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Mathematics Matters in Education

Essays in Honor of Roger E. Howe

Springer

Advances in STEM Education

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The gathering of many mathematicians, mathematics educators, and other scholars made the workshop both memorable and productive. With a follow-up conference that already took place, and more planned for the future, the workshop was a catalyst for furthering connections and communications among mathematicians, mathematics educators, and practitioners of mathematics. We would like to acknowledge and thank all workshop participants for their participation and support.

We also take this opportunity to acknowledge and thank all those who have been involved in preparing and contributing to this volume. This book would not have been possible without the dedicated group of 20 contributors from more than 15 universities or institutes across the United States, many of whom also volunteered their time to review chapters. Their collective efforts help ensure this book's quality. The resulting book is a wonderful Festschrift in honor of Roger Howe for his contributions to mathematics education as an interdisciplinary field.

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Finally, we want to express our deep appreciation to the workshop's host institution, the College of Education and Human Development at Texas A&M University and especially the Department of Teaching, Learning and Culture, for their great support that made the workshop possible. The selfless support from many dedicated staff and graduate students was beyond the call of duty.

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Contents

Part I Introduction

1 "Mathematics Matters in Education" to Roger E. Howe and to All: An Introduction 3
Yeping Li, W. James Lewis, and James J. Madden

2 About Roger E. Howe and His Contributions to Mathematics Education 9
Yeping Li and W. James Lewis

3 Cultural Knowledge for Teaching Mathematics 19
Roger E. Howe

Part II Knowing and Connecting Mathematics in Teaching and Teacher Education

4 The Content Knowledge Mathematics Teachers Need 43
Hung-Hsi Wu

5 Knowing Ratio and Proportion for Teaching 93
James J. Madden

6 How Future Teachers Reasoned with Variable Parts and Strip Diagrams to Develop Equations for Proportional Relationships and Lines 117
Sybilla Beckmann and Torrey K. Kulow

7 Giving Reason and Giving Purpose 149
Yvonne Lai, Mary Alice Carlson, and Ruth M. Heaton

8 Who Are the Experts? 173
John Ewing

Part III Identifying and Structuring Core Ideas and Practices in K-12 Mathematics Curriculum

9 Building on Howe's Three Pillars in Kindergarten to Grade 6 Classrooms 185
Karen C. Fuson

10 Is the Real Number Line Something to Be Built or Occupied? 209
Hyman Bass

11 What Content Knowledge Should We Expect in Mathematics Education? 219
Richard Askey

12 Approaching Euclidean Geometry Through Transformations 233
Zalman Usiskin

13 Curricular Coherence in Mathematics 245
Al Cuoco and William McCallum

Part IV Supporting and Engaging Mathematicians in K-12 Education

14 Attracting and Supporting Mathematicians for the Mathematical Education of Teachers 259
Amy Cohen

15 The Contributions of Mathematics Faculty to K-12 Education: A Department Chair's Perspective 277
Solomon Friedberg

16 Supporting Education and Outreach in a Research Mathematics Department 287
Jerry Dwyer and Lawrence Schovanec

Erratum E1

Index 301

Part I
Introduction

Chapter 1
"Mathematics Matters in Education" to Roger E. Howe and to All: An Introduction

Yeping Li, W. James Lewis, and James J. Madden

Abstract Roger Howe is one of a small group of prominent mathematicians in the United States who has acted vigorously and productively out of concern for the quality of K-12 mathematics education. Moreover, his work has encouraged and supported the engagement of other mathematicians in this important endeavour. This book is a Festschrift to recognize Howe for his more than 20 years of exemplary efforts in addressing important issues in mathematics education and promoting the development of mathematics education as an interdisciplinary field. It brings together mathematicians and mathematics educators to demonstrate not only the possibility but also the importance of joint efforts in improving the quality of mathematics education.

1.1 Introduction

This volume grew out of a workshop held in 2015 on the campus of Texas A&M University, College Station, in honor of Roger Howe's 70th birthday. At that time, Howe was a faculty fellow of the Texas A&M University Institute for Advanced Study. Conceived with the theme "mathematics matters in education," the workshop aimed to highlight the importance of mathematics not only as a scientific discipline but also as an essential component of school education. The theme reflects Howe's career, which began in pure research and expanded to accommodate his deep concern for the quality of mathematics education and teacher education (see Li and Lewis, in this book).

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Mathematicians, mathematics educators, and other scholars from many higher education institutions and professional organizations across the United States attended the workshop. The broad participation demonstrates the significance of the workshop's theme and represents a tribute to Howe's work and efforts in mathematics education. This book is a collection of contributions developed from the workshop that show how other mathematicians and mathematics educators have connected to, learned from, or built upon several aspects of Howe's work. As a scholarly product of the workshop, this book is a celebration in print of Howe's contributions to mathematics education as an interdisciplinary field.

The workshop was structured with three topic areas in which Howe has invested time and effort for more than 20 years. To make the workshop an occasion for promoting further scholarly exchanges, Deborah Ball (also a co-organizer of the workshop) helped to formulate these topic areas as the following questions for workshop participants to share and discuss:

1. What is the state of our understanding of *mathematical knowledge needed for teaching*? Howe has been engaged in the question of what teachers need to know to teach in mathematically serious ways. We will explore what progress has been made in identifying what teachers need to know in order to teach mathematics with integrity.
2. What are some of the *core ideas and practices in the K-12 mathematics curriculum*? Bringing coherence to the broad span of the school curriculum depends on the identification of fundamental mathematical structures that underlie the proliferation of disconnected topics and skills that muddle many US textbooks. Howe has worked on this, focusing especially on the whole-number curriculum, and has worked to bring the critical structural components of the curriculum in sharp focus.
3. What can be done to *support mathematicians who want to engage productively in K-12 mathematics education*, but who will not become full-time mathematics educators? This is a question that Howe has been asking for over 15 years, and it inspired the Institute for Mathematics Education, led by Bill McCallum at the University of Arizona, to start a series of annual professional development workshops for mathematicians. What other models may support mathematicians' engagement in education? What do mathematicians need to understand about education research or teaching teachers mathematics? What are the incentives? What are the contextual issues that mathematicians must appreciate?

Howe's work and leadership have helped draw together a diverse group of mathematicians, mathematics educators, and other scholars. The spirit of interdisciplinary discussion and collaboration among them is evident, not only in terms of the joint participation of the workshop itself but also in terms of their complementary contributions to this book. The workshop's speakers and discussion panellists were invited because they had worked with Howe or used his work in their own research in these topic areas, but there are certainly many other mathematicians and mathematics educators who have worked with him. The contributors of this book by no means exhaust the community of those who have collaborated or cooperated with Howe in mathematics education over the years.

This book demonstrates in a unique way the importance and benefits of joint efforts of mathematicians and mathematics educators in addressing problems in mathematics education. Although there have been a growing number of articles and books that exemplify such joint efforts in recent years (e.g. Bass, 2005; Dewar, Hsu, & Pollatsek, 2016; Fried & Dreyfus, 2014), more collaborations are needed. We hope that the publication of this book will help serve as a call for further collaborations among scholars of numerous backgrounds to improve mathematics education in the United States and beyond.

This book is also the inaugural volume of the new international book series on *Advances in STEM Education*, the first book series on science, technology, engineering, and mathematics (STEM) education published by Springer. This new series aims to provide a venue for sharing the research, policy, and practice of STEM education and to promote cross-disciplinary collaborations in STEM education at all school levels as well as through teacher education around the world. The spirit of cross-disciplinary collaboration evident in this volume provides a starting point for this book series to promote and advance multidisciplinary and interdisciplinary research in STEM education across the globe.

1.2 Structure of the Book

This book is designed to offer mathematicians and mathematics educators with the opportunity to share their work and reflect on how their work connects with or builds upon Roger Howe's efforts in mathematics education. It is organized in four parts that parallel the structure of the workshop: (I) Introduction, (II) Knowledge of mathematics for teaching and teacher education, (III) Core ideas and practices in K-12 mathematics, and (IV) Supporting and engaging mathematicians in K-12 education.

Part I contains three chapters. The present introductory chapter provides readers with an overview of the book. Chapter 2 reviews Howe's contributions to mathematics education, and Chap. 3 describes Howe's views about selected challenges faced by elementary school mathematics in the United States.

Part II focuses on the nature of the knowledge and expertise that teachers need for effective teaching and the means by which they obtain it. Scholarly inquiries and research about these questions have gone on for years, yet much remains unclear. This is a problem that Howe has been interested in exploring, both in terms of the mathematics itself, as well as the design of teacher preparation. Five chapters are included in this part. In Chap. 4, Wu discusses the mathematical content knowledge that teachers need in order to achieve a basic level of competence in mathematics teaching and contrasts this with the incomplete form of understanding that is perpetuated by what he calls "Textbook School Mathematics" (TSM).

Wu emphasizes the importance of mathematical integrity, precision, and coherence in teachers' knowing of mathematics for teaching. In the next chapter, Madden reviews the history of ratio and proportion attempting to isolate those ideas that are most relevant to teaching. He suggests that not only teachers but also textbook authors and curriculum supervisors ought to be aware of the ways in which very old traditions have shaped the way we treat these topics. In Chap. 6, Beckmann and Kulow discuss preservice teachers' learning of mathematics for teaching. They provide a case study of six preservice middle-grade teachers' reasoning related to proportional relationships. Their results highlight the importance of supporting preservice teachers' learning by identifying and responding to their learning difficulties. Lai, Carlson, and Heaton (Chap. 7) consider the importance of "knowing why" and "knowing what" in teaching in order to help students make meaningful connections. They emphasize the importance of "giving reason" (why) and "giving purpose" (what) in teaching and planning instruction. They analyse one first-grade teacher's classroom work to illustrate how these components relate to one another. Ewing (Chap. 8) highlights the importance of learning from K-12 mathematics teachers. In particular, he offers a dramatic call for all involved in mathematics education to respect K-12 mathematics teachers' expertise and learn from them when thinking about teaching and its improvement.

Part III takes on core ideas and practices in the K-12 mathematics curriculum. This is a topic that has fascinated Howe for many years, as he has worked meticulously to design proposals for change and improvement in elementary school mathematics. The five chapters, contributed by both mathematicians and mathematics educators, treat topics spanning kindergarten to high school. In Chap. 9, Fuson builds upon Howe's proposed three pillars for first-grade mathematics and beyond (Howe, 2014), to propose visual models that can help students learn the aspects of mathematics identified by Howe. Bass (Chap. 10) contrasts two different perspectives on number concept development using the number line: the *occupation narrative* and the *construction narrative*. The first is the dominant approach in current teaching practice, which begins by learning to count and then placing the counting numbers and subsequently the fractions and the reals on the number line. The second, exemplified by the Danilov curriculum, gives priority to reasoning about quantity rather than the symbolic expressions for numbers. In Chap. 11, Askey argues that some important topics in the curriculum can and should be presented and taught in a manner that is more precisely motivated by mathematics. He illustrates his ideas with the topic of geometric measurement and fractions. Usiskin (Chap. 12) reflects on his work in developing a high school geometry course based on transformations. This is closely connected to Howe's work on continuous symmetries in Euclidean geometry (Barker & Howe, 2007). Usiskin discusses the similarities and differences in these approaches. In Chap. 13, Cuoco and McCallum present a definition of curricular coherence with two aspects: coherence of content, which deals with the arrangement of topics in curriculum, and coherence of practice, which focuses on the habits of mind the curriculum fosters in students. They also distinguish curriculum and standards and illustrate the two aspects of curricular coherence with examples.

Part IV focuses on ways of supporting and engaging mathematicians in K-12 school education. Over many years, Howe has actively advocated and supported work in this important area, using his status as a leader in research to increase the level of attention and respect for such work in the whole mathematical community. The three chapters included here take perspectives at different administrative levels. In Chap. 14, Cohen presents extensive evidence gleaned from her own experiences concerning how mathematicians can work productively with teachers. There is a significant intellectual challenge in finding ways to support teachers' mathematical work. But this must be joined with a respect for the broader intellectual work that teachers engage in, which is of a different character from what mathematicians are likely to be familiar with. Friedberg in Chap. 15 presents a department chair's perspective. He suggests ways to encourage conversations within the mathematics department and beyond and reminds us that if the contributions of mathematicians to education are to be valued, they must be recognized and evaluated in a sustainable, professional way. Finally, Dwyer and Schovanec in Chap. 16 share their perspectives and insights about getting support for mathematicians at different administrative levels, with a specific emphasis on how to include output in education and outreach in procedures for evaluation, merit recognition, tenure, and promotion.

