Using Digital Environments to Address Students' Mathematical Learning Difficulties

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Abstract The need to deal with different cognitive necessities of students in the mathematical classroom, and in particular of students who persistently fail in mathematics, frequently referred to as "having mathematical learning difficulties or disabilities" (MLD), has become an important topic of research in mathematics education and in cognitive psychology. Though frameworks for analyzing students' difficulties and/or for designing inclusive activities are still quite fragmentary, the literature rather consistently suggests that technology can support the learning of students with different learning characteristics. The focus of this chapter is on providing insight into this issue by proposing analyses of specific software with a double perspective. We will analyze design features of the selected software, based on the potential support these can provide to students' learning processes, in particular those of students classified as having MLD. We will also analyze some interactions that actually occurred between students and the software, highlighting important qualitative results from recent studies in which we have been involved.

1 Introduction

Since we will be discussing software with respect to students "with mathematical learning difficulties (MLD)" it is necessary to first explain how *unclear* the situation actually is around the issue of low achievement in mathematics and MLD. This will be done in the first section of this chapter, immediately followed by our opinion on ways in which software can address specific MLD. The rest of the chapter is divided into three other sections: one in which we describe the theoretical background we will be using to analyze the proposed examples of software; one in

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which we explore the design of specific digital environments to which we have contributed; and a last section presenting selected results from studies we have conducted with students using the previously analyzed software.

1.1 The Murky Notion of "Students with MLD"

When exploring persistent low achievement in mathematics from a cognitive point of view, most of the literature from the field of psychology investigates typical development of basic number processing, introducing terms for describing atypical situations. Terms used to refer to students in such situations include "developmental dyscalculia", "mathematical learning disability (or disorder)", among many others (e.g., Butterworth, 2005; Passolunghi & Siegel, 2004; Piazza et al., 2010). The definitions of these terms are still a topic of debate (e.g., Mazzocco, 2008), and the ways in which they are used in different studies is inconsistent. For example, Mazzocco and Räsänen (2013) note that "math learning disability (MLD) has been used as synonymous with DD [Developmental Dyscalculia] […], but also as distinct from DD when MLD is used to refer to the larger category of mathematics difficulties (MD)" (ibid., p. 66). Even the use of the acronym MLD is not consistent, in that the "D" in some cases stands for "disabilities" and in others for "difficulties" (ML stands for "Mathematical Learning" in all cases). We attribute this, at least in part, to a problem described by Heyd-Metzuyanim (2013), according to which the "learning disability" construct does not afford to differentiate between difficulties that signal a stable disability in mathematics and those that are a result of inadequate teaching experiences or lack of sufficient exposure (also see González & Espínel, 1999; Mazzocco & Myers, 2003).

The bulk of studies conducted within the field of cognitive psychology use tests of different cognitive abilities (either cognitive domain specific or general) and investigate how scores derived from those tests correlate with students mathematical performance on standardized achievement tests (e.g., Geary, 1994, 2004; Nunez & Lakoff, 2005; Piazza et al., 2010; Andersson & Östergren, 2012; Szucs et al., 2013; Bartelet, Ansari, Vaessen, & Blomert, 2014). In this scenario it is not surprising that the cut off scores for diagnosing MLD vary from the 3rd to the 32nd percentile (Mussoli, 2009), and prevalence is reported between 1.3 and 13.8% of the population (see, for example, Kaufmann et al., 2013; Mazzocco & Räsänen, 2013; Watson & Gable, 2013).

It is beyond the scope of this chapter to delve deeper into these issues; for our purposes it suffices to consider students "having MLD" as students with persistent low achievement in mathematics (this is what the "D" in the acronym MLD will refer to in this chapter), who are at risk of being labelled by clinicians as "having a learning disability" or who have been diagnosed clinically with such a condition.[1] So any of these conditions are what we imply when using the acronym MLD in this chapter.

In Italy the percentage of these students diagnosed with learning disabilities is estimated between 3 and 5% (MIUR, 2011a) and over the last few years the percentages have been persistently increasing (MIUR, 2011b). Because of this phenomenon and because in Italy classrooms are completely inclusive,[2] it has become a more and more pressing issue to study and develop didactical practices appropriate for *all* students (Ianes, 2006; Ianes & Demo, 2013). Though frameworks for analyzing students' difficulties and/or for designing inclusive activities are still quite fragmentary, the literature rather consistently suggests that technology can support the learning of students with different learning characteristics (Edyburn, 2005; Baccaglini-Frank & Robotti, 2013; Robotti, Antonini, & Baccaglini-Frank, 2015), also in *inclusive* teaching settings, such as the Italian classrooms (Robotti & Ferrando, 2013).

1.2 How Can Software "Address" Specific MLD?

We must ask ourselves what it means to "address" students' learning difficulties. Once we will have agreed upon a meaning for this, we will be able to discuss how software can do it.

The paradigm used (at least in Italy) in special needs education, as has recently been argued by the Santi and Baccaglini-Frank (2015), is such that the teaching activity strives to allow the "special needs" student to reach as much as possible, according to his/her possibilities, the same objectives of "normal" students, thereby disregarding his/her identity and being "special" from many points of view (cognitive, social, communicative, emotional, perceptive…). The stand point behind this approach is that thinking and learning is purely in the functioning of the mind (or, according to neurosciences, in the brain) and that a deficit provokes a dysfunction that has to be recovered resorting to a variety of supports: technological, didactical, psychological and social. This leads to a homogeneization of all students' contributions, that tends to not take into account or value in any way alternative insight brought to the classroom by the special needs student. To overcome such approach, the authors proposed a paradigm shift: "Educational activity should aim at fostering a mode of existence in mathematics, i.e., being and becoming with others to make sense of the world also through mathematics. The aim of education should be to

[1]There are four types of learning disabilities recognized at the moment in Italy: dyslexia, dyscalculia, dysgraphia, dysorthographia (LEGGE 8 ottobre 2010, n. 170, Nuove norme in materia di disturbi specifici di apprendimento in ambito scolastico).

[2]In some "extreme" cases Italy grants a special education teacher to the student in need, who will sit next to the student during given hours of the student's regular school schedule.

allow all students to make sense of the world in spite of their particular conditions" (ibid., p. 222).

The described approach and the proposed paradigm shift are useful to keep in mind when considering two main directions in which the development of technological tools for MLD students seems to be going (at least in Italy): some software aim at strengthening a particular cognitive or mathematical ability, through repetitive tasks, designed for a one-to-one student-computer interaction, in an environment with constrained types of input and feedback—we will refer to this approach as for "rehabilitation"; while other software are designed to propose fundamental mathematical content (e.g., the notion of "variable" or "function") in ways that take advantage of particular hardware and software affordances. We will refer to this approach as "radical", because didactical material developed within it may propose (although they do not have to necessarily), more or less explicitly, radical changes in the mathematical curriculum and/or in the modalities in which certain content is proposed. Interactions with software designed according to the "radical" approach are frequently less constrained: tasks within the environment need to be designed by an educator (as they might not be part of the software), input and feedback may be given in various ways, and the role of the teacher becomes fundamental in mediating the meanings developed by the students within the environment.

Neither the "rehabilitation" nor the "radical" approach are necessarily one "better" than the other—of course to make any judgment of this sort we would have to make explicit the criteria according to which we are making such judgment—and both could be useful in supporting the learning of students with MLD. However, if our aim is to provide means for as many students as possible to make sense of the world, through mathematics, in spite of their particular conditions, it is inevitable to embrace, at least some of the time, the latter approach, when teaching. This approach is somewhat innovative in education, at least in the Italian panorama.

