Chapter 5
Knowing Ratio and Proportion for Teaching

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Abstract Ratio and proportion have been part of school mathematics since the earliest manifestations of anything like school math in the Middle Ages. In this paper, I compile and comment on statements from primary sources of the last 2300 years to exhibit ideas that appear to have influenced the treatment of these topics in schoolbooks today. Historical sources clarify many points about the contemporary curriculum, supporting the contention that an understanding of history of ideas concerning ratio and proportion is an important component of knowledge of mathematics for teaching.

5.1 Introduction

Some of the occasionally puzzling things that we read in school mathematics textbooks, or find in discussions about standards or in commentaries about school math, can best be explained by reference to the long, complicated history of the curriculum. When we read that the quantities used in forming a ratio must be of the same kind, we are catching an echo of Definition 3 of Book V of Euclid's *Elements*: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind." Similarly, the statement that a proportion is an equality between two ratios refers back to Definition 6 of the same book: "Let magnitudes which have the same ratio be called proportional." Euclid, and two millennia of scholarly writings on Euclid, have influenced the way we speak about proportion today.

Another powerful influence, largely independent of the classical tradition, developed with the emergence of mercantilism in Europe in the Middle Ages. The rule of three is a method for solving the proportions that arise in trade, such as deducing the cost of one amount of a commodity from the cost of another amount, assuming that the conditions of the sale remain the same. The rule was known in antiquity and was described in texts such as al-Khwārizmī's *Algebra* (c. 820 CE) and Fibonacci's *Liber Abaci* (1202 CE). It was always closely associated with numerical computations and the use of units. From the thirteenth to the sixteenth century, the

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method was taught in the so-called abbaco schools that sprang up in northern Italy to provide the training in calculation required for the trades. As the Renaissance spread northward from Italy across Europe, the curriculum of these schools spread with it (Bjarnadóttir, 2014; Heeffer, 2009), eventually making its way to the New World and helping to shape the American mathematics curriculum in the eighteenth and nineteenth centuries. Abraham Lincoln wrote of learning the rule of three in the brief autobiography he prepared for his 1860 presidential campaign. Even today, one finds on the Internet many problems that are modern versions of schoolbook exercises from this tradition.

Ideas and habits that shape teaching practice tend to persist from generation to generation by a kind of cultural replication (Stigler & Hiebert, 2009). With respect to ratio and proportion, it is not merely that artifacts of the past survive. An examination of history shows continuity over more than a millennium. Some traditions have imprinted the practices of the present to such an extent that it can be difficult to make sense of the latter without reference to the former. A deep, mathematically informed understanding of the history of mathematics—not at the higher levels of scholarship, which is where most histories tend to focus, but at the level of the classroom—has practical relevance for teachers, teacher-educators, textbook authors, curriculum specialists, and anyone else who might influence mathematics education.

This essay assembles evidence in support of this thesis. In Sect. 5.2, I collect numerous statements of the rule of three from the seventh century up to the present. As we shall see, the formulations of this rule remained stable over more than ten centuries. Clearly, there has been a tendency for teachers (or at least textbooks) to perpetuate certain ways of thinking and behaving mathematically.

In Sect. 5.3, I return to Book V of the *Elements*. This work influenced scholarly writing about ratio and proportion from the Middle Ages onward. Wallis and Newton referred to Euclidean proportion in explaining the new conception of the number system that was beginning to emerge in the seventeenth century. Wallis discussed explicitly the reasons why ratios (as comparisons of magnitudes) could only be formed between magnitudes of the same kind. Galileo used patterns of reasoning borrowed from Euclid in considering the proportional relationships that arise in physics. Thinking in terms of magnitudes and their classical ratios—rather than numbers and operations on them—is characteristic of physics from the early Enlightenment up till today. In addition, important influences of Book V on contemporary mathematics occur in the theory of measurement (see Hölder, 1901 and Michell, 1999) and in the mathematics of ordered algebraic structures (Bigard, Keimel, & Wolfenstein, 1977). The Common Core Standards for grades K-5 treat measurement in a manner that is consistent with the Euclidean approach. However Book V shows a profound connection between the topic of ratio and proportion and the topic of measurement that seems to be overlooked in contemporary school math.

In Sect. 5.4, I examine some comments by mathematicians and mathematics educators about ratio and proportion in the contemporary curriculum. I believe that they sacrifice some aspects of the Euclidean theory that are meaningful in the sciences. In Sect. 5.5, I present a modern interpretation of a central idea of *Elements*, Book V. The point here is to show that the contents of this book are neither quaint nor outdated. It contains ideas that are relevant to how we think about the modern curriculum. I have tried to present the most important ideas in a way that will be accessible to a broad audience. Finally, in the last section, I gather together the conclusions that I think can be gleaned from this historical sight-seeing tour, and I share some final thoughts.

This essay is not intended to be a contribution to scholarship on the history of mathematics in the usual sense. I wish to suggest the relevance of the history of the mathematics curriculum to modern problems of teaching and instructional design. I hope that the reader will conclude that significant "discursive formations" (to borrow a phrase from M. Foucault) can be identified in the history of mathematics teaching and that they illuminate the structure of the modern curriculum. This paper will have achieved its goal if readers come away convinced that there is something to be gained by studying what one might call the "archeology of the mathematics curriculum." The task for the future is to pursue this in a disciplined and systematic way, with the aim of contributing to knowledge of mathematics for teaching.

5.2 The Rule of Three

Early in 1859, Abraham Lincoln's friend, Jesse Fell, asked Lincoln to prepare an autobiography, hoping to use it to help generate publicity for the potential presidential nominee. Lincoln's response, a letter of four paragraphs,[1] was used as a basis for an article that appeared in the *Chester County Times*, February 11, 1860, the day before Lincoln's fifty-first birthday. In the second paragraph, Lincoln described his boyhood in Spencer County, Indiana, where his family moved in 1816:

> . . . It was a wild region, with many bears and other wild animals still in the woods. There I grew up. There were some schools, so called; but no qualification was ever required of a teacher, beyond "readin, writin, and cipherin" to the Rule of Three. If a straggler supposed to understand latin, happened to sojourn in the neighborhood, he was looked upon as a. . . wizzard. There was absolutely nothing to excite ambition for education. Of course when I came of age I did not know much. Still somehow, I could read, write, and cipher to the Rule of Three, but that was all.

In pioneer communities like Lincoln's, families contracted a schoolmaster and paid a fee for each child. When he was around eleven, Lincoln attended Azel Dorsey's school in Little Pigeon Creek, Indiana, a mile and a half from the Lincoln cabin. "It was built of unhewn logs, and had holes for windows, in which greased paper served for glass. The roof was just high enough for a man to stand erect" (Lamon, 1872, p. 33). At the schools he attended during the next several years, he kept a "ciphering book" much of which his stepmother preserved. After Lincoln's

[1]The Library of Congress holds authenticated reproductions of the original letter, which may be viewed on the internet.

death, his law partner, William Herndon, acquired it. Ten of the surviving pages are reproduced in the first volume of the Rutgers edition of Lincoln's collected works (1953). One page bears the title "The Single Rule of Three."