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Chapter 2
About Roger E. Howe and His Contributions to Mathematics Education

Yeping Li and W. James Lewis

Abstract Roger Howe is one of a few notable research mathematicians who is also well respected in the mathematics education community nationally and internationally. In this paper, we outline three aspects of his achievements related to mathematics education: his accomplishments as an educator, his contributions and achievements as a leader and his contributions and achievements as a scholar in mathematics education.

2.1 Introduction

Born in 1945, Roger Howe began his career as a research mathematician. After obtaining his bachelor's degree in mathematics from Harvard College in 1966, he pursued doctoral study in mathematics at the University of California at Berkeley, earning a PhD. in 1969. He became an assistant professor and later associate professor of mathematics at the State University of New York at Stony Brook from 1969 to 1974 and then moved to New Haven as a full professor at the Yale University from 1974 to 2016. He served as chair of the Mathematics Department from 1992 to 1995. He was the inaugural Frederick Phineas Rose Professor (1997–2002) and then the William Kenan Jr. Professor of Mathematics (2002–2016). He became interested in issues in mathematics education in the early 1990s and has been involved in the field of mathematics education nationally and internationally ever since. Among his numerous honorary and visiting positions, Dr. Howe was selected and appointed as a faculty fellow (class 2013–2014) at the Texas A&M University Institute for Advanced Study, which allowed him to study issues of curriculum and teacher preparation in mathematics. He was subsequently invited

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to join Texas A&M University in 2015 to improve the efforts in preservice teachers' preparation in mathematics, which he accepted. He has been a full professor and the holder of Curtis D. Robert Endowed Chair in Mathematics Education in the Department of Teaching, Learning and Culture since 2016 and an affiliated professor of mathematics in the Mathematics Department, Texas A&M University.

Trained as a mathematician, Dr. Howe built his legacy for his achievements in mathematical research since the early stage of his professional career. He is best known for his breakthroughs in representation theory, which allows mathematicians to translate problems from abstract algebra into linear algebra, thus making the problems easier to manage. He first introduced the concept of the reductive dual pair – often referred to as a "Howe pair" – in a preprint during the 1970s, followed by a formal paper in 1989. This and other significant contributions to mathematics research earned him election as a member of the American Academy of Arts and Sciences and as a member of the US National Academy of Sciences in 1994. He is also a recipient of the Lester R. Ford Award from the American Mathematical Society (AMS), and he was a member of the inaugural class of AMS Fellows in 2013. Today, he continues to work on representation theory, as well as other applications of symmetry, including harmonic analysis, automorphic forms and invariant theory.

As a mathematician who has made many contributions to mathematics education, Dr. Howe has championed national initiatives to advance mathematics education through engaging mathematicians and contributing to issues in mathematics curriculum, teaching and teacher education. His professional service on numerous committees and panels has allowed him to help shape the direction of important developments in mathematics education nationally and internationally, and strengthen connections between mathematics and mathematics education. At the same time, such service helped inform his writing and thinking over many issues in mathematics education. He is a great thinker and is also widely recognized as a scholar. Together with his professional interest and care about educating future generations of mathematicians, he has kept investing time and effort in the study and improvement of K-12 mathematics education as an essential part of his obligation as a mathematician and educator.

Dr. Howe's achievements have been recognized in the awards he has received. In addition to numerous recognitions he has received in mathematics, including a Guggenheim Fellowship in 1984, he received the American Mathematical Society Award for Distinguished Public Service in 2006 for his "multifaceted contributions to mathematics and to mathematics education" and the Texas A&M University Inaugural Award for Excellence in Mathematics Education in 2015 "for his more than forty years of sustained and distinguished lifetime achievement in mathematics research, work impacting mathematics education and promoting interdisciplinary collaboration in mathematics education".

The following sections outline three aspects of Dr. Howe's achievements: his accomplishments as an educator, his contributions and achievements as a leader and his contributions and achievements as a scholar in mathematics education. We describe each of them in a bit more detail and point to references that will be helpful to those who want to learn more.

2.2 Accomplishments as an Educator

Dr. Howe's thinking about mathematics education started with his own teaching of undergraduate mathematics courses at Yale. His interest in engaging students more in mathematical thinking and learning in classrooms led him to develop and use more effective pedagogy. Instead of simply giving lectures in an undergraduate course, he put more questions in his presentations to get students to think about the mathematical and conceptual issues they were addressing. This helped focus student attention and created mathematical dialogue in real time. The process gave students more time to think and understand the ideas under discussion and at the same time provided the instructor feedback about what students knew and were thinking. His teaching was clearly appreciated by more and more undergraduate students over the years. In 1997, Yale presented him with the Yale College/Dylan Hixon '88 Prize for Teaching Excellence in the Natural Sciences. In part, his citation reads, "... if mathematics is a language, you certainly speak it beautifully. Fortunately for those who are not themselves native speakers, you have demonstrated a gift for making fundamental concepts in the structure of mathematics become familiar and intelligible".

As a world-class mathematician, Dr. Howe's dedication in nurturing future generations of mathematicians certainly includes graduate education and mentoring of junior faculty on and off campus. He is making a qualitative difference through his work as mentor. The majority of his graduate students have gone on to solid academic positions and are regularly achieving tenure and advancement to full professor at research-intensive institutions in the USA and other countries, including China, Hong Kong, Israel and Singapore. A substantial number of the PhDs who studied with him have, themselves, come to wield significant influence in mathematics both nationally and internationally. This is illustrated by the multiple conferences that were organized and held in his honour. In January 2006, an international conference on harmonic analysis, group representations, automorphic forms and invariant theory was organized and held in Singapore on the occasion of his 60th birthday (see http://www.ims.nus.edu.sg/activities/rogerhoweconf/). A research volume was produced and published out of the conference (Li, Tan, Wallach, & Zhu, 2007). In June 2015, another international conference on representation theory, number theory and invariant theory was organized and held at Yale as an occasion to celebrate his 70th birthday (see http://math.mit.edu/conferences/howe/). Research volumes will also soon be published out of this conference.

Dr. Howe's mentoring of younger mathematicians goes well beyond the campus of Yale. He is an easily accessed and beloved mathematician by many nationally and internationally. He regularly interacts and mentors junior faculty in many other institutions in the USA and other countries. In fact, another conference on representation theory and applications was organized and held in his honour in Istanbul, Turkey, in June, 2013 (see http://dauns.math.tulane.edu/~mcan/Istanbul.html).

Dr. Howe is dedicated to education and mentoring not only in mathematics but also in mathematics education. Now at Texas A&M University, he regularly sits in undergraduate courses for preservice elementary teachers and occasionally teaches such classes. He has started to mentor graduate students in mathematics education as well, serving as chair/co-chair for two PhD students in mathematics education. At the time when we organized and held this “Mathematics Matters in Education” workshop in honour of his 70th birthday, we had great responses and participations from many different universities in Texas and across the nation. Indeed, he has a popular following in mathematics education as well.

2.3 Contributions and Achievements as a Leader

Dr. Howe’s research and scholarship in mathematics earned him numerous recognitions and professional leadership positions in mathematics and mathematics education. As noted earlier, he was elected as a member of the US National Academy of Sciences in 1994 and as an inaugural fellow of the American Mathematical Society in 2013.

As one of a few well-respected mathematicians in the USA working simultaneously, and continuously, in the two “distant” areas of mathematical research and mathematics education, Dr. Howe has worked hard to bridge the discipline of mathematics and mathematics education through professional service. In so doing, he emphasizes the importance of education and encourages other mathematicians to get involved in mathematics education.

Dr. Howe served on the Mathematical Sciences Education Board (MSEB) from 1995 to 1998, as chair of the American Mathematical Society (AMS) Consultative Committee to the National Council of Teachers of Mathematics (NCTM) mathematics standards revision project in 1998, the AMS Committee on Education from 2000 to 2006 (as chair from 2000 to 2004), the Study Committee for the report *Adding It Up* of the National Academy of Sciences on the state of US mathematics education, the Steering Committee for the first Conference Board of the Mathematical Sciences (CBMS) report on *The Mathematical Education of Teachers* and several committees for the College Board. Currently he is on the Education Advisory Committee of the Mathematical Sciences Research Institute (MSRI), among many others. He also served on the Steering Committee of the Park City Mathematics Institute as its undergraduate program coordinator from 2000 to 2007, and he served on the planning board of the Institute for Mathematics and Education (IME) at the University of Arizona. He has also been instrumental in helping establish IME’s workshop series: Mathematicians in Mathematics Education.

Internationally, Dr. Howe served for 6 years (2006–2012) on the US National Commission on Mathematics Instruction (USNC/MI) and on the Executive Committee of the International Commission on Mathematics Instruction (ICMI) from 2008 to 2016. When serving on the Executive Committee of ICMI, he proposed to carry out a series of ICMI studies on school mathematics, starting on elementary mathematics. This led to the first successful ICME study in this direction: ICMI Study 23, primary mathematics study on whole number, which was held in Macao, June 2015. Indeed, with his leadership roles in mathematics and mathematics education in the USA and internationally, Dr. Howe has promoted communication, understanding and collaboration between mathematicians and mathematics educators.

Through his committee services, Dr. Howe has made many important contributions to influential committee reports in mathematics education. They include: *Adding It Up*, published by the National Academies Press, 2001; *The Mathematical Education of Teachers*, CBMS Issues in Mathematics Education, volume 11, published by American Mathematical Society, 2001; *Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education* (report of the Rand Mathematics Study Panel), published by RAND Corporation, 2003; *Focus in High School Mathematics: Reasoning and Sense-Making*, published by National Council of Teachers of Mathematics, 2009. As a member of the USNC/MI, he also helped to convene a workshop comparing teaching careers in the USA and China. The proceedings were published by the National Academies Press as *The Teacher Development Continuum in the United States and China* in 2010.

2.4 Contributions and Achievements as a Scholar in Mathematics Education

Although Dr. Howe's involvement with mathematics education started as primarily a service activity, it has gradually evolved in a more scholarly direction. His critical thinking of information about mathematics education, together with his professional interests in pondering over many unresolved questions and issues, has led him to become increasingly involved in mathematics education and to think critically in searching for possible solutions. His passion for preparing future generations of mathematicians has evolved to thinking about ways to elevate the quality of mathematics education in K-12, focusing on curriculum, teaching and teacher preparation. In particular, he has focused on elementary mathematics education. We highlight here his work in curriculum and teacher education at the elementary school level.

2.4.1 Identifying and Articulating Core Ideas and Practices in K-12 Mathematics Curriculum, Especially in Elementary School Mathematics

Around the time when Dr. Howe served on the Mathematical Sciences Education Board (MSEB) of the National Research Council (NRC) in the mid-1990s, the results of the Third International Mathematics and Science Study (TIMSS) was released and alarmed many educators, policy makers and the general public about the unsatisfactory performance of US students in school mathematics. The results led to many further discussions and reflections about the state of mathematics education and possible ways for improvements in the USA among mathematics educators, mathematicians and those who care about school education. Specifically, questions and concerns about mathematics teaching and curriculum were raised and discussed. Dr. Howe paid close attention to problems related to subject matter coverage, especially on such topics and ideas that were clearly germane but seemed not to get as much attention as they might need.

The first topic Dr. Howe identified was place value. He has written several essays on this topic. The first piece was for Harcourt Publishers, when he was asked to review for them for an edition of their elementary mathematics textbooks. His remarks about place value were included in “Teacher Pages” in one of the books of the set. His initial intention of broadening such ideas led to more extensive and serious work about place value. He worked together with Susanna Epp of the DePaul University on this topic and published a rather long and detailed essay (Howe & Epp, 2008). In this article, they made connections among arithmetic of whole numbers, then of decimals and fractions and later of rational expressions through a systematic emphasis on place-value structure in the base 10 number system. The article helps make the study of arithmetic more unified and conceptual.