Since researchers in psychology and neuroscience have been designing, conducting and publishing research with rehabilitation software (e.g., Wilson, Revkin, Cohen, Cohen, & Dehaene, 2006; Wilson, Dehaene, Pinel, Revkin, Cohen, & Cohen, 2006; Butterworth & Laurillard, 2010), in this chapter we would like to focus mostly on software developed within the radical approach, which is innovative because it characterizes not only software design but also a general line of research regarding the development of didactical material that seems to be appropriate for inclusive mathematics education (see, for example Baccaglini-Frank & Poli, 2015a, b; Robotti, 2017). Software designed and adopted within the radical approach can also offer the student with MLD specific compensatory tools embedded within it, to alleviate the cognitive load of particular tasks in order for the student to be able to devote as many resources as possible to fundamental mathematical reasoning involved in the activity. However, these environments are not designed *only to compensate* certain cognitive difficulties. Within a software designed according to the radical approach there may exist sub-environments in which, through repetitive exercises, a specific ability or set of abilities may be strengthened. On the other hand, software developed primarily to strengthen a specific ability through repetitive exercise can be more difficult to use for fostering the development mathematical content within the radical approach. This is the case also because the closed, and in many cases fast, interaction between student and software does not leave much space for teacher-guided interventions.

In general there is no clear boundary between software designed according to either approach: we prefer to think of a spectrum with "radical" and "rehabilitation" designs at the extremes. Most software we can think of would be situated along the spectrum, more towards one or the other extreme. Moreover, there are significant variables, such as how the software is actually used or what role the teacher decides to play, that can contribute to shifting the software's placement within the spectrum, in either direction. In this sense, it can be possible to also use rehabilitation software within the innovative approach to special education presented above.

The perspective we are taking on how software can address specific MLD provides our rationale for analyzing how digital resources can support students in learning mathematics. The analyses will be carried out using a composite framework emerging from the notions of "Universal Design for Learning" and theories on channels for accessing and producing mathematical information.

2 Theoretical Background

In the field of mathematics education a number of frameworks have been developed, on one hand, to explain phenomena like "students experiencing learning difficulties in mathematics" from different perspectives, and others have provided tools for analyzing teaching-learning activity within technological settings (e.g., Lagrange, Artigue, Laborde, & Trouche, 2003; Noss & Hoyles, 1996; Bartolini Bussi & Mariotti, 2008). However, these theoretical tools are still quite fragmentary and very few have been adequately adapted and/or integrated to take into account findings (both practical and theoretical) from neighbouring fields such as cognitive psychology and neuroscience that have also been very active in investigating such phenomena. Notable exceptions are studies by the Unit of Instructional Psychology and Technology in Leuven, directed by Lieven Verschaffel (e.g., Vamvakoussi, Dooren, & Verschaffel, 2013); studies by Mulligan and her team based in Australia (e.g., Mulligan & Mitchelmore, 2013); and the work of the Center for Applied Special Technology (CAST), elaborating on the concept of Universal Design for Learning (Edyburn, 2005), which we will present in Sect. 2.2. Also, recent work of Karagiannakis and his colleagues contributes to establishing common grounds, at a cognitive level, attempting to transpose relevant aspects of the cognitive psychology literature into the field of mathematics education (Karagiannakis, Baccaglini-Frank, & Papadatos, 2014; Karagiannakis & Baccaglini-Frank, 2014; Karagiannakis, Baccaglini-Frank, & Roussos, 2017).

In particular in the Italian context, we have been active in trying to elaborate theoretical grounding for research on MLD students when teaching and learning include physical and digital artifacts (e.g., Baccaglini-Frank & Robotti, 2013; Baccaglini-Frank & Scorza, 2013; Robotti & Ferrando, 2013; Baccaglini-Frank, Antonini, Robotti, & Santi, 2014; Robotti et al., 2015; Santi & Baccaglini-Frank, 2015; Robotti et al., 2015; Baccaglini-Frank & Bartolini Bussi, 2016). In the two following sections we will review some notions from the theoretical background of cognitive psychology that will be useful for the analyses in this chapter (Sect. 2.1), and review some principles and guidelines from the framework elaborated by CAST that will also be insightful in the analyses proposed in the rest of the chapter (Sect. 2.2). The relationship between these different frameworks will allow us to analyse how and why the use of technology can foster mathematical learning in all students who present MLD.

2.1 Means of Information Access and Production, with Particular Attention to Mathematical Information

Research in cognitive psychology has identified four basic channels of access to and production of information: the visual-verbal channel (verbal written code), the visual non-verbal channel (visual-spatial code), the auditory channel (verbal oral code), and the kinaesthetic-tactile channel (Mariani, 1996).

Italian research has indicated that most students with specific learning difficulties (or disabilities), not only in mathematics, encounter greatest difficulties in using the visual-verbal channel, especially those with dyslexia, and this conditions their development for preferring different channels (Stella & Grandi, 2011).

The importance of these different channels to access and produce information shifts the focus from simply “being able or not” to solve a certain task, to different paths and strategies adopted by the individual (whether successful or not) for approaching the task. This allows to explain mathematical difficulties not only in terms of “lacking abilities” but also in terms of necessity to use certain preferred modalities that lead the student to access, elaborate and/or produce information in a certain way.

Moreover, various studies in cognitive science point to a correlation between mathematical achievement, working memory (Raghubar, Barnes, & Hecht, 2010; Mammarella, Lucangeli, & Cornoldi, 2010; Mammarella, Giofrè, Ferrara, & Cornoldi, 2013; Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013), and non verbal intelligence (DeThorne & Schaefer, 2004; Szucs et al., 2013). These findings suggest that non-verbal intelligence may partially depend on spatial skills (Rourke & Conway, 1997) and these can potentially be important in mathematical achievement, where explicit or implicit visualization is required.

We have found other theoretical stances advanced in mathematics education that are in line with the idea that means of access to and production of information, different from the visual-verbal one, can be very important in learning. In particular, these have pointed to the importance of experiences of a sensorial, perceptive, tactile and kinaesthetic nature for the formation of mathematical concepts (Arzarello, 2006; Gallese & Lakoff, 2005; Radford, 2003; see also Chap. The Coordinated Movements of a Learning Assemblage: Secondary School Students

Exploring Wii Graphing Technology). For example, Arzarello (2006), quoting Nemirovsky, points to how recent research in math education suggests that the paradigm of multimodality implies that "the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuomotor activities, which become more or less active depending of the context" (Nemirovsky, 2003, p. 108). Also Radford (2003, 2006) highlights that the understanding of relationships between bodily actions carried out through artifacts (objects, technological tools, etc.) and linguistic and symbolic activity is essential in order to understand human cognition and mathematical thinking in particular.

A new framework for teaching and learning in the context of "special needs" has been developed, taking into account many of the perspectives advanced above, and suggesting that technology can facilitate all students' learning. The framework is built around the concept of *Universal Design for Learning*.

2.2 Universal Design for Learning

The Center for Applied Special Technology (CAST) has developed a comprehensive framework around the concept of Universal Design for Learning (UDL), with the aim of focusing research, development, and educational practice on understanding diversity and applying technology to facilitate learning (Edyburn, 2005). UDL includes a set of Principles, articulated in *Guidelines and Checkpoints*[3] that arise from CAST's review of current studies on how to reduce barriers in learning and to increase access to curriculum for all the students, including those with disability, giving all individuals equal opportunities to learn. The research grounding UDL's framework is that "learners are highly variable in their response to instruction. […] individual differences are not only evident in the results; they are prominent. However, these individual differences are usually treated as sources of annoying error variance as distractions from the more important "main effects"".[4] In contrast, UDL treats these individual differences as an equally important focus of attention. The UDL framework considers these findings to be fundamental to understanding and designing effective instruction.