Besides the so-called subscription schools like those that Lincoln attended, there were other forms of basic education at that time. Many boys in their early teens entered indentures, or contracts of apprenticeship, which obligated them to work for a master and reciprocally obligated the master to care for and educate the young person. Often, these contracts required the master to teach his charge to read, write, and "cipher to [or as far as, or through] the rule of three." This meant learning arithmetic with whole numbers and fractions, conversions of units of measure, weight and currency, and techniques for basic proportions. Numerous examples of such contracts can be found in genealogical databases, accessible via the Internet. One web site, for example, contains a copy of the Apprentice Bonds from Cumberland County, North Carolina. There we read that on December 6, 1823, the orphan Leonord Cason, about 14 years of age (thus sharing his birth year with Lincoln), was bound to a certain David D. Salmon, "To learn the art and trade of a saddler and harness maker and to be taught to read, write and cypher through the rule of three." Other boys were bound in apprenticeships as carpenters, bricklayers, coopers, blacksmiths, chair-makers, cabinetmakers, hatters, shoemakers, tailors, carriage-makers, farmers, millwrights, clerks, accountants, printers, bookbinders, and so on—all of them to be trained to "read, write and cipher through the rule of three." Lincoln's words echo the formulaic language of the learning standards of his day. In writing what he did in his autobiography, Lincoln was saying that his education met the typical requirements for the education of young man preparing for a trade.

What, then, is the rule of three? In modern algebraic notation, the rule is expressed as follows:

$$\text{for any positive numbers } a, b, c, x \text{ if } \quad \frac{a}{b} = \frac{c}{x} \quad \text{then} \quad x = \frac{b\,c}{a}\,.$$

However, the algebraic formulation tells us little about the teaching and use of the rule. Every application of the rule requires recognizing the roles of the numbers involved, including the units of measure in which the problem is stated and the units required for the answer. This is the pragmatic dimension of the rule, distinct from the symbolic structure by which the rule is expressed and the idea it encapsulates, but essential in teaching, learning, and using the rule.

In early sources, the rule was presented as a procedure for finding the value of x, given a, b, and c and an understanding of the roles they play in a transaction. For example, given that b shillings are paid for a ounces, if you want to find what must be paid for c ounces, multiply b and c and divide this number by a. This is a number of shillings, which must be expressed as a number of pounds, a number of shillings, and a number of pence before the solution is complete.

Smith (1958, p. 483) states that the name for the rule originated in India, though similar rules were stated in older materials from other places. Brahmagupta, who wrote around 630 CE, says (in translation):

> In the rule of three, argument, fruit and requisition [are names of the terms]: the first and last terms must be similar. Requisition, multiplied by the fruit, and divided by the argument is the produce. (Colebrooke, 1817, p. 287)

The procedure was described by al-Khwārizmī in his *Treatise on Algebra*, written early in the ninth century in Baghdad. The brief *Chapter on Transactions* (which is more or less independent of the other major chapters of the work) contains the following statement:

> Know that all transactions between people, be they sales, purchases, exchange, hire, or any others, take place according to two modes, and according to four numbers pronounced by the enquirer: the evaluative quantity, the rate, the price, and the evaluated quantity. ... [A]mong these four numbers, three are always obvious and known, and one of them is unknown ... You examine the three obvious numbers. Among them it is necessary that there be two, of which each is not proportional to its associate. You multiply [them] and divide the product by the other obvious number ...; what you get is the unknown number sought .... (Rasheed, 2009, p. 196)

The words *evaluative quantity*, *rate*, *price*, and *evaluated quantity* are translations of Arabic words that were used to differentiate the roles of the numbers. In modern notation, according to the translator, these numbers stand in the following relationship:

$$\frac{\text{evaluative quantity}}{\text{rate}} = \frac{\text{evaluated quantity}}{\text{price}}.$$

The chapter contains only a few, trivial examples to illustrate the application of the rule. It seems to be a report on widely used commercial practices, rather than guide for teaching them.

Now we jump ahead several centuries and shift focus from the East to pre-Renaissance Italy. Here, between 1200 and 1300 CE, as the mercantile revolution gathered momentum, communal and independent schools grew up to meet the needs of increasing numbers of young men headed for commercial and civic careers. The mathematics required for commerce was taught in the abbaco schools, which first appeared in northern Italy after 1250 (Goldthwaite, 1972). Historian Jens Høyrup has argued that the curriculum of the abbaco schools was derived from a culture of practical mathematics based on the Hindu-Arabic system that was well established in northern Africa, Spain, and southern France by the twelfth century (Høyrup, 2005). Presumably, this had been carried from the East by the expansion of Islamic civilization.

Late in the twelfth century, the young Fibonacci traveled through northern Africa, absorbing the mathematics used there and recording it in his *Liber Abaci* (1202). Fibonacci's book is often cited as a source for the emerging abbaco curriculum, but Fibonacci clearly was not the only conduit (Høyrup, 2005). In the abbaco schools, boys (roughly) between the ages of 10 and 13 learned how to write numbers with Hindu-Arabic numerals; how to perform the basic algorithms for whole-number addition, subtraction, multiplication, and division; and how to calculate with fractions. After this, they studied "commercial mathematics (in varying order): the rule of three, monetary and metrological conversions, simple and composite interest and reduction to interest per day, partnership, simple and composite discounting, alloying, the technique of a single false position and area measurement" (Høyrup, 2014, page 120). In short, boys in the abbaco schools learned to cipher to the rule of three—plus some.

Chapter 8 of the *Liber Abaci* opens with a paragraph that echoes al-Khwārizmī:

> In all commercial transactions, four proportional numbers are always found, of which three are known, but the remaining one is unknown. The first of the three known numbers is the number of units sold, be they bundles, or weights, or measures. A bundle might be, for example, a hundred hides or a hundred goatskins, or similar things: a weight might be a *cantarum*, or a *centum*, or a *libra* or an *uncia*, or something similar. A measure might be a *metra* of oil, or a *sestario* of corn, or a *canne* of cloth. The second number is the price of the sale to which the first number refers, and it may be a quantity of denari, or of bezants, or of tareni or some other monetary unit. The third is another quantity of the same merchandise as in the sale, and the fourth is the unknown price [to be determined]. (Boncompagni, 1857, page 83; free translation by JJM, aided by Sigler, 2012)

The first example in the chapter asks, *"If 100 rolls cost 40 pounds, how many rolls can I buy for 2 pounds?"* (a roll is a unit of weight.) Solutions were found by means of what Fibonacci called the "Principal Method," which goes as follows. Write the number of items of the first sale in the upper right of a square and in the upper left write the price paid; in the lower left, write the price in the second sale and to the right, leave a blank:

40	100
2	?

Multiply the two numbers that lie in the ascending diagonal and then divide by the number in the upper left. The result is the price to be paid.