Continuing on this topic, Dr. Howe also thought about its treatment and placement in school mathematics curriculum. He worked together with Harold Reiter of the University of North Carolina at Charlotte to publish an article of “the five stages of place value” (Howe & Reiter, 2012). Most recently, he has written another article that emphasizes the importance of studying the underlying structure of place value, “The most important thing for your child to learn about arithmetic” (Howe, 2015). His thinking and work on such topics have allowed him to make important and thoughtful suggestions on the development of school mathematics curriculum, such as the Common Core State Standards in Mathematics (CCSSM). In fact, CCSSM gives place-value ideas more attention than most of the state mathematics standards that it replaced.

According to Dr. Howe, he developed his work in mathematics education using approaches of inquiry similar to what he does in mathematical research. He thinks about questions and issues in mathematics that arise in reading, or are presented in talks, or most importantly after a certain point is raised by his previous work. Likewise, his work in mathematics education has primarily been guided by thinking about questions and issues that he observed through his involvement in committee services, from talks or through reading. In particular, his extensive participations in numerous committees at the national and international levels have allowed him not only to jointly set up the directions of some important developments in mathematics education but also to further his thinking, clarify his ideas about mathematics education and inform his writing. For example, with his involvement in the formulation of the CCSSM and working with teachers through the Yale Teachers Institute (YTI), he wrote an article, “Three pillars of first grade mathematics, and beyond” (Howe, 2014), describing some key topics to be covered and a coherent way of teaching these essential topics of first grade mathematics and beyond. Fuson (in this book) further illustrates how visual models can be built upon each of these three pillars to help students learn these aspects identified by Dr. Howe.

Based on his work with teachers through YTI, Dr. Howe developed several briefs on other important content topics in elementary school mathematics, such as fractions and the transition from arithmetic to algebra (Howe, 2010). Indeed, his writing on mathematics education seeks to illuminate and clarify the ideas underlying key stages of mathematical learning.

Dr. Howe often takes an advanced perspective when looking at questions and issues in elementary school mathematics. His work is not only thorough and rigorous but also conceptual, which often makes others keep his work as a regular reference. This approach extends to the work he has done on geometry and his advocacy that it needs to be a greater part of the undergraduate mathematics curriculum. He has worked together with William Barker of Bowdoin College to produce a textbook on euclidean geometry from the transformational point of view. The original intent of this book was to show how the transformational approach allowed them to connect classical euclidean geometry with Einstein's special theory of relativity. However, just the euclidean part ended up filling a full textbook (see Barker & Howe, 2007). Usiskin (in this book) indicated that "... it is valuable to mathematics teaching at all levels because it provides a mathematical grounding for approaching euclidean geometry via transformations that is sorely needed in today's environment".

2.4.2 Developing and Improving Mathematics Training for Preservice Elementary Teachers and In-Service Teachers

Dr. Howe has been very interested in learning and examining school mathematics curriculum and practices in some high-achieving education systems, especially those in East Asia. For example, he was fascinated to learn about what Chinese teachers know and are able to do, in comparison to their counterparts in the USA, through reading Ma's book (1999) of "Knowing and teaching elementary mathematics". He thus wrote a book review that was first published in *Notices of the American Mathematical Society* (Howe, 1999), then reprinted in *Journal for Research in Mathematics Education*. The book review helped call for scholars' attention, especially mathematicians, to this book, and also reinforced the idea that mathematics teachers need to have strong training in mathematics.

Starting in 2004, Dr. Howe has run seminars for the Yale Teachers Institute (YTI). YTI offers an unusual form of professional development for teachers. Teachers, known as fellows, participate in seminars run by Yale faculty. Instead of showing mastery of the material through an examination, teacher fellows are required to write a curriculum unit for their class based on the theme of the seminar. According to Dr. Howe, this is a big challenge for teacher fellows but they respond with alacrity, and their overall response to the YTI experience is amazing enthusiasm. This makes it very rewarding to run YTI seminars, and he has done so roughly every 2 years since starting. Preparing and giving those seminars to teachers have afforded Dr. Howe great opportunities to refine his ideas about mathematics education. For example, some of the ideas for the essay *Three Pillars of First Grade Mathematics* (Howe, 2014) were developed in the course of running YTI seminars. He has also written several brief articles for *On Common Ground*, the publication of the Yale Teachers Institute, in connection with seminars he has led for the institute. They are *Leading the Seminar on the Craft of Word Problems*, #12, Spring 2008; *Making Estimation Precise*, #13, Spring 2009; *The Mathematics of Wallpaper*, #14, Fall 2011; and *Can We Teach the Common Core Standards in Mathematics?* #15, Fall 2015.

Building upon his reflection on issues in mathematics education and working with teachers, Dr. Howe believes that a different approach needs to be used in elementary teacher preparation in mathematics, an approach that develops teachers' in-depth understanding of some key ideas in school mathematics. It is strongly mathematical but pays attention to the step-by-step growth of mathematical constructs over time. The hope is that such an approach can help teachers absorb and understand the mathematics better and also give them a better idea of what and how to teach their students. The progression of such mathematical constructs aligns with school mathematics curriculum standards, such as the CCSSM. As an example, he wants to ensure that preservice elementary teachers know and understand, but not simply memorize, all five expressions of the base 10 number place-value notation system (see below) and their corresponding placement in school mathematics.

$$
\begin{array}{lll}
256 & & \\
= 200 & +50 & +6 \\
= 2 \times 100 & +5 \times 10 & +6 \times 1 \\
= 2 \times (10 \times 10) & +5 \times 10 & +6 \times 1 \\
= 2 \times 10^2 & +5 \times 10^1 & +6 \times 10^0
\end{array}
$$

He emphasizes the importance for preservice elementary teachers to understand how these various expressions underlie the familiar computation algorithms of arithmetic and, especially, how all the algorithms reflect the basic strategy of the place-value system, which is to break up numbers as sums of place-value pieces. Preservice teachers need to learn how to think in terms of the pieces and also how to help their students do so, at whatever grade level they find themselves teaching. The mathematical training should enable preservice teachers to learn how to integrate such ideas into their thinking about number and operations in the curriculum and how to design instruction to help students learn.

Now at the Texas A&M University, to provide preservice elementary teachers' systematic training in mathematics, Dr. Howe is working with a group of faculty from both the College of Education and the Mathematics Department to redesign and develop three new courses as a sequence. The sequence is planned to take a developmental approach, as described above, with a close alignment with curriculum standards. Careful consideration is also given to linking topics that should interact, but frequently don't, in the current curriculum. For example, careful attention is paid to the measurement aspects of the number line and to using manipulatives such as base 10 blocks to create length models of numbers and of addition and subtraction, so that linear measurement and whole number arithmetic can develop in tandem, and the number line model can be available to deal with fractions in productive ways. Moreover, each of these three new courses combines mathematics content and pedagogical considerations directly related to that content.

2.5 Summary

Dr. Howe is one of a few notable research mathematicians who is also well respected in mathematics education community nationally and internationally. His achievements include not only a remarkable amount of distinguished contributions to mathematical research but also his contributions and dedications to connecting mathematics and mathematics education and the improvement of mathematics education at the regional, national and international levels. He has provided important leadership in prestigious professional associations and joint research endeavours, both nationally and internationally, as a chair or co-chair of a panel or task force, a committee or board member, a faculty fellow at prestigious research institutes, a visiting professor in several institutions and an invited speaker at numerous conferences across the globe.

To a research mathematician, mathematics matters as one's own passion. And Dr. Howe knows well that one needs to be smart and also love mathematics in order to make possible breakthrough contributions in mathematical research. As a mathematics educator, he takes another great challenge, that is, to help others to become smart and love mathematics. To Dr. Howe, mathematics matters not only to himself but many others in education. Even more, he is working to help teachers to learn how to help their students to become smart and love mathematics. It is a new territory with many unknowns, and he is willing to take such challenges as next chapter of his legacy.

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Chapter 3
Cultural Knowledge for Teaching Mathematics

Roger E. Howe

Abstract This paper discusses selected topics of elementary mathematics that have been problematic in the US mathematics curriculum. It notes some ways that the CCSSM promises to improve on previous practice and offers suggestions for other possibilities for improvement. The main topics mentioned are place value, the concept of number, the notion of unit, linear measurement and the number line, and symmetry in geometry.

Deborah Ball (1989, 1991, etc.), among others (Hill, Ball, Sleep & Lewis (2007), Hill & Ball (2009), Hill (2010), etc.), has emphasized that teaching mathematics is a special kind of applied mathematics, distinct from, say, engineering or statistics or operations research or other fields that use mathematics heavily in order to produce practical results. Correspondingly, it calls on specialized knowledge different from the mathematical knowledge of those other fields or of pure mathematics. She and her coworkers have produced test items that a university-based mathematician might find difficult to answer correctly but that, when given to teachers, distinguish effective teachers of mathematics from less effective ones.

This is a very important insight and one that provokes many follow-up questions. What are some of the key mathematical skills and insights that promote effective mathematics teaching? How can we identify them and promote their use? And, in light of the study of Chinese teachers by Liping Ma (1999) combined with the continuing mediocre performance of US students on a variety of international comparisons of mathematics achievement (Mullis, Martin, Foy, & Arora (2012), NCES (2013)), one is provoked to wonder: Are there some important aspects of mathematics or the teaching of it that none (or almost none) of us in the USA know? In other words, do we have cultural deficits in our knowledge for teaching mathematics? Although not originally explicit, this latter question can be seen as an impetus for much of my thinking about mathematics education in the last roughly 15 years. Here I would like to describe some guesses and some further questions.

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3.1 Place Value

The first topic that caught my attention as problematic in the US curriculum was place value or, more completely, the base ten place value system that we use to write numbers. This is a marvelous notational device that lets us express numbers compactly (with a unique expression for each whole number) and compute with them efficiently, including estimation and approximation, which are crucial for real-world use. Its virtues have led to its essentially worldwide use for dealing with whole numbers and their arithmetic. However, it accomplishes its wonders by using clever conventions that harness a huge amount of mathematical structure in service of its very compact notation. In a rather long essay (Howe & Epp, 2008) on the basic ideas of computation with the base ten place value system, Susanna Epp and I identify "five stages of place value," meaning five levels of interpretation of the standard positional notation. The levels are expressed through the following sequence of equations:

$$
\begin{aligned}
356 &= 300 && +50 && +6 \\
&= 3 \times 100 && +5 \times 10 && +6 \times 1 \\
&= 3 \times (10 \times 10) && +5 \times 10 && +6 \times 1 \\
&= 3 \times 10^2 && +5 \times 10^1 && +6 \times 10^0.
\end{aligned}
\tag{3.1}
$$

When writing these equations for (Howe & Epp, 2008), I thought of them as simply a straightforward exegesis of what the standard compact notation signified in more conceptual mathematical terms. However, as I realized later, each of these five stages represents a substantial intellectual advance over the previous one, and, in toto, they constitute a long intellectual development for children learning mathematics. The second stage is presented in the second and sometimes even first grade, under the rubric of "expanded form." However, the third stage requires understanding of multiplication, which, in the Common Core State Standards for Mathematics (CCSSM), is not introduced until third grade. The further decomposition of the base ten units (e.g., 100) into multiple products of 10s would require greater comfort with multiplication, probably not achieved before fourth grade. And the use of exponents in the final stage would mean that it could not be presented before sixth grade in a CCSSM-based curriculum. The exact time of introduction of the required concepts might vary from curriculum to curriculum, but the overall thesis, that it takes a substantial part of the elementary to middle school curriculum to develop all the ideas required to understand the five stages of place value identified by the sequence of equations (3.1), is evident. A corollary would be that it would take a highly coherent curriculum to ensure that students did come to understand all the five stages and the consequent relationship of base ten place value notation to polynomial algebra. Curricula in the USA, however, have, at least for several decades and perhaps longer (Schmidt, (online)), been far from coherent. The result has been that many students have not come anywhere near understanding the structure that enables our workhorse place value notation. Indeed, a recent study (Thanheiser, 2009) showed that of preservice teachers in an important teacher preparation program, only a small minority thought even in terms of the third stage.