As a matter of fact, "individuals bring a huge variety of skills, needs, and interests to learning. Neuroscience reveals that these differences are as varied and unique as our DNA or fingerprints. Three primary brain networks come into play:"[5] Recognition Networks, which refer to recognition tasks such as: How we gather facts and categorize what we see, hear, and read, Identifying letters, words; Strategic Networks, which refer to strategic tasks such as solve a math problem;

[3]For a complete list of the principles, guidelines and checkpoints and a more extensive description of CAST's activities, visit http://www.udlcenter.org.

[4]See http://www.udlcenter.org/aboutudl/udlevidence.

[5]See http://www.udlcenter.org/aboutudl/whatisudl.

Affective Networks, which refer to the affective dimension: How learners get engaged and stay motivated, How they are challenged, excited, or interested.

Linked to each of these brain networks, UDL advances three foundational Principles[6]: (1) provide multiple means of representation, (2) provide multiple means of action and expression, (3) provide multiple means of engagement. In particular, guidelines within the first principle have to do with means of perception involved in receiving certain information, and of "comprehension" of the information received. Instead, the guidelines within the second principle take into account the elaboration of information/ideas and their expression. Finally, the guidelines within the third principle deal with the domain of "affect" and "motivation", also essential in any educational activity. For our analyses in this chapter we will focus in particular on specific guidelines within the three Principles.[7]

Guidelines and checkpoints within Principle 1 (provide multiple means of representation), suggest proposing different options for perception and offering support for decoding mathematical notation and symbols (checkpoints 1.2, 1.3, 2.3). We will give examples of how this can be realized through different software. Moreover, guidelines suggest the importance of providing options for comprehension highlighting patterns, critical features, big ideas, and relationships among mathematical notions (checkpoint 3.2). We will identify various of such options in the remainder of the chapter. Finally, our analyses will give examples of how software can guide information processing, visualization, and manipulation, in order to maximize transfer and generalization (checkpoints 3.3 and 3.4).

Moreover, our analyses will provide examples of how guidelines from Principle 2 (provide multiple means of action and expression) can be incorporated into technology-based mathematical learning, in particular how different options for expression and communication supporting planning and strategy development can be offered (checkpoints 4.2 and 6.2). Finally, our analyses will show how certain software can recruit students' interest, optimizing individual choice and autonomy, and minimizing threats and distractions (checkpoints 7.1 and 7.3).

In the two following sections we will analyze specific examples of software, classifying them by the type of mathematical learning they are designed to address. The analyses highlight which kinds of compensatory tools each software offers the student and which kind of tasks could be designed in order for the student to be able to devote as many resources as possible to fundamental mathematical reasoning involved in the activity.

Each software will be introduced by a section looking into research around the particular way of thinking or concept or tool being targeted. The rationale for choosing the software presented is that each one was used by one of the authors in studies carried out in the context of special needs or inclusive mathematics

[6]For further details see: http://www.udlcenter.org/aboutudl/whatisudl/3principles.

[7]The items are taken from the interactive list at http://www.udlcenter.org/research/researchevidence.

education, focus of this chapter. In some cases one of the authors was also directly involved in the software design process, while in other cases a particular software was chosen among other existing ones because of its fit with the UDL principles.

3 Examples of Digital Environments to Promote the Development of Number Sense and Spatial Orientation

In this session we analyse different software promoting number sense and we report on results from a case study on learning special orientation by interacting with the software *Mak-Trace*.

3.1 Software to Promote "Number Sense"

According to various studies a characterizing feature of students with MLD is a lack of "number sense". Although there is no monolithic interpretation of *number sense* across the communities of cognitive scientists and of mathematics educators, and not even within the community of mathematics educators alone (e.g., Berch, 2005), there seems to be a certain consensus about its importance in mathematics education. Indeed the development of number-sense is seen as a necessary condition for learning formal arithmetic at the early elementary level (e.g., Griffin, Case, & Siegler, 1994; Verschaffel & De Corte, 1996) and it is critical to early algebraic reasoning, particularly in relation to perceiving the "structure" of number (Mulligan & Mitchelmore, 2013). Some crucial aspects upon which number sense is seen to rely, are: recognition of part-whole relationships, appropriate uses of fingers, and the development of a mental number line. We will describe these and explain how they can be promoted through software applications.

Part-whole relationships arise from what Resnick et al. (1991) have described as protoquantitative part-whole schemas that "organize children's knowledge about the ways in which material around them comes apart and goes together" (ibid., p. 32). The interiorization of the part-whole relation between quantities entails understanding of addition and subtraction as dialectically interrelated actions that arise from such relation (Schmittau, 2011), and recognizing that numbers are abstract units that can be partitioned and then recombined in different ways to facilitate numerical (also mental) calculation.

Literature from the fields of neuroscience, developmental psychology, and mathematics education indicate that using fingers for counting and representing numbers (Brissiaud, 1992), but also for accomplishing tasks that have no apparent connection to mathematics (Butterworth, 2005; Gracia-Bafalluy & Noel, 2008), can have a positive effect on the development of numerical abilities and of number-sense. The importance of the role attributed to the use of fingers in the development of number-sense by the quoted literature is highly resonant with the frame of embodied cognition, mentioned in Sect. 2.1. For example, hands and fingers can be used to foster development of the part-whole relation, in particular with respect to 5 and 10, in a naturally embodied way.

Development of a mental number line

Number sense has also been put in relationship with the development of an internal representation of the number line. A number of studies have explored a relationship between space and the processing of numbers (e.g., Pinel et al., 2004; Seron et al., 1992), suggesting that the (mental) number line model corresponds to an intuitive representation and to a natural translation of the sequence of (natural) numbers into a spatial dimension. This model can be used in more abstract (and potentially more general) processes compared to that of counting existing sets of objects, because, for example, it opens to the possibility of counting *any* number of objects and *any* object. The number line model is not a static representation, nor is it necessarily innate,[8] instead studies suggest that it evolves as the subject develops cognitively, and such evolution depends on cultural influences (see, for example, Zorzi, Priftis, & Umiltà, 2002).

Moreover, studies suggest that a solid mental representation of the number line provides students with a rapid and successful means of access to numerical information necessary for the development of a variety of arithmetical skills. The number line can also be an appropriate tool not only for calculation (mostly addition and subtraction) with numbers within 10 (which can also be done using hands and fingers) but also for dealing with numbers beyond 10, when hands and fingers no longer are sufficient.[9]

Finally, the number line is not made up of only natural numbers, but also all other real numbers, which include, for instance, fractions. However, frequently the position of numbers on the number line can become a cognitive obstacle: for example, placing fractions on the number line (mathematically this involves ordinal properties and the density in the field of rational numbers) is notoriously a difficult task for many students (Robotti et al., 2015).

Given these considerations on fundamental aspects that have been identified as promoters of number sense, we can assume that software designed to promote these aspects, may be used in one of two ways: to help prevent the emergence of MLD in young students (younger than 8), or to strengthen weaker "number sense" abilities of older students who have developed MLD. In the sections below we will describe two innovative examples of these kinds of software.

[8]For a more complete discussion see volume 42(4) of the *Journal of Cross-Cultural Psychology*.

