality). Moreover, $\langle u \rangle \in (0, 1 + 1/m]$ implies

$$J_m[u] \leq -\alpha \langle u \rangle^{-m} + \alpha^{m/(m+1)} \langle u \rangle^{1-m} + \langle u \rangle^{-m} - \langle u \rangle^{1-m} \leq 0,$$

using $\alpha \geq 1$. Hence, if $J_m[u] = 0$, then we must have $\beta = \alpha^{m/(m+1)}$, and so $u \equiv 1$. The converse is trivial by substituting into the expression for $J_m[u]$. Taking u to be the Brenier map ψ'', we conclude that $\frac{d}{dt} \mathbf{W}(\rho(t), \bar{\rho}_k)^2 = 0$ if and only if $\rho = \bar{\rho}_k$. □

The utility of the previous result for understanding the asymptotic behaviour of solutions depends of course on the set of initial data for which solutions satisfy the stability estimate (1.23) at all times. This set is rather difficult to characterise, and we do not know its size.

Let us now explore what we can say about the long-time behaviour of solutions in the general case. The first insight consists in calculating the evolution of the second

moment. It follows from homogeneity that

$$\frac{d}{dt}\mathcal{V}[\rho(t)] = 2(m-1)\mathcal{F}_k[\rho(t)]\,. \tag{1.25}$$

Identity (1.25) implies that the second moment is non-decreasing, and it converges to some value $\mathcal{V}_\infty \in \mathbb{R}_+ \cup \{+\infty\}$. Following [12] we discuss the dichotomy of $\mathcal{V}_\infty < +\infty$ and $\mathcal{V}_\infty = +\infty$. Let $\rho(t) \in \mathcal{Y}$ be a solution of (1.4) such that $|x|^2\rho(t) \in L^1_+(\mathbb{R})$ for all $t > 0$. Let $\bar\rho_k$ be a stationary state of (1.4) according to Definition 1.2.1. Note that $\mathcal{V}[\bar\rho_k] < \infty$ since $\bar\rho_k$ is compactly supported by Calvez et al. [24, Corollary 3.9].

Case 1: $\mathcal{V}_\infty < +\infty$ If the second moment $\mathcal{V}[\rho(t)]$ converges to $\mathcal{V}_\infty < +\infty$, then we deduce from (1.25) that the energy functional $\mathcal{F}_k[\rho(t)]$ converges to $\mathcal{F}_k[\bar\rho_k] = 0$ since $\mathcal{F}_k$ is non-increasing along trajectories. This is however not enough to conclude convergence of $\rho(t)$ to $\bar\rho_k$ and the question remains open. Note further that in order to have convergence, we need to choose a dilation of $\bar\rho_k$ with second moment equal to $\mathcal{V}_\infty$. For any dilation $\bar\rho_k^\lambda$ of $\bar\rho_k$, we have $\mathcal{V}[\bar\rho_k^\lambda] = \mathcal{V}[\bar\rho_k]/\lambda^2$, and so there exists a unique λ_* such that $\mathcal{V}[\bar\rho_k^{\lambda_*}] = \mathcal{V}_\infty$. This would be the natural candidate for the asymptotic behaviour of the solution $\rho(t)$.

Case 2: $\mathcal{V}_\infty = +\infty$ If the second moment $\mathcal{V}[\rho(t)]$ diverges to $\mathcal{V}_\infty = +\infty$ however, the discussion is more subtle and we can give some further intuition. First of all, let us remark that one has to seek a convergence other than in Wasserstein distance since $\infty = \mathcal{V}_\infty \neq \mathcal{V}[\bar\rho_k] < \infty$. We can not exclude this case a priori however since a convergence in another sense may be possible in principle. We use the homogeneity properties of the flow to derive refined inequalities. To do this, we renormalise the density as in (1.21), but now with a time dependency in σ:

$$\hat\rho(t,y) = \sigma(t)\rho(t,\sigma(t)y)\,, \quad \sigma(t)^2 = \mathcal{V}[\rho(t)] = \int_{\mathbb{R}} |x|^2\rho(t,x)\,dx\,. \tag{1.26}$$

Then $\hat\rho$ satisfies the equation

$$\begin{aligned}\partial_t\hat\rho(t,y) =& \sigma(t)\partial_t\rho(t,x) + \dot\sigma(t)\left(\rho(t,x) + x\cdot\partial_x\rho(t,x)\right)\\ =& \sigma(t)\left\{\sigma(t)^{-2-m}\partial_{yy}\hat\rho(t,y)^m + 2\chi_c(k)\sigma(t)^{-3+k}\partial_y\left(\hat\rho(t,y)\partial_y(W_k(y)*\hat\rho(t,y))\right)\right\}\\ &+ \frac{\dot\sigma(t)}{\sigma(t)}\left(\hat\rho(t,y) + y\cdot\partial_y\hat\rho(t,y)\right)\,.\end{aligned}$$

By homogeneity of $\mathcal{F}_k$, we have

$$\mathcal{F}_k[\rho(t)] = \sigma(t)^{1-m}\mathcal{F}_k[\hat\rho(t)]\,, \tag{1.27}$$

and so it follows from (1.25) that $2\sigma(t)\dot\sigma(t) = 2(m-1)\mathcal{F}_k[\rho(t)] = 2(m-1)\sigma(t)^{1-m}\mathcal{F}_k[\hat\rho(t)]$. We deduce

$$\begin{aligned}\partial_t\hat\rho(t,y) =&\sigma(t)^{-1-m}\left\{\partial_{yy}\hat\rho(t,y)^m + 2\chi_c(k)\partial_y\left(\hat\rho(t,y)\partial_y(W_k(y)*\hat\rho(t,y))\right)\right\}\\ &+\sigma(t)^{-1-m}(m-1)\mathcal{F}_k[\hat\rho(t)]\left(\hat\rho(t,y)+y\cdot\partial_y\hat\rho(t,y)\right).\end{aligned}$$

Alternatively, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_k[\hat\rho(t)] &= \frac{d}{dt}\left\{\sigma(t)^{m-1}\mathcal{F}_k[\rho(t)]\right\}\\ &= -\sigma(t)^{m-1}\int_{\mathbb{R}}\rho(t,x)\left|\partial_x\left(\frac{m}{m-1}\rho(t,x)^{m-1}+2\chi_c(k)W_k(x)*\rho(t,x)\right)\right|^2dx\\ &\quad+(m-1)^2\sigma(t)^{m-2}\sigma(t)^{-m}\mathcal{F}_k[\hat\rho(t)]\mathcal{F}_k[\rho(t)]\\ &= \sigma(t)^{-1-m}\mathcal{G}[\hat\rho]\,,\end{aligned}\tag{1.28}$$

where

$$\mathcal{G}[\hat\rho] := -\int_{\mathbb{R}}\left|\partial_y\left(\frac{m}{m-1}\hat\rho(y)^{m-1}+2\chi_c(k)W_k(y)*\hat\rho(y)\right)\right|^2\hat\rho(y)\,dy+(m-1)^2\mathcal{F}_k[\hat\rho]^2\,.$$

Proposition 1.4.3 *The functional $\mathcal{H}$ defined by $\mathcal{H}[\rho] := \mathcal{G}[\hat\rho]$ on $\mathcal{Y}_2$ is zero-homogeneous, and everywhere non-positive. Moreover, $\mathcal{H}[\rho] = 0$ if and only if ρ is a stationary state of Eq.* (1.4).

Proof Homogeneity follows from the very definition of $\mathcal{H}$. Non-positivity is a consequence of the Cauchy-Schwarz inequality:

$$\begin{aligned}|(m-1)\mathcal{F}_k[\hat\rho]|^2 &= \left|-\int_{\mathbb{R}}y\cdot\partial_y\left(\frac{m}{m-1}\hat\rho(y)^{m-1}+2\chi_c(k)W_k(y)*\hat\rho(y)\right)\hat\rho(y)\,dy\right|^2\\ &\le\left(\int_{\mathbb{R}}|y|^2\hat\rho(y)\,dy\right)\left(\int_{\mathbb{R}}\left|\partial_y\left(\frac{m}{m-1}\hat\rho(y)^{m-1}\right.\right.\right.\\ &\quad\left.\left.\left.+2\chi_c(k)W_k(y)*\hat\rho(y)\right)\right|^2\hat\rho(y)\,dy\right).\end{aligned}\tag{1.29}$$

If ρ is a stationary state of Eq. (1.4), so is $\hat\rho$ and it follows from (1.28) that $\mathcal{G}[\hat\rho] = 0$. Conversely, if $\mathcal{G}[\hat\rho] = 0$, then we can achieve equality in the Cauchy-Schwarz inequality (1.29) above, and so the two functions y and

$$\partial_y\left(\frac{m}{m-1}\hat\rho(y)^{m-1}+2\chi_c(k)W_k(y)*\hat\rho(y)\right)$$

are proportional to each other. In other words, there exists a constant $\hat{\pi}$ such that for all $y \in \mathbb{R}$,

$$\partial_y \left(\frac{m}{m-1} \hat{\rho}(y)^{m-1} + 2\chi_c(k) W_k(y) * \hat{\rho}(y) \right) + \hat{\pi} y = 0 \,. \tag{1.30}$$

This equation is the Euler-Langrange condition of the gradient flow given by the energy functional $\mathcal{F}_k + \hat{\pi}\mathcal{V}$:

$$\partial_t u = \partial_y \left(u \, \partial_y \left(\frac{\delta}{\delta u} \left(\mathcal{F}_k + \hat{\pi}\mathcal{V} \right)[u] \right) \right) , \tag{1.31}$$

and since $\hat{\rho}$ satisfies (1.30), it is a stationary state of Eq. (1.31). Testing this equation against $y\hat{\rho}(y)$, we obtain

$$\hat{\pi} = (m-1)\mathcal{F}_k[\hat{\rho}] \geq 0 \,.$$

Non-negativity of $\hat{\pi}$ follows from the variant of the HLS inequality Theorem 1.3.1 since $\mathcal{F}_k[\rho] \geq 0$ for any $\rho \in \mathcal{Y}$ if $\chi = \chi_c(k)$. We will show $\hat{\pi} = 0$ by contradiction. Assume $\hat{\pi} > 0$. Applying Theorem 1.3.4 for $\mathcal{F}_k[\cdot] + \hat{\pi}\mathcal{V}[\cdot]$ instead of $\mathcal{F}_k[\cdot] + \frac{1}{2}\mathcal{V}[\cdot]$, we deduce that $\hat{\rho}$ is a minimiser of the rescaled energy $\mathcal{F}_k[\cdot] + \hat{\pi}\mathcal{V}[\cdot]$. In particular, this means that we have for any $u \in \mathcal{Y}_2$,

$$\mathcal{F}_k[u] + \hat{\pi}\mathcal{V}[u] \geq \mathcal{F}_k[\hat{\rho}] + \hat{\pi}\mathcal{V}[\hat{\rho}] = \hat{\pi}/(m-1) + \hat{\pi} > \hat{\pi} \,.$$

However, [24, Proposition 3.4(i), Corollary 3.9] and homogeneity of $\mathcal{F}_k$ provide a stationary state $\bar{\rho}_{k,1} \in \mathcal{Y}_2$ with unit second moment, which is also a global minimiser by Calvez et al. [24, Theorem 2.8]. Then choosing $u = \bar{\rho}_{k,1}$ in the above inequality yields $\mathcal{F}_k[\bar{\rho}_{k,1}] + \hat{\pi}\mathcal{V}[\bar{\rho}_{k,1}] = 0 + \hat{\pi}$, a contradiction. Therefore we necessarily have $\hat{\pi} = 0$ and so $\mathcal{F}_k[\hat{\rho}] = 0$. By (1.27), $\mathcal{F}_k[\rho] = 0$ and this implies that ρ is a global minimiser of $\mathcal{F}_k$ by Theorem 1.3.1, and consequently it is a stationary state of (1.4) by Calvez et al. [24, Theorem 2.8]. □

It would be desirable to be able to show that $\mathcal{H}[\rho(t)] \to \mathcal{H}[\bar{\rho}_{k,1}]$ as $t \to \infty$ to make appropriate use of the new energy functional $\mathcal{H}$. But even then, similar to the first case, we are lacking a stability result for $\mathcal{H}$ to prove that in fact $\hat{\rho}(t)$ converges to $\bar{\rho}_{k,1}$. Here, in addition, we do not know at which rate the second moment goes to $+\infty$.

We conjecture that only the first case $\mathcal{V}_\infty < +\infty$ is admissible. The motivation for this claim is the following: $\mathcal{F}$ and $\mathcal{H}$ have both constant signs, and vanish only when $\hat{\rho} = \bar{\rho}_{k,1}$. If the stability inequality

$$\eta \mathcal{F}_k[\hat{\rho}] \leq -\mathcal{H}[\rho], \quad \forall \rho \tag{1.32}$$

were satisfied for some $\eta > 0$, then we would be able to prove that $\mathcal{V}_\infty < +\infty$. To see this, we derive a second-order differential inequality for $\omega(t) := \sigma(t)^{m+1}$. We

have

$$\dot{\omega}(t) = (m+1)\sigma(t)^m \dot{\sigma}(t) = (m+1)(m-1)\mathcal{F}_k[\hat{\rho}(t)] \geq 0\,,$$

and so by (1.28),

$$\ddot{\omega}(t) = (m+1)(m-1)\omega(t)^{-1}\mathcal{H}[\rho(t)] \leq 0\,.$$

Here, non-positivity of $\ddot{\omega}(t)$ follows from Proposition 1.4.3. Therefore, the stability estimate (1.32), if true, would imply that $\ddot{\omega}(t) \leq -\eta\omega(t)^{-1}\dot{\omega}(t)$, hence

$$\dot{\omega}(t) \leq C - \eta \log \omega(t).$$

Consequently, $\omega(t)$ would be bounded, and so we arrive at a contradiction with the assumption $\mathcal{V}_\infty = +\infty$.

1.4.1.2 The Sub-Critical Case $\chi < \chi_c$

We know that in the logarithmic case ($m = 1, k = 0$), solutions to (1.4) converge exponentially fast towards a unique self-similar profile as $t \to \infty$, provided that the parameter χ is sub-critical ($\chi < 1$) [19]. A similar argument works in the porous medium regime $k \in (-1, 0)$ under certain regularity assumptions as we will show below. Surprisingly enough, convergence is uniform as the rate of convergence does not depend on the parameter χ. In particular, it was shown in [19] for $k = 0$ that we have uniform convergence in Wasserstein distance of any solution $\rho(t)$ for the rescaled system (1.7) to the equilibrium distribution $\bar{\rho}_0$ of (1.7),

$$\frac{d}{dt}\mathbf{W}(\rho(t), \bar{\rho}_0)^2 \leq -2\mathbf{W}(\rho(t), \bar{\rho}_0)^2\,.$$

A similar result has been obtained in two dimension in [26].

Studying the long-time behaviour of the system in the porous medium case $k < 0$ is more subtle than the logarithmic case and we cannot deduce exponentially fast convergence from our calculations without assuming a uniform stability estimate, which coincides with (1.24). But as in the critical case, we do not know how many initial data actually satisfy this condition. Note also that due to the additional confining potential, homogeneity has been broken, and so we cannot renormalise the second moment of minimisers as we did in the critical case. As in the critical case, stationary states of the rescaled equation (1.7) are compactly supported by Calvez et al. [24, Corollary 3.9].

Proposition 1.4.4 *For sub-critical interaction strength $0 < \chi < \chi_c(k)$, let $\rho(t)$ be a solution to (1.7) in the porous medium case $k \in (-1, 0)$, $m = 1 - k$*

and $\bar\rho_k$ a stationary state of (1.7). If the transport map ψ given by $\rho(t,x)dx = \partial_x\psi(t,x)\#\bar\rho_k(x)dx$ satisfies the uniform stability estimate (1.23), then

$$\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 \le -2\mathbf{W}(\rho(t),\bar\rho_k)^2\,,$$

where equality holds if and only if $\rho(t)$ is a dilation of $\bar\rho_k$. It follows that

$$\lim_{t\to\infty}\mathcal{V}[\rho(t)] = \mathcal{V}[\bar\rho_k]\,.$$

Proof We compute the evolution of the Wasserstein distance along the gradient flow similar to the proof of Proposition 1.4.2, denoting by ϕ the inverse transport map, $\partial_x\phi(t,x) = \partial_x\psi(t,x)^{-1}$,

$$\begin{aligned}\frac12\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 \le &-\int_{\mathbb{R}}\phi''(t,x)\rho(t,x)^m\,dx\\ &+\chi\iint_{\mathbb{R}\times\mathbb{R}}\left(\frac{\phi'(t,x)-\phi'(t,y)}{x-y}\right)|x-y|^k\rho(t,x)\rho(t,y)\,dxdy\\ &+\int_{\mathbb{R}}\rho(t,x)^m\,dx-\chi\iint_{\mathbb{R}\times\mathbb{R}}|x-y|^k\rho(t,x)\rho(t,y)\,dxdy\\ &+\frac12\iint_{\mathbb{R}\times\mathbb{R}}(\phi'(t,x)-\phi'(t,y))(x-y)\rho(t,x)\rho(t,y)\,dxdy\\ &-\int_{\mathbb{R}}|x|^2\rho(t,x)\,dx\,,\end{aligned}$$

where we have used the fact that the centre of mass is zero at all times to double the variables:

$$\int_{\mathbb{R}}\phi'(t,x)x\rho(t,x)\,dx = \frac12\iint_{\mathbb{R}\times\mathbb{R}}(\phi'(t,x)-\phi'(t,y))(x-y)\rho(t,x)\rho(t,y)\,dxdy\,.$$

This rewrites as follows in terms of the transport map ψ':

$$\begin{aligned}\frac12\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 \le &-\int_{\mathbb{R}}\big(\psi''(t,a)\big)^{-m}\,\bar\rho_k(a)^m\,da\\ &+\chi\iint_{\mathbb{R}\times\mathbb{R}}\big\langle\psi''(t,(a,b))\big\rangle^{-m}|a-b|^{1-m}\bar\rho(a)\bar\rho_k(b)\,dadb\\ &+\int_{\mathbb{R}}\big(\psi''(t,a)\big)^{1-m}\,\bar\rho_k(a)^m\,da\\ &-\chi\iint_{\mathbb{R}\times\mathbb{R}}\big\langle\psi''(t,(a,b))\big\rangle^{1-m}|a-b|^{1-m}\bar\rho_k(a)\bar\rho_k(b)\,dadb\end{aligned}$$

$$+\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left\langle\psi''(t,(a,b))\right\rangle|a-b|^2\bar\rho_k(a)\bar\rho_k(b)\,dadb$$
$$-\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left\langle\psi''(t,(a,b))\right\rangle^2|a-b|^2\bar\rho_k(a)\bar\rho_k(b)\,dadb\,.$$

Using the characterisation (1.10), we obtain for any $\gamma\in\mathbb{R}$,

$$\int_{\mathbb{R}}\left(\psi''(t,a)\right)^{-\gamma}\bar\rho_k(a)^m\,da$$
$$=\iint_{\mathbb{R}\times\mathbb{R}}\left(\chi|a-b|^{1-m}+\frac{|a-b|^2}{2}\right)\left\langle\psi''(t,(a,b))^{-\gamma}\right\rangle\bar\rho_k(a)\bar\rho_k(b)\,dadb\,.$$

Hence, the dissipation of the distance to equilibrium can be written as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2\le\chi\iint_{\mathbb{R}\times\mathbb{R}}|a-b|^k\Big\{&-\left\langle\psi''(t,(a,b))^{-m}\right\rangle+\left\langle\psi''(t,(a,b))^{1-m}\right\rangle\\&+\left\langle\psi''(t,(a,b))\right\rangle^{-m}-\left\langle\psi''(t,(a,b))\right\rangle^{1-m}\Big\}\bar\rho_k(a)\bar\rho_k(b)\,dadb\\&+\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}}|a-b|^2\Big\{-\left\langle\psi''(t,(a,b))^{-m}\right\rangle+\left\langle\psi''(t,(a,b))^{1-m}\right\rangle\\&+\left\langle\psi''(t,(a,b))\right\rangle-\left\langle\psi''(t,(a,b))\right\rangle^2\Big\}\bar\rho_k(a)\bar\rho_k(b)\,dadb\,.\end{aligned}$$

We now examinate the signs of the microscopic functionals $J_m[u]$ and $J_{m,2}[u]$ defined as follows for non-negative functions $u:(0,1)\to\mathbb{R}_+$,

$$J_m[u]:=-\left\langle u^{-m}\right\rangle+\left\langle u^{1-m}\right\rangle+\left\langle u\right\rangle^{-m}-\left\langle u\right\rangle^{1-m}\,,\tag{1.33}$$
$$J_{m,2}[u]:=-\left\langle u^{-m}\right\rangle+\left\langle u^{1-m}\right\rangle+\left\langle u\right\rangle-\left\langle u\right\rangle^{2}\,.\tag{1.34}$$

The first two terms in the functionals J_m and $J_{m,2}$ are common. We can rewrite them as

$$-\left\langle u^{-m}\right\rangle+\left\langle u^{1-m}\right\rangle=-\alpha\left\langle u\right\rangle^{-m}+\beta\left\langle u\right\rangle^{1-m}\,,$$

where $\alpha=\left\langle u\right\rangle^m\left\langle u^{-m}\right\rangle$ and $\beta=\left\langle u\right\rangle^{m-1}\left\langle u^{1-m}\right\rangle$. By Jensen's inequality we have $\alpha\ge1$, $\beta\ge1$, and by interpolation we have $\beta\le\alpha^{m/(m+1)}$. Therefore,

$$J_m[u]\le j_m(\langle u\rangle):=\max_{\alpha\ge1}g(\alpha)+\left\langle u\right\rangle^{-m}-\left\langle u\right\rangle^{1-m}\,,$$
$$J_{m,2}[u]\le j_{m,2}(\langle u\rangle):=\max_{\alpha\ge1}g(\alpha)+\left\langle u\right\rangle-\left\langle u\right\rangle^{2}\,,$$

where

$$g(\alpha) := -\alpha\langle u\rangle^{-m} + \alpha^{m/(m+1)}\langle u\rangle^{1-m}\,.$$

We can compute explicitly the maximal value of g, and as before the first order condition gives

$$\alpha_{max} = \left(\frac{m}{m+1}\langle u\rangle\right)^{m+1}\,.$$

It is straight forward to see that

$$\max_{\alpha\geq 1} g(\alpha) = g(1) \qquad \text{for } \langle u\rangle \leq 1 + 1/m\,,$$

and hence we obtain

$$j_m(\langle u\rangle) = \begin{cases} 0, & \text{if} \quad \langle u\rangle \leq 1 + \dfrac{1}{m} \\ \left(\dfrac{m}{m+1}\right)^m \dfrac{1}{m+1}\langle u\rangle + \langle u\rangle^{-m} - \langle u\rangle^{1-m}, & \text{if} \quad \langle u\rangle \geq 1 + \dfrac{1}{m} \end{cases}\,, \tag{1.35}$$

$$j_{m,2}(\langle u\rangle) = \begin{cases} -\langle u\rangle^{-m} + \langle u\rangle^{1-m} + \langle u\rangle - \langle u\rangle^2, & \text{if} \quad \langle u\rangle \leq 1 + \dfrac{1}{m} \\ \left(\dfrac{m}{m+1}\right)^m \dfrac{1}{m+1}\langle u\rangle + \langle u\rangle - \langle u\rangle^2, & \text{if} \quad \langle u\rangle \geq 1 + \dfrac{1}{m} \end{cases}\,. \tag{1.36}$$

We have $\lim_{+\infty} j_m = +\infty$, and $\lim_{+\infty} j_{m,2} = -\infty$. In addition, the function $j_{2,m}$ is non-positive and uniformly strictly concave:

$$\begin{aligned} \forall\, \langle u\rangle \in \left(0, 1 + \frac{1}{m}\right] \quad j''_{m,2}(\langle u\rangle) &= m\,\langle u\rangle^{-m-2}\left(-(m+1) + (m-1)\,\langle u\rangle\right) - 2 \\ &\leq -(m+1)\,\langle u\rangle^{-m-2} - 2\,. \end{aligned}$$

Thus, $\forall\, \langle u\rangle \in \mathbb{R}_+, j''_{m,2}(\langle u\rangle) \leq -2$ and so the following coercivity estimate holds true:

$$\forall\, \langle u\rangle \in \left(0, 1 + \frac{1}{m}\right], \quad j_{m,2}(\langle u\rangle) \leq -\left(\langle u\rangle - 1\right)^2\,. \tag{1.37}$$

Furthermore, the function j_m is everywhere non-negative. The above analysis allows us to rewrite the dissipation in Wasserstein distance as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t), \bar{\rho}_k)^2 \leq &\iint_{\mathbb{R}\times\mathbb{R}} \chi|a-b|^k J_m[\psi''(t,(a,b))]\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \\ &+ \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2 J_{m,2}[\psi''(t,(a,b))]\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb \end{aligned}$$

$$\leq \iint_{\mathbb{R}\times\mathbb{R}} \chi|a-b|^k j_m[\left|\psi''(t,(a,b))\right|]\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb$$
$$+\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2 j_{m,2}[\left|\psi''(t,(a,b))\right|]\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb$$

to finally conclude that

$$\frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t),\bar{\rho}_k)^2 \leq -\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2\left(\left|\psi''(t,(a,b))\right|-1\right)^2\,\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb,$$

where the last inequality follows from (1.35) and the coercivity property (1.37) thanks to the stability estimate (1.24). This concludes the proof,

$$\frac{d}{dt}\mathbf{W}(\rho(t),\bar{\rho}_k)^2 \leq -\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2\left(\left|\psi''(t,(a,b))\right|-1\right)^2\,\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb$$
$$= -\iint_{\mathbb{R}\times\mathbb{R}} \left(\psi'(a)-a-\left(\psi'(b)-b\right)\right)^2\,\bar{\rho}_k(a)\bar{\rho}_k(b)\,dadb,$$
$$= -2\int_{\mathbb{R}} \left(\psi'(a)-a\right)^2\,\bar{\rho}_k(a)\,da, = -2\,\mathbf{W}(\rho(t),\bar{\rho}_k)^2,$$

using the fact that $\rho(t)$ and $\bar{\rho}_k$ both have zero centre of mass. □

Remark 1.4.1 (Non-existence of Stationary States) Proposition 1.4.4 motivates the rescaling in the sub-critical case since it means that there are no stationary states in original variables. Indeed, assume $\bar{u}$ is a stationary states of Eq. (1.4), then its rescaling $\rho(t,x) = e^t\bar{u}(e^t x)$ is a solution to (1.7) and converges to δ_0 as $t \to \infty$. Calvez et al. [24, Proposition 3.4(ii)] on the other hand provides a stationary state $\bar{\rho}_k$, and the transport map $\partial_x\psi(t,x)$ pushing forward $\bar{\rho}_k$ onto $\rho(t,x)$ can be written as $\psi(t,x) = e^{-t}\phi(x)$ for some convex function ϕ. Hence, for large enough $t > 0$, $\psi(t,x)$ satisfies the stability estimate (1.23) and so eventually $\rho(t,x)$ converges to $\bar{\rho}_k$ by Proposition 1.4.4 which is not possible.

1.4.1.3 The Super-Critical Case $\chi > \chi_c$

Here, we investigate the possible blow-up dynamics of the solution in the super-critical case. In contrast to the logarithmic case ($m = 1, k = 0$), for which all solutions blow-up when $\chi > \chi_c$, provided the second momentum is initially finite, see [9], the picture is not so clear in the fair-competition regime with negative homogeneity $k < 0$. There, the key identity is (1.25), which states in particular that the second momentum is a concave function.

It has been observed in [12] that if the free energy is negative for some time t_0, $\mathcal{F}_k[\rho(t_0)] < 0$, then the second momentum is a decreasing concave function for

$t > t_0$. So, it cannot remain non-negative for all time. Necessarily, the solution blows up in finite time. Whether or not the free energy could remain non-negative for all time was left open. In [82], the author proved that solutions blow-up without condition on the sign of the free energy at initial time, but for the special case of the Newtonian potential, for which comparison principles are at hand.

In [21], a continuous time, finite dimensional, Lagrangian numerical scheme of [10] was analysed. This scheme preserves the gradient flow structure of the equation. It was proven that, except for a finite number of values of χ, the free energy necessarily becomes negative after finite time. Thus, blow-up seems to be a generic feature of (1.4) in the super-critical case. However, we could not extend the proof of [21] to the continuous case for two reasons: firstly, we lack compactness estimates, secondly, the set of values of χ to be excluded gets dense as the number of particles in the Lagrangian discretisation goes to ∞.

Below, we transpose the analysis of [21] to the continuous level. We highlight the missing pieces. Let us define the renormalised density $\hat{\rho}$ as in (1.26). The following statement is the analogue of Proposition 1.4.3 in the super-critical case.

Proposition 1.4.5 *The functional $\mathcal{H}$ defined by $\mathcal{H}[\rho] := \mathcal{G}[\hat{\rho}]$ on $\mathcal{Y}_2$ is zero-homogeneous, and everywhere non-positive. Moreover, it cannot vanish in the cone of non-negative energy:*

$$(\mathcal{F}[\rho] \geq 0) \Longrightarrow (\mathcal{H}[\rho] < 0) \,. \tag{1.38}$$

Proof We proceed as in the proof of Proposition 1.4.3. Zero-homogeneity follows from the definition of $\mathcal{H}$, and non-positivity is a direct consequence of the Cauchy-Schwarz inequality. It remains to show (1.38). Assume that ρ is such that $\mathcal{F}[\rho] \geq 0$ and $\mathcal{H}[\rho] = 0$. The latter condition ensures that there exists a constant $\hat{\pi}$ such that $\hat{\rho}$ is a critical point of the energy functional $\mathcal{F} + \hat{\pi}\mathcal{V}$:

$$\partial_y \Big(\frac{m}{m-1} \hat{\rho}(y)^{m-1} + 2\chi W_k(y) * \hat{\rho}(y) \Big) + \hat{\pi} y = 0 \,.$$

Testing this equation against $y\hat{\rho}(y)$, we obtain

$$\hat{\pi} = (m-1)\mathcal{F}_k[\hat{\rho}] = (m-1)\sigma(t)^{m-1}\mathcal{F}_k[\rho] \geq 0 \,.$$

Applying as in the proof of Proposition 1.4.3 a variant of Theorem 1.3.4, we obtain that $\hat{\rho}$ is a global minimiser of the energy functional $\mathcal{F} + \hat{\pi}\mathcal{V}$. Here, the amplitude of the confinement potential $\hat{\pi}$ plays no role, but the sign $\hat{\pi} \geq 0$ is crucial. By Calvez et al. [24, Theorem 2.8], there exists a stationary state $\bar{\rho} \in \mathcal{Y}_2$ for critical interaction strength $\chi = \chi_c(k)$. If $\chi > \chi_c(k)$, we have $\mathcal{F}_k[\bar{\rho}] = \mathcal{U}_m[\bar{\rho}] + \chi\mathcal{W}_k[\bar{\rho}] < \mathcal{U}_m[\bar{\rho}] + \chi_c(k)\mathcal{W}_k[\bar{\rho}] = 0$. Taking mass-preserving dilations of $\bar{\rho}$, we see immediately that the functional $\mathcal{F} + \hat{\pi}\mathcal{V}$ is not bounded below in the super-critical case. This is a contradiction with $\hat{\rho}$ being a minimiser. Hence, $\mathcal{H}[\rho] < 0$ and (1.38) holds true. □

As in Sect. 1.4.1.1, the following non-linear function of the second momentum,

$$\omega(t) = \sigma(t)^{m+1} = \left(\int_{\mathbb{R}} |x|^2 \rho(t,x)\, dx \right)^{\frac{m+1}{2}} ,$$

satisfies the second order differential inequality,

$$\ddot{\omega}(t) = (m^2 - 1)\omega(t)^{-1} \mathcal{H}[\rho(t)] \leq 0 . \tag{1.39}$$

In view of the property (1.38) of the zero-homogeneous functional $\mathcal{H}$, it seems natural to ask whether there exists a positive constant $\delta > 0$, such that

$$(\mathcal{F}[\rho] \geq 0) \Longrightarrow (\mathcal{H}[\rho] < -\delta) . \tag{1.40}$$

If this would be the case, then (1.39) could be processed as follows: assume that $\dot{\omega}(t) \geq 0$ for all t. This is equivalent to say that the free energy remains non-negative for all $t \geq 0$ using (1.25). Hence, assuming (1.40) holds, (1.39) becomes

$$\ddot{\omega}(t) < -\delta(m^2 - 1)\omega(t)^{-1} < 0. \tag{1.41}$$

Multiplying by $\dot{\omega}(t) \geq 0$, and integrating between 0 and T, we would get

$$\frac{1}{2}\dot{\omega}(T)^2 + \delta(m^2 - 1)\log(\omega(T)) \leq \frac{1}{2}\dot{\omega}(0)^2 + \delta(m^2 - 1)\log(\omega(0)) .$$

Hence, for any $t > 0$,

$$\omega(t) \leq \omega(0) \exp\left(\frac{\dot{\omega}(0)^2}{2\delta(m^2 - 1)} \right) .$$

Back to estimate (1.41), we would conclude that ω is uniformly concave,

$$\ddot{\omega}(t) \leq -\left(\frac{\delta(m^2 - 1)}{\omega(0)} \right) \exp\left(-\frac{\dot{\omega}(0)^2}{2\delta(m^2 - 1)} \right) < 0 .$$

Therefore, $\frac{d}{dt}\mathcal{V}[\rho(t)]$ would become negative in finite time. This would be a contradiction with the everywhere non-negativity of the free energy by (1.25). As a conclusion, the existence of positive $\delta > 0$ as in (1.40) implies unconditional blow-up. In [21], existence of such δ is proven for a finite dimensional Lagrangian discretisation of $\mathcal{F}_k$, and accordingly $\mathcal{H}$, except for a finite set of values for χ. Numerical simulations using the numerical scheme proposed in [10] clearly show that the energy has the tendency to become negative, even for positive initial data. Proving (1.40) remains an open problem.

1.4.2 Fast Diffusion Asymptotics

In the fast diffusion case $k > 0$, we are able to show a much stronger result: every stationary state of (1.7) is in fact a global attractor for any choice of interaction strength $\chi > 0$. Investigating the evolution of the Wasserstein distance to equilibrium yields exponential convergence with an explicit rate which is independent of the interaction strength $\chi > 0$. In contrast to the porous medium case, where we required a stability estimate on Brenier's map, we do not need such an estimate here. As a consequence, we obtain an alternative proof of uniqueness of stationary states by a dynamical argument.

Proposition 1.4.6 (Long-Time Asymptotics) *For $k \in (0,1)$ and $m = 1-k$, if $\rho(t)$ has zero centre of mass initially and satisfies (1.7), then the evolution of the Wasserstein distance to the stationary states $\bar{\rho}_k$ of (1.7) can be estimated by*

$$\frac{d}{dt}\mathbf{W}(\rho(t),\bar{\rho}_k)^2 \le -2\mathbf{W}(\rho(t),\bar{\rho}_k)^2 \tag{1.42}$$

for any interaction strength $\chi > 0$. As a consequence, stationary states are unique if they exist.

Proof We compute the evolution of the Wasserstein distance along the gradient flow, denoting by ϕ the inverse transport map, $\partial_x\phi(t,x) = \partial_x\psi(t,x)^{-1}$. Proceeding as in the proof of Proposition 1.4.4, we can write the dissipation of the distance to equilibrium as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t),\bar{\rho}_k)^2 \le \chi \iint_{\mathbb{R}\times\mathbb{R}} |a-b|^k \Big\{ &-\big\langle \psi''(t,(a,b))^{-m}\big\rangle + \big\langle \psi''(t,(a,b))^{1-m}\big\rangle \\ &+\big\langle \psi''(t,(a,b))\big\rangle^{-m} - \big\langle \psi''(t,(a,b))\big\rangle^{1-m}\Big\}\,\bar{\rho}_k(a)\bar{\rho}_k(b)\,da\,db \\ + \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2 \Big\{ &-\big\langle \psi''(t,(a,b))^{-m}\big\rangle + \big\langle \psi''(t,(a,b))^{1-m}\big\rangle \\ &+\big\langle \psi''(t,(a,b))\big\rangle - \big\langle \psi''(t,(a,b))\big\rangle^{2}\Big\}\,\bar{\rho}_k(a)\bar{\rho}_k(b)\,da\,db\,.\end{aligned}$$

We now examine the signs of the microscopic functionals $J_m[u]$ and $J_{m,2}[u]$ defined as in (1.33) and (1.34) for non-negative functions $u : (0,1) \to \mathbb{R}_+$ by

$$\begin{aligned} J_m[u] &:= -\big\langle u^{-m}\big\rangle + \big\langle u^{1-m}\big\rangle + \big\langle u\big\rangle^{-m} - \big\langle u\big\rangle^{1-m}\,, \\ J_{m,2}[u] &:= -\big\langle u^{-m}\big\rangle + \big\langle u^{1-m}\big\rangle + \big\langle u\big\rangle - \big\langle u\big\rangle^{2}\,. \end{aligned}$$

However, since $m < 1$ we now have by convexity $\big\langle u\big\rangle^{-m} - \big\langle u^{-m}\big\rangle \le 0$ and $\big\langle u^{1-m}\big\rangle - \big\langle u\big\rangle^{1-m} \le 0$, hence

$$J_m[u] \le 0, \quad m \in (0,1). \tag{1.43}$$

For the functional $J_{m,2}$, the first two terms can be written as

$$-\langle u^{-m}\rangle + \langle u^{1-m}\rangle = -\alpha\langle u\rangle^{-m} + \beta\langle u\rangle^{1-m}\,,$$

where $\alpha = \langle u\rangle^{m}\langle u^{-m}\rangle$ and $\beta = \langle u\rangle^{m-1}\langle u^{1-m}\rangle$. As opposed to the proof of Proposition 1.4.4, we now have $\beta \le 1 \le \alpha$ by Jensen's inequality since $m < 1$, and therefore,

$$\forall\, \langle u\rangle \in \mathbb{R}_+, \quad J_{m,2}[u] \le j_{m,2}(\langle u\rangle) := -\langle u\rangle^{-m} + \langle u\rangle^{1-m} + \langle u\rangle - \langle u\rangle^2\,.$$

Note that $\lim_{+\infty} j_{m,2} = -\infty$. In addition, the function $j_{2,m}$ is non-positive and uniformly strictly concave:

$$\forall\, \langle u\rangle \in \mathbb{R}_+, \quad j''_{m,2}(\langle u\rangle) = -m(1+m)\,\langle u\rangle^{-m-2} - m(1-m)\,\langle u\rangle^{-m-1} - 2 \le -2\,,$$

and hence

$$\forall\, \langle u\rangle \in \mathbb{R}_+, \quad j_{m,2}(\langle u\rangle) \le -\left(\langle u\rangle - 1\right)^2\,. \tag{1.44}$$

From these estimates, we can deduce the exponential speed of convergence for the stationary state $\bar\rho_k$ by rewriting the dissipation to equilibrium as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 &\le \iint_{\mathbb{R}\times\mathbb{R}} \chi|a-b|^k J_m[\psi''(t,(a,b))]\bar\rho_k(a)\bar\rho_k(b)\,dadb \\
&\quad + \iint_{\mathbb{R}\times\mathbb{R}} \frac{1}{2}|a-b|^2 J_{m,2}[\psi''(t,(a,b))]\bar\rho_k(a)\bar\rho_k(b)\,dadb \\
&\le \iint_{\mathbb{R}\times\mathbb{R}} \frac{1}{2}|a-b|^2 j_{m,2}[\langle\psi''(t,(a,b))\rangle]\bar\rho_k(a)\bar\rho_k(b)\,dadb \\
&\le -\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2\left(\langle\psi''(t,(a,b))\rangle - 1\right)^2\,\bar\rho_k(a)\bar\rho_k(b)\,dadb,
\end{aligned}$$

where the last inequality follows from (1.43) and (1.44). This concludes the proof,

$$\begin{aligned}
\frac{d}{dt}\mathbf{W}(\rho(t),\bar\rho_k)^2 &\le -\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2\left(\langle\psi''(t,(a,b))\rangle - 1\right)^2\,\bar\rho_k(a)\bar\rho_k(b)\,dadb \\
&= -\iint_{\mathbb{R}\times\mathbb{R}} \left(\psi'(a) - a - \left(\psi'(b)-b\right)\right)^2\,\bar\rho_k(a)\bar\rho_k(b)\,dadb, \\
&= -2\int_{\mathbb{R}} \left(\psi'(a)-a\right)^2\,\bar\rho_k(a)\,da, = -2\,\mathbf{W}(\rho(t),\bar\rho_k)^2,
\end{aligned}$$

using the fact that $\rho(t)$ and $\bar\rho_k$ both have zero centre of mass. □

Remark 1.4.2 (Non-existence of Stationary States) This result also provides a dynamical proof for the non-existence of stationary states for $k \in (0, 2/3)$ in original variables. Indeed, if $\bar{u}$ were a stationary state of Eq. (1.4), then its rescaled density $\rho(t, x)$ would converge to δ_0 for large times. This contradicts the existence of a stationary state in rescaled variables [24, Theorem 4.10] for $k \in (0, 2/3)$ together with exponential convergence to equilibrium Proposition 1.4.6.

1.5 Numerical Simulations

There exists an illuminating way to rewrite the energy functional $\mathcal{F}_k[\rho]$ due to the particular form of the transport map. We use the Lagrangian transformation $\rho \mapsto X$, where $X : (0, 1) \to \mathbb{R}$ denotes the pseudo-inverse of the cumulative distribution function (cdf) associated with ρ [10, 19, 54, 81],

$$X(\eta) = F^{-1}(\eta) := \inf\{x : F(x) \geq \eta\}, \quad F(x) := \int_{-\infty}^{x} \rho(y)\, dy.$$

We introduce the parameter $r \in \{0, 1\}$ as we are interested in both original ($r = 0$) and rescaled ($r = 1$) variables. Integrating Eqs. (1.4) and (1.7) over $(-\infty, X(t, \eta))$ with respect to the space variable yields

$$\partial_t \int_{-\infty}^{X(t,\eta)} \rho(t, y)\, dy = \left[\partial_x \rho^m + 2\chi\rho\partial_x (W_k * \rho) + rx\rho\right]|_{x=X(t,\eta)}. \tag{1.45}$$

Differentiating the identity $F(t, X(t, \eta)) = \eta$ with respect to η twice yields

$$\rho(t, X(t, \eta)) = \left(\partial_\eta X(t, \eta)\right)^{-1} \quad \text{and} \quad \partial_x \rho(t, X(t, \eta)) = -\partial_{\eta\eta} X(t, \eta) / \left(\partial_\eta X(t, \eta)\right)^3.$$

Differentiating with respect to time, we obtain $\partial_t F(t, X(t, \eta)) = -\partial_t X(t, \eta)/\partial_\eta X(t, \eta)$. This allows us to simplify (1.45),

$$\begin{aligned}\partial_t X(t, \eta) = &-\partial\eta \left(\left(\partial_\eta X(t, \eta)\right)^{-m}\right) \\ &-2\chi \int_0^1 |X(t, \eta) - X(t, \tilde{\eta})|^{k-2} \left(X(t, \eta) - X(t, \tilde{\eta})\right) d\tilde{\eta} - rX(t, \eta).\end{aligned}$$

Similarly, the functionals $\mathcal{G}_{k,0} := \mathcal{F}_k$ and $\mathcal{G}_{k,1} := \mathcal{F}_{k,\mathbf{resc}}$ read equivalently

$$\begin{aligned}\mathcal{G}_{k,r}[X] = &\frac{1}{m-1} \int_0^1 (\partial_\eta X(\eta))^{1-m}\, d\eta \\ &+ \chi \int_0^1 \int_0^1 \frac{|X(\eta) - X(\tilde{\eta})|^k}{k}\, d\eta d\tilde{\eta} + \frac{r}{2} \int_0^1 |X(\eta)|^2\, d\eta.\end{aligned}$$

for $k \in (-1,1)\backslash\{0\}$, and

$$\mathcal{G}_{0,r}[X] = -\int_0^1 \log\left(\frac{dX}{d\eta}(\eta)\right) d\eta + \chi \int_0^1 \int_0^1 \log|X(\eta) - X(\tilde{\eta})|\, d\eta d\tilde{\eta} + \frac{r}{2}\int_0^1 |X(\eta)|^2\, d\eta\,.$$

in the logarithmic case $k = 0$. Intuitively, X encodes the position of particles with respect to the partial mass $\eta \in (0,1)$, and the same homogeneity is preserved: $\mathcal{G}_{k,0}[\lambda X] = \lambda^k \mathcal{G}_{k,0}[X]$.

In Sect. 1.3, we showed uniqueness of minimisers of the rescaled energy functional $\mathcal{F}_{k,\mathbf{resc}}[\rho]$ for $0 < k < 2/3$ and any $\chi > 0$ (Corollary 1.3.11) and also for the sub-critical porous medium case $-1 < k < 0$, $\chi < \chi_c(k)$ (Corollary 1.3.6). One may take these results as an indication that $\mathcal{F}_{k,\mathbf{resc}}[\rho]$ could in fact be displacement convex. As discussed in Sect. 1.2.3, $\mathcal{F}_{k,\mathbf{resc}}[\rho]$ is a sum of displacement convex and concave contributions and we do not know its overall convexity properties. We recall that the functionals related to the classical Keller-Segel models in two dimensions are displacement convex once restricted to bounded densities [36]. We will give some heuristics for the power-law potential case. If $\mathcal{G}_{k,1}[X]$ were convex, then $\mathcal{F}_{k,\mathbf{resc}}[\rho]$ would be displacement convex [30, 81] and uniqueness of minimisers directly follows [65]. Taylor expanding $\mathcal{G}_{k,1}$ around X yields for any test function $\varphi \in C_c^\infty([0,1])$,

$$\mathcal{G}_{k,1}[X + \epsilon\varphi] = \mathcal{G}_{k,1}[X] + \epsilon D_\varphi \mathcal{G}_{k,1}[X] + \frac{\epsilon^2}{2} D_\varphi^2 \mathcal{G}_{k,1}[X] + O(\epsilon^3),$$

where $D_\varphi \mathcal{G}_{k,1}[X] = \int_0^1 \delta\mathcal{G}_{k,1}[X](\eta)\,\varphi(\eta)\,d\eta$ with the first variation $\frac{\delta \mathcal{G}_{k,1}}{\delta X}[X](\eta)$ given by

$$\frac{\delta \mathcal{G}_{k,1}}{\delta X}[X](\eta) = \partial_\eta\left(\left(\partial_\eta X\right)^{-m}\right) + 2\chi \int_0^1 |X(\eta) - X(\tilde{\eta})|^{k-2}\,(X(\eta) - X(\tilde{\eta}))\, d\tilde{\eta} + X(\eta)$$

for $k \in (-1,1)/\{0\}$. However, the Hessian

$$D_\varphi^2 \mathcal{G}_{k,1}[X] = m\int_0^1 \left(\partial_\eta \varphi(\eta)\right)^2 \left(\partial_\eta X(\eta)\right)^{-(m+1)}\, d\eta + \chi(k-1)\int_0^1\int_0^1 |X(\eta) - X(\tilde{\eta})|^{k-2}\,(\varphi(\eta) - \varphi(\tilde{\eta}))^2\, d\eta d\tilde{\eta} + \int_0^1 \varphi(\eta)^2\, d\eta$$

does not have a sign. In other words, we cannot use this strategy to conclude overall convexity/concavity properties of the rescaled energy functional $\mathcal{F}_{k,\mathbf{resc}}$. It is an interesting problem to explore convexity properties of $\mathcal{G}_{k,r}$ in a restricted set of densities such as bounded densities as in [36, 49].

1.5.1 Numerical Scheme

To simulate the dynamics of X we use a numerical scheme which was proposed in [10, 21] for the logarithmic case, and generalised to the one-dimensional fair-competition regime for the porous medium case $k \in (-1, 0)$ in [22]. It can easily be extended to rescaled variables adding a confining potential, and works just in the same way in the fast diffusion case $k \in (0, 1)$. We discretise the energy functional via a finite difference approximation of $X(\eta)$ on a regular grid. If $(X_i)_{1\le i\le n}$ are the positions of n ordered particles sharing equal mass $\Delta\eta = 1/n$ such that $X_1 < X_2 < \cdots < X_n$, then we define the discretised energy functional by

$$\mathcal{G}^n_{k,r}[(X_i)] = \frac{(\Delta\eta)^m}{m-1}\sum_{i=1}^{n-1}(X_{i+1}-X_i)^{1-m} + \chi\,(\Delta\eta)^2\sum_{1\le i\ne j\le n}\frac{\left|X_j-X_i\right|^k}{k} + r\frac{\Delta\eta}{2}\sum_{i=1}^{n}|X_i|^2$$

for $k \in (-1, 1)\backslash\{0\}$, and by

$$\mathcal{G}^n_{0,r}[(X_i)] = -\Delta\eta\sum_{i=1}^{n-1}\log\left(\frac{X_{i+1}-X_i}{\Delta\eta}\right) + \chi\,(\Delta\eta)^2\sum_{1\le i\ne j\le n}\log\left|X_j-X_i\right| + r\frac{\Delta\eta}{2}\sum_{i=1}^{n}|X_i|^2$$

in the logarithmic case $k = 0$. The Euclidean gradient flow of $\mathcal{G}^n_{k,r}$ writes for $1 < i < n$

$$\begin{aligned}\dot{X}_i = &-(\Delta\eta)^{m-1}\left((X_{i+1}-X_i)^{-m} - (X_i-X_{i-1})^{-m}\right)\\ &-2\chi\Delta\eta\sum_{1\le j\ne i\le n}\operatorname{sign}(i-j)\left|X_i-X_j\right|^{k-1} - rX_i\,,\end{aligned} \tag{1.46}$$

complemented with the dynamics of the extremal points

$$\dot{X}_1 = -(\Delta\eta)^{m-1}(X_2-X_1)^{-m} + 2\chi\Delta\eta\sum_{j\ne 1}\left|X_j-X_1\right|^{k-1} - rX_1\,, \tag{1.47}$$

$$\dot{X}_n = (\Delta\eta)^{m-1}(X_n-X_{n-1})^{-m} - 2\chi\Delta\eta\sum_{j\ne n}\left|X_j-X_n\right|^{k-1} - rX_n\,. \tag{1.48}$$

Equations (1.47)–(1.48) follow from imposing $X_0 = -\infty$ and $X_{n+1} = +\infty$ so that the initial centre of mass $\sum_{i=1}^n X_i = 0$ is conserved. Working with the pseudo-inverse of the cumulative distribution function of ρ also has the advantage that we can express the Wasserstein distance between two densities ρ and $\tilde{\rho}$ in a more tractable way. More precisely, if ψ' is the optimal map which transports $\tilde{\rho}$ onto ρ, then the Monge-Ampére equation (1.13) is an increasing rearrangement. Let F and $\tilde{F}$ be the cumulative distribution function of ρ and $\tilde{\rho}$ respectively, with pseudo-inverses

X and $\tilde{X}$. Then we have

$$\tilde{F}(x) = \int_{-\infty}^{x} \tilde{\rho}(y)\, dy = \int_{-\infty}^{\psi'(x)} \rho(y)\, dy = F \circ \psi'(x)\,.$$

Hence the transport map is given explicitly by $\psi' = F^{-1} \circ \tilde{F}$, and we have for the Wasserstein distance

$$\mathbf{W}(\rho, \tilde{\rho})^2 = \int_0^1 \left|\tilde{F}^{-1}(\eta) - F^{-1}(\eta)\right|^2 d\eta = \int_0^1 \left|\tilde{X}(\eta) - X(\eta)\right|^2 d\eta = ||\tilde{X} - X||_2^2\,. \tag{1.49}$$

This means that this numerical scheme can be viewed formally as the time discretisation of the abstract gradient flow equation (1.6) in the Wasserstein-2 metric space, which corresponds to a gradient flow in $L^2((0,1))$ for the pseudo-inverse X,

$$\dot{X}(t) = -\nabla_{L^2} \mathcal{G}_{k,r}[X(t)]\,.$$

Discretising (1.46)–(1.47)–(1.48) by an implicit in time Euler scheme, this numerical scheme then coincides with a Jordan-Kinderlehrer-Otto (JKO) steepest descent scheme (see [10, 70] and references therein). The solution at each time step of the non-linear system of equations is obtained by an iterative Newton-Raphson procedure.

1.5.2 Results

For the logarithmic case $k = 0, m = 1$, we know that the critical interaction strength is given by $\chi_c = 1$ separating the blow-up regime from the regime where self-similar solutions exist [8, 9, 50]. As shown in [24], there is no critical interaction strength for the fast diffusion regime $k > 0$, however the dichotomy appears in the porous medium regime $k < 0$ [12, 24]. It is not known how to compute the critical parameter $\chi_c(k)$ explicitly for $k < 0$, however, we can make use of the numerical scheme described in Sect. 1.5.1 to compute $\chi_c(k)$ numerically.

Figure 1.2 gives an overview of the behaviour of solutions. In the grey region, we observe finite-time blow-up of solutions, whereas for a choice of (k, χ) in the white region, solutions converge exponentially fast to a unique self-similar profile. The critical regime is characterised by the black line $\chi_c(k)$, $-1 < k \leq 0$, separating the grey from the white region. Note that numerically we have $\chi_c(-0.99) = 0.11$ and $\chi_c(0) = 1$. Figure 1.2 has been created by solving the rescaled equation (1.7) using the numerical scheme described above with particles equally spaced at a distance $\Delta\eta = 10^{-2}$. For all choices of $k \in (-1, 0)$ and $\chi \in (0, 1.5)$, we choose as initial condition a centered normalised Gaussian with variance $\sigma^2 = 0.32$, from where we let the solution evolve with time steps of size $\Delta t = 10^{-3}$. We terminate the

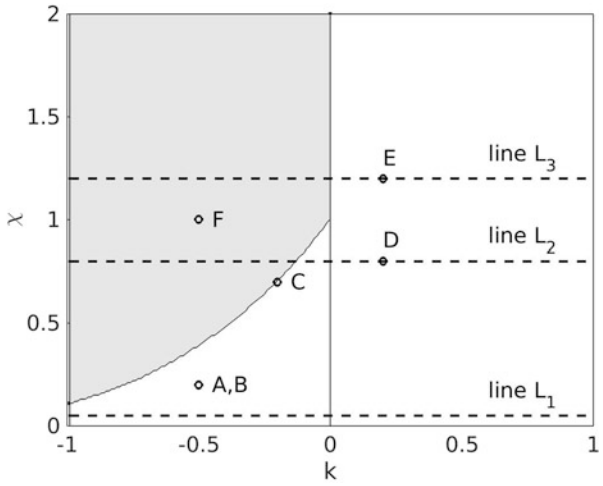

Fig. 1.2 Regions of blow-up (*grey*) and convergence to self-similarity (*white*). The notation refers to subsequent figures as follows: Lines L_1, L_2 and L_3 show the asymptotic profiles over the range $k \in (-1, 1)$ for $\chi = 0.05$, $\chi = 0.8$ and $\chi = 1.2$ respectively (Fig. 1.3). Point A shows the density evolution at $(k, \chi) = (-0.5, 0.2)$ in original variables (Fig. 1.4), and Point B for the same choice of parameters $(k, \chi) = (-0.5, 0.2)$ in rescaled variables (Fig. 1.5). Points C, D and E correspond to simulations at $(-0.2, 0.7)$ (Fig. 1.6), $(0.2, 0.8)$ (Fig. 1.7) and $(0.2, 1.2)$ (Fig. 1.8) respectively in the parameter space (k, χ), all in rescaled variables. Point F corresponds to simulations at $(k, \chi) = (-0.5, 1.0)$ in original variables (Fig. 1.9)

time evolution of the density distribution if one of the following two conditions is fulfilled: either the L^2-error between two consecutive solutions is less than a certain tolerance (i.e. we consider that the solution converged to a stationary state), or the Newton-Raphson procedure does not converge for $\rho(t, x)$ at some time $t < t_{max}$ because the mass is too concentrated (i.e. the solution sufficiently approached a Dirac Delta to assume blow-up). We choose t_{max} large enough, and $\Delta\eta$ and Δt small enough so that one of the two cases occurs. For Fig. 1.2, we set the maximal time to $t_{max} = 10$ and the tolerance to 10^{-5}. For a fixed k, we start with $\chi = 0.01$ and increase the interaction strength by 0.01 each run until $\chi = 1.5$. This is repeated for each k from -0.99 to 0 in 0.01 steps. For a given k, the numerical critical interaction strength $\chi_c(k)$ is defined to be the largest χ for which the numerical solution can be computed without blow-up until the L^2-error between two consecutive solutions is less than the specified tolerance. In what follows, we investigate the behaviour of solutions in more detail for chosen points in the parameter space Fig. 1.2.

1.5.2.1 Lines L_1, L_2 and L_3

Apart from points $A - F$ shown in Fig. 1.2, it is also interesting to observe how the asymptotic profile changes more globally as we move through the parameter

space. To this purpose, we choose three different values of χ and investigate how the stationary profile in rescaled variables changes with k. Three representative choices of interaction strengths are given by lines L_1, L_2 and L_3 as indicated in Fig. 1.2, where L_1 corresponds to $\chi = 0.05$ and lies entirely in the self-similarity region (white), L_2 corresponds to $\chi = 0.8$ and captures part of the sub-critical region in the porous medium regime $k < 0$ (white), as well as some of the blow-up regime (grey), and finally line L_3 which corresponds to $\chi = 1.2$ and therefore captures the jump from the self-similarity (white) to the blow-up region (grey) at $k = 0$. Note also that points D and E are chosen to lie on lines L_2 and L_3 respectively as to give a more detailed view of the behaviour on these two lines for the same k-value. The asymptotic profiles over the range $k \in (-1, 1)$ for lines L_1, L_2 and L_3 are shown in Fig. 1.3, all with the same choice of parameters using time step size $\Delta t = 10^{-3}$ and equally spaced particles at distance $\Delta\eta = 10^{-2}$.

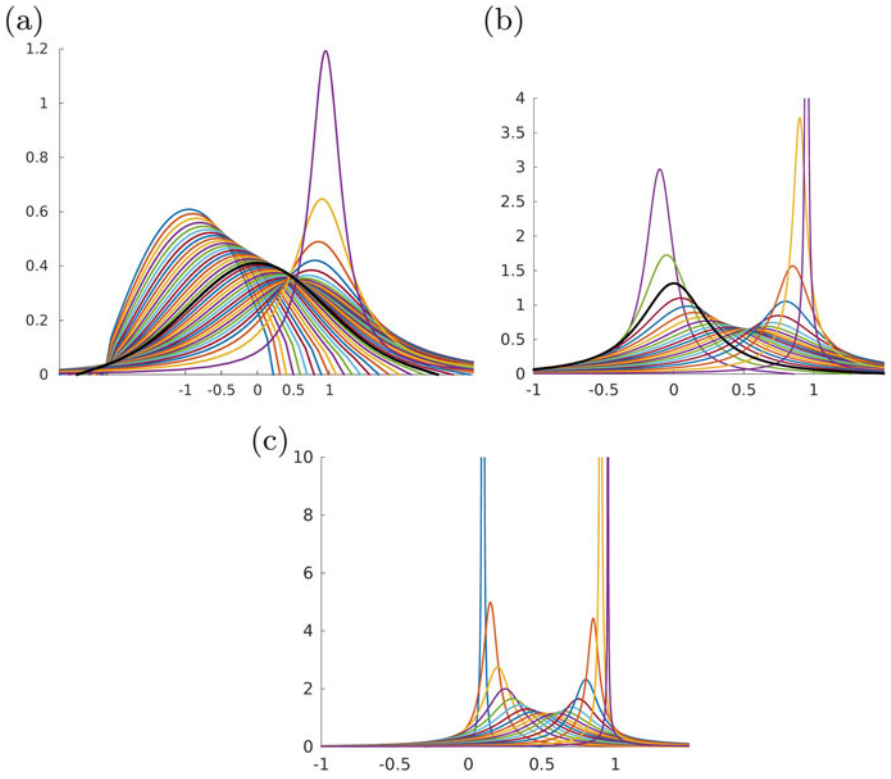

Fig. 1.3 Profiles of stationary states in rescaled variables ($r = 1$) corresponding to lines L_1, L_2 and L_3 in Fig. 1.2 for (**a**) $\chi = 0.05$, (**b**) $\chi = 0.8$ and (**c**) $\chi = 1.2$ with k ranging from 0.95 to (**a**) -0.95, (**b**) -0.1 and (**c**) 0.1 in 0.05 steps respectively. All stationary states are centered at zero, but are here displayed shifted so that they are centered at their corresponding value of k. The *black curve* indicates the stationary state for $k = 0$

For each choice of interaction strength χ, we start with $k = 0.95$ and decrease k in 0.05 steps for each simulation either until $k = -0.95$ is reached, or until blow-up occurs and (k, χ) lies within the grey region. For each simulation, we choose as initial condition the stationary state of the previous k-value (starting with a centered normalised Gaussian distribution with variance $\sigma^2 = 0.32$ for $k = 0.95$). As for Fig. 1.2, we terminate the time evolution of the density distribution for a given choice of k and χ if either the L^2-error between two consecutive solutions is less than the tolerance 10^{-5}, or the Newton-Raphson procedure does not converge. All stationary states are centered at zero. To better display how the profile changes for different choices of k, we shift each stationary state in Fig. 1.3 so that it is centered at the corresponding value of k. The black curve indicates the stationary profile for $k = 0$.

In Fig. 1.3a, we observe corners close to the edge of the support of the stationary profiles for $k < 0$. This could be avoided by taking $\Delta\eta$ and Δt smaller, which we chose not to do here, firstly to be consistent with Fig. 1.2 and secondly to avoid excessive computation times. For interaction strength $\chi = 0.8$, the smallest k for which the solution converges numerically to a stationary state is $k = -0.1$ (see Fig. 1.3b). This fits with what is predicted by the critical curve $\chi_c(k)$ in Fig. 1.2 (line L_2).

In Fig. 1.3b, c, we see that the stationary profiles become more and more concentrated for k approaching the critical parameter $k = k^*$ with $\chi = \chi_c(k^*)$, which is to be expected as we know that the stationary state $\bar{\rho}_k$ converges to a Dirac Delta as k approaches the blow-up region. In fact, for $\chi = 1.2$ the numerical scheme stops converging for $k = 0.05$ already since the mass is too concentrated, and so we only display profiles up to $k = 0.1$ in Fig. 1.3c. Further, in all three cases $\chi = 0.05$, $\chi = 0.8$ and $\chi = 1.2$ we observe that the stationary profiles become more and more concentrated as $k \to 1$. This reflects the fact that attractive forces dominate as the diffusivity m converges to zero. Finally, note that we have chosen here to show only a part of the full picture for Fig. 1.3b, c, cutting the upper part. More precisely, the maximum of the stationary state for $k = 0.95$ and $\chi = 0.8$ in Fig. 1.3b lies at 75.7474, whereas it is at 3216.8 for parameter choices $k = 0.95$ and $\chi = 1.2$ shown in Fig. 1.3c.

1.5.2.2 Points *A*-*F*

Let us now investigate in more detail the time-evolution behaviour at the points A–F in Fig. 1.2. For $k = -0.5$ in the porous medium regime and sub-critical $\chi = 0.2$ (point A in Fig. 1.2), the diffusion dominates and the density goes pointwise to zero as $t \to \infty$ in original variables. Figure 1.4a, b show the inverse cumulative distribution function and the density profile for $(k, \chi) = (-0.5, 0.2)$ respectively, from time $t = 0$ (black) to time $t = 100$ (red) in time steps of size $\Delta t = 10^{-3}$ and with $\Delta\eta = 10^{-2}$. We choose a centered normalised Gaussian with variance $\sigma^2 = 0.32$ as initial condition. Figure 1.4c shows the evolution of the free energy (1.1) over time, which continues to decay as expected.

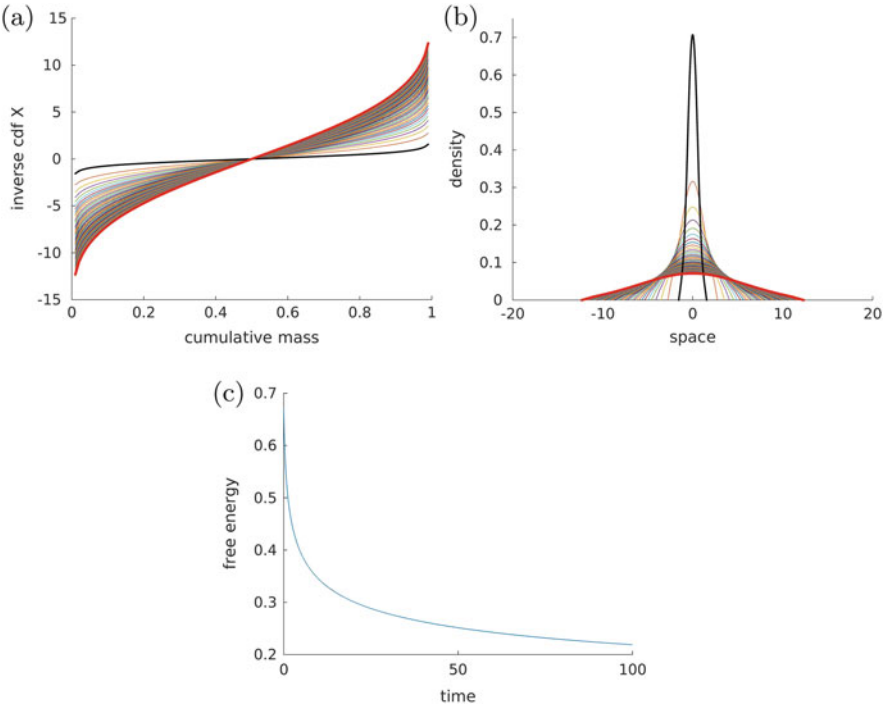

Fig. 1.4 Point A: $\chi = 0.2$, $k = -0.5$, $r = 0$. (**a**) Inverse cumulative distribution function, (**b**) solution density, (**c**) free energy

For exactly the same choice of parameters $(k, \chi) = (-0.5, 0.2)$ and the same initial condition we then investigate the evolution in rescaled variables (point B in Fig. 1.2), and as predicted by Proposition 1.4.4, the solution converges to a stationary state. See Fig. 1.5a, b for the evolution of the inverse cumulative distribution function and the density distribution with $\Delta t = 10^{-3}$ and $\Delta\eta = 10^{-3}$ from $t = 0$ (black) to the stationary state $\bar{\rho}$ (red). Again, we terminate the evolution as soon as the L^2-distance between the numerical solution at two consecutive time steps is less than a certain tolerance, chosen at 10^{-5}. We see that the solution converges very quickly both in relative energy $|\mathcal{F}_k[\rho(t)] - \mathcal{F}_k[\bar{\rho}]|$ (Fig. 1.5c) and in terms of the Wasserstein distance to the solution at the last time step $\mathbf{W}(\rho(t), \bar{\rho})$ (Fig. 1.5e). To check that the convergence is indeed exponential as predicted by Proposition 1.4.4, we fit a line to the logplot of both the relative free energy (between times $t = 0$ and $t = 0.9$), see Fig. 1.5d, and to the logplot of the Wasserstein distance to equilibrium, see Fig. 1.5f. In both cases, we obtain a fitted line $y = -a * t + b$ with some constant b and rate $a = 7.6965$ for the relative free energy and rate $a = 4.392$ for the Wasserstein distance to equilibrium. Recall that the L^2-error between two solutions $X(\eta)$ and $\tilde{X}(\eta)$ is equal to the Wasserstein distance between

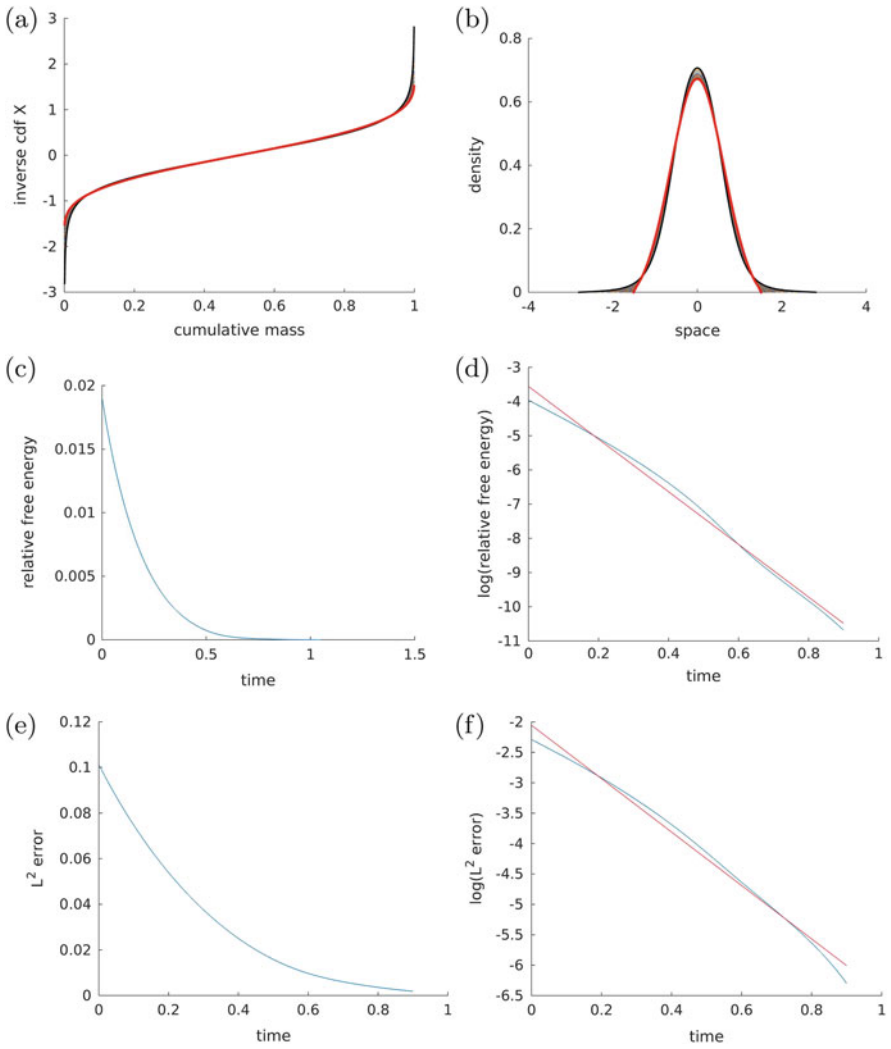

Fig. 1.5 Point B: $\chi = 0.2$, $k = -0.5$, $r = 1$. (**a**) Inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**b**) solution density from initial condition (*black*) to the profile at the last time step (*red*), (**c**) relative free energy, (**d**) log(relative free energy) and fitted line between times 0 and 0.9 with slope -7.6965 (*red*), (**e**) L^2-error between the solutions at time t and at the last time step, (**f**) log(L^2-error) and fitted line with slope -4.392 (*red*)

the corresponding densities $\rho(x)$ and $\tilde{\rho}(x)$ as described in (1.49). We observe a rate of convergence that is in agreement with [19, 26, 51] for the logarithmic case $k = 0$.