In any case, what the second stage of place value, aka expanded form, reveals is that the key idea of the place value system is to break up a number into a sum of pieces of a very special sort. While working on the paper (Howe & Epp, 2008), I came to realize that mathematics education does not have a standard short name for these pieces. They can be described in mathematical terminology as "a digit times a power of 10," but this is long and invokes the idea of exponents, which, as noted above, comes rather late in a student's mathematical career. I have been campaigning, with very limited success so far, for the adoption of such a term. In my own writing, I have taken up using "base ten pieces." It is not very descriptive, but it is short.

I would argue that "base ten pieces" or some other short name for these numbers, plus another short name, like "base ten unit" for the powers of 10, could be quite useful in mathematics education, especially in the training of teachers. In particular, it would afford short descriptions of what is going on in the various operations of arithmetic.

One further piece of terminology that refines the idea of base ten pieces enhances its affordances. Each base ten piece is a digit times a power of 10. Sometimes one would like to specify which power of 10. In formal mathematical terminology, this would be the "exponent" of the power, but since, for many applications, their most salient feature is their size, the term *order of magnitude* of a base ten piece will be used here to refer to the number of zeroes in its standard base ten representation, which, in algebraic terms, is the exponent in the power of 10. Thus, 1 has magnitude zero, 10 has magnitude one, and so on. Note that a base ten piece and its associated base ten unit have the same order of magnitude.

With these ideas, we can give succinct descriptions of addition and multiplication, or, more precisely, the procedures for expressing the result of adding or multiplying two base ten numbers as a new base ten number, as follows.

For addition:

(i) To add two base ten numbers, for each order of magnitude, take the base ten pieces of that order of magnitude from the two numbers, and add them together.
(ii) The sum of two base ten pieces of the same order of magnitude is the sum of their digits times their common base ten unit.
(iii) If the sum of the digits for a particular order of magnitude is greater than 10, convert 10 of that unit to 1 of the next larger unit, and combine with the sum for that unit.

For multiplication:

(i) To multiply two base ten numbers, multiply each base ten piece of one number with each base ten piece of the other number, and add all the products.

(ii) The product of two base ten pieces is the product of their digits times the product of their base ten units. This latter product is another base ten unit, of order of magnitude equal to the sum of the orders of magnitude of the two factor units.

Such succinct recipes might be of considerable value to teachers, both for helping them achieve a higher-level understanding of what is entailed in the standard algorithms and as a focus of discussion, to check whether they have indeed achieved that higher-level understanding. For example, from the description of addition, one can argue that:

(a) Starting from magnitude zero (the "ones place") and continuing in order to larger orders of magnitude will produce in succession the correct base ten pieces at each place.
(b) The procedure for regrouping (aka "carrying") is exactly the same at each order of magnitude and reduces to knowing the "addition facts" – the sums of digits.

Also, although the descriptions above are on the spare side, they can be elaborated until they give a full description of a commonly used algorithm. Discussion of the details that should be added could be very beneficial for teachers. In addition, these summary descriptions could serve as a point of reference for comparing algorithms, e.g., the standard US multiplication algorithm versus the lattice method.

3.2 Units and Numbers

Just as the base ten units and their associated base ten pieces should be recognized by suitable terminology, I would claim that units in general need more attention in US mathematics education. This is related to the concept of number. To quote Herb Gross (online) in his works on "Mathematics as a Second Language,"[1]. "A number is an adjective that modifies a noun." This was a mantra adopted to benefit adult learners of arithmetic. For teachers, it might be worthwhile being more specific, and saying what sort of adjective a number is: that a number expresses a quantity relationship, and that it tells you how large one quantity is in (multiplicative) comparison to another quantity, which plays the role of a unit. Effectively, it is a ratio. In particular, a number by itself does not specify a quantity; one must know the unit to which the number refers. The lesson for teaching is that students should be reminded to specify the units attached to the numbers they report, until it becomes a habit.

According to Klein (1992) and Bashmakova and Smirnova (1999), this point of view on number was not part of classical Greek thinking. They separated number, by which they meant the counting numbers, or the cardinality of a multitude, from

[1]Adapted to form the core course for the Vermont Mathematics Initiative, which, in turn, has influenced other significant professional development programs, e.g., Nebraska Math.

the idea of ratio, which they used in geometry to compare lengths, or areas, or other geometric quantities. Thus, in particular, they did not have the number π nor the formula $A = \pi r^2$ for the area of a circle. Instead, Euclid stated that "Two circles are to each other as the squares on their diameters." And by "squares on their diameters," he did not mean d^2; he was referring to the geometric figures with the diameter as side length. Thus, while we now express proportional relationships in the form

$$y = cx \qquad \text{or} \qquad \frac{y}{x} = c,$$

where c is the "constant of proportionality" the Greeks would consistently express them in the form

$$\frac{y_2}{x_2} = \frac{y_1}{x_1}$$

where x_1, x_2 and y_1, y_2 are corresponding values of two proportionally varying quantities x and y. Our modern view of number, and the coalescence of the notions of number and ratio, developed gradually and was especially stimulated by the development of symbolic algebra by François Viète and others around the turn of the seventeenth century. One of the first formal declarations of this conception of number was given by Newton in his book *Arithmetica Universalis* on arithmetic. He declared:

> By number we understand not so much a multitude of unities, as the abstracted ratio of any quantity, to another quantity of the same kind, which we take for unity. And this is threefold, integer, fracted, and surd: an integer is what is measured by unity; a fraction, that which a submultiple part of unity measures; and a surd, to which unity is incommensurable. (Bashmakova & Smirnova, 1999)

Note that the main statement, the description of the number concept, is essentially identical to the formulation advocated here. He then goes on to articulate three types of number, of increasing order of difficulty to describe.

This convention that, in addition, all numbers must refer to the same unit is especially relevant to fractions, but it even is relevant to whole number arithmetic. For example, many students feel that the recipes for adding base ten whole numbers and for adding decimal fractions are "different." For whole numbers, the recipe is:

Put the numbers under each other, right justified, and add the columns (with carrying, aka regrouping, as needed).

For decimal fractions, the recipe is:

Put the numbers under each other, with the decimal points lined up, and add the columns (with carrying, aka regrouping, as needed).

A teacher in command of the ideas of base ten piece and base ten unit could explain that the purpose of both recipes is to make sure that the digits referring to the same base ten unit are added together and that is what is the same between the two. In other words, both recipes ensure that the base ten pieces of the same order of magnitude are combined.

The teacher should also make the broader point that, in performing addition, there is a, generally unspoken, convention that all numbers must refer to the same unit. If you add together numbers that refer to different units, you get what we would consider to be nonsense. A favorite example of Herb Gross (online) is

$$3 + 4 = 2, \text{(not!)}$$

which violates what we think of as addition but makes perfect sense if you include suitable units:

$$3 \text{ dimes} + 4 \text{ nickels} = 2 \text{ quarters}.$$

It is especially important to attend to units when dealing with fractions. The popular procedure for adding fractions, by adding the numerators and adding the denominators, commits the same sin, of having each term refer to a different unit. For example, if we talk about fractions as "so many out of so many," which is a popular way of describing them, we need to make sure that the student understands that the second "so many" is the unit. Thus, if we represent ½ as

$$\{X \quad O\}$$

and $\frac{1}{3}$ as

$$\{X \;\; O \;\; O\}$$

and think of adding as "putting together," then we might conclude that we can represent $\frac{1}{2} + \frac{1}{3}$ as

$$\{X \quad X \quad O \quad O \quad O\},$$

which would correspond to the symbolic relationship

$$\frac{1}{2} + \frac{1}{3} = \frac{2}{5} \text{ (not!)}.$$

But just as in Herb Gross' example with coins, each number in this "equation" is referring to a different unit. The ½ refers to a set of two elements as its unit, the $\frac{1}{3}$ to a set of three elements, and the $\frac{2}{5}$ to a set of five elements. If we settle on a common unit, for example, a single element, then we should write the equation as

$$\frac{1}{2} \times 2 + \frac{1}{3} \times 3 = \frac{2}{5} \times 5,$$

which is a complicated way of writing the unremarkable equation $1 + 1 = 2$.

Failure to attend to units is the deeper reason why the popular method is wrong, beyond the fact that it gives what we regard as the wrong answer. This same sin, of ignoring units, gets serious attention in statistics, where it has earned the title of "Simpson's paradox" (Wikipedia-Simpson, online).

Batting averages afford examples of Simpson's paradox. Suppose there are two baseball players, A and B, and A's batting average for the first half of the season is .333, and B's average is .500. Then for the second half, they have slumps, with A going to a .200 average and B going to .250. Since B has a better batting average than A in both halves of the season, you expect that B has a better batting average than A for the whole season, but it may not be so. Suppose that A hit 30 out of 90 times at bat in the first half and B hit 10 out of 20 times and then for the second half, A hit only 2 out of 10 times at bat, while B hit 20 out of 80 times. Then for the whole season, A has 32 hits out of 100 times at bat, for a .320 batting average, while B has 30 hits out of 100 times at bat, for a .300 average. The point here that is relevant for us is that the unit to which B's batting average in the first half refers is much smaller than the unit to which A's refers, while in the second half season, it is the reverse, with A's total times at bat being much smaller than B's in the second half.

More attention to the function of numbers as describing quantity relationships might be useful in teacher preparation. In a number of teacher preparation textbooks (e.g., Van de Walle, 2006), one can read about fractions, that they have five aspects or that there are five "fraction constructs" (Kieren, 1976; Post, Harel, Behr, & Lesh, 1988):

(i) Part-whole relationships
(ii) Measurement
(iii) Division
(iv) Operator
(v) Ratio

I would argue that these are all encompassed in the idea that a number expresses how large something is in relation to a unit. More precisely, they are manifestations of this basic idea in different contexts.

Part-whole relationships: If you want to compare something that is in fact part of a whole to the whole, then the relevant fraction would indeed be a part-whole relationship. Thus, this is a special (albeit important) case of the general relationship. It could only apply to "proper fractions," that is, fractions between 0 and 1. Even a proper fraction, however, need not refer to an actual part of the unit. For example, a pint is half of a quart, whether or not a particular pint is actually part of a particular quart or completely separate. It is the size relationship that is described by the 1/2, not belongingness.[2]

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[2]Here a secondary issue of units is relevant: the substance in the pint and the quart should be the same. Whether a pint of cream is half of a quart of water would depend very much on the context (and perhaps usually would not be so).

Measurement: If you have your quantity and do not know how large it is relative to a unit, but want to know, then you need to measure it in terms of that unit. The measurement process allows you to find the size relationship. The size relationship is, in some sense, playing a passive role here. You have the two quantities, and you are asking, "What is the size relation between them?" Finding the answer is the process of measurement. It is the basis for defining length, or area, or volume, or many other amounts that depend on comparing a given quantity to a unit of the same type. Whole number relationships appear when the given quantity is exactly equal to some number of copies of the unit. Fractional relationships appear when the given quantity is exactly equal to some number of copies of (to quote Newton) a "submultiple part" (i.e., a unit fraction – see below) of the unit.

Division: Measurement, especially length measurement, is closely related to division. Measuring one quantity relative to another is asking, "How many of the second quantity does it take to make the first quantity?" This is what we also mean by division – for example, 24 divided by 4 is 6 because 6 4s make 24. Indeed, if both quantities are lengths, then division amounts exactly to seeing how many of the smaller lengths are needed to compose the longer one, and this can literally be pictured as a measurement process. The situation here is that both quantities are expressed in relation to a third quantity, which had been taken as unit, and now you seek the direct relation between them, and this relation is found by division. Precisely, if both quantities are known multiples of the unit, then the relation between them is found by dividing one multiple by the other. The simplest situation is when one quantity is a whole number multiple of the other, and then the two processes coincide in the most naive way. When the multiple is a more complicated number, the process needs appropriate adaptation. One must form unit fractions of the quantity functioning as the unit and repeat the measurement process using these. If one imagines the whole process, it can become quite cumbersome, but this only serves to highlight the complexity of the idea we call "number."

Division is also implicated in the definition of fractions via the unit fraction approach (described below). There we see that a unit fraction $\frac{1}{d}$ can be thought of as what you get when you divide the unit into d equal parts, and take one of them. It is still a definite size relationship: It is the inverse of being d times as large. General fractions can also be thought of as the result of division: $\frac{3}{4}$ is what you get when you divide three units into four equal pieces. This is a very important alternative interpretation of fractions. (Note: However, this is not immediate in the unit fraction approach; it must be demonstrated.)