[9]Sometimes fingers are used also to represent numbers larger than 10, but in this case the meanings referred to by different fingers must be different (for example 4 and 13 might be represented raising the same fingers: 1 on one hand and 3 on the other) which can be confusing for children.

3.1.1 Software Promoting Number Sense Through Fingers

Technology offers the possibility of embedding a number of features into software that can be significant in promoting number sense through the use of fingers. For example, thanks to touch and multi-touch screens, input may be given in terms of a number of fingers placed simultaneously on the screen, as a number of sequential taps (possibly on items in the stimulus), or as particular gestures (swipe, pinch, lasso/capture, …). Here we give an example of software that exploits such innovative potential.

TouchCounts,[10] an application for the iPad, is made up of two environments (Sinclair & Pimm, 2014; Sinclair & Zaskis, in press; see also Chap. Returning to Ordinality in Early Number Sense: Neurological, Technological and Pedagogical Considerations). Here we will briefly analyze the "Operating world" with respect to its design and potential of fostering development of number sense through fingers. In this environment the student can create autonomous numbered sets, here referred to as herds, by placing one or several fingers on the screen. This immediately creates a large disc encompassing all the fingers and including, in the middle, a numeral corresponding to the total number of fingers touching the screen. At the same time, every one of the fingers on the screen creates its own much smaller (and unnumbered) disc, centred on each fingertip. When the fingers are lifted off the screen, the numeral is spoken aloud and the smaller discs are then lassoed into a herd and arranged regularly around the inner circumference of the big disc. This design offers four representations (UDL Principle 1) of a number: visual non verbal (or analogical), symbolic (the numeral in the herd), auditory, and of course gestural (the number is represented by the number of fingers placed on the screen simultaneously). Moreover, the student is guided to perceive the herd a single entity made up of units through the movement of the small discs all together in either a clockwise or counter-clockwise direction.

The software also offers multiple means of action and expression (UDL Principle 2) because the student can act on the herds in different ways. For example, s/he can interactively drag herds, either to move them around on the screen or to operate upon them. After two or more herds have been produced they can either be pinched together (a metaphor for addition) or 'unpinched' (metaphor for subtraction or partition). When herds are pinched together they then become one herd that contains the small discs from each previous herd. The new herd is labelled with the associated numeral of the sum, which *TouchCounts* announces aloud. Moreover, the new herd keeps differentiated colors for the small discs coming from the previous herds. Similarly, the student can do an inverse pinch gesture to decompose a given herd into two herds. The gesture supports the idea of partitioning, or 'taking out' or 'removing', which, in turn, supports the idea of subtracting. The further the swipe travels, the more will be taken out from the starting herd. When the swiping finger is lifted, two new herds are formed and *TouchCounts* announces the number that has been taken out.

[10]See https://itunes.apple.com/us/app/touchcounts/id897302197?mt=8.

Students can engage with this software through different means (UDL Principle 3)—using gestures, through listening, visually—as they freely explore or approach a variety of tasks proposed by a nearby educator (e.g., "make *n* all at once", "count by *n*", "make the herds equal", "how many different ways can you make *n*?"). Indeed, the environment allows proposing many different types of tasks that can foster the development of number sense in different ways, through a "radical" approach.

3.1.2 Software Promoting Number Sense Through the Number Line

There are many software applications that propose representations of the number line: some are discrete containing only natural numbers, others continuous with marks such as those on the ruler, some are static and designed only for responding to specific tasks implemented within the application, while others are dynamic and allow various user interactions.

A first example we would like to analyze is *Motion Math*: *Fractions*,[11] an application for tablets. At the moment it is designed only for promoting processes involved in the estimation of fractions, exploiting both epistemological and cognitive analyses of fractions (Riconscente, 2013), emphasizing, on the one hand, the importance of using the number line to give coherence to the study of fractions and of whole numbers and, on the other hand, the neurological evidence of the mental number line discussed above (Zorzi et al., 2002).

Within this environment a number line appears on what looks like the "ground" together with a ball that can bounce (completely elastically) and that can be controlled by the gravity accelerator of the tablet that is, it responds to physically tilting the tablet, as if the ball had a weight. A fraction appears within the ball, which needs to be placed correctly on the line. The fraction is presented in different representational formats: it may be in the form *n*/*m*, or a decimal number, a percentage, or a shaded section of a circle. Successive hints are given if the user makes mistakes in positioning the fraction on the line. The app is designed as a game (the user gets points, passes levels, and "dies" when a mistake has been made even after all the hints). The ball's regular bounces constrain the user's response time, forcing each placement choice to be planned and executed in pre-determined and regular time intervals.

The application appears to be in line with a number of the UDL principles outlined in Sect. 2.2: multiple means of representation are provided and integrated (fractions are presented in different forms: as "*n*/*m*", as decimal numbers, as percentages, as parts of a whole, and as numbers on the number line), support is offered in the form of successive hints for finding the position of the given fraction on the number line, the successive hints highlight critical features of the relationship between the given representations of fractions and their position on the number line, no verbal skills are necessary because the channels activated for input and output of

[11]See the app *Motion Math: Fractions* at http://motionmathgames.com/motion-math-game/.

information are visual and kinaesthetic, distractions are minimized by the need to plan and give successive input according to pre-determined and regular time intervals.

Moreover, *Motion Math: Fractions* can be seen to exploit embodied learning and, in particular, the integrated perceptual-motor approach (Nemirovsky et al., 2012) in the development of the mental number line.

It is possible (and, we believe, advisable in many cases) to complement a student-software interaction with verbal guidance and successive discussion of each playing session. For example, in the episode presented in Bartolini, Baccaglini-Frank, and Ramploud (2014) the student, who had been diagnosed with various learning difficulties, including severe dyscalculia, was significantly helped by the introduction of a different way of reading the fractions in the falling ball. The teacher (second author) suddenly exclaimed: "Let's name the fractions as Chinese do!…[1/2 *falls*] Of two parts, take one!…[3/4 *falls*] Four parts, three!" and the student improved his performance very quickly, especially on unitary fractions (e.g., 1/5). Similar episodes have since been observed with other low achievers.

In this example we can observe that providing options for mathematical expressions and symbols by language and different linguistic expression, can be effective for overcoming some difficulties in math comprehension (according to Principles 1 and 2 of UDL framework). We note that in the case described above the verbal expression that identifies the fraction expresses at the same time a process for constructing (and thus placing) the fraction that follows a same order.

3.2 *Spatial Orientation and Non-verbal LD*

A possible source of difficulties in mathematical learning is what has been referred to as a non-verbal (or visual-spatial) LD (e.g., Mammarella et al., 2010; Andersson & Östergren, 2012; Mammarella et al., 2013). An ability that may be weaker in these students is perspective-taking (Piaget & Inhelder, 1967; Clements 1999), that is being able to embrace different frames of reference based on one's self or on external points of reference, is fundamental both in everyday life and in instruction. The importance of such ability is declared, for example, in the Italian National Curriculum Indications (MIUR, 2011a, b) relative to mathematical learning about *Space and Figures*. Developing the perspective-taking ability may not be straightforward: it involves a transition from "perceptual space" to "representational space" (Piaget & Inhelder, 1967), as well as "connecting different viewpoints" (Clements, 1999, p. 3).

While children showing typical development seem to have acquired such ability by the end of primary school, in some children with MLD—including developmental dyscalculia (e.g., Mazzocco & Räsänen, 2013)—the development of perspective-taking, among other abilities, may be delayed and/or deficient.