For parameter choices $k = -0.2$ and $\chi = 0.7$ (point C in Fig. 1.2), we are again in the sub-critical regime where solutions converge to a stationary state in

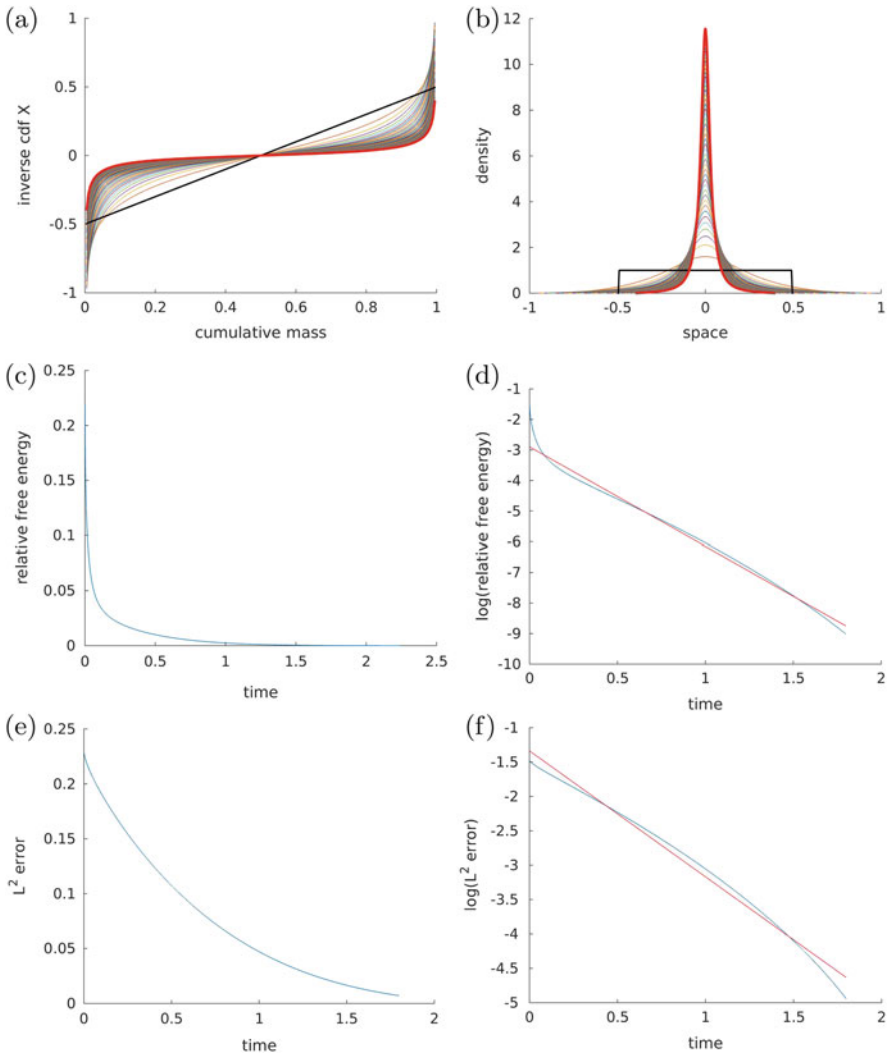

Fig. 1.6 Point C: $\chi = 0.7$, $k = -0.2$, $r = 1$. (**a**) Inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**b**) solution density from initial condition (*black*) to the profile at the last time step (*red*), (**c**) relative free energy, (**d**) log(relative free energy) and fitted line between times 0 and 1.8 with slope -3.2522 (*red*), (**e**) L^2-error between the solutions at time t and at the last time step, (**f**) log(L^2-error) and fitted line with slope -1.8325 (*red*)

rescaled variables according to Proposition 1.4.4, see Fig. 1.6a, b. However, point C is closer to the critical interaction strength $\chi_c(k)$ than point B (numerically, we have $\chi_c(-0.2) = 0.71$), and as a result we can observe that the stationary density $\bar{\rho}$ in Fig. 1.6b (red) is more concentrated than in Fig. 1.5b. Here, we choose as initial

condition a characteristic function supported on the ball centered at zero with radius $1/2$ (black, Fig. 1.6b), and fix $\Delta t = 10^{-3}$, $\Delta\eta = 5 \times 10^{-3}$ with tolerance 10^{-5}. We observe that the solution converges very quickly to a stationary state both in relative free energy $|\mathcal{F}_k[\rho(t)] - \mathcal{F}_k[\bar{\rho}]|$ (Fig. 1.6c) and in terms of the Wasserstein distance to equilibrium $\mathbf{W}(\rho(t), \bar{\rho})$ (Fig. 1.6e). To investigate the exponential rate of convergence, we fit again a line to the logplot of both the relative free energy (here between times $t = 0$ and $t = 1.8$) see Fig. 1.6d, and the Wasserstein distance to equilibrium, see Fig. 1.6f. We obtain fitted lines $y = -a * t + b$ with some constant b and rate $a = 3.2407$ for the relative free energy, whereas the rate is $a = 1.8325$ for the Wasserstein distance to equilibrium.

Next, we are looking at point D in Fig. 1.2, which corresponds to the choice $(k, \chi) = (0.2, 0.8)$ and is part of line L_2 (see Fig. 1.3b). Since point D lies in the fast diffusion regime $k > 0$, no critical interaction strength exists [24], and so we look at convergence to self-similarity. Figure 1.7a, b display the evolution of the inverse cumulative distribution function and the density distribution from $t = 0$ (black) to the stationary state $\bar{\rho}$ (red) in rescaled variables including the solutions at 50 intermediate time steps. We start with a characteristic function supported on a centered ball of radius $1/2$. Choosing $\Delta t = 10^{-3}$ and $\Delta\eta = 10^{-2}$ is enough. The density instantaneously becomes supported on the whole space for any $t > 0$ as shown in the proof of [24, Corollary 4.4], which cannot be fully represented numerically since the tails are cut by numerical approximation, see Fig. 1.7a, b. Again, we observe very fast convergence both in relative energy (Fig. 1.7c, d) and in Wasserstein distance to equilibrium (Fig. 1.7e, f) as predicted by Proposition 1.4.6. A logplot of the relative free energy (Fig. 1.7d) and the Wasserstein distance to equilibrium (Fig. 1.7f) show exponential rates of convergence with rates $a = 3.6904$ and $a = 1.9148$ respectively for the fitted line $y = -a * t + b$ with some constant b and for times $0.2 \le t \le 3.8$.

For the same choice of $k = 0.2$ in the fast diffusion regime, but with higher interaction strength $\chi = 1.2$ (point E in Fig. 1.2, which is part of line L_3, see Fig. 1.3c), we obtain a similar behaviour. Figure 1.8a, b show the inverse cumulative distribution function and the density distribution, both for the initial data (black), a characteristic supported on the centered ball of radius $1/2$, and for the stationary state $\bar{\rho}$ (red). Here we choose as before $\Delta t = 10^{-3}$ and $\Delta\eta = 10^{-2}$. We observe that the stationary state for $\chi = 1.2$ (Fig. 1.8b) is more concentrated than for $\chi = 0.8$ (Fig. 1.7b), which is exactly what we would expect for decreasing k as $\bar{\rho}$ approaches a Dirac Delta for $k \to 0$ if $\chi = 1.2$, whereas it becomes compactly supported if $\chi = 0.8$ as k crosses the χ-axis (see [24, Corollary 3.9]). Again, we observe very fast convergence both in relative energy (Fig. 1.8c, d) and in Wasserstein distance to equilibrium (Fig. 1.8e, f) as predicted by Proposition 1.4.6. A logplot of the relative free energy (Fig. 1.8d) and the Wasserstein distance to equilibrium (Fig. 1.8f) show exponential rates of convergence with rates $a = 3.6898$ and $a = 1.9593$ respectively for the fitted lines $y = -a * t + b$ and some constant b between times $0.3 \le t \le 3.5$.

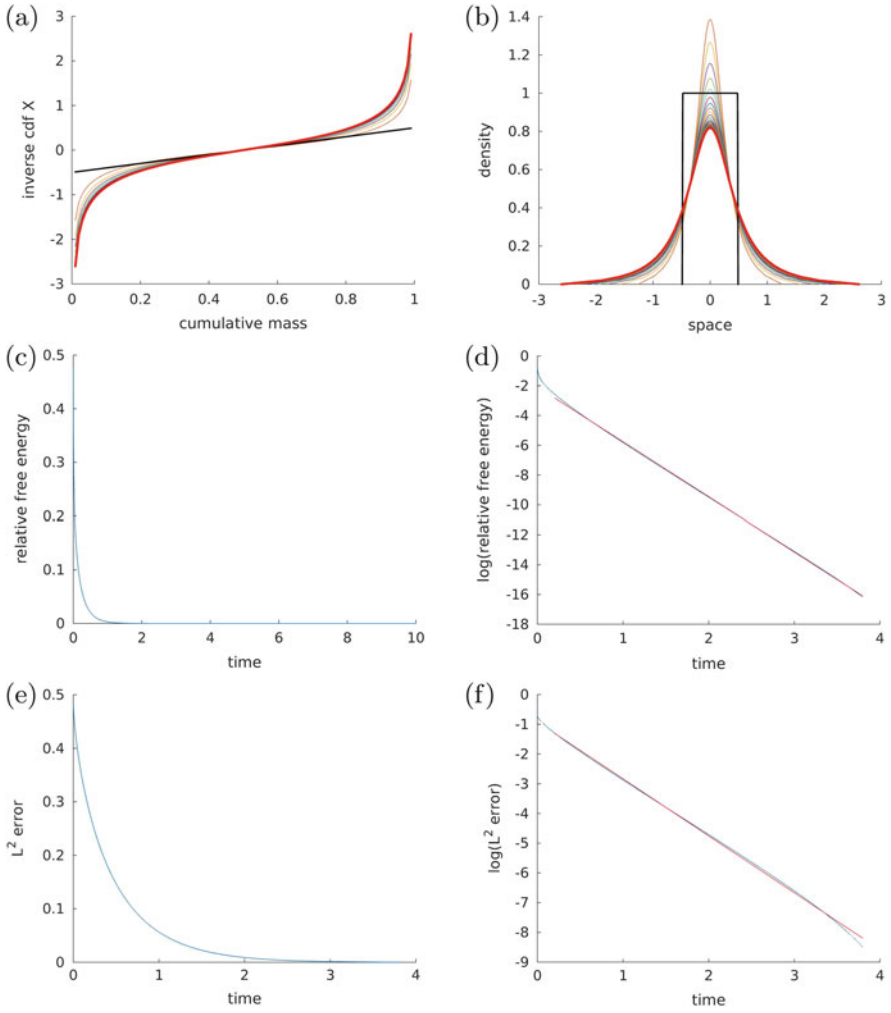

Fig. 1.7 Point D: $\chi = 0.8$, $k = 0.2$, $r = 1$. (**a**) Inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**b**) solution density from initial condition (*black*) to the profile at the last time step (*red*), (**c**) relative free energy, (**d**) log(relative free energy) and fitted line between times 0.2 and 3.8 with slope -3.6904 (*red*), (**e**) L^2-error between the solutions at time t and at the last time step, (**f**) log(L^2-error) and fitted line between times 0.2 and 3.8 with slope -1.9148 (*red*)

Finally, let us investigate the behaviour for $(k, \chi) = (-0.5, 1)$ in original variables (point F in Fig. 1.2). Point F lies in the porous medium regime and we expect blow-up as $\chi_c(-0.5) < 1$, see Sect. 1.4.1.3. If the mass becomes too concentrated, the Newton-Raphson procedure does not converge and the simulation stops. We have therefore adapted the numerical scheme to better capture the blow-up. We fix $\Delta t = 10^{-3}$ and $\Delta\eta = 10^{-3}$ and take a centered normalised Gaussian

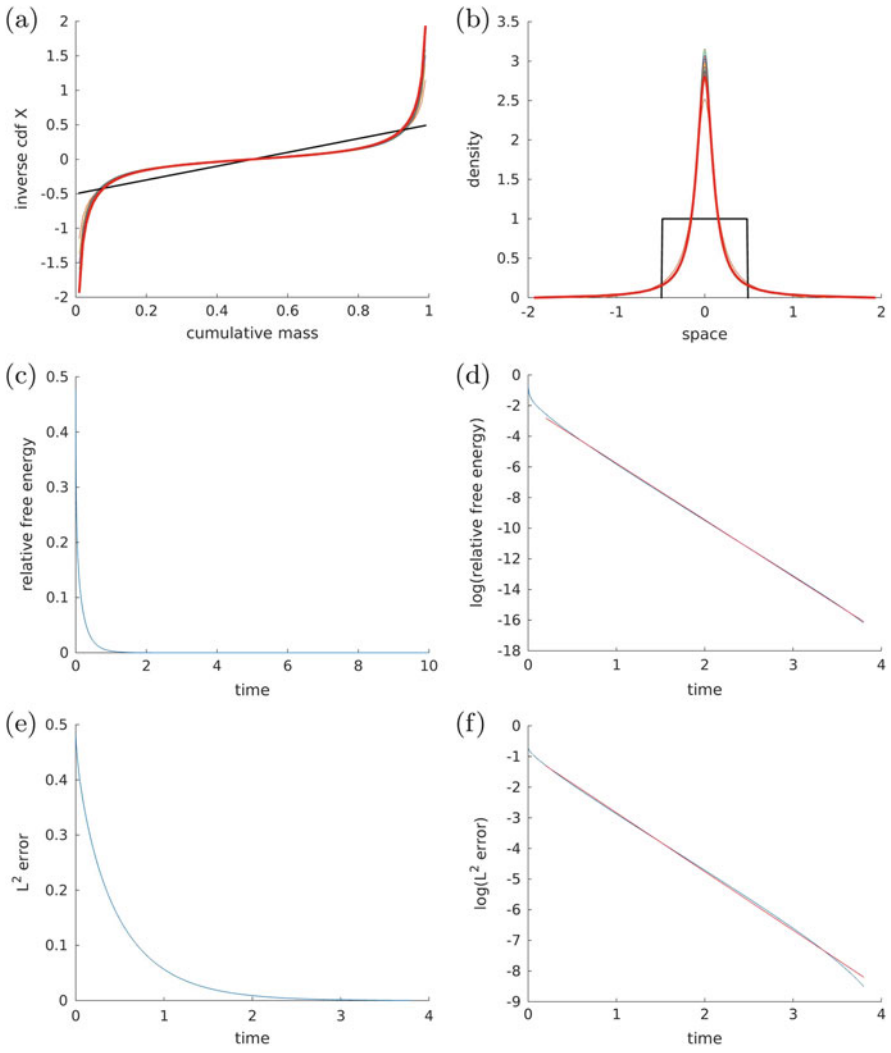

Fig. 1.8 Point E: $\chi = 1.2$, $k = 0.2$, $r = 1$. (**a**) Inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**b**) solution density from initial condition (*black*) to the profile at the last time step (*red*), (**c**) relative free energy, (**d**) log(relative free energy) and fitted line between times 0.3 and 3.5 with slope -3.6898 (*red*), (**e**) L^2-error between the solutions at time t and at the last time step, (**f**) log(L^2-error) and fitted line between times 0.3 and 3.5 with slope -1.9593 (*red*)

with variance $\sigma^2 = 0.32$ as initial data. When the simulation stops, we divide the time step size Δt by two and repeat the simulation, taking as initial condition the last density profile before blow-up. This process can be repeated any number of times, each time improving the approximation of an emerging Dirac Delta. The

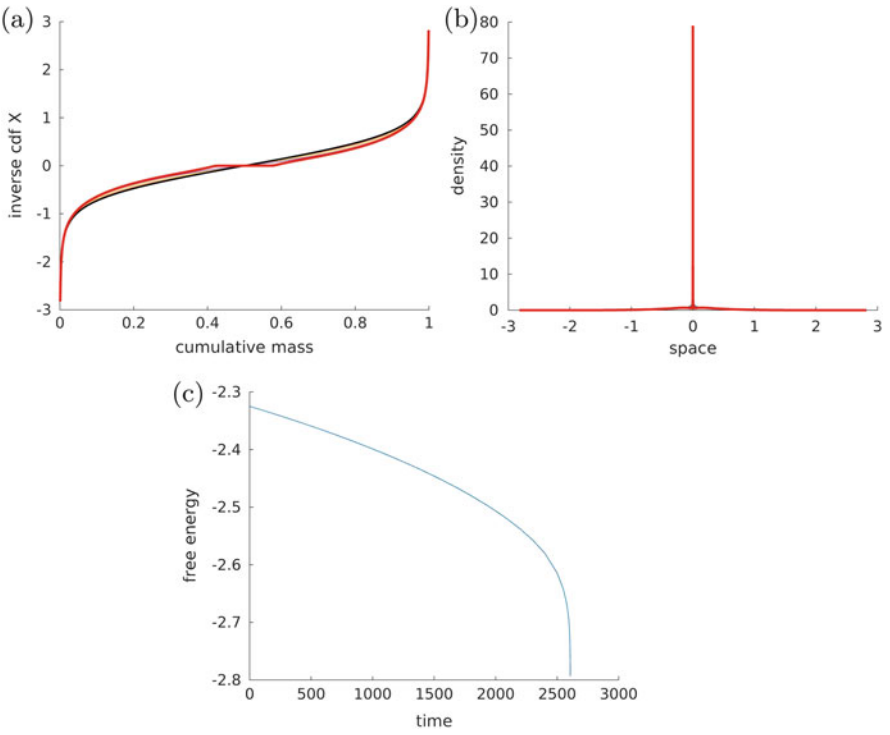

Fig. 1.9 Point F: $\chi = 1$, $k = -0.5$, $r = 0$. (**a**) Inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**b**) solution density from initial condition (*black*) to the profile at the last time step (*red*), (**c**) free energy

formation of a Dirac Delta in Fig. 1.9b corresponds to the formation of a plateaux in Fig. 1.9a. As expected from the analysis in Sect. 1.4.1.3, the free energy diverges to $-\infty$ (Fig. 1.9c).

1.6 Explorations in Other Regimes

1.6.1 Diffusion-Dominated Regime in One Dimension

The numerical scheme described here gives us a tool to explore the asymptotic behaviour of solutions for parameter choices that are less understood. For example, choosing $\chi = 0.3$, $k = -0.5$ and $m = 1.6$ in original variables ($r = 0$), we observe convergence to a compactly supported stationary state, see Fig. 1.10. This

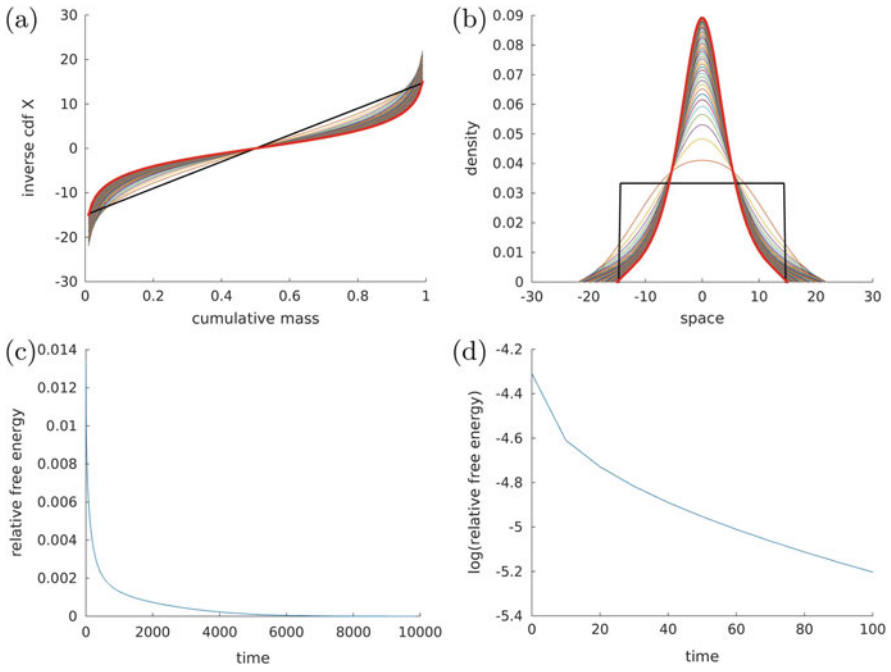

Fig. 1.10 Diffusion-dominated regime: $\chi = 0.3$, $k = -0.5$, $m = 1.6$, $r = 0$. (**a**) Inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**b**) solution density from initial condition (*black*) to the profile at the last time step (*red*), (**c**) relative free energy, (**d**) log(relative free energy)

choice of parameters is within the diffusion-dominated regime since $m + k > 1$ (see Definition 1.2.6). We choose as initial condition a normalised characteristic function supported on $B(0, 15)$ from where we let the solution evolve with time steps of size $\Delta t = 10^{-2}$ and particles spaced at $\Delta\eta = 10^{-2}$. We let the density solution evolve until the L^2-error between two consecutive solutions is less than 10^{-7}. Note that here $m + k = 1.1$ is close to the fair-competition regime, for which $\chi_c(-0.5) = 0.39$ (see Fig. 1.2).

1.6.2 Attraction-Dominated Regime in Any Dimension

In the attraction-dominated regime $N(m-1) + k < 0$ (corresponding to Definition 1.2.6) both global existence of solutions and blow-up can occur in original variables in dimension $N \geq 1$ depending on the choice of initial data [6, 20, 42, 43, 48, 63, 74, 77]. Using the numerical scheme introduced in Sect. 1.5, we can demonstrate this change of behaviour numerically in one dimension, see Figs. 1.11 (dispersion) and 1.12 (blow-up).

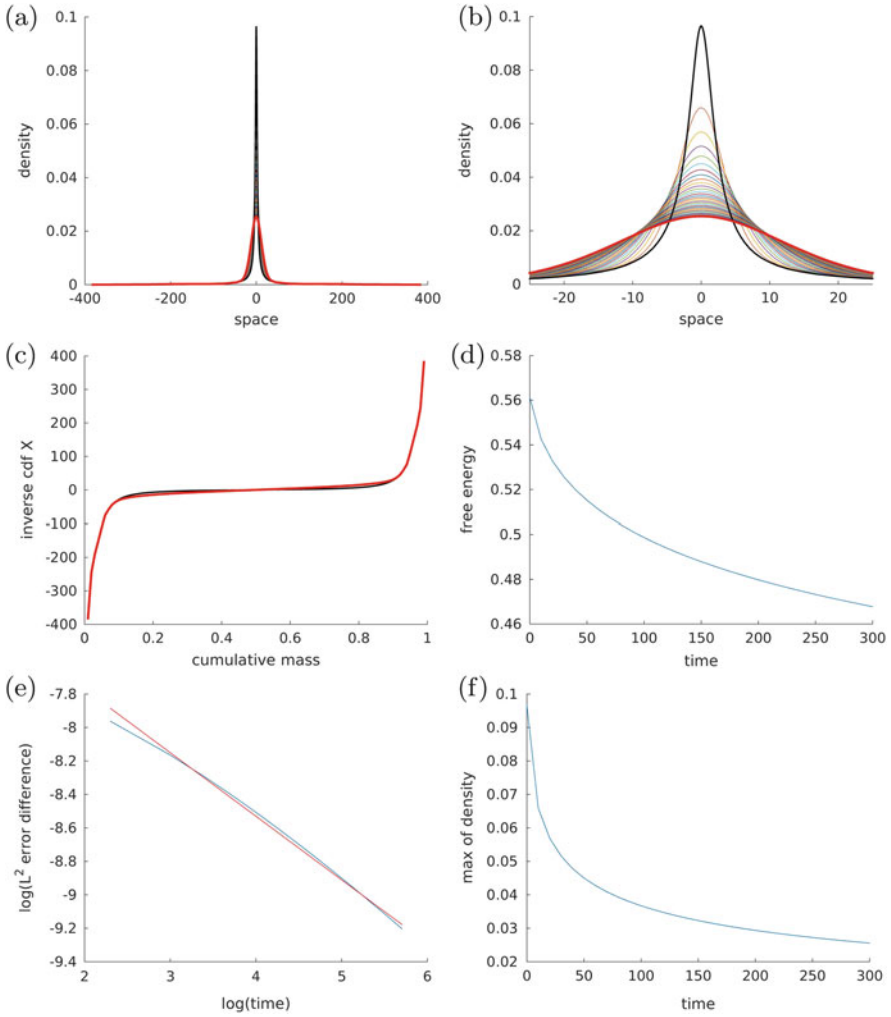

Fig. 1.11 Attraction-dominated regime: $\chi = 0.35$, $k = -1/2$, $m = 4/3$, $r = 0$ with initial data $\rho(t = 0, x) = \rho_{HLS,\lambda_0,c_0}(x) < \rho_{HLS,\lambda_0,c^*}(x)$ for all $x \in \mathbb{R}$ with $c_0 = 0.4c^*$. (**a**) Solution density from initial condition (*black*) to the profile at the last time step (*red*), (**b**) zoom of Figure (**a**), (**c**) inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**d**) free energy, (**e**) log-log plot of the L^2-error difference between two consecutive solutions and fitted line with slope -0.37987, (**f**) time evolution of $\max_x \rho(t, x)$

We will now investigate in more detail a special parameter choice (m, k) that belongs to the attraction-dominated regime. Instead of fixing m and k such that attractive and repulsive forces are in balance ($N(m-1)+k=0$), one may choose instead to investigate the regime where the free energy functional (1.1) is conformal invariant, corresponding to $m = 2N/(2N+k)$. For $k < 0$, this corresponds to the case $p = q = m$ in the HLS inequality (1.16) for which the optimisers ρ_{HLS} and the optimal constant C_{HLS} are known explicitly [61]. We have the following existence result:

Theorem 1.6.1 *Let $\chi > 0$, $k \in (-N, 0)$ and $m = 2N/(2N+k) \in (1, 2)$. Then the free energy functional $\mathcal{F}_k$ admits a critical point in $\mathcal{Y}$.*

Proof Following the approach in [43], we rewrite the free energy functional (1.1) as a sum of two functionals

$$\mathcal{F}_k[\rho] = \mathcal{F}_k^1[\rho] + \mathcal{F}_k^2[\rho]\,,$$

where

$$\begin{aligned}\mathcal{F}_k^1[\rho] :=& \frac{1}{N(m-1)}||\rho||_m^m\left(1-\chi C_{HLS}\frac{N(m-1)}{(-k)}||\rho||_m^{2-m}\right)\\ =& \frac{2N+k}{N(-k)}||\rho||_m^m\left(1-\chi C_{HLS}\frac{N}{2N+k}||\rho||_m^{2-m}\right),\end{aligned} \tag{1.50}$$

and

$$\mathcal{F}_k^2[\rho] := \frac{\chi}{(-k)}\left(C_{HLS}||\rho||_m^2 - \iint_{\mathbb{R}^N\times\mathbb{R}^N}|x-y|^k\rho(x)\rho(y)\,dxdy\right). \tag{1.51}$$

By the HLS inequality (1.16), the second functional (1.51) is bounded below for any $\chi > 0$,

$$\mathcal{F}_k^2[\rho] \geq 0\,, \quad \forall \rho \in \mathcal{Y}\,,$$

and by Lieb [61, Theorem 3.1], there exists a family of optimisers $\rho_{HLS,\lambda,c}$,

$$\rho_{HLS,\lambda,c}(x) = c\left(\frac{\lambda}{\lambda^2+|x|^2}\right)^{N/m}, \quad \lambda > 0, c > 0 \tag{1.52}$$

satisfying $\mathcal{F}_k^2[\rho_{HLS,\lambda,c}] = 0$ with the optimal constant C_{HLS} given by

$$C_{HLS} := \pi^{-k/2}\left(\frac{\Gamma\left(\frac{N+k}{2}\right)}{\Gamma\left(N+\frac{k}{2}\right)}\right)\left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right)^{-(N+k)/N}.$$

The parameter $\lambda > 0$ in (1.52) corresponds to the scaling that leaves the L^m-norm of $\rho_{HLS,\lambda,c}$ invariant. Since the first variation of the functional $\mathcal{F}_k^1$ defined in (1.50) is given by

$$\frac{\delta \mathcal{F}_k^1}{\delta \rho}[\rho](x) = \frac{2}{(-k)} \left(1 - \chi C_{HLS} ||\rho||_m^{2-m}\right) \rho^{m-1}(x)$$

and since the L^m-norm of the optimiser can be calculated explicitly,

$$||\rho_{HLS,\lambda,c}||_m = c \left(\frac{2^{1-N} \pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} \right)^{1/m},$$

there exists a unique choice of $(\lambda, c) = (\lambda^*, c^*)$ for each $\chi > 0$ such that

$$\frac{\delta \mathcal{F}_k^1}{\delta \rho}[\rho_{HLS,\lambda^*,c^*}](x) = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_{HLS,\lambda^*,c^*}(x)\, dx = 1$$

given by

$$c^*(\chi) := \left(\frac{2^{1-N} \pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} \right)^{-1/m} (\chi C_{HLS})^{1/(m-2)}, \quad \lambda^*(\chi) := \left(\int_{\mathbb{R}^N} \rho_{HLS,1,c^*(\chi)}(x)\, dx \right)^{2/k}. \tag{1.53}$$

Hence ρ_{HLS,λ^*,c^*} is a critical point of $\mathcal{F}_k$ in $\mathcal{Y}$. □

We can choose to leave $\lambda > 0$ as a free parameter in (1.52), only fixing $c = c^*(\chi)$ so that ρ_{HLS,λ,c^*} is a critical point of $\mathcal{F}_k$ with arbitrary mass. We conjecture that a similar result to [43, Theorem 2.1] holds true for general $k \in (-N, 0)$ and $m = 2N/(2N + k)$ for radially symmetric initial data:

Conjecture 1.6.2 (Global Existence vs Blow-Up) Let $\chi > 0$, $k \in (-N, 0)$ and $m = 2N/(2N + k)$ in dimension $N \geq 1$. Assume the initial datum $\rho_0 \in \mathcal{Y}$ is radially symmetric.

(i) If there exists $\lambda_0 > 0$ such that

$$\rho_0(r) < \rho_{HLS,\lambda_0,c^*}(r), \quad \forall\, r \geq 0,,$$

then any radially symmetric solution $\rho(t, r)$ of (1.4) with initial datum $\rho(0, r) = \rho_0(r)$ is vanishing in $L^1_{loc}\left(\mathbb{R}^N\right)$ as $t \to \infty$.

(ii) If there exists $\lambda_0 > 0$ such that

$$\rho_0(r) > \rho_{HLS,\lambda_0,c^*}(r)\,, \quad \forall\, r \geq 0\,,$$

then any radially symmetric solution $\rho(t,r)$ of (1.4) with initial datum $\rho(0,r) = \rho_0(r)$ must blow-up at a finite time T^* or has a mass concentration at $r = 0$ as time goes to infinity in the sense that there exist $R(t) \to 0$ as $t \to \infty$ and a positive constant $C > 0$ such that

$$\int_{B(0,R(t))} \rho(t,x)\,dx \geq C\,.$$

Further, we expect the following to be true analogous to [43]:

Conjecture 1.6.3 (Unstable Stationary State) For any $\chi > 0$, the density $\rho_{HLS,\lambda^*,c^*} \in \mathcal{Y}$ with (λ^*, c^*) given by (1.53) is an unstable stationary state of Eq. (1.4).

Numerically, we indeed observe the behaviour predicted in Conjecture 1.6.2 for $N = 1$. Using the scheme introduced in Sect. 1.5, we choose as initial data the density ρ_{HLS,λ_0,c_0} given by the optimisers of the HLS inequality (1.52). For any choice of $c_0 > 0$, we fix $\lambda_0 > 0$ such that ρ_{HLS,λ_0,c_0} has unit mass and is therefore in $\mathcal{Y}$. Note that ρ_{HLS,λ_0,c_0} is not a critical point of $\mathcal{F}_k$ unless $c_0 = c^*$. Comparing with the stationary state ρ_{HLS,λ_0,c^*}, we have

$$\operatorname{sign}\left(c^* - c_0\right) = \operatorname{sign}\left(\rho_{HLS,\lambda_0,c^*}(x) - \rho_{HLS,\lambda_0,c_0}(x)\right)\,, \qquad \forall x \in \mathbb{R}\,.$$

Note that the mass of the stationary state ρ_{HLS,λ_0,c^*} is given by

$$\int_{\mathbb{R}^N} \rho_{HLS,\lambda_0,c^*(\chi)}(x)\,dx = \lambda_0^{-k/2} \int_{\mathbb{R}^N} \rho_{HLS,1,c^*(\chi)}(x)\,dx\,,$$

which is equal to one if and only if $\lambda_0 = \lambda^*$, that is $c_0 = c^*$. If we choose $c_0 < c^*$, then $\rho_0 := \rho_{HLS,\lambda_0,c_0} < \rho_{HLS,\lambda_0,c^*}$ and according to Conjecture 1.6.2(i), we would expect the solution $\rho(t,r)$ to vanish in $L^1_{loc}(\mathbb{R})$. This is exactly what can be observed in Fig. 1.11 for the choice of parameters $\chi = 0.35$, $k = -1/2$, $m = 4/3$ in original variables ($r = 0$) and with $c_0 = 0.4\,c^*$. Here, we chose time steps of size $\Delta t = 10^{-2}$ and particles spaced at $\Delta\eta = 10^{-2}$. We let the density solution evolve until the L^2-error between two consecutive solutions is less than 10^{-4} (plotting every 1000 iterations).

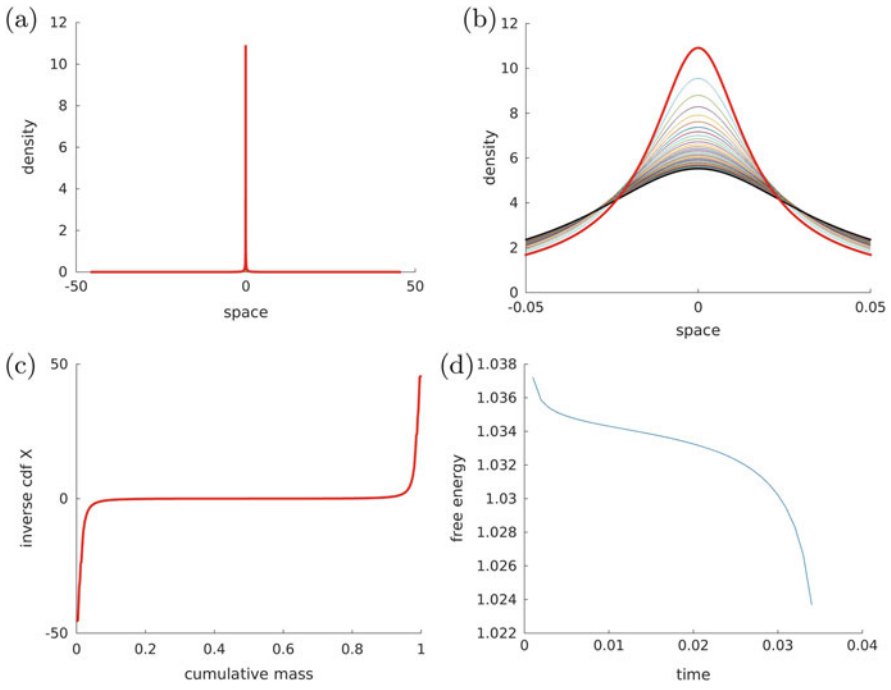

Fig. 1.12 Attraction-dominated regime: $\chi = 0.35$, $k = -1/2$, $m = 4/3$, $r = 0$ with initial data $\rho(t = 0, x) = \rho_{HLS,\lambda_0,c_0}(x) > \rho_{HLS,\lambda_0,c^*}(x)$ for all $x \in \mathbb{R}$ with $c_0 = 1.1c^*$. (**a**) Solution density from initial condition (*black*) to the profile at the last time step (*red*), (**b**) zoom of Figure (**a**), (**c**) inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**d**) free energy

For the same choice of initial data, but with $c_0 = 1.1\,c^* > c^*$ we observe numerically that the solution density concentrates at $x = 0$ as predicted by Conjecture 1.6.2(ii), see Fig. 1.12. The Newton-Raphson procedure stops converging once the mass it too concentrated. Here, we chose time steps of size $\Delta t = 10^{-3}$ and particles spaced at $\Delta\eta = 2 * 10^{-3}$.

One may also take as initial condition exactly the steady state $\rho_0 = \rho_{HLS,\lambda^*,c^*}$, see Fig. 1.13. However, the numerical approximation of the initial data is only accurate up to $\Delta\eta = 10^{-2}$ and we observe indeed pointwise convergence to zero, in accordance with the statement of Conjecture 1.6.3 that the stationary state ρ_{HLS,λ^*,c^*} is unstable. Again, we let the Newton-Raphson procedure evolve with time steps of size $\Delta t = 10^{-2}$ until the L^2-error between two consecutive solutions is less than the tolerance 10^{-4}.

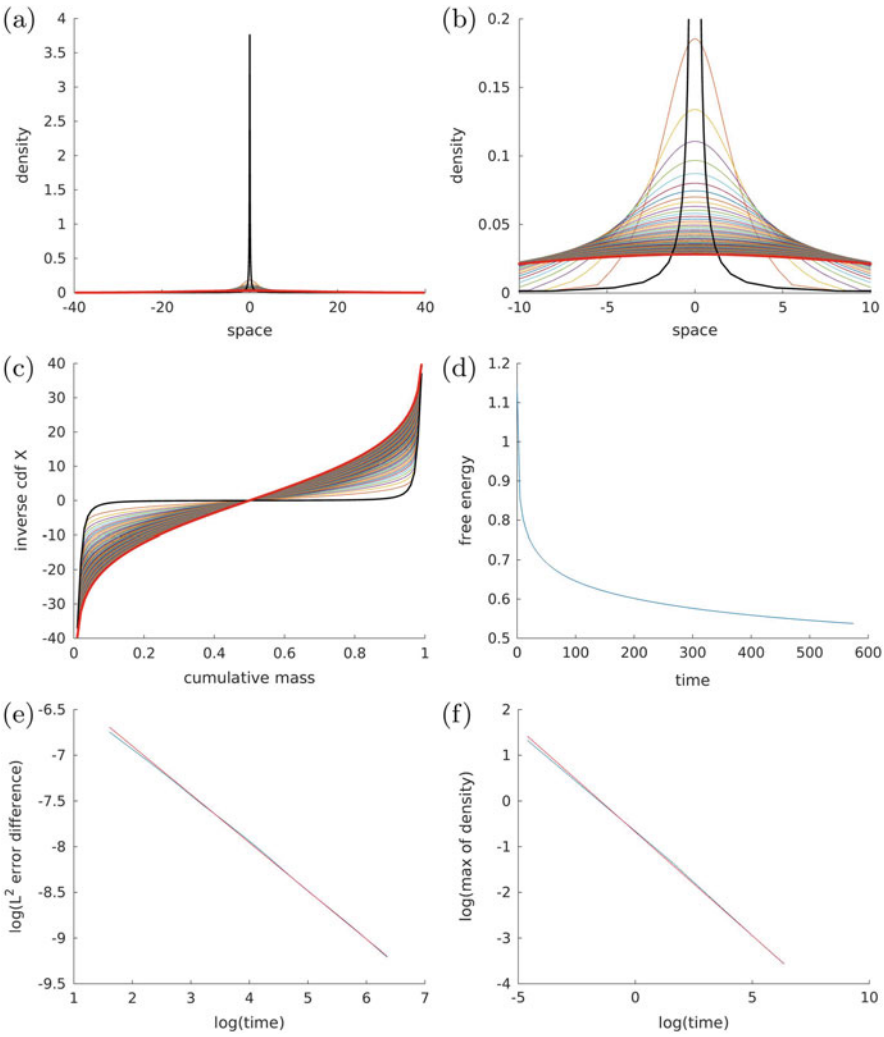

Fig. 1.13 Attraction-dominated regime: $\chi = 0.35$, $k = -1/2$, $m = 4/3$, $r = 0$ with initial data $\rho(t = 0, x) = \rho_{HLS,\lambda^*,c^*}(x)$ given in (1.52). (**a**) Solution density from initial condition (*black*) to the profile at the last time step (*red*), (**b**) zoom of Figure (**a**), (**c**) inverse cumulative distribution function from initial condition (*black*) to the profile at the last time step (*red*), (**d**) free energy, (**e**) log-log plot of the L^2-error difference between two consecutive solutions and fitted line with slope -0.52817, (**f**) log-log plot of $\max_x \rho(t, x)$ and fitted line with slope -0.45431

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Chapter 2
Bubbling Blow-Up in Critical Parabolic Problems

Manuel del Pino

Abstract These lecture notes are devoted to the analysis of blow-up of solutions for some parabolic equations that involve *bubbling phenomena*. The term *bubbling* refers to the presence of families of solutions which at main order look like scalings of a single stationary solution which in the limit become singular but at the same time have an approximately constant energy level. This arise in various problems where critical loss of compactness for the underlying energy appears. Three main equations are studied, namely: the Sobolev critical semilinear heat equation in $\mathbb{R}^n$, the harmonic map flow from $\mathbb{R}^2$ into S^2, the Patlak-Keller-Segel system in $\mathbb{R}^2$.

2.1 Introduction

These notes are devoted to the analysis of blow-up of solutions for some parabolic equations, classical in the literature, that involve so-called *bubbling phenomena*. The term *bubbling* in a variational problem refers to the presence of families of solutions which at main order look like scalings of a single stationary solution which in the limit become singular but at the same time have an approximately constant energy level. This arise in various problems where critical loss of compactness for the underlying energy appears. In time dependent versions of these problems, one expects that *blow-up by bubbling* in finite or infinite time for specific solutions appears. Those solutions are usually asymptotically not self-similar and, while not generic, their presence is among the most important features of the full dynamics since they correspond to threshold solutions between different generic regimes. In these lectures we will consider the following three problems:

1. *The Sobolev critical semilinear heat equation in* $\mathbb{R}^n$.

$$u_t = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \times (0, \infty), \tag{2.1}$$

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$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty),$$
$$u(\cdot,0) = u_0 \quad \text{in } \Omega.$$

Here Ω designates a smooth domain in $\mathbb{R}^n$, $n \geq 3$ and u_0 is a positive, smooth initial datum.

2. *The harmonic map flow from* $\mathbb{R}^2$ into S^2

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0,T) \\ u &= \varphi \quad \text{on } \partial\Omega \times (0,T) \\ u(\cdot,0) &= u_0 \quad \text{in } \Omega \end{aligned} \tag{2.2}$$

for a function $u : \Omega \times [0,T) \to S^2$, where Ω be a bounded smooth domain in $\mathbb{R}^2$ and S^2 denotes the standard 2-sphere. Here $u_0 : \bar{\Omega} \to S^2$ is a given smooth map and $\varphi = u_0\big|_{\partial\Omega}$.

3. *The Patlak-Keller-Segel system in* $\mathbb{R}^2$.

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u \nabla v) \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\ v &= (-\Delta)^{-1} u := \frac{1}{2\pi} \log \frac{1}{|\cdot|} * u \\ u(\cdot,0) &= u_0 > 0 \quad \text{in } \mathbb{R}^2 \end{aligned} \tag{2.3}$$

A salient common feature of these problems is the presence of *Lyapunov functionals*. In fact we let

$$E_1(u) = \frac{1}{2}\int |\nabla u|^2 dx - \frac{1}{p+1}\int |u|^{p+1} dx, \quad p = \frac{n+2}{n-2}$$

where the spacial domain is understood in the integral symbol. Then we compute, for sufficiently smooth solutions $u(x,t)$ of (2.1)

$$\partial_t E_1(u(\cdot,t)) = -\int |u_t|^2 dx.$$

Similarly, for solutions of $u(x,t)$ of (2.2) we have that

$$\partial_t E_2(u(\cdot,t)) = -\int |u_t|^2 dx.$$

where

$$E_2(u) = \frac{1}{2}\int |\nabla u|^2 dx.$$

These functionals are therefore decreasing along trajectories. In fact, we can interpret Eqs. (2.1) and (2.2) as negative L^2-gradient flows respectively for the energies E_1 and E_2.

Problem (2.3) also has a Lyapunov functional, which is less obvious. Let us write (2.3) in divergence form as

$$u_t = \nabla \cdot (u \nabla (\log u - (-\Delta)^{-1} u))$$

Then setting

$$E_3(u) := \int u(\log u - (-\Delta)^{-1} u)\, dx$$

we see that for a solution of (2.3) with sufficient regularity and space decay,

$$\partial_t E_3(u(\cdot, t)) = -\int u |\nabla (\log u - (-\Delta)^{-1} u)|^2 dx \leq 0.$$

Problem (2.3) can also be interpreted as a negative gradient flow for E_3 with respect to Wasserstein's metric, see [4].

The three problems above also have in common the presence of a *continuum of energy invariant steady states in entire space* which is what is behind the possibility of bubbling blow-up phenomena. Indeed, the steady-state equation for (2.1) in entire space is

$$\Delta u + u^{\frac{n+2}{n-2}} = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n.$$

It is solved by

$$U(x) = \alpha_n \left(\frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}}, \alpha_n = (n(n-2))^{\frac{1}{n-2}},$$

and so are solutions the scalings

$$U_{\lambda, x_0}(x) = \frac{1}{\lambda^{\frac{n-2}{2}}} U\left(\frac{x - x_0}{\lambda} \right) = \alpha_n \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}. \tag{2.4}$$

We have that

$$E_1(U_{\lambda, x_0}) = E_1(U) \quad \text{for all} \quad \lambda, x_0.$$

Similarly, for (2.2) the steady state problem in $\mathbb{R}^2$ is

$$\Delta u + |\nabla u|^2 u = 0, \quad |u| = 1 \quad \text{in } \mathbb{R}^2$$

which is solved by the *1-corrotational harmonic map*

$$U(x) = \begin{pmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{pmatrix}, \quad x \in \mathbb{R}^2.$$

We observe that the equation is also solved by

$$U_{\lambda,x_0,Q}(x) = QU\left(\frac{x-x_0}{\lambda}\right) \tag{2.5}$$

with Q a linear orthogonal transformation of $\mathbb{R}^3$. We see that

$$E_2(U_{\lambda,x_0,Q}) = E_2(U) \quad \text{for all} \quad \lambda, x_0.$$

Similarly, for (2.3) we have that the equation

$$\Delta u - \nabla \cdot (u \nabla (-\Delta)^{-1} u) = 0 \quad \text{in } \mathbb{R}^2$$

is solved by

$$U(x) = \frac{8}{(1+|x|^2)^2} \quad x \in \mathbb{R}^2,$$

and also by the scalings, singular as $\lambda \to 0$,

$$U_{\lambda,x_0}(x) = \lambda^{-2} U\left(\frac{x-x_0}{\lambda}\right) = \frac{8\lambda^2}{(\lambda^2+|x-x_0|^2)^2}. \tag{2.6}$$

We see that

$$E_3(U_{\lambda,x_0}) = E_3(U) \quad \text{for all} \quad \lambda, x_0.$$

The presence of these steady states represents loss of compactness for the respective energies, for as $\lambda \to 0^+$ they become singular, so that their limits do not belong to the natural energy space. In this way, we have the presence of non-convergent Palais-Smale sequences for the respective energies.

Our purpose in the remaining of these notes is to construct solutions of the time-dependent problems (2.1)–(2.3) that at main order look like one of the associated scalings, around one or more points, with time-dependent parameters, so that the scaling $\lambda(t)$ becomes zero in the limit. This is a *bubbling blow-up solution*. We will set up an adequate framework for each of the problems, with a common scheme. We shall do this in a rather detailed manner in Problem (2.1) and present a sketch in the case of Problems (2.2) and state the corresponding result in (2.3). In Problem (2.1) we will construct solutions with infinite time blow-up around a given arbitrary

number of points of the domain. In Problem (2.2) we construct such objects but in a finite time. In (2.3) we consider the so-called critical-mass case for a fast-decay initial condition and analyze the infinite-time singularity created.

2.2 Infinite-Time Blow-Up in the Critical Heat Equation

In this section we will construct infinite time blow-up solutions to Problem (2.1).

2.2.1 Discussion and Statement of Main Result

We begin with a discussion on the blow-up topic for the more general problem

$$\begin{aligned} u_t &= \Delta u + u^p \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) &= u_0 > 0 \quad \text{in } \Omega. \end{aligned} \tag{2.7}$$

Here Ω be a smooth domain in $\mathbb{R}^n$, $n \geq 1$, $p > 1$ and $0 < T \leq +\infty$.

The role of the exponent $p_S := \frac{n+2}{n-2}$ when $n \geq 3$ is fundamental in the different phenomena arising in this equation. Let us first review the steady state problem

$$\begin{aligned} \Delta u + u^p &= 0 \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.8}$$

When $1 < p < p_S$, Problem (2.8) is always solvable. In fact the best Sobolev constant

$$S_p(\Omega) = \inf_{0 \neq u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}$$

is achieved by a positive function which solves (2.8) thanks to the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. An alternative way to find a solution of Problem (2.8) is as a *mountain pass* of the energy functional

$$E_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}. \tag{2.9}$$

This functional satisfies the *Palais-Smale condition*: if $u_n \in H_0^1(\Omega)$ is such that

$$E(u_n) \to c \in \mathbb{R}, \quad \nabla E(u_n) \to 0 \quad \text{in } H^{-1}(\Omega)$$

then u_n has a convergent subsequence in $H_0^1(\Omega)$. When $p = p_S$ we have that

$$S_{p_S}(\Omega) = S_{p_S}(\mathbb{R}^n) =: S_n > 0$$

and it is not attained if $\Omega \subsetneq \mathbb{R}^n$. In $\mathbb{R}^n$. The best Sobolev constant S_n is achieved precisely by (scalar multiples of) the functions

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right) = \alpha_n \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{\frac{n-2}{2}}. \tag{2.10}$$

where

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{1}{n-2}}. \tag{2.11}$$

called the Aubin-Talenti bubbles, see [34]. By a result of Caffarelli-Gidas-Spruck, these functions correspond to all positive solutions of the equation

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n.$$

namely positive critical points of the energy

$$E(u) := E_{p_S}(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 - \frac{n-2}{2n}\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}.$$

The family (2.10) is energy invariant: for all μ, ξ,

$$E(U_{\mu,\xi}) = E(U) =: S_n > 0. \tag{2.12}$$

The presence of this asymptotically singular family of critical points of E in entire space precisely reflect its loss of compactness in a bounded domain Ω:

A Palais-Smale sequence in $H_0^1(\Omega)$ must asymptotically be, passing to a subsequence, of the form (called *the bubble resolution*)

$$u_n = u_\infty + \sum_{i=1}^{k} U_{\mu_n^i, \xi_n^i} + o(1), \tag{2.13}$$

for some $k \geq 0$, a critical point $u_\infty \in H_0^1(\Omega)$ of E, $\xi_n^i \in \Omega$, $\mu_n^i \to 0$ after a result by Struwe [31].

The fact that the Sobolev constant is not achieved is not just a technical obstruction for existence of solutions of (2.8): If the domain Ω is star-shaped around

a point $x_0 \in \Omega$ an u solves (2.8) then Pohozaev's identity [25] yields

$$\left(\frac{2-n}{2} - \frac{n}{p+1}\right)\int_\Omega u^{p+1} = \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x - x_0)\cdot \nu > 0$$

hence necessarily $p < \frac{n+2}{n-2} = p_S$ and thus no solutions at all exist if $p \geq p_S$. In particular $u_\infty = 0$ in the bubble-resolution.

As for the problem in entire space,

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n,$$

we have that no positive entire solution exists if $p < p_S$, while if $p = p_S$ all solutions are the Aubin-Talenti bubbles For $p > p_S$ there are positive solutions (in fact radially symmetric with $u(x) \sim |x|^{-\frac{2}{p-1}}$) but they do not have finite energy.

Coming back to the parabolic problem (2.7), let us consider a function $\varphi(x)$ positive and smooth function in Ω, with boundary value zero. Let $u_\alpha(x,t)$ be the unique (local) smooth solution of (2.7) with initial condition

$$u_\alpha(x,0) = \alpha\varphi(x), \quad \alpha > 0.$$

We claim that for all sufficiently small ε we have that $u(x,t) \to 0$ uniformly as $t \to \infty$. To see this, let λ_1 be the first eigenvalue of $-\Delta$ in Ω under Dirichlet boundary conditions and ϕ_1 a positive first eigenfunction. Let $\varepsilon > 0$ and consider the function

$$\bar{u}(x,t) = \varepsilon e^{-\gamma t}\phi_1(x)$$

where $0 < \gamma < \lambda_1$. Then we see that

$$\bar{u}_t - \Delta\bar{u} - \bar{u}^p = \varepsilon\phi_1 e^{-\gamma t}[(\lambda_1 - \gamma) - \varepsilon^{p-1}\phi_1^{p-1}] > 0$$

and hence $\bar{u}(x,t)$ is a supersolution of (2.7) if ε is fixed sufficiently small. Then for all $0 < \alpha < \alpha_0$, we have that $u_\alpha(x,0) = \alpha\varphi(x) \leq \bar{u}(x,0)$ and hence

$$u_\alpha(x,t) \leq Ce^{-\gamma t} \to 0 \quad \text{as} \quad t \to +\infty.$$

The claim is proven. Next, we claim that for all large α, $u_\alpha(x,t)$ *blows-up in finite time*. Indeed, assume that $u_\alpha(x,t)$ is defined in $[0,T)$ and use $\phi_1(x)$ as a test function. We get

$$\frac{\partial}{\partial t}\int_\Omega \phi_1 u_\alpha(\cdot,t) = -\lambda_1\int_\Omega \phi_1 u_\alpha(\cdot,t) + \int_\Omega \phi_1 u_\alpha(\cdot,t)^p.$$

If we let $q(t) = \int_\Omega \phi_1 u_\alpha(\cdot, t)$ then

$$q'(t) \ \geq \ -\lambda_1 q(t) + Cq(t)^p.$$

On the other hand

$$q(0) = \alpha \int_\Omega \phi_1 \varphi \, dx$$

Then for any $\alpha > \alpha_1$ we have that $-\lambda_1 q(0) + Cq(0)^p > 0$. Then we easily see that $q'(t) > 0$ for all t and hence

$$T \leq \int_{q(0)}^{\infty} \frac{dq}{-\lambda_1 q + Cq^p} \ < \ +\infty.$$

Two types of finite-time blow-up are present. Blow-up at time T for a solution $u(x, t)$ to (2.7) is said to be

- *Type I* if

$$\lim_{t \to T} (T - t)^{\frac{1}{p-1}} \|u(\cdot, t)\|_\infty \ < \ +\infty$$

- *Type II* if

$$\lim_{t \to T} (T - t)^{\frac{1}{p-1}} \|u(\cdot, t)\|_\infty \ = \ +\infty.$$

Type I means a blow-up rate that goes along with the natural scaling of (2.7)

$$\lambda \mapsto \lambda^{-\frac{1}{p-1}} u(\lambda^{-\frac{1}{2}} x, \lambda^{-1} t).$$

Said in a different way, Type I blow-up is one in which reaction overtakes diffusion effect, so that the blow-up mechanism is driven by the ODE

$$\frac{du}{dt}(t) = 0 + u(t)^p$$

whose solution with blow up at time T is precisely

$$u(t) = c_p (T - t)^{-\frac{1}{p-1}}, \quad c_p = (p - 1)^{-\frac{1}{p-1}}.$$

The following facts are known:

- Type I blow-up is the only one that can arise in the subcritical case $p < p_S$ (at least for convex domains). Giga and Kohn [19].

- Type II blow-up is rare but it exists. (A radial example was found by Herrero and Velazquez [21].) One needs $p > p_{JL}$ where $p_{JL} > p_S$ is the *Joseph-Lundgren exponent*, a number which is only well-defined for dimension $n \geq 11$.
- Matano-Merle [23] proved that the set of values α for which type-II blow-up in $u_\alpha(x,t)$ may exists is just finite in the radially symmetric case.
- Still in the radial case, the condition $p > p_{JL}$ is necessary for blow-up type II to occur [17]. As seen by Galaktionov and Vázquez, type II radial blowing-up can be naturally continued beyond blow-up time for $p > p_{JL}$. In fact this radial blow-up can only take place for a given radial solution a finite number of times.

2.2.1.1 The Threshold Solution

In summary, $u_\alpha(x,t)$ blows-up in finite time for all α large and it goes to zero for small α. As a conclusion, the following number is well-defined:

$$\alpha_* = \sup\{\alpha > 0 \,/\, \lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty = 0\},$$

in fact $0 < \alpha_* < +\infty$. Ni et al. [24] found that $u_{\alpha_*}(x,t)$ is a well-defined L^1-weak solution of (2.7).

u_{α_*} is a type of solution which loosely speaking lies in the dynamic threshold between solutions globally defined in time and those that blow-up in finite time.

It is not clear that u_{α^*} will be smooth at all times. In fact, it will not be in general the case.

- When $1 < p < p_S$, $u_{\alpha*}(x,t)$ is smooth. Indeed it is uniformly bounded (Cazenave-Lions [7]), and up to subsequences it converges to a (positive) solution of Eq. (2.8).
- When $p > p_S$, Ω a ball and radially symmetric solutions, it turns out that $u_{\alpha^*}(x,t) \to 0$ as $t \to +\infty$, see Quittner-Souplet [28].
- The case $p = p_S$ *is special*. In fact in this case the threshold solution u_{α_*} in the radial case does have *infinite-blow up time*, namely

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_\infty = +\infty.$$

We can mention in addition that by results by Du and Suzuki [13, 33]: along sequences $t_n \to +\infty$, $u_{\alpha_*}(x,t)$ does have a *bubble resolution* of the type of a Palais-Smale sequence when $p = p_S$.

$$u_{\alpha_*}(x,t_n) = u_\infty + \sum_{i=1}^{k} U_{\mu_n^i,\xi_n^i} + o(1), \tag{2.14}$$

for some $k \geq 0$, a critical point $u_\infty \in H_0^1(\Omega)$ of, $\xi_n^i \in \Omega$, $\mu_n^i \to 0$.

For any $p > 1$ the energy

$$E_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

defines a Lyapunov functional for Eq. (2.7). Indeed, we readily compute, for a solution $u(x,t)$ of (2.7),

$$\frac{d}{dt}E_p(u(\cdot,t)) = -\int_\Omega |u_t|^2 dx$$

In the case $p = p_S$ the k-bubble resolution would yield

$$\lim_{n\to+\infty} E(u(\cdot,t_n)) = kS_n + E(u_\infty).$$

where we recall, necessarily $u_\infty = 0$ if Ω is star-shaped.

Galaktionov and Vázquez [17] found that in the case that $\Omega = B(0,1)$ and the threshold solution $u_{\alpha *}$ is radially symmetric, then no finite time singularities for $u_{\alpha *}(r,t)$ occur and it must become unbounded as $t \to +\infty$, thus exhibiting infinite-time blow up. Galaktionov and King discovered in [16] that this radial blow-up solution does have a bubbling asymptotic profile as $t \to +\infty$ of the form

$$u_{\alpha_*}(r,t) \approx \alpha_n \left(\frac{\mu(t)}{\mu(t)^2 + r^2}\right)^{\frac{n-2}{2}}, \quad r = |x|. \tag{2.15}$$

where for $n \geq 5$, $\mu(t) \sim t^{-\frac{1}{n-4}} \to 0$. For $\alpha > \alpha_*$ blow-up in finite time of $u_{\alpha_*}(r,t)$ occurs while, it goes to zero when $\alpha < \alpha_*$. Understanding this threshold phenomenon is central in capturing the global dynamics of Problem (2.1). These solutions are unstable, while intuitively codimension-one stable in the space of initial conditions containing $\alpha_* \varphi$.

Nothing seems to be known however on existence of infinite-time bubbling solutions in the nonradial case, or about their degree of stability. Our main goal is to build solutions with single or multiple blow-up points as $t \to +\infty$ in problem (2.1) when Ω is arbitrary and $n \geq 5$, providing precise account of their asymptotic form and investigate their stability.

Our construction unveils the interesting role played by the (elliptic) Green function of the domain Ω. In what follows we denote by $G(x,y)$ Green's function for the boundary value problem

$$-\Delta_x G(x,y) = c_n \delta(x-y) \quad \text{in } \Omega, \quad G(\cdot,y) = 0 \quad \text{on } \partial\Omega,$$

where $\delta(x)$ is the Dirac mass at the origin and c_n is the number such that

$$-\Delta_x \Gamma(x) = c_n \delta(x), \quad \Gamma(x) = \frac{\alpha_n}{|x|^{n-2}}, \tag{2.16}$$

namely $c_n = (n-2)\omega_n\alpha_n$ with ω_n the surface area of the unit sphere in $\mathbb{R}^n$ and α_n the number in (2.15). We let $H(x, y)$ be the regular part of $G(x, y)$ namely the solution of the problem

$$-\Delta_x H(x, y) = 0 \quad \text{in } \Omega, \quad H(\cdot, y) = \Gamma(\cdot - y) \quad \text{in } \partial\Omega. \tag{2.17}$$

The diagonal $H(x, x)$ is called the Robin function of Ω. It is well known that it satisfies

$$H(x, x) \to +\infty \quad \text{as dist}\,(x, \partial\Omega) \to 0. \tag{2.18}$$

Let $q = (q_1, \ldots, q_k)$ be an array of k distinct points in Ω, and define the $k \times k$ matrix

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\ -G(q_1, q_2) & H(q_2, q_2) & -G(q_2, q_3) \cdots & -G(q_3, q_k) \\ \vdots & & \ddots & \vdots \\ -G(q_1, q_k) & \cdots & -G(q_{k-1}, q_k) & H(q_k, q_k) \end{bmatrix} \tag{2.19}$$

Our main result states that a global solution to (2.1) which blows-up at exactly k given points q_j exists if q lies in the open region of Ω^k where the matrix $\mathcal{G}(q)$ is positive definite.

Theorem 1 ([11]) *Assume $n \geq 5$. Let $q_1, \cdots, q_k$ be distinct points in Ω such that the matrix $\mathcal{G}(q)$ is positive definite. Then there exist an initial datum u_0 and smooth functions $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \ldots, k$, such that the solution u_q of Problem (2.1) has the form*

$$u_q(x, t) = \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}} - \mu_j(t)^{\frac{n-2}{2}} H(x, q_j) + \mu_j(t)^{\frac{n-2}{2}} \theta(x, t), \tag{2.20}$$

where $\theta(x, t)$ is bounded, and $\theta(x, t) \to 0$ as $t \to +\infty$, uniformly away from the points q_j. In addition, for certain positive constants β_j depending on q.

$$\mu_j(t) = \beta_j t^{-\frac{1}{n-4}}(1 + o(1)), \quad \xi_j(t) - q_j = O(t^{-\frac{2}{n-4}}) \quad \text{as } t \to +\infty$$

Our construction of the solution $u_q(x, t)$ in Theorem 1 yields the codimension k-stability of its bubbling phenomenon.

Our construction of the solution $u_q(x, t)$ yields the codimension k-stability of its bubbling phenomenon in the following sense.

Corollary 2.2.1 *There exists a codimension k manifold in* $C^1(\bar{\Omega})$ *that contains* $u_q(x,0)$ *such that if* $u(x,0)$ *lies in that manifold and it is sufficiently close to* $u_q(x,0)$*, then the solution* $u(x,t)$ *of (2.1) has exactly k bubbling points* $\tilde{q}_j$*,* $j = 1,\dots,k$ *which lie close to the* q_j*.*

Positive definiteness of $\mathcal{G}(q)$ trivially holds if $k = 1$. For $k = 2$ this condition holds if and only if

$$H(q_1,q_1)H(q_2,q_2) - G(q_1,q_2)^2 > 0,$$

in particular it does not hold if both points q_1 and q_2 are too close to a given point in Ω. Given $k > 1$, using that

$$H(x,x) \to +\infty \quad \text{as dist}\,(x,\partial\Omega) \to 0.$$

we can always find k points where $\mathcal{G}(q)$ is positive definite.

The proof of the above result consists of building a first approximation to the solution, then solving for a small remainder by means of linearization and fixed point arguments. First we construct a first approximation of the desired form. We shall compute the error and see that in order to **improve the approximation** we need certain solvability conditions for the elliptic linearized operator around the bubble. These relations yield a system of ODEs for the scaling parameters, of which we find a suitable solution.

2.2.2 *Construction of the Approximate Solution and Error Computations*

We consider the Talenti bubbles

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}},$$

and

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right), \quad \mu > 0, \quad \xi \in \mathbb{R}^n.$$

Given k points $q_1,\dots,q_k \in \mathbb{R}^n$, we want to find a solution $u(x,t)$ of equation (P) with

$$u(x,t) \approx \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x) \tag{2.21}$$

where $\xi_j(t) \to q_j$ and $\mu_j(t) \to 0$ as $t \to \infty$ for each $j = 1, \dots, k$. The functions $\xi_j(t)$ and $\mu_j(t)$ cannot of course be arbitrary.

We assume that the vanishing speed of all functions $\mu_j(t)$ is the same. More precisely that for a function $\mu_0(t) \to 0$ and positive constants $b_1, \dots, b_k$ we have

$$\mu_j(t) = b_j \mu_0(t) + O(\mu_0^2(t)) \quad \text{as} \quad t \to \infty.$$

Also, we assume

$$\xi_j(t) - q_j = O(\mu_0^2(t)) \quad \text{as} \quad t \to \infty.$$

If a solution to (2.1) satisfies $u(x,t) \approx \sum_{j=1}^k U_{\mu_j,\xi_j}(x)$ then

$$u_t \approx \Delta u + \sum_{j=1}^k U_{\mu_j,q}(x)^p$$

Besides, we see that

$$\int_\Omega U_{\mu_j,q}(x)^p dx \approx \mu_j^{\frac{n-2}{2}} a_n, \quad a_n := \int_{\mathbb{R}^n} U(y)^p dy,$$

and hence away from the points q_j

$$u_t \approx \Delta u + c_n \mu_0^{\frac{n-2}{2}} \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0,\infty)$$

where δ_q designates the Dirac mass at the point q.

Letting $u = \mu_0^{\frac{n-2}{2}} v(x,t)$ we get

$$v_t \approx \Delta v - \frac{n-2}{2} \mu_0^{-1} \dot{\mu}_0 v + c_n \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0,\infty).$$

We assume that $\mu_0^{-1} \dot{\mu}_0 \to 0$, so that

$$v_t \approx \Delta v + a_n \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0,\infty),$$
$$v = 0 \quad \text{on } \partial\Omega \times (0,\infty).$$

So that away from the q_j we should have

$$v(x,t) \approx a_n \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} G(x,q_j),$$

$$u(x,t) \approx \sum_{j=1}^{k} \frac{\alpha_n \mu_j^{\frac{n-2}{2}}}{|x-q_j|^{n-2}} - \mu_j^{\frac{n-2}{2}} H(x,q_j).$$

Observing that for x away from the point q_j, we precisely have

$$U_{\mu_j,\xi_j}(x) \approx \frac{\alpha_n \mu_j^{\frac{n-2}{2}}}{|x-q_j|^{n-2}}$$

we see that a better global approximation to a solution $u(x,t)$ to our problem is given by the corrected k-bubble

$$u_{\xi,\mu}(x,t) := \sum_{j=1}^{k} u_j(x,t), \quad u_j(x,t) := U_{\mu_j,\xi_j}(x) - \mu_j^{\frac{n-2}{2}} H(x,q_j). \tag{2.22}$$

We have obtained this correction term out of a rough analysis to what is happening away from the blow-up points. Let us now analyze the region near them. That will allow us to identify the function $\mu_0(t)$ and the constants b_j. It is convenient to write

$$S(u) := -u_t + \Delta_x u + u^p.$$

We consider the error of approximation $S(u_0)$. We have

$$S(u_{\mu,\xi}) = -\sum_{i=1}^{k} \partial_t u_i + \left(\sum_{i=1}^{k} u_i\right)^p - \sum_{i=1}^{k} U^p_{\mu_i,\xi_i}.$$

We obtain the following estimate near a given concentration point q_j, from where the formal asymptotic derivation of the unknown parameters will be a rather direct consequence.