Operator: If we don't have the quantity, but want to produce it, the number tells us what we have to multiply the unit by to get the quantity we want; this can be thought of as operating on the unit to make the quantity. It is an active way of thinking about the number, with measurement being the passive way of thinking about it.

Ratio: Finally, ratio is indeed just another name for the multiplicative relationship between two like quantities.

Thus, rather than requiring five distinct constructs, the idea of fraction is essentially captured in the idea of size relationship, with the five commonly mentioned constructs describing how to operationalize this idea in different familiar contexts where numbers are used. One could hope that this conception of fraction (or of number in general), with an underlying unity being operationalized in somewhat differing ways to fit different contexts, might help teachers and students think about this basic idea. In particular, one could hope that it could help them conceptualize fractions in a more productive way than what seems currently to be the case for many.

Although the conception of number as quantity or size relationship is one that I would like to be understood by teachers, it is plausible that it is not easily taught directly to young students. One needs pedagogical approaches that ease students into thinking in this way. Indeed, from a pedagogical point of view, one needs a way of saying specifically what size or quantity relationship is specified by a given fraction. Since students being introduced to fractions have so far only met whole numbers, the relationship should be couched in terms of whole numbers. For the typical fraction, say $\frac{3}{5}$, this relationship is somewhat awkward. To say that

$$A = \frac{3}{5} B$$

means that five copies of A equal three copies of B. Grasping this requires holding the three quantities, A, B, and $C = 5A = 3B$ in the mind at one time, which can be a mental challenge for many young children.

However, the special case of unit fractions, with numerator 1, is significantly simpler than the general case. To say that $A = \frac{1}{5}B$ is the same as saying that 5 copies of A make B, or $5A = B$; which can easily be translated to: if you partition B into five equal pieces, i.e., divide it by 5, then one piece is equal to A. This is much more readily visualizable. Also, somewhat later on, when multiplication by fractions is considered, this relationship can be reformulated as: multiplication by $\frac{1}{5}$ is the same thing as division by 5. This of course is a special case of the fundamental relationship between division and multiplication – that division by a number is the same operation as multiplying by its reciprocal (aka “multiplicative inverse”). It can be a step toward establishing the principle in general.

These considerations would suggest that approaching fractions via unit fractions, that is, starting by describing the quantity relationship defined by a unit fraction and then defining a general fraction as a whole number multiple of a unit fraction, might provide a more accessible path to learning. I would have been happy to have proposed this approach but instead have been delighted that it has been adopted by the CCSSM. One can hope that the approach to fractions through unit fractions, together with instilling the habit of specifying the unit to which any number refers, will improve our students’ ability to understand and work with fractions.

Here it should also be mentioned that the importance of units was also appreciated by Scott Baldridge, who learned it by studying the Singapore mathematics curriculum. He has incorporated this viewpoint into the Eureka Math/Engage New York curriculum (https://greatminds.org/math), whose development he supervised and which has found widespread use. The elementary part of this curriculum has the subtitle, A Story of Units.

3.3 Length and the Number Line

The number line is a beloved object for mathematicians and central to modern mathematics. Lines, of course, are basic geometric objects, and the number line provides the marriage of geometry with number. In the 1600s, Descartes showed how to use two number lines to create the coordinate plane, and since then extensions of his construction have been used to create spaces of arbitrarily many, and even infinite, dimensions.

The issue of coordinating the line to create the number line, although done in tentative ways in the late Middle Ages, was in fact a substantial intellectual challenge, finally met to modern satisfaction only in the late nineteenth century. The ideas invoked are still a source of provocative thinking (G. Chaitin 1998, 1999). It stimulated Cantor to investigate the idea of infinity and led him to the conclusion that the number of points on the line must be considered to be larger than the number of whole numbers, raising issues that continue to occupy logicians.

In view of the central role that the number line plays in mathematics, and the status of the line in geometry and the real world, many mathematicians have wondered why the number line does not play a more prominent role in the K-12 mathematics curriculum. It is used in the early years as a model for one- and sometimes two-digit arithmetic, and then it more or less goes into hibernation. One can see video clips of classrooms where a class struggles to put ½ or some similarly simple fraction on the number line. What lies behind such difficulties? While observing some demonstration classes of Deborah Ball, in which the number line was being discussed, I noticed that some of her students (who were rising fifth graders) appeared to think about the number line in ways that were not connected with length or distance. It was as if the number line was simply the number row – the whole numbers lined up, one after the other. That they happened all to be at equal distances from their neighbors was either incidental (or perhaps only approximate, since, for example, two-digit numbers take more space to write, so they may appear to be closer together). Similarly, in video clips I have seen about placing fractions on the number line, length or distance does not seem often to be invoked in the arguments of students. In a similar vein, I have heard several mathematics educators remark that students seem to pay more attention to the tick marks separating intervals on the line, rather than to the intervals themselves.

All this suggests that we are failing to teach a prerequisite idea: that the number line is about length and distance, and that length is an arena where number can be applied. It raises the question whether more explicit attention to length issues before working with the number line, and recalling them at the start of study of the number line, would improve student understanding of the line – whether they would have more success in placing fractions and perhaps be able to use the line to think about other issues.

The line seemed particularly attractive to me as a means of providing insight into addition and subtraction of fractions. In the arena of length measurement, addition and subtraction have very attractive, very physical, and visual interpretations. Addition amounts to putting bars together end to end, to create a longer bar, and subtraction amounts to comparing bars – putting them side by side and measuring the overhang of the longer one. These physical operations carry over to addition and subtraction of bars of any length. In particular, the addition and subtraction of fractions can be represented concretely in exactly the same way as with whole numbers. The length versions of addition and subtraction can be understood in a very physical and symbol-free way.

But to add two numbers by combining lengths, one must of course have the bars of those lengths. This reverts back to the issue of placing fractions on the number line. To address this issue, the expression of the key idea in a compact form gradually emerged, in what I have come to call the

Measurement Principle:
The number labeling a point on the number line
tells the distance of the point from the origin,
as a multiple of the unit distance.

This statement identifies the key ideas involved in number placement. It helps both to appreciate the complexity of the act and to break it down into its components. One could hope that explicit teaching of this principle with preparation by representing whole numbers as lengths, especially trains of ten rods and one cubes, as sketched in (Howe, 2014), and with lots of examples, could help students figure out how to place fractions on the number line.

Indeed, this principle fits very well with the approach to fractions through unit fractions. It should be relatively easy to argue that ½ goes where it does, right in the middle of the unit interval, because moving twice the distance from 0 to ½ would then get you to 1, as it should, according to the definition of $\frac{1}{2}$. Likewise, $\frac{1}{3}$ should go where it does because then moving 3 times from 0 to $\frac{1}{3}$ will get you to 1. Similar reasoning will apply to unit fractions with larger denominators. From this, it is easy to place general fractions, since these are just whole number multiples of unit fractions. If we fix a denominator d and look at all its whole number multiples, we get pictures like the following: the multiples of $\frac{1}{d}$ form an evenly spaced sequence that look very much like the whole numbers but are closer together – there are d intervals of length $\frac{1}{d}$ in every unit interval. We show in Fig. 3.1 these pictures for the multiples of ½ and $\frac{1}{5}$.

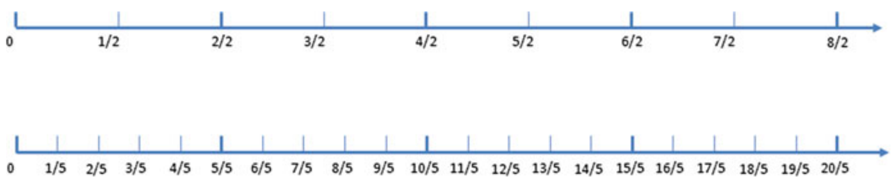


Fig. 3.1 Number lines marked in half-unit intervals and one-fifth-unit intervals

One could hope that such pictures are sufficiently compelling visually to make students comfortable both with the placement process for fractions and with adding and subtracting fractions with a fixed denominator. Beyond that, it also seems possible that using the length model for addition and subtraction, probably supplemented by a better treatment of renaming/equivalence of fraction, can help students improve their grasp of this issue.

Greater success with addition and subtraction of fractions was one of my main hopes for more effective use of the number line. However, two teachers (Jeffrey Rossiter and Aaron Bingea, from the Chicago Public Schools), who participated in a seminar I led for the Yale Teachers Institute, have broadened my perspective on the possibilities for the number line to contribute to student understanding. Jeff and Aaron had noticed that their students tended to "silo," i.e., compartmentalize, the various types of numbers they had been presented with. Whole numbers, integers, fractions, and decimals all occupied their own worlds, with minimal interactions between the different types. It is true that each of these classes of numbers has special symbolism adapted to dealing with that specific class, and if students deal with numbers primarily symbolically, the differences in symbolism apparently lead students to put them in different conceptual bins. Jeff and Aaron had the idea that showing students how all these numbers could live together happily on the number line would promote a unified concept of number. So far, they have carried out their plan with whole numbers and integers and have found that students talk in a much more flexible way about how to determine the sign of a sum of integers and about the fundamental idea that subtraction amounts to addition of the additive inverse. Results from Aaron's class were sufficiently encouraging that his lessons are being adapted by the math coach for his cluster of schools for purposes of professional development. All this provides encouragement for the mathematician's hope that the number line could contribute much more to mathematical understanding than it now does.

Summarizing to this point, consideration of these various topics – place value, the idea of number and its relation to units and to measurement, the relationship between number and linear measurement, and the role of the number line in mathematics instruction – seems to reveal fairly important areas in which the approaches and practices of the US mathematical education community as a whole have not been as productive as possible. Our relative ineffectiveness in these areas is probably reflected in US performance on the various international benchmark exams – TIMMS, PISA, etc. The bright side of this picture is that identification of areas where our practice is problematic offers promise of improvement. Especially, the greater emphasis the CCSSM puts on place value ideas, and on the approach to fractions through unit fractions, gives encouragement that we will improve results on those topics.

3.4 Symmetry

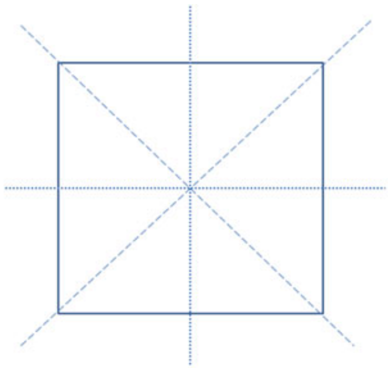
Since Felix Klein pointed out in his Erlanger Programm the intimate connection between geometry and symmetry, symmetry has been gradually receiving more and more attention in geometry instruction. Indeed, the CCSSM calls for basing the study of geometry on symmetry.[3]

Still, there are some topics that currently appear well before any formal treatment of geometry and to which fairly simple symmetry considerations bring substantial insight. Incorporating symmetry ideas into the treatment of such topics might help students gain intuition for the transformations themselves and provide an analog for geometry of "early algebra."

One such topic is the standard nomenclature for quadrilaterals; we have squares, rectangles, rhombuses (rhombi to an earlier generation more attuned to Greek derivations), parallelograms, trapezoids, and kites (probably a neologism). It should be pretty easy for students to appreciate that quadrilaterals in these various classes have symmetries.

A square has four lines of reflection symmetry: the two diagonals and the two perpendicular bisectors of pairs of opposite sides. In addition (or as a consequence, one could say after some study of transformations), it has rotational symmetries. It is unchanged by rotation around its center (which is the common point of intersection of all the lines of symmetry) by 90°, 180°, 270°, and 360° (which in fact accomplishes the same thing as rotation by 0°, i.e., doing nothing; this is the identity transformation). Counting the identity transformation, this is in all eight symmetries. A square and its lines of symmetry are shown in Fig. 3.2.

Fig. 3.2 A square has four lines of symmetry and eight symmetries



[3]On the other hand, it stops short of dealing with the idea of composition of transformations. Admittedly, this is a difficult idea. However, it is the source of the power of the transformational approach to geometry.