Software environments that seem particularly appropriate for addressing perspective taking are microworlds, such as *Logo* (Papert, 1980). The potential of

Logo-like microworlds for fostering learning in students with persistent difficulties in mathematics is documented in the literature. In particular, Vasu and Tyler found that Logo may foster the development of spatial abilities and of critical thinking skills (Vasu & Tyler, 1997), and various other researchers have reported several potential benefits of using Logo with students who have learning difficulties (Atkinson, 1984; Maddux, 1984; Michayluk & Saklofske, 1988; Russell, 1986), especially using a more structured, mediated approach (Ratcliff & Anderson, 2011).

Below we describe design features of a Logo-like microworld, Mak-Trace, an environment we used to analyze cognitive processes involved in juggling different frames of reference of students with non-verbal difficulties.

3.2.1 The Logo-like Microworld Mak-Trace

Mak-Trace is an environment in which a character can be programmed to move and draw on a grid. The grid is 10×15 and the character can only be programmed to go forwards (F) or backwards (B) (of the distance of one side of a square of the grid at the time) or to turn 90° clockwise (R) or counterclockwise (L). The characters can be dragged on the grid with a finger to choose a starting position and then they will, by default, leave a trace mark as they move according to the commands in the programmed sequence (see Fig. 1). It is also possible to program the character so that it does not leave a trace mark on the grid, by inserting appropriate commands in the programmed sequence. The commands appear as icons that have to be dragged and placed on a vertical bar that represents the programmed sequence. This design proposes different representations (UDL Principle 1) corresponding to the

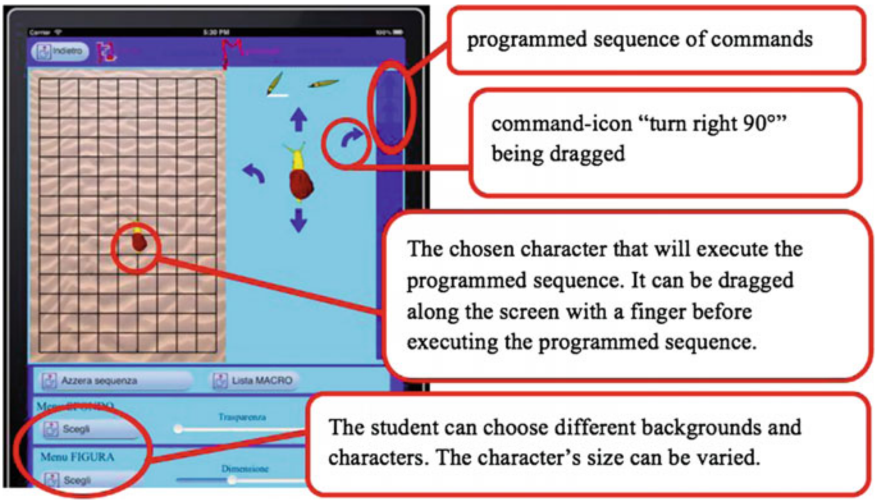

Fig. 1 Main screen in Mak-Trace, where the student can program his/her character

movements of the snail on the grid: a "draggable" arrow-symbol, a movement of the character on the grid, a segment (or point) traced on the grid.

An aim in designing Mak-Trace was to create an environment accessible to young children, or students with learning difficulties or disabilities, especially of a visual-spatial nature, by offering an intuitive iconic programming language. Students can act on the environment in different ways (UDL Principle 2): dragging the character on the grid with their finger, or dragging command icons to into a sequence to make a "program". Of course the student can also interact verbally with a nearby educator.

The fact that the command-icons can be treated as objects can make it natural to assign symbolic names to each of them in order to quickly describe a programmed sequence, orally or by writing on paper (Principle 2 of the UDL framework). This practice can be proposed and pursued by an educator using Mak-Trace with her students, and it may help students make use of a pre-algebraic language that can be quite useful in certain tasks involving generalization.

Another design choice is that Mak-Trace gives no feedback in terms of movements of the character until the student touches "GO". At this point the character executes the whole list of commands in the constructed sequence. To change the constructed sequence, the student has to go back to the "programming mode": automatically the character goes back to its original position and all trace marks are cleaned off the screen. This choice was made to foster planning and spatial orientation abilities. In particular, the student has to visualize what the character will do as she is programming, and where the character will be at each step of the programmed sequence, before actually executing the sequence. These design choices were made in accordance with the UDL Checkpoints 4.2 ("Optimize access to tools and assistive technologies") and 6.2 ("Support planning and strategy development").

In Mak-Trace the perspective-taking ability consists in embracing the character's moving frame of reference. To exemplify how working in this environment can be beneficial to students who experience difficulties in perspective-taking, we will revisit some critical episodes from a case study (Baccaglini-Frank et al., 2014; Santi & Baccaglini-Frank, 2015).

3.2.2 The Case of Filippo

Filippo was 15 years old and had been diagnosed by clinicians as having MLD including dyscalculia and severe dyslexia. From the accounts of his special education teacher, he also was not able to read maps or to give directions, however he did not show difficulties in recognizing or naming his left and right hands. He had a short attention span and little—if any—interest in the activities proposed during math class. Furthermore he suffered from very low self-esteem and sense of self-efficacy. We developed a protocol so that Filippo would work with Mak-Trace when he met with his special education teacher, for five weeks, either once or twice each week. The tasks were designed based on two hypotheses: we expected Filippo's perspective-taking ability to be weak at least initially, but all the same we expected that interacting with the software under supervision of the teacher could enhance his abilities to plan, visualize, and give directions, potentially through means different than his perspective-taking ability. Here we briefly report on the two tasks Filippo carried out: (1) describe the relationship between sequences of commands in Mak-Trace, and the movements and trace mark left by the snail; (2) program the snail to draw a square.

During the first task Filippo initially thought that the arrow commands "go forward", "go backward", "turn right", "turn left" (F, B, R, L) made the snail go forward, backward, right, and left, where these directions are relative to Filippo's front, back, left and right, or possibly to "absolute" directions, like north, south, east, west. Therefore Filippo was not able to construct a sequence of commands to make the snail draw a given path. For over half an hour he struggled to relate the brief sequences of commands he programs to their representation on the grid. He did not seem to be aware of any reference frames other than his own until the teacher intervened, in the interaction that follows.

Filippo: it went backwards, not upwards […]
Teacher: so what do the little arrows refer to?
Filippo: it depends on how the snail is oriented.

This was a decisive moment which lay the foundations for Filippo's conception of the snail's perspective. However, Filippo still mostly relied on trial and error, embracing the snail's perspective as long as the snail is not oppositely oriented, which he was confronted with in the task of making the snail draw a square.

The first time Filippo tried to program the snail he was able to program the sequence correctly for the first two sides of the square, then he uses (incorrectly) the commands B and R, correct in his frame of reference, but not in the snail's; while the fourth side, horizontal in Filippo's frame, is programmed correctly. It is interesting that he used opposite commands for the first and third sides (F and B, respectively), while for the second and fourth he used the same command (F). The effect of this programmed sequence is shown in Fig. 2a.

The second time Filippo tried to program the sequence, he composed: FFFFLFFFFL [hesitated, inserted L, erased it, and with the index of his right hand made the gesture of a counter clockwise turn] FFFF [he said: "I have to always keep the" and made another counter clockwise turn gesture with his right hand] RFFFF (Fig. 2b). The feedback from Mak-Trace (snail moving on the screen and leaving a mark on the grid) confirmed that three sides were now correctly programmed. However Filippo made a mistake again on the rotation when the snail is oppositely oriented. This behavior suggests that indeed Filippo had a weak perspective-taking ability.