Given j, assuming that

$$|x-q_j| \le \frac{1}{2} \min_{i \neq l} |q_i - q_l|$$

and setting

$$y_j := \frac{x-\xi_j}{\mu_j},$$

we have

$$S(u_{\mu,\xi}) = \mu_j^{-\frac{n+2}{2}}\big(\mu_j E_{0j} + \mu_j E_{1j} \ + \ \mathcal{R}_j\big)$$

$$E_{0j} = pU(y_j)^{p-1}\big[-\mu_j^{n-3}H(q_j,q_j) \ + \ \sum_{i\neq j}\mu_j^{\frac{n-4}{2}}\mu_i^{\frac{n-2}{2}}G(q_j,q_i)\big]$$
$$+\dot{\mu}_j\Big[y_j\cdot\nabla U(y_j) + \frac{n-2}{2}U(y_j)\Big],$$

and

$$E_{1j} = pU(y_j)^{p-1}\big[-\mu_j^{n-2}\nabla_x H(q_j,q_j)$$
$$+\sum_{i\neq j}\mu_j^{\frac{n-2}{2}}\mu_i^{\frac{n-2}{2}}\nabla_x G(q_j,q_i)\,\big]\cdot y_j + \dot{\xi}_j\cdot\nabla U(y_j)$$

and $\mathcal{R}_j$ contains smaller order terms.

To see this, we write, for $y_i = \frac{x-\xi_i}{\mu_i}$,

$$u_{\mu,\xi}(x,t) = \sum_{i=1}^{k}\mu_i^{-\frac{n-2}{2}}U(y_i) - \mu_i^{\frac{n-2}{2}}H(x,q_i), \quad \text{and} \quad S(u_{\mu,\xi}) = S_1 + S_2$$

where

$$S_1 \ := \ \sum_{i=1}^{k}\mu_i^{-\frac{n}{2}}\dot{\xi}_i\cdot\nabla U(y_i) + \mu_i^{-\frac{n}{2}}\dot{\mu}_i Z_{n+1}(y_i) + \frac{n-2}{2}\mu_i^{\frac{n-4}{2}}\dot{\mu}_i H(x,q_i), \tag{2.23}$$

and

$$S_2 \ := \ \left(\sum_{i=1}^{k}\mu_i^{-\frac{n-2}{2}}U(y_i) - \mu_i^{\frac{n-2}{2}}H(x,q_i)\right)^p - \sum_{i=1}^{k}\mu_i^{-\frac{n+2}{2}}U(y_i)^p. \tag{2.24}$$

Then at main order we have that near q_j,

$$S_2 \approx \mu_j^{-\frac{n+2}{2}}\left[\,\big(U(y_j)+\Theta_j\big)^p - U(y_j)^p\big]\right.,$$

with

$$\Theta_j = -\mu_j^{n-2}H(x,q_j) + \sum_{i\neq j}\big(\mu_j\mu_i^{-1}\big)^{\frac{n-2}{2}}U(y_i) - (\mu_j\mu_i)^{\frac{n-2}{2}}H(x,q_i). \tag{2.25}$$

Taylor expanding we get

$$S_2 \approx \mu_j^{-\frac{n+2}{2}} pU(y_j)^{p-1}\Theta_j\,.$$

We make some further expansion. We have, for $i \neq j$,

$$U(y_i) \;=\; \frac{\alpha_n \mu_i^{n-2}}{(|\mu_j y_j + \xi_j - \xi_i|^2 + \mu_i^2)^{\frac{n-2}{2}}} \;\approx\; \frac{\alpha_n \mu_i^{n-2}}{|\mu_j y_j + q_j - q_i|^{n-2}}.$$

Hence we get the approximation

$$\Theta_j \approx -\mu_j^{n-2} H(q_j + \mu_j y_j, q_j) + \sum_{i\neq j} (\mu_i\mu_j)^{\frac{n-2}{2}} G(q_j + \mu_j y_j, q_i).$$

Further expanding, we get

$$\begin{aligned}\Theta_j \;\approx\; & -\mu_j^{n-2} H(q_j, q_j) + \sum_{i\neq j} (\mu_i\mu_j)^{\frac{n-2}{2}} G(q_j, q_i) \\ & + \Big[-\mu_j^{n-2}\nabla_x H(q_j, q_j) + \sum_{i\neq j} (\mu_i\mu_j)^{\frac{n-2}{2}} \nabla_x G(q_j, q_i) \Big] \cdot \mu_j y_j.\end{aligned}$$

We also approximate

$$S_1 \;\approx\; \mu_j^{-\frac{n}{2}} \dot\xi_j \cdot \nabla U(y_j) + \mu_j^{-\frac{n}{2}} \dot\mu_j \Big[\frac{n-2}{2} U(y_j) + y_j \cdot \nabla U(y_j)\Big],$$

2.2.3 The Choice of the Parameters at Main Order

We are looking for a solution of our equation of the form

$$u(x,t) = u_{\mu,\xi}(x,t) + \tilde\phi(x,t)$$

where $\tilde\phi$ is globally smaller. We see that

$$0 = S(u_{\mu,\xi} + \tilde\phi) = -\partial_t\tilde\phi + \Delta_x\tilde\phi + p u_{\mu,\xi}^{p-1}\tilde\phi + S(u_{\mu,\xi}) + \tilde N_{\mu,\xi}(\tilde\phi)$$

where $\tilde N_{\mu,\xi}(\tilde\phi) = (u_{\mu,\xi} + \tilde\phi)^p - u_{\mu,\xi}^p - p u_{\mu,\xi}^{p-1}\tilde\phi$.

We rewrite

$$\tilde\phi(x,t) = \mu_j^{-\frac{n-2}{2}} \phi\left(\frac{x-\xi_j}{\mu_j}, t\right)$$

so that

$$0 = \mu_j^{\frac{n+2}{2}} S(u_{\mu,\xi} + \tilde{\phi}) \approx \tag{2.26}$$

$$-\mu_j^2 \partial_t \phi + \Delta_y \phi + pU(y)^{p-1}\phi + \mu_j^{\frac{n+2}{2}} S(u_{\mu,\xi}) + A[\phi] \tag{2.27}$$

where the terms in $A(\phi)$ are all of smaller order. It is reasonable to assume that $\phi(y,t)$ decays in the y variable.

Considering the largest term E_0 in the expansion of the error $\mu_j^{\frac{n+2}{2}} S(u_{\mu,\xi})$ we find that $\phi(y,t)$ should equal at main order a solution $\phi_{0j}(y,t)$ of the elliptic equation

$$\Delta_y \phi_{0j} + pU^{p-1}\phi_{0j} = -\mu_{0j} E_{0j} \quad \text{in } \mathbb{R}^n, \quad \phi_0(y,t) \to 0 \quad \text{as} \quad |y| \to \infty. \tag{2.28}$$

where we recall

$$E_{0j} = pU(y_j)^{p-1}\Big[-\mu_j^{n-3} H(q_j,q_j) + \sum_{i\neq j} \mu_j^{\frac{n-4}{2}} \mu_i^{\frac{n-2}{2}} G(q_j,q_i)\Big]$$
$$+ \dot{\mu}_j \Big[y_j \cdot \nabla U(y_j) + \frac{n-2}{2} U(y_j)\Big],$$

2.2.3.1 Basic Linear Elliptic Theory

We recall some standard facts on a linear equation of the form

$$L_0(\psi) := \Delta_y \psi + pU^{p-1}\psi = h(y) \quad \text{in } \mathbb{R}^n, \quad \psi(y) \to 0 \quad \text{as} \quad |y| \to \infty.$$

It is well known that all bounded solutions of the equation $L_0(\psi) = 0$ in $\mathbb{R}^n$ consist of linear combinations of the functions $Z_1, \ldots, Z_{n+1}$ defined as

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \ldots, n, \quad Z_{n+1}(y) := \frac{n-2}{2} U(y) + y \cdot \nabla U(y).$$

If $h(y) = O(|y|^{-m})$, $m > 2$, then the problem is solvable iff

$$\int_{\mathbb{R}^n} h(y) Z_i(y)\, dy = 0 \quad \text{for all} \quad i = 1, \ldots, n+1.$$

Since $n \geq 5$, we can solve

$$\Delta_y \phi_{0j} + pU^{p-1}\phi_{0j} = -\mu_{0j} E_{0j} \quad \text{in } \mathbb{R}^n, \quad \phi_0(y,t) \to 0 \quad \text{as} \quad |y| \to \infty. \tag{2.29}$$

provided that

$$\int_{\mathbb{R}^n} E_{0j}(y,t)\, Z_{n+1}(y)\, dy \;=\; 0 \quad \text{for all} \quad j = 1, \dots, k \tag{2.30}$$

We compute

$$\int_{\mathbb{R}^n} E_{0j}(y,t)\, Z_{n+1}(y)\, dy \;=\; c_1 \Big[\mu_j^{n-3} H(q_j, q_j) - \sum_{i \neq j} \mu_j^{\frac{n-4}{2}} \mu_i^{\frac{n-2}{2}} G(q_i, q_j) \Big] \;+\; c_2\, \dot{\mu}_j\,, \tag{2.31}$$

where c_1 and c_2 are the positive constants given by

$$c_1 = -p \int_{\mathbb{R}^n} U^{p-1} Z_{n+1} = \frac{n-2}{2} \int_{\mathbb{R}^n} U^p, \quad c_2 = \int_{\mathbb{R}^n} |Z_{n+1}|^2. \tag{2.32}$$

We observe that $c_2 < +\infty$ thanks to the assumed fact $n \geq 5$.

These relations define a nonlinear system of ODEs for which a solution can be found as follows: we write

$$\mu_j(t) = b_j \mu_0(t)$$

and arrive at the relations

$$b_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} (b_i b_j)^{\frac{n-2}{2}} G(q_i, q_j) \;+\; c_2 c_1^{-1}\, b_j^2 \mu_0^{3-n} \dot{\mu}_0(t) = 0$$

so that $\mu_0^{3-n} \dot{\mu}_0(t)$ should equal a constant, which is necessarily negative since μ_0 decays to zero. This constant can be scaled out, hence it can be chosen arbitrarily to the expense of changing accordingly the values b_i. We impose

$$\dot{\mu}_0 = -\frac{2 c_1 c_2^{-1}}{n-2} \mu_0^{n-3}, \tag{2.33}$$

which yields after a suitable translation of time,

$$\mu_0(t) \;=\; \gamma_n t^{-\frac{1}{n-4}}, \quad \gamma_n = (2^{-1}(n-4)^{-1}(n-2) c_1^{-1} c_2)^{\frac{1}{n-4}} \tag{2.34}$$

and therefore the positive constants b_j (in case they exist) must solve the nonlinear system of equations

$$b_j^{n-3} H(q_j, q_j) - \sum_{i \neq j} b_i^{\frac{n-2}{2}} b_j^{\frac{n-2}{2}-1} G(q_i, q_j) = \frac{2 b_j}{n-2} \quad \text{for all} \quad j = 1, \dots, k. \tag{2.35}$$

This system has a solution (which is unique) if the matrix $\mathcal{G}(q)$ defined in is positive definite. System (2.35) can be written as a variational problem. Indeed, it is equivalent to $\nabla_b I(b) = 0$ where

$$I(b) := \frac{1}{n-2}\left[\sum_{j=1}^{k} b_j^{n-2} H(q_j, q_j) - \sum_{i\neq j} b_i^{\frac{n-2}{2}} b_j^{\frac{n-2}{2}} G(q_i, q_j) - \sum_{j=1}^{k} b_j^2\right]$$

Writing $\Lambda_j = b_j^{\frac{n-2}{2}}$ the functional becomes

$$(n-2)\, I(b) = \tilde{I}(\Lambda) = \sum_{j=1}^{k} H(q_j, q_j)\Lambda_j^2 - \sum_{i\neq j} G(q_i, q_j)\Lambda_i\Lambda_j - \sum_{j=1}^{k} \Lambda_j^{\frac{4}{n-2}}.$$

Let us assume that the matrix $\mathcal{G}(q)$ is positive definite. Then the functional $\tilde{I}(\Lambda)$ is strictly convex in the region where all $\Lambda_j > 0$. It clearly has a global minimizer with all components positive. This yields the existence of a unique critical point b of $I(b)$ with positive components which what we needed.

From the choice of the parameters μ_0, b_j we have

$$\mu_{0j} E_{0j}[\bar{\mu}_0, \dot{\mu}_{0j}] = -\gamma_j \mu_0(t)^{n-2}\, q_0(y) \tag{2.36}$$

where γ_j is a positive constant and

$$q_0(y) := p\, U^{p-1}(y) c_2 + c_1 Z_{n+1}(y),$$

so that $\int_{\mathbb{R}^n} q_0(y)\, Z_{n+1}(y)\, dy = 0$.

The problem

$$\Delta\phi_{0j} + pU(y)^{p-1}\phi_{0j} = -\gamma_j \mu_0(t)^{n-2} q_0(y), \quad \phi_{0j}(y,t) \to 0 \quad \text{as} \quad y \to \infty.$$

has a radially symmetric solution which we can describe from the variation of parameters. Let $\tilde{Z}_{n+1}(r)$ so that $L_0(\tilde{Z}_{n+1}) = 0$ with

$$\tilde{Z}_{n+1}(r) \sim r^{2-n} \quad \text{as} \quad r \to 0, \quad \tilde{Z}_{n+1}(r) \sim 1 \quad \text{as} \quad r \to \infty,$$

and the radial solution $p_0 = p_0(|y|)$ of $L_0(p_0) = q_0$ described as

$$p_0(r) = cZ_{n+1} \int_0^r \tilde{Z}_{n+1}(s) q_0(s) s^{n-1}\, ds - c\tilde{Z}_{n+1} \int_0^r Z_{n+1}(s) q_0(s) s^{n-1}\, ds.$$

p_0 satisfies

$$p_0(|y|) = O(|y|^{-2}) \quad \text{as} \quad |y| \to \infty. \tag{2.37}$$

Then a solution $\phi_{0j}(y,t)$ is simply given by the function

$$\phi_j(y,t) = \gamma_j \mu_0(t)^{n-2} p_0(y).$$

This leads us to the following corrected approximation,

$$u^*_{\mu,\xi}(x,t) := u_{\mu,\xi}(x,t) + \tilde{\Phi}(x,t), \quad \tilde{\Phi}(x,t) := \sum_{j=1}^{k} \mu_j^{-\frac{n-2}{2}} \phi_{0j}\left(\frac{x-\xi_j}{\mu_j}, t\right). \tag{2.38}$$

2.2.3.2 Total Expansion of the Error

Expansion for the error $S(u^*_{\mu,\xi})$ near each q_j. Write

$$\mu_j = b_j \mu_0 + \lambda_j, \quad |\lambda| \le \mu_0^{1+\sigma}$$

Then setting $x = \xi_j + \mu_j y_j$, we get

$$\begin{aligned} S(u^*_{\mu,\xi}) \approx \sum_{j=1}^{k} \mu_j^{-\frac{n+2}{2}} \Big\{ & \mu_{0j} \dot{\lambda}_j Z_{n+1}(y_j) - \mu_{0j} \mu_0^{n-4} p U(y_j)^{p-1} \sum_{i=1}^{k} M_{ij} \lambda_i \\ & + \mu_j \dot{\xi}_j \cdot \nabla U(y_j) + p U(y_j)^{p-1} \big[-\mu_j^{n-2} \nabla_x H(q_j, q_j) \\ & + \sum_{i \neq j} \mu_j^{\frac{n-2}{2}} \mu_i^{\frac{n-2}{2}} \nabla_x G(q_j, q_i) \big] \cdot y_j \Big\} \end{aligned}$$

where M_{ij} is a certain positive definite matrix depending on the points.

2.2.4 The Inner-Outer Gluing Procedure

$$\partial_t u = \Delta u + u^p \quad \text{in } \Omega \times [0,\infty), \qquad u = 0 \qquad \text{on } \partial\Omega \times [0,\infty). \tag{2.39}$$

We solve with u of the form

$$u = u^*_{\mu,\xi} + \tilde{\phi}, \tag{2.40}$$

where $\tilde{\phi}(x,t)$ is a smaller term. We construct the function $\tilde{\phi}$ by means of what we call the *inner-outer gluing* procedure.

This procedure consists in writing

$$\tilde{\phi}(x,t) = \psi(x,t) + \phi^{in}(x,t) \quad \text{where} \quad \phi^{in}(x,t) := \sum_{j=1}^{k} \eta_{j,R}(x,t)\tilde{\phi}_j(x,t)$$

with

$$\tilde{\phi}_j(x,t) := \mu_{0j}^{-\frac{n-2}{2}} \phi_j\left(\frac{x-\xi_j}{\mu_{0j}}, t\right), \quad \mu_{0j}(t) = b_j\mu_0(t)$$

and

$$\eta_{j,R}(x,t) = \eta\left(\frac{x-\xi_j}{R\mu_{0j}}\right).$$

Here $\eta(s)$ is a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$.

In terms of $\tilde{\phi}$, Problem (2.1) reads as

$$\partial_t\tilde{\phi} = \Delta\tilde{\phi} + p(u^*_{\mu,\xi})^{p-1}\tilde{\phi} + \tilde{N}(\tilde{\phi}) + S(u^*_{\mu,\xi}) \quad \text{in } \Omega \times [0,\infty), \tag{2.41}$$

$$\tilde{\phi} = -u^*_{\mu,\xi} \quad \text{on } \partial\Omega \times [0,\infty), \tag{2.42}$$

where

$$\tilde{N}_{\mu,\xi}(\tilde{\phi}) = (u^*_{\mu,\xi} + \tilde{\phi})^p - (u^*_{\mu,\xi})^p - p(u^*_{\mu,\xi})^{p-1}\tilde{\phi},$$

$$S(u^*_{\mu,\xi}) = -\partial_t u^*_{\mu,\xi} + \Delta u^*_{\mu,\xi} + (u^*_{\mu,\xi})^p.$$

We decompose

$$S(u^*_{\mu,\xi}) \approx \sum_{j=1}^{k} S_{\mu,\xi,j} \tag{2.43}$$

where, for $y_j = \frac{x-\xi_j}{\mu_j}$,

$$\begin{aligned} S_{\mu,\xi,j} = \mu_j^{-\frac{n+2}{2}} \Big\{ & \mu_{0j}\left[\dot{\lambda}_j Z_{n+1}(y_j) - \mu_0^{n-4} pU(y_j)^{p-1} \sum_{i=1}^{k} M_{ij}\lambda_i\right] \\ & + \mu_j\Big[\dot{\xi}_j \cdot \nabla U(y_j) + pU(y_j)^{p-1}\big[-\mu_j^{n-2}\nabla_x H(q_j,q_j) \\ & + \sum_{i\neq j} \mu_j^{\frac{n-2}{2}} \mu_i^{\frac{n-2}{2}} \nabla_x G(q_j,q_i)\,\big]\cdot y_j\Big]\Big\}. \end{aligned}$$

Let

$$V_{\mu,\xi} = p\sum_{j=1}^{k}((u^*_{\mu,\xi})^{p-1} - (\mu_j^{-\frac{n-2}{2}}U(\frac{x-\xi_j}{\mu_j}))^{p-1})\eta_{j,R} + p\big(1-\sum_{j=1}^{k}\eta_{j,R}\big)(u^*_{\mu,\xi})^{p-1}. \tag{2.44}$$

A main observation is the following: $\tilde{\phi}$ solves the problem if (ψ,ϕ) solves the following system:

Outer problem:

$$\begin{aligned}\partial_t\psi &= \Delta\psi + V_{\mu,\xi}\psi \\ &\quad + \sum_{j=1}^{k}[2\nabla\eta_{j,R}\nabla_x\tilde{\phi}_j + \tilde{\phi}_j(\Delta_x-\partial_t)\eta_{j,R}] \\ &\quad + \tilde{N}_{\mu,\xi}(\tilde{\phi}) + S^o_{\mu,\xi} \quad \text{in } \Omega\times[t_0,\infty), \\ \psi &= -u^*_{\mu,\xi} \quad \text{on } \partial\Omega\times[t_0,\infty),\end{aligned} \tag{2.45}$$

where

$$S^o_{\mu,\xi} = S^{(2)}_{\mu,\xi} + \sum_{j=1}^{k}(1-\eta_{j,R})S_{\mu,\xi,j},$$

and for all $j=1,\dots,k$, the *Inner problems:*

$$\partial_t\tilde{\phi}_j = \Delta\tilde{\phi}_j + pU_j^{p-1}\tilde{\phi}_j + pU_j^{p-1}\psi + S_{\mu,\xi,j} \quad \text{in } B_{2R\mu_{0j}}(\xi_j)\times[t_0,\infty).$$

The Inner problems in terms of $\phi_j(y,t)$, $y\in B_{2R}(0)$ become

$$\begin{aligned}\mu_{0j}^2\partial_t\phi_j &= \Delta_y\phi_j + pU(y)^{p-1}\phi_j + \mu_{0j}^{\frac{n+2}{2}}S_{\mu,\xi,j}(\xi_j+\mu_{0j}y,t) \\ &\quad + p\mu_{0j}^{\frac{n-2}{2}}\frac{\mu_{0j}^2}{\mu_j^2}U^{p-1}(y)\psi(\xi_j+\mu_{0j}y,t) + B_j[\phi_j] + B_j^0[\phi_j]\end{aligned}$$

where $B_j[\phi_j]$ is a smaller order linear operator.

We proceed as follows. For given parameters $\lambda,\xi,\dot{\lambda},\dot{\xi}$ and functions ϕ_j fixed in a suitable range, we solve for ψ the outer problem (2.45). Indeed, in the form of a (nonlocal) operator

$$\psi = \Psi(\lambda,\xi,\dot{\lambda},\dot{\xi},\phi)$$

Then we replace this ψ in the inner equations and solve them by a fixed point argument involving a suitable inverse of the main part of the linear operators in ϕ_j. Let us explain how to do so.

Recall that the elliptic linear operator $L_0(\phi) := \Delta\phi + pU^{p-1}(y)\phi$ has an $n+1$ dimensional bounded kernel generated by the bounded functions

$$Z_i(y) = \frac{\partial U}{\partial y_i}, \quad i = 1, \dots, n, \quad Z_{n+1}(y) = \frac{n-2}{2} U(y) + \nabla U(y) \cdot y.$$

If we consider the model problem for (I), in which now we do not neglect the term corresponding to time derivative, and we consider it on the whole $\mathbb{R}^n$

$$\mu_{0j}^2 \partial_t \phi = L_0(\phi) + E(y,t), \tag{2.46}$$

we observe that $\mu_{0j}^2 \partial_t \phi = L_0(\phi)$ when ϕ is any linear combination of the functions $Z_i(y)$, $i = 1, \dots, n, n+1$. This fact suggests that solvability depends on whether the right hand side $E(y,t)$ does have component in the directions *spanned* by the $Z_i(y)$'s.

In other words, one expects solvability of (2.46) provided that some orthogonality conditions like

$$\int_{\mathbb{R}^n} E(y,t) Z_i(y)\, dy = 0, \quad i = 1, \dots, n+1, \quad \text{for all} \quad t$$

are fulfilled. Since we have k of these conditions, for any $j = 1, \dots, k$, this system takes the form of a nonlinear, nonlocal system of $(n+1)k$ ODEs in the $(n+1)k$ parameter functions $\lambda_1, \dots, \lambda_k$ and $\xi_1, \dots, \xi_k$. It is at this point that we choose the parameters λ and ξ (as functions of the given ϕ) in such a way that these orthogonality (or solvability) conditions are satisfied.

Another known fact about the elliptic $L_0(\phi)$: L_0 has a positive radially symmetric bounded eigenfunction Z_0 associated to the only negative eigenvalue λ_0 to the problem

$$L_0(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n).$$

Furthermore, λ_0 is simple and Z_0 decays like

$$Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{|\lambda_0|}\,|y|} \quad \text{as} \quad |y| \to \infty.$$

Let $e(t) := \int_{\mathbb{R}^n} \phi(y,t) Z_0(y)\, dy$, the projection of $\phi(y,t)$ in the direction $Z_0(y)$.

Integrating equation (2.46) in $\mathbb{R}^n$, using that $\mu_{0j}(t)^2 = b_j^2 t^{-\frac{2}{n-4}}$ we get

$$b_j^2 t^{-\frac{2}{n-2}} \dot{e}(t) - \lambda_0 e(t) = f(t) := \Big(\int_{\mathbb{R}^n} Z_0(y)^2\, dy\Big)^{-1} \int_{\mathbb{R}^n} E(y,t) Z_0(y)\, dy.$$

Hence, for some $a > 0$,

$$e(t) = \exp(at^{\frac{n-2}{n-4}}) \Big(e(t_0) + \int_{t_0}^{t} s^{\frac{2}{n-4}} f(s) \exp(-as^{\frac{n-2}{n-4}}) ds \Big).$$

The only way in which $e(t)$ does not grow exponentially in time (and hence $\phi(y,t)$ does not growth exponentially in time) is for the specific value of initial condition

$$e(t_0) = \int_{\mathbb{R}^n} \phi(y,t_0) Z_0(y)\, dy = -\int_{t_0}^{\infty} s^{\frac{2}{n-4}} f(s) \exp(-as^{\frac{n-2}{n-4}}) ds.$$

This argument suggests that the (small) initial condition required for ϕ should lie on a certain manifold locally described as a translation of the hyperplane orthogonal to $Z_0(y)$. Since we have k of these hyperplanes, for any $j = 1, \ldots, k$ in (I), these constraints define a *codimension k manifold* of initial conditions which describes those for which the expected asymptotic bubbling behavior is possible.

A central point of the full proof is to design a linear theory that allows us to solve the final system by means of a contraction mapping argument. For a large number $R > 0$ we shall construct a solution to an initial value problem of the form

$$\phi_\tau = \Delta\phi + pU(y)^{p-1}\phi + h(y,\tau) \quad \text{in } B_{2R} \times (\tau_0, \infty) \tag{2.47}$$

$$\phi(y,\tau_0) = e_0 Z_0(y) \quad \text{in } B_{2R}.$$

We define

$$\|h\|_{\nu,a} := \sup_{\tau > \tau_0} \sup_{y \in B_{2R}} \tau^{\nu} (1 + |y|^a)\, |h(y,\tau)|. \tag{2.48}$$

for a suitable number ν.

The following is a central step in the proof:

Lemma 2.2.1 *Let $0 < a < 1$, $\nu > 0$. Then, for all sufficiently large $R > 0$ and any $h = h(y,\tau)$ with $\|h\|_{\nu,2+a} < +\infty$ that satisfies for all $j = 1, \ldots, n+1$*

$$\int_{B_{2R}} h(y,\tau)\, Z_j(y)\, dy \;=\; 0 \quad \textit{for all} \quad \tau \in (\tau_0, \infty) \tag{2.49}$$

there exist $\phi = \phi[h]$ and $e_0 = e_0[h]$ which solve Problem (2.47). They define linear operators of h that satisfy the estimates

$$|\phi(y,\tau)| \;\lesssim\; \tau^{-\nu} \frac{R^{n+1-a}}{1 + |y|^{n+1}} \|h\|_{\nu,2+a}. \tag{2.50}$$

and

$$|e_0[h]| \;\lesssim\; \|h\|_{\nu,2+a}. \tag{2.51}$$

After this lemma is proven, the remaining argument roughly goes as follows: We solve (this is rather straightforward) the outer problem (2.45) for given ϕ_j's in the class of the estimated in Lemma 2.2.1 and parameter functions, in the form $\psi = \Psi(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi)$. Then replacing these into the inner problems we get for the ϕ_j's the system of equations

$$\partial_\tau \phi_j = \Delta_y \phi_j + pU(y)^{p-1}\phi_j + E_j(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi)$$

where by definition $\partial_\tau = \mu_{0j}^2 \partial_t$. $E_j(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi)$ is an operator with the property that the conditions

$$\int_{B_{2R}} E_j(y, \tau)\, Z_l(y)\, dy = 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty)$$

for all j, l, amount to an explicit system of first order differential equations for the tuple (λ, ξ) (which involve small nonlinear, nonlocal terms). The dependence in ϕ makes the operator E_j a contraction mapping in its dependence in the operator in Lemma 2.2.1 and the result then follows from a fixed point argument. We refer the reader to [11] for the complete argument.

To establish Lemma 2.2.1 we will make use of the following basic, key lemma regarding the quadratic form associated to the linear operator $L_0 = \Delta + pU^{p-1}$,

$$Q(\phi, \phi) := \int \left[|\nabla \phi|^2 - pU^{p-1}|\phi|^2 \right]. \tag{2.52}$$

The next result provides an estimate of the associated second L^2-eigenvalue in a ball B_{2R} with large radius under zero boundary conditions.

There exists a constant $\gamma > 0$ such that for all sufficiently large R and all radially symmetric function $\phi \in H_0^1(B_{2R})$ with $\int_{B_{2R}} \phi Z_0 = 0$ we have

$$\frac{\gamma}{R^{n-2}} \int_{B_{2R}} |\phi|^2 \le Q(\phi, \phi). \tag{2.53}$$

To prove this, we let H_R be the linear space of all radial functions $\phi \in H_0^1(B_{2R})$ that satisfy the orthogonality condition $\int_{B_{2R}} \phi Z_0 = 0$, and

$$\lambda_R := \inf \left\{ Q(\phi, \phi) \,/\, \phi \in H_R, \int_{B_{2R}} |\phi|^2 = 1 \right\}. \tag{2.54}$$

A standard compactness argument yields that λ_R in (2.54) is achieved by a radial function $\phi_R(x) = \psi_R(r) \in H_R$ with $\int_{B_{2R}} \phi_R^2 = 1$, which satisfies the equation

$$L_0[\phi_R] + \lambda_R \phi_R = c_R Z_0 \quad \text{in } B_{2R}, \quad \phi_R = 0 \quad \text{on } \partial B_{2R}, \tag{2.55}$$

for a suitable Lagrange multiplier c_R.

We have that $\lambda_R \geq 0$. Indeed, the radial eigenvalue problem in $\mathbb{R}^n$

$$\mathcal{L}_0[\psi] + \lambda\psi = 0, \quad \psi'(0) = \psi(+\infty) = 0 \tag{2.56}$$

where

$$\mathcal{L}_0[\psi] := \psi'' + \frac{n-1}{r}\psi' + pU(r)^{p-1}\psi$$

has just one negative eigenvalue, as it follows from maximum principle, using the fact that $\mathcal{L}_0[Z] = 0$ with $Z = Z_{n+1}$, and the fact that this function changes sign just once. It follows that the associated quadratic form must be positive in $H^1(\mathbb{R}^n)$-radial, subject to the L^2-orthogonality condition with respect to Z_0. This implies $\lambda_R \geq 0$.

Thus, to establish (2.53), we assume by contradiction that

$$\lambda_R = o(R^{2-n}) \quad \text{as} \quad R \to +\infty. \tag{2.57}$$

Let χ be a smooth cut-off function with

$$\chi(s) = 1 \text{ for } s < 1 \text{ and } \chi(s) = 0 \text{ for } s > 2. \tag{2.58}$$

Testing against $Z_0(y)\eta_R(|y|)$ where $\eta_R(s) = \chi(s - \frac{R}{2})$, we get

$$c_R \int_{B_{2R}} Z_0^2 \eta_R = \int_{B_{2R}} \phi_R[Z_0 \Delta\eta_R + 2\nabla\eta_R \nabla Z_0].$$

Since $\|\phi_R\|_{L^2(B_{2R})} = 1$, it follows that, for some $\sigma > 0$, $c_R = O(e^{-\sigma R})$.

On the other hand, again using that $\|\phi_R\|_{L^2(B_{2R})} = 1$, standard elliptic estimates yield that $\|\phi_R\|_{L^\infty(B_{2R})} \lesssim 1$.

Let us represent $\phi_R(x) = \psi_R(r)$ using the variation of parameters formula. The function ψ_R satisfies the ODE

$$\mathcal{L}_0[\psi_R] = h_R(r), \quad r \in (0, R), \quad \psi_R'(0) = \psi_R(R) = 0 \tag{2.59}$$

where $h_R(r) = -\lambda_R\psi_R(r) + c_R Z_0(r)$. Furthermore, it is uniformly bounded in R.

Letting $Z = Z_{n+1}$, we consider a second, linearly independent solution $\tilde{Z}(r)$ of this problem, namely $\mathcal{L}_0[\tilde{Z}] = 0$, normalized in such a way that their Wronskian satisfies

$$\tilde{Z}'Z(r) - \tilde{Z}Z'(r) = \frac{1}{r^{n-1}}.$$

Since $Z(r) \sim 1$ near $r = 0$ and $Z(r) \sim r^{2-n}$ as $r \to \infty$, we see that $\tilde{Z}(r) \sim r^{2-n}$ near $r = 0$ and $\tilde{Z}(r) \sim 1$ as $r \to \infty$.

The formula of variation of parameters then yields the representation

$$\psi_R(r) = \tilde{Z}(r)\int_0^r h_R(s)\,Z(s)\,s^{n-1}\,ds + Z(r)\int_r^{2R} h_R(s)\,\tilde{Z}(s)\,s^{n-1}\,ds - A_R Z(r) \tag{2.60}$$

where A_R is such that $\psi_R(2R) = 0$, namely

$$A_R = Z(2R)^{-1}\tilde{Z}(2R)\int_0^{2R} h_R(s)\,Z(s)\,s^{n-1}\,ds.$$

We observe that $\|h_R\|_{L^2(B_{2R})} \lesssim \lambda_R + e^{-\sigma R}$. Then we estimate

$$\left|\int_0^r h_R(s)\,Z(s)\,s^{n-1}\,ds\right| \;\le\; \|Z\|_{L^2(B_{2R})}\|h_R\|_{L^2(B_{2R})} \;\lesssim\; (\lambda_R + e^{-\sigma R})\|Z\|_{L^2(B_{2R})}$$

and

$$\left|\int_r^{2R} h_R(s)\,\tilde{Z}(s)\,s^{n-1}\,ds\right| \;\lesssim\; R^{\frac{n}{2}}(\lambda_R + e^{-\sigma R}).$$

Hence we have for instance,

$$\|A_R Z\|_{L^2(B_{2R})} \;\lesssim\; R^{n-2}(\lambda_R + e^{-\sigma R})\|Z\|_{L^2(B_{2R})},$$

and estimating the other two terms we obtain at last,

$$\|\phi_R\|_{L^2(B_{2R})} \le R^{n-2}(\lambda_R + e^{-\sigma R})\|Z\|_{L^2(B_{2R})}. \tag{2.61}$$

At this point we notice $\|Z\|_{L^2(\mathbb{R}^n)} < +\infty$. This, the fact that $\lambda_R = o(R^{2-n})$ and then $\|\phi_R\|_{L^2(B_{2R})} \to 0$. This is a contradiction and estimate (2.53) is thus proven.

How is this fact used in the complete argument? We solve first a projected problem of the form (just in the radial case for now)

$$\phi_\tau = \Delta\phi + pU(r)^{p-1}\phi + h(r,\tau) - c(\tau)Z_0 \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{2.62}$$

$$\phi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R}.$$

where h decays fast and the function $c(\tau)$ is such that

$$\int_{B_{2R}} \phi(\cdot,\tau)\,Z_0 \;=\; 0 \quad \text{for all} \quad \tau\in(\tau_0,\infty). \tag{2.63}$$

We obtain the relation

$$\partial_\tau \int_{B_{2R}} \phi^2 + Q(\phi,\phi) = \int_{B_{2R}} g\phi, \quad g = h_0 - c(\tau)Z_0.$$

Using the estimate (2.53) we get that for some $\gamma > 0$,

$$\partial_\tau \int_{B_{2R}} \phi^2 + \frac{\gamma}{R^{n-2}} \int_{B_{2R}} \phi^2 \lesssim R^{n-2} \int_{B_{2R}} g^2. \tag{2.64}$$

Using that $\tilde{\phi}(\cdot,\tau_0) = 0$ and Gronwall's inequality, we readily get from the L^2-estimate

$$\|\phi(\cdot,\tau)\|_{L^2(B_{2R})} \lesssim \tau^{-\nu} R^{n-2} K, \quad K := \left[\|h\|_{0,\nu} + e^{-\gamma R}\|\nabla_y \tilde{\phi}\|_{0,\nu}\right]. \tag{2.65}$$

Using standard parabolic estimates, we obtain the desired result in Lemma 2.2.1.

2.2.5 *The Cases of Dimensions $n = 4$ and $n = 3$*

The construction above corresponds to dimension $n \geq 5$. Here we state the form the bubbling solutions take in lower dimensions $n = 3, 4$. For $n = 4$ the statement is similar, with different blow-up rates, but it qualitatively changes in dimension 3.

Let start with $n = 4$. In this case our result reads as follows.

Theorem 2 *Assume $n = 4$. Let $q_1, \cdots, q_k$ be distinct points in Ω such that the matrix $\mathcal{G}(q)$ is positive definite. Then there exist smooth functions $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \ldots, k$, and a solution of Problem (2.1) of the form*

$$u(x,t) = \sum_{j=1}^{k} \alpha_4 \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right) + \theta(t,x),$$

where $\|\theta(\cdot,t)\|_\infty \to 0$ as $t \to +\infty$. The functions $\mu_j(t)$ satisfy

$$\mu_j(t) = \beta_4 \Lambda_j e^{-\beta_4^{-2} t^{\frac{1}{2}}} t^{\frac{1}{4}} + o(e^{-\beta_4^{-2} t^{\frac{1}{2}}} t^{\frac{1}{4}}) \quad \text{as } t \to +\infty$$

for a certain (explicit) positive constant β_4. Here $\Lambda_j = \frac{v_j}{\|v\|} > 0$, where $v = (v_1, \ldots, v_k)$ is an eigenvector associated to the first positive eigenvalue γ_1 of the matrix $\mathcal{G}(q)$.

To state the result in dimension $n = 3$, we need to introduce another Green's function. Let λ_1 be the first (positive) eigenvalue of the Laplace operator, with zero Dirichlet boundary condition on Ω and let γ be a fixed number with $\gamma \in [0, \lambda_1)$, we

denote by G_γ the Green's function for the boundary value problem

$$\begin{aligned} -\Delta_x G_\gamma(x,y) - \gamma G_\gamma(x,y) &= \alpha_3\omega_3\delta(x-y), \quad x\in\Omega, \\ G(\cdot,y) &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.66}$$

where ω_3 is the area of the unit sphere in $\mathbb{R}^3$ and α_3 the number in (2.15). We let $H_\gamma(x,y)$ be the regular part of $G_\gamma(x,y)$ namely the solution of the problem

$$\Delta_x H_\gamma(x,y) + \gamma H_\gamma(x,y) = \gamma\frac{\alpha_3}{|x-y|} \quad \text{in } \Omega, \quad H_\gamma(\cdot,y) = \frac{\alpha_3}{|x-y|} \quad \text{in } \partial\Omega. \tag{2.67}$$

Let $q_1,\dots,q_k$ be given distinct points in Ω so that the matrix $\mathcal{G}(q)$ is positive definite. Define now a new matrix $\mathcal{G}_\gamma(q)$ as

$$\mathcal{G}_\gamma(q) = \begin{bmatrix} H_\gamma(q_1,q_1) & -G_\gamma(q_1,q_2) & \cdots & -G_\gamma(q_1,q_k) \\ -G_\gamma(q_1,q_2) & H_\gamma(q_2,q_2) & -G_\gamma(q_2,q_3)\cdots & -G_\gamma(q_3,q_k) \\ \vdots & & \ddots & \vdots \\ -G_\gamma(q_1,q_k) & \cdots & -G_\gamma(q_{k-1},q_k) & H_\gamma(q_k,q_k) \end{bmatrix} \tag{2.68}$$

Observe that $\mathcal{G}_0(q) = \mathcal{G}(q)$. Since, for any i, $H_\gamma(q_i,q_i)\to-\infty$, as $\gamma\uparrow\lambda_1$, the matrix $\mathcal{G}_\gamma(q)$ becomes positive definite as $\gamma\uparrow\lambda_1$. We define

$$\gamma^*(q) = \sup\{\gamma>0 : \mathcal{G}_\gamma(q) \quad \text{is positive definite}\}. \tag{2.69}$$

Clearly $0<\gamma^*<\lambda_1$. Furthermore, there exists a vector $b=(b_1,\dots,b_k)$ such that

$$\mathcal{G}_{\gamma^*}(q)[b] = 0, \quad \text{and} \quad b_i>0 \quad \text{for all} \quad i. \tag{2.70}$$

Indeed, by definition of γ^* we see that there exists $b\in\mathbb{R}^k$ with $\|b\|=1$ such that

$$b^T\mathcal{G}_{\gamma^*}[q]b = \inf_{x\in\mathbb{R}^k,\,\|x\|=1} x^T\mathcal{G}_{\gamma^*}[q]x = 0.$$

We observe first that all components of b are positive, $b_i\ge 0$, $i=1,\dots,k$. If not, we consider $\tilde b=(|b_1|,\dots,|b_k|)$ and observe that

$$b^T\mathcal{G}_{\gamma^*}[q]b \ge \tilde b^T\mathcal{G}_{\gamma^*}[q]\tilde b.$$

In order to show that $b_i>0$ for all i, we assume the contrary, we call $I=\{i : b_i=0\}$ and we define the vector $b^\varepsilon=(b_1^\varepsilon,\dots,b_k^\varepsilon)$ with

$$b_i^\varepsilon = \varepsilon, \quad \text{if} \quad i\in I, \quad b_i^\varepsilon = b_i \quad \text{otherwise},$$

for $\varepsilon > 0$ fixed. A direct computation gives that

$$(b^\varepsilon)^T \mathcal{G}_{\gamma^*}[q] b^\varepsilon = b^T \mathcal{G}_{\gamma^*}[q] b + \varepsilon^2 \sum_{i \in I} H(q_i, q_i) - \varepsilon \sum_{i \in I, j \notin I} G(q_i, q_j) < b^T \mathcal{G}_{\gamma^*}[q] b$$

if ε is chosen small enough. We thus reach a contradiction and the claim is proven.

We can now state our result.

Theorem 3 *Assume $n = 3$. Let $q_1, \cdots, q_k$ be distinct points in Ω such that the matrix $\mathcal{G}(q)$ is positive definite. Then there exist smooth functions $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \ldots, k$, and a solution of Problem (2.1) of the form*

$$u(x,t) = \sum_{j=1}^{k} \alpha_3 \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{1}{2}} + \theta(t,x),$$

where $\|\theta(\cdot,t)\|_\infty \to 0$ as $t \to +\infty$. The functions $\mu_j(t)$ satisfy

$$\mu_j(t) = \frac{1}{2\sqrt{\gamma_*}} b_j e^{-2\gamma_* t} + o(e^{-2\gamma_* t}) \quad \text{as } t \to +\infty.$$

In the case Ω is the unit ball in $\mathbb{R}^3$, the number γ^* is explicit, it is given by $\gamma_* = \frac{\pi^2}{4}$. Recalling that, in this case, the first eigenvalue λ_1 of the Laplace operator with zero Dirichlet boundary condition is given by π^2, the previous asymptotic becomes

$$\ln \|u\|_\infty \sim \frac{\pi^2}{4} t = \frac{\lambda_1}{4} t,$$

and we recover the asymptotics in [16] obtained in the radial case.

2.3 The Harmonic Map Flow from $\mathbb{R}^2$ into S^2

The results presented in this section for Problem (2.2) correspond to joint work with Juan Dávila and Juncheng Wei.

2.3.1 Preliminaries and Statement of Main Result

We summarize some characteristics of the flow given by Eq. (2.2), some of them we already commented in the introductory section.

- Local existence and uniqueness of a classical solution of (2.2) was established in the works by Eeels-Sampson [14], Struwe [32] and Chang [8]. In fact, a solution of the equation satisfies $|u(x,t)| = 1$ at all times if initial and boundary conditions do.
- Problem (2.2) is the negative L^2-gradient flow for the Dirichlet energy $E(u) := \int_\Omega |\nabla u|^2 dx$. along smooth solutions $u(x,t)$:

$$\frac{d}{dt}E(u(\cdot,t)) \;=\; -\int_\Omega |u_t(\cdot,t)|^2 \;\leq 0\,.$$

- The problem has blowing-up families of **energy invariant steady states** in entire space (entire harmonic maps). Harmonic maps in $\mathbb{R}^2$ are solutions of

$$\Delta u + |\nabla u|^2 u = 0, \quad |u| = 1 \text{in } \mathbb{R}^2$$

for which the simplest nontrivial example is the inverse of the stereographic map,

$$U_0(x) = \begin{pmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{pmatrix}, \quad x \in \mathbb{R}^2.$$

The *1-corrotational harmonic maps* are given by

$$U_{\lambda,x_0,Q}(x) = QU_0\left(\frac{x-x_0}{\lambda}\right)$$

with Q a linear orthogonal transformation of $\mathbb{R}^3$.

$$E_2(U_{\lambda,x_0,Q}) = E(U) = 4\pi \quad \text{for all} \quad \lambda, x_0.$$

- Struwe [32] proved the following important result: there exists a global H^1-weak solution of (2.2), where just for a finite number of points in space-time loss of regularity occurs. In fact, at those times jumps down in energy occur. This solution is unique within the class of weak solutions with degreasing energy [15].

  If $T > 0$ designates the first instant at which smoothness is lost, we must have

$$\|\nabla u(\cdot,t)\|_\infty \;\to\; +\infty$$

Several works have clarified the possible blow-up profiles as $t \uparrow T$.

The following fact follows from results in the works [12, 22, 26, 27, 32]:

Along a sequence $t_n \to T$ and points $q_1, \ldots, q_k \in \Omega$, not necessarily distinct, $u(x,t_n)$ blows-up occurs at exactly those k points in the form of *bubbling*. Precisely,

we have

$$u(x,t_n) \;-\; u_*(x) - \sum_{i=1}^{k} [\, U_i\left(\frac{x-q_i^n}{\lambda_i^n}\right) - U_i(\infty)\,] \to 0 \quad \text{in } H^1(\Omega)$$

where $u_* \in H^1(\Omega)$, $q_i^n \to q_i$, $0 < \lambda_i^n \to 0$, satisfy for $i \neq j$,

$$\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \;\to\; +\infty.$$

The U_i's are entire, finite energy harmonic maps, namely solutions $U : \mathbb{R}^2 \to S^2$ of the equation

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty.$$

After stereographic projection, U lifts to a conformal smooth map in S^2, so that its value $U(\infty)$ is well-defined. It is known that U is in correspondence with a complex rational function or its conjugate. Its energy corresponds to the absolute value of the degree of that map times the area of the unit sphere, and hence

$$\int_{\mathbb{R}^2} |\nabla U|^2 = 4\pi m, \quad m \in \mathbb{N}.$$

In particular, $u(\cdot,t_n) \rightharpoonup u_*$ in $H^1(\Omega)$ and for some positive integers m_i, we have

$$|\nabla u(\cdot,t_n)|^2 \;\rightharpoonup\; |\nabla u_*|^2 + \sum_{i=1}^{k} 4\pi m_i\, \delta_{q_i}$$

δ_q denotes the Dirac mass at q.

A *least energy* entire, non-trivial harmonic map is given by

$$U_0(x) \;=\; \frac{1}{1+|x|^2} \begin{pmatrix} 2x \\ |x|^2 - 1 \end{pmatrix}, \; x \in \mathbb{R}^2,$$

which satisfies

$$\int_{\mathbb{R}^2} |\nabla U_0|^2 = 4\pi, \quad U_0(\infty) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Very few examples are known of solutions which exhibit the singularity formation phenomenon, and all of them concern single-point blow-up in radially symmetric *corrotational* classes.

When Ω is a disk or the entire space, a 1-corrotational solution of (2.2) is one of the form

$$u(x,t) = \begin{pmatrix} e^{i\theta} \sin v(r,t) \\ \cos v(r,t) \end{pmatrix}, \quad x = re^{i\theta}.$$

Problem (2.2) then reduces to the simple looking scalar equation under radial symmetry,

$$v_t = v_{rr} + \frac{v_r}{r} - \frac{\sin v \cos v}{r^2}.$$

We observe that the function $w(r) = \pi - 2\arctan(r)$ is a steady state corresponding to to the harmonic map U_0:

$$U_0(x) = \begin{pmatrix} e^{i\theta} \sin w(r) \\ \cos w(r) \end{pmatrix}.$$

Chang et al. in 1991 [9] found the first example of a blow-up solution of Problem (2.2) (which was previously conjectured not to exist). It is a 1-corrotational solution in a disk with the blow-up profile $v(r,t) \sim w\left(\frac{r}{\lambda(t)}\right)$ or

$$u(x,t) \sim U_0\left(\frac{x}{\lambda(t)}\right).$$

and $0 < \lambda(t) \to 0$ as $t \to T$. No information is provided on $\lambda(t)$.

Topping [36] estimated the general blow-up rates as

$$\lambda_i = o(T-t)^{\frac{1}{2}}$$

(valid in more general targets), namely blow-up is of "type II": it does not occur at a self-similar rate. Angenent et al. [1] estimated the blow-up rate of 1-corrotational maps as $\lambda(t) = o(T-t)$.

Using formal analysis, van den Berg et al. [38] demonstrated that this rate for 1-corrotational maps should generically be given by

$$\lambda(t) \sim \kappa \frac{T-t}{|\log(T-t)|^2}$$

for some $\kappa > 0$.

Raphael and Schweyer [29] succeeded to rigorously construct a 1-corrotational solution with this blow-up rate in entire $\mathbb{R}^2$. Their proof provides the **stability** of the blow-up phenomenon within the radially symmetric class.

A natural, important question is the nonradial case: find nonradial solutions, single and multiple blow-up in entire space or bounded domains and analyze their stability.

Our main result: For any given finite set of points of Ω and suitable initial and boundary values, then a solution with a simultaneous blow-up at those points exists, with a profile resembling a translation, scaling and rotation of U_0 around each bubbling point. Single point blow-up is **codimension-1 stable**.

The functions

$$U_{\lambda,q,Q}(x) := QU_0\left(\frac{x-q}{\lambda}\right).$$

with $\lambda > 0$, $q \in \mathbb{R}^2$ and Q an orthogonal matrix are least energy harmonic maps:

$$\int_{\mathbb{R}^2} |\nabla U_{\lambda,q,Q}|^2 = 4\pi.$$

For $\alpha \in \mathbb{R}$ we denote

$$Q_\alpha \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} e^{i\alpha}(y_1 + iy_2) \\ y_3 \end{bmatrix},$$

the α-rotation around the third axis.

Theorem 4 *Given $T > 0$, $q = (q_1, \ldots, q_k) \in \Omega^k$, there exists initial and boundary data such the solution $u_q(x,t)$ of (HMF) blows-up as $t \uparrow T$ in the form*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_i^*}\Big[\, U_0\left(\frac{x-q_i}{\lambda_i}\right) - U_0(\infty)\,\Big] \to 0$$

in the H^1 and uniform senses where $u_* \in H^1(\Omega) \cap C(\bar{\Omega})$,

$$\lambda_i(t) = \frac{\kappa_i^*(T-t)}{|\log(T-t)|^2}.$$

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 4\pi \sum_{j=1}^{k} \delta_{q_j}$$

Raphael and Schweyer [29] proved the stability of their solution *within the 1-corrotational class*, namely perturbing slightly its initial condition in the associated radial equation the same phenomenon holds at a slightly different time. On the other hand, numerical evidence led van den Berg and Williams [37] to conjecture that this radial bubbling *loses its stability* if special perturbations off the radially symmetric class are made. Our construction shows so at a linear level.

Theorem 5 *For $k = 1$ there exists a manifold of initial data with codimension* 1, *that contains $u_q(x, 0)$, which leads to the solution of (2.2) to blow-up at exactly one point close to q, at a time close to T.*

A natural question is that of *Continuation after blow-up.*

Struwe defined a global H^1-weak solution of (2.2) by dropping the bubbles appearing at the blow-up time and then restarting the flow. This procedure modifies the topology of the image of $u(\cdot, t)$ across T. On the other hand, Topping [35] built a continuation of Chang-Ding-Ye solution by *attaching a bubble with opposite orientation* after blow-up (this does not change topology and makes the energy values "continuous"). This procedure is called reverse bubbling. The reverse bubble is by definition

$$\bar{U}_0(x) = \frac{1}{1+|x|^2}\begin{pmatrix} -2x \\ |x|^2 - 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta}\sin \bar{w}(r) \\ \cos \bar{w}(r) \end{pmatrix}, \quad \bar{w}(r) = -w(r).$$

Our result is the following.

Theorem 6 *The solution u_q can be continued as an H^1-weak solution in $\Omega \times (0, T+\delta)$, with the property that $u_q(x, T) = u_*(x)$*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_i^*}\Big[\, \bar{U}_0\left(\frac{x-q_i}{\lambda_i}\right) - U_0(\infty)\,\Big] \to 0 \quad \text{as } t \downarrow T,$$

in the H^1 and uniform senses in Ω, where

$$\lambda_i(t) = \kappa_i^* \frac{t-T}{|\log(t-T)|^2} \quad \text{if } t > T.$$

It is reasonable to think that the blow-up behavior obtained is generic. Is it possible to have bubbles other than those induced by U_0 or $\bar{U}_0$, and or decomposition in several bubbles at the same point? Evidence seems to indicate the opposite:

In fact, no blow-up is present in the higher corrotational class (Guan et al. [20]) and no *bubble trees* in finite time exist in the 1-corrotational class, van der Hout [39]. In infinite time they do exist and their elements have been classified (Topping [36]).

2.3.2 Sketch of the Construction of Bubbling Solution for $k = 1$

Here we present the main elements present in the construction of a first approximation. Given a $T > 0$, $q \in \Omega$, we want

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u = 0 \quad \text{in } \Omega \times (0, T)$$

with

$$u(x,t) \approx U(x,t) := Q_{\alpha(t)} U_0\left(\frac{x - x_0(t)}{\lambda(t)}\right)$$

The functions $\alpha(t),\ \lambda(t),\ x_0(t)$ are continuous functions up to T and satisfy

$$\lambda(T) = 0, \quad x_0(T) = q$$

Error of approximation:

$$S(U) = -U_t = \frac{\dot\lambda}{\lambda} Q_\alpha \nabla U_0(y) \cdot y - \dot\alpha(\partial_\alpha Q_\alpha) U_0(y) + Q_\alpha \nabla U_0(y) \cdot \frac{\dot x_0}{\lambda},$$

$$y = \frac{x - x_0}{\lambda}.$$

Then $S(U) \perp U$. We recall

$$U_0(y) = \begin{pmatrix} e^{i\theta} \sin w(\rho) \\ \cos w(\rho) \end{pmatrix}, \quad w(\rho) = \pi - 2\arctan(\rho), \quad y = \rho e^{i\theta},$$

$$E_1(y) = \begin{pmatrix} -e^{i\theta} \cos w(\rho) \\ \sin w(\rho) \end{pmatrix}, \quad E_2(y) = \begin{pmatrix} i e^{i\theta} \\ 0 \end{pmatrix},$$

constitute an orthonormal basis of the tangent space to S^2 at the point $U_0(y)$.

$$\begin{aligned} S(U)(x,t) = Q_\alpha [& \frac{\dot\lambda}{\lambda} \rho w_\rho E_1 \ + \ \dot\alpha \rho w_\rho E_2] \\ & + \frac{\dot x_{01}}{\lambda} w_\rho Q_\alpha [\cos\theta E_1 + \sin\theta E_2] \\ & + \frac{\dot x_{02}}{\lambda} w_\rho Q_\alpha [\sin\theta E_1 - \cos\theta E_2]. \end{aligned}$$

For a small function φ, we compute

$$S(U+\varphi) = -\varphi_t + L_U(\varphi) + N_U(\varphi) + S(U).$$

$$L_U(\varphi) = \Delta\varphi + |\nabla U|^2 \varphi + 2(\nabla U \nabla\varphi) U$$

$$N_U(\varphi) = |\nabla\varphi|^2 U + 2(\nabla U \nabla\varphi)\varphi + |\nabla\varphi|^2 \varphi.$$

A useful observation: if φ with $|U+\varphi| = 1$ solves

$$-U_t - \partial_t \varphi + L_U(\varphi) + N_U(\varphi) + b(x,t)U = 0$$

for some scalar function $b(x,t)$ and $|\varphi| \le \frac{1}{2}$, then $u = U + \varphi$ solves (2.2) namely $S(u) = 0$. Indeed,

$$S(u) + bU = 0$$

hence

$$-b(x,t)\, U \cdot u \;=\; S(u) \cdot u \;=\; -\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2}\Delta(|u|^2) = 0.$$

which implies $b = 0$ since $U \cdot u > 0$.

We must have $|U+\varphi|^2 = 1$, namely

$$2U \cdot \varphi + |\varphi|^2 = 0.$$

If φ is small, this approximately means

$$U \cdot \varphi = 0.$$

If we neglect $N_U(\varphi)$ which is quadratic in φ, we look for canceling the linear part (up to terms along U). Thus we want:

$$-\varphi_t + L_U(\varphi) + S(U) + b(x,t)U \approx 0, \quad \varphi \cdot U = 0.$$

We describe a way of finding such a φ for suitable choices of the parameter functions.

For a function φ we write

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi \cdot U)U.$$

We want to find a small function φ^* such that

$$-\partial_t \Pi_{U^\perp}\varphi^* + L_U(\Pi_{U^\perp}\varphi^*) + S(U) + b(x,t)U \approx 0.$$

φ^* will be made out of different pieces. For simplicity we fix

$$x_0 \equiv q, \quad \alpha \equiv 0.$$

Step 1 Concentrating the error. The outer problem: Far away from the concentration point the largest part of the error becomes

$$S(U)(x,t) \;\approx\; \mathcal{E}_0 \;=\; \frac{\dot\lambda}{\lambda}\rho w_\rho(\rho)\, E_1(y) \quad y = \frac{x-x_0}{\lambda} = \rho e^{i\theta}, \quad \rho = |y|.$$

So that we have

$$\mathcal{E}_0 \approx -\frac{2}{r}\begin{bmatrix} e^{i\theta}\dot\lambda \\ 0 \end{bmatrix}, \quad x = q + re^{i\theta}.$$

Set

$$\varphi^0(x,t) = \begin{bmatrix} \phi(r,t)e^{i\theta} \\ 0 \end{bmatrix}$$

so that $\Pi_{U^\perp}\varphi^0 \approx \varphi^0$ away from q.

$$-\partial_t \Pi_{U^\perp}\varphi^0 + L_U[\Pi_{U^\perp}\varphi^0] + \mathcal{E}_0 \approx -\varphi_t + \Delta_x\varphi^0 - \frac{2}{r}\begin{bmatrix} e^{i\theta}\dot\lambda \\ 0 \end{bmatrix}.$$

So we require

$$\phi_t = \phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2} - \frac{2\dot\lambda}{r} \;=\; 0.$$

We solve this equations setting

$$\phi^0 = \phi_0[-2\dot\lambda]$$

where for a continuous function $p(t)$, $t \in [0,T)$, $\phi = \phi_0[p]$ is the unique solution of the Cauchy problem

$$\phi_t = \phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2} + \frac{p(t)}{r} \;=\; 0, \quad (r,t) \in (0,\infty)\times(0,T),$$
$$\phi(r,0) = 0, \quad \phi(0,t) = 0 = \phi(+\infty,t).$$

With the aid of Duhamel's formula, we find

$$\phi_0[p](r,t) = \int_0^t p(s)\frac{1 - e^{-\frac{r^2}{4(t-s)}}}{2r}\, ds.$$

and modify the error as

$$\tilde{\mathcal{E}}_0 \;:=\; -\partial_t \Pi_{U^\perp}\varphi^0 + L_U(\Pi_{U^\perp}\varphi^0) + \mathcal{E}_0 \;=$$

At main order we get

$$\tilde{\mathcal{E}}_0 \approx \lambda^{-2}\left[\frac{8\phi^0}{(1+\rho^2)^2} + \frac{2\lambda\dot{\lambda}}{\rho(1+\rho^2)}\right] E_1 \; + \; \phi^0\,\lambda^{-1}\dot{\lambda}\rho w_\rho\, U.$$

Step 2 We add to $\Pi_{U^\perp}\varphi^0$ a small function $\Pi_{U^\perp}Z^*(x,t)$. We consider a small smooth function $z^*(x,t) = z_1^*(x,t) + iz_2^*(x,t)$ which solves the heat equation,

$$\begin{aligned} z_t^* &= \Delta z^*, \quad \text{in } \Omega\times(0,T),\\ z(x,t) &= z_0(x) \quad \text{in } \partial\Omega\times(0,T),\\ z(x,0) &= z_0(x) \quad \text{in } \partial\Omega\ . \end{aligned}$$

And on $z_0^*(x)$ we assume the following. For a point q_0 close to q,

$$\begin{aligned} \operatorname{div} z_0(q_0) &= \partial_{x_1} z_{01}(q_0) + \partial_{x_2} z_{02}(q_0) \;<0\\ \operatorname{curl} z_0(q_0) &= \partial_{x_1} z_{02}(q_0) - \partial_{x_2} z_{01}(q_0) \;=\; 0\\ z_0(q_0) &= 0, \quad Dz_0(q_0) \text{ non-singular.} \end{aligned}$$

We write

$$Z^*(x,t) = \begin{bmatrix} z^*(x,t)\\ 0\end{bmatrix} = \begin{bmatrix} z_1^* + iz_2^*\\ 0\end{bmatrix}$$

and compute the linear error

$$\begin{aligned} &-\partial_t\Pi_{U^\perp}Z^* + L_U(\Pi_{U^\perp}Z^*)\\ &-\frac{1}{\lambda}\rho w_\rho^2\big[\operatorname{div} z^* E_1 + \operatorname{curl} z^*\, E_2\big]\\ &\frac{1}{\lambda}\rho w_\rho^2\big[\operatorname{div}\bar{z}^*\cos 2\theta + \operatorname{curl}\bar{z}^*\sin 2\theta\,\big]E_1\\ &\frac{1}{\lambda}\rho w_\rho^2\big[\operatorname{div}\bar{z}^*\sin 2\theta - \operatorname{curl}\bar{z}^*\cos 2\theta\,\big]E_2\\ &+\; O(\rho^{-2}) \end{aligned}$$

Step 3 The improvement of approximation gets then reduced to finding φ with $\varphi\cdot U = 0$ and

$$\begin{aligned} &-\partial_t(\Pi_{U^\perp}(\varphi^0+Z^*)+\varphi) + L_U(\Pi_{U^\perp}(\varphi^0+Z^*)+\varphi) + \mathcal{E} + bU\\ &\quad\approx -\partial_t\varphi + L_U(\varphi) + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + bU = 0 \end{aligned}$$

where

$$\mathcal{E}_1 = \left[\lambda^{-2} \frac{4}{(1+\rho^2)^2} \Big[\phi_0[-2\dot{\lambda}] + \lambda\rho \,\mathrm{div}\, z^* \Big] + \frac{2\lambda^{-1}\dot{\lambda}}{\rho(1+\rho^2)} \right] E_1$$

$$\mathcal{E}_2 = \frac{4\lambda^{-1}\rho}{(1+\rho^2)^2} \Big\{ \big[d_1 \cos 2\theta + d_2 \sin 2\theta \big] E_1 + \big[d_1 \sin 2\theta - d_2 \cos 2\theta \big] E_2 \Big\}$$

$$\mathcal{E}_3 = \frac{4\lambda^{-1}\rho}{(1+\rho^2)^2} \,\mathrm{curl}\, z^* \, E_2 + (U \cdot \tilde{z}^*) \frac{2\lambda^{-1}\dot{\lambda}\rho}{1+\rho^2} E_1 + b(x,t)U + O(\rho^{-2})$$

We recall:

$$z^*(q,0) = 0,$$

$$\mathrm{curl}\, z^*(q,0) = 0.$$

$$\mathrm{div}\, z^*(q,0) < 0,$$

In order to find φ which cancels at main order $\mathcal{E}_1$ we consider the problem of finding φ which decays away from the concentration point and satisfies

$$L_U(\varphi) + \mathcal{E}_1 = 0 \quad \varphi \cdot U = 0.$$

the following is a necessary (and sufficient!) condition. We need the orthogonality condition

$$\int_{\mathbb{R}^2} \mathcal{E}_1 \cdot Z_{01} = 0$$

where

$$Z_{01} = \rho w_\rho E_1$$

which satisfies $L_U[Z_{01}] = 0$.

After some computation the equation for $\lambda(t)$ becomes approximately

$$\int_0^{t-\lambda^2} \frac{\dot{\lambda}(s)}{t-s}\, ds = 4\mathrm{div}\, z^*(q,t)\ .$$

Assuming that $\log\lambda \sim \log(T-t)$ the equation is well-approximated by

$$-\dot{\lambda}(t)\log(T-t) + \int_0^t \frac{\dot{\lambda}(s)}{T-s}\, ds + 4\mathrm{div}\, z^*(q,t) = 0.$$

which is explicitly solved as

$$\dot{\lambda}(t) = -\frac{\kappa}{\log^2(T-t)}(1+o(1))$$

The value of κ is precisely that for which

$$\kappa \int_0^T \frac{ds}{(T-s)\log^2(T-s)} \;=\; -4\mathrm{div}\, z^*(q,T).$$

Then if T is small we get the approximation

$$\dot{\lambda}(t) \;\approx\; \dot{\lambda}_0(t) := \frac{4|\log T|}{\log^2(T-t)}\,\mathrm{div}\, z^*(q,T)$$

Since λ decreases to zero as $t \to T^-$, this is where we need the assumption

$$\mathrm{div}\, z^*(q,T) < 0.$$

With this procedure we then get a true reduction of the total error by solving $L_U[\varphi] + \mathcal{E}_j = 0, j = 1, 2$.

At last we find a new approximation of the solution of the type

$$U_*(x,t) = U_0\left(\frac{x-q}{\lambda}\right) + \Pi_{U^\perp}[\phi_0[-2\dot{\lambda}] + Z^*(x,t)] + \varphi_*(x,t)$$

where $\varphi_*(x,t)$ is a decaying solution to

$$L_U[\varphi_*] = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3, \quad \varphi_* \cdot U = 0.$$

To solve the full problem we consider consider

$$\lambda(t) = \lambda_0(t) + \lambda_1(t), \quad \alpha(t) = 0 + \alpha_1(t), \quad x_0(t) = q_0 + x_1(t).$$

The true perturbations λ_1, α_1 approximately solve linear equations of the type

$$\int_0^{t-\lambda_0^2} \frac{\dot{\lambda}(s)}{t-s}\,ds \;=\; p_1(t)$$

$$\int_0^{t-\lambda_0^2} \frac{\dot{\alpha}_1(s)\lambda_0(s)}{t-s}\,ds \;=\; p_2(t)$$

which are approximated by

$$-\dot{\lambda}_1(t)\log(T-t) + \int_0^t \frac{\dot{\lambda}_1(s)}{T-s}\,ds = p_1(t).$$

$$\dot{\alpha}_1(t)\lambda_0 \log(T-t) = p_2(t).$$

These equations are actually a weakly coupled system. In particular the value of $\alpha_1(0)$ turns out to depend of the data, at main order in linear way. This sets a constraint in the solution which yields the codimension-one stability of the solution, a situation that confirms the non-radial instability conjecture in [37]. Actually it is so determined $\lambda(0)$, but the degree of freedom given by moving T allows to choose it a priori as an arbitrary small number. That degree of freedom is lost in α. This is what yields the codimension 1 statement. The full construction follows the same lines as that for the critical equation while it is of harder technical nature.