The simple problem in the previous paragraph illustrates the procedure, but it is not at all representative of the kinds of problems that Fibonacci discusses. In the third problem, for example, the price of 27 rolls is sought, given that 100 rolls sells for 13 pounds. This is found to be $3 + {}^{51}/_{100}$ pounds, by the Principal Method. Fibonacci goes on to express the result in the form that would be needed in an actual transaction, as a number of pounds plus a number of soldi plus a number of denari. There being 20 soldi in a pound and 12 denari in a soldo, the result is 3 pounds, 10 soldi, and $2 + {}^{4}/_{10}$ denari. Chapter 8 contains nearly 150 examples illustrating the rule and, in virtually all cases, the units of measure, and the monetary denominations require attention. Eventually, in some problems, four different units are used in stating the problem, and much of the effort in finding a solution goes into making the required conversions. The problems, we can assume, are typical of those that merchants encountered in an age when different regions had their own systems of weights and measures and their own coinage.

During the 1300s, the abbaco curriculum acquired a stable, durable form. From the earliest times, many abbaco masters prepared handwritten manuscripts recording problems and solutions, as they may have been used in instruction. About 250 of these survive in libraries all over the world (van Egmond, 1981). As the new technology for printing spread across Europe after 1450, printed textbooks in practical mathematics began to appear. It is said that the first of these was the *Treviso Arithmetic*, of 1478. (A translation of this anonymous work is in Swetz, 1987.) A large part of this text is devoted to examples of applications of the rule of three. As in Fibonacci's work, there is much attention to units, unit conversions, and expressing monetary amounts in mixed denominations.

As Renaissance culture spread north through Europe, the mathematical culture of the abbaco schools spread with it. The *Bamberger Rechenbuch* by Ulrich Wagner (1483) contains a section on the rule, though here it is called the golden rule. Robert Recorde's *The Ground of Artes* (1543), one of the earliest printed books on arithmetic in the English language, has a chapter entitled "The golden rule, and the backer rule with divers questions therto belongynge." Recorde's book does not appear to me to be a manual for instruction, but more an exposition for a literate audience. Only a few illustrative examples of the rule are provided, but as in the works already mentioned, they require careful attention to the roles of the numbers involved and to the required unit conversions.

The rule was featured in Cocker's famous *Arithmetick*, which first appeared in 1677. (Several editions can be viewed complete on Google Books.) In the 48th edition (1736), Chapter 10 is entitled *The Single Rule of Three Direct*. It begins on page 87 as follows:

> 1. The Rule of Three (not undeservedly called the Golden Rule) is, that by which we find out a fourth Number, in Proportion unto three given Numbers, so as this fourth Number that is sought may bear the same Rate, Reason, or Proportion to the third (given) Number, as the second doth to the first, from whence it is also called the Rule of Proportion.

A few paragraphs later, we read:

> 6. In the Rule of Three, the greatest Difficulty is to discover the Order of the 3 Terms of the Question propounded, *viz.*, which is the first, second and third; which that you may understand; observe that of the Three given Numbers, two always are of one Kind, and the other [is] of the same Kind, with the proportional Number that is sought . . .
>
> 7. . . . to find out the fourth number . . ., multiply the second Number by the third, and divide the Product thereof by the first . . .

Following a page of explanation, there are 15 examples worked in detail, each filling about a page. These are very much like the problems in the *Liber Abaci*, in that they require discerning the roles of the quantities, converting units, and expressing the answer in a form appropriate for trade. In the following example, *C.* stands for a hundredweight, which consists of 4 quarters (*qrs.*), each being 28 pounds (*l.*) in weight. A pound (money) (*l.*) is 20 shillings (*s.*).

> *Quest. 10.* If 3 *C.* 1 *qr.* 14 *l.* of Raisins cost 9 *l.* 9 *s.* what will 6 *C.* 3 *qrs.* 14 *l.* cost?

Thomas Dilworth's *Schoolmaster's Assistant*, first published in London in 1743 and surely influenced by Cocker, became one of the most popular early arithmetic texts in the United States, with numerous North American printings between 1770 and 1820. Dilworth begins his presentation of the rule of three with the following catechism:

> Q. *By what is the* Single Rule *of* Three *known?*
> A. By *three Terms*, which are always given in the Question to find a *Fourth*.
> ...
> Q. *What do you observe concerning the* first *and* third Terms?
> A. They must be of the same Name and Kind.
> Q. *What do you observe concerning the* fourth Term?
> A. It must be of the same Name and Kind with the *Second*.
> ...
> Q. *How is the fourth* Term *in* Direct Proportion *found?*
> A. By multiplying the second and third Terms together and dividing that Product by the first Term.

The first problem following the instructional part reads, "If 3 *Oz*. of silver cost 17*s*. what will 48 *Oz*. cost?" The answer is worked out by multiplying 48 and 17 to get 816 and then dividing by 3 to get 272. This number of shillings is then converted to pounds and shillings by dividing by 20 (the number of shillings in a pound) to get the final answer: 13*l*. and 12*s*. The questions and answers from Dilworth quoted above are written out verbatim in Lincoln's ciphering book. Here we also read in Lincoln's hand the statement of the problem of the 3 *Oz*. of silver and its solution, as well as several other problems from Dilworth. Lincoln's teacher must have been using a copy of the *Schoolmaster's Assistant*. (Much more information about Lincoln's mathematics education, and his ciphering book in particular, can be found in Ellerton and Clements, 2014.)

In the late nineteenth century, schoolbooks began incorporating modern algebraic notation; see White (1870), for example. Rather than a cipher with four numbers, students would write an equation between two ratios, e.g., $12/30 = 42/x$. In a 1921 manual for teachers (Klapper, 1921), we read:

> A proportion is merely one method of writing a simple equation, and with the use of the letter x allowed, the equation form is likely to replace that of proportion. ... For example, consider this problem: If a shrub 4 ft. high casts a shadow 6 ft. long at a time that a tree casts one 54 ft. long, how high is the tree? Here we may write a proportion in the form
>
> $$6 \text{ ft.} : 4 \text{ ft.} = 54 \text{ ft.} : (?),$$
>
> not attempting to explain it, but applying only an arbitrary rule. This is the old plan. Or we may put the work into equation form,
>
> $$\frac{x}{54} = \frac{4}{6},$$
>
> and deduce the rule for dividing the product of the means by the given extreme ... (page 183)

No less than before, students need to discern the roles of the various numbers in order to place them in the appropriate graphical schema. Operationally, the method remains as it has been all along. The connection to algebra clarifies some points that would not have been evident from the rule for the manipulation of numbers. For example, various cancelations that are justified by the algebraic content can be used to simplify the calculation.

We have seen the use of the word "proportion" in the historical sources. The quotes from Cocker (and other remarks made by Cocker in the chapter we looked at) give the impression that he regarded proportion as a more theoretical topic that provided the justification for the rule of three. By the time of Cocker, Euclid was widely studied and written about by English scholars, and the connections to practical mathematics were probably recognized. The language in the teachers' manual suggests that the author understood the phrase "a proportion" to refer a problem of the type that the rule of three was meant to dispatch and that he expected students to deal with such problems by well-practiced but poorly grasped routines.