However, our second hypothesis was also confirmed, as Filippo, on his own, interacting with the software, developed alternative strategies for managing the different frames of reference. A first strategy is developed to finally solve the square drawing task. This time Filippo re-wrote the sequence: FFFFLFFFF [he made the gesture of a counter clockwise turn with his right hand] LFFFF…[he rotated the

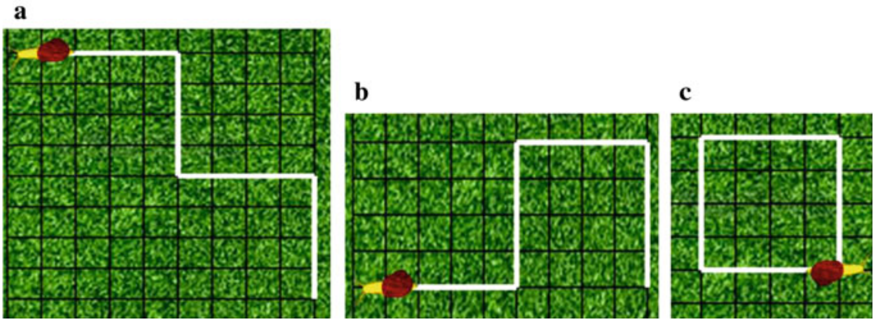

Fig. 2 Effect of Filippo's first, second and final programmed sequence

iPad so that his frame coincided with the snail's, observing the screen he rotated his right hand counter clockwise]. Then he completed the last turn and side.

> Filippo: Done, I found it […] no, I got…lost […] when it is turned around…it goes opposite [*clockwise* rotation gesture with the right hand] so…if I want it to go here [horizontal gesture from left to right with the left hand] … oh, I don't know, I'll try this [RFFFF]… no wait, because this otherwise is like before [he substitutes R with L].

The sequence was correct (Fig. 2c).

Rotating the iPad is a gesture that reveals how Filippo is now aware that he should consider the snail's frame of reference, and that this frame is oppositely oriented with respect to his (at the moment of the rotation). It is as if Filippo was aware of not being able to *feel* the snail's frame of reference when it is "too different" from his own (oppositely oriented), so he figured out a way of physically making the frame of reference of his body match the one of the snail. This allowed him to overcome his disorientation and to successfully complete the task.

4 An Example of Digital Environment Promoting Algebraic Abilities

We now briefly discuss learning difficulties in algebra. In this discussion algebra will be the chosen learning object (Principle 1 of UDL framework), and we analyse potentialities of the software *AlNuset*, showing how they played out during a case study. In this sense, according to Principles 1 and 2 of the UDL framework, we will analyse how *AlNuset* introduces both multiple means of representation and multiple means of actions and expression in order to help students grasp the meaning of some algebraic notions. The analysis will be focused, in particular on the MLD students' difficulties.

With a significant percentage of students, the current teaching of algebra seems not to be sufficient to effectively develop skills and knowledge to master this domain of knowledge (Sfard & Linchevski, 1992; Kieran, 2006). Here, we focus on the students' difficulties in algebra considering, in particular, students with MLD. These students can have severe difficulties in arithmetic, (Butterworth, 2005), however, there are also areas of mathematics, which do not depend so much on manipulating numbers, such as algebra, geometry and topology.

Indeed, some studies on MLD students have shown that there is dissociation between the recovery ability of arithmetic facts, which is compromised, and algebraic manipulations, which are intact (Hittmair-Delazer, Sailer, & Benke, 1995; Dehaene, 1997). Thus, there is evidence for the existence of two independent processing levels of mathematics: a formal-algebraic level and an arithmetic-numeric level (Dehaene, 1997). Moreover, neuroimaging results, focusing on the algebraic transformations, have highlighted how the visual-spatial areas of the brain are activated at the expense of those devoted to language. For example, it has been shown that when we solve equations, the expressions are manipulated mentally by means of a visual elaboration rather than through verbal means (Landy & Goldstone, 2010). Such neuroscientific results can help us analyze the difficulties of students with MLD in algebra.

Many students' difficulties in algebra, including difficulties in controlling algebraic manipulation (e.g., Robotti & Ferrando, 2013), seem to be due to a lack of grasp on the meaning of the notions involved (Arzarello, Bazzini, & Chiappini, 1994). Recent studies in math education have suggested that the construction of mathematical knowledge, as a cognitive activity, should be supported by the sensori-motor system activated in suitable contexts (Arzarello, 2006). Indeed, according to Nemirovsky (2003), the understanding of a mathematical concept spans diverse perceptuomotor activities, which become more or less active depending of the context. Thus, the construction of meaning can be seen as based on a rich interplay among three different types of semiotic sets: speech, gestures and written representations (Radford, 2003, 2006). Studies concerning both the algebraic domain (Chiappini, Robotti, & Trgalova, 2009; Chaachoua et al., 2012) and the geometrical domain (Goldenberg, Cuoco, & Mark, 1998) suggest using educational tools through which images can be constructed and managed (dynamically or statically), exploiting mainly visual non-verbal rather than (or together with) verbal means. This is in accordance with the UDL principle of providing multiple means of action and expression (Principle 2).

We will show how the software AlNuSet (Algebra of Numerical Sets) can be used to make algebraic notions explicit, and to construct their meanings dynamically, while involving all the students in a classroom, as much as possible (Baccaglini-Frank & Robotti, 2013). In particular we will look at how AlNuSet can be used in relation to the algebraic notions of variable, unknown, algebraic expression, equation and solution of an equation, and the formal solution of an equation can be addressed with the support of AlNuSet.

4.1 AlNuSet to Construct Algebraic Meanings: Examples to Inclusive Education

AlNuSet was designed for secondary school students (from age 12–13 to age 16–17) and it is made up of three separate environments that are tightly integrated: the Algebraic Line, the Algebraic Manipulator, and the Cartesian Plane. We will describe some features of these environments, with particular attention to the Algebraic Line and the Algebraic Manipulator, through examples of activities,[12] stressing their support for the conceptualization of algebraic notions in MLD students.

Variable and dependent expressions

On the Algebraic Line it is possible to place variables and expressions that depend from them. To do this, the user has to type a letter, for example, "x", and a mobile point will appear on the line. The point can vary within the chosen set of numbers (natural, whole, rational, or real[13]) and variation can be controlled directly by the user through dragging. This feature was designed so that important aspects of the notion of *variable* could become embodied. Moreover, it is possible to construct expressions on the line that depend on a chosen variable, for example, $2x + 1$. This dependent expression cannot be acted upon directly, but it will move as a consequence when x is dragged. The dependent expression will assume the positions on the line that correspond to the values it takes on when the dependent variable takes on the value it is dragged to (Fig. 3).

We note that the functionalities described propose different representations (UDL Principle 1) and they are designed to foster for the user a mediation of the algebraic concepts of *variable* and *dependent expression*, through a dynamic model that can be acted upon (UDL Principle 2). The mediation can occur thanks to visual and kinaesthetic channels, without the need of visual verbal means (written language). The construction of the concept realized as so may allow students, and especially students with MLD, to find mnemonic references that are appropriate for their cognitive style. This allows them to start using representations of the fundamental algebraic concepts at stake, and possibly to place and retrieve them from long term memory in a more effective way. AlNuSet allows to address "typical" topics in the secondary school algebra curriculum; in particular, in the following section we will analyze how *equations* can be addressed.

Equations

Let us consider a common task: "Solve the Eq. $3x-5 = 13$", or—stated in a possibly less common way—"Find the values of x for which the expression $3x-5$ is equal to 13".