2.4 Infinite Time Blow-up in the Critical Mass Platak-Keller-Segel Equation

We state a result corresponding to joint work with J. Dávila, J. Dolbeault, M. Musso and J. Wei where the same general scheme of the problems in the previous two sections has been followed. For the Platak-Keller-Segel equation (2.3), our main result is existence and stability of the **critical mass solution.**

$$u_t = \Delta u - \nabla\cdot(u\nabla(-\Delta)^{-1}u), \quad u > 0 \quad \text{in } \mathbb{R}^2\times(0,\infty)$$

Assuming that $u(x,0) \in L^1(\mathbb{R}^2)$, the following is known:

- If $\int_{\mathbb{R}^2} u(x,0)dx > 8\pi$ then finite-time blow-up always takes place. On the other hand, if $\int_{\mathbb{R}^2} u(x,0)dx < 8\pi$ then the solution is globally defined in time, and it goes to zero uniformly as $t\to\infty$ with a self-similar profile. See Blanchet et al. [2]. Bubbling blow-up behavior in the radial case with exact rates when mass is close from above to 8π have been built by Raphael and Schweyer [30].
- The case of *critical mass* $\int_{\mathbb{R}^2} u(x,0)dx = 8\pi$ is delicate concerning its asymptotic behavior. The solution is globally defined in time and it may or may not blow-up. If the second moment of the initial condition is finite, namely $\int_{\mathbb{R}^2} |x|^2 u(x,0)dx < +\infty$, then the solution blows-up in infinite time, with a bubbling behavior, see Carlen and Figalli [6], Blanchet et al. [3, 4]. Formal rates of bubbling when mass equals 8π have been studied by Chavanis and Sire [10] and by Campos [5]. In the very recent preprint by Ghoul and Masmoudi [18], a radial solution with exact rates has been built. Stability is proven within the radial class and the method does not yield it in general.

Theorem 7 *There exists a solution $u(x,t)$ of Problem (2.3) with fast-decay initial condition of mass 8π, which blows-up in infinite time, with a profile which at main order is*

$$u(x,t) \approx \frac{8\lambda(t)^2}{(\lambda(t)^2 + |x|^2)^2}$$

where

$$\lambda(t) \sim \frac{1}{\sqrt{\log t}}.$$

All positive initial conditions (not necessarily radial) with fast decay and mass 8π suitably close to $u(x,0)$ lead to the same phenomenon.

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Chapter 3
Regularity Theory for Local and Nonlocal Minimal Surfaces: An Overview

Matteo Cozzi and Alessio Figalli

Abstract These notes record the lectures for the CIME Summer Course held by the second author in Cetraro during the week of July 4–8, 2016. The goal is to give an overview of some classical results for minimal surfaces, and describe recent developments in the nonlocal setting.

3.1 Introduction

Let $1 \leqslant k \leqslant n-1$ be two integers and $\Gamma \subset \mathbb{R}^n$ be a $(k-1)$-dimensional, smooth manifold without boundary. The classical *Plateau problem* consists in finding a k-dimensional set Σ with $\partial\Sigma = \Gamma$ such that

$$\text{Area}(\Sigma) = \min\Big\{\text{Area}(\Sigma') : \partial\Sigma' = \Gamma\Big\}. \tag{3.1}$$

Here, with the notation Area$(\cdot)$ we denote a general "area-type functional" that we shall specify later. We will consider two main examples: one where the area is the standard Hausdorff k-dimensional measure, and one in which it represents a recently introduced notion of nonlocal (or fractional) perimeter.

We stress that we do not have a well-defined nonlocal perimeter for k-dimensional manifolds with $k \leqslant n-2$. Moreover, even in the codimension 1 case, we need Σ to be the boundary of some set in order to be able to define its fractional perimeter. Therefore, to make the parallel between the local and the nonlocal theories more evident, we shall always focus on the setting

$$k = n-1 \qquad \text{and} \qquad \Sigma = \partial E, \text{ with } E \subset \mathbb{R}^n \text{ } n\text{-dimensional.}$$

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In the forthcoming sections, we will outline several issues and solutions relevant to this minimization problem.

Most of these notes will be devoted to presenting the main ideas involved in the case of the traditional area functional. Then, in the last section, we will briefly touch on the main challenges that arise in the nonlocal setting.

Remark 3.1.1 In these notes, we shall say that a surface is "minimal" if it *minimizes* the area functional. This notation is not universal: some authors call a surface "minimal" if it is a critical point of the area functional, and call it "area minimizing" when it is a minimizer.

3.1.1 The Minimization Problem

Given a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary and a $(n-2)$-dimensional smooth manifold Γ without boundary, we want to find a set $E \subset \mathbb{R}^n$ satisfying the boundary constraint

$$\partial E \cap \partial\Omega = \Gamma,$$

and minimizing

$$\text{Area}(\partial E \cap \Omega),$$

among all sets $E' \subset \mathbb{R}^n$ such that $\partial E' \cap \partial\Omega = \Gamma$.

Note that it is very difficult to give a precise sense to the intersection $\partial E \cap \partial\Omega$ when E has a rough boundary. In order to avoid unnecessary technical complications related to this issue, we argue as follows. For simplicity, we shall assume from now on that Ω is equal to the unit ball B_1, but of course this discussion can be easily extended to the general case.

Fix a smooth n-dimensional set $F \subset \mathbb{R}^n$ such that $\partial F \cap \partial B_1 = \Gamma$. Instead of prescribing the boundary of our set E on ∂B_1, we will require it to coincide with F on the complement of B_1. That is, we study the equivalent minimization problem

$$\min\Big\{\text{Area}(\partial E \cap B_1) : E \setminus B_1 = F \setminus B_1\Big\}. \tag{3.2}$$

Still, there is another issue. It may happen that a non-negligible part of ∂E is not inside B_1, but on the boundary of B_1 (see Fig. 3.1). As a consequence, this part would either contribute or not contribute to $\text{Area}(\partial E \cap B_1)$, depending on our understanding of B_1 as open or closed.

In order to overcome this ambiguity, we consider the slightly different minimization problem

$$\min\Big\{\text{Area}(\partial E \cap B_2) : E \setminus B_1 = F \setminus B_1\Big\}. \tag{3.3}$$

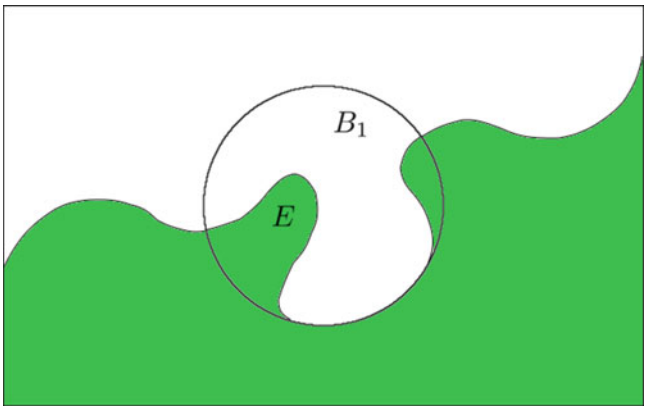

Fig. 3.1 An example of a boundary ∂E that *sticks* to the sphere ∂B_1 on a non-negligible portion of it

Observe that, in contrast to (3.2), we are now minimizing the area inside the larger open ball B_2. In this way, we do not have anymore troubles with sticking boundaries. On the other hand, we prescribe the constraint outside of the smaller ball B_1. Hence, in (3.3) we are just adding terms which are the same for all competitors, namely the area of ∂F inside $B_2 \setminus B_1$. Notice that (3.3) is equivalent to

$$\min\Big\{\text{Area}(\partial E \cap \overline{B}_1) : E \setminus B_1 = F \setminus B_1\Big\}.$$

However, as we shall see later, (3.3) is "analytically" better because the area inside an open set will be shown to be lower-semicontinuous under L^1_{loc}-convergence (see Proposition 3.2.3).

Now, the main question becomes: what is the area? For smooth boundaries, this is not an issue, since there is a classical notion of surface area. On the other hand, if in (3.3) we are only allowed to minimize among smooth sets, then it is not clear whether a minimizer exists in such class of sets. Actually, as we shall see later, minimizers are not necessarily smooth! Thus, we need a good definition of area for non-smooth sets.

3.2 Sets of Finite Perimeter

The main idea is the following: If E has smooth boundary, then it is not hard to verify that

$$\text{Area}(\partial E) = \sup\left\{\int_{\partial E} X \cdot \nu_E : X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ |X| \leqslant 1\right\}, \tag{3.4}$$

where ν_E denotes the unit normal vector field of ∂E, pointing outward of E. Indeed, if ∂E is smooth, one can extend ν_E to a smooth vector field N defined on the whole $\mathbb{R}^n$ and satisfying $|N| \leqslant 1$. By setting $X = \eta_R N$, with η_R a cutoff function supported inside the ball B_R, and letting $R \to \infty$, one is easily led to (3.4).

Since, by the divergence theorem,

$$\int_{\partial E} X \cdot \nu_E = \int_E \mathrm{div} X,$$

we can rewrite (3.4) as

$$\mathrm{Area}(\partial E) = \sup \left\{ \int_E \mathrm{div} X : X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n),\ |X| \leqslant 1 \right\}.$$

Notice that we do not need any regularity assumption on ∂E for the right-hand side of the formula above to be well-defined. Hence, one can use the right-hand side as the *definition* of perimeter for a non-smooth set.

More generally, given any open set $\Omega \subseteq \mathbb{R}^n$, the same considerations as above show that

$$\mathrm{Area}(\partial E \cap \Omega) = \sup \left\{ \int_E \mathrm{div} X : X \in C_c^1(\Omega; \mathbb{R}^n),\ |X| \leqslant 1 \right\}. \tag{3.5}$$

Again, this fact holds true when E has smooth boundary. Conversely, for a general set E, we can use (3.5) as a definition.

Definition 3.2.1 Let $\Omega \subseteq \mathbb{R}^n$ be open and $E \subset \mathbb{R}^n$ be a Borel set. The *perimeter* of E inside Ω is given by

$$\mathrm{Per}(E; \Omega) := \sup \left\{ \int_E \mathrm{div} X : X \in C_c^1(\Omega; \mathbb{R}^n),\ |X| \leqslant 1 \right\}.$$

When $\Omega = \mathbb{R}^n$, we write simply $\mathrm{Per}(E)$ to indicate $\mathrm{Per}(E; \mathbb{R}^n)$.

Note that $\mathrm{Per}(E; \Omega)$ is well-defined for any Borel set, but it might be infinite. For this reason, we will restrict ourselves to a smaller class of sets.

Definition 3.2.2 Let $\Omega \subseteq \mathbb{R}^n$ be open and $E \subset \mathbb{R}^n$ be a Borel set. The set E is said to have *finite perimeter* inside Ω if $\mathrm{Per}(E; \Omega) < +\infty$. When $\Omega = \mathbb{R}^n$, we simply say that E has finite perimeter.

With these definitions, the minimization problem becomes

$$\min \Big\{ \mathrm{Per}(E; B_2) : E \setminus B_1 = F \setminus B_1 \Big\}. \tag{3.6}$$

Of course, since F is a competitor and $\mathrm{Per}(F, B_2) < +\infty$, in the above minimization problem it is enough to consider only sets of finite perimeter.

In the remaining part of this section we examine two fundamental properties of the perimeter that will turn out to be crucial for the existence of minimizers: lower semicontinuity and compactness.

3.2.1 Lower Semicontinuity

In this subsection, we show that the perimeter is lower semicontinuous with respect to the L^1_{loc} topology. We recall that a sequence of measurable sets $\{E_k\}$ is said to converge in $L^1(\Omega)$ to a measurable set E if

$$\chi_{E_k} \longrightarrow \chi_E \text{ in } L^1(\Omega),$$

as $k \to +\infty$. Similarly, the convergence in L^1_{loc} is understood in the above sense.

The statement concerning the semicontinuity of Per is as follows.

Proposition 3.2.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\{E_k\}$ be a sequence of Borel sets, converging in $L^1_{\text{loc}}(\Omega)$ to a set E. Then,*

$$\text{Per}(E;\Omega) \leqslant \liminf_{k\to+\infty} \text{Per}(E_k;\Omega).$$

Proof Clearly, we can assume that each E_k has finite perimeter inside Ω. Fix any vector field $X \in C^1_c(\Omega;\mathbb{R}^n)$ such that $|X| \leqslant 1$. Then,

$$\int_E \text{div}X = \int_\Omega \chi_E \,\text{div}X = \lim_{k\to+\infty} \int_\Omega \chi_{E_k}\,\text{div}X = \lim_{k\to+\infty} \int_{E_k} \text{div}X.$$

Since by definition

$$\int_{E_k} \text{div}X \leqslant \text{Per}(E_k;\Omega) \qquad \text{for any } k \in \mathbb{N},$$

this yields

$$\int_E \text{div}X \leqslant \liminf_{k\to+\infty} \text{Per}(E_k;\Omega).$$

The conclusion follows by taking the supremum over all the admissible vector fields X on the left-hand side of the above inequality. □

Lower semicontinuity is the first fundamental property that one needs in order to prove the existence of minimal surfaces. However, alone it is not enough. We need another key ingredient.

3.2.2 Compactness

Here we focus on a second important property enjoyed by the perimeter. We prove that a sequence of sets having perimeters uniformly bounded is precompact in the L^1_{loc} topology. That is, the next result holds true.

Proposition 3.2.4 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of Borel subsets of Ω such that*

$$\text{Per}(E_k;\Omega) \leqslant C \tag{3.7}$$

for some constant $C > 0$ independent of k. Then, up to a subsequence, E_k converges in $L^1_{\text{loc}}(\Omega)$ to a Borel set $E \subseteq \Omega$.

The proof of the compactness result is more involved than that of the semicontinuity. We split it in several steps.

First, we recall the following version of the Poincaré's inequality. We denote by $(u)_A$ the integral mean of u over a set A with finite measure, that is

$$(u)_A := \fint_A u = \frac{1}{|A|}\int_A u.$$

Also, Q_r denotes a given (closed) cube of sides of length $r > 0$.

Lemma 3.2.5 *Let $r > 0$ and $u \in C^1(Q_r)$. Then,*

$$\int_{Q_r} |u - (u)_{Q_r}| \leqslant C_n r \int_{Q_r} |\nabla u|,$$

for some dimensional constant $C_n > 0$.

Proof Up to a translation, we may assume that $Q_r = [0, r]^n$. Moreover, we initially suppose that $r = 1$.

We first prove the result with $n = 1$. In this case, note that for any $x, y \in [0, 1]$, we have

$$|u(x) - u(y)| \leqslant \int_x^y |\nabla u(z)|\, dz \leqslant \int_0^1 |\nabla u(z)| dz.$$

Choosing $y \in [0, 1]$ such that $u(y) = (u)_{[0,1]}$ (note that such a point exists thanks to the mean value theorem) and integrating the inequality above with respect to $x \in [0, 1]$, we conclude that

$$\int_0^1 |u - (u)_{[0,1]}| \leqslant \int_0^1 |\nabla u|,$$

which proves the result with $C_1 = 1$.

Now, assume by induction that the result is true up to dimension $n-1$. Then, given a C^1 function $u : [0,1]^n \to \mathbb{R}$, we can define the function $\bar{u} : [0,1]^{n-1} \to \mathbb{R}$ given by

$$\bar{u}(x') = \int_0^1 u(x', x_n) dx_n.$$

With this definition, the one-dimensional argument above applied to the family of functions $\{u(x', \cdot)\}_{x' \in [0,1]^{n-1}}$ shows that

$$\int_0^1 |u(x', x_n) - \bar{u}(x')| dx_n \leqslant \int_0^1 |\partial_n u(x', x_n)| dx_n \quad \text{for any } x' \in [0,1]^{n-1}.$$

Hence, integrating with respect to x', we get

$$\int_{[0,1]^n} |u - \bar{u}| \leqslant \int_{[0,1]^n} |\partial_n u|. \tag{3.8}$$

We now observe that, by the inductive hypothesis,

$$\int_{[0,1]^{n-1}} |\bar{u} - (\bar{u})_{[0,1]^{n-1}}| \leqslant C_{n-1} \int_{[0,1]^{n-1}} |\nabla_{x'} \bar{u}|. \tag{3.9}$$

Noticing that

$$(\bar{u})_{[0,1]^{n-1}} = (u)_{[0,1]^n} \quad \text{and} \quad \int_{[0,1]^{n-1}} |\nabla_{x'} \bar{u}| \leqslant \int_{[0,1]^n} |\nabla_{x'} u|,$$

combining (3.8) and (3.9) we get

$$\int_{[0,1]^n} |u - (u)_{[0,1]^n}| \leqslant \int_{[0,1]^n} |\partial_n u| + C_{n-1} \int_{[0,1]^n} |\nabla_{x'} u|,$$

which proves the result with $C_n = 1 + C_{n-1}$.

Finally, the general case follows by a simple scaling argument. Indeed, if $u \in C^1(Q_r)$, then the rescaled function $u_r(x) := u(rx)$ belongs to $C^1(Q_1)$. Moreover, we have that

$$\int_{Q_r} |u - (u)_{Q_r}| = r^n \int_{Q_1} |u_r - (u_r)_{Q_1}|,$$

and

$$\int_{Q_r} |\nabla u| = r^{n-1} \int_{Q_1} |\nabla u_r|.$$

The conclusion then follows from the case $r = 1$ applied to u_r. □

We now plan to deduce a Poincaré-type inequality for the characteristic function χ_E of a bounded set E having finite perimeter. Of course, $\chi_E \notin C^1$ and Lemma 3.2.5 cannot be applied directly to it. Instead, we need to work with suitable approximations.

Given $r > 0$, consider a countable family of disjoint open cubes $\{Q^j\}$ of sides r such that $\cup_j \overline{Q^j} = \mathbb{R}^n$. We order this family so that

$$\begin{aligned} |Q^j \cap E| &\geqslant \frac{|Q^j|}{2} \text{ for any integer } j = 1, \dots, N, \\ |Q^j \cap E| &< \frac{|Q^j|}{2} \text{ for any integer } j > N, \end{aligned} \tag{3.10}$$

for some uniquely determined $N \in \mathbb{N}$. Notice that such N exists since E is bounded. We then write

$$T_{E,r} := \bigcup_{j=1}^{N} Q^j, \tag{3.11}$$

see Fig. 3.2.

Lemma 3.2.6 *Let $r > 0$ and $E \subset \mathbb{R}^n$ be a bounded set with finite perimeter. Then,*

$$\|\chi_E - \chi_{T_{E,r}}\|_{L^1(\mathbb{R}^n)} \leqslant C_n r \operatorname{Per}(E),$$

with C_n as in Lemma 3.2.5.

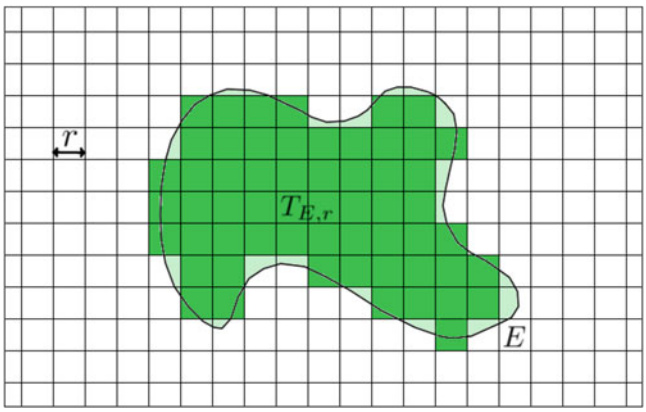

Fig. 3.2 The grid made up of cubes of sides r, the set E (in *light green*) and the resulting set $T_{E,r}$ (in *dark green*)

Proof Consider a family $\{\rho_\varepsilon\}$ of radially symmetric smooth convolution kernels, and define $u_\varepsilon := \chi_E * \rho_\varepsilon$. Clearly, $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to \chi_E$ in $L^1(\mathbb{R}^n)$, as $\varepsilon \to 0^+$. Furthermore, by considerations analogous to the ones at the beginning of Sect. 3.2, it is not hard to see that

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| = \sup\left\{ -\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot X : X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ |X| \leqslant 1 \right\}.$$

Integrating by parts and exploiting well-known properties of the convolution operator, we find that

$$\begin{aligned} -\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot X &= \int_{\mathbb{R}^n} u_\varepsilon \,\mathrm{div} X = \int_{\mathbb{R}^n} (\chi_E * \rho_\varepsilon)\,\mathrm{div} X \\ &= \int_{\mathbb{R}^n} \chi_E(\rho_\varepsilon * \mathrm{div} X) = \int_E \mathrm{div}(X * \rho_\varepsilon). \end{aligned}$$

Since $|X| \leqslant 1$, it follows that $|X * \rho_\varepsilon| \leqslant 1$. Therefore, by taking into account Definition 3.2.1, we obtain

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \leqslant \mathrm{Per}(E). \tag{3.12}$$

Recall now the partition (up to a set of measure zero) of $\mathbb{R}^n$ into the family of cubes $\{Q^j\}$ introduced earlier. Applying the Poincaré's inequality of Lemma 3.2.5 to u_ε in each cube Q^j, we get

$$C_n r \int_{\mathbb{R}^n} |\nabla u_\varepsilon| = C_n r \sum_{j\in\mathbb{N}} \int_{Q^j} |\nabla u_\varepsilon| \geqslant \sum_{j\in\mathbb{N}} \int_{Q^j} |u_\varepsilon - (u_\varepsilon)_{Q^j}|. \tag{3.13}$$

On the other hand, for any $j \in \mathbb{N}$,

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \int_{Q^j} |u_\varepsilon - (u_\varepsilon)_{Q^j}| &= \int_{Q^j} |\chi_E - (\chi_E)_{Q^j}| = \int_{Q^j} \left|\chi_E - \frac{|Q^j \cap E|}{|Q^j|}\right| \\ &= |Q^j \cap E| \frac{|Q^j| - |Q^j \cap E|}{|Q^j|} + |Q^j \setminus E| \frac{|Q^j \cap E|}{|Q^j|} \\ &= 2\frac{|Q^j \cap E||Q^j \setminus E|}{|Q^j|}. \end{aligned}$$

Using this in combination with (3.12) and (3.13), we obtain that

$$C_n r\,\mathrm{Per}(E) \geqslant 2 \sum_{j\in\mathbb{N}} \frac{|Q^j \cap E||Q^j \setminus E|}{|Q^j|}.$$

But then, recalling (3.10) and (3.11), we conclude that

$$\begin{aligned} C_n r \operatorname{Per}(E) &\geqslant \sum_{j=1}^{N} \left(2\frac{|Q^j \cap E|}{|Q^j|}\right) |Q^j \setminus E| + \sum_{j=N+1}^{+\infty} \left(2\frac{|Q^j \setminus E|}{|Q^j|}\right) |Q^j \cap E| \\ &\geqslant \sum_{j=1}^{N} |Q^j \setminus E| + \sum_{j=N+1}^{+\infty} |Q^j \cap E| = |T_{E,r} \setminus E| + |E \setminus T_{E,r}| \\ &= \|\chi_E - \chi_{T_{E,r}}\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

which concludes the proof. □

By virtue of Lemma 3.2.6, we see that $T_{E,r}$ converges to E in $L^1(\mathbb{R}^n)$, as r goes to 0, with a rate that is controlled by $\operatorname{Per}(E)$. Knowing this fact, we are now in position to deal with the proof of Proposition 3.2.4. The main step is represented by the next:

Lemma 3.2.7 *Let $C, R > 0$ be fixed. Let $\{E_k\}$ be a sequence of sets such that*

$$E_k \subseteq B_R, \tag{3.14}$$

and

$$\operatorname{Per}(E_k) \leqslant C,$$

for any $k \in \mathbb{N}$. Then, up to a subsequence, E_k converges in $L^1(\mathbb{R}^n)$ to a set E.

Notice that this result is slightly weaker than the one claimed by Proposition 3.2.4 (with $\Omega = \mathbb{R}^n$), since the E_k's are supposed to be uniformly bounded sets.

Proof of Lemma 3.2.7 Consider the following class of sets

$$\mathcal{X}_{R,C} := \Big\{F \subseteq B_R : F \text{ is Borel, } \operatorname{Per}(F) \leqslant C\Big\},$$

and endow it with the metric defined by

$$d(E, F) := \|\chi_E - \chi_F\|_{L^1(\mathbb{R}^n)}, \qquad \text{for any } E, F \in \mathcal{X}_{R,C}.$$

Observe that the lemma will be proved if we show that the metric space $(\mathcal{X}_{R,C}, d)$ is compact.

We first claim that

$$(\mathcal{X}_{R,C}, d) \text{ is complete.} \tag{3.15}$$

Note that $(\mathcal{X}_{R,C}, d)$ may be seen as a subspace of $L^1(\mathbb{R}^n)$, via the identification of a set E with its characteristic function χ_E. Therefore, it suffices to prove that $\mathcal{X}$ is closed in $L^1(\mathbb{R}^n)$. To see this, let $\{F_k\} \subset \mathcal{X}_{R,C}$ be a sequence such that χ_{F_k}

converges to some function f in $L^1(\mathbb{R}^n)$. Clearly $f = \chi_F$ for some set $F \subseteq B_R$, since a subsequence of $\{\chi_{F_k}\}$ converges to f a.e. in $\mathbb{R}^n$. In addition, Proposition 3.2.3 implies that

$$\mathrm{Per}(F) \leqslant \liminf_{k\to+\infty} \mathrm{Per}(F_k) \leqslant C.$$

This proves that $F \in \mathcal{X}_{R,C}$, hence $\mathcal{X}_{R,C}$ is closed in $L^1(\mathbb{R}^n)$ and (3.15) follows.

We now claim that

$$(\mathcal{X}_{R,C}, d) \text{ is totally bounded.} \tag{3.16}$$

To check (3.16), we need to show the existence of a finite ε-net. That is, for any $\varepsilon > 0$, we need to find a finite number of sets $F_1, \dots, F_{N_\varepsilon}$, for some $N_\varepsilon \in \mathbb{N}$, such that, for any $F \in \mathcal{X}_{R,C}$,

$$d(F, F_i) < \varepsilon, \text{ for some } i \in \{1, \dots, N_\varepsilon\}.$$

Fix $\varepsilon > 0$ and set

$$r_\varepsilon := \frac{\varepsilon}{2CC_n},$$

with C_n as in Lemma 3.2.5. Given any $F \in \mathcal{X}_{R,C}$, we consider the set T_{F,r_ε} introduced in (3.11). By Lemma 3.2.6, we have that

$$d(F, T_{F,r_\varepsilon}) = \|\chi_F - \chi_{T_{F,r_\varepsilon}}\|_{L^1(\mathbb{R}^n)} \leqslant C_n r_\varepsilon \,\mathrm{Per}(F) \leqslant CC_n r_\varepsilon < \varepsilon.$$

Since the cardinality of

$$\left\{T_{F,r_\varepsilon} : F \in \mathcal{X}_{\mathcal{R},\mathcal{C}}\right\},$$

is finite (as a quick inspection of definition (3.11) reveals), we have found the desired ε-net and (3.16) follows.

In view of (3.15) and (3.16), we know that $(\mathcal{X}_{R,C}, d)$ is closed and totally bounded. It is a standard fact in topology that this is in turn equivalent to the compactness of $(\mathcal{X}_{R,C}, d)$. Hence, Lemma 3.2.7 holds true. □

With the help of Lemma 3.2.7, we can now conclude this subsection by proving the validity of our compactness statement in its full generality.

Proof of Proposition 3.2.4 We plan to obtain the result combining Lemma 3.2.7 with a suitable diagonal argument. To do this, consider first $\{\Omega_\ell\}$ an exhaustion of Ω made of open bounded sets with smooth boundaries, so that, in particular, the perimeter of each set Ω_ℓ is finite. Moreover, we may assume without loss of generality that $\Omega_\ell \subset B_\ell$.

For any $\ell \in \mathbb{N}$, we define

$$E_k^\ell := E_k \cap \Omega_\ell.$$

For any fixed ℓ, it holds $E_k^\ell \subseteq \Omega_\ell \subset B_\ell$ for any $k \in \mathbb{N}$. In particular, E_k^ℓ satisfies (3.14) with $R = \ell$. Moreover, using (3.7), it is not hard to check that

$$\mathrm{Per}(E_k^\ell) \leqslant \mathrm{Per}(E_k; \Omega_\ell) + \mathrm{Per}(\Omega_\ell) \leqslant \mathrm{Per}(E_k; \Omega) + \mathrm{Per}(\Omega_\ell) \leqslant C_\ell,$$

for some constant $C_\ell > 0$ independent of k.

In light of these facts, the sequence $\{E_k^\ell\}_{k\in\mathbb{N}}$ satisfies the hypotheses of Lemma 3.2.7. Hence, we infer that, for any fixed ℓ, there exists a diverging sequence $\mathcal{K}^\ell = \{\varphi^\ell(j)\}_{j\in\mathbb{N}}$ of natural numbers such that $E^\ell_{\varphi^\ell(j)}$ converges in $L^1(\mathbb{R}^n)$ to a set $E^\ell \subseteq \Omega_\ell$, as $j \to +\infty$. By a diagonal argument we can suppose that $\mathcal{K}^m \subseteq \mathcal{K}^\ell$ if $\ell \leqslant m$. Furthermore, it is easy to see that $E^m \cap \Omega_\ell = E^\ell$, if $\ell \leqslant m$. We then define

$$E := \bigcup_{\ell\in\mathbb{N}} E^\ell,$$

and notice that $E \cap \Omega_\ell = E^\ell$ for any ℓ. Set $k_\ell := \varphi^\ell(\ell)$, for any $\ell \in \mathbb{N}$. Clearly, $\{k_\ell\}$ is a subsequence of each $\mathcal{K}^m$, up to a finite number of indices ℓ. Hence, for any fixed $m \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{\ell\to+\infty} \|\chi_{E_{k_\ell}} - \chi_E\|_{L^1(\Omega_m)} &= \lim_{\ell\to+\infty} \|\chi_{E^m_{k_\ell}} - \chi_{E^m}\|_{L^1(\mathbb{R}^n)} \\ &= \lim_{j\to+\infty} \|\chi_{E^m_{\varphi^m(j)}} - \chi_{E^m}\|_{L^1(\mathbb{R}^n)} \\ &= 0.\end{aligned}$$

This proves that $E_{k_\ell} \to E$ in $L^1_{\mathrm{loc}}(\Omega)$ as $\ell \to +\infty$, completing the proof. □

3.3 Existence of Minimal Surfaces

With the help of the lower semicontinuity of the perimeter and the compactness property established in the previous section, we can now easily prove the existence of a solution to the minimization problem (3.6).

Theorem 3.3.1 *Let F be a set with finite perimeter inside B_2. Then, there exists a set E of finite perimeter inside B_2 such that $E \setminus B_1 = F \setminus B_1$ and*

$$\mathrm{Per}(E; B_2) \leqslant \mathrm{Per}(E'; B_2)$$

for any set E' such that $E' \setminus B_1 = F \setminus B_1$.

Proof Our argument is based on the direct method of the calculus of variations. Set

$$\alpha := \inf\Big\{P(E'; B_2) : E' \setminus B_1 = F \setminus B_1\Big\}. \tag{3.17}$$

Note that α is finite since $\alpha \leqslant P(F; B_2)$.

Take a sequence $\{E_k\}$ of sets of finite perimeter such that $E_k \setminus B_1 = F \setminus B_1$ for any $k \in \mathbb{N}$ and

$$\lim_{k\to+\infty} \mathrm{Per}(E_k; B_2) = \alpha.$$

Clearly, we can assume without loss of generality that

$$\mathrm{Per}(E_k; B_2) \leqslant \alpha + 1 \quad \text{for any } k \in \mathbb{N}.$$

Therefore, by Proposition 3.2.4, we conclude that there exists a subsequence E_{k_j} converging to a set E in $L^1_{\mathrm{loc}}(B_2)$, as $j \to +\infty$. Consequently, Proposition 3.2.3 yields

$$\mathrm{Per}(E; B_2) \leqslant \lim_{j\to+\infty} \mathrm{Per}(E_{k_j}; B_2) = \alpha.$$

Since $E \setminus B_1 = \lim_{j\to+\infty} E_{k_j} \setminus B_1 = F \setminus B_1$, the set E is admissible in (3.17) and we conclude that

$$\mathrm{Per}(E; B_2) = \alpha.$$

The set E is thus the desired minimizer. □

In the following sections, our goal will be to show that the minimizers just obtained are more than just sets with finite perimeter. That is, we will develop an appropriate regularity theory for minimal surfaces. However, to do that, we first need to describe some important facts about sets of finite perimeter.

3.4 Fine Properties of Sets of Finite Perimeter

In this section, we introduce a different concept of boundary for sets of finite perimeter: the *reduced boundary*. As we shall see, up to a "small" component, this new boundary is always contained in a collection of $(n-1)$-dimensional hypersurfaces of class C^1. Moreover, through this definition, one can compute the perimeter of a set in a more direct way via the Hausdorff measure.

We begin by recalling the definition of Hausdorff measure.

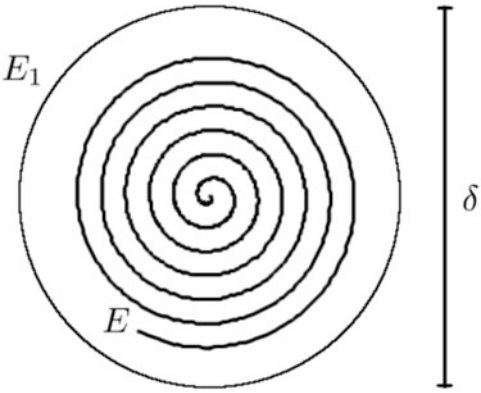

Fig. 3.3 The spiral-like set E is covered by the ball $E_1 = B_{\delta/2}$. If δ is comparable to the diameter of E, the covering consisting only of the set E_1 cannot capture the geometry of E

3.4.1 Hausdorff Measure

The aim is to define a σ-dimensional surface measure for general non-smooth subsets of the space $\mathbb{R}^n$.

Fix $\sigma \geqslant 0$ and $\delta > 0$. Given a set E, we cover it with a countable family of sets $\{E_k\}$ having diameter smaller or equal than δ. Then, the quantity

$$\sum_{k\in\mathbb{N}} (\mathrm{diam}(E_k))^\sigma$$

represents more or less a notion of σ-dimensional measure of E, provided we take δ sufficiently small. Of course, if δ is not chosen small enough, we might lose the geometry of the set E (see Fig. 3.3).

We give the following definition.

Definition 3.4.1 Let $\sigma \geqslant 0$ and $\delta > 0$. Given any $E \subseteq \mathbb{R}^n$, we set

$$\mathcal{H}^\sigma_\delta(E) := \inf\left\{ \omega_\sigma \sum_{k\in\mathbb{N}} \left(\frac{\mathrm{diam}(E_k)}{2}\right)^\sigma : E \subseteq \bigcup_{k\in\mathbb{N}} E_k,\ \mathrm{diam}(E_k) \leqslant \delta \right\},$$

where

$$\omega_\sigma := \frac{\pi^{\frac{\sigma}{2}}}{\Gamma(\frac{\sigma}{2}+1)},$$

and Γ is Euler's Gamma function. Then, we define the *s-dimensional Hausdorff measure of* E by

$$\mathcal{H}^\sigma(E) := \lim_{\delta\to 0^+} \mathcal{H}^\sigma_\delta(E).$$

The factor ω_σ is a normalization constant that makes the Hausdorff measure consistent with the standard Lebesgue measure of $\mathbb{R}^n$. In particular, ω_n is precisely the volume of the n-dimensional unit ball.

It is immediate to check that the limit defining the Hausdorff measure $\mathcal{H}^\sigma$ always exists. Indeed, since $\mathcal{H}^\sigma_\delta$ is non-increasing in δ,

$$\mathcal{H}^\sigma(E) = \sup_{\delta>0} \mathcal{H}^\sigma_\delta(E).$$

Finally, it can be proved that, when $k \geqslant 0$ is an integer, $\mathcal{H}^k$ coincides with the classical k-dimensional measure on smooth k-dimensional surfaces of $\mathbb{R}^n$ (see for instance [20, Sects. 3.3.2 and 3.3.4.C] or [25, Chap. 11]).

3.4.2 De Giorgi's Rectifiability Theorem

Having recalled the definition of Hausdorff measure, we may now present the main result of this section, referring to [25, Chap. 15] for a proof.

Theorem 3.4.2 (De Giorgi's Rectifiability Theorem) *Let E be a set of finite perimeter. Then, there exists a set $\partial^* E \subseteq \partial E$, such that:*

(*i*) *we have*

$$\partial^* E \subseteq \bigcup_{i\in\mathbb{N}} \Sigma_i \cup N,$$

for a countable collection $\{\Sigma_i\}$ of $(n-1)$-dimensional C^1 hypersurfaces and some set N with $\mathcal{H}^{n-1}(N) = 0$;

(*ii*) *for any open set A, it holds*

$$\mathrm{Per}(E; A) = \mathcal{H}^{n-1}(\partial^* E \cap A).$$

Notice that, thanks to (3.4.2), we have now an easier way to compute the perimeter of any set.

The object $\partial^* E$ introduced in the above theorem is usually called *reduced boundary*. Typically, it differs from the usual topological boundary, which may be very rough for general Borel sets.

Example 3.4.3 Let $\{x_k\}$ be a sequence of points dense in $\mathbb{R}^n$. For $N \in \mathbb{N}$, define

$$E_N := \bigcup_{k=1}^{N} B_{2^{-k}}(x_k).$$

Then,

$$\mathrm{Per}(E_N) \leqslant \sum_{k=1}^{N} \mathrm{Per}(B_{2^{-k}}) = c_n \sum_{k=1}^{N} 2^{-k(n-1)} \leqslant \tilde{c}_n,$$

for some dimensional constants $c_n, \tilde{c}_n > 0$. Since

$$E_N \to E_\infty := \bigcup_{k=1}^{+\infty} B_{2^{-k}}(x_k) \qquad \text{in } L^1(\mathbb{R}^n),$$

Proposition 3.2.3 implies that

$$\mathrm{Per}(E_\infty) \leqslant \tilde{c}_n,$$

that is E_∞ is a set of finite perimeter. On the other hand, the topological boundary of E_∞ is very large: indeed, while

$$|E_\infty| \leqslant \sum_{k=1}^{+\infty} |B_{2^{-k}}| < +\infty,$$

since E_∞ is dense in $\mathbb{R}^n$ we have $\bar{E}_\infty = \mathbb{R}^n$, thus $|\partial E_\infty| = +\infty$. Also, although it does not follow immediately from the definition, it is possible to prove that

$$\partial^* E_\infty \subseteq \bigcup_{N=1}^{+\infty} \partial^* E_N \subseteq \bigcup_{k=1}^{+\infty} \partial B_{2^{-k}}(x_k).$$

This example shows that the topological boundary may be a very *bad* notion in the context of perimeters.

Luckily, this is not always the case for minimizers of the perimeter. In fact, we will shortly prove partial regularity results (i.e., smoothness outside a lower dimensional set) for the topological boundary of minimizers of the perimeter.

3.5 Regularity of Minimal Graphs

After the brief parenthesis of Sect. 3.4, we now focus on the regularity properties enjoyed by the minimizers of problem (3.6), whose existence has been established in Theorem 3.3.1.

We first restrict ourselves to minimal surfaces which can be written as graphs with respect to one fixed direction.

Consider the cylinder

$$C_1 := B_1^{n-1} \times \mathbb{R},$$

with

$$B_1^{n-1} := \Big\{(x', 0) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1\Big\}.$$

Given $g : \partial B_1^{n-1} \to \mathbb{R}$, we denote by $\Gamma \subset \partial C_1$ the graph of g.

The following result shows that minimizing the area among graphs is the same as minimizing the area among all sets.

Lemma 3.5.1 *Let $\Sigma = \text{graph}(u)$ for some $u : \bar{B}_1^{n-1} \to \mathbb{R}$ such that $u = g$ on ∂B_1^{n-1}. Then, Σ is a minimal surface if and only if it satisfies*

$$\mathcal{H}^{n-1}(\Sigma) \leqslant \mathcal{H}^{n-1}(\text{graph}(v)),$$

for any $v : \bar{B}_1^{n-1} \to \mathbb{R}$ such that $v = g$ on ∂B_1^{n-1}.

Sketch of the Proof Clearly, we just need to show that if Σ is a minimizer among graphs, then it also solves problem (3.1).

Let K be a convex set and denote with $\pi_K : \mathbb{R}^n \to \mathbb{R}^n$ the projection from $\mathbb{R}^n$ onto K. It is well-known that π_K is 1-Lipschitz (see for instance [21, Lemma A.3.8]). Hence, since distances (and therefore also areas) decrease under 1-Lipschitz maps,

$$\mathcal{H}^{n-1}(\pi_K(\Sigma')) \leqslant \mathcal{H}^{n-1}(\Sigma')$$

(see for instance [21, Lemma A.7] applied with $L = 1$). By applying this with $K = C_1$, it follows that we can restrict ourselves to consider only competitors Σ' which are contained in C_1.

We now show that the area decreases under vertical rearrangements. To explain this concept, we describe it in a simple example. So, we suppose for simplicity that $n = 2$ and Σ' is as in Fig. 3.4, so that

$$\Sigma' \cap C_1 = \text{graph}(f_1) \cup \text{graph}(f_2) \cup \text{graph}(f_3),$$

for some smooth functions $f_i : [-1, 1] \to \mathbb{R}$, $i = 1, 2, 3$. Then it holds.

$$\mathcal{H}^1(\Sigma') = \sum_{i=1}^{3} \int_{-1}^{1} \sqrt{1 + (f_i')^2}.$$

Consider now the function $h := f_1 - f_2 + f_3$. Note that h is geometrically obtained as follows: given $x \in [-1, 1]$, consider the vertical segment $I_x := \{x\} \times [f_2(x), f_3(x)]$ and shift it vertically unit it touches $\{x\} \times (-\infty, f_1(x)]$. Then the set constructed in this way coincides with the epigraph of h.

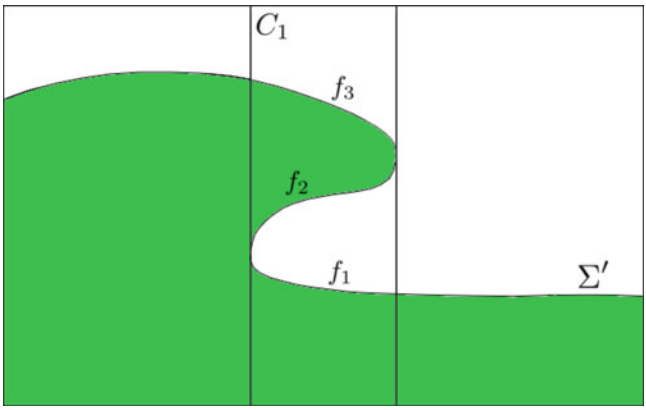

Fig. 3.4 The curve Σ', given by the union of the graphs of f_1, f_2 and f_3

We note that, thanks to the numerical inequality

$$\sqrt{1+(a+b+c)^2} \leqslant \sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2} \quad \text{for any } a,b,c>0,$$

it follows that

$$\begin{aligned}
\mathcal{H}^1(\text{graph}(h)\cap C_1) &= \int_{-1}^{1}\sqrt{1+(f_1'-f_2'+f_3')^2}\\
&\leqslant \int_{-1}^{1}\sqrt{1+\left(|f_1'|+|f_2'|+|f_3'|\right)^2}\\
&\leqslant \sum_{i=1}^{3}\int_{-1}^{1}\sqrt{1+(f_i')^2}\\
&= \mathcal{H}^1(\Sigma').
\end{aligned}$$

In other words, the area decreases under vertical rearrangement.

We note that this procedure can be generalized to arbitrary dimension and to any set $E\subset C_1$, allowing us to construct a function $h_E : B_1^{n-1}\to\mathbb{R}$ whose epigraph has boundary with less area than ∂E. However, to make this argument rigorous one should notice that the function h_E may jump at some points (see Fig. 3.5). Hence, one needs to introduce the concept of BV functions and discuss the area of the graph of such a function. Since this would be rather long and technical, we refer the interested reader to [24, Chaps. 14–16]. □

In view of the above result, we may limit ourselves to minimize area among graphs, i.e., we may restrict to the problem

$$\min\left\{\int_{B_1^{n-1}}\sqrt{1+|\nabla u|^2} : u=g \text{ on } \partial B_1^{n-1}\right\}. \tag{3.18}$$

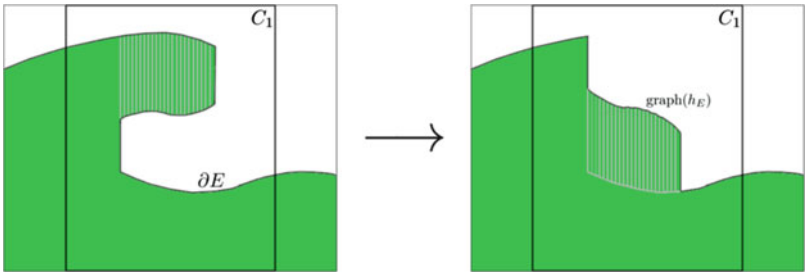

Fig. 3.5 On the *left* is the original set E, while on the *right* is its vertical rearrangement, given as the epigraph of the function h_E. In *gray* are depicted the segments I_x (on the *left*), and their vertical translations (on the *right*). As it is clear from the picture, the graph of h_E may have jumps

Note that the existence of a solution to such problem is not trivial, as the functional has linear growth at infinity, which may determine a lack of compactness since the Sobolev space $W^{1,1}$ is not weakly compact. We shall not discuss the existence problem here and we refer to [24] for more details.

The following comparison principle is easily established.

Lemma 3.5.2 *Suppose that g is bounded. Then, the solution u to the minimizing problem* (3.18) *is bounded as well, and it holds*

$$\|u\|_{L^\infty(B_1^{n-1})} \leqslant \|g\|_{L^\infty(\partial B_1^{n-1})}.$$

Proof Let $M := \|g\|_{L^\infty(\partial B_1^{n-1})}$. Then, $u_M := (u \wedge M) \vee -M = g$ on ∂B_1^{n-1}. Also, since $1 = \sqrt{1+|\nabla u_M|} \leqslant \sqrt{1+|\nabla u|}$ inside $\{|u| \geqslant M\}$,

$$\begin{aligned}\mathcal{H}^{n-1}(\text{graph}(u_M)) &= \int_{\{-M<u<M\}} \sqrt{1+|\nabla u|^2} + |\{|u| \geqslant M\}| \\ &\leqslant \int_{B_1^{n-1}} \sqrt{1+|\nabla u|^2} \\ &= \mathcal{H}^{n-1}(\text{graph}(u)).\end{aligned}$$

By the minimality of u, it follows that the above inequality is in fact an identity. Hence, $|u| \leqslant M$. □

Starting from this, the regularity theory for minimal graphs can be briefly described as follows. First of all, the well-known gradient estimate of Bombieri et al. [6] ensures that minimizers are locally Lipschitz functions.

Theorem 3.5.3 *Let u be a bounded solution to the minimizing problem* (3.18). *Then, u is locally Lipschitz inside B_1^{n-1}.*

Knowing that u is locally Lipschitz, we may differentiate the area functional to infer more information on the smoothness of u. Fix $\varphi \in C_c^\infty(B_1^{n-1})$. By the minimality of u, we have that

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{B_1^{n-1}} \sqrt{1 + |\nabla u + \varepsilon \nabla \varphi|^2}.$$

From this, we deduce that

$$\int_{B_1^{n-1}} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla \varphi = 0 \text{ for any } \varphi \in C_c^\infty(B_1^{n-1}),$$

which is the weak formulation of the Euler-Lagrange equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0. \tag{3.19}$$

Write now $F(p) := \sqrt{1 + |p|^2}$ for any $p \in \mathbb{R}^{n-1}$. Since $DF(q) = q/\sqrt{1 + |q|^2}$, we see that (3.19) may be read as

$$\operatorname{div}\left(DF(\nabla u)\right) = 0.$$

By differentiating this equation with respect to the direction e_ℓ, we get[1]

$$\operatorname{div}\left(D^2F(\nabla u) \cdot \nabla(\partial_\ell u)\right) = 0,$$

for any $\ell = 1, \dots n-1$. Setting now $A(x) := D^2F(\nabla u(x))$ and $v := \partial_\ell u$, the above equation becomes

$$\operatorname{div}\left(A(x)\nabla v\right) = 0.$$

Note that, because u is locally Lipschitz, given any ball $B_r(x) \subset B_1^{n-1}$, there exists a constant $L_{x,r}$ such that $|\nabla u| \leqslant L_{x,r}$ inside $B_r(x)$. Hence, since

$$0 < \lambda_{x,r}\mathrm{Id}_{n-1} \leqslant D^2F(q) \leqslant \Lambda_{x,r}\mathrm{Id}_{n-1} \quad \text{for any } |q| \leqslant L_{x,r},$$

we deduce that

$$\lambda_{x,r}\mathrm{Id}_{n-1} \leqslant A(y) = D^2F(\nabla u(y)) \leqslant \Lambda_{x,r}\mathrm{Id}_{n-1} \quad \text{for any } y \in B_r(x).$$

[1] Of course, to make this rigorous one should first check that $u \in W^{2,2}$. This can be done in a standard way, starting from Eq. (3.19) and exploiting the Lipschitz character of u to prove a Caccioppoli inequality on the incremental quotients of ∇u.

This proves that A is measurable and uniformly elliptic, therefore we may apply the De Giorgi-Nash-Moser theory [13, 26, 27] and conclude that $\partial_\ell u = v \in C^{0,\alpha}_{\text{loc}}$, for some $\alpha \in (0,1)$. Hence, $u \in C^{1,\alpha}_{\text{loc}}$ and consequently $A = D^2F(\nabla u) \in C^{0,\alpha}_{\text{loc}}$. Then, by Schauder theory (see e.g. [23]), we get that $v \in C^{1,\alpha}_{\text{loc}}$, i.e. $u \in C^{2,\alpha}_{\text{loc}}$. Accordingly, $A \in C^{1,\alpha}_{\text{loc}}$ and we can keep iterating this procedure to show that u is of class C^∞. Actually, by elliptic regularity, one can even prove that u is analytic. Hence, we obtain the following result.

Theorem 3.5.4 *Let $u : \bar{B}_1^{n-1} \to \mathbb{R}$ be a bounded solution to problem (3.18). Then u is analytic inside B_1^{n-1}.*

We have therefore proved that minimal graphs are smooth. This is no longer true for general minimal sets, as we will see in the next section.

3.6 Regularity of General Minimal Surfaces

We deal here with the regularity of minimal sets which are not necessarily graphs.

Let E be a minimal surface. By De Giorgi's rectifiability theorem (Theorem 3.4.2), we have the tools to work as if ∂E were already smooth (of course, there are technicalities involved, but the philosophy is the same). Thus, for simplicity we shall make computations are if ∂E were smooth, and we will prove estimates that are independent of the smoothness of ∂E.

3.6.1 Density Estimates

In this subsection we show that, nearby boundary points, minimal sets occupy *fat* portions of the space, at any scale. That is, we rule out the behavior displayed in Fig. 3.6.

Lemma 3.6.1 *There exists a dimensional constant $c_\star > 0$ such that*

$$|B_r(x) \cap E| \geqslant c_\star r^n \quad \text{and} \quad |B_r(x) \setminus E| \geqslant c_\star r^n, \tag{3.20}$$

for any $x \in \partial E$ and any $r > 0$.

Proof First, recall the isoperimetric inequality: there is a dimensional constant $c_n > 0$ such that

$$c_n \operatorname{Per}(F) \geqslant |F|^{\frac{n-1}{n}}, \tag{3.21}$$

for any bounded set $F \subset \mathbb{R}^n$. One can show (3.21) via Sobolev inequality. Indeed, let $\{\varphi_\varepsilon\}$ be a family of smooth convolution kernels and apply e.g. [19, Sect. 5.6.1,Theorem 1] to the function $\chi_F * \rho_\varepsilon$, for any $\varepsilon > 0$. Recalling also (3.12),

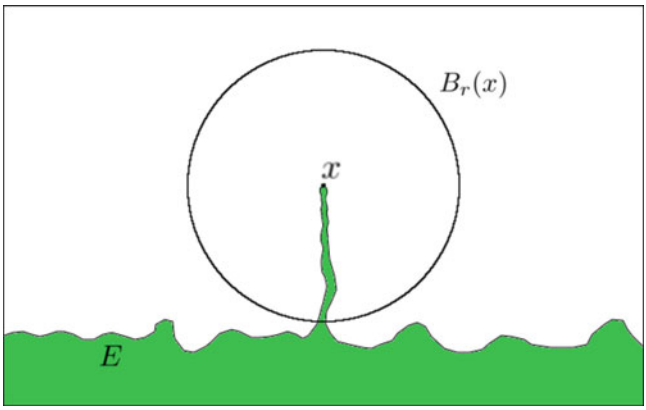

Fig. 3.6 An example of a set E which cannot be area minimizing. In fact, the measure of $E \cap B_r(x)$ is too small

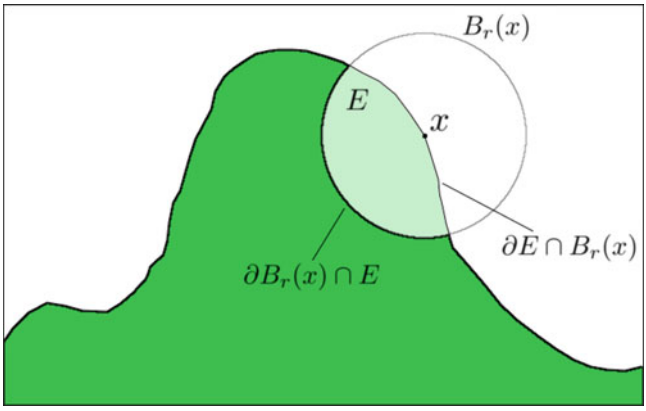

Fig. 3.7 The set $E \cap B_r(x)$ (in *light green*) and $E \setminus B_r(x)$ (in *dark green*). Then, one uses $E \setminus B_r(x)$ as competitor in the minimality of E. The boundary of $B_r(x) \cap E$ is the union of the two sets $\partial B_r(x) \cap E$ and $\partial E \cap B_r(x)$

we get

$$\|\chi_F * \rho_\varepsilon\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leqslant c_n \|\nabla(\chi_F * \rho_\varepsilon)\|_{L^1(\mathbb{R}^n)} \leqslant c_n \operatorname{Per}(F).$$

Inequality (3.21) follows by letting $\varepsilon \to 0^+$.

Let $V(r) := |B_r(x) \cap E|$. By the minimality of E, we have that (see Fig. 3.7)

$$\mathcal{H}^{n-1}(B_r(x) \cap \partial E) \leqslant \mathcal{H}^{n-1}(\partial B_r(x) \cap E).$$

Therefore, by this and (3.21), we obtain

$$\begin{aligned} V(r)^{\frac{n-1}{n}} &\leqslant c_n \left[\mathcal{H}^{n-1}(B_r(x)\cap\partial E) + \mathcal{H}^{n-1}(\partial B_r(x)\cap E)\right] \\ &\leqslant 2c_n \mathcal{H}^{n-1}(\partial B_r(x)\cap E). \end{aligned} \tag{3.22}$$

Using polar coordinates, we write

$$V(r) = \int_0^r \mathcal{H}^{n-1}(\partial B_s(x)\cap E)\, ds.$$

Accordingly,

$$V'(r) = \mathcal{H}^{n-1}(\partial B_r(x)\cap E),$$

and hence, by (3.22), we are led to the differential inequality

$$V(r)^{\frac{n-1}{n}} \leqslant 2c_n V'(r).$$

By this, we find that

$$\left(V^{\frac{1}{n}}(r)\right)' = \frac{1}{n}\frac{V'(r)}{V^{\frac{n-1}{n}}(r)} \geqslant \frac{1}{2nc_n},$$

and thus, since $V(0) = 0$, we conclude that

$$V^{\frac{1}{n}}(r) \geqslant \frac{r}{2nc_n}.$$

This is equivalent to the first estimate in (3.20). The second one is readily obtained by applying the former to $\mathbb{R}^n \setminus E$ (note that if E is minimal, so is $\mathbb{R}^n \setminus E$). □

An immediate corollary of the density estimates of Lemma 3.6.1 is given by the following result.

Corollary 3.6.2 *Let $\{E_k\}$ be a sequence of minimal surfaces, converging in $L^1_{\rm loc}$ to another minimal surface E. Then E_k converges to E in $L^\infty_{\rm loc}$.*

Notice that convergence in $L^\infty_{\rm loc}$ means that the boundaries of E_k and E are (locally) uniformly close.

Proof of Corollary 3.6.2 Fix a compact set $K \subset \mathbb{R}^n$. We need to prove that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$K \cap \partial E_k \subset \left\{x \in K : \operatorname{dist}(x, \partial E) < \varepsilon\right\}$$

and

$$K \cap \partial E \subset \left\{ x \in K : \operatorname{dist}(x, \partial E_k) < \varepsilon \right\}$$

for any $k \geqslant N$. We just prove the first inclusion, the proof of the second one being analogous.

We argue by contradiction, and suppose that there exist a diverging sequence $\{k_j\}$ of integers and a sequence of points $\{x_j\} \subset K$ such that $x_j \in \partial E_{k_j}$ and $\operatorname{dist}(x_j, \partial E) \geqslant \varepsilon$, for any $j \in \mathbb{N}$. Up to a subsequence, $\{x_j\}$ converges to a point $\bar{x} \in K$. Clearly, $\operatorname{dist}(x, \partial E) \geqslant \varepsilon$ and thus, in particular,

$$\text{either} \quad B_{\varepsilon/2}(x) \subset \mathring{E} \quad \text{or} \quad B_{\varepsilon/2}(x) \subset \mathbb{R}^n \setminus \bar{E}.$$

Suppose without loss of generality that the latter possibility occurs, i.e., that

$$B_{\varepsilon/2}(x) \subset \mathbb{R}^n \setminus \bar{E}.$$

By this, Lemma 3.6.1, and the L^1_{loc} convergence of the E_k's, we get

$$c_\star \left(\frac{\varepsilon}{2}\right)^n \leqslant \lim_{j \to +\infty} |B_{\varepsilon/2}(x_j) \cap E_{k_j}| = \lim_{j \to +\infty} \int_{B_{\varepsilon/2}(x_j)} \chi_{E_{k_j}} = \int_{B_{\varepsilon/2}(x)} \chi_E = 0,$$

which is a contradiction. The proof is therefore complete. □

The regularity theory in this case does not proceed as the one for minimal graphs (see Sect. 3.5). In fact, we need a more refined strategy.

3.6.2 ε-Regularity Theory

The aim of this subsection is to prove the following deep result, due to De Giorgi [14].

Theorem 3.6.3 *There exists a dimensional constant $\varepsilon > 0$ such that, if*

$$\partial E \cap B_r \subseteq \{|x_n| \leqslant \varepsilon r\}$$

for some radius $r > 0$ and $0 \in \partial E$, then

$$\partial E \cap B_{r/2} \text{ is a } C^{1,\alpha} \text{graph}$$

for some $\alpha \in (0, 1)$.

Theorem 3.6.3 ensures that, if a minimal surface is sufficiently flat in one given direction, then it is a $C^{1,\alpha}$ graph. The proof presented here is based on several ideas

contained in the work [28] by Savin. The key step is represented by the following lemma.

Lemma 3.6.4 *Let $\rho \in (0,1)$. There exist $\eta, \rho \in (0,1)$ and $\varepsilon_0 > 0$ such that, if*

$$\partial E \cap B_1 \subseteq \{|x_n| \leqslant \varepsilon\},$$

for some $\varepsilon \in (0, \varepsilon_0)$, and $0 \in \partial E$, then

$$\partial E \cap B_\rho \subseteq \{|x \cdot e| \leqslant \eta\rho\varepsilon\},$$

for some unit vector $e \in \mathbb{S}^{n-1}$.

Lemma 3.6.4 yields a so-called *improvement of flatness* for the minimal surface ∂E. Indeed, it tells that, shrinking from the ball B_1 to the smaller B_ρ, the oscillation of ∂E around some hyperplane is dumped by a factor η smaller than 1, possibly changing the direction of the hyperplane under consideration. Of course, even if ∂E is a smooth surface, its normal at the origin may not be e_n. Hence, we really need to tilt our reference frame in some new direction $e \in \mathbb{S}^{n-1}$ in order to capture the $C^{1,\alpha}$ behavior of ∂E at the origin.

We now suppose the validity of Lemma 3.6.4 and show how Theorem 3.6.3 can be deduced from it.

Sketch of the Proof of Theorem 3.6.3 First of all, we only consider the case of $r = 1$, as one can replace E with $r^{-1}E$. To this regard, observe that the minimality of a set is preserved under dilations.

We then suppose for simplicity that the rotation that sends e_n to e may be avoided in Lemma 3.6.4. That is, we assume that we can prove that

$$\partial E \cap B_1 \subseteq \{|x_n| \leqslant \varepsilon\} \text{ implies } \partial E \cap B_\rho \subseteq \{|x_n| \leqslant \eta\rho\varepsilon\}, \tag{3.23}$$

provided that $\varepsilon \leqslant \varepsilon_0$. As pointed out before, this clearly cannot be true. Nevertheless, we argue supposing the validity of (3.23), since the general case may be obtained using the same ideas and only slightly more care.

Thanks to the hypothesis of the theorem, we may apply (3.23) and deduce that

$$\partial E \cap B_\rho \subseteq \{|x_n| \leqslant \eta\rho\varepsilon\}.$$

Consider now the rescaled set $E_1 := \rho^{-1}E$. The previous inclusion can be read as

$$\partial E_1 \cap B_1 \subseteq \{|x_n| \leqslant \eta\varepsilon\}.$$

Since $\rho\varepsilon \leqslant \varepsilon \leqslant \varepsilon_0$, we can apply (3.23) to $E_1 := \rho^{-1}E$, and we get

$$\partial E_1 \cap B_\rho \subseteq \left\{|x_n| \leqslant \eta^2\rho\varepsilon\right\}.$$

Getting back to E, this becomes

$$\partial E \cap B_{\rho^2} \subseteq \left\{|x_n| \leqslant \eta^2 \rho^2 \varepsilon\right\}.$$

By iterating this procedure, we find that, for any $k \in \mathbb{N}$,

$$\partial E \cap B_{\rho^k} \subseteq \left\{|x_n| \leqslant \eta^k \rho^k \varepsilon\right\} = \left\{|x_n| \leqslant \rho^{(1+\alpha)k} \varepsilon\right\}, \tag{3.24}$$

where $\alpha > 0$ is chosen so that $\rho^\alpha = \eta$. Now, given $s \in (0, 1)$, there exists $k \in \mathbb{N}$ such that $\rho^k \leqslant s \leqslant \rho^{k-1}$. Hence

$$\begin{aligned}\partial E \cap B_s \subseteq \partial E \cap B_{\rho^{k-1}} &\subseteq \left\{|x_n| \leqslant \rho^{(1+\alpha)(k-1)} \varepsilon\right\} \\ &= \left\{|x_n| \leqslant \rho^{-(1+\alpha)} \rho^{(1+\alpha)k} \varepsilon\right\} \subseteq \left\{|x_n| \leqslant \rho^{-(1+\alpha)} s^{1+\alpha} \varepsilon\right\}\end{aligned}$$

Thus, we deduce that

$$\partial E \cap B_s \subseteq \left\{|x_n| \leqslant C \varepsilon s^{1+\alpha}\right\},$$

for any $s \in (0, 1]$, where $C := \rho^{-(1+\alpha)}$.

As mentioned above, this estimate is obtained forgetting about the fact that one needs to tilt the system of coordinates. If one takes into account such tilting, instead of (3.24) one would obtain an inclusion of the type

$$\partial E \cap B_{\rho^k} \subseteq \left\{|x \cdot e^k| \leqslant \rho^{(1+\alpha)k} \varepsilon\right\},$$

for some sequence $\{e^k\} \subset \mathbb{S}^{n-1}$. However, at each step the inner product $e^{k+1} \cdot e^k$ cannot be too far from 1 (see Fig. 3.8). By obtaining a quantification of this defect,

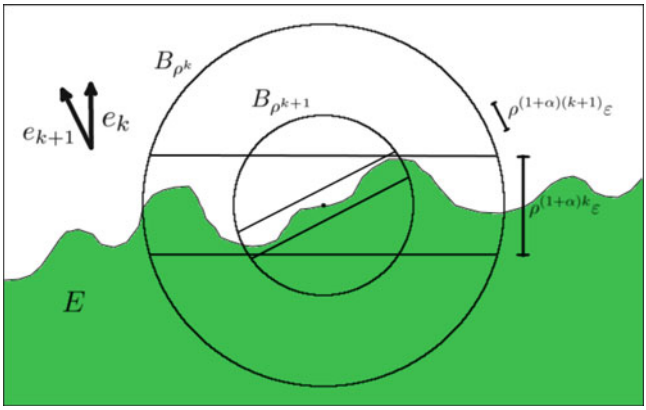

Fig. 3.8 The boundary of a minimal set E may be trapped in slabs of different orientations inside balls of different radii. However, the discrepancy between these orientations cannot be too large

one can show that the tiltings $\{e^k\}$ converge at some geometric rate to some unit vector e_0. Hence, the correct bound is

$$\partial E \cap B_s \subseteq \left\{|x \cdot e_0| \leqslant C\varepsilon s^{1+\alpha}\right\} \qquad \text{for any } s \in (0,1], \text{ for some } e_0 \in \mathbb{S}^{n-1}. \tag{3.25}$$

Let now z be any point in $B_{1/2} \cap \partial E$. As $B_{1/2}(z) \subset B_1$, we clearly have that

$$\partial E \cap B_{1/2}(z) \subseteq \{|x_n| \leqslant \varepsilon\}.$$

Assume that $\varepsilon \leqslant \varepsilon_0/2$. Then the set $E_z := 2(E - z)$ satisfies

$$\partial E_z \cap B_1 \subseteq \{|x_n| \leqslant 2\varepsilon\}.$$

This allows us to repeat the argument above with E_z in place of E and conclude that

$$\partial E \cap B_s(z) \subseteq \left\{|(x-z) \cdot e_z| \leqslant 2C\varepsilon s^{1+\alpha}\right\} \qquad \text{for some } e_z \in \mathbb{S}^{n-1},$$

for any $z \in B_{1/2} \cap \partial E$ and any $s \in (0, 1/2)$. With this in hand, one can then conclude that ∂E is a $C^{1,\alpha}$ graph. □

In order to finish the proof of the ε-regularity theorem, we are therefore only left to show the validity of Lemma 3.6.4. We do this in the remaining part of the subsection.

To prove Lemma 3.6.4 we argue by contradiction and suppose that, given two real numbers $\rho, \eta \in (0, 1)$ to be fixed later, there exist an infinitesimal sequence $\{\varepsilon_k\}$ of positive real numbers and a sequence of minimizers $\{E_k\}$ for which $0 \in \partial E_k$,

$$\partial E_k \cap B_1 \subseteq \{|x_n| \leqslant \varepsilon_k\},$$

but

$$\partial E_k \cap B_\rho \not\subseteq \{|x \cdot e| \leqslant \eta\rho\varepsilon_k\} \qquad \text{for any } e \in \mathbb{S}^{n-1}. \tag{3.26}$$

Consider the changes of coordinates $\Psi_k : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Psi_k(x', x_n) := \left(x', \frac{x_n}{\varepsilon_k}\right),$$

and define $\tilde{E}_k := \Psi_k(E_k)$. Observe that the new sets $\tilde{E}_k$ are not minimizers, as stretching in one variable does not preserve minimality. However, thanks to the following result, the surfaces $\tilde{E}_k$ are precompact:

Lemma 3.6.5 (Savin [28]) *Up to a subsequence, the surfaces $\{\partial\tilde{E}_k\}$ converge in L^∞_{loc} to the graph of some function u.*

What can we say about u? Let us deal with the easier case in which the original boundaries ∂E_k are already the graphs of some functions u_k. This is of course not always the case, but the Lipschitz approximation theorem for minimal surfaces (see for instance [25, Theorem 23.7]) tells that a flat minimal surface is a Lipschitz graph at many points (the measure of the points being more and more as the surface gets flatter and flatter).

Under this assumption, we have that $\partial \tilde{E}_k = \text{graph}(\tilde{u}_k)$, with $\tilde{u}_k := \varepsilon_k^{-1} u_k$. Observe that $|\tilde{u}_k| \leqslant 1$, since $\partial \tilde{E}_k \cap B_1 \subseteq \{|x_n| \leqslant 1\}$. In view of the minimality of ∂E_k, we compute

$$0 = \frac{1}{\varepsilon_k} \operatorname{div}\left(\frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}} \right) = \operatorname{div}\left(\frac{\nabla \tilde{u}_k}{\sqrt{1+\varepsilon_k^2 |\nabla \tilde{u}_k|^2}} \right).$$

Assuming that $|\nabla \tilde{u}_k|$ is bounded, by taking the limit as $k \to +\infty$ in the above expression, we find that u solves

$$\begin{cases} \Delta u = 0 \text{ in } B_1^{n-1} \\ \|u\|_{L^\infty(B_1^{n-1})} \leqslant 1. \end{cases} \tag{3.27}$$

(In order to rigorously obtain the claimed equation for u, one needs to use the concept of viscosity solutions that we shall not discuss here. We refer to [9, 28] for more details.)