Fig. 3.3 A typical rectangle has two lines of symmetry, connecting opposite sides, and four symmetries

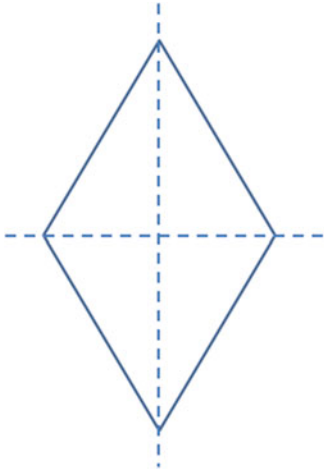


Fig. 3.4 A rhombus has two lines of symmetry, connecting opposite vertices, and four symmetries

A rectangle has two lines of reflection symmetry: the two perpendicular bisectors of the pairs of opposite sides. Note that these two lines of symmetry are mutually perpendicular (Fig. 3.3).

If you think of a rectangle as resulting from a square by stretching (equally) one of the pairs of opposite sides, you can think that the symmetries of the square arising from the perpendicular bisectors of these sides are preserved during the deformation, while the symmetries across the diagonals are destroyed. Although they are much less prominent than in the case of the square, the rectangle also has two rotational symmetries, the rotation of 180° around its central point, the intersection of the two lines of symmetry, and the identity transformation. If we count all of these, we get four symmetries.

A rhombus also has two lines of symmetry: the diagonals. Like the pair of perpendicular bisectors of the opposite sides of a rectangle, these diagonals are mutually perpendicular (Fig. 3.4).

In close analogy to the comments in the previous paragraph about rectangles, if you think of a rhombus as resulting from a square by deforming the angles while keeping the sides equal, then the diagonals survive as lines of symmetry, while the lines bisecting a pair of opposite sides do not. Again, in addition to the reflections across the diagonals, the rhombus is unchanged by rotation by 180°

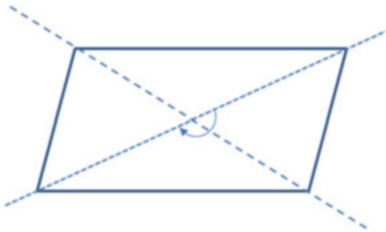


Fig. 3.5 A parallelogram has no lines of symmetry but has a center of symmetry

around its center, the point of intersection of the diagonals; of course, the identity transformation will preserve it. Again counting all of these, we find four symmetries of a rhombus.

A parallelogram can be thought of as resulting from a square by performing both of the deformations that produce rectangles and rhombuses. First one can push on the corners to turn a square into a rhombus, without changing the side lengths, and then one can stretch one pair of mutually parallel opposite sides by the same factor, keeping the other pair unchanged in length. This double deformation destroys all lines of symmetry, so the typical parallelogram has no lines of symmetry.

Yet the parallelogram does have a nontrivial symmetry. A well-known fact about a parallelogram is that its diagonals bisect each other. From this, it is easy to see that rotating by 180° around this point of intersection will preserve the diagonals and therefore preserve the set of their endpoints, which are the vertices of the parallelogram; therefore, this rotation preserves the parallelogram. Counting also the identity transformation, this gives two symmetries for a typical parallelogram (Fig. 3.5).

Continuing the survey of quadrilateral terminology, we can look at kites, which can be thought of as deformations of rhombuses. Take one of the two mutually orthogonal, mutually bisecting diagonals, and slide it in the direction of the other diagonal, keeping it perpendicular and keeping its midpoint on the other diagonal. The result is a shape that is still symmetric across the second diagonal but with sides of two different lengths. This is the shape in which many kites are made, hence the name. It has reflection in the second diagonal as a symmetry. Counting the identity, it has two symmetries (Fig. 3.6).

We can also deform rectangles by stretching (or shrinking) one of a pair of parallel sides while leaving the other unchanged. This will keep the pair of sides parallel, but the other pair will no longer be parallel. The resulting quadrilaterals constitute the class of trapezoids. If the stretching is performed symmetrically on both sides of the perpendicular bisector of the sides that remain parallel, the symmetry in this line will remain as a symmetry, and the result is usually referred to as an *isosceles trapezoid*. However, in general, trapezoids will not have any nontrivial (meaning, other than the identity) symmetry (Fig. 3.7).

This survey of the commonly distinguished classes of quadrilaterals has emphasized the symmetries possessed by each class. These symmetries are not usually pointed out explicitly, but they are probably recognized implicitly by many people.

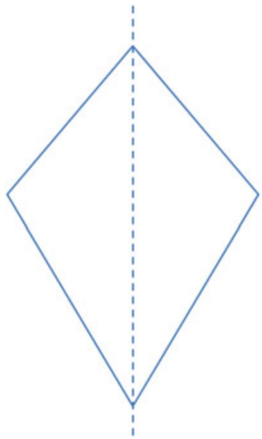


Fig. 3.6 A kite has one line of symmetry, through a pair of opposite vertices

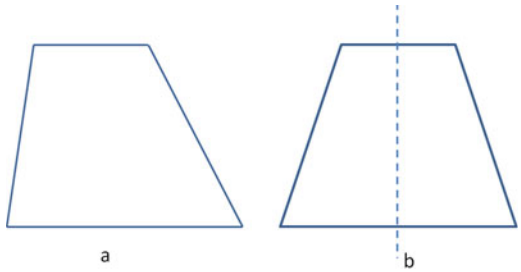


Fig. 3.7 An isosceles trapezoid has one line of symmetry, through a pair of opposite sides

A deeper consequence of describing the symmetries explicitly is that it sets the stage for recognizing that the symmetries of these various classes of quadrilaterals actually *characterize* the classes: any quadrilateral with the symmetries noted for the various classes above will in fact belong to that class.

This statement requires some justification. For example, suppose a quadrilateral has a line of symmetry. If the line goes through a vertex, then, since the set of vertices must be mapped to itself by the reflection across the line and since the line either exchanges pairs of points not on the line or leaves points on the line fixed, one of the other vertices must also be on the line. So the line of symmetry is a diagonal of the quadrilateral. The other two vertices are symmetric across this diagonal, and the line segments connecting a vertex on the diagonal to these other two will be mutual reflections across the diagonal. In particular, they will be equal. Thus, the quadrilateral is a kite, with two pairs of equal adjacent sides.

On the other hand, suppose the line of symmetry does not go through a vertex. Then it must intersect a side of the quadrilateral, and its continuation to the inside of the quadrilateral will have to exit by a second side. Since the quadrilateral is symmetric across the line, each of these sides that intersect the line of symmetry must be perpendicular to it. Otherwise, their reflections would be different lines, intersecting the originals on the line of symmetry, so the quadrilateral would have its sides intersecting in pairs on the line of symmetry. This is clearly impossible. (Notice here that we are not allowing a quadrilateral to have sides that intersect anywhere but at the vertices. That is, part of our understanding of what a quadrilateral is includes the condition that sides intersect only at vertices.)

So, the quadrilateral has two sides that are perpendicular to the lines of symmetry. These sides are, therefore, parallel to each other. Also, the line of symmetry must be the perpendicular bisector of each of these sides. The other two sides of the quadrilateral will then be mutual reflections across the line of symmetry. In particular, the angles of intersection of both of these sides with one of the invariant sides will be the same. Thus, this quadrilateral is an isosceles trapezoid.

Now suppose that the quadrilateral has another line of symmetry. If both lines of symmetry are diagonals, then each side will be equal to both of its adjacent sides. This means that all sides are equal, and we have a rhombus. If both lines go through a pair of sides, then both pairs of opposite sides are mutually parallel, and the angles at the vertices on each side are equal. Therefore all angles are equal, so are all 90°, and we have a rectangle.

If one line is a diagonal and one line goes through the middle of a pair of opposite sides, then the angles at the end of either of these sides are equal, and, also, the two angles at the pair of vertices that are symmetric across the diagonal are equal. Thus, all angles are equal, and we again have a rectangle. This means that opposite sides are equal, but also, pairs of adjacent sides across the diagonal of symmetry are equal. Thus, all sides are equal, so we have a rectangle with equal sides, i.e., a square.

The above shows that all quadrilaterals that have one or more lines of symmetry belong to the appropriate class. In other words, each of those classes could be defined as the class of quadrilaterals with the relevant kinds of lines of symmetry, instead of the more pedestrian definitions/descriptions usually given.

The case we have not yet covered is the class of parallelograms. This is, in some sense, the subtlest case, because the typical parallelogram does not have any line of symmetry. Instead, parallelograms have a center of symmetry and are taken into themselves by a rotation of 180° around that center. We can again argue that, conversely, any quadrilateral that has a center of symmetry must be a parallelogram. We will not give the details of the argument, but it is not hard to show that, in a quadrilateral with central symmetry, opposite sides must be equal and parallel, which characterizes parallelograms.

The symmetry of a parallelogram around its center is an elegant way of rolling all the standard geometric theorems about parallelograms into one tidy package. Moreover, it provides an elegant connection between parallelograms and trapezoids, which are the one commonly recognized class of quadrilateral whose members typically have no symmetry, and, therefore, cannot be separated from generic quadrilaterals by symmetry considerations.

Take a parallelogram ABCD. We know it has a central point, P. Draw any straight line l through P. Assume that l is not one of the diagonals of ABCD. Then it will exit ABCD through two of the opposite sides of ABCD. Let E and F be the points of intersection of l with the edges of ABCD. See Fig. 3.8.

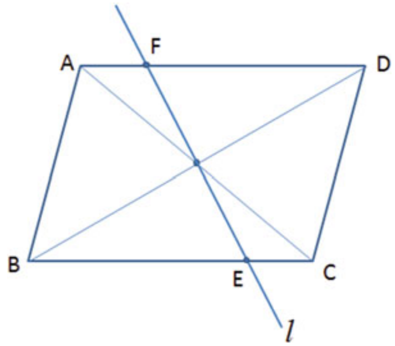


Fig. 3.8 A typical line through the center of a parallelogram divides it into two congruent trapezoids

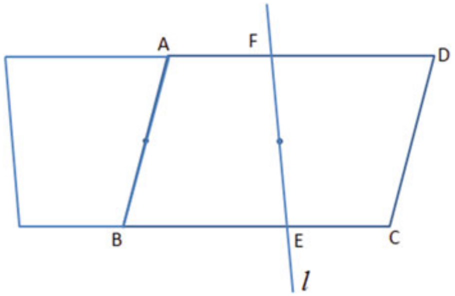


Fig. 3.9 A trapezoid can be half of two different parallelograms

The line l divides ABCD into two pieces, ABEF and FECD. Both of these are clearly quadrilaterals. Moreover, each has a pair of parallel sides, namely, AF and BE in ABEF, and FD and EC in FECD. Thus, both pieces are trapezoids.

Furthermore, the trapezoids ABEF and FECD are congruent. Indeed, they are mapped to each other by the 180° rotation around the center P of ABCD. We can summarize this by saying that cutting a parallelogram in half by a straight line, other than one of the diagonals, produces two congruent trapezoids, which are interchanged by the symmetry of 180° rotation around P.

In fact, we can produce any trapezoid by this construction. If you start with a given trapezoid ABEF, and if you rotate ABEF by 180° around the midpoint P of EF, then the transformed trapezoid FECD will fit together with ABEF to form a parallelogram ABCD. Then ABEF appears as the half of ABCD produced by cutting ABCD along the line $l = \text{EF}$.

We see then that a typical trapezoid ABEF will arise as half of a parallelogram in two ways. In one way, the side EF will play the role of the dividing line, and in the other, the side AB will play this role. See Fig. 3.9.

The two parallelograms from which ABEF can be obtained by cutting in half will be congruent exactly when ABEF is an isosceles trapezoid.

In summary, then, although typical trapezoids do not in themselves have any symmetry, they are linked to parallelograms by symmetry considerations. In this way, they are more special than general quadrilaterals.

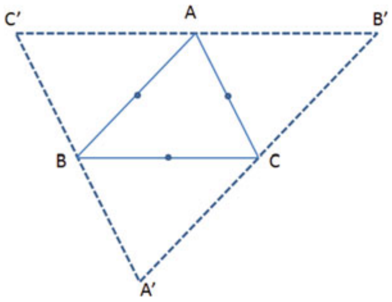


Fig. 3.10 A triangle can be half of three different parallelograms

We should complete this discussion by looking at what happens when the line l through the center P of the parallelogram ABCD happens to be a diagonal of ABCD. The line l still separates ABCD into two congruent halves, but these halves are now triangles instead of trapezoids. From a given parallelogram, one can get two triangles, depending on which diagonal one takes.