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[12]For a more detailed description of these environments see www.alnuset.com.

[13]Of course the representations of the numerical sets are accomplished on a computer, so the sets are actually finite and discrete, but they simulate—with some limitations—the properties of the number sets they represent.

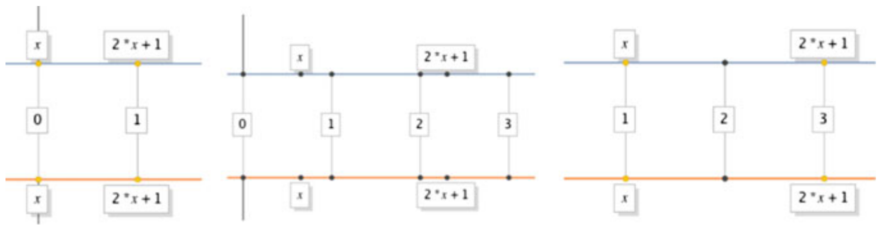


Fig. 3 The movement of the variable x on the Algebraic Line produces the movement of the dependent expression $2x + 1$ on the line

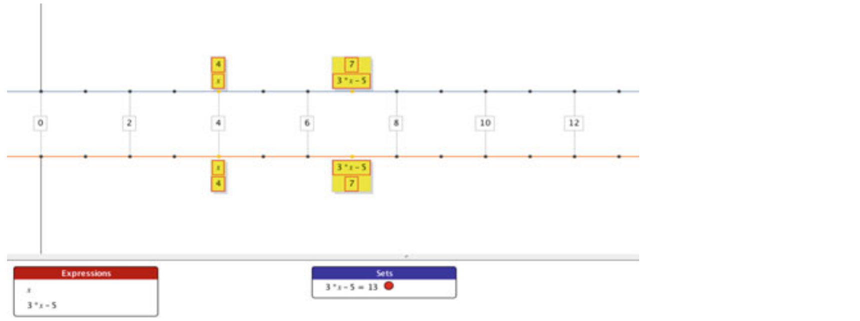
Solution on the Algebraic Line

Solving this equation on the Algebraic Line requires observing for which values of x the expression $3x-5$ (represented as a mobile point on the line) coincides with the number 13. When trying to verify the equality of expressions, dragging x is accomplished with a specific objective: that of trying to make the expressions coincide, that is, to make them take on a same value, becoming thus associated to a same "post-it" (yellow rectangle in Fig. 4). If the dragging is done with this objective, the variable assumes a meaning similar to that of *unknown*, that is of letter of which values need to be found in order to make the equality true. This allows students to act on the representations in different way, according to UDL Principle 2.

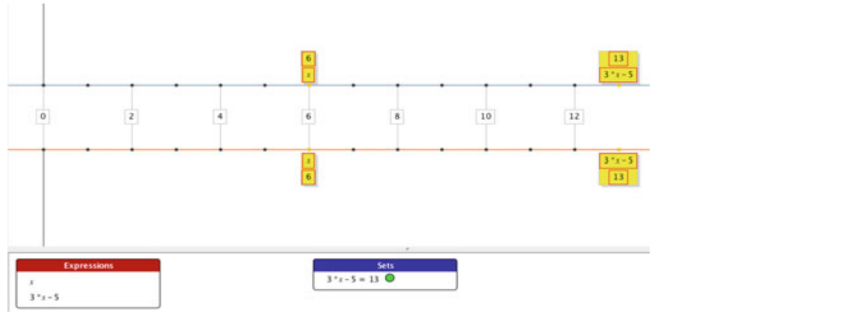
In Fig. 4 we can observe what happens on the Algebraic line as the point "x" is dragged.

The possibility of solving an equation through a perceptive kinaesthetic approach (dragging x along the line) without directly using a solution algorithm can help students concentrate their attention on the meaning of equation and its solutions. The Algebraic Line in AlNuSet was designed with this aim, which it attempts to reach through specific signs and functionalities embedded in it. Among these there is the possibility of dragging the point corresponding to "x", the visualization of "post it" markers containing values on the line and the constructed expressions that correspond to them (Fig. 4), the color of the dot corresponding to the equation (Fig. 4a and b). In particular this last feature is an example of how a visual non-verbal channel is used to give feedback to the student, guiding his/her construction of meaning of *solution of an equation*.

Features like the dot changing color and the yellow "post-it" signs, supporting the comprehension and the construction of meaning for algebraic notion and relationships involved, are examples of how AlNuSet's design seems to be well in line with the UDL principle advocating "multiple means of representation" (Principle 1). Indeed, they support perception providing the representations for algebraic notions through different modalities (e.g., through vision, dynamic image, touch…); and in a way that will allow for adjustability by the user (e.g., dragging the point corresponding to x as often as the user wants). Such multiple representations not only ensure that algebraic notion is accessible to MLD student, but also easier to comprehend for many others.



a Moving x along the line, the expression moves as a consequence. For example, when x is on 4, the expression 3x-5 is on 7. Notice that in the window "Sets" the equation is followed by a red dot that indicates that the value currently assumed by x does not make the equality true.



b When x in on 6, the equality is true because the expression 3x-5 is on 13. Notice that the dot associated to the equation in this case turns green.

Fig. 4 A way of solving the Eq. $3x-5 = 13$ on the Algebraic Line

Another functionality of AlNuSet that can be useful for the construction of a solid meaning of solution of an equation is the command "E = 0", in the environment Algebraic Line. This command allows the student to ask the system to calculate the roots of a polynomial (to read more about this functionality visit www.alnuset.com). This functionality can help the student tackle the "truth value of an equation", alleviating his/her cognitive resources from the burden of calculation procedures associated with the solution algorithms of an equation. This can be appreciated, for example, thinking about the cognitive load—excessive for some students—associated with the application of quadratic equations. Indeed, many students with MLD have trouble both with arithmetic calculations and with memorization and execution of procedures. The more complex a procedure is, greater are the difficulties for these students to retrieve the steps involved and to execute them. The "E = 0" functionality of AlNuSet allows these students to focus their attention on the cognitive task related to the meaning of solving an equation, in terms of searching for truth values of the equation, as opposed to dispersing their cognitive resources only on the calculation, loosing track of most (or all) meaning (Robotti, 2014; Robotti & Ferrando, 2013).

Given these features and ways in which their use can be integrated in approaching mathematical situations, the Algebraic Line can be used as a tool that can help lighten the cognitive burden involved retrieving and carrying out procedures, and allow the student to focus most of his/her cognitive efforts on the construction of the algebraic meanings at stake, favoring autonomy in approaching algebra. This is in agreement with UDL Principle 2 and, in particular, with the idea to provide option for comprehension: guiding information processing, visualization, and manipulation; maximizing transfer and generalization.

4.2 The Case of Eleonora

We now present the case of a student we will call Eleonora using the Algebraic Line of AlNuSet, carried out by the first author. She was 26 years old at the time of the study and had obtained her first diagnosis of dyscalculia the same year.

Before proposing the use of AlNuSet, one of the questions the interviewer asked Eleonora was the following: "*When 3 is added to 3 times a certain number, the sum is 28; find the number*".

Eleonora did not set up an equation, but proceeded by subtracting 3 from 28 (obtaining 25) and then dividing by 3, "undoing" the operations stated in the problem text. She then tried to prove the arithmetical equality (in Fig. 5) through "trial and error", approximating the value of $\frac{25}{3}$ to 8.333… She preferred to do this in spite of what she had been taught in various algebra classes where many examples of verbal texts of this type had been given and transformed into equations, such as $3x + 3 = 28$.