From (3.27), it follows by regularity theory for harmonic functions that $\|u\|_{C^2(B_{3/4})} \leqslant \bar{C}_n$, for some dimensional constant $\bar{C}_n \geqslant 0$. Therefore,

$$|u(x') - u(0) - \nabla u(0) \cdot x'| \leqslant \bar{C}_n \rho^2 \quad \text{for any } x' \in B_{2\rho}^{n-1}.$$

Taking $\rho \leqslant \eta/(4\bar{C}_n)$ and observing that $u(0) = 0$, this becomes

$$|u(x') - \nabla u(0) \cdot x'| \leqslant \frac{\eta \rho}{2} \quad \text{for any } x' \in B_{2\rho}^{n-1}.$$

As $\partial \tilde{E}_k$ converges uniformly to graph(u) in L^∞_{loc}, the above estimate implies that

$$\partial \tilde{E}_k \cap \left(B_\rho^{n-1} \times \mathbb{R} \right) \subseteq \{|x \cdot \tilde{v}| \leqslant \eta \rho\}, \qquad \text{with } \tilde{v} := (-\nabla u(0), 1),$$

for $k \gg 1$. Dilating back we easily obtain

$$\partial E_k \cap B_\rho \subseteq \{|x \cdot \tilde{e}_k| \leqslant \eta \rho \varepsilon_k\} \qquad \text{with } \tilde{e}_k := \frac{(-\varepsilon_k \nabla u(0), 1)}{\sqrt{1+\varepsilon_k^2 |\nabla u(0)|^2}} \in \mathbb{S}^{n-1},$$

in contradiction with (3.26).

We have therefore proved Theorem 3.6.3 in its entirety (up to the compactness result in Lemma 3.6.5, and some small technical details). By this result, we know that if a minimal surface is sufficiently flat around a point, then it is locally the graph of a $C^{1,\alpha}$ function. Note that, by the regularity theory discussed in Sect. 3.5, such a function will actually be analytic.

In order to proceed further in the understanding of the regularity theory, the next question becomes: at how many points minimal surfaces are flat?

An answer to this question is provided via the so-called *blow-up procedure*.

3.6.3 Blow-Up Technique

The idea is to look at points of ∂E from closer and closer. More precisely, for a fixed $x \in \partial E$, we define the family of minimal surfaces $\{E_{x,r}\}$ as

$$E_{x,r} := \frac{E - x}{r}, \tag{3.28}$$

for any $r > 0$. By taking the limit as $r \to 0^+$ of such close-ups, one reduces to problem of counting flat points to that of classifying limits of blow-ups.

In order to rigorously describe the above anticipated blow-up procedure, we first need some preliminary results.

We recall that a set C is said to be a cone with respect to a point x if

$$y \in C \text{ implies that } \lambda(y - x) \in C - x \text{ for any } \lambda > 0.$$

Theorem 3.6.6 (Monotonicity Formula) *The function*

$$\Psi_E(r) := \frac{\mathcal{H}^{n-1}(\partial E \cap B_r(x))}{r^{n-1}},$$

is monotone non-decreasing in r.

Proof Let Σ_r be the cone centered at x and such that

$$\Sigma_r \cap \partial B_r(x) = \partial E \cap \partial B_r(x). \tag{3.29}$$

Set $f(r) := \mathcal{H}^{n-1}(\partial E \cap B_r(x))$. By minimality,

$$f(r) \leqslant \mathcal{H}^{n-1}(\Sigma_r \cap B_r(x)).$$

Using polar coordinates, the fact that Σ_r is a cone, and again (3.29), we compute

$$\begin{aligned}\mathcal{H}^{n-1}(\Sigma_r \cap B_r(x)) &= \int_0^r \mathcal{H}^{n-2}(\Sigma_r \cap \partial B_s(x))\, ds \\ &= \frac{\mathcal{H}^{n-2}(\Sigma_r \cap \partial B_r(x))}{r^{n-2}} \int_0^r s^{n-2}\, ds \\ &= \frac{r\mathcal{H}^{n-2}(\partial E \cap \partial B_r(x))}{n-1}.\end{aligned}$$

As $\mathcal{H}^{n-2}(\partial E \cap \partial B_r(x)) = f'(r)$, we conclude that

$$f(r) \leqslant \frac{r}{n-1} f'(r).$$

This in turn implies that

$$\Psi_E'(r) = \left(\frac{f(r)}{r^{n-1}}\right)' = \frac{rf'(r) - (n-1)f(r)}{r^n} \geqslant 0,$$

and the monotonicity follows. □

By a more careful inspection of the proof, one can show that Ψ_E is constant if and only if E is a cone with respect to the point x.

Proposition 3.6.7 *There exists an infinitesimal sequence $\{r_j\}$ of positive real numbers such that $\{E_{x,r_j}\}$ converges to a set F in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, as $j \to +\infty$. Furthermore,*

(*i*) *∂F is a minimal surface;*
(*ii*) *F is a cone.*

Sketch of the Proof By scaling and Theorem 3.6.6, given $R > 0$, for any $r \in (0, 1/R]$ we estimate

$$\mathrm{Per}(E_{x,r}; B_R) = \mathcal{H}^{n-1}(\partial E_{x,r} \cap B_R) = \frac{\mathcal{H}^{n-1}(\partial E \cap B_{rR}(x))}{r^{n-1}} \leqslant R^{n-1}\mathcal{H}^{n-1}(\partial E \cap B_1(x)).$$

This proves that the perimeter of $\partial E_{x,r}$ in B_R is uniformly bounded for all $r \in (0, 1/R]$. Accordingly, by Proposition 3.2.4 the family $\{E_{x,r}\}$ is compact in $L^1_{\mathrm{loc}}(B_R)$. Since this is true for any $R > 0$, a diagonal argument yields the existence of an infinitesimal sequence $\{r_j\}$ such that

$$E_{x,r_j} \longrightarrow F \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

for some set $F \subseteq \mathbb{R}^n$. Since the sets $E_{x,r}$ are minimal, exploiting the lower semicontinuity of the perimeter (see Proposition 3.2.3) it is not difficult to show that ∂F is a minimal surface and that $\mathcal{H}^{n-1}(\partial E_{x,r_j} \cap B_s) \to \mathcal{H}^{n-1}(\partial F \cap B_s)$ for a.e. s (see, for instance, [24, Lemma 9.1]).

We now prove that F is a cone. For any $s > 0$, we have

$$\Psi_F(s) = \frac{\mathcal{H}^{n-1}(\partial F \cap B_s)}{s^{n-1}} = \lim_{j\to+\infty} \frac{\mathcal{H}^{n-1}(\partial E_{x,r_j} \cap B_s)}{s^{n-1}}$$
$$= \lim_{j\to+\infty} \frac{\mathcal{H}^{n-1}(\partial E \cap B_{r_j s}(x))}{(r_j s)^{n-1}} = \lim_{\rho\to 0+} \frac{\mathcal{H}^{n-1}(\partial E \cap B_\rho(x))}{\rho^{n-1}}.$$

Thus, $\Psi_F(s)$ is constant, which implies that F is a cone. □

We have thus established that the blow-up sequence (3.28) converges to a minimal cone. Notice now that halfspaces are particular examples of cones. Also, if F is a halfspace and $E_{x,r}$ is close to F in L^1_{loc} (and hence in L^∞_{loc}, see Corollary 3.6.2), then $\partial E_{x,r}$ becomes flatter and flatter as $r \to 0^+$. In particular, we may apply Theorem 3.6.3 to $E_{x,r}$ for some r sufficiently small to deduce the smoothness of ∂E around x. Hence, the goal now is to understand whether minimal cones are always halfplanes or not. The desired classification result is given by the following theorem.

Theorem 3.6.8 *If $n \leqslant 7$, all minimal cones are halfplanes. If $n \geqslant 8$, then there exist minimal cones which are not halfplanes.*

Theorem 3.6.8 has been obtained by De Giorgi [15] for $n = 3$, by Almgren [2] for $n = 4$ and, finally, by Simons [30] in any dimension $n \leqslant 7$. The counterexample in dimension $n = 8$ is given by the so-called Simons cone

$$C := \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| < |y|\}.$$

Simons conjectured in [30] that the above set was a minimal cone in dimension 8, and this was proved by Bombieri et al. [5].

Notice that the case $n = 2$ of Theorem 3.6.8 is trivial. Indeed, the cone on the left of Fig. 3.9 cannot be minimal, as the competitor showed on the right has less perimeter (by the triangle inequality).

In conclusion, the discussion above shows that minimal surfaces are smooth up to dimension 7. Although in higher dimension minimal surfaces may develop a singular set $\mathcal{S}$, an argument due to Federer (called "Federer reduction argument")

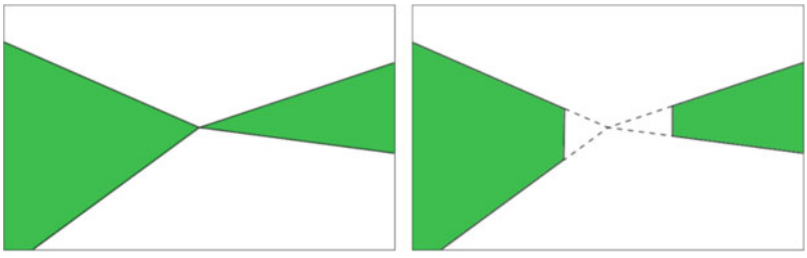

Fig. 3.9 The cone on the *left* is not minimal, since the perturbation on the *right* has lower perimeter

allows one to exploit the absence of singular minimal cones in dimension 7 to give a bound on dimension of $\mathcal{S}$. We can summarize this in the following result (see [24, Chap. 11] or [25, Chap. 28] for more details):

Corollary 3.6.9 *Let $E \subset \mathbb{R}^n$ be minimal. We have:*

(*i*) *if $n \leqslant 7$, then ∂E is analytic;*
(*ii*) *if $n \geqslant 8$, then there exists $\mathcal{S} \subset \partial E$ such that $\mathcal{S}$ is closed, $\partial E \setminus \mathcal{S}$ is analytic, and $\mathcal{H}^\sigma(\mathcal{S}) = 0$ for any $\sigma > n - 8$.*

3.7 Nonlocal Minimal Surfaces

In this last section, we consider a different *nonlocal* notion of area, introduced by Caffarelli et al. in [9]. After briefly motivating its definition, we discuss which of the results and approaches described up to now can be carried over to this new setting.

To begin with, we should ask ourselves why we study perimeters. Of course, perimeters model surface tension, as for example in soap bubbles. Moreover, perimeters naturally arise in phase transition problems. Suppose that we have two different media (e.g. water and ice, or water and oil) that are put together in the same container. Of course, the system pays an energy for having an interface between them. Since nature tends to minimize such an energy, interfaces must be (almost) minimal surfaces (e.g. spheres of oil in water, planar regions, etc.).

Hence, perimeters are useful for interpreting in simple ways several complex events that take place in our world. In general, perimeters give good *local* descriptions of intrinsically *nonlocal* phenomena. We now address the problem of establishing a truly nonlocal energy that may hopefully better model the physical situation.

Let E be a subset of $\mathbb{R}^n$, representing the region occupied by some substance. In order to obtain an energy that incorporates the full interplay between E and its complement—that we think to be filled with a different composite—we suppose that each point x of E interacts with each point y of $\mathbb{R}^n \setminus E$. Of course, we need to weigh this interaction, so that closer points interact more strongly than farther ones. Moreover, E must not interact with itself, and similarly for its complement. Finally, because the regularity theory only depends on the interaction for extremely close-by points, it is natural to consider energies that have some scaling invariance. In the end, one comes up with the following notion of a *fractional perimeter* of E:

$$\mathrm{Per}_s(E) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy = 2\int_E\int_{\mathbb{R}^n\setminus E} \frac{dx\,dy}{|x-y|^{n+s}},$$

for any fixed $s \in (0,1)$. Notice that, since we chose a homogeneous weight, rescalings of minimal surfaces are still minimal, as for the standard perimeter. But why did we restrict to the above range for the power s?

To answer this question, we first need to define a preliminary restricted version of the fractional perimeter. Consider the quantity

$$\mathrm{Per}_s^{B_1}(E) := \int_{B_1}\int_{B_1} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy. \tag{3.30}$$

Observe that $\mathrm{Per}_s^{B_1}(E)$ sums up all the interactions between E and its complement that occur inside B_1. If we take $s<0$, then

$$\mathrm{Per}_s^{B_1}(E) \leqslant 2\int_{B_1}\int_{B_1} \frac{dx\,dy}{|x-y|^{n+s}} \leqslant C_n \int_0^2 \frac{d\rho}{\rho^{1+s}} < +\infty,$$

that is, $\mathrm{Per}_s^{B_1}(E)$ is always finite, no matter how rough the boundary of E is. Hence, this would lead to a too weak notion of perimeter.

On the other hand, suppose that $s \geqslant 1$. Then, if we take as E the upper halfspace $\{x_n > 0\}$, a simple computation reveals that

$$\begin{aligned}
\mathrm{Per}_s^{B_1}(E) &= 2\int_{B_1\cap\{x_n>0\}}\int_{B_1\cap\{x_n<0\}} \frac{dx\,dy}{|x-y|^{n+s}} \\
&\geqslant 2\int_0^{\frac{\sqrt{2}}{2}}\int_{-\frac{\sqrt{2}}{2}}^0 \int_{B^{n-1}_{\frac{\sqrt{2}}{2}}}\int_{B^{n-1}_{\frac{\sqrt{2}}{2}}} \frac{dx'\,dy'\,dx_n\,dy_n}{\left[|x'-y'|^2+(x_n-y_n)^2\right]^{\frac{n+s}{2}}} \\
&\geqslant c_n \int_0^{\frac{\sqrt{2}}{4}} \frac{dt}{t^{1+s}} = +\infty.
\end{aligned}$$

Thus, halfspaces have infinite s-perimeter in the ball B_1 if $s \geqslant 1$. As halfspaces represents the simplest examples of surfaces, this is clearly something we do not want to allow for. Consequently, we restrict ourselves to consider weights corresponding to $s \in (0,1)$.[2]

It can be easily seen that $\mathrm{Per}_s(E) = +\infty$ if E is a halfspace, even for $s \in (0,1)$. This is due to the fact that Per_s takes into account also interactions coming from infinity (actually this happens also in the case of classical perimeters, as the perimeter of a halfspace in the whole $\mathbb{R}^n$ is not finite). Therefore, we need to restrict our definition (3.30) to bounded containers.

[2] Although the choice $s=0$ is in principle admissible for the restricted perimeter $\mathrm{Per}_s^{B_1}$, we discard it anyway. In fact, it determines a weight with too fat tails at infinity, which would not be suitable for the full fractional perimeter Per_s.

Fix an open set Ω, and prescribe E outside Ω, i.e., suppose that $E \setminus \Omega = F \setminus \Omega$ for some given set F. Then,

$$\begin{aligned}\mathrm{Per}_s(E) &= \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy + 2\int_\Omega \left(\int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\,dy\right) dx \\ &\quad + \int_{\mathbb{R}^n\setminus\Omega}\int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy \\ &= \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy + 2\int_\Omega \left(\int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\,dy\right) dx \\ &\quad + \int_{\mathbb{R}^n\setminus\Omega}\int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_F(x) - \chi_F(y)|}{|x-y|^{n+s}}\,dx\,dy.\end{aligned}$$

Notice that the last integral only sees outside of B_1 and is hence independent of E once the boundary datum F is fixed. Thus, when minimizing $\mathrm{Per}_s(E)$, it is enough to restrict ourselves to the two other terms. Thus, given a bounded open set Ω, we define

$$\mathrm{Per}_s(E;\Omega) := \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy + 2\int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x--y|^{n+s}}\,dx\,dy. \tag{3.31}$$

One can check that, with this definition, halfspaces have finite s-perimeters inside any bounded set Ω.

Accordingly, we have the following notion of minimal surface for Per_s.

Definition 3.7.1 (Caffarelli et al. [9]) Given a bounded open set Ω, a measurable set $E \subseteq \mathbb{R}^n$ is said to be a *nonlocal s-minimal surface* inside Ω if

$$\mathrm{Per}_s(E;\Omega) \leqslant \mathrm{Per}_s(E';\Omega)$$

for any measurable E' such that $E' \setminus \Omega = E \setminus \Omega$.

In the following subsections, we proceed to investigate some important properties shared by nonlocal minimal surfaces.

3.7.1 Existence of s-Minimal Surfaces

We begin by showing the existence of s-minimal surfaces. Assuming for simplicity that $\Omega = B_1$, we have the following result.

Theorem 3.7.2 *Let F be a set with locally finite s-perimeter. Then, there exists a s-minimal surface E in B_1 with $E \setminus B_1 = F \setminus B_1$.*

As in Sect. 3.3, the proof of the existence of minimal surfaces is based on the semicontinuity of Per_s and on a compactness result similar to Proposition 3.2.4. The lower semicontinuity of Per_s in L^1_{loc} can be easily established right from definition (3.31), using for instance Fatou's lemma. On the other hand, the needed compactness statement amounts to show that

$$\mathrm{Per}_s(E_k; B_1) \leqslant C \text{ implies that } \{E_k\} \text{ is precompact in } L^1(B_1). \tag{3.32}$$

To check this fact, we first notice that

$$\begin{aligned}\mathrm{Per}_s(F; B_1) &\geqslant \int_{B_1}\int_{B_1} \frac{|\chi_F(x) - \chi_F(y)|}{|x-y|^{n+s}}\,dx\,dy \\ &= \int_{B_1}\int_{B_1} \frac{|\chi_F(x) - \chi_F(y)|^2}{|x-y|^{n+s}}\,dx\,dy \\ &= [\chi_F]^2_{H^{s/2}(B_1)},\end{aligned}$$

where $[\cdot]_{H^{s/2}}$ denotes the Gagliardo seminorm of the fractional Sobolev space $H^{s/2}$. By the compact fractional Sobolev embedding (see e.g. [16, Theorem 7.1]), the uniform boundedness of $\{\chi_{E_k}\}$ in $H^{s/2}(B_1)$ implies that, up to a subsequence, it converges in $L^1(B_1)$ to χ_F, for some measurable set F. Hence, (3.32) is true.

3.7.2 Euler-Lagrange Equation

Suppose that E is a nonlocal minimal surface in B_1 and let $\{E_\varepsilon\}$ be a continuous family of perturbations of E, with $E_\varepsilon \setminus B_1 = E \setminus B_1$ for any ε. From the minimality of E, we have that

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{Per}_s(E_\varepsilon; B_1).$$

Eventually, we are led to the equation

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n\setminus E}(y)}{|x-y|^{n+s}}\,dy = 0 \text{ for any } x \in \partial E \cap B_1 \tag{3.33}$$

(see [9, Sect. 5]). Heuristically, this means that

$$\int_E \frac{dy}{|x-y|^{n+s}} = \int_{\mathbb{R}^n\setminus E} \frac{dy}{|x-y|^{n+s}},$$

at any point $x \in \partial E \cap B_1$. In other words, each point $x \in \partial E$ interacts in the same way both with E and with $\mathbb{R}^n \setminus E$. However, the above identity cannot be interpret in

a rigorous way, as both integrals do not converge. Hence, (3.33) must be understood in the principal value sense, that is

$$0 = \text{P.V.} \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x-y|^{n+s}}\, dy = \lim_{\delta \to 0^+} \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x-y|^{n+s}}\, dy.$$

When E is the (global) subgraph of a function $u : \mathbb{R}^{n-1} \to \mathbb{R}$, this can be written as a nonlocal equation for u: more precisely, if we assume that u is small enough so that we neglect nonlinear terms, we find that

$$0 = I[u](x) \simeq (-\Delta)^{\frac{1+s}{2}} u(x) = \text{P.V.} \int_{\mathbb{R}^{n-1}} \frac{u(x) - u(y)}{|x-y|^{(n-1)+(1+s)}}\, dy,$$

where I denotes a suitable integral operator (cp. [9, Lemma 6.11] and [4, Sect. 3]). The fact that $I[u]$ is close to the fractional Laplacian of order $\frac{1+s}{2}$ when the Lipschitz norm of u is small should be compared with the classical mean curvature operator appearing in (3.19), which is close to the classical Laplacian when ∇u is small.

3.7.3 *s*-Minimal Graphs

As we did before, we begin by addressing the problem of obtaining regularity results for minimal surfaces in the case when they are (locally) the graph of a function u. So, we consider a s-minimal surface E in the infinite cylinder $C_1 = B_1^{n-1} \times \mathbb{R}$ such that

$$E \cap C_1 = \Big\{(x', x_n) \in C_1 : x_n < u(x')\Big\}, \tag{3.34}$$

for some function $u : B_1^{n-1} \to \mathbb{R}$, with $u(0) = 0$.

In Sect. 3.5, we saw that bounded classical minimal graphs are smooth functions. The first step in the proof of this result was the gradient estimate of [6], which established their Lipschitz character. From this, additional regularity then followed by the De Giorgi-Nash-Moser and Schauder theories.

In the nonlocal setting, we are still missing the initial step of this argument. In fact, we can propose the following open problem.

Open Problem *Suppose that u is bounded. What can be said of the regularity of u in the ball $B_{1/2}^{n-1}$? Is it locally Lipschitz?*

When u is already Lipschitz, then its smoothness follows. This is achieved in two essential steps. First, we have:

Theorem 3.7.3 (Figalli-Valdinoci [22]) *If u is Lipschitz, then u is $C^{1,\alpha}$ for any $\alpha < s$.*

Then, the following Schauder-type result allows one to conclude:

Theorem 3.7.4 (Barrios et al. [4]) *If u is $C^{1,\alpha}$ for some $\alpha > s/2$, then u is C^∞.*

At the moment it is not known whether smooth s-minimal graphs are actually analytic. The results in [1] show that they enjoy some Gevrey regularity.

We conclude the subsection by observing that s-minimal surfaces with graph properties as (3.34) indeed exist, for instance when their boundary data are graphs too.

Theorem 3.7.5 (Dipierro et al. [18]) *Suppose that E is a s-minimal surface in C_1 such that*

$$E \setminus C_1 = \left\{(x', x_n) \in (\mathbb{R}^n \setminus B_1^{n-1}) \times \mathbb{R} : x_n < v(x')\right\},$$

for some bounded, continuous function $v : \mathbb{R}^{n-1} \to \mathbb{R}$. Then, (3.34) holds true for some continuous function $u : \bar{B}_1^{n-1} \to \mathbb{R}$.

3.7.4 Regularity of General s-Minimal Sets

The regularity theory for nonlocal minimal surfaces established in [9] follows an analogous strategy to that outlined in Sect. 3.6.

The density estimates follow via the same argument of the proof of Lemma 3.6.1, using the fractional Sobolev inequality in place of the isoperimetric inequality (see [9, Sect. 4]).

The ε-regularity theory is also similar [9, Sect. 6], but we need to check what happens with the behavior at infinity of the s-minimal surface. The key step is represented by the following improvement of flatness result.

Lemma 3.7.6 *Let E be a s-minimal surface in B_1. For any fixed $\alpha \in (0, s)$, there exists $k_0 \in \mathbb{N}$ such that if*

$$\partial E \cap B_{2^{-k}} \subseteq \left\{|x \cdot e_k| \leqslant 2^{-k(1+\alpha)}\right\},$$

for some unit vector $e_k \in \mathbb{S}^{n-1}$ and for any $k = 0, \dots, k_0$, then

$$\partial E \cap B_{2^{-k_0-1}} \subseteq \left\{|x \cdot e_{k_0+1}| \leqslant 2^{-(k_0+1)(1+\alpha)}\right\},$$

for some $e_{k_0+1} \in \mathbb{S}^{n-1}$.

Lemma 3.7.6 tells that if ∂E is sufficiently flat for a sufficiently large number of geometric scales, then it is flatter and flatter at all smaller scales. Compare this with Lemma 3.6.4: in the local case it was sufficient to check the flatness of the boundary of E at only one scale to deduce its improvement at smaller scales.

We can rephrase the above statement by rescaling everything by a factor 2^{k_0} (in other words, replacing E by $2^{k_0}E$). Lemma 3.7.6 is then equivalent to prove that

$$\partial E \cap B_{2^j} \subseteq \left\{|x \cdot e_j| \leqslant \varepsilon (2^j)^{1+\alpha}\right\},$$

for any $j = 0, \dots, k_0$ and with $\varepsilon = 2^{-k_0\alpha}$, implies that

$$\partial E \cap B_{1/2} \subseteq \left\{ |x \cdot \bar{e}| \leqslant \varepsilon 2^{-1-\alpha} \right\}.$$

From this formulation, the role played by the nonlocality of Per_s is even more evident: to obtain information inside the ball $B_{1/2}$ we need to have it already in $B_{2^{k_0}}$, with k_0 sufficiently large.

To prove the improvement of flatness that we just stated, we argue by contradiction. As in Sect. 3.6.2, we pick two sequences of s-minimal surfaces E_m and positive real numbers ε_m, with $\varepsilon_m \to 0$. We suppose that each E_m violates the implication above, with $\varepsilon = \varepsilon_m$ and $k_0 = |\log \varepsilon_m|/(\alpha \log 2)$. It can be shown that suitable rescalings of the sets E_m (analogue to the rescaling in Sect. 3.6.2) converge to the graph of a function u that satisfies

$$\begin{cases} (-\Delta)^{\frac{1+s}{2}} u = 0 \text{ in } \mathbb{R}^{n-1} \\ |u(x)| \leqslant C\left(1 + |x|^{1+\alpha}\right) \text{ for any } x \in \mathbb{R}^{n-1}, \end{cases} \tag{3.35}$$

for some $C > 0$. The conclusive step of the proof of Lemma 3.7.6 is then provided by the next general Liouville-type result.

Lemma 3.7.7 *Suppose that u satisfies (3.35) for some $\alpha \in (0, s)$ and $s \in (0, 1]$. Then, u is affine.*

Sketch of the Proof We include the proof of the lemma in the classical case $s = 1$. The argument for the fractional powers of the Laplacian is analogous (see [9, Proposition 6.7]).

Fix $R \geqslant 1$ and set $u_R(x) := R^{-1-\alpha} u(Rx)$. Clearly, $\Delta u_R = 0$ and $\|u_R\|_{L^\infty(B_1)} \leqslant C$. Consequently, by elliptic regularity, we have that $\|D^2 u_R\|_{L^\infty(B_{1/2})} \leqslant C_n C$. But

$$D^2 u_R(x) = R^{1-\alpha} D^2 u(Rx),$$

and therefore we get that

$$\|D^2 u\|_{L^\infty(B_{R/2})} \leqslant \frac{C_n C}{R^{1-\alpha}}.$$

The result follows by letting $R \to +\infty$. □

In view of the ε-regularity theory outlined above, we know that flat s-minimal surfaces are smooth.

The next step is then to use blow-ups in order to understand at how many points a nonlocal minimal surface is flat. To this aim, we first need an appropriate monotonicity formula, as in Theorem 3.6.6. Instead of working with the nonlocal perimeter Per_s as defined in (3.30), we consider a slightly different energy coming from the so-called *extension problem* (see [7, 9]).

Let $\mathbb{R}^{n+1}_+$ denote the upper halfspace $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\}$ and $u : \bar{\mathbb{R}}^{n+1}_+ \to \mathbb{R}$ be the unique solution to the problem

$$\begin{cases} \operatorname{div}_{\mathbb{R}^{n+1}} \left(y^{1-s} \nabla_{\mathbb{R}^{n+1}} u \right) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u|_{y=0} = \chi_E - \chi_{\mathbb{R}^n \setminus E} & \text{on } \mathbb{R}^n. \end{cases}$$

Then, define

$$\Phi_E(r) := \frac{1}{r^{n-s}} \int_{B_r^+} y^{1-s} \left| \nabla_{\mathbb{R}^{n+1}} u \right|^2,$$

for any $r > 0$. The notation B_r^+ is used here to indicate the upper half-ball of radius r, centered at the origin of $\mathbb{R}^{n+1}$, i.e. $B_r^+ := B_r^{n+1} \cap \mathbb{R}^{n+1}_+$. We have the following:

Theorem 3.7.8 (Caffarelli et al. [9]) *The function Φ_E is monotone non-decreasing in r.*

With the help of this monotonicity result, we can successfully perform the standard blow-up procedure.

Proposition 3.7.9 *Let E be a s-minimal surface and let $x \in \partial E$. For small $r > 0$, set $E_{x,r} := r^{-1}(E - x)$. Then, up to a subsequence,*

$$E_{x,r} \longrightarrow F \text{ in } L^1_{\text{loc}},$$

as $r \to 0^+$, with F a s-minimal cone.

As in the classical case, to complete our investigation on the regularity properties of minimal surfaces we are left with the problem of classifying minimal cones. This task turns out to be not trivial at all, even in the plane. In fact, here one cannot argue as easily as for the standard perimeter (recall Fig. 3.9). However, a more refined approach can be developed to show that in $\mathbb{R}^2$ there are no non-trivial s-minimal cones.

Theorem 3.7.10 (Savin-Valdinoci [29]) *If E is a s-minimal cone in $\mathbb{R}^2$, then E is a halfspace. In particular, s-minimal surfaces in $\mathbb{R}^2$ are smooth.*

This result has been recently improved, via quantitative flatness estimates, in [10].

Another way to attack the problem of the regularity for s-minimal surfaces, when s is close to 1, is by taking advantage of the classical regularity theory. First, we recall the following result due to Davila [11] (see also [3, 8]).

Theorem 3.7.11 *There exists a dimensional constant $c_\star > 0$ such that*

$$(1 - s) \operatorname{Per}_s(E; B_1) \longrightarrow c_\star \operatorname{Per}(E; B_1),$$

as $s \to 1^-$.

In view of the above theorem, (a suitable rescaling of) the nonlocal perimeter converges to the standard one as $s \to 1^-$. Similarly, nonlocal minimal surfaces approaches classical ones in the same limit. Hence, as we already know that classical minimal surfaces are smooth up to dimension $n = 7$, the same is true for s-minimal surfaces, provided s is sufficiently close to 1. More precisely, the following result holds as a consequence of Theorem 3.7.11:

Corollary 3.7.12 *Let $n \geqslant 2$, and let $E \subset \mathbb{R}^n$ be s-minimal. There exists $s_n \in (0, 1)$ close to 1 such that, if $s \geqslant s_n$, then:*

(1) if $n \leqslant 7$, then $\partial E \in C^\infty$ (in particular, the only s-minimal cones are halfspaces);
(2) if $n \geqslant 8$, then there exists $\mathcal{S} \subset \partial E$ such that $\mathcal{S}$ is closed, $\partial E \setminus \mathcal{S}$ is smooth, and $\mathcal{H}^\sigma(\mathcal{S}) = 0$ for any $\sigma > n - 8$.

On the contrary, as $s \to 0^+$, a suitable rescaling of Per_s converges to the volume [17]. In this respect, Per_s is a very natural way to interpolate between the volume and the perimeter.

We note that, if s is small, there is an example of a cone $F \subset \mathbb{R}^7$ such that, for any continuous family $\{F_\varepsilon\}$ of perturbations of F, it holds

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{Per}_s(F_\varepsilon; B_1) = 0 \quad \text{and} \quad \frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathrm{Per}_s(F_\varepsilon; B_1) \geqslant 0.$$

That is, F is a stable solution of (3.33). If one could prove that F actually minimizes the s-perimeter, then one would have found a counterexample to the above corollary when s is far from 1. We refer the interested reader to [12] for more details on this construction.

In conclusion, the regularity theory for nonlocal minimal surfaces that we just described is often based on ideas that also work for classical ones. Often these methods are simpler and work better in the local scenario, but there are some tools and techniques that are naturally better suited for nonlocal objects.

For instance, as we saw in Sect. 3.5 the proof that classical Lipschitz minimal graphs are $C^{1,\alpha}$ is based on the De Giorgi-Nash-Moser theory for elliptic PDEs with bounded measurable coefficients. On the other hand, this strategy does not seem to work for nonlocal minimal surfaces. Conversely, a new geometric argument can be successfully applied and the same regularity result is true [22].

As one can see, several important questions in this theory are still open (the most fundamental one being the classification of minimal cones). We hope that new results will come in the next few years.

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Chapter 4
Short Tales from Nonlinear Calderón-Zygmund Theory

Giuseppe Mingione

Abstract Nonlinear Calderón-Zygmund Theory aims at reproducing, in the nonlinear setting, the classical linear theory originally developed by Calderón and Zygmund. This topic has large intersections with Nonlinear Potential Theory. We survey here the main results of this theory.

4.1 The Classical Linear Theory

The classical work of Calderón and Zygmund [26, 27] in the fifties introduced a wealth of new techniques allowing to analyze the behaviour of singular integrals in the multidimensional case. This, in turn, opened the way to the analysis of optimal integrability and differentiability properties of solutions to linear elliptic and parabolic equations. See for instance the classical papers of Agmon et al. [4]. The path, that we are going to briefly describe in a few lines below, is the following: linear problems admit fundamental solutions. These in turn allow to write explicit representation formulae for solutions to general equations and their derivatives via singular integral operators. These can be finally analyzed using the methods of Calderón and Zygmund, that actually belong to a nowadays gigantic field called Harmonic Analysis. It is difficult to give an account of the developments of such theories and methods for linear partial differential equations and we shall not even make an attempt in this direction. It is on the other hand clear that, since the above approach uses fundamental solutions and related representation formulae, it is restricted to linear equations. There have been different and by now classical proofs of the L^p-estimates by Campanato and Stampacchia [28, 109] that avoid singular integrals, but they again work for linear equations, since they strongly rely on interpolation methods. On the other hand, in the last years there have been a series of developments leading to a set of results replicating the classical linear theorems

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in the nonlinear setting. All such results and methods concur to build what could be called *Nonlinear Calderón-Zygmund Theory* (Nonlinear CZ-theory, in short). In these notes, stemming from the series of lectures given at the CIME school at Cetraro in June 2016, we shall try to give an overview of the results nowadays available, emphasizing the connections with other branches of the regularity theory of nonlinear differential equations, such as Nonlinear Potential Theory. This is about all those classical regularity and fine properties of solutions to linear elliptic and parabolic equations, that can be recast, in one way or another, in the nonlinear setting. Its first official appearance dates back to the fundamental paper of Havin and Maz'ya [60] and the reader may consult the classical monograph [62] of Heinonen et al. for the basics; see also [3, 61, 93]. As a matter of fact, several main results in Nonlinear Potential Theory, like for instance pointwise potential estimates, find their roots in De Giorgi's methods [39] for establishing a priori regularity estimates for solutions to linear elliptic equations with measurable coefficients. Such methods are indeed nonlinear in nature, and they involve estimates that perfectly emulate the presence of the otherwise non-existent fundamental solutions. Starting from such basic estimates, it is indeed possible to reconstruct, in the nonlinear setting, a whole wealth of properties which are typical of the linear one. The same thing happens in Nonlinear CZ-theory, giving, if needed, yet another proof of the deepness and generality of De Giorgi's original techniques.

We conclude this section by giving a brief account of some classical linear results that the Nonlinear CZ-theory, and Nonlinear Potential Theory, aim at reproducing in the nonlinear setting. Let us therefore consider the Poisson equation

$$-\triangle u = -\operatorname{div} Du = \mu \tag{4.1}$$

in $\mathbb{R}^n$ for $n \geq 2$. The classical representation formula via convolution with the fundamental solution $G(\cdot)$ is

$$u(x_0) = \frac{1}{|B_1|n(n-2)} \int_{\mathbb{R}^n} G(x, x_0)\, d\mu(x)\,, \qquad G(x, x_0) := \frac{1}{|x - x_0|^{n-2}}\,. \tag{4.2}$$

In (4.2), for simplicity, we consider $n \geq 3$ (otherwise $G(x, x_0) \approx \log|x - x_0|$ for $n = 2$), $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, with u being the unique solution u to (4.1) that decays to zero at infinity.

Differentiating (4.2) and considering a suitable definition of integrals involving so called principal values, we arrive at a representation formula of the type

$$D^2u(x_0) = \int_{\mathbb{R}^n} K(x, x_0)\, d\mu(x)\,, \qquad |K(x, x_0)| \approx \frac{1}{|x - x_0|^n}\,, \tag{4.3}$$

where at the right-hand side of the first expression we find a singular integral. The kernel $K(\cdot)$ is indeed not locally integrable but enjoys special and decisive cancelation properties. The mapping properties of such an operator, described in

[26, 27] and based on the aforementioned cancelation properties, lead to establish that

$$\mu \in L^q \Longrightarrow Du \in W^{1,q} \qquad \text{whenever } 1 < q < \infty\,. \tag{4.4}$$

The result in (4.4) fails for $q = 1, \infty$ and it is obviously sharp. The same approach works for solutions to equations with right-hand side in divergence form, that is

$$\triangle u = \operatorname{div} F \tag{4.5}$$

for which the result becomes

$$F \in L^q \Longrightarrow Du \in L^q \qquad \text{whenever } 1 < q < \infty\,. \tag{4.6}$$

Another approach to (4.4) and (4.6) has been later on developed by Campanato and Stampacchia [28, 109]; this avoids the use of singular integrals. The idea is replacing them with suitable interpolation theorems involving the BMO spaces of John and Nirenberg [66] (see (4.16) below). Cancellation properties that are at the core of the analysis of singular do not disappear but are now incorporated in the definition of the space BMO. Linearity of the equation considered is anyway still a fundamental ingredient at this stage, allowing for the use of interpolation methods.

The identity in (4.2) allows to reconstruct the pointwise properties of solutions via classical Riesz potentials. These are described in the following

Definition 4.1.1 Let $\beta \in (0, n]$; the linear operator defined by

$$I_\beta(\mu)(x_0) := \int_{\mathbb{R}^n} \frac{d\mu(x)}{|x - x_0|^{n-\beta}}\,,$$

is called the β-Riesz potential of μ, where μ is a Borel measure defined on $\mathbb{R}^n$. Estimate (4.2) implies the following pointwise inequalities:

$$|u(x_0)| \lesssim |I_2(\mu)(x_0)| \qquad \text{and} \qquad |Du(x_0)| \lesssim cI_1(|\mu|)(x_0)\,. \tag{4.7}$$

(The first one actually holds when $n \geq 3$.) Eventually, by a standard approximation argument the previous formulae still hold for general Borel measures with locally finite total mass. By means of (4.7), and using the basic regularizing properties of the Riesz potential, it is then possible to infer a priori estimates for u and Du in various function spaces in terms of the assumed integrability of μ. This allows to obtain regularity properties of solutions without directly using the equation. As in the case of Nonlinear CZ-theory, estimates in (4.7) seem to be linked to the fact that we are dealing with a specific linear equation. Nevertheless they admit a sharp reformulation for solutions to nonlinear equations. This fact is at the core of Nonlinear Potential Theory and will be described in detail starting from Sect. 4.11.

4.2 Notation

Constants are generically denoted by c; these are larger or equal than one; dependence on parameters is indicated using parenthesis. In the following $B_R(x_0) := \{x \in \mathbb{R}^n : |x-x_0| < R\}$ denotes the open ball with center x_0 and radius $R > 0$. When not important, or when it will be clear from the context, we shall omit denoting the center as follows: $B_R \equiv B_R(x_0)$. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable subset with positive measure, and with $f\colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being an integrable map, we shall denote by

$$(f)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} f\,dx := \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} f(x)\,dx$$

its integral average; here $|\mathcal{B}|$ denotes the Lebesgue measure of $\mathcal{B}$. In the following Ω will denote an arbitrary open subset of $\mathbb{R}^n$, with $n \geq 2$. We shall identify $L^1_{\mathrm{loc}}(\Omega)$-functions μ with measures, thereby denoting

$$|\mu|(\mathcal{B}) = \int_{\mathcal{B}} |\mu|\,dx \qquad \text{for every measurable subset } \mathcal{B} \Subset \Omega\ .$$

Moreover, we shall denote by $\mathbf{M}_{\mathrm{loc}}(\Omega)$ the space of Borel (signed) measures with locally finite total mass defined on Ω. This means that $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$ iff $|\mu|(\mathcal{K}) < \infty$ for every compact subset $\mathcal{K} \subset \Omega$. Accordingly, we denote by $\mathbf{M}(\Omega)$ the space of Borel measures with finite total mass over Ω.

4.3 Nonlinear CZ-Theory: Energy Estimates

We start presenting some basic results dealing with that part of Nonlinear CZ-theory covering the case in which the solutions of the equations considered are in the natural energy space associated to the operator defining the equation itself. See assumptions (4.10) below. Specifically, here we are considering distributional solutions u to scalar equations of the type

$$\operatorname{div} A(x, Du) = \operatorname{div}(|F|^{p-2}F) \qquad \text{in } \Omega \subset \mathbb{R}^n \tag{4.8}$$

with Ω being an open subset and $n \geq 2$. Notice that, for the choice $A(x, Du) \equiv Du$ and $p = 2$, the last equation reduces to (4.5), for which the standard, linear CZ-theory prescribes that the implication (4.6).

First of all, let us specify the main assumptions we are going to work with. The vector field $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to $C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$-regular with

respect to the gradient variable z and initially satisfies assumptions

$$\begin{cases} |A(x,z)| + |\partial A(x,z)||z| \le L|z|^{p-1} \\ \nu|z|^{p-2}|\xi|^2 \le \langle \partial A(x,z)\xi, \xi\rangle \\ |A(x_1,z) - A(x_2,z)| \le L\omega(|x_1 - x_2|)|z|^{p-1} \end{cases} \tag{4.9}$$

for every choice of $z \in \mathbb{R}^n \setminus \{0\}, \xi \in \mathbb{R}^n$, $x, x_1, x_2 \in \Omega$, and for fixed ellipticity constants $0 < \nu \le 1 \le L$. Here the symbol ∂ always refers to the differentiation with respect to the gradient variable z. Moreover, we assume that the map $\partial A(\cdot)$ is Carathéodory-regular; this means that

$$\begin{cases} \text{the map } x \mapsto \partial A(x,z) \text{ is measurable for every } z \in \mathbb{R}^n \\ \text{the map } z \mapsto \partial A(x,z) \text{ is continuous for almost every } x \in \Omega\ . \end{cases}$$

Unless otherwise specified, we shall always deal with the case $p > 1$. The function $\omega\colon [0,\infty) \to [0,1]$ is a modulus of continuity. This means that is, a continuous, bounded and non-decreasing function such that $\omega(0) = 0$; with no loss of generality it can be taken to be concave. Assumption (4.9) basically serves to describe the continuous dependence on the variable x (the coefficients) of the renormalised vector field

$$\mathbb{R}^n \setminus \{0\} \ni z \mapsto \frac{A(x,z)}{|z|^{p-1}}\ .$$

Conditions (4.9) are classical since the work of Ladyzhenskaya and Uraltseva [88] and they are modelled on the p-Laplacean operator, i.e.,

$$A(x,z) = |z|^{p-2}z\ .$$

For basic regularity results concerning the p-Laplacean operator we refer for instance to [42, 50, 52, 93, 94]. In this section our permanent assumptions on u and F are such that

$$u \in W^{1,p}_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad F \in L^p_{\mathrm{loc}}(\Omega)\ . \tag{4.10}$$

Distributional solutions u satisfying $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ are usually called (local) weak solutions or (local) energy solutions. This terminology stems from the fact that they belong, at least locally, to the natural function space associated to Eq. (4.8) under assumptions (4.9), which is $W^{1,p}$. In the same way, the integrability assumption $F \in L^p_{\mathrm{loc}}(\Omega)$ guarantees that, F belongs to the dual space $W^{-1,p'}(\Omega')$ (which is the dual of $W^{1,p}_0(\Omega')$), whenever $\Omega' \Subset \Omega$ is an open subset. The weak formulation of (4.8), defining the concept of weak solutions, is as follows:

Definition 4.3.1 A function $u \in W^{1,p}_{\rm loc}(\Omega)$ is a weak (energy) to the equation in (4.8) under assumptions (4.9), if and only if

$$\int_\Omega \langle A(x, Du), D\varphi \rangle \, dx = \int_\Omega \langle |F|^{p-2}F, D\varphi \rangle \, dx \qquad \forall \ \varphi \in C^\infty_0(\Omega) \ . \tag{4.11}$$

In other words, an energy solution is nothing but a distributional solution satisfying the additional energy condition $u \in W^{1,p}_{\rm loc}(\Omega)$.

A basic theorem in Nonlinear CZ-theory, that in the following local version can be obtained by the methods of Iwaniec [63], is the following:

Theorem 4.3.2 ([63]) *Let $u \in W^{1,p}_{\rm loc}(\Omega)$ be a weak solution to the equation in (4.8), under assumptions (4.9). Then*

$$|F|^p \in L^\gamma_{\rm loc}(\Omega) \Longrightarrow |Du|^p \in L^\gamma_{\rm loc}(\Omega) \qquad \text{for every } \gamma > 1 \ . \tag{4.12}$$

Moreover, for every $\gamma > 1$ there exist a radius $r \equiv r(n, p, \nu, L, \gamma, \omega(\cdot))$ and constants $c \equiv c(n, p, \nu, L)$ and $c(\gamma) \equiv c(n, p, \nu, L, \gamma)$, such that the estimate

$$\left(\fint_{B_{R/2}} |Du|^{p\gamma} \, dx \right)^{1/\gamma} \leq c \fint_{B_R} |Du|^p \, dx + c(\gamma) \left(\fint_{B_R} |F|^{p\gamma} \, dx \right)^{1/\gamma} \tag{4.13}$$

holds for every ball $B_R \Subset \Omega$ such that $R \leq r$.

The original proof of Theorem 4.3.2 following the methods of [63] rests on a clever use of Harmonic Analysis tools such as sharp maximal operators (see Sect. 4.9 below) and a priori estimates regularity estimates for solutions w to homogeneous equations with frozen coefficients of the type

$$-\operatorname{div} A(x_0, Dw) = 0 \ . \tag{4.14}$$

These serve to overcome the lack or representation formulae and the possibility of using singular integral operators. In particular, the approach in [63] uses sharp maximal operators to replace singular integrals, while local estimates for solutions to (4.14) serve as a local replacement of the representation formulae.

DiBenedetto and Manfredi [44] extended Iwaniec's original results to the case of systems. Caffarelli [24] found a beautiful approach to the nonlinear theory in the case of fully nonlinear equations, a class of problems we are not considering here. Partially relying on Caffarelli's original ideas, Caffarelli and Peral [25] found a different approach to integral estimates, still using maximal operators. This approach inspired several subsequent developments [115], including those featuring irregular boundary value problems and irregular coefficients [21, 22], and more general structures [71]. Finally, a maximal function free-proof, based solely on PDE estimates, has been later found [2] in the setting of parabolic problems; for this we refer to Sect. 4.5 below.

The assumption of continuity of $A(\cdot)$ with respect to the coefficients, i.e., with respect to x can be relaxed in assuming so called VMO dependence (Vanishing Mean Oscillations). Specifically, we define

$$\tilde{\omega}(\varrho) := \left[\sup_{\substack{z\in\mathbb{R}^n\setminus\{0\}\\ B_\varrho\subseteq\Omega}} \fint_{B_\varrho} \left(\frac{|A(x,z)-(A)_{B_\varrho}(z)|}{|z|^{p-1}} \right)^2 dx \right]^{1/2},$$

where the averaged vector field $(A)_{B_\varrho}(z)$ is defined as

$$(A)_{B_\varrho}(z) := \fint_{B_\varrho} A(x,z)\,dx\,.$$

The main assumption, playing the role of VMO-dependence on x is then

$$\lim_{\varrho\searrow 0} \tilde{\omega}(\varrho) = 0\,.$$

Under such a condition Theorem 4.3.2 still holds. On the other hand, well-known counterexamples tell that if the dependence on x of the vector field $A(\cdot)$ is merely measurable Theorem 4.3.2 does not hold.

4.4 Two Extensions

What about systems instead of equations? This means we are considering solutions with values in $\mathbb{R}^N$, $N \geq 1$, and a vector field $A\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ that satisfies assumptions similar to those in (4.9), but recast for the vector-valued case, that is,

$$\begin{cases} |A(x,z)| + |\partial A(x,z)||z| \leq L|z|^{p-1} \\ \nu|z|^{p-2}|\xi|^2 \leq \langle \partial A(x,z)\xi,\xi\rangle \\ |A(x_1,z)-A(x_2,z)| \leq L\omega(|x_1-x_2|)|z|^{p-1}\,, \end{cases} \tag{4.15}$$

hold for every choice of $z \in \mathbb{R}^{N\times n}\setminus\{0\}, \xi \in \mathbb{R}^{N\times n}$, $x, x_1, x_2 \in \Omega$, and again $0 < \nu \leq 1 \leq L$ are fixed constants. Here we are again requiring that A is C^1-regular with respect to $z \in \mathbb{R}^{N\times n}\setminus\{0\}$.

In the vectorial case local energy solutions are in general not regular (see [99] for a through discussion), and can be even unbounded [111], already for simpler systems of the type $-\mathrm{div}\, A(Du) = 0$. Therefore Theorem 4.3.2 cannot hold as such unless additional assumptions are made on the vector field $A(\cdot)$, while, when considering general systems, weaker results are available. And in fact,

Theorem 4.3.2 continues to hold in the case of the model case system

$$\operatorname{div}(Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F) \qquad \text{in } \Omega \subset \mathbb{R}^n .$$

The result for this model case has been originally proved by DiBenedetto and Manfredi [44], who furthermore obtained the following borderline result:

$$|F|^{p-2}F \in \mathrm{BMO}_{\mathrm{loc}} \Longrightarrow Du \in \mathrm{BMO}_{\mathrm{loc}}$$

that in case linear case $p = 2$ is a classical fact due to Campanato (see [58, Chap. 10]). Recall that an $L^1(\Omega)$-function f belongs to $\mathrm{BMO}_{\mathrm{loc}}(\Omega)$ provided

$$\sup_{B \subset \Omega'} \fint_B |f - (f)_B| \, dx < \infty \tag{4.16}$$

for every open subset $\Omega' \Subset \Omega$ and where the sup is taken over all possible balls.

When passing to considering general systems some form of CZ-theory survives. Indeed we have

Theorem 4.4.1 ([71]) *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to the system*

$$\operatorname{div} A(x, Du) = \operatorname{div}(|F|^{p-2}F)$$

under assumptions (4.15). There exists a number $\chi_m \equiv \chi_m(n, N, p, \nu, L) > 1$ such that

$$\begin{cases} \chi_m > \frac{n}{n-2} & \text{if } n > 2 \\ \chi_m \text{ is any positive number} & \text{if } n = 2 \end{cases}$$

and such that (4.12) holds provided $1 \le \gamma < \chi_m$. In particular for every $\gamma \in [1, \chi_m)$ there exist a radius $r \equiv r(n, N, p, \nu, L, \gamma, \omega(\cdot))$ and constants $c \equiv c(n, N, p, \nu, L)$ and $c(\gamma, \chi_m) \equiv c(n, N, p, \nu, L, \gamma, \chi_m)$, such that the estimate

$$\left(\fint_{B_{R/2}} |Du|^{p\gamma} \, dx \right)^{1/\gamma} \le c \fint_{B_R} |Du|^p \, dx + c(\gamma, \chi_m) \left(\fint_{B_R} |F|^{p\gamma} \, dx \right)^{1/\gamma}$$

holds for every ball $B_R \Subset \Omega$ such that $R \le r$.

As shown in [71], the previous result plays a significant role in the estimate of Hausdorff dimension of the singular set of minima of vector-valued variational problems. It is also important in order to establish a few boundary partial regularity results [54, 72].

The second extension we are presenting here concerns obstacle problems. We consider the constrained minimization problem

$$\min_{v\in K}\int_\Omega |Dv|^p\,dx \tag{4.17}$$

where

$$K := \{v \in W^{1,p}_0(\Omega) \,:\, v \geq \psi \text{ a.e.}\} \qquad \psi \in W^{1,p}_0(\Omega)\,.$$

The integrability result available is then the following natural and optimal one:

Theorem 4.4.2 ([17]) *Let $u \in W^{1,p}(\Omega)$ be the unique variational solution to the obstacle problem (4.17), where Ω is a bounded domain in $\mathbb{R}^n$. Then*

$$|D\psi|^p \in L^\gamma_{\rm loc}(\Omega) \Longrightarrow |Du|^p \in L^\gamma_{\rm loc}(\Omega) \qquad \text{for every } \gamma > 1\,.$$

Moreover, for every $\gamma > 1$ there exist a constant $c \equiv c(n,p,\nu,L)$ and $c(\gamma) \equiv c(n,p,\nu,L,\gamma)$, such that the estimate

$$\left(\fint_{B_{R/2}} |Du|^{p\gamma}\,dx\right)^{1/\gamma} \leq c \fint_{B_R} |Du|^p\,dx + c(\gamma)\left(\fint_{B_R} |D\psi|^{p\gamma}\,dx\right)^{1/\gamma}$$

holds for every ball $B_R \Subset \Omega$.

The previous theorem is obviously optimal, as it follows by considering the gradient integrability of u on the contact set $\{u \equiv \psi\}$, where Du and $D\psi$ coincide almost everywhere. This result has been extended in several directions; see for instance [23].

4.5 Parabolic Nonlinear CZ-Theory

The problem of extending Theorem 4.3.2 to the parabolic case

$$u_t - \operatorname{div} A(Du) = -\operatorname{div}(|F|^{p-2}F) \qquad \text{in } \Omega_T := \Omega\times(-T,0) \subset \mathbb{R}^{n+1} \tag{4.18}$$

has remained open until [2]. The main obstruction was in the fact that the techniques developed in [63] and [25] rely on the use of maximal operators, which are ruled out in parabolic problems as long as $p \neq 2$. This is basically due to the fact that equations of the type

$$w_t - \operatorname{div}(|Dw|^{p-2}Dw) = 0 \qquad \text{in } \Omega_T\,, \tag{4.19}$$

are such that if $c \in \mathbb{R}$ is a constant, then cw is no longer a solution of the same equation as long as $p \neq 2$. This lack of scaling prevents the validity of homogeneous a priori estimates for solutions to (4.19) which are a basic ingredient in the proof of Theorem 4.3.2. More precisely, considering for simplicity the case $p \geq 2$, for homogeneous solutions to (4.19) the estimate

$$\sup_{Q_{R/2}} |Dw| \lesssim \left[\fint_{Q_R} |Dw|^p \, dx \, dt + 1 \right]^{p/2}$$

holds for every parabolic cylinder $Q_R \equiv B_R(x_0) \times (t_0 - R^2, t_0) \Subset \Omega_T$. This estimate is obviously not homogeneous as long as $p \neq 2$. Homogeneous estimates can be recast by using the so-called *called intrinsic parabolic cylinders* (in the original language of DiBenedetto). These are cylinders of the type

$$Q_R^\lambda \equiv Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - \lambda^{2-p} R^2, t_0) \tag{4.20}$$

where $\lambda > 0$ is determined by the solution w in the same cylinder. Specifically, conditions of the type

$$\fint_{Q_R^\lambda} |Dw|^p \, dx \, dt \lesssim \lambda^p$$

are required to be satisfied *on the very same cylinder* Q_ϱ^λ. This motivates the use of the word intrinsic. Notice that when $p = 2$ intrinsic cylinders reduce to the standard ones. The use of such *intrinsic geometries* been introduced in the fundamental work of DiBenedetto, see [43]. As a matter of fact, on intrinsic cylinders the following homogeneous estimate holds:

$$\fint_{Q_R^\lambda} |Dw|^p \, dx \, dt \lesssim \lambda^p \Longrightarrow \sup_{Q_{R/2}^\lambda} |Dw| \lesssim \lambda \, .$$

The drawback is that since the cylinders in question depend on the solution itself, the use of maximal operators are ruled out since these need an a priori assigned family of balls or cylinders to be defined.

To overcome this gap a direct, maximal function-free proof has been introduced in [2] and this allows to give a direct approach to Calderón-Zygmund estimates which is only based on PDE estimates. No maximal operators or other Harmonic Analysis tools are required.

Theorem 4.5.1 ([2]) *Let $u \in C(-T, 0; L^2_{\mathrm{loc}}(\Omega)) \cap L^p(-T, 0; W^{1,p}_{\mathrm{loc}}(\Omega))$ be a weak solution to the parabolic equation (4.18), where Ω is a bounded domain in $\mathbb{R}^n$, and the $C^1(\mathbb{R}^n \setminus \{0\})$-vector field $A \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies (4.9) (recast for the case with no x) dependence. Assume also that*

$$p > \frac{2n}{n+2} \, . \tag{4.21}$$

Then

$$|F|^p \in L^\gamma_{\mathrm{loc}}(\Omega_T) \Longrightarrow |Du|^p \in L^\gamma_{\mathrm{loc}}(\Omega_T) \qquad \text{for every } \gamma > 1\ . \tag{4.22}$$

Moreover, for every $\gamma > 1$, there exists a constant $c \equiv c(n, p, \nu, L, \gamma)$ such that the estimate

$$\left(\fint_{Q_{R/2}} |Du|^{p\gamma}\,dx\,dt\right)^{1/\gamma} \leq c\left[\fint_{Q_R} |Du|^p\,dx\,dt + \left(\fint_{Q_R} |F|^{p\gamma}\,dx\,dt\right)^{1/\gamma} + 1\right]^d \tag{4.23}$$

holds for every parabolic cylinder $Q_R \equiv B_R(x_0) \times (t_0 - R^2, t_0) \Subset \Omega_T$. Here d is the scaling deficit of the system, i.e.

$$d := \begin{cases} \frac{p}{2} & \text{if} \quad p \geq 2 \\ \frac{2p}{p(n+2)-2n} & \text{if } \frac{2n}{n+2} < p < 2\ . \end{cases}$$

The above result is sharp. Indeed, we remark that assuming the lower bound in (4.21) is essential in order to prove (4.22); see the counterexamples to regularity in [43] which apply already to the case $F \equiv 0$ when $1 < p \leq 2n/(n+2)$. The appearance of the exponent d in (4.23) reflects the fact that the parabolic p-Laplacean operator is not homogeneous unless $p = 2$. Therefore a priori estimates cannot be homogeneous as well. Indeed we notice that $d = 1$ if and only if $p = 2$. Moreover, we notice that $d \to \infty$ as $p \to 2n/(n+2)$. The result of Theorem 4.5.1 extends to the case of the parabolic p-Laplacean system as well, and, again, the presence of VMO-coefficients can be allowed; we refer to [2, 55, 73, 74] for more details and various cases. The techniques of [2] are flexible enough to be extended to more general rearrangement invariant function spaces, as for instance Lorentz spaces (see Definition 4.9.4 below); for this see the results of Baroni [7]. We also refer to the important work [70].

4.6 Non-uniformly Elliptic Operators

We now consider the case of non-uniformly elliptic operators. Here non-uniform ellipticity means that we are considering equations as

$$-\mathrm{div}\, A(x, Du) = 0\ ,$$

where the ratio between the highest and the lowest eigenvalue of $\partial A(x, z)$ is not bounded. Observe that this ratio is bounded when assumptions in (4.15) are in force.

For non-uniformly elliptic operators a complete nonlinear CZ-theory is still lacking, due to the fact that such operators can have completely different structure properties from each other. We therefore focus on two particular, yet significant cases, related to the following variational integrals:

$$\begin{aligned} \mathcal{V}(w,\Omega) &:= \int_\Omega |Dw|^{p(x)}\,dx\,, \quad 1<\gamma_1\le p(x)\le \gamma_2<\infty \\ \mathcal{P}_{p,q}(w,\Omega) &:= \int_\Omega (|Dw|^p + a(x)|Dw|^q)\,dx\,, \quad 0\le a(x)\le L\,, \quad 1<p<q\,. \end{aligned} \tag{4.24}$$

The common point of the two functionals is the fact that the growth, and therefore the ellipticity properties, of the integrand with respect to the gradient variable depends on the point x. This fact is at the origin of the non-uniformly elliptic nature of the Euler-Lagrange equations associated to the functionals in (4.24). These are of the type

$$-\operatorname{div} A(x,Du)=0$$

where

$$A(x,z)=p(x)|z|^{p(x)-2}z \qquad \text{and} \qquad A(x,z)=|z|^{p-2}z+(q/p)a(x)|z|^{q-2}z\,,$$

respectively. The integrals in (4.24) are related to the seminal work of Marcellini [95, 96] on functionals with non-standard growth conditions and to the work of Zhikov in the setting of Homogenization theory [116–118].

As for the functional $\mathcal{V}(\cdot)$, the main, sharp assumption is the one of log-Hölder continuity on the variable exponent $p(x)$:

$$\lim_{\varrho\to 0}\omega(\varrho)\log\left(\frac{1}{\varrho}\right)=0\,, \tag{4.25}$$

where $\omega(\cdot)$ is the modulus of continuity of $p(\cdot)$

$$|p(x)-p(y)|\le \omega(|x-y|) \qquad \forall\, x,y\,.$$

We have then the following

Theorem 4.6.1 ([1]) *Let $u\in W^{1,1}(\Omega;\mathbb{R}^N)$ be a distributional solution to the system*

$$\operatorname{div}(Du|^{p(x)-2}Du)=\operatorname{div}(|F|^{p(x)-2}F) \qquad \text{in } \Omega\subset\mathbb{R}^n\,,$$

such that $|Du|^{p(x)}, |F|^{p(x)}\in L^1_{\rm loc}(\Omega)$. Assume that (4.25) holds. Then

$$|F|^{p(x)}\in L^\gamma_{\rm loc}(\Omega)\Longrightarrow |Du|^{p(x)}\in L^\gamma_{\rm loc}(\Omega) \qquad \text{for every } \gamma>1\,.$$

Moreover, for every $\gamma > 1$ *there exists a constant* $c \equiv c(n, N, p, \gamma)$ *such that the estimate*

$$\left(\fint_{B_{R/2}} |Du|^{p(x)\gamma}\,dx\right)^{1/\gamma} \leq c \fint_{B_R} |Du|^{p(x)}\,dx + c\left(\fint_{B_R} |F|^{p(x)\gamma}\,dx\right)^{1/\gamma} \tag{4.26}$$

holds for every ball $B_R \Subset \Omega$.

We just remark that the assumption (4.25) is necessary and it serves to bound the rate of non-uniform ellipticity of the problems considering via the bound on the oscillations of $p(x)$. Notice that in the case the variable exponent is constant, i.e., $p(x) \equiv p$, estimate (4.26) gives back (4.13).

Theorem 4.6.2 ([36]) *Let* $u \in W^{1,1}(\Omega)$ *be a local minimiser of the functional* $\mathcal{V}$ *and assume that* $p(\cdot)$ *is locally Hölder continuous in* Ω. *Then Du is locally Hölder continuous in* Ω.

We next switch to the functional $\mathcal{P}_{p,q}(\cdot)$ and consider related equations of the type

$$\operatorname{div} A(x, Du) = \operatorname{div} A(x, F) \qquad \text{in } \Omega \subset \mathbb{R}^n \tag{4.27}$$

where

$$A(x, z) := (|z|^{p-2} + a(x)|z|^{q-2})z = \frac{H(x, z)z}{|z|^2}$$

and

$$H(x, z) := |z|^p + a(x)|z|^q\,.$$

The main assumption we are considering is the following *balance* between the gap q/p and the regularity of $a(\cdot)$, that is

$$a(\cdot) \in C^{0,\alpha}(\Omega)\,, \qquad \frac{q}{p} < 1 + \frac{\alpha}{n}\,. \tag{4.28}$$

This condition plays the role condition (4.25) has for functionals with a variable growth exponent of the type $\mathcal{V}(\cdot)$. We shall expand on this more later. We then have the following, optimal:

Theorem 4.6.3 ([35]) *Let* $u \in W^{1,1}(\Omega)$ *be a distributional solution to Eq.* (4.27) *such that* $H(x, Du), H(x, F) \in L^1(\Omega)$, *under the assumptions* (4.28). *Then*

$$H(x, F) \in L^{\gamma}_{\mathrm{loc}}(\Omega) \Longrightarrow H(x, Du) \in L^{\gamma}_{\mathrm{loc}}(\Omega) \qquad \textit{holds for every } \gamma \geq 1\,.$$

Moreover, for every $\gamma \geq 1$ there exist a positive radius r and constants $c, c(\gamma)$

$$\begin{cases} r & \equiv r(n,p,q,\nu,L,[a]_{0,\alpha},\|H(x,Du)\|_{L^1},\gamma) > 0 \\ c & \equiv c(n,p,q,\nu,L,[a]_{0,\alpha},\|H(x,Du)\|_{L^1}) > 0 \\ c(\gamma) & \equiv c(n,p,q,\nu,L,[a]_{0,\alpha},\|H(x,Du)\|_{L^1},\gamma) > 0 \end{cases}$$

such that the estimate

$$\left(\fint_{B_{R/2}} [H(x,Du)]^{\gamma}\,dx\right)^{1/\gamma} \leq c \fint_{B_R} [H(x,Du)]\,dx + c(\gamma)\left(\fint_{B_R} [H(x,F)]^{\gamma}\,dx\right)^{1/\gamma} \tag{4.29}$$

holds for every ball $B_R \subset \Omega$ such that $R \leq r$.

Notice that, also in this case, estimate (4.29) perfectly reduces to (4.13) when $a(\cdot) \equiv 0$, or when $p = q$. The operator in (4.27) has been originally introduced by Zhikov in the context of Homogenization [116–118], in order to provide models for strongly anisotropic materials. In this respect the exponents p and q represent the hardening exponents of two materials forming a composite whose geometry is dictated by the coefficient $a(\cdot)$, or, more precisely, by its zero set $\{a(x) = 0\}$. The bound in (4.28) reflects in a sharp way the subtle interaction between the different kinds of ellipticity properties of the operator—given by the numbers p and q—and the regularity of the coefficient $a(\cdot)$ that dictates the phase transition. More precisely, this in turn relates to the kind of non-uniform ellipticity of the Euler-Lagrange equation of the functional, which is

$$-\operatorname{div}\left[|Du|^{p-2}Du + (q/p)a(x)|Du|^{q-2}Du\right] = 0\,. \tag{4.30}$$

The non-uniform ellipticity of equation in display (4.30), when evaluated on the specific solution u, is measured by the potential blow-up of the ratio

$$\frac{\text{highest eigenvalue of } \partial_z A(x,Du)}{\text{lowest eigenvalue of } \partial_z A(x,Du)} \approx 1 + a(x)|Du|^{q-p}\,. \tag{4.31}$$

Around the phase transition, that is the zero set $\{a(x) = 0\}$, the ratio in (4.31) exhibits a potential blow-up with respect to the gradient of rate $q-p$; to compensate, $a(x)$ is required to be suitably small. This means that, since we are close to $\{a(x) = 0\}$, then α must be large enough as prescribed in (4.28). This heuristic reasoning is confirmed when looking at the functional $\mathcal{V}$: in this case, when $p(\cdot)$ is continuous, the variability of x produces a small change in the growth exponent and therefore the non-uniform ellipticity of the related Euler-Lagrange equation is modest. Indeed, the correction needed in (4.25) is a regularity assumptions imposed on $p(x)$ which is weaker than the one imposed on $a(x)$ in (4.28).

As for the regularity of minimizers, we this time have

Theorem 4.6.4 ([33, 34]) *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional $\mathcal{P}_{p,q}$ and assume that (4.28) is in force. Then Du is locally Hölder continuous in Ω. The same conclusion holds under the condition*

$$0 \le a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \text{and} \qquad q \le p+1\,, \tag{4.32}$$

provided the local minimizer u is locally bounded.

We conclude this section by observing that the main assumption in (4.28), as well as the one in (4.32) when minimizers and/or solutions are locally bounded, are sharp. This indeed follows from a counterexample in [56] where it is shown the existence of a $W^{1,p}$-regular bounded local minimizer u of the functional $P_{p,q}(\cdot)$, with $0 \le a(\cdot) \in C^{0,\alpha}(\Omega)$, but with $q > n+\alpha > n > p$, such that $u \notin W^{1,q}_{\rm loc}$.

4.7 Estimates Below the Natural Growth Exponent

In Sect. 4.3 we have considered equations of the type (4.8) under the main assumption (4.10) on the solution and on the datum F. This means that we are in the so called *energy range*. We now want to briefly describe what happens when assumptions (4.10) are not considered and we therefore fall in the *subenergy range*. Here we again consider general equations of the type (4.8), but this time with measurable coefficients, i.e., the vector field $A(\cdot)$ is Carathéodory regular and the map $x \mapsto A(x,z)$ is just measurable. We moreover consider the weaker assumptions

$$|A(x,z)| \le L|z|^{p-1} \qquad \text{and} \qquad \nu|z|^p \le \langle A(x,z), z\rangle \tag{4.33}$$

for every $z \in \mathbb{R}^n$ and $x \in \Omega$, where $0 < \nu \le 1 \le L$ are fixed real numbers. To fix the ideas, we immediately observe that in order to give the distributional definition (4.11) sense it is sufficient to assume that

$$u \in W^{1,q}_{\rm loc}(\Omega),\ F \in L^q_{\rm loc}(\Omega) \qquad q > p-1\,. \tag{4.34}$$

A distributional solution that does not belong to $W^{1,p}_{\rm loc}(\Omega)$ is called a *very weak solution*. Such solutions were first considered in detail by Serrin [107], who showed that they might exist next to the standard energy solutions.