Looking back, we see that the rule of three has been a robust schema that for hundreds of years has been a stable part of the school mathematics experience. There have been changes in appearance, including more prominent reference to proportion and attempts to link the notation to algebra. Yet even today, if one searches the web for problems on ratio and proportion, much of what one finds reflects the ancient patterns with modern adaptations. The following comes from the Khan Academy web site: "Pamela drove her car 99 kilometers and used 9 liters of fuel. She wants to know how many kilometers (k) she can drive with 12 liters of fuel."

In the Common Core Standards for Mathematics, the rule of three is not mentioned, nor are the manipulations associated with it. In grade 6, students represent and reason about ratios and collections of equivalent ratios, and in grade 7 they learn to recognize proportional relationships between varying quantities and to represent them with an equation of the form $y = kx$, where k is a constant. The standards shift away from setting up and solving *proportions*, i.e., equations of the form $\frac{A}{B} = \frac{C}{x}$ with A, B, and C constants, and focus on *proportional relationships*, i.e., the relationships between variables x and y that are expressed by $y = kx$. It is not my intention to describe the vision of the new standards here, but I would like to draw attention to the fact that some observers of teachers have noted that the rule of three schema seems to have a powerful hold on the thinking of many, to the extent of creating the suspicion that it hinders the understanding of *proportional relationships* in the manner suggested by the standards; see (Stanley, 2014).

5.3 Euclidean Ratio and Proportion

A completely different tradition concerning proportion springs from Greek mathematics. This is described in Book V of Euclid's *Elements*. The terms A and B in the Euclidean ratio of A to B are not numbers but things, classically called "magnitudes." Lengths, areas, weights, and temporal durations are kinds of magnitudes. When forming a Euclidean ratio, the magnitudes must be of the same kind; otherwise, the sort of direct comparison required to make a ratio is not possible. According to Plato, understanding this was basic knowledge; those who believed that a line might measure a surface, or a surface a volume, exhibited an ignorance "more worthy of a stupid beast like the hog than of a human being" (*Laws*, 819d, A. E. Taylor, trans.).

The ancient Greek concept of number was fundamentally different from ours. In Greek mathematics, *number—ἀριθμός (arithmos)*—referred to a multitude composed of units. This idea encompassed the counting numbers 2, 3, 4, ... but none of the other things that today we call number. On the other hand, the ancient Greeks understood there to be nonnumerical quantities of many kinds: line segments, areas, volumes, weights, etc. The generic term for these was *μεγέθη (megethe)*, typically translated as *magnitudes*. These were not numbers and were not associated with numbers, but nonetheless one could operate upon magnitudes of a given kind by doing some (but not all) of the things we do with numbers. For example, given two different magnitudes of the same kind, one could determine which is larger by direct juxtaposition, or one could add them together by manipulations specific to the kind: in the case of lengths, by placing them end to end in a straight line; for polygonal areas, by cutting along lines and joining along edges; and for volumes of a liquid, by putting them in a single container. Most importantly (for us), given two magnitudes of the same kind, one could form a *ratio—λόγος (logos)*—between them. Magnitudes forming equivalent ratios were said to be *in proportion—ἀνάλογον (analogon)*. Because Greek ratios are not formed from numbers, but from magnitudes, the meaning of ratio and proportion in ancient Greek thought was different from present-day schoolbook notions, but it is nonetheless relevant to the modern curriculum in some unexpected ways, especially in measurement and in understanding quantity concepts.

Euclid's *Elements* was influential in European mathematics from the late Middle Ages. Translations into the vernacular languages of Europe were made in the sixteenth century, and in the seventeenth century, the study of Euclid was basic to a scientific education. During the seventeenth century, some English mathematicians strove to blend the Euclidean framework with the more modern number concepts that were then emerging. John Wallis (1685, page 79) described the idea of a Euclidean ratio as follows (with italics as in the original):

> [The] whole definition of λόγος (*Ratio, Rate*, or *Proportion*) ... [is] *that Relation of two Homogeneous Magnitudes* (or Magnitudes of the same kind,) *how the one stands related to the other, as to the* (Quotient, or) *Quantuplicity*: That is, *How many times*, (or How much of a time, or times,) *one of them contains the other*. The *English* word *How-many-fold*, doth in part answer it, ... but because beside these which are properly called *Multiple* or *Many-fold*, (such as the *Double, Treble*, &c. which are denominated by whole Numbers,) there be many others to be denominated by Fractions, (proper or improper,) or Surds, or otherwise; ... to which would answer (in *English*,) *How-much-fold*, (if we had such a word) ...

Ratio in the sense described here is not a relationship between numbers but is the means by which we pass from magnitudes of a nonnumerical kind to numbers.

Wallis was explicit on this point: "When a comparison in terms of ratio is made, the resultant ratio often (namely with the exception of the 'numerical genus' itself) leaves the genus of quantities compared, and passes into the numerical genus, whatever the genus of quantities compared may have been" (Wallis, 1968). This, of course, is not something that Euclid said; it is a new spin on Euclid, made possible by the new conception of number. The closest analog we have in modern thought to the Euclidean ratio of *A* to *B* is the *measure of A by B*. Indeed, Otto Hölder referred to Euclid in proposing an axiomatization for measurement in his fundamental paper of 1901 (Hölder, 1901). We shall return to Hölder's work in Sect. 5.5.

One of the most important applications of ratio in the *Elements* occurs at the beginning of Book VI, where Euclid shows that the ratio of the areas of two triangles with the same altitude is equal to the ratio of their bases. Although Euclidean ratios are relationships between magnitudes of the same kind, Euclid can compare a ratio between things of one kind to a ratio between things of another. The famous criterion for sameness of ratio is given in Definition 5 of Book V and is recognized as a precursor of the modern definition of real number. (We will say more about Definition 5 in §5, below.) As we have already mentioned, the term that Euclid used to describe equal ratios is ἀνάλογον, which is translated into English as "in proportion." The term is introduced in Definition 6: "Magnitudes which have the same ratio are said to be in proportion." That is to say, when the ratio of *A* to *B* is the same as the ratio of *C* to *D*, we say that the magnitudes form a *proportion*. (As we can see in the first quote from Wallis, the word "proportion" has in addition been used to refer to ratios.)

There is a powerful tradition related to Euclid's definitions. If one looks for "ratio and proportion" on the Internet, one finds numerous statements along the lines of the following:

> The ratio of two quantities of the same kind is the quotient of their measures. . . . An equality of two ratios is called a proportion. (1977, p. 38)

The influence of the Euclidean paradigm is evident, but the use of measurement to pass to numbers *before* taking ratios is a modern twist and a very peculiar—if not incoherent—one if viewed from a Euclidean perspective. Measurement itself is the formation of a ratio between nonnumerical inputs. Therefore, we cannot explain what a ratio is by reference to measurement. To do so would be circular. An orthodox modern Euclidean would explain the meaning of "in proportion" in the following way: "If the measure of *A* by *B* is the same as the measure of *C* by *D*, we say that the four quantities are in proportion." To repeat, the quantities *A*, *B*, *C*, and *D* are *not* themselves numbers, and no one of them is naturally associated with any number. It is only the measure of *A* by *B* and the measure of *C* by *D* that can be thought of as numbers.