The researcher (first author) advanced the hypothesis that Eleonora had not developed a strong enough (if any) mathematical meaning of the notion of equation, possibly also due to the fact that she had trouble managing the typical procedures given to her during regular courses for solving first and second degree equations. The intervention proposed to Eleonora therefore was planned as a sequence of activities with the Algebraic Line in AlNuSet aimed at developing the mathematical

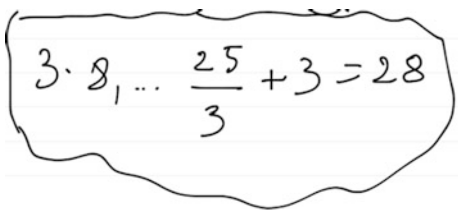


Fig. 5 Eleonora's attempt to solve the interviewer's question

meaning of equation and of solution of an equation. In the following excerpt we show Eleonora responding to the researcher's (R) question: "For which value of "a" is the expression 2 × a equal to 8?".

1. E: Right now we can see that "a" changes value,… it changes value if I drag it
2. R: For which value of "a" is the expression equal to 8?
3. E: The expression is equal to 8… that is 2*a is equal to 8…
4. E: If I move it along the line, I am looking for the right value, where the letter matches
5. E: For example, I discovered that if I place "a" on 3…if I give "a" the value 3… 2*a is 6
6. E: Instead, if I put "a" on 4, 2*a is 8… because I'm multiplying […]
7. R: What did you get? [Referring to the colored dot associated to the equation in Sets window]
8. E: A verification. It's a check, if I drag "a", the red dot shows that I make a mistake
9. E: …if I drag "a", if I change the value of "a", the red dot shows that I make a mistake
10. E: Because, in this moment, 2*a equal to 8 is not true
11. E: There isn't an equality. Because I'm on 2*a equal to 10, if I give "a" the value 5

The solution to the problem is developed through a visual-spatial kinaesthetic approach in AlNuSet. Here, new representations (algebraic expressions, post-it, colored dots…) and different ways to act on them are provided, as proposed by UDL Principle 1 and Principle 2. As matter of fact, manipulating the expression 2 × a on the line allows Eleonora to associate meaningful (to her) dynamic representations of the notions of *variable, unknown, equation and solution.*

Indeed we can observe that the verbal utterances used by Eleonora first refer to perceived aspects of the solution to the problem. Examples of such utterances are: "If I place "a" on 3…" (5) or "If I put "a" on 4…" (6). Later she seems to be attributing to "a" characteristics of an unknown: "if I give "a" the value 3…" (5), "in this moment, 2 × a equal to 8 is not true" (10).

In intervention (6), we can also observe that Eleanor manages to relate the truth of the equation obtained by assigning to "a" the value of 4, with the arithmetic operation in 2 × a, which guided her first solution strategy (in the pre-testing phase). Thus, dragging "a" along the line until the value 4, she finds a link between the "meaning of an equation solution" with the "arithmetic procedure".

The construction of these meanings seemed to become more and more stable throughout the intervention, that is Eleonora was able to access and retrieve the meanings constructed within the Algebraic Line environment even months after the end of the intervention. This suggests a transfer to long term memory. Referring to the UDL principles, this environment seems to have successfully provided for Eleonora multiple means of representation, in this case offering dynamic representations of algebraic objects on the Algebraic Line of AlnuSet. Moreover, it provided multiple means of action and expression, exploiting the various functionalities through which Eleonora could act on receiving instantaneous feedback from the system. Making sense of such feedback Eleonora was able to give meaning to and manage the process of the solution of equations.

5 Conclusion

Specific theoretical frameworks in mathematics education research for the use of technology for fostering mathematical learning of students with MLD are still quite fragmentary. Moreover, very few have been integrated with findings from fields such as cognitive psychology and neuroscience, fields that have also been very active in investigating such phenomena. Therefore we felt the need to turn to more general theoretical notions related to different research fields. Among them, the idea of different means of information access and production, related to research in cognitive psychology, the three primary design principles of the Universal Design for Learning framework, which we refer to specific software's' design, and the paradigm of multimodality, related to research in math education, according to which experiences of a sensorial, perceptive, tactile and kinaesthetic nature are essential for the formation of mathematical concepts.

If we turn back and think about the analyses of students' interactions with selected software, we can again trace down our effort of seeking out evidence, within each particular mathematical learning context, of the usefulness of design choices, interpreted as aligned with the general UDL framework. In the case of Filippo, use of Mak-Trace, mediated by the teacher, helped the student develop personal strategies to solve problems concerning perspective-taking ability that initially he found unsormountable. These strategies later were endorsed also by his regular mathematics teacher. The analysis pointed to specific instances in which the software allowed the student to avoid the use of symbolic language and to rely on his sensorimotor activity in an interplay between movement, gestures and language (multiple means of action and expression—Principle 2). Moreover, similarly to what has been described for Logo, Mak-Trace appeared to be highly engaging (Principle 3), helping the student to "remain absorbed in a task for a period of time; … tolerate a period of confusion (with appropriate support);… use errors as a source of information about what to try next" (Russell, 1986, p. 103). In the case of Eleonora we highlighted how the environment seemed to successfully provide her with multiple means of representation (Principle 1) of algebraic objects on the Algebraic Line (for example, mobile points representing variable, expressions or unknowns, or the "yellow square" indicating expressions that refer to the same value/point on the line), and multiple means of action and expression (Principle 2), leading to instantaneous feedback from the system (for example, the movement induced by dragging a point on the line).

In general, we showed how the software applications analyzed provide multiple means of representation (Principle 1 of UDL framework), multiple means of action and expression (Principle 2) and multiple means of engagement (Principle 3), meeting specific checkpoints within each of these principles. To complete the analyses of each environment we also felt the need to add discussions of important literature on the learning of the specific mathematical content involved. This is because it is well known in mathematics education that the learning of different concepts or ways of thinking in mathematics can involve the activation of different cognitive processes in the students; and for the learning to be promoted effectively, it implies specific pedagogical content knowledge for teaching (Ball, Lubienski, & Mewborn, 2001) which the context of MLD includes information on cognitive issues involved in the learning of the specific mathematical content.

For this analysis, we referred mostly to software developed within the "radical" approach, according to which new ways of approaching specific mathematical content can also lead to changes in the organization of the mathematical curriculum or in the ways in which certain content is proposed (see, for instance, the notions of *variable* or *unknown* addressed in AlNuSet). As of today, we have only taken some initial steps towards reaching a framework to analyze the use of technology for fostering mathematical learning of students with MLD, and we definitely have yet a way to go in this direction. Until now we have (1) looked for ways of implementing checkpoints from the UDL principles designing software we collaborated to produce, and we have (2) looked for evidence of the usefulness of such design choices analyzing students' interactions with the software. These two tasks are still far from straightforward and necessitate a good deal of discussion and interpretation of the checkpoints of the UDL framework, because these are stated in very general terms. This of course makes them applicable to a number of different learning contexts (other than mathematics), but it costs their meaningfulness within the domain of mathematical learning, or even within more specific contexts, like learning natural numbers, learning about geometrical figures, or learning to solve quadratic equations.

We believe it is yet premature to propose a new coherent framework through which to look at technology mediated learning in the presence of MLD, but at the moment we see the intertwining of the different theoretical notions used for the analyses of the software and of students' interactions with the software as effective in giving insight into how and why some innovative software can foster mathematical learning for students with MLD.

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