Understanding the basic properties of very weak solutions is not an easy task, since even establishing basic estimates becomes problematic. To highlight the main ideas without getting involved in misleading technicalities, we consider the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x,Du) = \operatorname{div}\left(|F|^{p-2}F\right) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4.35}$$

where, again for simplicity, Ω is just a ball (a smooth or even a Lipschitz domain would be fine as well). Now, assume for a moment to deal with an energy solution, that is (4.33) hold for some $q \geq p$. By a standard density argument we can use $\varphi \equiv u$ in (4.11), thereby getting, by means of (4.33), the basic *energy estimate*

$$\int_{\Omega} |Du|^p \, dx \leq c(p, \nu, L) \int_{\Omega} |F|^p \, dx \, .$$

The above density argument does not match with the growth properties in (4.33) in case (4.33) for $q < p$ and the choice $\varphi \equiv u$ is now no longer admissible in (4.11). Nevertheless, the previous estimate continues to hold even when $q < p$, provided the distance between p and q is not too large. This is a highly non-trivial fact and is contained in the following:

Theorem 4.7.1 ([65, 89]) *Let $u \in W^{1,q}_0(\Omega)$ be a distributional solution to the equation in (4.8), under assumptions (4.33) and (4.34). Then there exist numbers p_1, p_2 with*

$$p - 1 < p_1 < p < p_2 < \infty \, , \tag{4.36}$$

and depending only on n, p, ν, L, such that the estimate

$$\int_{\Omega} |Du|^q \, dx \leq c(n, p, q, \nu, L) \int_{\Omega} |F|^q \, dx \tag{4.37}$$

holds in the range $q \in (p_1, p_2)$.

When $q < p$ estimates of the type in (4.37) are for obvious reasons sometimes called estimates below the natural growth exponent. Theorem 4.7.1 is due to Iwaniec and Sbordone [65] and Lewis [89], who proved it independently by different means. The authors of [65] use a method based on the Hodge decomposition and its stability properties with respect to certain nonlinear perturbations. In [89], a method based on the truncation of certain maximal operators is instead used. Both techniques are applied to solutions in order to obtain test functions with improved integrability properties, recovering the initial lack of integrability.

In the rest of the section we shall briefly describe the approach of Iwaniec and Sbordone, that reveals some fascinating and deep features of the Hodge decomposition. We still recall that in the following lines Ω is assumed to be, for simplicity, a ball.

Given a vector field $F \in L^q(\Omega)$ with $q > 1$, the Hodge decomposition (actually also-called Helmotz decomposition) allows to express F as a gradient plus a divergence-free vector field, i.e.

$$F = Du + H \, , \qquad \operatorname{div} H = 0 \, .$$

This decomposition comes along with the natural estimate, that is

$$\|Du\|_{L^q(\Omega)} + \|H\|_{L^q(\Omega)} \leq c(q) \|F\|_{L^q(\Omega)} \, .$$

It is not very difficult to realize how this works, at least when $\Omega \equiv \mathbb{R}^n$; let us give a very informal sketch of the construction. One solves the equation $\triangle u = \operatorname{div} F$ via singular integrals (actually using a composition of Riesz transforms), and the standard Calderón-Zygmund theory provides us with an estimate of the type

$$\|Du\|_{L^q(\Omega)} \leq c(q)\|F\|_{L^q(\Omega)} \,. \tag{4.38}$$

Then one concludes with $H := F - Du$. If F is a gradient, we can obviously take $H = 0$ and in fact, for general F, Du is nothing but the closest gradient vector field to F with respect to the distance in L^2. The key point is that this construction, together with the related estimate (4.38), is stable with respect to certain natural nonlinear perturbations. Indeed, we have

Theorem 4.7.2 ([64, 65]) *Let $w \in W^{1,q}_0(B)$ for $q > 1$ and let $\varepsilon \in (-1, q-1)$. There exists $\varphi \in W^{1,q/(1+\varepsilon)}_0(B)$ and a divergence free vector field*

$$H \in L^{q/(1+\varepsilon)}(B;\mathbb{R}^n) \,, \qquad \operatorname{div} H = 0$$

such that

$$|Dw|^{\varepsilon} Dw = D\varphi + H \tag{4.39}$$

and

$$\|H\|_{L^{q/(1+\varepsilon)}(B)} \leq c(n,q)|\varepsilon|\|Dw\|^{1+\varepsilon}_{L^q(B)} \tag{4.40}$$

hold. The constant $c(q)$ depends on q but it is stable as long as q varies in a compact subset of $(1,\infty)$.

It is now rather clear how to use the above result to the proof of estimate (4.37): the idea is to apply Theorem 4.7.2 to u for some negative ε. If ε is sufficiently small, then the remainder H is small too thanks to (4.40) and can be re-absorbed in the estimates, finally leading to (4.37). The smallness of ε fixes the range of the exponents appearing in (4.36). This method can be used in several other contexts; see for instance [59].

4.8 Measure Data Problems

In this section we start considering equations with general measure data of the type

$$-\operatorname{div} A(x, Du) = \mu \qquad \text{in } \Omega \subset \mathbb{R}^n \,. \tag{4.41}$$

Here μ is a Borel (signed) measure concentrated on Ω with locally finite total mass, i.e. $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$. The starting assumptions we shall consider throughout this section

on the Carathéodory vector field $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are

$$\begin{cases} |A(x,z)| \le L|z|^{p-1} \\ \nu\left(|z_1|^2+|z_2|^2\right)^{\frac{p-2}{2}}|z_1-z_2|^2 \le \langle A(x,z_1)-A(x,z_2),z_1-z_2\rangle \end{cases} \tag{4.42}$$

for any choice of $z, z_1, z_2 \in \mathbb{R}^n$, for almost every $x \in \Omega$, and for fixed ellipticity constants $0 < \nu \le 1 \le L$. In particular, the dependence on x of $A(\cdot)$ is at the moment *just measurable*. The monotonicity inequality in $(4.42)_2$ is satisfied by the main model example given by the p-Laplacean operator $A(z) \equiv |z|^{(p-2)/2}z$. Notice that in the case $p \ge 2$, from $(4.42)_2$ we recast the familiar strict monotonicity property

$$\nu|z_1-z_2|^p \le \langle A(x,z_1)-A(x,z_2),z_1-z_2\rangle\ .$$

Remark 4.8.1 The assumptions in (4.42) are the starting ones we shall consider when looking at measure data problems. They are sufficient to get integrability results for the gradient. Later on, when considering higher regularity properties of solutions, we shall use stronger assumptions (see (4.52)–(4.53) below).
For measure data problems the plain notion of distributional solution is not sufficient to build a reasonable theory and suitably reinforced notions must be considered. These naturally stem from the available existence theory. Indeed, solutions to measure data problems can be obtained via approximation methods as for instance first shown by Boccardo and Gallouët in [12]. These methods in turn generate the most commonly used notion of solution, called SOLA, that here we report in its local version (as described in [104]). We remark that in the linear and semilinear cases the theory of measure data problems has been pioneered in [92] and [20], respectively.

Definition 4.8.2 (Local SOLA) A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ being an arbitrary open subset, is a local SOLA to the equation in (4.42) under assumptions (4.42) and with $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$, if and only if there exists a sequence of local energy solutions $\{u_k\} \subset W^{1,p}_{\mathrm{loc}}(\Omega)$ to the equations

$$-\mathrm{div}\, A(x,Du_k) = \mu_k \in L^\infty_{\mathrm{loc}}(\Omega)\,,$$

such that $u_k \rightharpoonup u$ weakly in $W^{1,1}_{\mathrm{loc}}(\Omega)$. Here the sequence $\{\mu_k\}$ converges to μ (locally) weakly* in the sense of measures and satisfies

$$\limsup_k |\mu_k|(B) \le |\mu|(\overline{B})$$

for every ball $B \Subset \Omega$.
Local SOLA, that are at the moment defined only via a convergence property, are in fact distributional solutions in the sense that they satisfy

$$\int_\Omega \langle A(x,Du),D\varphi\rangle\,dx = \int_\Omega \varphi\,d\mu \qquad \forall\ \varphi \in C^\infty_0(\Omega)\ .$$

but they are not necessarily energy solutions (see a few lines below). There are indeed SOLA that do not belong to the natural spaces $W^{1,p}$ and they are very weak solutions in the sense explained in Sect. 4.7; a typical example is provided in (4.45) below. Concerning local SOLA integrability properties, we have the following result, whose proof can be found in [104]. It extends to the local case some basic convergence and regularity results obtained in [12–14]:

Theorem 4.8.3 ([12, 104]) *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a local SOLA to the equation in (4.41) in the sense of Definition 4.8.2, with $p > 2 - 1/n$, and let $\{u_k\}$ the corresponding sequence of approximate solutions thereby introduced. Then, up to a not relabelled subsequence, it holds that*

$$u_k \to u \qquad \textit{strongly in } W^{1,q}_{\rm loc}(\Omega) \qquad \forall\, q < \min\left\{p, \frac{n(p-1)}{n-1}\right\}\,.$$

As a consequence, u is a distributional solution to the equation in (4.41).

The notion of local SOLA is consistent with the one of weak solution in the sense that every $W^{1,p}$-regular weak solution turns out to be a local SOLA; moreover when the measure μ belongs to the dual space $W^{-1,p'}(\Omega)$, then SOLA become energy solutions. We refer for instance to [104, Chap. 5] for more details.

By Theorem 4.8.3 it follows that every local SOLA u satisfies

$$Du \in L^q_{\rm loc}(\Omega;\mathbb{R}^n) \qquad \text{for every } q < \frac{n(p-1)}{n-1}\,, \qquad \text{when } p \le n\,. \tag{4.43}$$

Actually, the limiting integrability of SOLA can be described in terms of weak-Lebesgue spaces, i.e., Marcinkiewicz spaces, that is (for $p \le n$)

$$Du \in \mathcal{M}^{\frac{n(p-1)}{n-1}}_{\rm loc}(\Omega;\mathbb{R}^n) \Longleftrightarrow \sup_{0<\lambda} \lambda^{\frac{n(p-1)}{n-1}} |\{x \in \Omega' \,:\, |Du(x)| > \lambda\}| < \infty\,, \tag{4.44}$$

with the last inequality that holds for every open subset $\Omega' \Subset \Omega$. For this result see [11, 46]. This result is optimal, as shown by the so-called nonlinear fundamental solution

$$G_p(x) \approx \begin{cases} |x|^{\frac{p-n}{p-1}} & \text{if } 1 < p \neq n \\ -\log\,|x| \text{ if } \quad p = n\,, \end{cases} \qquad |x| \neq 0\,. \tag{4.45}$$

This is in fact the unique SOLA to the equation $-\triangle_p u = \delta$, where δ is the Dirac measure charging the origin. We again refer to [104, Chap. 5] for more details on this fact and on general measure data problems. Different notions of solutions have been proposed in the literature, see for instance [15, 37]. They turn out to be equivalent in the case of positive measures [69]. For results in the vectorial case see also [45, 46].

Finally, notice that the lower bound $p > 2 - 1/n$ essentially serves to guarantee that the SOLA belong to $W^{1,1}$

$$p > 2 - \frac{1}{n} \Longleftrightarrow \frac{n(p-1)}{n-1} > 1 \,. \tag{4.46}$$

4.9 Estimates on Level Sets and Maximal Operators

The result in (4.43) fixes the best regularity in terms of gradient integrability we can expect for solutions when dealing with general measure data equations. It is anyway only a starting point of a more general theory that in fact allows to get estimates in various function spaces for solutions, in terms of the integrability assumed on μ. In this respect, in this section we shall present a level sets inequality involving maximal operators and allowing to get a wealth of results in virtually all rearrangement invariant function spaces. We premise a few definitions.

Definition 4.9.1 (Fractional Maximal Operator) Let $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ being an open subset; with $x \in \Omega$ and $R < \mathrm{dist}(x, \partial\Omega)$, the function defined by

$$M_{\beta;R}(\mu)(x) := \sup_{0<\varrho<R} |B_\varrho(x)|^{\beta/n} \frac{|\mu|(B_\varrho(x))}{|B_\varrho(x)|} \,, \qquad 0 \le \beta \le n$$

is called the restricted (centered) fractional maximal function of μ.
When $\beta \equiv 0$ and $R = \infty$, we recast the usual maximal operator of Hardy and Littlewood

$$M(\mu)(x) := \sup_{B_\varrho(x)} \frac{|\mu|(B_\varrho(x))}{|B_\varrho(x)|} \,.$$

The non-centered version is instead defined as follows:

Definition 4.9.2 (Non-centered Fractional Maximal Operator) Let $\mu \in \mathbf{M}(\Omega)$ with $\mathcal{B} \Subset \Omega \subset \mathbb{R}^n$ being open subsets; the function defined by

$$M^*_{\beta;\mathcal{B}}(\mu)(x) := \sup_{\substack{x \in B \\ B \subset \mathcal{B}}} |B|^{\beta/n} \frac{|\mu|(B)}{|B|} \,, \qquad 0 \le \beta \le n$$

is called the non-centered restricted fractional maximal function of μ.

Definition 4.9.3 (Fractional Sharp Maximal Operator) Let $f \in L^1_{\mathrm{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ being an open subset, $\alpha \in [0, 1]$, $x \in \Omega$ and $R < \mathrm{dist}(x, \partial\Omega)$; the function defined by

$$M^{\#}_{\alpha;R}(f)(x) := \sup_{0<\varrho<R} \varrho^{-\alpha} \fint_{B_\varrho(x)} |f - (f)_{B_\varrho(x)}| \, dy$$

is called the restricted (centered) sharp fractional maximal function of f. In case $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we have the non-restricted version

$$M^{\#}_{\alpha}(f)(x) := \sup_{0<\varrho} \varrho^{-\alpha} \fint_{B_\varrho(x)} |f - (f)_{B_\varrho(x)}|\, dy\,.$$

The boundedness of maximal operators in various functions spaces as for instance Lebesgue and weak Lebesgue spaces is a well-known fact; see for instance [104]. A particular class of spaces which is of interest here is the one of Lorentz spaces.

Definition 4.9.4 (Lorentz Spaces) Let $q \in (0,\infty)$ and $\gamma \in (0,\infty)$ and let $\Omega \subseteq \mathbb{R}^n$ be an open subset; a measurable map $f\colon \Omega \to \mathbb{R}^k$, $k \in \mathbb{N}$, belongs to $L(q,\gamma)(\Omega;\mathbb{R}^k) \equiv L(q,\gamma)(\Omega)$ if and only if

$$\|f\|_{L(q,\gamma)(\Omega)} = \left(q\int_0^\infty (\lambda^q |\{x \in \Omega\,:\, |f(x)| > \lambda\}|)^{\gamma/q}\,\frac{d\lambda}{\lambda}\right)^{1/\gamma} < \infty\,.$$

The space $L(q,\infty)(\Omega)$ is finally defined setting

$$L(q,\infty)(\Omega) \equiv \mathcal{M}^q(\Omega)$$

with

$$\|f\|_{L(q,\infty)(\Omega)} := \|f\|_{\mathcal{M}^q(\Omega)}\,.$$

The local variant can are defined as usual, by saying that $f \in L(q,\gamma)_{\mathrm{loc}}(\Omega)$ iff $f \in L(q,\gamma)(\Omega')$ for every open subset $\Omega' \Subset \Omega$.
Observe that Lorentz spaces coincide with Lebesgue spaces when $q = \gamma$ and indeed we have $\|f\|_{L(q,q)(\Omega)} = \|f\|_{L^q(\Omega)}$. Lorentz spaces refine the standard Lebesgue spaces in the sense that the second index tunes the first in the sense that when Ω has finite measure, whenever $0 < q < t < r < \infty$ the following continuous embeddings take place:

$$L^r \equiv L(r,r) \subset L(t,q) \subset L(t,t) \subset L(t,r) \subset L(q,q) \equiv L^q\,,$$

while all the previous inclusions are strict. For later purposes it will be also convenient to introduce the following averaged quasinorms. For this we shall then assume that Ω has bounded measure. We then define

$$\nparallel f\|_{L(q,\gamma)(\Omega)} := \left(q\int_0^\infty \left(\lambda^q \frac{|\{x \in \Omega\,:\, |f(x)| > \lambda\}|}{|\Omega|}\right)^{\gamma/q}\,\frac{d\lambda}{\lambda}\right)^{1/\gamma}$$

if $\gamma < \infty$ and

$$\mathbb{f}f\|_{\mathcal{M}^q(\Omega)} \equiv \mathbb{f}f\|_{L(q,\infty)(\Omega)} := \left(\sup_{0<\lambda} \lambda^q \frac{|\{x \in \Omega \,:\, |f(x)| > \lambda\}|}{|\Omega|}\right)^{1/q} < \infty \,.$$

Notice that in the case $q = \gamma$ we have

$$\mathbb{f}f\|_{L(q,q)(\Omega)} = \left(\fint_\Omega |f|^q \, dx\right)^{1/q},$$

and this justifies the terminology of averaged quasinorm. Lorentz spaces can also be realized as interpolation spaces and therefore the boundedness of Maximal operators in Lorentz spaces follows by standard interpolations theory.

We now go back to solutions of measure data problems and state the main result of this section, which is

Theorem 4.9.5 ([101]) *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a local SOLA to the equation in (4.41) in the sense of Definition 4.8.2, and under assumptions (4.42) with $p > 2 - 1/n$. Let B be a ball such that $2B \Subset \Omega$. There exist constants $H \equiv H(n,p,\nu,L) > 1$, $\chi \equiv \chi(n,p,\nu,L) > 1$ and $c(n) \geq 1$ such that the following is true: For every $T \geq 1$ there exists $\varepsilon \equiv \varepsilon(n,p,\nu,L,T) \in (0,1)$ such that*

$$\begin{aligned} &\left|\left\{x \in r_1 B : M^*_{0;2B}(|Du|)(x) > HT\lambda\right\}\right| \\ &\qquad \leq \frac{1}{T^{p\chi}}\left|\left\{x \in r_2 B \,:\, M^*_{0;2B}(|Du|)(x) > \lambda\right\}\right| \\ &\qquad\qquad + \left|\left\{x \in r_1 B \,:\, [M^*_{1;2B}(\mu)]^{1/(p-1)} > \varepsilon\lambda\right\}\right| \end{aligned} \tag{4.47}$$

holds whenever

$$\lambda \geq \frac{c(n)T^{p\chi-1}}{(r_2 - r_1)^n} \fint_{2B} |Du| \, dx \quad \text{and} \quad 0 < r_1 < r_2 \leq 1 \,.$$

Theorem 4.9.5 allows to control the level sets of the gradient of solutions with the level sets of the measure μ, provided suitable maximal operators, reflecting the scaling properties of the equation, are considered. Moreover, there is a price to pay, that is the presence of the intermediate term in (4.47), that naturally prevents to get a degree of regularity for solutions that goes beyond the maximal one allowed in the case $\mu \equiv 0$. Note that without this term the inequality would be false since it would imply to much! Indeed, when considering homogeneous equations with measurable coefficients $-\operatorname{div} A(x, Dw) = 0$, the maximal regularity obtainable for the gradient is the following higher integrability result:

$$|Dw| \in L^{p\chi}_{\rm loc}(\Omega) \tag{4.48}$$

for some $\chi \equiv \chi(n,p,\nu,L) > 1$. This essentially follows from Gehring's theory. The exponent χ appearing in the line above is exactly the one appearing in Theorem 4.9.5, and this makes it sharp in this respect. On the other hand, with the aim to get integrability gradient estimates for solutions, Theorem 4.9.5 does not yield more than the integrability result in (4.48), even for smooth choices of μ. More in general, being formulated in terms of levels sets, Theorem 4.9.5 allows to sharply recover the integrability information solutions by that of the given datum μ in rearrangement invariant function spaces (recall that by their very definition, such spaces are defined via the decay properties of the level sets of their functions).

A first consequence of Theorem 4.9.5 is the following one:

Theorem 4.9.6 ([101]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.41) in the sense of Definition 4.8.2, and under assumptions (4.42) with $p > 2 - 1/n$. Then, for every couple of exponents (t,γ) satisfying*

$$1 \le t < \frac{p\chi}{p-1} \quad \text{and} \quad 0 < \gamma \le \infty\,, \tag{4.49}$$

there exists a constant $c \equiv c(n,p,\nu,L,t,\gamma)$ such that the maximal estimate

$$\begin{aligned}\|\!| M^*_{0;2B}(|Du|) \|\!|_{L(t(p-1),\gamma(p-1))(B/2)} \le c \fint_{2B} |Du|\,dx \\ + c\, \|\!| M^*_{1;2B}(\mu) \|\!|^{1/(p-1)}_{L(t,\gamma)(B)}\end{aligned}$$

holds whenever B is a ball such that $2B \Subset \Omega$. The constant $\chi \equiv \chi(n,p,\nu,L) > 1$ in (4.49) is the exponent appearing in Theorem 4.9.5.

Since the behaviour of maximal operators on Lorentz spaces is known we then get, as a corollary, theorems on solutions. For instance, it holds the following:

Theorem 4.9.7 ([101]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.41) in the sense of Definition 4.8.2, and under assumptions (4.42) with $p > 2 - 1/n$. Assume that $\mu \in L(q,\gamma)(\Omega)$ locally, with*

$$1 < q < \max\left\{1, \frac{np\chi}{np - n + p\chi}\right\} \quad \text{and} \quad 0 < \gamma \le \infty\,. \tag{4.50}$$

Then

$$Du \in L\left(\frac{nq(p-1)}{n-q}, \gamma(p-1)\right) \quad \text{holds locally in } \Omega\,.$$

Moreover, there exists a constant $c \equiv c(n,p,\nu,L,q,\gamma)$ such that the estimate

$$\|\!| Du \|\!|_{L\left(\frac{nq(p-1)}{n-q},\gamma(p-1)\right)(B_{R/2})} \le c \fint_{B_R} |Du|\,dx + cR\, \|\!| \mu \|\!|^{1/(p-1)}_{L(q,\gamma)(B_R)}$$

holds whenever $B_R \Subset \Omega$ is a ball. The constant $\chi \equiv \chi(n,p,\nu,L) > 1$ in (4.50) is the higher integrability exponent appearing in Theorem 4.9.5.
Many more examples are obtainable in various function spaces, in particular, in non-interpolation spaces as Morrey spaces. We refer to [101] for more on such aspects.

4.10 Limiting Calderón-Zygmund Theory

The results displayed in (4.43)–(4.44) and those of the last section deal with optimal gradient integrability results. What happens when looking at gradient differentiability? Using the results available for the Poisson equation (4.1) as a guide, the classical Calderón-Zygmund theory gives (4.4). This result fails for $q = 1$, nevertheless only a minimal amount of gradient differentiability is lost. This can be seen by using fractional Sobolev spaces. Indeed for solutions to (4.1) it can be proved that

$$\mu \in L^1_{\rm loc}(\Omega) \Longrightarrow Du \in W^{\sigma,1}_{\rm loc}(\Omega;\mathbb{R}^n) \qquad \text{whenever } 0 < \sigma < 1\,. \tag{4.51}$$

In the above display the fractional Sobolev space $W^{\sigma,1}$ appears; (4.51) essentially means that

$$\int_{\Omega'}\int_{\Omega'} \frac{|Du(x)-Du(y)|}{|x-y|^{n+\sigma}}\,dx\,dy < \infty$$

holds for every $\sigma \in (0,1)$ and every bounded open subset $\Omega' \Subset \Omega$. In general, with $\alpha \in (0,1)$, $q \in [1,\infty)$, $k \in \mathbb{N}$, the fractional Sobolev space $W^{\alpha,q}(\Omega;\mathbb{R}^k)$ is defined prescribing that $f\colon \Omega \to \mathbb{R}^k$ belongs to $W^{\alpha,q}(\Omega;\mathbb{R}^k)$ if and only if the following Gagliardo-type norm is finite:

$$\|f\|_{W^{\alpha,q}(\Omega)} := \left(\int_\Omega |f(x)|^q\,dx\right)^{1/q} + \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy\right)^{1/q}.$$

The local variant $W^{\alpha,q}_{\rm loc}(\Omega;\mathbb{R}^k)$ is defined by requiring that $f \in W^{\alpha,q}_{\rm loc}(\Omega;\mathbb{R}^k)$ if and only if $f \in W^{\alpha,q}_{\rm loc}(\Omega';\mathbb{R}^k)$ for every open subset $\Omega' \Subset \Omega$.

To get differentiability results in the nonlinear case we have to use a suitable reinforcement of the assumptions that have been considered in (4.42). More precisely we shall consider for simplicity equations with no dependence on the x-variable, that is of the type

$$-\operatorname{div} A(Du) = \mu \qquad \text{in } \Omega \subset \mathbb{R}^n\,. \tag{4.52}$$

using assumptions as those in (4.9), but suitably recast or the case under consideration. That is, we assume

$$\begin{cases} A(\cdot) \in C^1(\mathbb{R}^n \setminus \{0\}) \\ |A(z)| + |\partial A(z)||z| \le L|z|^{p-1} \\ \nu|z|^{p-2}|\xi|^2 \le \langle \partial A(z)\xi, \xi \rangle \ . \end{cases} \tag{4.53}$$

These assumptions in fact imply those in (4.42) (modulo a change of the constants) when no x-dependence of $A(\cdot)$ is considered.

Now, the point is that the conclusion of (4.51) still holds in the case μ is just a Borel measure with finite total mass.

Theorem 4.10.1 ([100]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \ge 2$. Then*

$$Du \in W^{\frac{\sigma}{p-1},p-1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{for every } \ \sigma \in (0,1) \tag{4.54}$$

holds. Moreover, for every $\sigma \in (0,1)$ there exists a constant $c \equiv c(n,p,\nu,L,\sigma)$ such that the fractional Caccioppoli type inequality

$$\fint_{B_{R/2}} \int_{B_{R/2}} \frac{|Du(x)-Du(y)|^{p-1}}{|x-y|^{n+\sigma}}\,dx\,dy \le \frac{c}{R^\sigma} \fint_{B_R} |Du|^{p-1}\,dx + \frac{c}{R^\sigma}\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]$$

holds for every ball $B_R \Subset \Omega$.

In (4.54) one cannot allow $\sigma = 1$. Indeed the fractional Sobolev embedding

$$W^{\alpha,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \hookrightarrow L^{nq/(n-\alpha q)}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{provided } \alpha q < n \ , \tag{4.55}$$

would then give $Du \in L^{n(p-1)/(n-1)}_{\mathrm{loc}}(\Omega)$, which is clearly not verified by the fundamental solution G_p defined in (4.45). The same argument using Sobolev embedding theorem allows to infer (4.43) from (4.54). Differentiability results are also available in the case $2 - 1/n < p < 2$, see [103, 104]. For the case $p = 2$, a fractional differentiability result for solutions to parabolic equations with measure data is available in [10].

Let us now exploit some closer connections to the standard classical case (4.1). When $p = 2$, (4.54) gives that

$$Du \in W^{\sigma,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \ \sigma \in (0,1) \tag{4.56}$$

which is exactly what happens in (4.51). The parallel with the linear can be pushed further, investigating the differentiability of the intrinsic quantity $A(Du)$, that naturally stems from the equation. Surprisingly enough, it holds the following result, which is in some sense the limiting case of the Calderón-Zygmund theory for nonlinear problems:

Theorem 4.10.2 ([5]) *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2 - 1/n$; moreover assume also that the matrix $\partial A(\cdot)$ is symmetric when $2 - 1/n < p < 2$. Then*

$$A(Du) \in W^{\sigma,1}_{\rm loc}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \sigma \in (0,1)\ . \tag{4.57}$$

Moreover, for every $\sigma \in (0,1)$, there exists a constant $c \equiv c(n,p,\nu,L,\sigma)$ such that the following fractional Caccioppoli type inequality

$$\fint_{B_{R/2}} \int_{B_{R/2}} \frac{|A(Du(x)) - A(Du(y))|}{|x-y|^{n+\sigma}}\,dx\,dy \leq \frac{c}{R^\sigma} \fint_{B_R} |A(Du)|\,dx + \frac{c}{R^\sigma}\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]$$

holds for every ball $B_R \Subset \Omega$.
In the case of the p-Laplacean operator, (4.57) is

$$|Du|^{p-2}Du \in W^{\sigma,1}_{\rm loc}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \sigma \in (0,1)\ ,$$

that reduces to (4.56) when $p = 2$. Theorem 4.10.2 in turn implies a family of differentiability estimates showing a sort of principle of uniformization of singularities: raising a solution to a power larger than one, increases its differentiability (although it naturally decreases its local integrability). In particular, it allows to recast the result of Theorem 4.10.1. We indeed have

Theorem 4.10.3 ([5]) *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2$. Then for every γ such that $0 \leq \gamma \leq p-2$*

$$|Du|^\gamma Du \in W^{\sigma\frac{\gamma+1}{p-1},\frac{p-1}{\gamma+1}}_{\rm loc}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \sigma \in (0,1)\ . \tag{4.58}$$

This means there is a trade between integrability and differentiability of various nonlinear quantities of the gradient, which is always sharp. In fact, observe that, for every choice of $\gamma \in [0, p-2]$, in (4.58) we can never allow for $\sigma = 1$, otherwise the fractional version of Sobolev embedding theorem (4.55) applied to $|Du|^\gamma Du$, would give that $Du \in L^{n(p-1)/(n-1)}_{\rm loc}$. On the other hand, again applying Sobolev embedding theorem, for every choice of $\gamma \in [0, p-2]$, we see that (4.58) implies (4.43), that is

the maximal regularity of Du in terms of Lebesgue spaces. We notice that the case $\gamma = p-2$ of (4.58) essentially corresponds to Theorem 4.10.2, while the case $\gamma = 0$ gives (4.54). Again, when $\gamma = 0$ and when $\mu = 0$, Theorem 4.10.2 relates to some classical fractional differentiability results of Simon [108]; see also [18, 98].

4.11 Nonlinear Potential Estimates

Here we approach the nonlinear extension of the linear potential estimates (4.7). For simplicity we shall confine ourselves to equations of the type in (4.52), under assumptions (4.53), immediately observing that these can be in several cases relaxed. We remark that the main point here is not the treatment of degenerate equations as (4.52) as such, but is extending (4.7) to the case of nonlinear equations, where representation formulae are not available. Indeed the forthcoming results are totally non-trivial already in the case one assumes (4.53) with $p = 2$. Then, in this framework, treating degenerate equations create additional and highly non-trivial difficulties. To proceed, we still need the next

Definition 4.11.1 (Truncated Riesz Potentials) Let $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ being an open subset; the (truncated) Riesz potential $\mathbf{I}^{\mu}_{\beta}$ is defined by

$$\mathbf{I}^{\mu}_{\beta}(x_0, R) := \int_0^R \frac{|\mu|(B_{\varrho}(x_0))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}, \qquad \beta > 0,$$

whenever $B_R(x_0) \Subset \Omega$.
The relation between truncated Riesz potentials and the classical ones in Definition 4.1.1 is very simple:

$$\mathbf{I}^{\mu}_{\beta}(x_0, R) \le c(n) I_{\beta}(|\mu|)(x_0) \qquad \text{for every } R > 0\,. \tag{4.59}$$

The truncated Riesz potentials are the natural local counterpart of the classical For obvious scaling reasons, when $p \neq 2$, estimates of the type in (4.7) cannot hold. It is therefore necessary to consider also different types of potentials, incorporating the scaling exponent of the equations considered. These are known in the literature as Wolff potentials, and they have been first introduced and used in [60].

Definition 4.11.2 (Wolff Potentials) Let $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ being an open subset; the nonlinear Wolff potential $\mathbf{W}^{\mu}_{\beta,p}$ is defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x_0, R) := \int_0^R \left(\frac{|\mu|(B_{\varrho}(x_0))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}, \qquad \beta > 0$$

whenever $B_R(x_0) \Subset \Omega$.
Wolff potentials play a basic role in Nonlinear Potential Theory and in the analysis of fine properties of Sobolev functions and solutions. They intervene in basic

issues such as the pointwise behaviour of Sobolev functions, the Wiener criterion for degenerate equations [67, 68, 91, 97], existence theory for non-homogeneous equations [105, 106]. Comparing Definition 4.11.2 with Definition 4.11.1, we see that

$$\mathbf{W}^{\mu}_{1,2}(x_0,R)=\mathbf{I}^{\mu}_{2}(x_0,R) \quad \text{and} \quad \mathbf{W}^{\mu}_{1/2,2}(x_0,R)=\mathbf{I}^{\mu}_{1}(x_0,R)\,, \tag{4.60}$$

so that Wolff potentials actually reduce to Riesz potentials when $p=2$.

Wolff potentials sharply replace Riesz potentials when considering degenerate equations ($p\neq 2$). This is phenomenon is displayed in

Theorem 4.11.3 ([53, 67, 68, 81]) *Let $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p\geq 2-1/n$. Let $B_R(x_0)\Subset\Omega$ be a ball. If $\mathbf{W}^{\mu}_{1,p}(x_0,R)<\infty$, then x_0 is Lebesgue point of u, in the sense that the limit*

$$\lim_{\varrho\searrow 0}(u)_{B_\varrho(x_0)}=:u(x_0)$$

exists and thereby defines the precise representative of u at x_0. Moreover, the pointwise Wolff potential estimate

$$|u(x_0)|\leq c\mathbf{W}^{\mu}_{1,p}(x_0,R)+c\fint_{B_R(x_0)}|u|\,dx \tag{4.61}$$

holds with a constant c depending only on n,p,ν,L.

This theorem is essentially a very fundamental contribution of Kilpeläinen and Malý [67, 68], who established estimate (4.61) for positive measures in the full range $p>1$, provided a suitable notion of solutions is considered for the case $1<p\leq 2-1/n$ to overcome the fact that SOLA do not belong to $W^{1,1}$ on this range (recall (4.46)); for such definition of solutions see also [62, 90]. An alternative approach is in [112]. The case of general measures is treated in [53], where yet another approach is given. The Lebesgue point criterion first appeared in [81]. Theorem 4.11.3 allows to recover several regularity properties of solutions to measure data problems from the analysis of Wolff potentials, whose mapping properties are indeed known in many relevant function spaces. Their behaviour can be indeed recovered from that of Riesz potentials via so-called Havin-Maz'ya potentials $\mathbf{V}_{\beta,p}(|\mu|)(x_0)$. The inequality

$$\mathbf{W}^{\mu}_{\beta,p}(x_0,\infty)\lesssim I_\beta\left\{\left[I_\beta(|\mu|)\right]^{1/(p-1)}\right\}(x_0)=:\mathbf{V}_{\beta,p}(|\mu|)(x_0)\,.$$

holds provided $\beta p<n$ [29, 60]. When $p=2$ and $\Omega\equiv\mathbb{R}^n$, assume that

$$\liminf_{R\to\infty}\fint_{B_R(x_0)}|u|\,dx=0\,.$$

Letting $R \to \infty$ in (4.61), and recalling (4.59)–(4.60), we find the first estimate in (4.7), that this time holds in the nonlinear setting.

The Lebesgue point criterion of Theorem 4.11.3 let us forecast that Wolff potentials can be used to control the oscillations of solutions. A sharp form of this phenomenon is given in the following:

Theorem 4.11.4 ([68, 81]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2-1/n$. If*

$$\lim_{\varrho \searrow 0} \mathbf{W}^{\mu}_{1,p}(x,\varrho) = 0 \ \textit{holds locally uniformly in } \Omega \textit{ w.r.t. } x\,,$$

then u is continuous in Ω.

Theorem 4.11.4 is due to Kilpeläinen and Malý [67, 68], who considered the case of positive measures. A proof in the general case is given in [81]. A remarkable point here is that estimate (4.61) is sharp, and the nonlinear potential $\mathbf{W}^{\mu}_{1,p}$ cannot be replaced by any other smaller potential. This is in fact reported in the following:

Theorem 4.11.5 ([67, 68]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2 - 1/n$ and with $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$ being a positive measure. Then there exists a constant $c \equiv c(n,p,\nu,L)$ such that the following pointwise estimate holds whenever $B(x_0,2R) \Subset \Omega$ and the Wolff potential is finite:*

$$c^{-1}\mathbf{W}^{\mu}_{1,p}(x_0,R) \leq u(x_0) \leq c\mathbf{W}^{\mu}_{1,p}(x_0,2R) + c \inf_{B_R(x_0)} u\,. \tag{4.62}$$

We refer to [87] for a fractional version of the last two theorems. The nonlinear extension of the second estimate in (4.7) has remained an open problem after the works [67, 68, 112]. In the non-degenerate case $p = 2$ the first result has been obtained in [102]. As for the degenerate case $p > 2$, the first result that appeared in the literature involved Wolff potentials:

Theorem 4.11.6 ([53]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2$. Then the pointwise Wolff potential estimate*

$$|Du(x_0)| \leq c\mathbf{W}^{\mu}_{1/p,p}(x_0,R) + c \fint_{B_R(x_0)} |Du|\,dx \tag{4.63}$$

holds with a constant c depending only on n,p,ν,L, and for every ball $B_R(x_0) \Subset \Omega$ such that x_0 is Lebesgue point of Du.

We recall that

$$\mathbf{W}^{\mu}_{1/p,p}(x_0,R) = \int_0^R \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}}\right]^{1/(p-1)} \frac{d\varrho}{\varrho}$$

and this coincides with the usual linear Riesz potential $\mathbf{I}_1^{\mu}(x_0,R)$ for $p=2$. Theorem 4.11.6 follows the standard orthodoxy of Nonlinear Potential Theory: Wolff potentials replace linear Riesz potentials whenever $p \neq 2$. Now we shall see a *change of paradigm*. Consider the equation $-\mathrm{div}\,(|Du|^{p-2}Du)=\mu$ and formally decouple it as

$$\begin{cases} \mathcal{D} := |Du|^{p-2}Du \\ -\mathrm{div}\,\mathcal{D} = \mu\,. \end{cases} \tag{4.64}$$

We now formally apply the Riesz potential $\mathbf{I}_1$, i.e. $\mathcal{D} \approx -\mathbf{I}_1^{\mu}$ (this is actually very rough but it is what could be done when trying to solve an equation as in $(4.64)_2$). This suggests the validity of the estimate

$$|Du(x_0)|^{p-1} = |\mathcal{D}(x_0)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(x)}{|x-x_0|^{n-1}}\,. \tag{4.65}$$

It indeed holds the following:

Theorem 4.11.7 ([51, 78]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2-1/n$. Let $B_R(x_0) \Subset \Omega$ be a ball. If $\mathbf{I}_1^{\mu}(x_0,R) < \infty$, then x_0 is Lebesgue point of the gradient, in the sense that the limit*

$$\lim_{\varrho \searrow 0} (Du)_{B_\varrho(x_0)} =: Du(x_0)$$

exists and thereby defines the precise representative of Du at x_0. Moreover, the pointwise Riesz potential estimate

$$|Du(x_0)| \leq c\left[\mathbf{I}_1^{\mu}(x_0,R)\right]^{1/(p-1)} + c\fint_{B_R(x_0)} |Du|\,dx \tag{4.66}$$

holds with a constant c depending only on n,p,ν,L.

As for a comparison between (4.66) and (4.63), we have

$$\left[\mathbf{I}_1^{\mu}(x_0,R)\right]^{1/(p-1)} \lesssim \mathbf{W}^{\mu}_{1/p,p}(x_0,2R) \quad \text{if} \quad p \geq 2$$

$$\mathbf{W}^{\mu}_{1/p,p}(x_0,R) \lesssim \left[\mathbf{I}_1^{\mu}(x_0,2R)\right]^{1/(p-1)} \text{ if } 1<p<2\,.$$

This means that Theorem 4.11.7 improves Theorem 4.11.6. Let us mention that the decoupling and linearization principle explained in (4.64) is very general and applied to very large classes of operators in divergence from as shown by Baroni [9].

Riesz potentials can be also used to control the local oscillations of the gradient; indeed we have

Theorem 4.11.8 ([78, 104]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2-1/n$. If*

$$\lim_{\varrho \searrow 0} \mathbf{I}_1^{\mu}(x,\varrho) = 0 \ \textit{holds locally uniformly in } \Omega \textit{ w.r.t. } x\,, \tag{4.67}$$

then Du is continuous in Ω.

Several typical facts of the linear theory can now be reproduced verbatim since the behavior of Riesz potentials is completely known in rearrangement invariant function spaces. See for instance the techniques in [29]. Moreover, assume this time that $\Omega \equiv \mathbb{R}^n$ and that

$$\liminf_{R\to\infty} \fint_{B_R(x_0)} |Du|\,dx = 0\,.$$

Letting $R \to \infty$ in (4.11.7), and recalling (4.59), we find exactly (4.65), which is in turn the second estimate in (4.7) when $p = 2$.

We finally mention a remarkable application of Theorem 4.11.8. A celebrated result of Stein [110] claims that if $v \in W^{1,1}$ is a Sobolev function defined in $\mathbb{R}^n$ with $n \geq 2$, then $Dv \in L(n,1)$ implies that v is continuous. From Definition 4.9.4 we recall that the Lorentz space $L(n,1)$ (over a subset Ω) is defined as the set of measurable maps $g\colon \Omega \to \mathbb{R}$ such that

$$\int_0^{\infty} |\{x \in \Omega\,:\,|g(x)| > \lambda\}|^{1/n}\,d\lambda < \infty\,. \tag{4.68}$$

Stein's theorem is, in a sense, the limiting case of Sobolev-Morrey embedding theorem, as on finite measure spaces we have $L^{n+\varepsilon} \subset L(n,1) \subset L^n$ for every $\varepsilon > 0$. A dual way to state Stein's theorem can be obtained when looking at the gradient regularity of solutions u to the Poisson equation (4.1), and amounts to observe that $\triangle u \in L(n,1)$ implies the continuity of Du. This follows by Steins' result and classical Calderón-Zygmund theory, since Lorentz spaces are actually interpolation spaces ($-\triangle u \in L(n,1)$ implies $D^2u \in L(n,1)$). The point is that now this fact transports verbatim to the nonlinear case, and we have the following nonlinear Stein theorem:

Theorem 4.11.9 ([78]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2-1/n$ and such that $\mu \in L(n,1)$ locally in Ω. Then Du is continuous in Ω.*

This last result is a corollary of (4.11.8), since it can be proved that condition (4.68) implies (4.67). Actually, but by different means, Theorem 4.11.9 continues to hold in the whole range $p > 1$ when considering the p-Laplacean system, and therefore vector-valued solutions. For this result we refer to [83]. The space $L(n,1)$ plays a role for more general operators and in this respect recent results dealing with non-uniformly elliptic equations can be found in [6]. For further a priori estimates see also [30–32, 77].

4.12 Systems

The scalar estimates of the previous section find a fully satisfying analog in the vectorial case, provided, of course, the right class of systems is considered. The problem of proving potential estimates for systems, that is when both $u\colon \Omega \to \mathbb{R}^N$ and μ are vector-valued, has remained open since the original work of Kilpeläinen and Malý [67, 68] and after the vectorial existence theory has been established in the work of Dolzmann et al. in [45–47]. This issue is delicate and needs different approaches, since its nature changes. Indeed, already in the case $\mu \equiv 0$ as in (4.83), solutions to vectorial problems develop singularities, no matter how data are smooth, and this is a well-known fact; we refer for instance to [71, 99] for a discussion of this topic. On the other hand, as noticed starting from the classical work of Uhlenbeck [113], under special structure conditions, solutions to vectorial problems do not exhibit singularities. This is the case of systems with so-called quasi-diagonal structure of the type

$$-\operatorname{div}(g(|Du|)Du) = \mu\,, \tag{4.69}$$

where $g(t) \approx t^{p-2}$. Indeed, when $\mu \equiv 0$, Uhlenbeck showed that solutions are $C^{1,\alpha}$-regular, for some α depending only on n, N, p and the ellipticity properties of $g(\cdot)$. In the recent paper [86] a rather satisfying Nonlinear Potential Theory has been established for solutions to systems as in (4.69) for the case $p \geq 2$. In particular, a suitable notion of SOLA is considered (essentially the one introduced in [45, 46]) and the whole content of Theorems 4.11.3, 4.11.4, 4.11.7 and 4.11.8 is established in the vectorial case too. The main results, for u and Du, respectively, are in the following two theorems and are stated directly for the model case given by the p-Laplacean system. Notice that we have not given the definition of local SOLA in the vectorial case, for which we refer to [86]. We just say that, with no surprise, this definition allows to recover the final estimate for SOLA from a priori estimates for energy solutions (i.e., $W^{1,p}$-regular) to systems with regular data ($\mu \in L^\infty$). So, when reading the statement of the next two theorems, the reader can think for simplicity to this last case and look at the shape of the estimates. The essence of the content will not be lost.

Theorem 4.12.1 ([86]) *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a local SOLA to the system*

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu \tag{4.70}$$

where μ is a vector-values Borel measure with finite total mass and for $p \geq 2$. Let $B_r(x_0) \subset \Omega$ be a ball. If

$$\lim_{\varrho \to 0} \frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-p}} = 0\,,$$

then u has vanishing mean oscillations at x_0, i.e.,

$$\lim_{\varrho\to 0} \fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|\,dx = 0\,.$$

Furthermore, if $\mathbf{W}^{\mu}_{1,p}(x_0,r)$ is finite, then x_0 is a Lebesgue point of u and

$$|u(x_0)-(u)_{B_r(x_0)}| \le c\mathbf{W}^{\mu}_{1,p}(x_0,r) + c\fint_{B_r(x_0)} |u-(u)_{B_r(x_0)}|\,dx \tag{4.71}$$

holds with a constant $c \equiv c(n,N,p)$. Finally, assume that

$$\lim_{\varrho\to 0} \sup_{x\in B_r(x_0)} \mathbf{W}^{\mu}_{1,p}(x,\varrho) = 0\,.$$

Then u is continuous in $B_r(x_0)$.

Notice that estimate (4.71) easily implies (4.61), and therefore gives the vector-valued analog of the scalar results of Kilpelainen and Malý [67, 68]. As for the gradient, we have the following:

Theorem 4.12.2 ([86]) *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a local SOLA to the system in (4.70), where μ is a vector-values Borel measure with finite total mass and for $p \ge 2$. Let $B_r(x_0) \subset \Omega$ be a ball. If*

$$\lim_{\varrho\to 0} \frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}} = 0\,,$$

then Du has vanishing mean oscillations at x_0, i.e.,

$$\lim_{\varrho\to 0} \fint_{B_\varrho(x_0)} |Du-(Du)_{B_\varrho(x_0)}|\,dx = 0\,.$$

Moreover, if $\mathbf{I}_1^{|\mu|}(x_0,r)$ is finite, then x_0 is a Lebesgue point of Du and

$$\begin{aligned} |Du(x_0)-(Du)_{B_r(x_0)}| &\le c\left[\mathbf{I}_1^{|\mu|}(x_0,r)\right]^{1/(p-1)} \\ &\quad + c\fint_{B_r(x_0)} |Du-(Du)_{B_r(x_0)}|\,dx \end{aligned} \tag{4.72}$$

holds for a constant $c \equiv c(n,N,p)$. Finally, if

$$\lim_{\varrho\to 0} \sup_{x\in B_r(x_0)} \mathbf{I}_1^{|\mu|}(x,\varrho) = 0$$

holds, then Du is continuous in $B_r(x_0)$.

The proof of the potential estimates and continuity criteria in the vectorial case requires approaches that do strongly differ from those employed in the scalar one. Ultimately, absence of maximum principles and related truncation techniques plays a decisive role here. The proofs involve a combination of methods from Nonlinear Potential Theory and from the partial regularity theory for elliptic systems, in turn originally stemming from Geometric Measure Theory. In particular, some of the methods of classical Uhlenbeck's paper [113] are used here; these are in turn inspired by the linearization techniques introduced by De Giorgi [40] in the setting of Geometric Measure Theory, minimal surfaces and partial regularity. These tools are then embedded in the context of measure data problems, building a bridge between two different branches of regularity theory, that is partial regularity theory and Nonlinear Potential Theory. An example of such an interaction is given by the following instrumental lemma, that could be useful in different settings.

Lemma 4.12.3 ([86]) *Let $p > 2 - 1/n$ and $w \in W^{1,p}(B_r; \mathbb{R}^N)$ satisfy*

$$\fint_{B_r} |w| \, dx \le Mr \quad \text{for some } M \ge 1 \, ,$$

where $B_r \subset \mathbb{R}^n$ is a ball with radius $r > 0$. Let

$$1 < q < \min \left\{ p, \frac{n(p-1)}{n-1} \right\} \, , \quad \varepsilon > 0 \, .$$

There exists a positive constant $\delta \equiv \delta(n, N, p, q, M, \varepsilon)$, $\delta \in (0, 1]$, such that if

$$\left| \fint_{B_r} |Dw|^{p-2} \langle Dw, D\varphi \rangle \, dx \right| \le \frac{\delta}{r} \|\varphi\|_{L^\infty(B_r)}$$

for every $\varphi \in W^{1,p}_0(B_r; \mathbb{R}^N) \cap L^\infty(B_r; \mathbb{R}^N)$, then there exists a p-harmonic map $h \in W^{1,p}(B_{r/2}; \mathbb{R}^N)$, that is a solutions to the system $-\mathrm{div}\,(|Dh|^{p-2}Dh) = 0$, satisfying

$$\left(\fint_{B_{r/2}} |Dw - Dh|^q \, dx \right)^{1/q} \le \varepsilon \, .$$

We notice that similar compactness lemmas already appear in the literature. They find their origins in the seminal work of De Giorgi on minimal surfaces, where the Laplacean operator is considered ($p = 2$). For lemmas involving the p-Laplacean operator we refer to [48, 57]. The main new and essential feature of Lemma 4.12.3 is that it allows to prove a comparison estimate via compactness methods without assuming energy bounds on u in the natural space $W^{1,p}$, but just in L^1. In other words, it works in a lower energy regime which is tailored to measure data problems, whose solutions are indeed not in the natural energy space.

We finally spend a few words on the case of general systems of the type

$$-\operatorname{div} A(Du) = \mu \ , \qquad \text{in } \Omega \subset \mathbb{R}^n \ . \tag{4.73}$$

In this case it is still possible to get results of the type of Theorem 4.11.9, but, since for such systems singularities occur anyway even when $\mu \equiv 0$, then a kind of partial regularity statement holds. Let us now describe the assumptions in detail. We consider the vector field $A\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ which is assumed to be C^1-regular and satisfying the following ellipticity and growth assumptions:

$$\begin{cases} |A(z)| + |\partial A(z)||z| \le L|z|^{p-1} \\ \nu|z|^{p-2}|\xi|^2 \le \langle \partial A(z)\xi, \xi\rangle \\ |\partial A(z_2) - \partial A(z_1)| \le L\omega\left(\dfrac{|z_2 - z_1|}{|z_1| + |z_2|}\right)(|z_1| + |z_2|)^{p-2} \\ p \ge 2 \ , \end{cases} \tag{4.74}$$

for every choice of $z, z_1, z_2, \xi \in \mathbb{R}^{N\times n}$, $|z_1| + |z_2| \neq 0$. Here $n, N \ge 2$, $0 < \nu \le L$, and $\omega : \mathbb{R}^+ \to [0,1]$ is a modulus of continuity i.e. a bounded, concave, and non-decreasing function such that $\omega(0) = 0$. As it is clear from the above assumptions, we are allowing the vector field $A(\cdot)$ to be degenerate elliptic at the origin. Specifically, we assume that $A(\cdot)$ is asymptotically close to the p-Laplacean operator at the origin in the sense that the limit

$$\lim_{t\to 0} \frac{A(tz)}{t^{p-1}} = |z|^{p-2}z \tag{4.75}$$

holds locally uniformly with respect to $z \in \mathbb{R}^{N\times n}$. This means that there exists a function $\eta\colon (0,\infty) \to (0,\infty)$ with the property

$$|z| \le \eta(s) \Longrightarrow \left|a(z) - |z|^{p-2}z\right| \le s|z|^{p-1} \quad \text{for every } z \in \mathbb{R}^{N\times n} \text{ and } s > 0 \ .$$

Finally, with $B_\varrho(x) \subset \Omega$ being a given ball, we introduce the so called excess functional defined by

$$E(Du, B_\varrho(x)) := \left(\fint_{B_\varrho(x)} \left||Du|^{p/2-1}Du - (|Du|^{p/2-1}Du)_{B_\varrho(x)}\right|^2 dy\right)^{1/2} \ ,$$

which is bound to give an integral measure of the oscillations of the gradient in $B_\varrho(x)$.

We now have the following partial regularity analog of the nonlinear Stein Theorem 4.11.9:

Theorem 4.12.4 ([85]) *Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a solution to the general system (4.73) under assumptions (4.74)–(4.75). If*

$$\mu \in L(n,1)\ ,$$

then there exists an open subset $\Omega_u \subset \Omega$ such that

$$|\Omega \setminus \Omega_u| = 0 \qquad \text{and} \qquad Du \in C^0(\Omega_u;\mathbb{R}^{N\times n})\ . \tag{4.76}$$

Moreover, there exist a positive constant ε_s and a positive radius ϱ_s, such that

$$\Omega_u = \left\{x \in \Omega : \exists\, B_\varrho(x) \Subset \Omega \text{ with } \varrho \le \varrho_s \ : E(Du, B_\varrho(x)) < \varepsilon_s\right\}\ . \tag{4.77}$$

The constant ε_s depends only on $n, N, p, \nu, L, \omega(\cdot)$ and $\eta(\cdot)$, while ϱ_s depends on the same parameters and additionally on μ.

Theorem 4.12.4 builds a bridge between the classical partial regularity theory—that prescribes to prove regularity of solutions outside closed subsets with zero measure as in (4.76)—and the nonlinear potential theory. Indeed, it rests on a series of potential estimates employing certain nonlinear potentials that also plays a role in the analysis of fully nonlinear equations [38]. We also notice that, for $\mu \equiv 0$, Theorem 4.12.4 coincides with the kind of partial regularity results obtained [49]. For more pointwise estimates we also refer to [19].

4.13 Universal Potential Estimates

In this section we want to describe an approach allowing to control oscillation properties of solutions via potentials. This fact is already partially displayed in estimates (4.71)–(4.72); here we shall present precise quantitative versions with respect to assigned moduli of continuity. The final outcome is a set of estimates, that we indeed call *universal potential estimates*, virtually allowing to describe any type of regularity of solutions via potentials. Moreover, the potential estimates in (4.61), (4.63) and (4.66) will be then recovered as special occurrences of such a new family. Again, let's start from the familiar Poisson equation (4.1). The elementary inequality

$$\begin{aligned}&\left||x_1-x_0|^{2-n}-|x_2-x_0|^{2-n}\right|\\ &\quad\lesssim \left||x_1-x_0|^{2-n-\alpha}+|x_2-x_0|^{2-n-\alpha}\right||x_1-x_2|^{\alpha}\ ,\end{aligned}$$

which is valid whenever $x_1, x_2, x_0 \in \mathbb{R}^n$, $0 \le \alpha < 1$, and estimate (4.2), then imply

$$|u(x_1) - u(x_2)| \lesssim [I_{2-\alpha}(|\mu|)(x_1) + I_{2-\alpha}(|\mu|)(x_2)]\,|x_1 - x_2|^{\alpha}$$

and

$$|Du(x_1) - Du(x_2)| \le c\,[I_{1-\alpha}(|\mu|)(x_1) + I_{1-\alpha}(|\mu|)(x_2)]\,|x_1 - x_2|^{\alpha}\,.$$

These two estimates allow to control the oscillations of u and of Du, respectively, in Hölder type spaces, via the use of potentials. Sharp nonlinear analogs of such inequalities actually hold. For this, we need a few preliminaries to fix the setting, starting by a remarkable feature of the fractional sharp operators from Definition 4.9.3.

Proposition 4.13.1 ([41]) *Let $f \in L^1(B_{2R})$; for every $\alpha \in (0,1]$ the inequality*

$$|f(x_1) - f(x_2)| \le \frac{c}{\alpha}\left[M^{\#}_{\alpha,R}(f)(x_1) + M^{\#}_{\alpha,R}(f)(x_2)\right]|x_1 - x_2|^{\alpha} \tag{4.78}$$

holds whenever $x_1, x_2 \in B_{R/4}$, for a constant c depending only on n. More precisely, x_1 and x_2 are Lebesgue points of f whenever $M^{\#}_{\alpha,R}(f)(x_1)$ and $M^{\#}_{\alpha,R}(f)(x_2)$ are finite, respectively. Therefore, whenever the right-hand side in (4.78) is finite, the values of f are defined via the precise representative as follows:

$$f(x_1) := \lim_{\varrho \searrow 0} (f)_{B_\varrho(x_1)} \in \mathbb{R} \qquad \text{and} \qquad f(x_2) := \lim_{\varrho \searrow 0} (f)_{B_\varrho(x_2)} \in \mathbb{R}\,.$$

As first realized by De Vore and Sharpley [41], the previous proposition allows to define a new, natural family of function spaces.

Definition 4.13.2 (Calderón Spaces) Let $\alpha \in (0,1]$, $q \ge 1$, $k \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be an open subset with $n \ge 2$. A measurable map $f\colon \Omega \to \mathbb{R}^k$, which is finite a.e. in Ω, belongs to the Calderón space $C^{\alpha}_{q}(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|f(x_1) - f(x_2)| \le [m(x_1) + m(x_2)]|x_1 - x_2|^{\alpha} \tag{4.79}$$

holds for almost every couple $(x_1, x_2) \in \Omega \times \Omega$.

This is just another way to say that f has "fractional derivatives". The advantage is that, with the above definition, the nonlocal character of fractional derivatives is reduced to a minimal status: only two points are considered in (4.79). Calderón spaces are closely related to the usual fractional Sobolev spaces $W^{\alpha,q}$ seen in Sect. 4.10 and the function m plays the role of a generalized fractional derivative of f of order α in the L^q-sense. Definition 4.13.2 is implicit in the work of DeVore and Sharpley [41], where, thanks to Proposition 4.13.1, the authors fix the canonical choice $m = M^{\#}_{\alpha,\infty}(f)$ from Definition 4.9.3, when f is locally integrable on $\mathbb{R}^n$. With this notion of fractional differentiability at our disposal, it is possible to give

a suitable formulation of potential estimates, controlling in a quantitative way the rate of oscillations in Hölder type spaces. To shorten the presentation we restrict to the simpler case of (4.52) and when $p \geq 2$, referring to the [76] for the subquadratic case. We first state a new estimate, involving fractional operators, and being interesting in itself.

Theorem 4.13.3 ([76, 81]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2$. Then, for every ball $B_R(x_0) \Subset \Omega$, the estimate*

$$M^{\#}_{\alpha,R}(u)(x_0) \leq c\left[\mathbf{I}^{\mu}_{p-\alpha(p-1)}(x_0,R)\right]^{1/(p-1)} + cR^{1-\alpha}\fint_{B_R(x_0)}|Du|\,dx \tag{4.80}$$

holds uniformly with respect to $\alpha \in [0,1]$, with $c \equiv c(n,p,\nu,L)$.

Applying (4.78) together with (4.80) yields a pointwise estimate on the oscillations of solutions to (4.41), that is the first universal potential estimate the title of this section is referring to.

Theorem 4.13.4 ([76, 81]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2$. The inequality*

$$\begin{aligned}|u(x_1)-u(x_2)| \leq \frac{c}{\alpha}\left[\mathbf{I}^{\mu}_{p-\alpha(p-1)}(x_1,R)+\mathbf{I}^{\mu}_{p-\alpha(p-1)}(x_2,R)\right]^{1/(p-1)}|x_1-x_2|^{\alpha}\\ +c\fint_{B_R}|u|\,dx\cdot\left(\frac{|x_1-x_2|}{R}\right)^{\alpha}\end{aligned} \tag{4.81}$$

holds uniformly in $\alpha \in [0,1]$, whenever $B_R \Subset \Omega$ is a ball and $x_1, x_2 \in B_{R/4}$, provided the right-hand side is finite. The constant c depends only on n, p, ν, L.

Estimate (4.81) gives back (4.66) when $\alpha = 1$, and extends it to the whole range of differentiability $\alpha \in (0,1]$; information deteriorates when $\alpha \to 0$. In view of Definition 4.13.2, we can interpret estimate (4.81) as

$$|\partial^{\alpha}u(x)|^{p-1} \lesssim I_{p-\alpha(p-1)}(|\mu|)(x)\,, \qquad 0<\alpha\leq 1\,.$$

Strong abuse of notation is made here, of course. The case $\alpha = 0$ is not included in Theorem 4.13.4. Indeed, the validity of (4.81) would ultimately contradict the optimality of the Wolff estimate displayed in (4.62) since

$$\left[\mathbf{I}^{\mu}_{p}(x_0,R)\right]^{1/(p-1)} \lesssim \mathbf{W}^{\mu}_{1,p}(x_0,2R)\,, \qquad \text{for } p\geq 2\,.$$

On the other, using Wolff potentials we can prove an estimate that works uniformly in the whole range $\alpha \in [0,1]$; this includes both the case $\alpha = 0$ (that is (4.61)) and

recovers (4.63) when $\alpha = 1$ (which is slightly less sharp than (4.66)). This is the content of

Theorem 4.13.5 ([76]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2$. The inequality*

$$|u(x_1) - u(x_2)| \leq c\left[\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x_1, R) + \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x_2, R)\right]|x_1 - x_2|^{\alpha} + c\fint_{B_R} |u|\,dx \cdot \left(\frac{|x_1 - x_2|}{R}\right)^{\alpha} \tag{4.82}$$

holds uniformly with respect to $\alpha \in [0, 1]$, whenever $B_R \Subset \Omega$ is a ball and $x_1, x_2 \in B_{R/4}$, provided the right-hand side is finite. The constant c depends only on n, p, ν, L.

The main use of estimates (4.81) and (4.82) is rather clear. In order to get regularity of solutions in spaces measuring the oscillations of functions, like Hölder spaces $C^{0,\alpha}$, fractional Sobolev spaces $W^{\alpha,q}$, and Calderón spaces C^{α}_{q}, it is sufficient to check the relevant mapping properties of potentials. This in fact encodes regularity theory via estimates as (4.81) and (4.82).

In view of the available gradient regularity theory for solutions to homogeneous equations as

$$\operatorname{div} A(Dw) = 0\,, \tag{4.83}$$

it is then natural to wonder if similar universal potential estimates hold for the oscillations of the gradient too. For this let us recall the basic information about the maximal regularity of $W^{1,p}$-solutions w to equations as (4.83). This theory goes back to the fundamental contribution of Ural'tseva [114]; see also [113] for the vectorial case. The outcome is the existence of a positive exponent $\alpha_M \in (0, 1)$, depending only on n, p, ν and L, such that $Dw \in C^{0,\alpha}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ takes places for every $\alpha < \alpha_M$ and

$$|Dw(x_1) - Dw(x_2)| \lesssim \fint_{B_R} |Dw|\,dx \cdot \left(\frac{|x_1 - x_2|}{R}\right)^{\alpha}$$

holds for every ball $B_R \Subset \Omega$, with $x_1, x_2 \in B_{R/2}$. Accordingly to this last inequality, in the non-homogeneous case we have

Theorem 4.13.6 ([76]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to the equation in (4.52) in the sense of Definition 4.8.2, under assumptions (4.53) with $p \geq 2$. Fix $\tilde{\alpha} < \min\{1/(p-1), \alpha_M\}$, then the inequality*

$$\begin{aligned} &|Du(x_1) - Du(x_2)| \\ &\quad \leq c\left[\mathbf{W}^{\mu}_{1-\frac{(1+\alpha)(p-1)}{p},p}(x_1, R) + \mathbf{W}^{\mu}_{1-\frac{(1+\alpha)(p-1)}{p},p}(x_2, R)\right]|x_1 - x_2|^{\alpha} \\ &\qquad + c\fint_{B_R} |Du - (Du)_{B_R}|\,dx \cdot \left(\frac{|x_1 - x_2|}{R}\right)^{\alpha} \end{aligned} \tag{4.84}$$

holds whenever $B_R \Subset \Omega$ *and* $x_1, x_2 \in B_{R/4}$, *and* $\alpha \in [0, \tilde{\alpha}]$, *provided the right-hand side is finite. The constant* c *depends only on* n, p, ν, L *and* $\tilde{\alpha}$.
Notice that estimate (4.84) catches-up the one in (4.63) "from above" (let $\alpha = 0$).

4.14 Parabolic Equations

We finally conclude with some results valid for parabolic problems, confining ourselves to a few significant cases. Proving potential estimates for solutions to nonlinear parabolic equations of the type

$$u_t - \operatorname{div} A(Du) = \mu \qquad \text{in } \Omega_T := \Omega \times (-T, 0) \subset \mathbb{R}^{n+1} \tag{4.85}$$

requires additional new ideas and a wealth of new technicalities; the assumptions considered for the vector field $A(\cdot)$ here are those considered in (4.53). As usual, $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega_T)$ denotes a Borel measure with locally finite total mass in Ω_T. For basic existence and regularity results concerning (4.85), we refer to [8, 16].

We shall treat the case of gradient potential estimates and we shall specialize to the case $p \geq 2$. When $p = 2$ gradient potential estimates have first been derived in [53], and in this case the approach is similar to the one used for the elliptic case. The situation drastically changes when $p \neq 2$. The basic idea here is to use the potential theoretic approach developed for the elliptic case together with the fundamental concept of intrinsic geometry introduced by DiBenedetto [43] and already described in Sect. 4.5. The key new point is to use the intrinsic cylinders defined in (4.20) to build a new type of "intrinsic potentials". These are (intrinsic) caloric Riesz potentials of the type

$$\mathbf{I}^{\mu}_{\beta,\lambda}(x_0, t_0; R) := \int_0^R \frac{|\mu|(Q^{\lambda}_{\varrho}(x_0, t_0))}{\varrho^{N-\beta}} \frac{d\varrho}{\varrho}, \tag{4.86}$$

where at the moment $\lambda > 0$ is just a free parameter, that will be eventually linked to a solution. Here $N := n + 2$ is the usual parabolic dimension. For $\lambda = 1$ or for $p = 2$, those in (4.86) reduce to the caloric Riesz potentials already used in [53] to get parabolic potential estimates when $p = 2$, and built using the standard parabolic cylinders, i.e.,

$$\mathbf{I}^{\mu}_{\beta}(x_0, t_0; R) := \int_0^R \frac{|\mu|(Q_{\varrho}(x_0, t_0))}{\varrho^{N-\beta}} \frac{d\varrho}{\varrho}. \tag{4.87}$$

The approach to parabolic potential estimates via intrinsic caloric potentials has been introduced and carried out in [79, 80, 82], with some anticipations in [75]. For the sake of simplicity we shall report results in the form of a priori estimates for

energy solutions. We therefore consider energy solutions

$$u \in C^0(-T,0;L^2_{\mathrm{loc}}(\Omega)) \cap L^p(-T,0;W^{1,p}_{\mathrm{loc}}(\Omega))$$

to parabolic equations of the type (4.85). From this it is then not difficult to get estimates for solutions to measure data parabolic problems when considering the natural definition of SOLA, for which we again refer to [80]. The following intrinsic Riesz potential bound for the (spatial) gradient now takes place:

Theorem 4.14.1 ([80, 82]) *Let u be an energy solution to (4.85), under assumptions (4.53) with $p \geq 2$. There exists a constant $c > 1$, depending only on n, p, ν, L, such that the implication*

$$\begin{aligned} &c\mathbf{I}^{\mu}_{1,\lambda}(x_0,t_0;R) + c\left(\fint_{Q^{\lambda}_R(x_0,t_0)} |Du|^{p-1}\,dx\,dt\right)^{1/(p-1)} \leq \lambda \\ &\Longrightarrow |Du(x_0,t_0)| \leq \lambda \end{aligned}$$

holds whenever $Q^{\lambda}_R(x_0,t_0) \Subset \Omega_T$ and (x_0,t_0) is Lebesgue point of Du.

This last estimate, which is homogeneous and actually a conditional one, then implies a bound on general standard parabolic cylinders, and making use of the standard caloric Riesz potential defined in (4.87).

Theorem 4.14.2 ([80, 82]) *Let u be an energy solution to (4.85), under assumptions (4.53) with $p \geq 2$. There exists a constant c, depending only on n, p, ν, L, such that*

$$|Du(x_0,t_0)| \leq c\mathbf{I}^{\mu}_1(x_0,t_0;R) + c\fint_{Q_R(x_0,t_0)} (|Du|+1)^{p-1}\,dx\,dt$$

holds whenever $Q_R(x_0,t_0) \Subset \Omega_T$ is a standard parabolic cylinder and (x_0,t_0) is Lebesgue point of Du.

The proof of Theorem 4.14.1 opens the way to an optimal continuity criterion for the gradient that involves classical (caloric) Riesz potentials and that, as such, is again independent of p.

Theorem 4.14.3 ([80]) *Let u be an energy solution to (4.85), under assumptions (4.53) with $p \geq 2$. If*

$$\lim_{\varrho \searrow 0} \mathbf{I}^{\mu}_1(x,t;\varrho) = 0 \ \textit{holds locally uniformly in } \Omega \textit{ w.r.t. } (x,t) \in \Omega_T\,,$$

then Du is continuous in Ω_T.

For the results in the case $p < 2$ we refer to [79]. See [84] for systems.