At this point, some of the statements made in this section may seem obscure. We will elaborate and clarify in the following sections.

5.4 Mathematicians on School Math

In recent years, many mathematicians have commented on the meaning of ratio in school mathematics. They seldom mention the Euclidean conception or take it seriously. The following passages are from texts or papers by mathematicians. Because I am quoting out of context, these snippets may not communicate accurately the intent of the author. Therefore I make no attributions. Regardless of the author's intent, these passages contain ideas about ratio that I think we can recognize and identify in many discussions of school math.

> Ratios are essentially just fractions, and understanding and working with ratios and proportions really just involves understanding and working with multiplication, division, and fractions. . . . To say that two quantities are in a ratio A to B means that for every A units of the first quantity there are B units of the second quantity.

> By definition, given two . . . [numbers] A and B, where $B \neq 0$ and both refer to the same unit (i.e., they are points on the same number line), the ratio of A to B, sometimes denoted by $A{:}B$, is the . . . [number] A/B.

> We say that the ratio between two quantities is $A : B$ if there is a unit so that the first quantity measures A units and the second measures B units. . . . Two ratios are equivalent if one is obtained from the other by multiplying or dividing all the measurements by the same nonzero number. . . . A proportion is a statement that two ratios are equal.

In one way or another, the authors of these passages all say that we form a ratio out of a pair of numbers or that a ratio is nothing but a pair of numbers. Notice that in all three statements, A and B stand for *numbers*. The things themselves—what I have been calling the *magnitudes*—are mentioned but never named. If we take these statements seriously, the term "ratio" is not essential part of mathematical vocabulary, but rather it is a word used to signal that the numbers that are involved originate as the measures of two things whose relationship is of concern. The words quoted above are suggestive of the notion that the vocabulary of mathematics includes words for numbers, for sets of numbers, for arrays of numbers, for relationships between numbers, and for operations on numbers but does *not* include words that refer to things in the world.

The sciences other than mathematics take a different view. The quantities of physics are not labeled numbers but magnitudes much as conceptualized by Euclid. The basic magnitudes are length, mass, and time, and other magnitudes are composites of the basic magnitudes, e.g., velocity is length/time, acceleration is velocity/time, force is mass $\cdot$ acceleration, energy is length $\cdot$ force, and power is energy/time. If a unit is chosen for each basic magnitude, then each instance of each magnitude has an associated number. But in physics, it is more productive to reason with the magnitudes than with the numbers assigned to them through a choice of unit. This is the position advocated in many physics textbooks. Physicist Sanjoy Mahajan explains as follows; see Mahajan (2010, page 4). The inclusion of units, such as feet or feet per second in a problem about a falling body, he says, "creates a significant problem. Because [if we are given that] the height is h feet, the variable h does not contain the units of height: h is therefore dimensionless." If the other variables in the given problem are also numerical, then they are also dimensionless, and likewise any combination of them is dimensionless. Consequently, no combination is favored. However, the kinds of the given quantities can guide us—and indeed they *will* guide us—if we use variables to stand directly for magnitudes. We should not pose the problem of a falling body by asking for "the number v of feet per second that the body is moving after falling h feet, given the acceleration a in feet per s^2." Instead, we should understand each variable to stand for a quantity with a kind (or "dimension"), and we should recognize that we may only combine and compare magnitudes in a manner that is consistent with their kinds. We benefit thereby, because the physical meaning is built in to the terms with which we reason. If we ask, "What is the velocity v after falling a distance h, given the acceleration a," then evidently, the only magnitude we can compound from h and a that has the same genus as v is the square root of $h \cdot a$, and so we can expect v to be proportional to the square root of $h \cdot a$. We double the velocity by increasing the height by a factor of 4. Readers of Newton's *Principia* will find it filled with passages where the reasoning is of this kind but far more sophisticated. Newton's arguments about complex proportional relationships typically do not mention units or numbers. He uses units and numbers only when presenting experimental data.

5.5 Euclidean Magnitudes and Measurement

Above, we quoted Wallis saying that when we take the ratio of two magnitudes (which need not be numbers) we create a number. Two centuries after Wallis, Hölder provided a systematic elaboration of this idea. In this section, I will give a simplified account of what Hölder said. This is based on a set of basic ideas about magnitudes that are implicit in Book V of the *Elements*, together with some modern ideas about number.

From the presentation in Book V, we can infer that Euclid assumed several things about the members of each specific kind of magnitude. Hölder carefully disentangled these assumptions and called them the axioms for measurement. We can express them as follows:

(1) *Compare.* Given two objects of one kind, either they are equivalent (as members of their kind), or if they are not equivalent, then one is larger than the other. Moreover, if A is larger than B and B is larger than C, then A is larger than C.

(2) *Add and subtract.* Given two objects of a kind, we can add them to make a larger thing of the same kind. For example, we can put things with length end to end, or we can bind two masses together, etc. (We are not adding numbers! We are operating directly on things, much as first-graders do in some curricula before they ever learn to make measurements and represent the sizes of things with numbers; see the contribution of H. Bass to the present volume.) A smaller

magnitude may be removed from a larger one of the same kind. Moreover, addition and subtraction of magnitudes have the following properties:

(a) Addition is not sensitive to the order in which the parts are joined or assembled (i.e., it is associative and commutative).
(b) Subtraction is the inverse of addition; that is to say, if we add B to A and then subtract it, then we get back to A. And if we subtract B from A and then add it back, we get back to A.
(c) Adding the same magnitude to two others preserves their order. In other words, if A is less than B, then $A + C$ is less than $B + C$. The same is true of subtraction; if A is less than B, then $A - C$ is less than $B - C$. If A is equivalent to B and the same magnitude is added to—or subtracted from—both, then the resulting magnitudes are equivalent.

(3) *Duplicate and form integer multiples.* We can make copies of a magnitude—as many as we like. We may add two, three, or any number of duplicates of a given magnitude to itself and thus double, or triple, or form any multiple we please of the given magnitude.

Remark Let us stop for a moment to note some important consequences of the first three items. If A is a magnitude and we add together m copies of A, we call the result mA. The properties of addition imply that *if* $A < B$ (respectively, $A = B$, $A > B$), *then* $mA < mB$ (respectively, $mA = mB$, $mA > mB$) for all m. By assumption (1), for any A and B, exactly one of the conditions $A < B$, $A = B$, $A > B$ holds. Therefore, *if* $mA < mB$ (respectively, $mA = mB$, $mA > mB$) for any particular m, *then* $A < B$ (respectively, $A = B$, $A > B$).

Infinitesimal magnitudes had no role in the Euclidean theory of ratio. On this point, Euclid was explicit. Definition 4 states, “Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.” Accordingly, we add the Archimedean axiom to the list of properties that the magnitudes of a given kind must possess:

(4) Given a lesser and a greater magnitude, some multiple of the lesser exceeds the greater.