Just as we can produce all trapezoids by cutting a parallelogram, we can get all triangles. Indeed, the typical triangle is obtained in three different ways as half a parallelogram. Given a triangle ABC, we can rotate it by 180° around the midpoint of any one of its sides. The original triangle and its rotated image will fit together to create a parallelogram, of which the original triangle is half. Three of the vertices of the parallelogram will belong to the original triangle, and the fourth vertex will be new. If we perform such operations on all three sides of a triangle, the three new vertices will form a new triangle, similar to the original triangle, but with side lengths twice as long. The vertices of the original triangle are then the midpoints of the sides of the new triangle. In other words, the original triangle is what is called the *medial triangle* of the new triangle. The other diagonals of the three parallelograms are the medians of the new triangle. They are concurrent in a point that is called the centroid, of both triangles. This configuration, usually arrived at from the opposite direction, is a staple of plane geometry courses. It is shown above (Fig. 3.10).

As a final remark, we note that symmetry is also relevant to triangle geometry in a more direct way. Indeed, the commonly recognized special classes of triangles, isosceles triangles and equilateral triangles, although not usually defined in terms of symmetries, could be so defined. A triangle is isosceles if and only if it has a line of symmetry, and a triangle is equilateral if and only if it has more than one line of symmetry. In the latter case, it will have exactly three lines of symmetry and, in addition, will be unchanged by rotations of 120°, 240°, and 360° (the identity), for a total of six symmetries. In comparison to the connections with quadrilateral geometry, as sketched above, these observations are fairly straightforward.

It would be absurd to equate this topic in seriousness to the issues of place value, units, and the length measurement – number connection provided by the number line – but hopefully, the above account makes a case for early attention to at least some simple geometric transformations (reflection in a line and 180° rotation around a point, sometimes also called “reflection in a point”). More important here, as with the other topics, are the connections. Study of the transformations can be interesting in itself, but the value added from connecting it with other more standard parts of the curriculum, specifically special types of quadrilaterals, substantially enriches both understanding of those topics and of the role of symmetry in understanding the world.

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Part II
Knowing and Connecting Mathematics in Teaching and Teacher Education

Chapter 4
The Content Knowledge Mathematics Teachers Need

Hung-Hsi Wu

Abstract We describe the mathematical content knowledge a teacher needs in order to achieve a basic level of competence in mathematics teaching. We also explain why content knowledge is essential for this purpose, how Textbook School Mathematics (TSM) stands in the way of providing teachers with this knowledge, and the relationship of this concept of content knowledge with pedagogical content knowledge (PCK).

4.1 Introduction

This is the first in a projected series of papers that examine the content knowledge that mathematics teachers need in order to achieve a basic level of competence in mathematics teaching.[1] We share the belief with Ball, Thames, and Phelps (2008) that "Teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher competency." (*Ibid.*, p. 404.) In subsequent articles, we will discuss specifically how to teach various topics from this perspective, such as long division, percent, ratio, rate, proportional reasoning, congruence and similarity, and slope.

I owe the reviewers of this article several useful suggestions for improvement. I wish to thank Katie Bunsey, Kyle Kirkman, and Rebecca Poon for going the extra mile to provide me with the needed data, and Dick Askey, Larry Francis, and Bob LeBoeuf for their corrections and suggestions. In particular, Larry Francis' devotion to this project—his willingness to put himself through the dreary task of reading multiple drafts—is beyond the call of duty or friendship.

[1] There should be no misunderstanding about what is being asserted: having this content knowledge is *necessary* for competent teaching.

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What mathematics teachers need to know for teaching is a contentious issue in mathematics education. It is indeed a tall order to prescribe the content knowledge—beyond what is in the standard *school mathematics*[2] curriculum—that would enable a teacher to teach "effectively" in a school classroom. It becomes all the more forbidding when the desired level of effectiveness is not specified.

Since 1998, I have been engaged in providing a detailed answer to a far simpler question: "What is the mathematical knowledge that teachers need in order to achieve *teaching competence on the most basic level*"? I will give a more precise description of "basic teaching competence" in the next section, but it is much easier to begin by describing several examples of teaching that I consider to be *below* this basic level.[3] One example is to teach a concept through several grades without ever giving that concept a precise definition, e.g., fraction, decimal, variable, slope, etc. This used to be the universal practice before the advent of the Common Core State Standards for Mathematics (**CCSSM** for short; see Common Core, 2010), but given the poor state of school textbooks, it is possible that this is still happening in many classrooms. Another is the failure to draw a sharp distinction between what is being *defined* and what is being *proved*, e.g., the assertion that a fraction is a division (of the numerator by the denominator), or the statement that $b^0 = 1$ (for a positive number b), or the statement that the graph of a quadratic function is a parabola (i.e., is this the definition of a "parabola" or is this a theorem that *proves* that the graph is a well-defined curve called a "parabola"?). Yet another is the careless blurring of the fine line between what is true and what is merely plausible. One of many such examples is the not uncommon attempt to show that, *without a definition of the division of fractions,* one can nevertheless arrive at the invert-and-multiply rule. Thus,

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{2}{3} \times \frac{5}{4}$$

because:

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times (3 \times 5)}{\frac{4}{5} \times (3 \times 5)} = \frac{2 \times 5}{4 \times 3} = \frac{2}{3} \times \frac{5}{4} \tag{4.1}$$

This chain of pseudo-reasoning is most seductive, but it suffers from a multitude of errors, the most glaring being the justification for the first equality: it is supposed to be based on equivalent fractions. Unfortunately, equivalent fractions only guarantees that if m and n are *whole numbers*, then

[2]By *school mathematics*, we mean the mathematics of K–12.

[3]These examples illustrate the almost universal bad practice forced on teachers by school textbooks from roughly 1970–2010; see Sect. 4.2.3 below.

$$\frac{m}{n} = \frac{m \times (3 \times 5)}{n \times (3 \times 5)} \tag{4.2}$$

What is needed to justify the first equality in (4.1), however, is for m and n in (4.2) to be equal to $\frac{2}{3}$ and $\frac{4}{5}$, respectively, and $\frac{2}{3}$ and $\frac{4}{5}$ are emphatically *not* whole numbers. As a final example of the kind of teaching that is below the most basic level, perhaps the failure to provide the reasoning for truly basic facts such as $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ or $(-a)(-b) = ab$ requires no further comment.

It is unfortunately a fact that, because of our collective dereliction of duty, most of our teachers have been forced to teach at a level below the basic level of competence (see, e.g., Wu, 2011b).

My effort to find out what content knowledge teachers need in order to achieve basic teaching competence took a practical turn when I began to provide professional development to preservice and in-service mathematics teachers of all grade levels in 2000; it lasted until 2013. Current practices in mathematics professional development have been to concentrate on instructional strategies (U.S. Department of Education, 2009, p. 89; Wu, 1999). Moreover, the teachers I taught have consistently told me that, whatever content-based professional development they got, it would be given in short workshops (half-day or one day, rarely two days) on specific topics. There have also been extended workshops lasting several weeks for teachers on "immersion in mathematics" devoted to problem-solving or doing mathematical research on topics sufficiently close to school mathematics (e.g., PCMI, 2016, or PROMYS, 2016). I made the decision from the beginning that I could better serve teachers by breaking with tradition. I would teach them, *systematically*, the mathematics they have to teach, but in a way that is both mathematically correct and adaptable to their classrooms. Such an endeavor requires long-term effort, e.g., 3-week institutes strictly devoted to the *mathematics* of one or two major topics, with follow-up sessions throughout the year, or course sequences in the mathematics departments of universities (see Wu, 1998, 2011a for elementary teachers; Wu, 2010a,b for middle school teachers; and the Appendix of Wu, 2011c for high school teachers).

It did not take me long to realize that these efforts will ultimately go nowhere unless we have on record at least one default model of a logical, coherent presentation of school mathematics *that is adaptable to the K–12 classroom*. Without such a presentation it is difficult to make the case that school mathematics, despite the need to be cognitively sensitive to the learning trajectory of school students, is nevertheless a discipline that respects mathematical integrity. In other words, the concepts and skills of school mathematics *can* be developed logically from one level to the next, and the transparency that one expects of mathematics proper is also attainable there. Without such a detailed presentation, our insistence that reasoning—and therewith problem-solving—must be everywhere in the school curriculum would also sound a bit hollow. Incidentally, an explanation of the need for such a presentation from the perspective of professional development will also be given on page 83.

For these reasons, I have embarked on a project of writing a series of textbooks for teachers that will cover all of school mathematics. Three have already appeared (Wu, 2011a, 2016a,b), and three more to round out the series will probably be in print by 2018 (Wu, to appear).[4] This article will attempt to explain from the vantage point of what may be called **principle-based mathematics** (to be explained in detail on page 56) the content knowledge that teachers need in order to carry out their basic duty of teaching mathematics. In the process, we will also make contact with Shulman's concept of pedagogical content knowledge (Shulman, 1986) and its refinement in Ball et al. (2008).

If there is one thing I have learned through my many years of involvement with teachers, it is the melancholic realization that—as of 2016—relatively few educators and mathematicians seem to be aware of the *urgency* of the need to provide this content knowledge to mathematics teachers (compare the last paragraph of Sect. 4.5). Our failure to do this has indirectly forced school students to memorize things that are unreasonable and incoherent and therefore ultimately *unlearnable*. Yet we expect students to be proficient in "sense making," "problem-solving," and attaining "conceptual understanding," and when such irrational expectations are not met, we evaluate these same students and pass judgment on their inability to learn. It is time to stop inflicting such cruel and unusual punishment on the young. There is another victim of this strange education philosophy too: the teachers. In my experience, many of them are unhappy with the limitations in their content knowledge and are *eager to expand their mathematical horizon*, only to be frustrated by the overwhelming scarcity of resources to help them. We have let our teachers down for far too long.

Let us take a modest first step to making amends by providing a better mathematical education for teachers.

This article is organized as follows. Section 4.2 describes, on the one hand, the mathematical knowledge base of most teachers at present (which we call TSM; see pp. 53 ff.) and, on the other, the minimum mathematical knowledge that teachers need in order to achieve basic teaching competence. We also provide some threadbare data that is available to show why this knowledge would be beneficial to student learning. Section 4.3 attempts to give a more detailed description of the chasm that separates the two kinds of content knowledge. Section 4.4 explains what we mean by "knowing" a concept or a skill, and Sect. 4.5 makes some comments on the state of professional development at present and the hard work that lies ahead if our goal of providing teachers with this minimum knowledge is to be achieved. The last section, Sect. 4.6, makes contact with pedagogical content knowledge.

[4]Although these professional development materials were written well before the CCSSM, they are compatible with the CCSSM because the first two served as a reference for the writing of the CCSSM. The CCSSM came to the same conclusion on numerous topics as these materials (fractions, rational numbers, use of symbols, middle school and high school geometry, etc.).

4.2 The Two Basic Requirements

What is the mathematical knowledge that teachers need in order to teach at a basic competence level, and how to assess whether teachers *know* it? We will postpone the answer to the latter question to Sect. 4.4 but will try to answer the former in this section. Broadly speaking, this knowledge should enable teachers to teach procedural knowledge as well as the reasoning that supports it. It therefore asks for a knowledge of the most basic facts (e.g., the standard algorithms, operations on fractions, standard algebraic identities, and foundational theorems such as the Pythagorean theorem or the angle sum of a triangle being 180°) as well as correct, grade-level appropriate *mathematical* explanations for them and the ability to "distinguish right from wrong," e.g., spot errors in "routine" situations related to these facts and be able to correct them. In particular, we will explicitly leave out from our considerations the more refined aspects of teaching (insofar as they are related to content knowledge) such as the ability to find more than one explanation for an assertion, give fruitful guidance to students' extemporaneous mathematical discussions, make up good examples or mathematical questions to pique students' interest, or make up good assessment items that probe students' understanding.

In the preceding section, it has already been mentioned in passing that the content knowledge that meets such a modest demand of basic teaching competence must satisfy, at least, both of the following requirements:

(1) It closely parallels what is taught in the school classroom.
(2) It respects the integrity of mathematics.