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Chapter 5
The Mathematical Theories of Diffusion: Nonlinear and Fractional Diffusion

Juan Luis Vázquez

Abstract We describe the mathematical theory of diffusion and heat transport with a view to including some of the main directions of recent research. The linear heat equation is the basic mathematical model that has been thoroughly studied in the last two centuries. It was followed by the theory of parabolic equations of different types. In a parallel development, the theory of stochastic partial differential equations gives a foundation to the probabilistic study of diffusion.

Nonlinear diffusion equations have played an important role not only in theory but also in physics and engineering, and we focus on a relevant aspect, the existence and propagation of free boundaries. Due to our research, we use the porous medium and fast diffusion equations as case examples.

A large part of the paper is devoted to diffusion driven by fractional Laplacian operators and other nonlocal integro-differential operators representing nonlocal, long-range diffusion effects. Three main models are examined (one linear, two nonlinear), and we report on recent progress in which the author is involved.

5.1 Introduction to Diffusion

There are a number of phenomena in the physical sciences that we associate with the idea of diffusion. Thus, populations of different kinds diffuse; particles in a solvent and other substances diffuse. Besides, heat propagates according to a process that is mathematically similar, and this is a major topic in applied science. We find many other instances of diffusion: electrons and ions diffuse; the momentum of a viscous fluid diffuses (in a linear way, if we are dealing with a Newtonian fluid). More recently, we even talk about diffusion in the financial markets.

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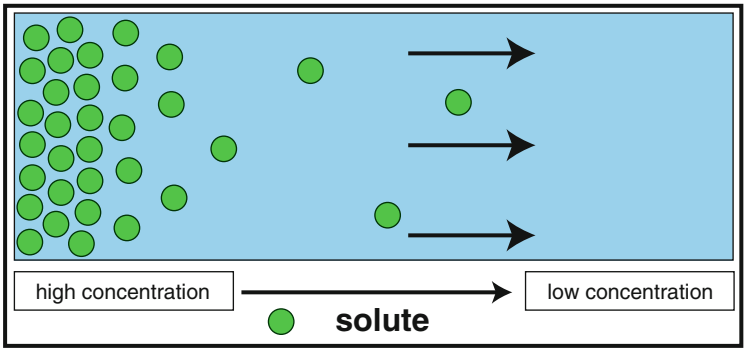
The word diffusion derives from the Latin *diffundere*, which means "to spread out". A substance spreads out by moving from an area of high concentration to an area of low concentration. This mixing behaviour does not require any bulk motion, a feature that separates diffusion from other transport phenomena like convection, or advection.



Source: Figure obtained from Google

This description can be found in *Wikipedia* [265], where we can also find a longer listing of more than 20 diffusive items, that are then developed as separate subjects. It includes all classical topics we mention here.

5.1.1 Diffusion in Mathematics

In this survey paper we want to present different topics of current interest in the mathematical theory of diffusion in a historical context. To begin with, we may ask if mathematics is really relevant in the study of diffusion process? The answer is that diffusion is a topic that enjoys superb mathematical modelling. It is a branch of the natural sciences that is now firmly tied to a number of mathematical theories that explain its working mechanism in a quite successful way. The quantity that diffuses can be a concentration, heat, momentum, information, ideas, a price,... every such process can be called a diffusion, and its evolution is governed by mathematical analysis.

Going into the details of how we actually explain diffusion with mathematics, it so happens that we may do it in a twofold way: roughly speaking, by means of the diffusion equation and its relatives, or by a random walk model and its relatives.

The older work concerns the description of heat propagation and mass diffusion by means of partial differential equations (PDEs), and this is the view that we are going to favor here. The type of PDEs used is the so-called parabolic equations, a family based on the properties of the most classical model, the linear Heat

Equation (HE), which is called in this context the diffusion equation. probabilistic On the other hand, according to probabilists diffusion is described by random walks, Brownian motion, and more generally, by stochastic processes, and this is a long and successful story in twentieth century mathematics, culminated by Itō's calculus. Let us recall that the connection between the two visions owes much to A.N. Kolmogorov.

Actually, an interesting question for the reader or the expert is 'How much of the mathematics of diffusion can be explained with *linear models,* how much is *essentially nonlinear*?' Linear models have priority when applicable by virtue of their rich theory and easier computation. But nonlinear models are absolutely necessary in many real-world contexts and most of our personal research has been based on them. Diffusion equations involving nonlinearities and/or nonlocal operators representing long-range interactions are the subject matter of the recent research that we want to report in this paper.

Outline of the Paper The declared intention is to make a fair presentation of the main aspects of Mathematical Diffusion as seen by an expert in PDEs, and then to devote preference to the work done by the author and collaborators, especially the more recent work that deals with free boundaries and with fractional operators. The main topics are therefore the heat equation, the linear parabolic theory and the fractional diffusion in a first block; the nonlinear models, with emphasis on those involving free boundaries come next; finally, the nonlocal and nonlinear models, and here we will focus on the two equations that have been most studied by the author in the last decade, both combine porous medium nonlinearities and fractional diffusion operators.

Lengthy details are not frequent, but we give some for very recent work of ours and our collaborators. On the other hand, we supply many important explicit solutions and comment on their role. Indeed, many such examples belong to the class *stable diffusive patterns*, that combine their surprising occurrence in numerous real-world applications with the beauty of pure mathematics.

A large number of connections with other topics is given in the text, as well as hints for further reading. More detail on the topics is to be found in the articles, monographs, or in our previous survey papers.

Disclaimer Let us comment on an important absence. The stationary states of diffusion belong to an important world, the *elliptic equations*. Elliptic equations, linear and nonlinear, appear in a large number of applications: diffusion, fluid mechanics, waves of all types, quantum mechanics, ... Elliptic equations are mathematically based on the Laplacian operator, $\Delta = \nabla^2$, the most important operator for our community. This is a huge world. We are not going to cover in any detail the many developments in elliptic equations related to diffusion in this paper, we will just indicate some important facts and connections here and there.

5.1.2 *Heat Equation: Main Model for Diffusion*

We begin our presentation with the linear heat equation (HE):

$$\boxed{u_t = \Delta u}$$

proposed by J. Fourier as a mathematical model for heat propagation ("Théorie Analytique de la Chaleur", 1822, [128], with a previous attempt in 1807), and the Fourier analysis that he promoted. For a long time the mathematical study of heat transport and diffusion was almost exclusively centered on the heat equation. In these two centuries, the mathematical models of heat propagation and diffusion have made great progress both in theory and application. Actually, they have had a strong influence on no less than six areas of Mathematics: PDEs, Functional Analysis, Infinite-Dimensional Dynamical Systems, Differential Geometry and Probability, as well as Numerics. And the theory has been influenced by its motivation from Physics, and in turn the concepts and methods derived from it have strongly influenced Physics and Engineering. In more recent times this influence has spread further away, to Biology, Economics, and the Social Sciences.

- The classical analysis of the heat flow is based on two main mathematical techniques: *integral representation* (convolution with a Gaussian kernel) and *Fourier analysis*, based on mode separation, analysis, and synthesis. Since this topic is well-known to the readers, see for instance [120], we will stress the points that interest us to fix some concepts and tools.

1. **The heat equation semigroup and Gauss**. When heat propagates in *free space* $\mathbb{R}^N$, the natural problem is the initial value problem

$$u_t = \Delta u, \qquad u(x,0) = f(x)\,, \tag{5.1}$$

which is solved by convolution with the evolution version of the Gaussian function

$$G(x,t) = (4\pi t)^{-N/2}\exp\left(-|x|^2/4t\right). \tag{5.2}$$

Note that G has all nice analytical properties for $t > 0$, but for $t = 0$ we have $G(x,0) = \delta(x)$, a Dirac mass. G works as a *kernel*, a mathematical idea that goes back to Green and Gauss.

The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot,t)$ form a *linear continuous semigroup* of contractions in all L^p spaces for all $1 \le p \le \infty$. This is pure Functional Analysis, a product of the twentieth century.

Asymptotic Behaviour as $t \to \infty$, Convergence to the Gaussian If u_0 is an integrable function and $M = \int u_0(x)\,dx \neq 0$, the following convergence is

proved

$$\lim_{t\to\infty} t^{N/2}(u(x,t) - M\,G(x,t)) = 0\,, \tag{5.3}$$

and the limit holds uniformly in the whole space. For convergence in L^p less is needed. So $u(\cdot,t)$ increasingly resembles (i.e. as t grows to infinity) a multiple of the Gaussian profile $G(\cdot,t)$. This is the famous *Central Limit Theorem* in its continuous form (famous in Probability with $M = 1$, but $M \neq 1$ makes no difference as long as M is not zero).

The Gaussian function is the most famous example of the many diffusive patterns that we will encounter, and the previous theorem shows that is not only stable but also asymptotic attractor of the heat flow (for finite-mass data). Note that the sharp convergence result needs renormalization in the form of the growth factor $t^{N/2}$.

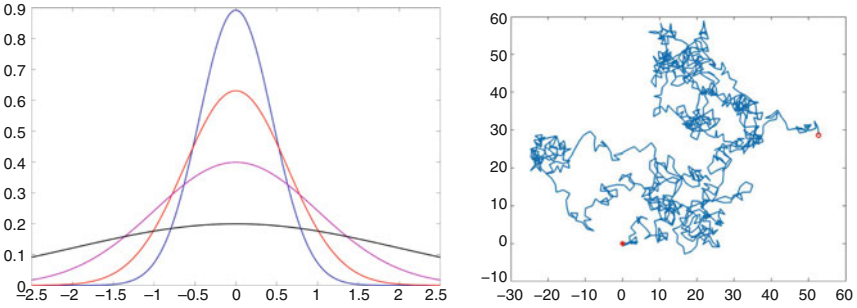
- These are two classical personalities of the diffusion equation.

J. Fourier and K.F. Gauss. *Source*: Figures obtained from Google and personal sources

2. **Matter diffusion**. This is an older subject in Physics, already treated by Robert Boyle in the seventeenth century with the study of diffusion in solids. After the work of Fourier in heat propagation, Adolf Fick proposed his law of diffusion of matter [126], where the mass flux is proportional to the gradient of concentration and goes in the direction of lower concentrations. This leads to the heat equation, HE, as mathematical model. He also pointed out the fundamental analogy between diffusion, conduction of heat, and also electricity. Actually, Fourier's law for heat conduction (1822), Ohm's law for electric current (1827), Fick's law for diffusion in solids (1855), and Darcy's [98] law for hydraulic flow (1856) have a similar mathematical gradient form.
3. **The connection with Probability**. The time iteration of independent random variables with the same distribution led to the theory of random walks, at the early times of the Bernoullis et al. Soon it was realized that this led to a large-time limit described (after renormalization) by the Gaussian distribution,

as textbooks in Probability and Stochastic Processes show. The connection of this evolution with the heat equation took place after the construction of the **Brownian motion** as a mathematically rigorous object in the form of *Wiener process*. In the 1930s Kolmogorov investigated the equivalence of the two view points, i.e., the PDE approach via the Heat Equation and the stochastic approach via Brownian motion. This topic is covered by many PDE authors, let us mention [122, 221].

Next we show two opposing diffusion graphs. They show the comparison of ordered dissipation in the heat equation view, as the spread of a temperature or concentration, versus the underlying chaos of the random walk particle approach, origin of the Brownian motion favoured by the probabilistic school.



Left, the nice HE evolution of a Gaussian. *Right*, a sample of random walk. *Source*: own sources

We will go back to the latter view in a while. The experimental observation of chaotic movement in Nature due to mechanical effects at the microscopic level is credited to Robert Brown (1827), see [57], hence the label 'Brownian motion'.

4. **The Fourier Analysis approach to heat flows in bounded domains**. The second classical scenario for heat flows occurs when heat propagates inside a bounded domain of space. The convolution approach does not work and other ideas have to be proposed. The Fourier approach proposes to look for a solution in the series form

$$u(x,t) = \sum T_i(t) X_i(x)\,, \tag{5.4}$$

and then the time factors are easily seen to be negative exponentials of t, while the space components $X_i(x)$ form the spectral sequence, solutions of the problems

$$\boxed{-\Delta X_i = \lambda_i X_i} \tag{5.5}$$

with corresponding eigenvalues λ_i. Boundary values are needed to identify the *spectral sequence* (λ_i, X_i), $i = 1, 2, \ldots$ This is the famous linear *eigenvalue*

problem, the starting point of the discipline of *Spectral Theory*. The time-space coupling implies then that $T_i(t) = e^{-\lambda_i t}$. This is nowadays one of the most celebrated and useful topics in Applied Mathematics and is covered in all elementary PDE books.

The scheme works for many other equations of the form $u_t = A(u)$ and in this way Fourier Analysis and Spectral Theory developed with great impetus, as well as Semigroup Theory. Fourier analysis also took a direction towards the delicate study of functions, a proper branch of pure mathematics, which is one of the most brilliant developments in the last two centuries. Through the work of Cantor this also motivated advances in Set Theory, since the sets of points where a Fourier series does not converge can be quite complicated.

5.1.3 Linear Heat Flows

We now consider a big step forward in the mathematical tools of diffusion. A more general family of models was introduced to represent diffusive phenomena under less idealized circumstances and this was done both in the framework of PDEs and Probability. This happened in several stages.

In the framework of PDEs, the heat equation has motivated the study of other linear equations, which now form the Theory of Linear Parabolic Equations. They are written in the form

$$\boxed{u_t = \sum_{i,j} a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f} \tag{5.6}$$

with variable coefficients $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t)$, and forcing term $f(x,t)$. Belonging to the parabolic family requires some *structure conditions* on such coefficients that will ensure that the solutions keep the basic properties of the heat equation. In practice, the main condition is the fact that the matrix $a_{ij}(x,t)$ has to be definite positive: there exists $\lambda > 0$

$$\sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \geq \lambda \sum_i \xi_i^2 . \tag{5.7}$$

This must be valid for all vectors $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and all x, t in the space-time domain of the problem. Let us point out that the theory is developed under some additional conditions of regularity or size on the coefficients, a common feature of all PDE analysis. Another prominent feature is that all coefficients can be submitted to different more or less stringent conditions that allow to obtain more or less regular solutions.

A more stringent uniform positivity condition is

$$\lambda \sum_i \xi_i^2 \le \sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \le \Lambda \sum_i \xi_i^2 \tag{5.8}$$

with $0 < \lambda < \Lambda$. Even the uniform condition (5.7) can be relaxed so that $\lambda > 0$ depends on x and t. When these conditions are relaxed we talk about degenerate or singular parabolic equations. This flexibility on the structure conditions makes for a big theory that looks like an ocean of results. It will be important later in the nonlinear models.

In fact, the parabolic theory was developed in the sequel of its more famous stationary counterpart, the theory for the elliptic equations,

$$\sum_{i,j} a_{ij}\partial_i\partial_j u + \sum_i b_i\partial_i u + c\,u + f = 0, \tag{5.9}$$

with variable coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ and forcing term $f(x)$. The main structure condition is again (5.7) or (5.8), which is usually called uniform ellipticity condition. In the time dependent case, (5.7) and (5.8) are called uniform parabolicity conditions.

Main steps in the Parabolic Theory are:

1. The first step is the classical parabolic theory in which a_{ij}, b_i, c, f are assumed to be continuous or smooth, as needed. Functional spaces are needed as framework of the theory, and these turn out to be C^α spaces (Hölder) and the derived spaces $C^{1,\alpha}$ and $C^{2,\alpha}$. This leads to a well-known theory in which existence and uniqueness results, continuous dependence on data are obtained after adding initial and boundary data to the problem. And the theory provided us with Maximum Principles, Schauder estimates, Harnack inequalities and other very precise estimates.

   This is a line of research that we would like to follow in all subsequent chapters when further models of diffusion will be treated, but unfortunately the direct application of the scheme will not work, and to be more precise, the functional setting will not be conserved and the techniques will change in a strong way.

   The results are extended into the disciplines of Potential Theory and Generation of Semigroups. These are also topics that will be pursued in the subsequent investigations.
2. A first extension of the classical parabolic theory concerns the case where the *coefficients are only continuous or bounded*. In the theory with bad coefficients there appears a bifurcation of the theory into Divergence and Non-Divergence Equations, which are developed with similar goals but quite different technology. The difference concerns the way of writing the first term with second-order derivatives.

The way stated before is called non-divergent form, while the divergent form is

$$\sum_{i,j} \partial_i(a_{ij}\partial_j u) + \sum b_i \partial_i u + cu + f = 0. \tag{5.10}$$

This form appears naturally in many of the derivations from physical principles. About the structure conditions, we assume the a_{ij} to be bounded and satisfy the uniform parabolicity condition. The basic functional spaces are the Lebesgue and Sobolev classes, L^p, $W^{1,p}$. Derivatives are understood as distributions or more often weak derivatives, and this motivates the label of *weak theory*. Existence, uniqueness and estimates in $W^{1,p}$ or $W^{2,p}$ norms are produced. Maximum Principles, Harnack inequalities work and C^α is often proved. A very important feature is the Calderón-Zygmund theory, basic to establish regularity in Sobolev spaces. Divergent form equations were much studied because of their appearance in problems of Science and Engineering. We refer to the books [139, 184] for the elliptic theory, and to [129, 185, 191] for the parabolic theory.

3. There is nowadays a very flourishing theory of elliptic and parabolic equations with bad coefficients in the non-divergence form (5.6), but we will not enter into it for reasons of space, since it does not affect the rest of our exposé.

5.1.4 The Stochastic Approach: SPDEs

The probabilistic way to address the previous field enlargement appears in the form of the *diffusion process*, which is a solution to a stochastic differential equation, SPDE. A diffusion is then a continuous-time Markov process with almost surely continuous sample paths. This is essentially a twentieth century theory, originated in the work of Bachelier, Einstein, Smoluchowski, then Kolmogorov, Wiener and Levy, and the last crucial step was contributed by Itō, Skorokhod, ... The stochastic equation reads

$$\boxed{dX = b\,dt + \sigma\,dW} \tag{5.11}$$

where W is the N-dimensional Wiener process, and b and σ are (vector and matrix valued respectively) coefficients under suitable conditions. In particular, σ must be a uniformly elliptic matrix. Derivatives are understood in the sense of Itō. Among the extensive literature we mention Bass [25], Friedman [130], Gihman-Skorohod [138] and Varadhan [244] for the relation between PDEs and Stochastic processes. Thus, Bass discusses the solutions of linear elliptic and parabolic problems by means of Stochastic processes in Chapter II. The stochastic equation (5.11) gives a formula

to solve the Cauchy problem for the evolution PDE $u_t = \mathcal{L}u$, where

$$\mathcal{L}u = \sum a_{ij}\partial_{ij}u + \sum b_i \partial_i u,$$

if the vector $b = (b_1, \dots, b_N)$ and the symmetric positive semi-definite matrix $a = (a_{ij})$ is given by $a = \sigma \cdot \sigma^T$. We call the functions σ and a the diffusion coefficients of the process X_t and $\mathcal{L}$, respectively, while b is the drift vector. Let us also mention [122, 221, 231].

A Comment About Real-World Practice If we consider a field of practical application like quantitative finance, one may ask the question about which of the two known approaches—PDEs versus martingales and SPDEs—is more important in the real practice of derivatives pricing. Here is a partial answer: since the Black-Scholes equation is a modified form of the Heat Equation, understanding PDEs is very important as a practical tool, see [266]. And the American options add a free boundary problem, a topic that we will find later in the text.

5.2 Fractional Diffusion

Replacing the Laplacian operator by fractional Laplacians is motivated by the need to represent processes involving anomalous diffusion. In probabilistic terms, it features long-distance interactions instead of the next-neighbour interaction of random walks and the short-distance interactions of their limit, the Brownian motion. The main mathematical models used to describe such processes are the fractional Laplacian operators, since they have special symmetry and invariance properties that makes for a richer theory. These operators are generators of stable Lévy processes that include jumps and long-distance interactions. They reasonably account for observed anomalous diffusion, with applications in continuum mechanics (elasticity, crystal dislocation, geostrophic flows,...), phase transition phenomena, population dynamics, optimal control, image processing, game theory, finance, and others. See [10, 28, 91, 140, 199, 200, 262, 267], see also Sect. 1.2 of [253].

After a very active period of work on problems involving nonlocal operators, there is now well established theory in a number of directions, like semilinear equations and obstacle problems, mainly of stationary type. We are interested here in evolution problems. Instead of the Heat Equation, the basic evolution equation is now

$$\boxed{u_t + (-\Delta)^s u = 0} \tag{5.12}$$

There has been intense work in Stochastic Processes for some decades on this equation, but not in Analysis of PDEs. My interest in the field dates from the

year 2007 in Texas in collaboration with Prof. Luis Caffarelli, who was one of the initiators, specially in problems related to nonlinear diffusion and free boundaries.

It is known that there is well defined semigroup associated with this equation for every $0 < s < 1$ that solves Cauchy problem (5.12) in the whole space or the typical initial and boundary value problems in a bounded domain, see more below. Though in the limit $s \to 1$ the standard heat equation is recovered, there is a big difference between the local operator $-\Delta$ that appears in the classical heat equation and represents Brownian motion on one side, and the nonlocal family $(-\Delta)^s$, $0 < s < 1$, on the other side. In the rest of the paper we are going to discuss some of those differences, both for linear and nonlinear evolution equations. We have commented on the origins and applications of the fractional Laplacian and other nonlocal diffusive operators in our previous survey papers [252, 253].

5.2.1 *Versions of the Fractional Laplacian Operator*

Before proceeding with the study of equations, let us examine the different approaches and defining formulas for the fractional Laplacian operator. We assume that the space variable $x \in \mathbb{R}^N$, and the fractional exponent is $0 < s < 1$.

- Fourier Approach. First, we may consider the pseudo-differential operator given by the Fourier transform:

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)\,. \tag{5.13}$$

  This allows to use the very rich theory of Fourier transforms, but is not very convenient for nonlinear analysis which is our final goal. Due to its symbol $|\xi|^{2s}$, the fractional Laplacian can be viewed as a symmetric differentiation operator of fractional order $2s$. Even when $2s = 1$, it is not the standard first derivative, just compare the Fourier symbols.
- Hyper-Singular Integral Operator. The formula reads

$$(-\Delta)^s u(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy\,. \tag{5.14}$$

  The kernel is not integrable near x and this motivates the need for the difference in the numerator of the integrand. The integral is understood as principal value. With this definition, the operator is the inverse of the Riesz integral operator $(-\Delta)^{-s}u$, which has a more regular kernel $C_1|x-y|^{N-2s}$, though not integrable at infinity. The fractional Laplacian operator is also called the *Riesz derivative*.
- Numerics and Stochastic Approach. Take the random walk for a processes with probability u_j^n at the site x_j at time t_n:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n\,, \tag{5.15}$$

where $\{P_{jk}\}$ denotes the transition function which has a *fat tail* (i.e., a power decay with the distance $|j-k|$), in contrast to the next-neighbour interaction of random walks. In a suitable limit of the space-time grid you get an operator A as the infinitesimal generator of a Lévy process: if X_t is the isotropic α-stable Lévy process we have

$$Au(x) = \lim_{h\to 0} \frac{1}{h}\mathbb{E}(u(x) - u(x + X_h)). \tag{5.16}$$

The set of functions for which the limit on the right side exists (for all x) is called the domain of the operator. We arrive at the fractional Laplacian with an exponent $s \in (0, 1)$ that depends on the space decay rate of the interaction $|j-k|^{-(N+2s)}$, $0 < s < 1$.

- The Caffarelli-Silvestre Extension. The α-harmonic extension: Find first the solution of the $(N + 1)$-dimensional elliptic problem

$$\nabla \cdot (y^{1-\alpha}\nabla U) = 0 \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}_+; \quad U(x, 0) = u(x), \quad x \in \mathbb{R}^N. \tag{5.17}$$

The equation is degenerate elliptic but the weight belongs to the Muckenhoupt A_2 class, for which a theory exists [123]. We may call U the extended field. Then, putting $\alpha = 2s$ we have

$$(-\Delta)^s u(x) = -C_\alpha \lim_{y\to 0} y^{1-\alpha}\frac{\partial U}{\partial y}\,. \tag{5.18}$$

When $s = 1/2$, i.e. $\alpha = 1$, the extended function U is harmonic (in $N + 1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^N$. The general extension was proposed in PDEs by Caffarelli and Silvestre [67], see also [227]. This construction is generalized to other differential operators, like the harmonic oscillator, by Stinga and Torrea, [240].

- Semigroup Approach. It uses the following formula in terms of the heat flow generated by the Laplacian Δ:

$$(-\Delta)^s f(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta}f(x) - f(x)\right)\frac{dt}{t^{1+s}}. \tag{5.19}$$

Classical references for analysis background on the fractional Laplacian operator in the whole space: the books by Landkof [187], Stein [239], and Davies [102]. The recent monograph by Bucur and Valdinoci [58] introduces fractional operators and more generally nonlocal diffusion, and then goes on to study a number of stationary problems. Numerical methods to calculate the fractional Laplacian are studied e.g. in [213].

5.2.1.1 Fractional Laplacians on Bounded Domains

All the previous versions are equivalent when the operator acts in $\mathbb{R}^N$. However, in order to work in a bounded domain $\Omega \subset \mathbb{R}^N$ we will have to re-examine all of them. For instance, using the Fourier transform makes no sense. Two main efficient alternatives are studied in probability and PDEs, corresponding to different way in which the information coming from the boundary and the complement of the domain is to be taken into account. They are called the restricted fractional Laplacian (RFL) and the spectral fractional Laplacian (SFL), and they are carefully defined in Sect. 5.8. And there are more alternatives that we will also discuss there.

5.2.2 *Mathematical Theory of the Fractional Heat Equation*

The basic linear problem is to find a solution $u(x,t)$ of

$$u_t + (-\Delta)^s u = 0\,, \qquad 0 < s < 1\,. \tag{5.20}$$

We will take $x \in \mathbb{R}^N$, $0 < t < \infty$, with initial data $u_0(x)$ defined for $x \in \mathbb{R}^N$. Normally, $u_0, u \geq 0$, but this is not necessary for the mathematical analysis. We recall that this model represents the linear flow generated by the so-called Lévy processes in Stochastic PDEs, where the transition from one site x_j of the mesh to another site x_k has a probability that depends on the distance $|x_k - x_j|$ in the form of an inverse power for $j \neq k$, more precisely, $c\,|x_k - x_j|^{-N-2s}$. The range is $0 < s < 1$. The limit from random walk on a discrete grid to the continuous equation can be read e. g. in Valdinoci's [243].

The solution of the linear equation can be obtained in $\mathbb{R}^N$ by means of convolution with the fractional heat kernel

$$u(x,t) = \int u_0(y) P_t(x-y)\, dy, \tag{5.21}$$

and the probabilists Blumental and Getoor proved in the 1960s [36], that

$$P_t(x) \asymp \frac{t}{\left(t^{1/s} + |x|^2\right)^{(N+2s)/2}}\,. \tag{5.22}$$

Here $a \asymp b$ means that a/b is uniformly bounded above and below by a constant. Only in the case $s = 1/2$ the kernel is known to be explicit, given precisely by the previous formula up to a constant. Note the marked difference with the Gaussian kernel G_t of the heat equation (case $s = 1$). The behaviour as x goes to infinity of the function P_t is power-like (with a so-called *fat tail*) while G_t has exponential spatial decay, see (5.2). This difference is expected in a theory of long-distance interactions. See more on this issue in [169].

Rather elementary analysis allows then to show that the convolution formula generates a contraction semigroup in all $L^p(\mathbb{R}^N)$ spaces, $1 \le p \le \infty$, with regularizing formulas of the expected type

$$\|u(t)\|_\infty \le C(N,s,p)\,\|u_0\|_p\; t^{-N/2sp}\,.$$

When the data and solutions are not assumed to be Lebesgue integrable, interesting questions appear. Such questions have been solved for the classical heat equation, where it is well-known that solutions exist for quite large initial data, more precisely data with square-quadratic growth as $|x| \to \infty$, see Widder [264]. The idea is that the convolution formula (5.21) still makes sense and can be conveniently manipulated.

Likewise, we may study the fractional heat equation in classes of (maybe) large functions and pose the question: given a solution of the initial value problem posed in the whole space $\mathbb{R}^N$, is it representable by the convolution formula? The paper [24] by Barrios et al., shows that the answer is yes if the solutions are suitable strong solutions of the initial value problem posed in the whole space $\mathbb{R}^N$, they are nonnegative, and the growth in x is no more that $u(x,t) \le (1+|x|)^a$ with $a < 2s$.

In the recent paper [51] by Bonforte, Sire, and the author, we look for optimal criteria. We pose the problem of existence, uniqueness and regularity of solutions for the same initial value problem in full generality. The optimal class of initial data turns out to be the class of locally finite Radon measures μ satisfying the condition

$$\int_{\mathbb{R}^N} (1+|x|)^{-(N+2s)}\, d\mu(x) < \infty\,. \tag{5.23}$$

We call this class $\mathcal{M}_s$. We construct weak solutions for such data, and we prove uniqueness of nonnegative weak solutions with nonnegative measure data. More precisely, we prove that there is an equivalence between nonnegative measure data in that class and nonnegative weak solutions, which is given in one direction by the representation formula, in the other one by the existence of an initial trace. So the result closes the problem of the Widder theory for the fractional heat equation posed in $\mathbb{R}^N$. We then review many of the typical properties of the solutions, in particular we prove optimal pointwise estimates and new Harnack inequalities. Asymptotic decay estimates are also found for the optimal class. Here is the general result in that direction. We want to estimate the behaviour of the constructed solution $u = P^t * \mu_0$ for $t > 0$ and prove that it is a locally bounded function of x and t with precise estimates. Here is the main result.

Theorem 5.2.1 *Let $u = S_t\mu_0$ the very weak solution with initial measure $\mu_0 \in \mathcal{M}_s^+$ and let $\|\mu_0\|_\Phi := \int_{\mathbb{R}^N} \Phi\, d\mu_0$. There exists a constant $C(N,s)$ such that for every $t > 0$ and $x \in \mathbb{R}^N$*

$$u(t,x) \le C\,\|\mu_0\|_\Phi\,(t^{-N/2s} + t)(1+|x|)^{N+2s}\,. \tag{5.24}$$

Here $\mathcal{M}_s^+$ are the nonnegative measures in the class $\mathcal{M}_s$ and $\|\mu_0\|_\Phi$ is the associated weighted norm with weight $\Phi(x) = (1 + |x|^2)^{-(N+2s)/2}$. See whole details in [51, Theorem 7.1]. The dependence on t cannot be improved. Under radial conditions a better growth estimate in x is obtained. Construction of self-similar solutions with growth in space also follows.

5.2.3 *Other Nonlocal Diffusive Operators*

- Equation (5.20) is the most representative example of a wide class of equations that are used to describe diffusive phenomena with nonlocal, possibly long-range interactions. We can replace the fractional Laplacian by a Lévy operator L which is the pseudo-differential operator with the symbol $a = a(\xi)$ corresponding to a certain convolution semigroup of measures, [169]. Popular models that are being investigated are integro-differential operators with irregular or rough kernels, as in [222], where the form is

$$u_t + b(x,t)\cdot\nabla u - \int_{\mathbb{R}^N} (u(x+h,t) - u(x,t))\, K(x,t,h)\, dh = f(x,t). \tag{5.25}$$

  See also [7, 54, 84, 109, 112, 125, 170, 228], among many other references.
- A different approach is taken by Nyström-Sande [210] and Stinga-Torrea [241], who define the fractional powers of the whole heat operator and solve

$$(\partial_t - \Delta)^s u(t,x) = f(t,x), \quad \text{for } 0 < s < 1. \tag{5.26}$$

  In this equation the random jumps are coupled with random waiting times. The authors find the space-time fundamental solution that happens to be explicit, given by

$$K_s(t,x) = \frac{1}{(4\pi t)^{N/2}|\Gamma(-s)|}\cdot\frac{e^{-|x|^2/4t}}{t^{1+s}} = \frac{1}{|\Gamma(-s)|t^{1+s}}G(x,t), \tag{5.27}$$

  for $x \in \mathbb{R}^N$, $t > 0$, where G is the Gaussian kernel. The limits $s \to 0, 1$ are singular. Motivations are given, extension methods are introduced, and regularity results proved.

5.3 Nonlinear Diffusion

The linear diffusion theory has enjoyed much progress, and is now solidly established in theory and applications. However, it was soon observed that many of the equations modeling physical phenomena without excessive simplification are

essentially nonlinear, and its more salient characteristics are not reflected by the linear theories that had been developed, notwithstanding the fact that such linear theories had been and continue to be very efficient for a huge number of applications. Unfortunately, the mathematical difficulties of building theories for suitable nonlinear versions of the three classical partial differential equations (Laplace's equation, heat equation and wave equation) made it impossible to make significant progress in the rigorous treatment of these nonlinear problems until the twentieth century was well advanced. This observation also applies to other important nonlinear PDEs or systems of PDEs, like the Navier-Stokes equations and nonlinear Schrödinger equations.

5.3.1 Importance of Nonlinear PDEs

The main obstacle to the systematic study of the Nonlinear PDE Theory was the perceived difficulty and the lack of tools. This is reflected in a passage by Nash [207]. In his seminal paper [207], he said

> The open problems in the area of nonlinear PDE are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

and he continues in a more specific way:

> Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

This is a grand project in pure and applied science and it is still going on. In order to start the work, and following the mathematical style that cares first about foundations, he set about the presumably humble task of proving the regularity of the weak solutions of the PDEs he was going to deal with. More precisely, the problem was to prove continuity (Hölder regularity) of the weak solutions of elliptic and parabolic equations assuming the coefficients a_{ij} to be uniformly elliptic (positive definite matrices) but only bounded and measurable as functions of $x \in \mathbb{R}^N$. In a rare coincidence of minds, this was done in parallel by Nash [206, 207] and the then very young Italian genius De Giorgi [103].[1] This was a stellar moment in the History of Mathematics, and the ideas turned out to be "a gold mine", in Nirenberg's words.[2] The results were then taken up and given a new proof by Moser, [202], who went on to establish the Harnack inequality, [203], a very useful tool in the sequel.

Once the tools were ready to start attacking Nonlinear PDEs in a rigorous way, it was discovered that the resulting mathematics are quite different from the linear

[1] Strictly speaking, priority goes to the latter, but the methods were different.

[2] Nash and Nirenberg shared the Abel Prize for 2015.

counterparts, they are often difficult and complex, they turn out to be more realistic than the linearized models in the applications to real-world phenomena, and finally they give rise to a whole set of new phenomena unknown in the linear world. Indeed, in the last decades we have been shown a multiplicity of new qualitative properties and surprising phenomena encapsulated in the nonlinear models supplied by the applied sciences. Some of them are very popular nowadays, like free boundaries, solitons and shock waves. This has kept generations of scientists in a state of surprise and delight. Nonlinear Science rests now on a firm basis and Nonlinear PDEs are a fundamental part of it.

5.3.2 Nonlinear Heat Flows, Nonlinear Diffusion

The general formula for the nonlinear diffusion models in divergence form is

$$\boxed{u_t = \sum \partial_i A_i(u, \nabla u) + \sum \mathcal{B}(x, u, \nabla u)\,,} \tag{5.28}$$

where $\mathcal{A} = (A_i)$ and $\mathcal{B}$ must satisfy some so-called structure conditions, the main one is again the ellipticity condition on the function $A(x, u, z)$ as a function of the vector variable $z = (z_i)$. This general form was already posed as a basic research project in the 1960s, cf. [15, 225]. Against the initial expectations, the mathematical theory turned out to be too vast to admit a simple description encompassing the stated generality. There are reference books worth consulting, like those by Ladyzhenskaya et al. [184, 185], Friedman [129], Lieberman [191], Lions-Magenes [193], and Smoller [230] are quite useful introductions. But they are only basic references.

Many specific examples, now considered the "classical nonlinear diffusion models", have been investigated separately to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins and types.

My personal experience with nonlinear models of diffusive type lies in two areas called respectively 'Nonlinear Diffusion with Free Boundaries' and 'Reaction-Diffusion PDEs'.

5.3.2.1 Pure Nonlinear Diffusion: The Free Boundary Models

The work on nonlinear parabolic equations in the mathematical research community to which I belonged focussed attention on the analysis of a number of paradigmatic models involving the occurrence of free boundaries, for which new tools were developed and tested. A rich theory originated that has nowadays multiple applications.

- The Obstacle Problem. This is the most famous free boundary problem and there is a huge literature for it, cf. [63, 66, 131, 174] and their references. It belongs

to the class of stationary problems, connected with elliptic equations, hence further away from our interests in this paper. Let us only say at this point that a free boundary problem is a mathematical problem in which we want to find the solution of a certain equation (normally, a PDE) as well as the domain of definition of the solution, which is also an unknown of the problem. Typically, there exists a fixed 'physical' domain D and the solution domain Ω that we seek is a subset of D, well-determined if we know the free boundary $\Gamma = \partial\Omega \cap D$.

Parabolic free boundaries may move in time. They appear in the 'four classical sisters' that we will introduce next:

- The Stefan Problem [186, 238] The problem typically describes the temperature distribution in a homogeneous medium undergoing a phase change (like ice and water). The heat equation must be solved in both separate media filling together a certain space $D \subset \mathbb{R}^N$, and the separation surface is allowed to move with time according to some transfer law. The mathematical formulation is thus

$$SE: \begin{cases} u_t = k_1 \Delta u \text{ for } u > 0, \\ u_t = k_2 \Delta u \text{ for } u < 0. \end{cases} \qquad TC: \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

(SE) means state equations, valid in the separate domains $\Omega_1 = \{(x,t) : u(x,t) > 0\}$ and $\Omega_2 = \{(x,t) : u(x,t) < 0\}$, which are occupied by two immiscible material phases (typically water for $u > 0$ and ice for $u < 0$). Of course, if the physical domain D is not the whole space then usual boundary conditions have to be given on the fixed boundary ∂D. The main mathematical feature is the existence of a *free boundary* or *moving boundary*[3] $\Gamma \subset \mathbb{R}^N \times \mathbb{R}$ that separates ice from water and there $u = 0$, see the monographs [198, 219]. This free boundary Γ moves in time and has to be calculated along with the PDE solution u, so that suitable extra information must be given to determine it: (TC) means transmission condition that applies at the free boundary Γ, and $\mathbf{v}$ is the normal advance speed of Γ. Physically, this formula is due to the existence of latent heat at the phase transition. We not only want to determine the location of Γ but we want to hopefully prove that it is a nice hypersurface in space-time.

Summing up, the combination of analysis of PDEs and variable geometry is what makes this problem difficult. The correct mathematical solution came only via the weak formulation [164] that allows to eliminate the geometry in a first step and concentrate in finding the so-called weak solution. The free boundary comes later as the zero level set of the weak solution, and finding it needs some regularity theory.

A simpler version is the **One-phase Stefan problem** where ice is assumed to be at zero degrees, roughly $u = 0$ in Ω_2. The free boundary is still there but the mathematical theory is much easier, hence better known.

[3] Also called *interface* in the literature.

- The Hele-Shaw Cell Problem. Hele-Shaw [153] and Saffman-Taylor [220] The problem is posed in a fixed spatial domain $D \subset \mathbb{R}^N$, and consists of finding $\Omega(t) \subset D$ and $u(x,t)$ such that

$$u > 0,\ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0,\ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial_f \Omega(t).$$

  Here the main unknown of the problem is the moving domain $\Omega(t) \subset D$, and $\partial_n u$ denotes normal derivative on the free boundary $\Gamma(t) = \partial_f \Omega(t)$, the part of the boundary of the set $\{x \in D : u(x,t) > 0\}$ that lies inside D. Additional conditions are to be given on the part of fixed boundary ∂D bounding $\Omega(t)$. Once $\Omega(t)$ is known, solving the Laplace equation for u is standard; notice that it is nontrivial because of the boundary conditions (sometimes there is a forcing term).

Mathematically, this is a simplified version of the previous model where there is only one phase, and besides the time derivative term disappears from the state equation. This increased simplicity comes together with beautiful analytical properties, some of them related to the theory of conformal transformations and complex variables when working in 2D, see [155, 216]. The Hele-Shaw flow appears in fluid mechanics as the limit of the Stokes flow between two parallel flat plates separated by an infinitesimally small gap, and is used to describe various applied problems. The weak formulation is studied in [118]. There are many examples of moving boundaries with interesting dynamics; thus, a peculiar complex variable pattern exhibiting a free boundary with a persistent pointed angle is constructed in [178] in 2D. In that example, the free boundary does not move until the pointed angle is broken, which happens in finite time. On the other hand, wider angles move immediately and the free boundary is then smooth.

- The Porous Medium Equation. This is an equation in the nonlinear degenerate parabolic category,

$$u_t = \Delta u^m, \quad m > 1.$$

  The equation appears in models for gases in porous media, underground infiltration, high-energy physics, population dynamics and many others. We will devote a whole section to review the free boundary and other nonlinear aspects of this equation, called PME for short, since it has served so much as a paradigm for the mathematics of nonlinear degenerate diffusion, see [12, 250, 251]. Actually, we see that the free boundary does not appear in the formulation, but it will certainly appear in the theory. It is a *hidden free boundary*.

The equation can be also considered for exponents $m < 1$, called *fast diffusion range*, and further generalized into the class of so-called *filtration equations* $u_t = \Delta \Phi(u)$, where Φ is a monotone increasing real function. This generality also allows to include the Stefan problem that can be written as a filtration equation with very degenerate Φ:

$$\Phi(u) = (u-1)_+ \qquad \text{for } u \geq 0, \qquad \Phi(u) = u \qquad \text{for } u < 0, \qquad p > 1.$$

- The p-Laplacian Equation. This is another model of nonlinear degenerate diffusion

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right). \tag{5.29}$$

Such a model appears in non-Newtonian fluids, turbulent flows in porous media, glaciology and other contexts. The mathematics of this equation turn out to be closely related to the PME: existence, regularity, free boundaries, and so on, but there are subtle differences. Here $p > 2$ is needed for a free boundary to appear, [114, 249]. Recent interest in the limit cases $p = 1$ (total variation flow, used in image analysis), or $p = \infty$ (appearing in geometry and transport), [121]. On the other hand, the equation can be generalized into the class of equations with *gradient-dependent diffusivity* of the general form

$$u_t - \nabla \cdot (a(|Du|)Du) = 0,$$

where a is a nonnegative real function with suitable growth assumptions to ensure degenerate parabolicity. Another extension is the *doubly non-linear diffusion equation* of the form

$$u_t = \nabla \cdot (|D(u^m)|^{p-2}D(u^m)).$$

Here the diffusivity takes the form $a(u, |Du|) = cu^{(p-1)(m-1)}|Du|^{p-2}$. We use the notations $\nabla u = Du$ for the spatial gradient.

5.3.2.2 The Reaction Diffusion Models

This is another important direction taken by Nonlinear Diffusion, in which the nonlinear features originate from a lower-order term with super-linear growth. This may create a mathematical difficulty in the form of *blow-up*, whereby a solution exists for a time interval $0 < t < T$ and then some norm of the solution goes to infinity as $t \to T$ (the blow-up time). In other cases the singular phenomenon is extinction (the solution becomes zero every where), or some other kind of singularity formation.

- The Standard Blow-Up Model: It is also called the Fujita model [133, 168]

$$u_t = \Delta u + u^p \qquad p > 1\,.$$

Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions may go to infinity in finite time. This depends on the domain and the initial data. For instance, if the space domain is $\mathbb{R}^N$ and the initial function is constant, then blow-up in finite time always happens. Hint: Integrate the ODE $u_t = u^p$. However, when the data are distributed in space then diffusion and reaction compete and the result is a

priori uncertain. This is how a large literature arose. Thus, if the initial data are bell-shaped (like the Gaussian function), the domain is bounded and boundary conditions are zero Dirichlet, then small data will not blow-up and large data will. For other configurations things depend on the exponent p: there exists a critical exponent p_F called the Fujita exponent, such that all positive solutions blow up if $p \in (1, p_F)$. See [133, 135, 168, 189].

A number of beautiful blow-up patterns emerge in such evolutions. Galaktionov and the author have constructed in [134] a particular one, called the *peaking solution*, that blows up in finite time T at a single point x_0 and then continues for later time as a bounded smooth solution, a clear example of the curious phenomenon called continuation after blow-up. However, the most common situation in reaction-diffusion systems of this diffusive type is *complete blow-up* at time T with no possible continuation (for instance, the numerical approximation goes to infinity everywhere for $t > T$). The intricate phenomenon of *bubbling* is studied by M. del Pino in another course of this volume [110].

As an extension of this elementary reaction-diffusion blow-up model there have been studies for many equations of the general form

$$u_t = \mathcal{A}(u) + f(u, Du)$$

where $\mathcal{A}$ is a linear or nonlinear diffusion operator, maybe of porous medium or p-Laplacian type. The studies also include systems. Some of them are systems of mixed type, one of the most popular ones is the chemotaxis system, where blow-up has a very interesting form that is still partially understood, [154].

- The Fisher-KPP Model and Traveling Waves: The problem goes back to Kolmogorov, Petrovskii and Piskunov, see [182], that present the most simple reaction-diffusion equation concerning the concentration u of a single substance in one spatial dimension,

$$\partial_t u = D u_{xx} + f(u)\,, \tag{5.30}$$

with an f that is positive between two zero levels $f(0) = f(1) = 0$. We assume that $D > 0$ is constant. The choice $f(u) = u(1-u)$ yields Fisher's equation [127] that was originally used to describe the spreading of biological populations. The celebrated result says that the long-time behavior of any solution of (5.30), with suitable data $0 \le u_0(x) \le 1$ that decay fast at infinity, resembles a traveling wave with a definite speed that can be explicitly calculated. The KPP traveling wave pattern is one of the most famous dynamic patterns in diffusive phenomena.

When considering Eq. (5.30) in dimensions $N \ge 1$, the problem becomes

$$u_t - \Delta u = f(u) \quad \text{in } (0, +\infty) \times \mathbb{R}^N, \tag{5.31}$$

This case has been studied by Aronson and Weinberger in [17, 18], where they prove the following result.

Theorem *Let u be a solution of (5.31) with $u_0 \neq 0$ compactly supported in $\mathbb{R}^N$ and satisfying $0 \le u_0(x) \le 1$. Let $c_* = 2\sqrt{f'(0)}$. Then,*

1. *if $c > c_*$, then $u(x,t) \to 0$ uniformly in $\{|x| \ge ct\}$ as $t \to \infty$.*
2. *if $c < c_*$, then $u(x,t) \to 1$ uniformly in $\{|x| \le ct\}$ as $t \to \infty$.*

In addition, problem (5.31) admits planar traveling wave solutions connecting 0 and 1, that is, solutions of the form $u(x,t) = \phi(x \cdot e + ct)$ with

$$-\phi'' + c\phi' = f(\phi) \text{ in } \mathbb{R}, \quad \phi(-\infty) = 0,\ \phi(+\infty) = 1.$$

This asymptotic traveling-wave behavior has been generalized in many interesting ways, in particular in nonlinear diffusion of PME or p-Laplacian type, [22, 104, 105]. Departing from these results, King and McCabe examined in [177] a case of fast diffusion, namely

$$u_t = \Delta u^m + u(1-u), \quad x \in \mathbb{R}^N, t > 0,$$

where $(N-2)_+/N < m < 1$, and showed that the problem does not admit traveling wave solutions and the long time behaviour is quite different. We will return to this question when dealing with fractional nonlinear diffusion in the work [233], in Sect. 5.7.

- In the last decades many other models and variants of diffusive systems have been proposed, in particular in the form of systems, like the various *cross-diffusion systems* [163]. Cross-diffusion gives rise to instabilities that attract much attention in population dynamics, since they allow to predict important features in the study of the spatial distribution of species. The seminal work in this field is due to Alan Turing [242]. In order to understand the appearance of certain patterns in nature with mathematical regularities like the Fibonacci numbers and the golden ratio, he proposed a model consisting of a system of reaction-diffusion equations.
- Blow-up problems have appeared in related disciplines and some of them have attracted in recent times the attention of researchers for their difficulty and relevance. We present two cases, a case still requiring more work, a case enjoying big success. The combination of diffusion with nonlinear reaction is in both cases very intricate and leading to the deepest mathematics.

The Fluid Flow Models: The *Navier-Stokes* or *Euler* equation systems for incompressible flow. The nonlinearity is quadratic and affects first order terms. Progress is still partial. There is also much work on the related topic of geostrophic flows. We will not enter into more details of such a relevant topic that has a different flavor.
The Geometrical Models: The *Ricci flow* describes the motion of the metric tensor of a Riemannian manifold by means of the Ricci matrix: $\partial_t g_{ij} = -2R_{ij}$. This is a nonlinear reaction-diffusion system, even if this information is not clear in the

succinct formula. Posed in the form of PDEs by R. Hamilton, 1982, it has become a Clay Millenium Problem. Its solution by G. Perelman in 2003 was one of biggest success stories of Mathematics in the twenty-first century, see [89, 201]. One of the main points in the proof is the study of the modes of blow-up of this system. In order to see that the evolution system of the Ricci flow is a type of nonlinear diffusion, it is convenient to recall the much simpler case of two-manifolds, since in that case it reduces to a type of fast diffusion called logarithmic diffusion, see below.

Let us finally mention the equations of movement by curvature to the list of geometrical models. Enormous progress has been made in that topic.

Basic Reading for This Chapter: On Nonlinear Diffusion: [99, 114, 251]. On free boundaries [113, 131]. Moreover, [95, 96]. Fully nonlinear equations are form a vast topic that we have not touched, see [59].

5.4 PME: Degenerate Diffusion and Free Boundaries

A very simple model of nonlinear diffusion in divergence form is obtained by means of the equation

$$\boxed{u_t = \nabla \cdot (D(u)\nabla u)} \tag{5.32}$$

where $D(u)$ is a diffusion coefficient that depends on the 'concentration variable' u. Strict parabolicity requires that $D(u) > 0$, and the condition can be relaxed to degenerate parabolicity if we make sure that $D(u) \geq 0$. Now, if we further assume that $D(u)$ is a power function, we get the simplest model of nonlinear diffusion equation in the form

$$u_t = \nabla \cdot (c_1 |u|^\gamma \nabla u) = c_2 \Delta(|u|^{m-1} u). \tag{5.33}$$

with $m = 1 + \gamma$ and $c_1, c_2 > 0$. Exponent m, in principle positive, will play an important role in the model, but the constants c_i are inessential, we may put for instance $c_2 = 1$, $c_1 = m$. The *concentration-dependent diffusivity* is then

$$D(u) = m|u|^{m-1}\,.$$

In many of the applications u is a density or concentration, hence essentially nonnegative, and then we may write the equation in the simpler form

$$u_t = \Delta(u^m)\,, \tag{5.34}$$

that is usually found in the literature (we have dispensed with useless constants). But there are applications in which u is for instance a height that could take negative values, and then version (5.33) is needed, since otherwise $D(u)$ would not be positive and the equation would not be parabolic.

The equation has enjoyed a certain popularity as a mathematical model for degenerate nonlinear diffusion, combining interesting and varied applications with a rich mathematical theory. The theory has many interesting aspects, like functional analysis in the existence and uniqueness theory, and geometry in the study of the free boundaries, as well as deep novelties in the long time asymptotics. Our monograph [251] gathers a large part of the existing theory up to the time of publication (2007). We will devote this section to review some of the main topics that affect the theory of fractional porous medium models of later nonlocal sections, and we will also present the very recent sharp results on the regularity and asymptotic behaviour of free boundaries, obtained in collaboration with Kienzler and Koch in [172].

5.4.1 The Porous Medium Equation

As we have already said, the value of exponent m is an important part of the model. Clearly, if $m = 1$ we have $D(u) = 1$, and we recover the classical heat equation, $u_t = \Delta u$, with its well-known properties, like the maximum principle, the C^∞ regularity of solutions, and the infinite speed of propagation of positive disturbances into the whole space, as well as the asymptotic convergence to a Gaussian profile for suitable classes of initial data.

- The first interesting nonlinear case is $m > 1$ where $D(u)$ degenerates at the level $u = 0$. This brings as a consequence the existence of weak solutions that have compact support in the space variable for all times, though that support expands. We refer to that situation as *Slow Diffusion*. As a consequence, free boundaries arise and a whole geometric theory is needed. All this is in sharp contrast with the heat equation.

The differences with the heat equation can be seen by means of an easy calculation for $m = 2$. In that case, and under the assumption that $u \geq 0$, the equation can be re-written as

$$\frac{1}{2} u_t = u\Delta u + |\nabla u|^2 ,$$

and we can immediately see that for values $u \gg 0$ the equation looks like a harmless nonlinear perturbation of the heat equation plus a lower order term, while for $u \sim 0$ the first term disappears and the equation looks like (a singular perturbation of) the eikonal equation

$$u_t = |\nabla u|^2 .$$

This last equation is not *parabolic*, but *hyperbolic*, with propagation along characteristics. The PME equation is therefore of mixed type near the critical value $u = 0$ where it degenerates, and it has therefore mixed properties.

The calculation may look very particular, for a specific value of m. But to the initial surprise of researchers, it extends to all value $m > 1$, of the slow diffusion range. The *pressure transformation* $v = cu^{m-1}$ allows us to get an equivalent equation for v:

$$v_t = (m-1)v\Delta v + |\nabla v|^2 , \tag{5.35}$$

where we have used the standard normalization $c = m/(m-1)$. Indeed, the apparent generality of this transformation goes further and there is a great unity in the theory developed for the PME in the whole range $m > 1$, see [251]. Indeed, it can be proved that in some weak sense the eikonal equation holds on the free boundary $\{u = 0\}$, and this implies that the support of the solution spreads with time, another property that can be rigorously proved.

- The pressure transformation is even more general, and can be applied to the filtration equation $u_t = \Delta\Phi(u)$. If we put $v = \int_1^u (\Phi'(s)/s)\, ds$, then we can get the pressure equation

$$v_t = \sigma(v)\Delta v + |\nabla v|^2 , \tag{5.36}$$

  where the function $\sigma(v) = \Phi'(u) \geq 0$, cf. [55].
- These pressure considerations apply under the assumption that $u \geq 0$, which is physically natural for most applications. It must be pointed out the existence and uniqueness theory has been done for signed solutions, according to the generality that is suitable in Functional Analysis. However, many of the estimates on which the qualitative theory is based do not apply for general signed solutions, and we will forsake them and assume $u \geq 0$ in the rest of the section unless mention to the contrary.
- When $m < 1$ the equation becomes singular at $u = 0$ in the sense that $D(u) \to \infty$. This range is called *Fast Diffusion*. We will return to that case in the next Sect. 5.5 since its properties show a remarkable difference with the PME range $m > 1$.

5.4.2 Applied Motivation: Fixing Some Physical Concepts

This application is maybe the best known and has played a role in developing the theory for the PME, a clear example of the influence of physics on the mathematics. According to [188, 205], the flow of gas in a porous medium (they were thinking of the petroleum industry) obeys the laws

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{V}) = 0, \\ \mathbf{V} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho), \end{cases}$$

where ρ is density, p is the averaged pressure and $\mathbf{V}$ is the seepage velocity. The first line is the usual continuity equation from fluid mechanics, and the second line left is the *Darcy law* for flows in porous media (Darcy 1856). Therefore, these porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales. We need a precise closure relation which is given by a gas law of the form $p = p_o\,\rho^\gamma$, with value of the exponent $\gamma = 1$ (isothermal gas) or $\gamma > 1$ (adiabatic gas flow), see details in [251]. Hence, we get

$$\rho_t = \mathrm{div}\,(\frac{k}{\mu}\rho\nabla p) = \mathrm{div}\,(\frac{k}{\mu}\rho\nabla(p_o\rho^\gamma)) = c\Delta\rho^{\gamma+1}.$$

In order to get the PME we put $u = \rho$, $m = 1 + \gamma$ (which happens to be equal or larger than 2) and we eliminate useless constants. We point out that the pressure is then

$$p = p_o u^{m-1},$$

just the variable that we called v in formula (5.35). No wonder that this equation is important. As for the local flow velocity we have $\mathbf{V} = -c\nabla v$ in our mathematical notation.

– There are many other applications, as described in the book [251]: underground water infiltration [53] with $m = 2$, plasma radiation with $m \geq 4$, (Zeldovich-Raizer [269], around 1950), spreading of populations (self-avoiding diffusion) $m \sim 2$ [150], thin films under gravity with no surface tension $m = 4$, and so on.

5.4.3 *Generalities: Planning of the Theory*

The way the nonlinear theory of the PME has developed is quite different from the way the linear heat equation is studied. Indeed, in the early years there were attempts to construct a perturbation theory putting $m = 1+\varepsilon$ in (5.34) and then perturbing the linear model, but the singular perturbation analysis was not successful. Fortunately, around 1958 when the theory started the serious development in Moscow [212], the tools of nonlinear functional analysis were ready, and in particular the concept of weak solution and the role of a priori estimates.

Here are the main topics of mathematical analysis (1958–2016):

– The precise meaning of solution. Since it was realized that classical solutions do not exist if there are free boundaries.
– The nonlinear approach: estimates; functional spaces.
– Existence of suitable solutions (like weak solutions). Uniqueness. Further in the theory, variant of the equation showed cases of non-existence or non-uniqueness.

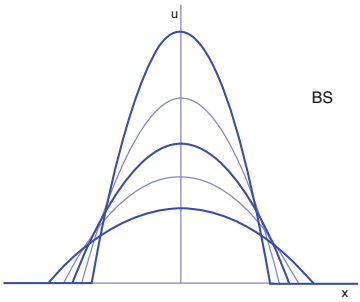
– Regularity of solutions: Are weak solutions indeed continuous functions? are they C^k for some k? which is the optimal k?
– Existence, regularity and movement of interfaces: are they C^k for some k?
– Asymptotic behaviour: is there something comparable to the Gaussian profile as a universal attractor? This is a question of *emerging patterns*. If there is convergence to a pattern we want to know that rate of convergence. We also want to know how universal that convergence is, in other words the basin of attraction of the asymptotic pattern.
– Comparison with other approaches like the probabilistic approach. Interesting new tools appear, like Wasserstein metrics and estimates.

The beauty of this plan is that it can be used *mutatis mutandis* on a huge number of related models: fast diffusion models, inhomogeneous media, anisotropic media, p-Laplacian models, applications to geometry or image processing; equations involving effects, like the chemotaxis models,...

5.4.4 Fundamental Solutions: The Barenblatt Profiles

These profiles are the alternative to the Gaussian profiles of the linear diffusion case. They are source solutions. *Source* means that $u(x,t) \to M\,\delta(x)$ as $t \to 0$. Explicit formulas exist for them (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)} \tag{5.37}$$



$\alpha = \frac{n}{2+N(m-1)}$
$\beta = \frac{1}{2+N(m-1)} < 1/2$
Maximum height $u = Ct^{-\alpha}$
Free boundary is located at distance $|x| = ct^{\beta}$

Source: Personal source

where $C > 0$ is a free constant and $k = k(m, N)$. Since Fourier analysis is not a way for find them, new ideas are needed. We observe that the solution obeys a scaling symmetry, it is self-similar. In other words, it is invariant under suitable scaling in space and time. This fact is the key to finding the expression, see [251, p. 63]. An important property for the applications is that (5.37) breaks with the Brownian space-time law: $|x| = ct^{1/2}$, so that it can be classified as *anomalous diffusion*.

If you look for the mathematical properties, we find a surprise with regularity. Put $m = 2$ for simplicity. $\mathbf{B}(x, t; M)$ does not satisfy the equation in a classical sense since u is not even C^1 continuous in space or time. The validity of this physical solution was a hot problem when it was discovered around 1950.

5.4.5 Concepts of Solution

Hence, there is a problem with the concept of solution that will satisfy the mathematical requirements (existence and uniqueness for a reasonable class of data, plus stability estimates) as well as the physical requirements (to reflect the behaviour that is expected from the evidence obtained in the applications). This problem did not exist for the main example of diffusion, the Heat Equation, since classical solutions could be found.

Many concepts of generalized solution have been used in developing the mathematical theory of the PME, and also in many related equations, not only in the parabolic theory:

- *Classical solution.* This is the most desirable option, and indeed it happens for non-degenerate situations, $u > 0$. But it cannot be expected if the Barenblatt solutions are to be included.
- *Limit solution.* This is the practical or computational remedy. To replace the equation by approximated problems with good physical or computational properties and then to pass to the limit. The catch is that the approximation may not converge, or we could be unable to prove it; even if the approximations do converge, the limit may depend on the approximation. Spurious solutions may appear when the approximation is not efficient, a quality difficult to tell a priori. On the positive side, limit solutions have been successfully used in the diffusive literature with the names of minimal solutions, maximal solutions, SOLAs (solutions obtained as limits of approximations), proper solutions, ...
- *Weak solution.* This was a very good solution to the problem of building a theory for the PME. The idea is to test the equation against a full set of smooth functions and to eliminate all or most of the derivatives prescribed by the equation on the unknown function. It was first implemented on the PME by Oleinik and collaborators [212] (1958). The simplest weak version reads

$$\int\int (u\,\eta_t - \nabla u^m \cdot \nabla \eta)\,dxdt + \int u_0(x)\,\eta(x,0)\,dx = 0\,,$$

while there is a second version, the *very weak solution*,

$$\int\int (u\,\eta_t + u^m\,\Delta\eta)\,dxdt + \int u_0(x)\,\eta(x,0)\,dx = 0.$$

This version is more relaxed than the first. In both cases functional spaces have to be chosen for the solutions to belong to so that the integrals in the formulation make sense and existence and uniqueness can be proved.

Once existence and uniqueness of a weak solution was proved for suitable initial data; that it was verified that all classical solutions are weak; and that the Barenblatt solutions are indeed solutions for $t > 0$ (and take the initial Dirac delta in a suitable sense) the theory of the PME could be conveniently framed as a theory of weak solutions. Even more, it was proved that suitable numerical approximations converge to the weak solution.

- *Better regularity. Strong solution.* The previous paragraph solves the problem of the correct setting in principle. But researchers want to have solutions that have good properties so that we can do calculus with them. Fortunately, weak solutions of the PME are better than weak, they are strong. In this context it means that all weak derivatives entering the original equation are L^p functions for some p.
- The search for an abstract method to solve a large number of evolution problems of diffusive type has led to a functional approach called *mild or semigroup solution*, that we discuss below.
- Solutions of more complicated diffusion-convection equations have motivated new concepts that can be translated to the PME:
  - *Viscosity solution*. Two different ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of [94]; adapted to PME by Caffarelli-Vázquez [69].
  - *Entropy solution* [183]. Invented for conservation laws; it identifies unique physical solution from *spurious* weak solutions. It is useful for general models with degenerate diffusion plus convection.
  - *Renormalized solutions* (by Di Perna-P.L. Lions), *BV solutions* (by Volpert-Hudjaev), *Kinetic solutions* (by Perthame,...).

5.4.6 Semigroup Approach: Mild Solution

Functional Analysis is a power tool for the expert in PDEs, and when used wisely it produces amazing result. Thus, when faced with the task of solving evolution equations of the type

$$u_t + Au = 0\,, \tag{5.38}$$

where A is a certain operator between function spaces, we may think about discretizing the evolution in time by using a mesh $t_0 = 0 < t_1, \dots t_K = T$ and posing the implicit problems

$$\frac{u(t_k) - u(t_{k-1})}{h_k} + A(u(t_k)) = 0, \quad h_k = t_k - t_{k-1}.$$

In other words, we want to find a discrete approximate solution $u = \{u_k\}_k$ such that

$$hA(u_k) + u_k = u_{k-1}\,, \tag{5.39}$$

where we have used equal time spacing $h_k = h > 0$ for simplicity. Of course, the approximate solution depends on the time step h, so that we should write $u^{(h)} = \{u_k^{(h)}\}_k$. This step is called *Implicit Time Discretization,* ITD. We start the iteration by assigning the initial value $u_0^{(h)} = u_{0h}$, where u_{0h} is the given initial data or an approximation thereof.

Parabolic to Elliptic The success of ITD depends on solving the iterated equation (5.39) in an iterative way. In fact, the iteration has always the same format

$$hA(u) + u = f\,, \tag{5.40}$$

since $f = u_{k-1}$ is the value calculated in the previous step. When this is used for the filtration equation $u_t - \Delta\Phi(u) = 0$, we get the stationary equation

$$-h\Delta\Phi(u) + u = f\,, \tag{5.41}$$

and the question reduces to solve for u if f is known. An easy change of variables $v = \Phi(u)$, $u = \beta(v)$ leads to

$$-h\Delta v + \beta(v) = f. \tag{5.42}$$

This is the semilinear elliptic problem that we must solve. We have reduced the theory of a (possibly nonlinear) parabolic problem to an elliptic problem with a specific form.

Accretive Operators: Semigroup Generation The rest of the story depends on the theory of accretive operators. If A is an m-accretive map in a Banach space X with densely defined domain, then the famous Crandall-Liggett Theorem [93] ensures not only existence of the solution of the iterated problems, but also that as $h \to 0$ the discrete solutions $u^{(h)}$ converge to a function $u(t) \in C([0,\infty) : X)$ that solves the evolution problem in a sense called *mild sense*. The solution is often termed the *semigroup solution*. Moreover, the set of solutions forms a semigroup of contractions in X.

But is this mild solution a solution in some more usual sense? In the case of the PME it is proved that the operator given by $A(u) = -\Delta(u^m)$ is m-accretive in the space $X = L^1(\mathbb{R}^N)$ when properly defined, [26], and also that the mild solution is a weak solution. We have explained the method in some detail in Chap. 10 of [251].

5.4.7 *Regularity Results for Nonnegative Solutions*

The next step in the theory is proving that under mild conditions on the data weak solutions of the PME are indeed continuous, and the free boundary is quite often a regular hypersurface in space-time, or in space for every fixed time. We recall that we are working with nonnegative solutions. We we also dealing with the Cauchy problem in the whole space to save effort and concentrate on the basics, but many results hold for locally defined solutions.

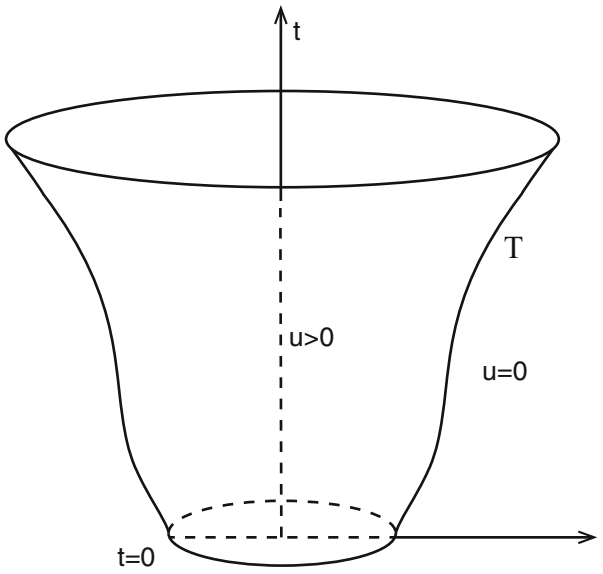
- The regularity theory for solutions relies on the existence of a rather miraculous a priori estimate, called the Aronson-Bénilan estimate [13], that reads:

$$\Delta v \geq -C/t,$$

  where $v = cu^{m-1}$ is the pressure and $C = (n(m-1)+2)^{-1}$. Nonnegative solutions with data in any L^p space are then proved to be bounded for positive times. A major step was then done by Caffarelli and Friedman [65] when they proved C^α regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is C^α continuous. This holds in all space dimensions.
- What happens to the free boundary? It was soon proved that free boundaries may be stationary for a while but eventually they must move to fill the whole domain as time passes. The movement is expansive, the positivity set keeps expanding in time and never recedes. Caffarelli and Friedman proved subsequently that if there is an interface Γ, it is also a C^α continuous set in space time (properly defined).
- How far can you go? The situation is understood in 1D. On the one hand, free boundaries can be stationary for a time (metastable) if the initial profile is quadratic near $\partial\Omega$: $v_0(x) = O(d^2)$, d being distance to the zero set. This time with lack of movement is called a *waiting time*. It was precisely characterized by the author in 1983; it is visually interesting in the experiments with thin films spreading on a table. In paper [19] we proved that metastable interfaces in 1D may start to move abruptly after the waiting time. This was called a *corner point*. It implies that the conjecture of C^1 regularity for free boundaries in 1D was false. But in 1D the problems with regularity stop here: 1D free boundaries are strictly moving and C^∞ smooth after the possible corner points. See [251] for full details.

5.4.8 *Regularity of Free Boundaries in Several Dimensions*

The situation is more difficult for free boundary behaviour in several space dimensions, and the investigation is still going on.