If we take conditions (1)–(4) together, they form a system of axioms. As we have said, they were first isolated by Hölder in (1901). Today, mathematicians will recognize them as an informal statement of the axioms for the positive part of an Archimedean totally ordered group. Math educators, on the other hand, will see here a collection of ideas that are closely related to the sequence of developmental benchmarks that children attain in mastering measurement. By age 5, children are able to identify measurable attributes, such as length and weight, to compare things with respect to length or weight and to use representations to make comparisons between objects that cannot be compared directly. After this, they acquire the ability to put several things in order with respect to a measurable attribute that they all share and to build up varying lengths by laying units end to end (or varying weights by combining weights in an appropriate way). Following this, the ability to compare and add are elaborated and refined, while the idea of using a number of identical units to represent an arbitrary length (or weight) develops (Sarama & Clements, 2009, pp. 289–292). The Common Core proposes standards for measurement in grades K-5 that reflect these stages. In grades K-2, children work with materials that directly mirror the abstract attributes of magnitudes that we listed.

At this point, we can continue the exposition in two ways. One way will be agreeable to mathematicians. It is brief and it states the mathematical content with great efficiency, but it is likely to be meaningless to many readers. The other way will be accessible to patient readers who have a modest mathematical background. It reveals historical connections and elaborates notions in the present-day curriculum concerning measurement, ratio, and proportion in interesting ways. We will go quickly through the first way and go carefully through the second.

For mathematicians, the heart of the matter is Hölder's theorem, which says the following. Suppose $\mathcal{G}$ is an Archimedean totally ordered group. (We will write $\mathcal{G}$ in additive notation.) Let $0 < B \in \mathcal{G}$. For each $0 < A \in \mathcal{G}$, define the real number $[A{:}B]$ by the following rule:

$$[A{:}B] := \sup \left\{ \frac{n}{m} \;\middle|\; mA \geq nB\,;\; m, n \in \mathbb{N} \setminus \{0\} \right\} \in \mathbb{R},$$

where sup means supremum, i.e., least upper bound. Evidently $[B:B] = 1$. It can be shown that for all $A, C \in \mathcal{G}_{>0}$, the following are true:

(i) $A \leq C \;\Rightarrow\; [A{:}B] \leq [C{:}B]$;
(ii) $[A + C{:}B] = [A{:}B] + [C{:}B]$.

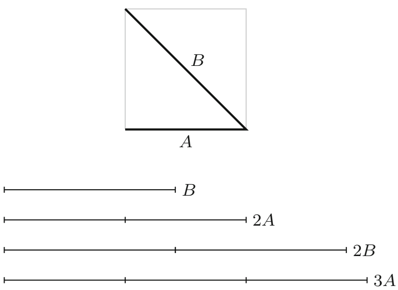
Furthermore, $A \mapsto [A : B]$ has a unique extension to an injective order-preserving group homomorphism from $\mathcal{G}$ to the additive real numbers. (Interestingly, Hölder's theorem does not require the hypothesis that $\mathcal{G}$ be commutative—the commutative property for $\mathcal{G}$ is implied by the other hypotheses; for background and a complete proof, the reader may consult (Bigard et al., 1977, pp. 48–50); see also Madden (2008) for elaborations relevant to measurement.) Notice that if $mA = nB$ for some positive integers m, n, then $[A : B] = n/m$. If there are no positive integers m, n such that $mA = nB$, then $[A : B]$ is an irrational number. We never assumed that a magnitude could be divided into equal parts—that is to say, we did not assume $\mathcal{G}$ to be divisible. If $\mathcal{G}$ *is* divisible, then $[A : B]$ is simply the supremum of the set of positive rational numbers q such that $qB \leq A$. In view of this, $[A : B]$ may reasonably be called "the ratio of A to B" or "the measure of A by B," because it has the properties that we expect of these things. In particular, to recall the words of Wallis, $[A{:}B]$ answers *How-much-fold* of B there is in A.

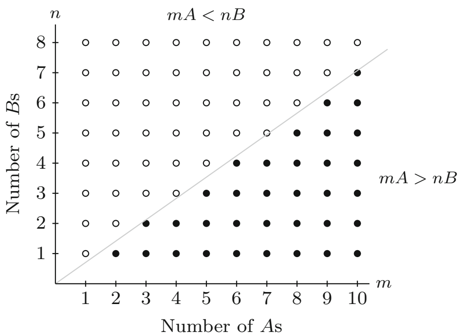
Now, let us examine the same material from a less technical vantage point. For concreteness, we will concentrate on lengths. Recall that we can (in principle) compare *any* two lengths by putting them side by side, lining them up at one end, and observing which goes further. If your pencil and mine line up at both ends, then *as lengths* they are the same. (Of course, the two pencils are different physical objects, but when we are concerned with them as lengths, this difference makes no difference; a philosopher might say that lengths are "equivalence classes" of objects of experience.) We can add any two lengths by putting them end to end. We can form multiples by duplicating and adding repeatedly. We can do all this with no need to measure or to assign numbers to the lengths, and these operations are well-behaved in the sense that the assumptions above are true of them. It is precisely because we can do these things, and because the outcomes are so governed, that we can form and compare ratios. How so? This will take some space to explain.

Suppose A and B are lengths. If we form a multiple of A and a multiple of B, then we can compare those multiples. Either they will be the same (with respect to length) or one will be larger than the other. Further, we need not stop with a single pair of multiples. We may consider *all* pairs, mA and nB, where m and n are allowed to range over all whole numbers. As we shall see, when we reason about the ratio of A to B, we must consider all such pairs of multiples.

Imagine all possible number pairs arrayed as a grid in the first quadrant of the coordinate plane, where (m, n) is the point lying m steps to the right of the origin and n steps above it. For each pair (m, n), just one of the following is true: $mA < nB$ or $mA = nB$ or $mA > nB$. Let us decorate the grid points according to which of the options we find. If $mA < nB$, we draw an open circle at (m, n). If $mA = nB$, we draw a red dot at (m, n). If $mA > nB$, we draw a black dot at (m, n). Note that we use a *square* grid—the horizontal steps are the same size as the vertical ones. Our grid is being used to record and label the number pairs only. We do not use the multiples of A and B in laying out the grid. We refer to the magnitudes only in deciding how to decorate the points with circles or dots.

The picture below shows the result of following this rule when A is the side of a square and B is its diagonal. The point $(2, 1)$ in the grid is black, because two As placed end-to-end exceed one B. Similarly, the point $(3, 2)$ is black, because three As exceed two Bs. We have drawn a light gray line through the origin in such a manner that it separates the black dots from the open circles. The point at $(7, 5)$ is above the line because $7A < 5B$ (though the circle around it happens to touch the line). There are no red dots in this picture, nor will there be any if the picture is extended, because we can never find a multiple of A that is equal to a multiple of B.





According to Euclid, the classification of the number pairs (m, n), which is illustrated in the grid diagram, tells us all there is to know about the ratio of A to B. That is, *if* we know *for which* (m, n) *it is true that* $mA < nB$, *and* we know *for which* (m, n) *it is true that* $mA = nB$, *and* we know *for which* (m, n) *it is true that* $mA > nB$, *then* we know *everything* about the ratio of A to B. All this is recorded in the diagram of dots and circles, since every (m, n) eventually gets marked with a circle or a red dot or a black one as the diagram is extended. Euclid says that if A and B are magnitudes of the same kind and C and D are magnitudes of the same kind as one another (but possibly of a different kind than A and B), then the ratio of A to B is the same as the ratio of C to D if and only if the diagram for A and B is the same as the diagram of C and D. This explains the meaning of the famous (and famously obscure) Definition 5 of Book V:

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Returning to our exposition of Hölder's ideas, we will use diagrams in place of the formal reasoning he employed. We will show that *every ratio of magnitudes has a real number associated with it in a canonical way*. We begin by listing some things that follow from conditions (1)–(4) concerning the diagram for a pair of magnitudes A, B. These are all things that Euclid would have understood, though of course he did not use dot diagrams.

First, if we draw a line through the origin and a point (m, n), then all the grid points on that line will be decorated in the same way—if one is circled, then they all are; if one is colored black (respectively, red), then they all are. This follows from the remarks after (3).

Second, in any diagram, there will be some circles and some black dots. This follows from (4), since given any m, there will be some n such that $mA > nB$. Thus, every column will have some circles. Symmetrically, given any n, there will be some m such that $nB < mA$. Thus, every row will have some black dots.

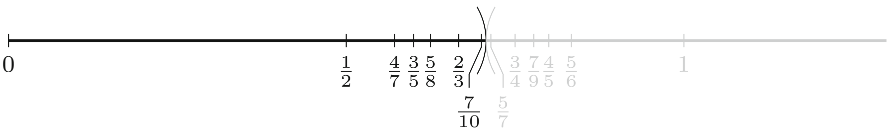
Third, if a line through the origin passes through a black grid point, then all the grid points below this line are black. Similarly, if a line through the origin passes through a circled grid point, then all the grid points above this line are circled. To see this, suppose (m, n) is black and ℓ is the line through $(0, 0)$ and (m, n). Suppose (m', n') lies below ℓ. Then $(m\,n', n\,n')$ lies on ℓ, and $(m'\,n, n'\,n)$ lies below $(m\,n', n\,n')$ on the same vertical line, so it's black. Since (m', n') and $(m'\,n, n'\,n)$ lie on the same line through the origin, the former is also black. The claim about circled points is seen by a similar argument.

Fourth, if a line through the origin passes through a red grid point, then all the grid points below that line are black and all the grid points above that line are circled. This can be seen by the same reasoning used in the previous observation.

Let us consider what happens if $mA = nB$ for some (m, n). In this case, the line through $(0, 0)$ and (m, n) marks the boundary between the black points and the circled points. Of course, all the grid points that lie on this line are red. According to

Euclid's Definition 5, the ratio of the numerical quantity n to the numerical quantity m is the same as the ratio of A to B. The reason for this is that the diagram for A and B is the same as the diagram for n and m since both diagrams have a red dot at (m, n). In this case, we associate the rational number n/m with the ratio of A to B. This takes care of the ratios that, as Wallis said, are "denominated by whole Numbers ... [or] by Fractions, (proper or improper)."

Now, let us consider what happens if $mA \neq nB$ for all (m, n). In this case, we need a different approach. With each grid point (m, n), we can associate the fractional number n/m. We may color the points on the fraction line in conformity with the decorations on the grid: color n/m black if the grid point (m, n) is black and color n/m gray if the grid point (m, n) is circled. Since $mA \neq nB$ for all (m, n), no points will be red. The first observation above shows that there is no ambiguity in the way we assign colorings. Condition (1) assures that every grid point is decorated, and therefore every positive fraction will be either black or gray. The second observation shows that some positive fractions will be colored black and some will be gray. Finally, the third observation shows that if n/m is colored black, then every positive fraction less than n/m will also be black, while if n/m is colored gray, then every positive fraction greater than n/m will also be gray. In particular every black number is less than every gray number. Hence, the decorated fraction line will appear as in the picture below. Here, we have marked the fractions corresponding to the grid points closest to the line we drew in the previous diagram.



The last step appeals to the modern definition of the real number system. A partition of the (positive) rational numbers into two sets with the properties of the black and gray sets above is called a Dedekind cut. To be precise, a Dedekind cut is a coloring of the rational numbers by two colors—black and gray, say—in such a manner that every rational number is colored, some rational numbers are black and some are gray and every black number is less than every gray number. The real number system has the property that for any Dedekind cut, there is a unique real number that is greater than or equal to each rational number colored black and less than or equal to each rational number colored gray. This is the number we associate with the ratio of A to B.

This has been rather long-winded, and essentially it has brought us to the definition of the function $A \mapsto [A : B]$ that we made above in a single line. On the other hand, the ideas and imagery that have entered this discussion might conceivably be incorporated in actual curriculum materials. For example, imagine a lab experiment where we attempt to measure the weight of a 10d common nail using a (lighter) 6d common nail as a unit. We might set up a balance and place various numbers of 10d nails in one pan and then add 6d nails to the other pan until the balance tips. We could record the data on a chart like what we made above, coloring the point (m, n) black if m 10d nails weigh more than n 6d nails. Specifically, beginning with a square grid of open circles in rows and columns labeled 1 through 20 (say), we could start by placing a 10*d* nail in one pan and then adding 6d nails to the other pan until the balance tips. Then we blacken the circles in the first column, up to the last one before the tipping point. Next, add a 10d nail to the first pan, and then add 6d nails to the other pan until tipping, and blacken dots in the second column by the same rule as the first. After moving through several columns, draw a line separating the black dots from the open ones. The slope is a good approximation of the measure of a 10d nail by a 6d one. The more columns we mark before making the line, the better the approximation, and since when there are m 10d nails in a pan, we are in essence measuring *one* 10d nail by mth parts of a 6d nail.

5.6 Conclusions

As we survey ideas related to ratio and proportion, a couple of things stand out. The first has to do with the meaning of the symbols that are used in reasoning about quantities. Even in the simplest of situations, such as the problem with the rolls, there is a problem in the world (How much should I receive for 2 pounds?), and there are symbols that we use to represent the problem and that we manipulate to find the answer. In several different contexts, we raised the question of whether the symbols refer to numbers or to objects in the world (or possibly to abstractions intermediate between the things we experience and the objects of the orthodox modern mathematical universe). In some respects, this does not seem to matter. The question might be dismissed as a philosophical concern with no implications for teaching, since it really makes no difference what the symbols mean in a metaphysical sense, but only what students do with them. But this assumes that the question makes no difference to the learners themselves. It very well might! The symbols that we use are present in our experience alongside everything else that we experience. That is, we are aware of the symbols themselves and are instinctively interested in how they work. When a child draws a picture, the picture itself becomes part of the world, and the child will speak about the picture, explain its parts, and develop and modify its meaning by talking about it (Woleck, 2001). In a manner that is not entirely different, learners are concerned about how meanings work in the symbol systems they use: "What refers to what? How do I recognize the connections? Why do I say or write this or that, and what does the result mean?" A good account of the meanings of things is not a philosophical indulgence but a solid support for student learning.