A regular free boundary in N-D. *Source*: Personal source

- Caffarelli, Wolanski, and the author proved in 1987 that if u_0 has compact support, then after some time $T > 0$ the interface is $C^{1,\alpha}$, and the pressure is also $C^{1,\alpha}$ in a lateral sense [72, 73]. Note the lateral regularity is the only option since the Barenblatt solutions are an example of solutions that exhibit a smooth profile that is broken at the FB. The general idea, taken from 1D, is that when the FB moves, the adjoining profile is always a broken profile, since the support of the solution moves forward only if the gradient of the pressure is nonzero (Darcy's law).
- In his excellent doctoral thesis (1997), Koch proved that if u_0 is compactly supported and transversal then the free boundary is C^∞ after some finite time and the pressure is "laterally" C^∞. This solved the problem of optimal regularity in many cases, though not all.
- The free boundary for a solution with a hole in 2D, 3D is the physical situation in which optimal regularity can be tested. Indeed, as the flow proceeds and the hole shrinks, it is observed that the part of the motion of the free boundary surrounding the hole accelerates, so that at the point and time where the hole disappears (this phenomenon is called focusing), the advance speed becomes infinite. The applied setup is a viscous fluid on a table occupying an annulus of radii r_1 and r_2. As time

passes $r_2(t)$ grows out while $r_1(t)$ goes to the origin. In a finite time T the hole disappears. The flow can be regular for $t < T$ but it was suspected from the numerical evidence that it was not at the focusing time $t = T$.

To prove this fact, a self-similar solution was constructed displaying the focusing behaviour. It has the form

$$U_f(x,t) = (T-t)^{\alpha} F(x/(T-t)^{\beta}).$$

with $(m-1)\alpha = 2\beta + 1$. The profile is such that $F(\xi) \sim |\xi|^{\gamma}$ near $\xi = 0$, with $\gamma = \alpha/\beta$. There is one free parameter, let us say β, that is not known a priori. We find a very interesting mathematical novelty, an *anomalous exponent*, or *similarity of the second kind* in the terminology popularized by Barenblatt [23]. The problem was solved by ODE analysis in 1993 by Aronson and Graveleau [14], and then further investigated by Aronson and collaborators, like [9, 16]. It is proved that $\gamma < 1$ so that the speed $\mathbf{v} \sim \nabla u^{m-1}$ blows-up at the focusing time $t = T$ at $x = 0$. Moreover, the limit profile $U_f(x,T)$ is not Lipschitz continuous at $x = 0$, it is only C^{γ} continuous. This is a counterexample to the hypotheses of higher regularity of heat equation and similar diffusive flows: a degenerate equation like PME has limited regularity for nonnegative solutions with moving free boundaries. Such a phenomenon is known to happen in the other typical evolution free boundary problems. Stefan, Hele-Shaw and p-Laplacian equation. For the latter see [20].

Summing up, higher regularity for PME flows has an obstacle. We may hope to prove higher regularity if we avoid it, like the situation of compactly supported solutions for large times.

5.4.9 Recent Results on Regularity and Asymptotics

The question remained for many years to know if we can prove regularity of the solutions and their free boundaries under some certain geometrical condition on the solution or the data less stringent than the conditions of compact support and initial transversality of papers [72, 73, 181].

- Much progress was done recently in paper with Koch and Kienzler [172], preceded by Kienzler [171]. Here is the main theorem proved in [172] about regularity of solutions that are locally small perturbations from a flat profile.

Theorem 5.4.1 *There exists $\delta_0 > 0$ such that the following holds:*
If u is a nonnegative δ-flat solution of the PME at $(0,0)$ on scale 1 with δ-approximate direction e_n and δ-approximate speed 1, and $\delta \le \delta_0$, then for all derivatives we have uniform estimates

$$|\partial_t{}^k \partial_x{}^{\alpha} \nabla_x (u^{m-1} - (x_n + t))| \le C\delta \tag{5.43}$$

at all points $(t,x) \in ([-1/2,0] \times \overline{B_{1/2}(0)}) \cap \mathcal{P}(u)$ *with* $C = C(N,m,k,\alpha) > 0$. *In particular,* ρ^{m-1} *is smooth up to the boundary of the support in* $(-\frac{1}{2},0] \times B_{1/2}$, *and*

$$|\nabla_x u^{m-1} - e_n|, \quad |\partial_t u^{m-1} - 1| \le C\delta\,. \tag{5.44}$$

Moreover, the level sets for positive values of u and the free boundary are uniformly smooth hypersurfaces inside $(-\frac{1}{2},0] \times B_{\frac{1}{2}}(0)$.

The technical assumption is being δ-flat, which means being very close to a flat travelling wave (the special solution that serves as model) in a certain space-time neighbourhood. See precise details in the paper. The size of this neighbourhood is taken to be unit, but this is not a restriction by the scale invariance of the equation. The very detailed form of the estimates allows us then to derive very strong results for large times, that we will explain in next subsection.

- Theorem 5.4.1 implies the eventual C^∞-regularity result for global solutions that we were looking for. The following result is Theorem 2 of [172]. We use the notation $R_B(t) = c_1(N,m)\,M^{(m-1)\lambda}t^\lambda$, with $\lambda = 1/(N(m-1)+2)$, for the Barenblatt radius for the solution with mass M located at the origin.

Theorem 5.4.2 *Let* $u \ge 0$ *be a solution of the PME posed for all* $x \in \mathbb{R}^N$, $N \ge 1$, *and* $t > 0$, *and let the initial data* u_0 *be nonnegative, bounded and compactly supported with mass* $M = \int u_0 dx > 0$. *Then, there exists a time* T_r *depending on* u_0 *such that for all* $t > T_r$ *we have:*

(i) Regularity. The pressure of the solution u^{m-1} *is a* C^∞ *function inside the support and is also smooth up to the free boundary, with* $\nabla u^{m-1} \ne 0$ *at the free boundary. Moreover, the free boundary function* $t = h(x)$ *is* C^∞ *in the complement of the ball of radius* $R(T_r)$.

(ii) Asymptotic approximation. There exists $c > 0$ *such that*

$$\begin{aligned} t^{-N\lambda}\Big(a^2M^{2(m-1)\lambda} - ct^{-2\lambda} - \frac{\lambda|x-x_0|^2}{2t^{2\lambda}}\Big)_+^{\frac{1}{m-1}} &\le u(t,x) \\ &\le t^{-N\lambda}\Big(a^2M^{2(m-1)\lambda} + ct^{-2\lambda} - \frac{\lambda|x-x_0|^2}{2t^{2\lambda}}\Big)_+^{\frac{1}{m-1}} \end{aligned} \tag{5.45}$$

where $x_0 = M^{-1}\int x\rho(x)dx$ *is the conserved center of mass, and a is a certain constant. Moreover,*

$$B_{R_B(t)-ct^{-\lambda}}(x_0) \subset \mathrm{supp}(u(\cdot,t)) \subset B_{R_B(t)+ct^{-\lambda}}(x_0) \tag{5.46}$$

In this way we are able to solve the problem posed in 1987, and improved by Koch in 1997. We use delicate flatness conditions, scalings, heat semigroups and harmonic analysis. We have eliminated the non-degeneracy condition on the initial data. The estimates are uniform. The result cannot be improved in a number of directions. Besides, some more information is available: if the initial function

is supported in the ball $B_R(0)$, then we can write the upper estimate of the regularization time as

$$T_r = T(N, m) M^{1-m} R^{\frac{1}{\lambda}}. \tag{5.47}$$

By scaling and space displacement we can reduce the proof to the case $M = 1$ and $x_0 = 0$. The fine asymptotic analysis uses also the results of Seis [223].

- **Nonlinear Central Limit Theorem revisited**

The last part of Theorem 5.4.2 refers to the way a general solution with compact support approaches the Barenblatt solution having the same mass. This kind of result is what we have called the PME version of the Central Limit Theorem.

It was proved in due time that the standard porous medium flow has an asymptotic stabilization property that parallels the stabilization to the Gaussian profile embodied in the classical Central Limit Theorem if we take as domain $\mathbb{R}^N$ and data $u_0(x) \in L^1(\mathbb{R}^n)$. The convergence result is

$$\|u(t) - B(t)\|_1 \to 0 \tag{5.48}$$

as $t \to \infty$, as well as

$$t^{N\lambda} \, |u(x,t) - B(x,t)| \to 0, \tag{5.49}$$

uniformly in $x \in \mathbb{R}^N$. Here, $B(x, t; M)$ be the Barenblatt with the asymptotic mass M. Note that the factor $t^{N\lambda}$ is just the normalization needed to work with relative errors since $B(x, t)$ decays like $O(t^{-N\lambda})$. Proofs are due to Kamin and Friedman [132] for compactly supported solutions, and the author (2001) in full generality. The result is reported with full detail in [251] and explained in [250, 253].

An improvement of the result to indicate a definite rate of convergence is due to Carrillo and Toscani [78]. It works for solutions with a finite second moment, $\int u_0(x) \, |x|^2 \, dx$, [78] and uses the powerful machinery of entropy methods, that become subsequently very popular in studies of nonlinear diffusion.

The result that we obtain above points out to a finer error rate for compactly supported solutions, that can be written as

$$t^{N(m-1)\lambda} \, |u^{m-1}(x,t) - B^{m-1}(x - x_0, t; M)| = O(t^{-2\lambda}) \, . \tag{5.50}$$

Seis' analysis and our paper show optimality of this rate. Note that $2\lambda < 1$.

Other Problems There are numerous studies of the PME in other settings, like bounded domains with Dirichlet or Neumann conditions, PME with forcing term:

$$u_t = \Delta u^m + f,$$

PME with variable coefficients or weights, generalized filtration equation, PME with convection and/or reaction, ...

Many of the above results have counterparts for the p-Laplacian flow. Thus, stabilization to the p-Laplacian version of the Barenblatt solution is proved by Kamin and the author in [166]. There are many studies but no comparable fine analysis of the FB has been done.

Further Reading for This Chapter On the PME: [250, 251]. On asymptotic behaviour: [247, 248]. About estimates and scaling: [249]. For entropy methods [11, 78, 79].

5.5 The Fast Diffusion Equation

We will consider now that range $m < 1$ for the model equation (5.33). The equation becomes singular at the level $u = 0$ in the sense that $D(u) \to \infty$. This range is called *Fast Diffusion Equation*, FDE, and we also talk about *singular diffusion*. The new range was first motivated by a number of applications to diffusive processes with fast propagation: plasma Physics (Okuda-Dawson law) [27, 211], material diffusion (dopants in silicon) [175], geometrical flows (Ricci flow on surfaces and the Yamabe flow), diffusive limit of kinetic equations, information theory, and others, see [249]. Once the mathematics started, it was seen that the FDE offers many interesting mathematical challenges, and some unexpected connections with other disciplines like Calculus of Variations.

The common denominations *slow* and *fast* for the parameter ranges $m > 1$ and $m < 1$ in (5.33) refer to what happens for $u \approx 0$. But when large values of u are involved, the names are confusing since the situation is reversed:

- $D(u) \to \infty$ as $u \to \infty$ if $m > 1$ ("slow case")
- $D(u) \to 0$ as $u \to \infty$ if $m < 1$ ("fast case")

Indeed, power functions are tricky. The pressure transformation can also be used for the FDE range, but then the factor $m-1$ changes sign and the equation that we obtain is different because of the sign changes. Putting $v = cu^{m-1}$ with $c = m/(1-m)$ we get a new pressure equation of the form

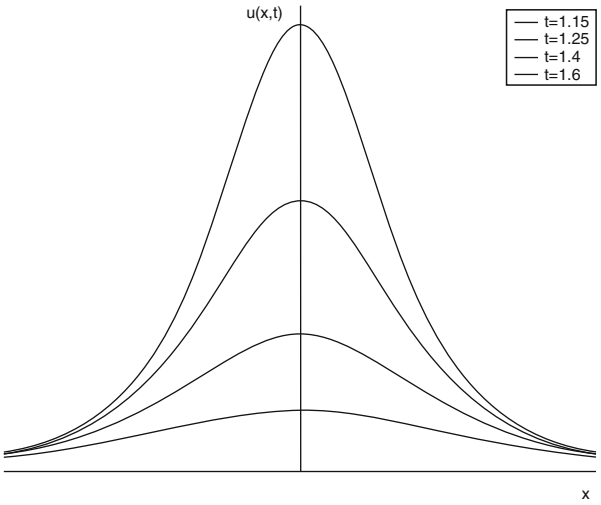
$$v_t = (1-m)v\Delta v - |\nabla v|^2\,, \tag{5.51}$$

so that the eikonal term is now an absorption term. Note that $u \to 0$ implies $v \to \infty$ in the FDE.

5.5.1 Barenblatt Solutions in the Good Range

We have well-known explicit formulas for source-type self-similar solutions called Barenblatt profiles, valid for with exponents m less than 1, but only if $1 > m > m_* = (N-2)/N$ if $N \geq 3$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = (C + k\xi^2)^{-1/(1-m)}.$$



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The decay rate and spreading rate exponents are

$$\alpha = \frac{N}{2 - N(1-m)}, \qquad \beta = \frac{1}{2 - N(1-m)} > 1/2\,.$$

Both exponents $\alpha, \beta \to \infty$ as m goes down to m_*. So the question is what happens for $m < m_*$? It is a long and complicated story, see a brief account further below.

The decay of the Barenblatt FDE profile for fixed time is $\mathbf{B} = O(|x|^{-2/(1-m)})$, a power-like decay that we have termed a *fat tail* in terms of probability distributions. The exponent ranges from N to ∞ in the range $m_* < m < 1$. Note that for $m_* < m < (N-1)/(N+1)$ the distribution $\mathbf{B}(\cdot, t)$ does not even have a first moment.

The exponent range $m_c < m < 1 <$ of the FDE where the Barenblatt solutions exist is called the "good fast diffusion range", since it has quite nice properties; though different from the linear heat equation, they nevertheless quite satisfying from many points of view, in particular from the point of view of existence, functional analysis, regularity and asymptotic behaviour. Thus, existence of a classical C^{∞} smooth solution is guaranteed for every nonnegative, locally bounded Radon measure as initial data (no growth conditions like the HE or the PME), the solution is unique, it is also positive everywhere even if the data are not (they must

be nontrivial), decay in time of the solutions depends in a predictable way from suitable norms of the data. Even unbounded Borel measures can be taken, see [85].

In particular, when $u_0 \in L^1(\mathbb{R}^N)$, a semigroup of contractions if generated, conservation of mass holds, and the solution converges for large time to the Barenblatt solution given above, and with the same expression of the Central Limit Theorem that we saw for the PME. The impressive Aronson-Bénilan estimate is now a two-sided universal estimate: $-C_1 u/t \le u_t(x,t) \le C_2 u/t$, which implies better and easier estimates for the rest of the theory. Of course, the absence of free boundaries makes it lose part of its power of attraction. We will not enter into the proofs of these results, than can be found in the literature, [99, 114, 249],...

5.5.2 Comparison of Anomalous Diffusions

The type of diffusion described by the Barenblatt solutions is called *anomalous diffusion* since it breaks the Brownian spread rate $|x| \sim t^{1/2}$ and space decay with an exponential rate. We have already seen that anomalous behaviour in the linear setting, as the fundamental solution $P_t(x)$ of the Fractional Heat Equation (FHE) $u_t + (-\Delta)^s u = 0$, cf. (5.22). In that case the spreading rate is $|x| \sim t^{1/2s}$. This leads to a formal (and partial) equivalence between anomalous diffusion of FHE and FDE types based on the spread strength. It reads $2s \sim 2 - N(1-m)$, hence

$$N(1-m) \sim 2(1-s),$$

which agrees for the classical heat equation: $m = 1$, $s = 1$. For the best known case $s = 1/2$ the equivalence gives $N = (N-1)/N$, a well-known exponent.

For the fundamental solution of the fractional Laplacian diffusion, the decay rate for fixed time is $P_t(x) = O(|x|^{-(N+2s)})$. This gives the formal equivalence between spatial decay rates: $N + 2s \sim 2/(1-m)$. It does not agree with the heat equation value in the limit: $m = 1$, $s = 1$.

One may wonder if we can get complete agreement of exponents and profile functions. This happens in dimension $N = 1$ for $s = 1/2$ and $m = 0$, a very exceptional case. The profile function is the Cauchy distribution

$$P(x) = \frac{1}{\pi} \frac{t}{t^2 + |x|^2}\,. \tag{5.52}$$

5.5.3 Subcritical, Logarithmic and Very Singular Fast Diffusion

The situation becomes much more involved once we cross the value $m_c = (N-2)/N$ for $N \ge 3$. In the range $m_c > m > 0$ many of the above properties do not hold. Thus,

there must be a condition on the initial data to guarantee that solutions becomes bounded, hence smooth, for all positive times. If $u_0 \in L^P_{loc}(\mathbb{R}^N)$ then we need $p > N(1-m)/2$, and the bound is sharp, otherwise regularization need not be true. When $u_0 \in L^1(\mathbb{R}^N)$, conservation of mass never holds, and in fact such solutions disappear in finite time, a phenomenon called extinction, that is discussed at length in our [249]. Obtaining valid versions of the Harnack inequality was challenging in this range [40, 41], see also the monograph [115].

- Logarithmic Diffusion. The limit $m \to 0$ makes sense if we slightly modify the constants in the equation and write it as

$$u_t = \nabla \cdot (u^{m-1}\nabla u) = (1/m)\Delta u^m. \tag{5.53}$$

  It is proved that the solutions $u_m(x,t)$ with fixed initial data, say bounded, converge as $m \to 0$ to a solution of the logarithmic diffusion equation

$$u_t = \nabla \cdot (\nabla u/u) = \Delta \log(u)\,, \tag{5.54}$$

  famous in 2D as a model for the evolution of the conformal matric by Ricci flow, as proposed by Hamilton [151] in 1988, where u is the conformal factor. A detailed study of the surprising mathematical theory is done in [249], where references are given. The following facts are remarkable: finite mass solutions are not uniquely determined by the initial data and moreover, they all lose at least 4π units of mass (which here means surface) per unit time. A very beautiful solution happens when we choose surface loss equal to 8π and the formula is

$$U(x,t) = \frac{8a(T-t)}{(a^2+|x|^2)^2}, \quad \text{with } a > 0. \tag{5.55}$$

  In the geometrical interpretation it describes the shrinking of a perfect 2D ball to a point in time $T > 0$. The ball is represented on the plane by stereographical projection. The solution qualifies as another beautiful diffusive pattern, this time it portrays extinction by Ricci flow. We ask the reader to note the difference with the Barenblatt FDE solutions, or with the Cauchy distribution (5.52).

Note that there is another natural limit as $m \to 0$, namely the equation $u_t = \Delta\,\mathrm{sign}(u)$. Though it means no flow for positive data, it has an interesting interpretation in terms of total variation flow for signed data, see [38] in 1D and compare with [217]. Total variation flow is an very important subject in itself, related to the p-Laplacian, cf. the monograph [8].

- Super-Fast Diffusion. Once we cover $m = 0$ the natural question is, can we cover $m < 0$. Actually, formula (5.53) makes perfect sense and a theory can be developed that extends much of what we have seen in the subcritical case $0 < m < m_c$. Some surprises arise in the form of nonexistence for integrable data

(which seem in principle the most natural), see [246]. The range is called very singular diffusion of super-fast diffusion.

- Subcritical Asymptotic Stabilization. The absence of Barenblatt solutions makes one wonder what happens for large times in the subcritical, logarithmic and very singular cases. This is a complicated topic, that needs lots of mathematics. We refer to [176, 249] for the earlier extinction analysis for so-called small solutions, and to [35, 45] for stabilization of solution with certain fat tails to the self-similar solutions called pseudo-Barenblatt solutions. The proofs are based on entropy-entropy dissipation methods. References to abundant related work are found.

5.5.4 Comments and Extensions

The subject FDE in the lower m ranges is quite rich. Part of the very interesting results concern problems posed in bounded domains, where the discussion is quite different. However, lack of space leads us not to continue the study of the Fast Diffusion range, and we refer to reader to monographs like [99, 249]. But let us just point out that there is no unity in the mathematics of the Fast Diffusion comparable to the Porous Medium range, and a number of critical exponents $m < 1$ appear. This is the source of many interesting functional developments and physical phenomena that researchers are still trying to understand.

The contents of the two last sections on PME and FDE can be translated to a large extent to the study of the evolution p-Laplacian equation (PLE), though some remarkable differences exist. We refer to the book [249] for an account of our ideas. There is even a transformation that maps all radial solutions of the PME to the corresponding class of the PLE, see [158]. Of course, $p = m + 1$ by dimensional considerations, but the transformation changes also the space dimension. If $N > 2$, then the corresponding PLE dimension is $N' = (N - 2)(m + 1)/2m$.

Some studies deal with the Doubly Nonlinear Equation $u_t = \nabla \cdot (|D(u^m)|^{p-2} D(u^m))$. See [232] for a recent work.

5.6 Nonlinear Fractional Diffusion: Potential Model

The combination of fractional diffusion and porous medium nonlinearities gives rise to interesting mathematical models that have been studied in the last decade both because of a number of scientific applications and for their mathematical properties. Two main models will be discussed below; a mechanical model has been developed in collaboration with Luis Caffarelli in Texas, and can be called *porous medium flow with fractional potential pressure* (or more generally, with nonlocal pressure); it has surprising properties. The other one has been developed later but it has better

analytical properties. For convenience we will call them here **PMFP** and **FPME**.[4] We will also examine models that interpolate between both.

5.6.1 Porous Medium Diffusion with Nonlocal Pressure

We devote this section to introduce model **PMFP**. It arises from the consideration of a continuum, say, a fluid, represented by a *density* distribution $u(x,t) \geq 0$ that evolves with time following a *velocity field* $\mathbf{v}(x,t)$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, and in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.

We still need a closure relation to relate p to u. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic. The PME follows. The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq's modelling. In that case we get the standard porous medium equation, $u_t = c\Delta(u^2)$. See Sect. 5.4 on the PME or [251] for these and many other applications.

The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the first steps of the derivation of the PME, but it differs by using a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel

$$k(x,y) = c|x-y|^{-(n-2s)}$$

(i.e., a Riesz operator). We have $(-\Delta)^s p = u$. This introduces long-distance effects in the model through the pressure, and we end up with a nonlocal model, given by the system

$$\boxed{u_t = \nabla \cdot (u\,\nabla p), \quad p = \mathcal{K}(u)} \tag{5.56}$$

[4] In [253] they were called Type I and Type II in reverse order.

where u is a function of the variables (x, t) to be thought of as a density or concentration, and therefore nonnegative, while p is the nonlocal pressure, which is related to u via a linear operator $\mathcal{K}$. We can write: $u_t = \nabla \cdot (u \nabla(-\Delta)^{-s} u)$. A technical observation: there are problems in defining $(-\Delta)^{-s} u$ in 1D since the kernel may be too singular, but then $\nabla(-\Delta)^{-s} u$ is always well defined, which is enough to perform the calculations that will be commented upon below.

The problem is posed for $x \in \mathbb{R}^N$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{5.57}$$

where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^N$.

5.6.2 Applied Motivation and Variants

- Particle Systems with Long Range Interactions. Equations of the more general form

$$u_t = \nabla \cdot (\sigma(u) \nabla \mathcal{L} u)$$

have appeared in a number of applications to the macroscopic evolution of particle systems. Thus, Giacomin and Lebowitz [136], 1997, consider a lattice gas with general short-range interactions and a Kac potential, and passing to the limit, the macroscopic density profile $\rho(r, t)$ satisfies the equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho) \nabla \frac{\delta F(\rho)}{\delta \rho} \right] \tag{5.58}$$

where $\sigma_s(\rho)$ may be degenerate. See also [137].

- Modeling Dislocation Dynamics as a Continuum. Following old modeling by Head in [152], Biler-Karch-Monneau [31] considered the one-dimensional case of model (5.56). By integration in x they introduced viscosity solutions à la Crandall-Evans-Lions. They prove that uniqueness holds, which is very satisfying property. But the corresponding mathematical model in several space dimensions looks quite different from (5.56).
- Hydrodynamic Limit for $s = 1$. This is a very interesting limit case. Putting $s = 1$ makes us lose the parabolic character of the flow that becomes hyperbolic. In 1D the situation is rather trivial since when we put $p = (-\Delta)^{-1} u$ we get $p_{xx} = -u_x$, and then

$$u_t = (u\, p_x)_x = u_x\, p_x - u^2$$

Moreover, if $v = -p_x = \int u\,dx$, we have

$$v_t = up_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + vv_x = 0$, which generates shocks in finite time but only if we allow for u to have two signs.

In several dimensions the issue becomes much more interesting because it does not reduce to a simple Burgers equation. We have

$$u_t = \nabla \cdot (u\,\nabla p) = \nabla u \cdot \nabla p - u^2, \qquad p = (-\Delta)^{-1} u\,, \tag{5.59}$$

A very close version to this model has appeared in superconductivity (the Chapman-Rubinstein-Schatzman-E model) see [192, 263], and Ambrosio-Serfaty [4]. In that application u describes the vortex density. Gradient flow structure for this example is established in [6].

- The PME Limit. If we take $s = 0$, $\mathcal{K}$ = the identity operator, we get the standard porous medium equation, whose behaviour is well-known. Therefore we can see the PMFP equation as a nonlinear interpolation between the PME and the hydrodynamic limit, $s = 1$.
- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

5.6.3 *Mathematical Results*

Early results on the PMFP have been reported in Proceedings from the Abel Symposium [252], and then in [253], so we will concentrate on general facts and only develop in more detail some of the new material. For applications of nonlinear nonlocal diffusion see also [64].

- In paper [70] Luis Caffarelli and the author established the existence of weak energy solutions, the basic properties of the solutions, like conservation of mass

$$\frac{d}{dt}\int u(x,t)\,dx = 0\,, \tag{5.60}$$

the two energy estimates

$$\frac{d}{dt}\int u(x,t)\log u(x,t)\,dx = -\int |\nabla Hu|^2\,dx\,, \tag{5.61}$$

where $H = (-\Delta)^{-s/2}$, and

$$\frac{d}{dt}\int |Hu(x,t)|^2\,dx = -2\int u|\nabla Ku|^2\,dx, \quad K = (-\Delta)^{-s}. \tag{5.62}$$

A number of usual properties in diffusive processes do hold here like conservation of positivity, as well as L^p decay. But we also found lack of a general comparison principle, a major difficulty in developing the theory (such a drawback will not be shared by the second model, FPME). And we could not prove uniqueness for general solutions in several space dimensions.

A main goal in the study of this model was to determine whether or not the *property of finite propagation* holds. The answer turned out to be yes. This is not clear in principle due to the competition between the slow propagation of the PME part with the infinite propagation of the fractional operator (amounting to long distance effects). The lack of plain comparison made the proof difficult, and the difficulty was surmounted by a novel use of the methods of viscosity solutions. Summing up, the degenerate character of the PME wins. On the contrary, infinite propagation was later proved to be true for FPME.

- In a second contribution [71] we explored the long-time behaviour in two steps. We first established the existence of self-similar profiles, so-called Fractional Barenblatt solutions

$$U(x,t) = t^{-\alpha}F(x\,t^{-\beta}), \qquad \beta = \frac{1}{N+2-2s},\ \alpha = N\beta,$$

The profile F is compactly supported, a clue to the finite propagation property, and is the solution of a certain fractional obstacle problem. A different proof in dimension 1 follows from paper [31]. The authors of [32] found the self-similar Barenblatt profiles in all dimensions with explicit formulas: $F(x) = (A - B\,|x|^2)_+^{1-s}$.

Then we introduced the renormalized Fokker-Planck equation and used a suitable entropy functional and proved stabilization of general solutions to the previous profiles that we called fractional Barenblatt profiles. All this is carefully explained in [252].

- The next issue in the programme was the regularity of the solutions. It was studied in a paper with Caffarelli and Soria, [75]. Proving boundedness for solutions with integrable data in L^p, $1 \le p \le \infty$ was an important step in the this theory. We can dispense with the extension method for fractional Laplacians by using energy estimates based on the properties of the quadratic and bilinear forms associated to the fractional operator, and then the iteration technique.

Theorem *Let u be a weak solution the initial-value problem for the PMFP with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive*

constant C such that for every $t > 0$

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^N)} \tag{5.63}$$

with $\alpha = N/(N+2-2s)$, $\gamma = (2-2s)/(N+2-2s)$. *The constant C depends only on N and s.*

The major step is then proving C^α regularity. The proof uses the DeGiorgi method with careful truncations together with very sophisticated energy methods that have to overcome the difficulties of both nonlinearity and nonlocality. A number of ideas come from Caffarelli-Vasseur [68] and [74] with difficult modifications due to the degenerate nonlinearity. The theory can be extended to data $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \ge 0$, giving global existence of bounded weak solutions.

5.6.4 Energy, Bilinear Forms and Fractional Sobolev Spaces

The previous results are obtained in the framework of weak energy solutions: The basis of the boundedness analysis is a property that goes beyond the definition of weak solution. The general energy property is as follows: for any real smooth function F and such that $f = F'$ is bounded and nonnegative, we have for every $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}\int F(u(t_2))\,dx - \int F(u(t_1))\,dx &= -\int_{t_1}^{t_2}\int \nabla[f(u)]u\nabla p\,dx\,dt \\ &= -\int_{t_1}^{t_2}\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx\,dt\,,\end{aligned}$$

where h is a function satisfying $h'(u) = uf'(u)$. We can write the last integral in terms of a bilinear form

$$\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx = \mathcal{B}_s(h(u),u)$$

This bilinear form $\mathcal{B}_s$ is defined as

$$\mathcal{B}_s(v,w) = C\iint \nabla v(x)\frac{1}{|x-y|^{N-2s}}\nabla w(y)\,dx\,dy = \iint \mathcal{N}_{-s}(x,y)\nabla v(x)\nabla w(y)\,dx\,dy$$

where $\mathcal{N}_{-s}(x,y) = C\,|x-y|^{-(N-2s)}$ is the kernel of operator $(-\Delta)^{-s}$. After some integrations by parts we also have

$$\mathcal{B}_s(v,w) = C_{n,1-s}\iint (v(x)-v(y))\frac{1}{|x-y|^{n+2(1-s)}}(w(x)-w(y))\,dx\,dy \tag{5.64}$$

since $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$. It is well known that $\mathcal{B}_s(u,u)$ is an equivalent norm for the fractional Sobolev space $W^{1-s,2}(\mathbb{R}^N)$. This is the way the fractional Sobolev spaces appear, as dissipated energies that will guarantee compactness in the arguments, see [75]. Fractional Sobolev spaces with a view to their use in PDEs are discussed in [116].

5.6.5 More Recent Work

- The particular value $s = 1/2$ of the fractional exponent turned out to be extremely delicate for the regularity analysis of [75] and needed a further article with new geometrical ideas, [71]. Briefly stated, there are some terms in the energy estimates that come from the tails of the solutions and cannot be suitably controlled in an iterative way. An iterated geometrical coordinate transformation allows to eliminate them at the cost of a *distorted geometry*.
- The Hydrodynamic Limit $s \to 1$ was studied by Serfaty and the author in [224]. We pass to the limit and construct a theory of existence, uniqueness and estimates for the hydrodynamic limit problem. It is interesting to note that the asymptotic attractor is a selfsimilar vortex of the form

$$U(x,t) = t^{-1} F(x/t^{1/N}),$$

  and F is the characteristic function of a ball. Therefore, even continuity is lost in the regularity of the solutions. This is not a contradiction since the limit equation is no longer parabolic. Our work is related to work on aggregation models by Bertozzi et al. [29].
- We posed the question of possible rates in the asymptotic convergence to selfsimilar solutions of Barenblatt type of papers [31, 71]. This question was partially solved in a paper [80] with Carrillo et al., where we showed exponential convergence towards stationary states for the Porous Medium Equation with Fractional Pressure in 1D. The many-dimensional case seems to be a difficult open problem, it is tied to some functional inequalities that are not known. Our analytical approach does not seem to apply either.

5.6.6 Additional Work, Open Problems

- The questions of uniqueness and comparison are solved in dimension $N = 1$ thanks to the trick of integration in space used by Biler et al. [31]. New tools are needed to make progress in several dimensions.

Recent uniqueness results are due to Zhou et al. [270]. They obtain local in time strong solutions in Besov spaces. Thus, for initial data in $B^{\alpha}_{1,\infty}$ if $1/2 \le s < 1$ and

$\alpha > N + 1$ and $N \geq 2$. Therefore, Besov regularity implies uniqueness for small times.

- The fractional Burgers connection was explored in [81] for $N = 1$, $s = 1/2$ where $\partial_x(-\Delta)^{-1/2} = -H$, the Hilbert transform.
- The study of the free boundary is in progress, but regularity is still open for small $s > 0$.
- The gradient flow structure of the **PMFM** flow in Wasserstein metrics has been recently established by Lisini et al. in [194]. For the general approach see the monograph [5]. Previous work in 1D was due to by Carrillo et al.
- The problem in a bounded domain with Dirichlet or Neumann data has not been studied, to our knowledge.
- Good numerical algorithms and studies are needed.

5.6.7 Elliptic Nonlinear Nonlocal Models

The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interactions, and this interest has been activated by the recent progress in the mathematical theory, in the form of a large number works on elliptic equations, mainly of the linear or semilinear type, as well as free boundary problems, like obstacle problems. There are so many works on the subject that we cannot refer them here. Let us mention the survey paper [64] by Caffarelli, that contains a discussion of the properties of solutions to several non-linear elliptic equations involving diffusive processes of non-local nature, including reference to drifts and game theory.

5.7 The FPME Model and the Mixed Models

Another natural model for the combination of fractional diffusion and porous medium nonlinearities is the equation that we will call fractional porous medium equation: $\partial_t u + (-\Delta)^s(u^m) = 0$. In order to be mathematically precise we write the equation as

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0} \tag{5.65}$$

with $0 < m < \infty$ and $0 < s < 1$. We will take initial data in $u_0 \in L^1(\mathbb{R}^N)$ unless mention to the contrary. Normally, $u_0, u \geq 0$. We will refer to this model as **FPME** for easy reference in this paper. Mathematically, it looks a more direct generalization of the linear fractional heat equation than the potential model **PMFP** studied in the previous section.

This model represents another type of nonlinear interpolation with parameter $s \in (0, 1)$, this time between the PME $u_t - \Delta(|u|^{m-1}u) = 0$ for $s = 1$ and the plain absorption ODE $u_t + |u|^{m-1}u = 0$ for $s = 0$.

We have written a detailed description of this model in the survey paper [253], where we give references to the physical motivations, among them [21, 159–161], the literature, and the mathematical developments until 2013 approximately. See also Appendix B of [41]. Therefore, we will mention the main items of the research, the references and general ideas, and then proceed to give notice of recent work, that covers different directions.

5.7.1 Mathematical Theory of the FPME

A complete analysis of the Cauchy problem posed for $x \in \mathbb{R}^N$, $t > 0$, with initial data in $L^1(\mathbb{R}^N)$ was done in two very complete papers coauthored with de Pablo F. Quirós, and Rodríguez: [106] in (2011) and [107] (2012). Using the Caffarelli-Silvestre extension method and the Bénilan-Brezis-Crandall functional semigroup approach, a weak energy solution is constructed, and $u \in C([0, \infty) : L^1(\mathbb{R}^N))$. Moreover, the set of solutions forms a semigroup of ordered contractions in $L^1(\mathbb{R}^N)$. This is the first instance of a 'better behaviour' than model **PMFP**.

- The second big difference is that *Nonnegative solutions have infinite speed of propagation for all m and s, so that there is no nonnegative solution with compact support (we mean, in the space variable).* Actually, a very important property of Model PMFP with Caffarelli is that solutions with compactly supported initial data do have the compact support property (i.e., they stay compactly supported for all times).

Even is propagation is always infinite in this model, we still use the name 'fractional FD' for the range $m < 1$ because of the many analogies with the usual FDE.

- On the other hand, some properties are similar in both models: Conservation of mass holds for all $m \geq 1$, and even for some $m < 1$ close to 1; the $L^1 - L^\infty$ smoothing effect works; and the C^α regularity holds also (unless m is near 0 and solutions are not bounded). Comparison of the models **PMFP** and **FPME** is quite interesting and has been pursued at all levels.
- The question of existence of classical solutions and higher regularity for the FPME and the more general model

$$\partial_t u + (-\Delta)^s \Phi(u) = 0$$

(where Φ is a monotone real function with $\Phi' \geq 0$) has been studied in two papers with the same authors (A.deP., F.Q., A.R., J.L.V.). The first paper, [108], treats the model case $\Phi(u) = \log(1 + u)$, which is interesting as a case of log-diffusion. The second treats general nonlinearities Φ and proves higher regularity

for nonnegative solutions of this fractional porous medium equation, [261]. This is a very delicate result. There is an extension of this result to prove C^∞ regularity to solutions in bounded domains by Bonforte et al. [49].

- Our paper [254] deals with the construction of what we call the fractional Barenblatt solution of the FPME, which has the also self-similar form:

$$U(x,t) = t^{-\alpha} F(xt^{-\beta}) \tag{5.66}$$

The construction works for $m > m_c = (N - 2s)/N$, a range that is optimal that reminds us of the Fast Diffusion Equation, Sect. 5.5. The difficulty is to find F as the solution of an elliptic nonlinear equation of fractional type. Such profile is not explicit as in the **PMFP** model (it is only for some very special exponents [156]). In any case, F has behaviour like a power tail

$$F(r) \sim r^{-(N+2s)}\,.$$

This is important for the applications and it the same as the one predicted by Blumental for the linear fractional kernel. This asymptotic spatial behaviour holds for all $m \geq 1$, and even for some $m < 1$, but not for some fast diffusion exponents $m_c < m < 1$ (see what happens then in [254]). The results are extended in [147].

Asymptotic behaviour as $t \to \infty$ follows, and this Barenblatt pattern is proved to be an attractor, as we were expecting from what has been seen along this whole text. The result holds for $m > m_c$. Open problem: Rates of convergence have not been found, and this is an interesting open problem.

Extinction in finite time is proved for exponents $0 < m < m_c$. The corresponding stabilization process must be studied.

- Another direction concerns regularity at the local or global level. In collaboration with M. Bonforte we have obtained a priori upper and lower estimates of intrinsic, local type for this problem posed in $\mathbb{R}^N$, [42]. Quantitative positivity and Harnack Inequalities follow. Against some prejudice due to the nonlocal character of the diffusion, we are able to obtain them here for fractional PME/FDE using a technique of weighted integrals to control the tails of the integrals in a uniform way. The novelty are the weighted functional inequalities. This also leads to existence of solutions in weighted L^1-space for the fast diffusion version FPME, a restriction that does not appear in the standard FDE.

More recent, very interesting work on bounded domains is reported in Sect. 5.8.

- Symmetrization (Schwarz and Steiner types). This is a project with Volzone [259, 260]. Applying usual symmetrization techniques is not easy and we have found a number of open problems. It turns out that Steiner symetrization works and it does much better for fractional FDE than for the fractional PME range. This work was followed by recent collaboration with Y. Sire and B. Volzone to apply the techniques to the fractional Faber-Krahn inequality, [229].

- We have also investigated very degenerate nonlinearities, like the *Mesa Problem*. This is the limit of FPME with $m \to \infty$. We have studied this limit in [255], and the limit flow characterized by the solution of a fractional obstacle problem, that is related to the obstacle problem for the PMFP that was described in [71].
- Numerics for the nonlinear nonlocal diffusion models is being done by a number of authors at this moment by : Nochetto et al. [208, 209], Teso [111].
- Fast diffusion and extinction. Very singular fast diffusion. Paper with Bonforte and Segatti [48], on non-existence due to instantaneous extinction, which is the common rule in very singular fractional fast diffusion as shown for standard diffusion in [246]. Paper [258] shows the existence of maximal solutions for some very singular nonlinear fractional diffusion equations in 1D in some borderline cases, this is an exception.
- We have looked at the phenomenon of KPP propagation in linear and nonlinear fractional diffusion with the particular reaction proposed by Kolmogorov-Piskunov-Petrovskii and Fisher [182]. In the case of standard linear diffusion travelling waves appear and serve as asymptotic attractors. Cabré and Roquejoffre [60, 61] studied the diffusion equation with linear fractional diffusion and KPP reaction and showed that there is no traveling wave propagation, and in fact the level sets move out at an exponential rate for large times. The results are extended to nonlinear fractional diffusion of the FPME type for all values of the exponents in work with Stan, [233].

5.7.2 *Mixed Models*

- The potential model **PMFP** given in (5.56) is generalized into **PMFP'**

$$u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u) \tag{5.67}$$

  with $m > 1$. This is an extension that accepts a general exponent m, so that the comparison of both models may take place on more equal terms.

The most interesting question seems to be deciding if there is finite and infinite propagation for **PMFP'**. Recent works with Stan and del Teso [234] and [236] show that finite propagation is true for $m \geq 2$ and propagation is infinite is $m < 2$. This is quite different from the standard porous medium case $s = 0$, where $m = 1$ is the dividing value of the exponent as regards propagation. The problem with existence is delicate for large m and is treated in a further paper [237].

An interesting and unexpected aspect of the theory is the existence of a transformation that maps self-similar solutions of the FPME with $m \geq 1$ into solutions of the same type for model **PMFP'** with exponent $1 < m < 2$. This applies in particular to the Barenblatt solutions constructed in [254]. The transformation is established in [235] and is quite useful in showing that **PMFP'** has infinite propagation in that range of parameters.

- Work by Biler-Imbert-Karch [33] deals with the variant

$$u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u^{m-1}) \,. \tag{5.68}$$

  They construct a family of nonnegative explicit compactly supported self-similar solutions which are a generalization of the well-known Barenblatt profiles for the classical porous-medium equation. They also establish the existence of sign-changing weak solutions to the initial-value problem, which satisfy sharp hypercontractivity L^1-L^p estimates.
- Reference [235] also treats on the double exponent model

$$\partial_t u + \nabla (u^{m-1} \nabla (-\Delta)^{-s} u^{n-1}) = 0 \tag{5.69}$$

  that generalizes all the previous models. The paper discusses self-similar transformations and finite propagation. The transformation of self-similar solutions indicates that finite propagation holds for $m \geq 2$, while $n > 1$ does not count. The graphic of parameters in [235] gives a very clear scheme of these transformations.

5.7.3 Some Related Directions

There is work on equations with other nonlocal linear operators, and also on equations with lower order terms, leading to reaction-diffusion and blow-up. Nonlinear diffusion and convection is treated in [1, 90]. The chemotaxis systems have been studied with nonlocal and/or nonlinear diffusion, like [30, 52, 119, 190]. We also have geometrical flows, like the fractional Yamabe problem (to be mentioned below). And there are a number of other options.

5.8 Operators and Equations in Bounded Domains

We have presented different definitions of the fractional Laplacian operator acting in $\mathbb{R}^N$ in Sect. 5.2, and we have mentioned that all these versions are equivalent. However, when we want to pose a similar operator in a bounded domain $\Omega \subset \mathbb{R}^N$ we have to re-examine the issue, and several non-equivalent options appear. This enlarges the theory of evolution equations of fractional type on bounded domains, and the recent literature has taken it into account. Actually, there is much recent progress in this topic and the next second subsection will describe our recent contributions. A large class of related nonlocal diffusive operators can be considered in the same framework.

5.8.1 The Linear Operators

There are a number of definitions that have been suggested for the fractional Laplacian operator (FLO) acting on a bounded domain Ω. The ones we consider here are naturally motivated, and they give rise to different operators. We will mention three basic options, two of them are mostly used.

The Restricted Fractional Laplacian operator (RFL) It is the simplest option. It acts on functions $g(x)$ defined in Ω and extended by zero to the complement, and then the whole hypersingular integral of the Euclidean case is used. Therefore, it is just the fractional Laplacian in the whole space "restricted" to functions that are zero outside Ω.

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,, \qquad \text{with supp}(g) \subset \overline{\Omega}\,. \tag{5.70}$$

Here, $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant. It is shown that, thus defined, $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum, with eigenvalues

$$0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots,$$

satisfying $\overline{\lambda}_j \asymp j^{2s/N}$, for $j \gg 1$. The corresponding eigenfunctions $\overline{\phi}_j$ are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$, [218].

An important issue is the way in which the additional conditions (formerly boundary conditions) are implemented for the RFL. It usually takes the form of exterior conditions:

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty)\times\left(\mathbb{R}^N \setminus \Omega\right). \tag{5.71}$$

The behavior of the Green function G plays an important role in the corresponding PDE theory. It satisfies a strong behaviour condition, that we call (K4) condition:

$$G(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \tag{K4}$$

where $\delta(x)$ is the distance from $x \in \Omega$ to the boundary. The exponent γ will play a role in the results derived from the kernel. In the RFL we have $\gamma = s$.

References There is an extensive literature on the RFL operator and the corresponding α-stable process in the probability literature. The interested reader is referred [47]where we have commented on relevant works in that direction.

The Spectral Fractional Laplacian Operator (SFL) It is defined by the two equivalent expressions

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x)\right) \frac{dt}{t^{1+s}}\,, \qquad (5.72)$$

where Δ_Ω is the classical Dirichlet Laplacian on the domain Ω, and $\hat{g}_j$ are the Fourier coefficients of f

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{L^2(\Omega)} = 1\,.$$

In this case the eigenfunctions ϕ_j are the same as in the Dirichlet Laplacian, smooth as the boundary of Ω allows. Namely, when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$. The eigenvalues are powers λ_j^s of the standard eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$. It is proved that the eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \ge 1$, [86].

Lateral boundary conditions for the SFL are different from previous case. They can be read from the boundary conditions of the Dirichlet Laplacian by the semigroup formula. They are often defined by means of the equivalent formulation that uses the Caffarelli-Silvestre extension defined in a cylinder adapted to the bounded domain, as done in [56, 62, 240]. If U is the extended function, then we impose $U = 0$ on the lateral boundary $x \in \partial\Omega$, $y > 0$.

The Green function of the SFL satisfies the strong assumption (K4), this time with exponent $\gamma = 1$.

Remarks Both SFL and RFL admit another possible definition using the so-called Caffarelli-Silvestre extension. They are the two best known options for a FLO. The difference between RFL and SFL seems to have been well-known to probabilists, it was discussed later in PDEs, see Servadei-Valdinoci [226], Bonforte and the author [43], and Musina-Nazarov [204]. In this last work the denomination *Navier fractional Laplacian* is used. The debate about the proper names to be used is not settled.

The Censored Fractional Laplacians (CFL) This is another option appearing in the probabilistic literature, it has been introduced in 2003 by Bogdan et al. [37]. The definition is

$$\mathcal{L}g(x) = \text{P.V.} \int_\Omega (g(x) - g(y)) \, \frac{a(x,y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with } \frac{1}{2} < s < 1\,, \qquad (5.73)$$

where $a(x, y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$. In the simplest case we put $a(x, y) =$ constant. On the other hand, [37] point out that in the

excluded range $s \in (0, 1/2]$ the censored 2s—stable process is conservative and will never approach the boundary. The CFL is also called *regional fractional Laplacian*.

The Green function $G(x, y)$ satisfies condition (K4) with $\gamma = s - \frac{1}{2}$, as proven by Chen et al. [88]. See also [149].

Note We have presented three models of Dirichlet fractional Laplacian. The estimates (K4) show that they are of course **not equivalent**. Our work described in the next subsection applies to those operators and a number of other variants, that are listed in [44, 50]. For instance, sums of operators of the above types and powers of said operators are included.

5.8.2 *Nonlocal Diffusion of Porous Medium Type on Bounded Domains*

We report here on very recent work done in collaboration with Bonforte, and also Sire and Figalli, on nonlinear evolution equations of porous medium type posed in bounded domains and involving fractional Laplacians and other nonlocal operators. The papers are [43, 44, 47, 49, 50].

We develop a new programme for nonlocal porous medium equations on bounded domains aiming at establishing existence, uniqueness, positivity, a priori bounds, regularity, and asymptotic behaviour for a large class of equations of that type in a unified way. We include the set of suitable versions of FLO in a bounded domain. The main equation is written in abstract form as

$$\partial_t u + \mathcal{L}\Phi(u) = 0\,, \tag{5.74}$$

where Φ a continuous and nondecreasing real function, most often a power function.

- A problem to be settled first is the suitable concept of solution. We use the "dual" formulation of the problem and the concept of *weak dual solution*, introduced in [43, Definition 3.4], which extends the concept of weighted very weak solution used before. In brief, we use the linearity of the operator $\mathcal{L}$ to lift the problem to a problem for the potential function

$$U(x,t) = \int_\Omega u(y,t)G(x,y)dy$$

  where G is the elliptic Green function for $\mathcal{L}$. Then $\partial_t U = -\Phi(u)$.
- Class of Operators. In our recent work we have extended the evolution theory to cover a wide class of linear operators $\mathcal{L}$ that satisfy the following conditions. $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and

sub-Markovian, more precisely, it satisfies (A1) and (A2):

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$;
(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$.

Moreover, the inverse operator $\mathcal{L}^{-1}$ can be written as

$$\mathcal{L}^{-1}[f](x) = \int_\Omega \mathbb{K}(x,y) f(y)\,\mathrm{d}y\,,$$

The kernel $\mathbb{K}$ is called the Green function and we assume that there exist constants $\gamma \in (0,1]$ and $c_{0,\Omega}, c_{1,\Omega} > 0$ such that, for a.e. $x, y \in \Omega$:

$$c_0\,\delta^\gamma(x)\,\delta^\gamma(y) \le \mathbb{K}(x,y) \le \frac{c_1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \tag{K2}$$

where we adopt the notation $\delta(x) := \mathrm{dist}(x, \partial\Omega)$. We will also use ϕ_1, the first eigenfunction of $\mathcal{L}$, and we know that $\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma$. Further assumptions will be made in each statement, depending on the desired result we want, in particular (K4) that we have already mentioned.

- Sharp Bounds. Under these assumptions, we obtain existence and uniqueness of solutions with various properties, like time decay in L^p spaces. We will not delve in this basic theory that is covered in the papers [44, 47]. We will stress here one of our main contributions in [50]: we prove sharp upper and lower pointwise bounds for nonnegative solutions, both at the interior and close to the boundary. Indeed, we must pay close attention to the boundary behaviour, that turns out to be different for different operators in this class. However, only some options appear, as we describe next. Let us introduce first an important exponent

$$\sigma = 1 \wedge \frac{2sm}{\gamma(m-1)}.$$

Notice that $\sigma = 1$ for the RFL and the CFL, but not always for the SFL unless $m = 1$. The results that follow are taken from the last work, [50]. In the next results (CDP) means the Cauchy-Dirichlet problem with zero lateral data, and solutions means dual weak solutions. We make the default assumptions (A1), (A2), and (K2) on $\mathcal{L}$.

Case 1. Nonlocal Operators with Nondegenerate Kernels We assume here moreover that the kernel of $\mathcal{L}$ is non degenerate at the boundary, namely

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,\mathrm{d}y\,, \qquad \text{with } \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,. \tag{5.75}$$

Under these assumptions we can prove the following first version of the Global Harnack Principle.

Theorem 5.8.1 *Let (A1), (A2), (K2), and (5.75) hold. Also, when $\sigma < 1$, assume that $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$ and that $\phi_1 \in C^\gamma(\Omega)$. Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\phi_1}(\Omega)$. Then, there exist constants $\underline{\kappa}, \overline{\kappa}$, so that the following inequality holds:*

$$\underline{\kappa}\left(\frac{t}{t+t_*}\right)^{\frac{m}{m-1}} \frac{\phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \tag{5.76}$$

for all $t > 0$ and all $x \in \Omega$.

Here, $t_* = \kappa_* \|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\phi_1}(\Omega)}$, and this time will appear in the other theorems. For large times both lower and upper bounds are similar. We point out that the result holds in particular for the Restricted and Censored Fractional Laplacians, but not for the Spectral Fractional Laplacian. The lower bound is false for $s = 1$ (in view of the finite speed of propagation of the standard PME).

Case 2. Matching Behaviour for Large Times We can prove that previous Global Harnack Principle for large times without using the non-degeneracy of the kernel, under the following conditions on σ : either

(i) $\sigma = 1$ (i.e., $2s > \gamma(m-1)/m$), or
(ii) $\sigma < 1$, and we have an improved version of (K2)

$$\mathbb{K}(x,y) \asymp \frac{c_1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \tag{K4}$$

and the initial data are not small: $u_0 \ge c\phi_1^{\sigma/m}$ for some $c > 0$.

Theorem 5.8.2 (Global Harnack Principle II) *Let (A1), (A2), and (K2) hold, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\phi_1}(\Omega)$. Assume that either (i) or (ii) above hold true. Then there exist constants $\underline{\kappa}, \overline{\kappa} > 0$ such that the following inequality holds:*

$$\underline{\kappa}\,\frac{\phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \ge t_* \text{ and all } x \in \Omega\,. \tag{5.77}$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

The conditions on σ are sharp. Actually, the proof in the case $\sigma = 1$ includes the classical PME (i.e., the non fractional equation, for which finite propagation holds, so that there can be no positive a priori lower bound for short times).

The Case of a Really Degenerate Kernel We assume moreover that we assume moreover that the kernel of $\mathcal{L}$ exists and can be degenerate at the boundary (actually, excluding the local case, this is the most general assumption) in the form

$$\mathcal{L}f(x) = P.V.\int_{\mathbb{R}^N} \big(f(x) - f(y)\big)K(x,y)\,\mathrm{d}y\,, \quad \text{with } K(x,y) \ge c_0\phi_1(x)\phi_1(y) \ \ \forall\, x, y \in \Omega\,. \tag{5.78}$$

This is an assumption that holds for the Spectral Fractional Laplacian operator. To our knowledge, precise information about the kernel of the SFL was not known before Lemma 3.1 of [50].

Note that, for small times, we cannot find matching powers for a global Harnack inequality (except for some special initial data), and such result is actually false for $s = 1$ (in view of the finite speed of propagation of the PME). Hence, in the remaining cases, we have only the following general result.

Theorem 5.8.3 (Global Harnack Principle III) *Let (A1), (A2), (K2), and (5.78) hold. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\phi_1}(\Omega)$. Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:*

$$\underline{\kappa} \left(\frac{t}{t + t_*} \right)^{\frac{m}{m-1}} \frac{\phi_1(x)}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa} \frac{\phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \tag{5.79}$$

for all $t > 0$ and all $x \in \Omega$.

This is what we call non-matching powers for the spatial profile at all times. The paper gives analytical and numerical evidence that such non matching behaviour does not happen in the associated elliptic problems, and came as a surprise to the authors. For some class of initial data, namely $u_0 \leq \varepsilon_0 \phi_1$ we can prove that for small times

$$\underline{\kappa}_0 \left(\frac{t}{T} \right)^{\frac{m}{m-1}} \frac{\phi_1(x)}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}_0 T^{\frac{1}{m-1}} \frac{\phi_1(x)}{t^{\frac{1}{m-1}}} \qquad \text{for all } 0 \leq t \leq T \text{ and all } x \in \Omega\,.$$

Numerics. This work has been improved in January 2017 with numerics done at BCAM Institute by my former student del Teso and collaborators, [97], 2017, that validates the different behaviour types.

- Asymptotic Behaviour. An important application of the Global Harnack inequalities of the previous section concerns the sharp asymptotic behavior of solutions. More precisely, we first show that for large times all solutions behave like the separate-variables solution $\mathcal{U}(t,x) = S(x)\, t^{-\frac{1}{m-1}}$. The profile S is the solution of an elliptic nonlocal problem. Then, whenever the Global Harnack Principle (GHP) holds, we can improve this result to an estimate in relative error.

Theorem 5.8.4 (Asymptotic Behavior) *Assume that $\mathcal{L}$ satisfies (A1), (A2), and (K2), and let S be as above. Let u be any weak dual solution to the (CDP). Then, unless $u \equiv 0$,*

$$\left\| t^{\frac{1}{m-1}} u(t,\cdot) - S \right\|_{\mathrm{L}^\infty(\Omega)} \xrightarrow{t \to \infty} 0\,. \tag{5.80}$$

We can exploit the (GHP) to get a stronger result, using the techniques of paper [47].

Theorem 5.8.5 (Sharp Asymptotic Behavior) *Under the assumptions of Theorem 5.8.4, assume that $u \not\equiv 0$. Furthermore, suppose that either the assumptions of Theorem 5.8.1 or of Theorem 5.8.2 hold. Set $\mathcal{U}(t,x) := t^{-\frac{1}{m-1}} S(x)$. Then there exists $c_0 > 0$ such that, for all $t \geq t_0 := c_0 \|u_0\|_{\mathrm{L}^1_{\phi_1}(\Omega)}^{-(m-1)}$, we have*

$$\left\| \frac{u(t,\cdot)}{\mathcal{U}(t,\cdot)} - 1 \right\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{2}{m-1} \frac{t_0}{t_0 + t}\,. \tag{5.81}$$

We remark that the constant $c_0 > 0$ only depends on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

Comments on Related Work Construction of the solutions of the FPME on bounded domains with the SFL was already used in [106, 107] as an approximation to the problem in the whole space, but the regularity or asymptotic properties were not studied. Kim and Lee in [173] study the Fast Diffusion range $m < 1$ with a fractional Laplacian in a bounded domain and prove Hölder regularity and asymptotic behaviour. There is a developing literature on nonlocal nonlinear diffusion equations on domains.

5.8.3 *Fractional Diffusion Equations of p-Laplacian Type*

We report here about our work [257]. It deals with a model of fractional diffusion involving a nonlocal version of the p-Laplacian operator, and the equation is

$$\boxed{\partial_t u + \mathcal{L}_{s,p} u = 0, \qquad \mathcal{L}_{s,p}(u) := \int_{\mathbb{R}^N} \frac{\Phi(u(y,t) - u(x,t))}{|x-y|^{N+sp}}\, dy = 0} \tag{5.82}$$

where $x \in \Omega \subset \mathbb{R}^N$, $N \geq 1$, $\Phi(z) = c|z|^{p-2}z$, $p \in (1,\infty)$ and $s \in (0,1)$. $\mathcal{L}_{s,p}$ is the Euler-Lagrange operator corresponding to a power-like functional with nonlocal kernel of the s-Laplacian type. The study of the equation is motivated by the recent increasing interest in nonlocal generalizations of the porous-medium equation. In the paper we cover the range $2 < p < \infty$. Note that for $p = 2$ we obtain the standard s-Laplacian heat equation, $u_t + (-\Delta)^s u = 0$, which was discussed before; on the other hand, it is proved that in the limit $s \to 1$ with $p \neq 2$, we get the well-known p-Laplacian evolution equation $\partial_t u = \Delta_p(u)$, after inserting a normalizing constant.

We consider the equation in a bounded domain $\Omega \subset \mathbb{R}^N$ with initial data

$$u(x,0) = u_0(x), \qquad x \in \Omega\,, \tag{5.83}$$

where u_0 is a nonnegative and integrable function. Moreover, we impose the homogeneous Dirichlet boundary condition that in the fractional Laplacian setting

takes the form

$$u(x,t) = 0 \qquad \text{for all } x \in \mathbb{R}^N, x \notin \Omega, \qquad \text{and all } t > 0. \tag{5.84}$$

When then apply the integral operator on the set of functions that vanish outside of Ω

The first result of this paper concerns the existence and uniqueness of a strong nonnegative solution to an initial-boundary value problem for (5.82) in bounded domain $\Omega \subset \mathbb{R}^N$, with zero Dirichlet data outside $\Omega \times (0, \infty)$. The boundedness of the solution is established after proving the existence of a special separating variable solution of the form

$$U(x,t) = t^{-1/(p-2)} F(x),$$

called the friendly giant. The profile function $F(x)$ of the friendly giant solves the interesting nonlocal elliptic problem

$$\int_{\mathbb{R}^N} \frac{\Phi(F(y) - F(x))}{|x-y|^{N+sp}} dy = c\, F(x).$$

The friendly giant solution provides a universal upper bound and also gives the large-time behaviour for all the nonnegative solutions of initial-boundary value problems with homogeneous Dirichlet boundary conditions.

The fractional p-Laplacian has recently attracted the attention of many researchers for its mathematical interest. See among other related works the papers by Caffarelli et al. [74], Mazón et al. [197], Puhst [214]. Another approach in the form of non-local gradient dependent operators is taken by Bjorland-Caffarelli-Figalli in [34]. The corresponding stationary equation is also studied in the literature, see previous references. Finally, the work [157] by Hynd and Lindgren deals with the doubly nonlinear model $|u_t|^{p-2} u_t + \mathcal{L}_{s,p} u = 0$, that has a special homogeneous structure. Regularity and asymptotic behaviour follow.

5.9 Further Work on Related Topics

5.9.1 Diffusion with Fractional Time Derivatives

Equations of the form

$$D_t^\alpha u = \mathcal{L} u + f$$

are another form of taking into account nonlocal effects. Here $\mathcal{L}$ represents the diffusion process with long-distance effects in the family of the fractional Laplacian operators. The symbol D_t^α denotes the fractional time derivative. There are a number

of variants of this concept, the most popular being maybe the *Caputo fractional derivative*, which was introduced by Caputo in 1967 [77] and reads

$$\boxed{{}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-n}}, \quad n-1 < \alpha \le n} \tag{5.85}$$

Indeed, fractional time derivatives are the most elementary objects of Fractional Calculus, a branch of mathematical analysis that studies the possibility of taking real number powers (real number fractional powers or complex number powers) of the differentiation operator $D = d/dx$, and the integration operator.

The foundations of the theory of fractional derivatives were laid down by Liouville in a paper from 1832. Different definitions use different kernels, but all them make weighted averages in time.

Some recent work: Dipierro and Valdinoci derive the linear time-fractional heat equation in 1D in a problem of neuronal transmission in cells, [117]; Allen, Caffarelli and Vasseur study porous medium flow with both a fractional potential pressure and a fractional time derivative,[2].

5.9.2 *Diffusion Equations on Riemannian Manifolds*

- The study of the heat equation posed on a Riemannian manifold comes from long time ago, the diffusive operator being the Laplace-Beltrami operator

$$\Delta_g(u) = \frac{1}{\sqrt{|g|}} \partial_i (g^{ij} \sqrt{|g|} \partial_j u)\,, \tag{5.86}$$

  with the usual Riemannian notations. Thus, the heat kernel was studied in [101, 143, 268], random walks and Brownian motion on manifolds are studied in [245]. There are lots of recent works, an example is [144].
- Generalization of the Caffarelli-Silvestre extension method allows to define extensions and boundary operators of the fractional Laplacian type when M is the boundary of a conformally compact Einstein manifold. Combining geometrical and PDE approaches, Chang and González in [83] related the original definition of the conformal fractional Laplacian coming from scattering theory to a Dirichlet-to-Neumann operator for a related elliptic extension problem for M, see also [141]. It is possible then to formulate fractional Yamabe-type problems for the conformally covariant operators P_γ, [142]. For more recent work in this problem see [100, 162].
- Work on the Porous Medium Equation on manifolds was done in the 2000s, like [39, 251]; fast diffusion was treated in [46]; general Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds were obtained in [195].

- Recently the author studied the PME on the hyperbolic space [256], and constructed the fundamental solution and proved the asymptotic convergence and free boundary propagation rates. The fact that the fundamental is not explicit or self-similar is not pleasant, and looking for some higher symmetry properties a remarkable object appeared to play an asymptotic role. Namely, there exists nonnegative weak solution of the PME defined on the whole of $\mathbb{H}^N$ for all $t > 0$, that has a strong algebraic structure. In the Poincaré upper half-space representation it is given by the formula

$$U(x, y, t)^{m-1} = a\frac{(\log(ct^{\gamma} y))_+}{t} \tag{5.87}$$

  with $m, N > 1$, $1/a = m(N-1)$ and $1/\gamma = (N-1)(m-1)$, and $x \in \mathbb{R}^{N-1}$, $y > 0$. Note that U has zero initial trace at $t = 0$ on the half-space, but it has a singularity as trace at $y = +\infty$, which corresponds to a singularity at the North Pole in the standard Poincaré ball representation. Therefore, we can say that the special solution (or geometrical soliton) U comes from the infinite horizon and expands to gradually to fill the whole space; for any $t > 0$ it has a support limited by a family of horospheres $\{(x, y) : y = (1/c)t^{-\gamma}\}$. Recall that geodesic distance is given by the formula $ds^2 = dy^2/y^2$. A detail for analysts: U represents an example of non-uniqueness of nonnegative solutions for the Cauchy Problem in hyperbolic space.

This family of special solutions is the pattern to which all solutions with compactly supported initial data are proved to converge as $t \to \infty$. Accordingly, we get the following sharp estimates for all solutions $u \geq 0$ with compact support:

$$\|u(\cdot, t)\|_\infty \sim ct^{-1/(m-1)} \log(t)^{1/(m-1)}, \quad S(t) \sim \gamma \log(t)$$

where $S(t)$ is the location of the free boundary measured in geodesic distance.

- The asymptotic analysis of PME flows is extended to more general manifolds in [148]. The work on the asymptotic behaviour on hyperbolic space is extended to fast diffusion by Grillo et al. in [145]. In another direction, Amal and Elliott [3] study fractional porous medium equations on evolving surfaces, a very novel subject.

5.9.3 Diffusion in Inhomogeneous Media

We have already mentioned the inhomogeneity of the medium as a reason for the introduction of coefficients in the passage from the heat equation to the parabolic class. In view of the important practical consequences, there is no surprise in finding coefficients appear in most of the models we have considered above, both linear and nonlinear, local and nonlocal. Let us just mention some well-known references

like [165, 167, 215], where it appears as diffusion with weights, see also the more recent [146] with two weights. There is an interesting connection between weighted diffusion in Euclidean space and Laplace-Beltrami diffusion on manifolds, that has been studied in [256] and is being further investigated.

5.9.4 *Drift Diffusion Equations with or Without Fractional Terms*

The main equation in this case is an equation for a scalar unknown θ driven by the equation

$$\partial_t \theta + \mathbf{v} \cdot \nabla_x \theta = \mathcal{L}(\theta)\,, \tag{5.88}$$

where $\mathbf{v} \cdot \nabla_x T$ is the convective term with velocity vector $\mathbf{v}$, and $\mathcal{L}(\theta)$ is the diffusion operator; that diffusion can be linear of nonlinear, local or nonlocal. A very important aspect is the relation of $\mathbf{v}$ to the rest of the variables. Thus, when $\mathbf{v} = \mathbf{v}(x,t)$ is a given function, no essential new problems arise if $\mathbf{v}$ is smooth. But serious difficulties happen for nonsmooth $\mathbf{v}$. All this is reflected for instance in the seminal paper by Caffarelli and Vasseur [68] where the equation is

$$\partial_t \theta + \mathbf{v} \cdot \nabla_x \theta + (-\Delta)^{1/2}(\theta) = 0$$

and $\mathbf{v}$ is a divergence-free vector field. In the popular quasi-geostrophic model $\mathbf{v}$ is given in terms of θ that makes the problem more involved, but the results stated in [68] do not depend upon such dependence. The proof of regularity needs to establish delicate local energy estimates, despite the fact that the diffusion operator $(-\Delta)^{1/2}$ is non-local. It also uses DeGiorgi's methods in an essential way. There is much work in geostrophic flows, like [179, 180, 196].

There is a vast literature on this important issue. In some cases $\mathbf{v}$ is given by Darcy's law in an incompressible fluid, and the papers refer to the problem as "flow in porous media", like [82]. Let us point out that this use is quite different from our use of "porous media" in the present paper, the difference being often stressed by calling their use "incompressible flow in porous media", [124].

5.9.5 *Other*

– Minimal surfaces are an important subject which uses many methods of the nonlinear elliptic and parabolic theory, Recently, it has developed a new branch, nonlocal minimal surfaces. Work on both aspects is reported in detail in another contribution to this volume by Cozzi and Figalli [92]. Related items are fractional

perimeters and fractional phase transition interfaces. We will not enter into that area.
– The study of the combined effects of diffusion and aggregation is a very active field where the methods of diffusion in its different forms must be combined with the counter mechanism of attraction. We refer to the contribution by Calvez et al. to this volume, [76].
– In the study of nonlinear diffusion we have chosen to present almost exclusively equations with diffusion terms in divergence form. There is a large body of work involving Fully Nonlinear Parabolic Equations (they are non-divergence equations), both elliptic and parabolic. We will not touch such theories here.

5.10 Addendum and Final Comment

Here is the complete Wikipedia list of diffusion topics:

Anisotropic diffusion, also known as the Perona-Malik equation, enhances high gradients; Anomalous diffusion, in porous medium; Atomic diffusion, in solids; Brownian motion, for example of a single particle in a solvent; Collective diffusion, the diffusion of a large number of (possibly interacting) particles; Eddy diffusion, in coarse-grained description of turbulent flow; Effusion of a gas through small holes; Electronic diffusion, resulting in electric current; Facilitated diffusion, present in some organisms; Gaseous diffusion, used for isotope separation; Heat flow, diffusion of thermal energy; Itō-diffusion, continuous stochastic processes; Knudsen diffusion of gas in long pores with frequent wall collisions; Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field; Osmosis is the diffusion of water through a cell membrane; Photon diffusion; Random walk model for diffusion; Reverse diffusion, against the concentration gradient, in phase separation; Self-diffusion; Surface diffusion, diffusion of adparticles on a surface; Turbulent diffusion, transport of mass, heat, or momentum within a turbulent fluid.

- The reader may wonder whether mathematical diffusion is a branch of applied mathematics? In principle it would seem that the answer is an obvious yes, and yet it is not so clear. As we have seen, the mathematical theories of diffusion have developed into a core knowledge in pure mathematics, that encompasses several branches, from analysis and PDEs to probability, geometry and beyond. We hope that the preceding pages will have convinced the reader of this trend, and will also motivate him/her to pursue some of many avenues open towards the future.

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