Laying out what the terms in a domain of knowledge refer to is a basic task of artificial intelligence. In order to develop a system for recording, filing, and systematically searching and retrieving medical information, for example, information engineers need a representation of the kinds of things that might be mentioned in a medical record and the kinds of relationships each might have to every other thing. A patient has a name, a date of birth, a weight, a pancreas, a prescription for eyeglasses, and innumerable other things. These things fall into classes and are related (or not) in ways dictated by the classes. Some things may change, some not. The weight may cause concern for the pancreas, but not for the eyeglasses. To sort these things out, the engineer will create what is called an ontology: a set of specifications about what there *is* in this knowledge domain, what the terminology refers to, and what properties and relations the objects may have to one another. The word "ontology" also has philosophical connections, but here we understand an ontology simply as a *very* explicit, practical specification of what a domain of discourse is about.

Our historical review of ratio and proportion has demonstrated that there are several competing ontologies for proportional reasoning. Up till now, no one has attempted to make the different competing ontologies fully explicit or to compare how the different alternatives might work out in a curriculum. The first step, clearly, should be to find an appropriate framework for sketching out the ontological alternatives. How to do this and how to put the final results to use are topics for future research.

The second thing that stands out is the intimate connection between measurement and proportional reasoning. It is interesting that in the Common Core Standards, the measurement and data domain spans kindergarten through grade 5, whence in sixth grade this domain vanishes and the ratio and proportional relationships domain appears. Reasoning with rates and proportions, I suggest, is more dependent upon the ability to understand the measurement process than widely acknowledged. The history of ratio and proportion bears this out. Of course, the ability to take measurements, calculate rates, put measure numbers into formulae, and "cancel units" at the appropriate times is important. But we need to attend to more than the mechanics. What is the *explanation* for a cancelation such as the following?

$$10{,}000\,\cancel{\text{feet}} \times \frac{0.3048\,\text{m}}{1\,\cancel{\text{foot}}} = 3048\,\text{m}$$

Perhaps you think that the words are just decorations to remind us that the 10,000 refers to feet, and the 0.3048 refers to meters. Or perhaps you prefer to think of the words as symbols for magnitudes that are here being multiplied by numbers. In either case, why is it that this cancelation procedure, which we have validated previously for numbers, can be carried over to this nonnumerical context? I cannot provide a complete rigorous answer that could be grasped in any seventh-grade classroom. I challenge readers to propose one. Most interesting proportional relationships involve heterogeneous quantities and a rate that relates the amount of one quantity in given units to the amount of the other, in other units. How do we change units—convert the driver's miles per hour to the runner's minutes per mile? We need a deep grasp of measurement in order to do this. It seems to me that the opportunity to produce a curriculum that ties measurement more closely to ratio and proportion is wide open and that the work to be done is great but has great potential.

I would like to close with some remarks of a broader nature. What teachers know and the knowledge that they value depends upon the knowledge and the values that are distributed throughout the systems that support teachers and teaching. The authors of textbooks; the people who train, observe, and evaluate math teachers; the people who develop and promote school policies; the people who compile standards; and the people who design and evaluate tests—all of them use special forms of mathematical knowledge and have their own mathematical priorities. At the system level, as opposed to the level of the individual teacher, "mathematical knowledge for teaching" becomes a matrix of meanings, understandings, habits of mind, and values that circulate among individuals in different roles in the organizations, agencies, and institutions that impact teaching. At this level, "mathematical knowledge for teaching" is more of a cultural entity than the set of understandings and abilities that we might find, or fail to find, in an individual. Culture is an emergent social phenomenon, not what is in someone's head.

Until the end of the twentieth century, the most powerful influencers of this culture were probably the traditions within the teaching community, the textbook writers and publishers, the professional organizations for teachers, and the university programs that prepared teachers. Textbooks, as concrete records of practice, were surely very influential. In the past several decades, new forces have come on the scene: the various systems of standards (created by the NCTM, the states themselves, and now the producers of the Common Core), the massive high-stake testing programs resulting from federal legislation, and increased use of test data in teacher evaluation. Within the last few years, there has been an explosion in the availability of curriculum materials on the Internet.

The scholarly discipline of mathematics has always been about creating and describing efficient, coherent systems of ideas. The same mindset ought to be applicable to school mathematics. It should be possible to lay out the content of school mathematics in good mathematical style, with rigorous definitions, clear logic, and appropriate, unambiguous symbolism. Roger Howe's essays on topics in school mathematics are examples of this. Historically, however, mathematicians have not been the chief architects of school mathematics—it has had no chief architects. It has developed like the ancient cities that Descartes contemplated in the *Discourse on Method*, which "from being at first only villages, have become, in course of time, large towns" and which, as a consequence, are "usually but ill-laid out compared with the regularly constructed towns which a professional architect has freely planned on an open plain." He added that "it is not customary to pull down all the houses of a town with the single design of rebuilding them differently, and thereby rendering the streets more handsome ...," and similarly, it would be "preposterous for a private individual to think of reforming a state by fundamentally changing it throughout, and overturning it in order to set it up amended ... [or to contemplate a] similar project for reforming the body of the Sciences, or the order of teaching them established in the Schools ...."

The culture of the curriculum, as sustained by the institutions described above, is traditional and syncretic. For whatever reasons and by whatever mechanisms, this culture preserves patterns of expression and habits of thought, meeting pressure for change by absorbing and transforming what is newly thrust upon it, forcing new things into the spaces between old structures, or on top of them, or within them.

It mixes and juxtaposes ideas, in much the same way popular culture samples and remixes styles, cuisines, icons, and beliefs. Knowledge for teaching *as it is* at the present time, rather than *as we might wish it to be*, resembles what knowledge for healing was 150 years ago: a mixture of folkways, craft wisdom, and science, shaped as much by social influence as by reason. To change it one would need to change the institutional conditions around teaching, . . . but all this is something to take up at another time. My main thesis here is that the knowledge for teaching that we have at present—no matter what anyone might envision as a replacement—is the result of cultural process spanning centuries. In many cases, the intellectual sources have been reasonable and coherent, though this is not always evident in the resulting hodgepodge. If we wish to replace what we presently have with something better, the first step should be to understand truly what we have.

I would like to end on an inspirational note that Dick Stanley mentioned to me. Descartes may have been quite right that it would be preposterous for an individual to undertake the rebuilding of a city. But the preposterous and the impossible are two different things. At the behest of Napoleon III, between 1853 and 1870, Georges-Eugène Haussmann led a massive renovation of Paris, tearing down vast tracts of ancient buildings and laying out the majestic city we know today. Though he was forced from his position as Prefect of the Seine in 1870 by political opponents, the project continued, reaching completion in 1927 with the opening of the Boulevard Haussmann. Might we, in the present century, achieve something analogous for the mathematics curriculum?

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