The first point should be self-evident: teachers should not be required to create new mathematics for their lessons, any more than violinists should compose the music they perform.[5] They should have a ready reference for what they teach. An additional reason for making this point explicit is that the mathematics community generally holds the conviction that teaching teachers the kind of mathematics *it* deems important will lead to educational improvement. The idea that, once teachers know the *good* stuff, they will somehow know the *elementary* stuff (school mathematics) better and therefore teach better[6] has led to the disastrous consequence that preservice teachers are typically not taught the mathematics of the K–12 curriculum in college. Another consequence is that many mathematicians, in their attempt to improve K–12 education, adopt the default position of teaching (preservice and in-service) teachers college topics that are elementary at the college level but are nevertheless too advanced for the K–12 curriculum, such as finite geometry, discrete mathematics, number theory, etc. There is as yet no widespread recognition that *the mathematics of the K–12 curriculum is not a proper subset*

[5] I am paraphrasing something said by Harold Stevenson at a TIMSS conference; see Math Forum@Drexel (1998).

[6] This is the Intellectual Trickle Down Theory as described on page 41 of Wu (2015).

of the mathematics taught in college (see p. 404 of Ball et al., 2008; Wu, 2006; and pp. 42–47 of Wu, 2015), and therefore preservice mathematics teachers need explicit instruction on school mathematics.

The second point about teachers' need for content knowledge that respects the integrity of mathematics is even more of a no-brainer. If the goal of mathematics education is to teach students *mathematics*, then it is incumbent on us not to teach them anything less than *correct* mathematics. Therefore teachers' content knowledge cannot afford to be polluted by any kind of mathematics that has no mathematical integrity. Those not familiar with school mathematics or the state of school mathematics textbooks may be shocked that one would consider something this obvious to be worthy of discussion. Unfortunately, the reality is that our teachers' content knowledge—due to reasons to be explained in Sect. 4.2.3—has been a very flawed version of mathematics for a long time. There is some reason to believe that this kind of flawed mathematical knowledge is also shared by many education researchers so that these flaws cease to be noticeable in the education literature after a while. We bring up the issue of mathematical integrity precisely because we wish to provide a proper context for a fresh analysis of this body of flawed mathematical knowledge. This analysis will also reveal why it is so difficult for teachers to acquire the content knowledge they need.

Because we are mainly concerned with the nature of the content knowledge mathematics teachers need for basic teaching competence, we will leave out any discussion about the scope of the content knowledge that *a teacher of a particular grade* needs for this purpose. Without getting into details, we can nevertheless agree with the recommendation of the National Mathematics Advisory Panel that "teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach" (National Mathematics Advisory Panel, 2008, p. 38).

In the first subsection below, we will propose a workable definition of *mathematical integrity.* Because the idea of emphasizing *definitions* is so new in K–12, we make a few additional comments in Sect. 4.2.2 on this advocacy to preclude any misunderstanding. Then using this definition of mathematical integrity, we briefly describe in Sect. 4.2.3 the current state of most teachers' content knowledge. In the last subsection (pp. 56 ff.), we give some indication of why teaching correct mathematics is beneficial to mathematics learning.

4.2.1 Five Fundamental Principles

Any detailed discussion of teachers' content knowledge requires first of all a definition of "mathematical integrity." Like the concept of "beauty" in art and music, it is not likely that there will ever be a comprehensive definition of "mathematical integrity" that is agreeable to everyone. Nevertheless, we can propose a usable and reasonably short definition that *most* working mathematicians would consider unobjectionable. With this in mind, here are five **fundamental principles** that we believe characterize mathematical integrity (see Wu, 2011b):

(A) Every concept is precisely defined.
(B) Every assertion is supported by reasoning.
(C) Every assertion is precise.
(D) The presentation of mathematical topics is coherent.
(E) The presentation of mathematical topics is purposeful.

Before we amplify on these principles, let it be mentioned that, strictly speaking, these are fundamental principles that undergird what is called *pure mathematics*. For so-called *applied mathematics*, each of these principles will acquire a slightly different flavor. Nevertheless, for reasons to be discussed in Appendix 1: Applied Mathematics (p. 85), it suffices to limit ourselves to (A)–(E) if our goal is to safeguard the mathematical integrity of *school mathematics*.

The first three principles, (A)–(C), are closely interrelated and therefore have to be discussed together. In mathematics, the starting point for any reasoning is a collection of *precise definitions* of concepts[7] and a collection of explicit assumptions or facts already known to be true. It is the unambiguous nature of the definitions, assumptions, or facts that enables them to serve as the foundation for correct logical deductions. The process of making logical deductions from precise definitions, assumptions, and facts in order to arrive at a desired conclusion is what we call *reasoning*, and reasoning is the vehicle that drives problem-solving.[8] It is therefore in the nature of mathematics that, without precise definitions, reasoning cannot get off the ground and therefore there will be no problem-solving. Those who lament students' inability to solve problems should look no further than the defective curricula around us that offer no precise (and correct) definitions for the most basic bread-and-butter concepts such as fractions, decimals, negative numbers, constant speed, slope, etc. (See Sects. 4.3.1 and 4.3.2 on pp. 60 and 65, respectively.)

It is easy to explain in everyday language why any mathematical discussion must rest on precise definitions. In a rational discourse, *we must know exactly what we are talking about,* and precise definitions serve the purpose of reminding us what we are talking about. Precision becomes even more critical when the discussion turns to *abstract* concepts and skills, which is what happens in the mathematics of middle school and high school. We need precision to minimize misunderstanding in the teaching and learning of mathematics because the precision helps to delimit, *exactly*, what each concept or assertion does or does not say. While human communication, being *human*, cannot maintain such precision at all times in a school classroom, there will come a time in any discussion of mathematics when such precision becomes absolutely indispensable. This is a persuasive argument that teachers should learn to judiciously nurture precision in the school classroom.

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[7] Or *undefined terms* at the beginning of an axiomatic development.

[8] In mathematics, there is no difference between *proving* and *problem-solving*.

Beyond definitions, precision manifests itself in school mathematics in almost every conceivable way, and there is no end of such examples. Thus the domain of definition of the function $\log x$ is not $\{x \geq 0\}$ but $\{x > 0\}$; indeed the difference between the two is only one number, namely, 0, but that is the difference between nonsense and being correct. Another example: it seems plausible that if we have an inequality between numbers, let us say $a < b$, and if c is another number, then we have $ca < cb$. As is well known, this is not correct because if c is negative, then the opposite is true, i.e., $ca > cb$, and if $c = 0$, then $ca = cb$. Therefore this assertion must be *precisely* announced as follows:

> Suppose two numbers a and b satisfy $a < b$. If $c > 0$, then $ca < cb$, but if $c < 0$, then $ca > cb$.

As a final example, if the three sides of a triangle are (of length) 20, 67.1, and 70, then it is not a right triangle. If you draw such a triangle using any unit of length (e.g., 20, 67.1, 70 cm) and measure the angles, you are most likely going to conclude, within the margin of error in measurements, that this is a right triangle. Yet, because $20^2 + 67.1^2 \neq 70^2$ (the left side is 4902.41, while the right side is 4900), we know by the Pythagorean theorem that this cannot be a right triangle.

As for the critical role of reasoning in mathematics education, suffice it to note that rote learning—the one quality in education that is universally decried—is nothing but the attempt to memorize in the absence of reasoning. When every assertion is seen to be supported by reasoning, students realize that mathematics is learnable after all because it is not faith-based and submission to another person's whimsical dictates is not required. For example, every elementary student has probably wondered *why* we cannot add fractions in the same simple way that we multiply fractions, i.e., why $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ but $\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$. If people in education had ever given serious thought to this question, they would have realized the urgency of defining precisely the meaning of adding and multiplying fractions and then *proving the addition and multiplication formulas for fractions* (see Wu, 1998). Such a realization might have changed the landscape of teaching fractions several decades earlier. Needless to say, the same goes for all the arithmetic operations for fractions and for *rational numbers*[9] and, indeed, for every assertion in school mathematics.

Next, let us turn to the concept of *coherence* in (D). The term "coherence" is often invoked in recent education discussions, but perhaps without realizing that it is quite subtle and can only be explained in terms of technical details. Roughly, it means that the body of knowledge we call mathematics, far from being a random collection of facts, is a tapestry in which all the concepts and skills are logically interwoven to form a single piece. For example, the concept of division, when presented correctly, *is* qualitatively the same for whole numbers, fractions, rational numbers, and real numbers (this fact is emphasized in Wu, 2011a) and even for complex numbers. Right there, we see why coherence is vital for the teaching and

[9] *Rational numbers* is the correct terminology for *fractions and negative fractions;* it should not be conflated with *fractions*. Fractions are nonnegative rational numbers.

learning of mathematics because it means that, if the concept of division is taught correctly the first time for whole numbers, it will spare learners the need to learn division anew on subsequent occasions. This message bears repeating because the division of fractions is still a much feared concept at the moment.[10] For another example, although the standard algorithms for whole numbers may seem to be four unrelated and unfathomable skills, they are all unified by a single idea: how to reduce all multi-digit computations to single-digit computations (cf. Wu, 2011a, Chapter 3). From this perspective, the success and the beauty of the standard algorithms are nothing short of stunning. They teach students the important lesson of reducing the complex to the simple, which is after all a main driving force behind all scientific investigations. Had this kind of coherence about the standard algorithms been widely understood among teachers and routinely taught in textbooks, it is doubtful that the math wars of the 1990s would have erupted at all. We can push this line of reasoning one step further: the four arithmetic operations on fractions may seem to be unrelated skills until one realizes that they are conceptually the same as those on the whole numbers (this fact is especially emphasized in Part 2 of Wu 2011a). Insofar as the whole of mathematics is coherent, there is no end of such examples, and some of them will naturally emerge in the discussions of the next section (Sect. 4.3). However, it should be obvious from this brief discussion that teachers must be aware of the coherence of mathematics if they want to be effective in the classroom.

Finally, the concept of *purposefulness* may also be a characteristic of mathematics that is hidden from a casual observer's view, but it is one of the main forces that shapes mathematics from the most elementary part to the most advanced. Mathematics is goal-oriented, and every concept or skill is therefore a *mathematical* purpose. This is especially true of school mathematics because the intense competition among the various topics to stay in the school curriculum naturally weeds out all but those that serve a compelling purpose. One of the most striking examples is the concept of *basic rigid motions* in the plane—translation, rotation, and reflection—that are standard topics in middle school. In TSM, these rigid motions are regarded as fun activities that shed light on the beauty of tessellations and Escher's prints (cf. p. 33 of Conference Board of the Mathematical Sciences, 2001), and they lead to so-called *transformational geometry*, a novelty whose charm quickly gets lost in the technicalities of high school mathematics. But when these basic rigid motions are properly realized as the cornerstone for the concept of congruence in the plane, the *mathematics* of these rigid motions comes to the forefront, and they become the thread that unifies middle and high school geometry (cf. Wu, 2010a, Chapter 4; CCSSM, 8.G and High School-Geometry; Wu, 2016a, Chapter 4). Another example is rational numbers (fractions and negative fractions). They are not just "another collection of numbers" that students must put up with, but are rather the agents that render the computations that one normally performs in solving linear equations, for example, entirely routine. (This may be likened to what the standard algorithms

[10]Ours is not to reason why, just invert and multiply.

do for computations with whole numbers.) A final example is the concept of place value. In the way it is commonly taught in schools, this is a concept that primary students must accept, *by rote*, at all costs. Would it not be more productive to explain to them, no matter how informally, the fact that we *need place value* in order to count (and write) to any number by using only ten symbols {0, 1, 2, ... 9} *and* to also make number computations manageable at the same time? (See Sections 1.1 and 1.2 in Wu, 2011a, and pp. 13–31 in Wu, 2013a.)

4.2.2 Two Caveats

Before we proceed further, we should clear up a common misunderstanding concerning the use of definitions in school mathematics. At present, there is great resistance to the idea of making the formulation of precise definitions a main focus of K–12 mathematics. Some textbook writers go so far as to refuse to let any reasoning be based on precise definitions because—as the saying goes—the definition of a concept emerges only *after* many explorations. Therefore some amplification on this idea is necessary.

Our insistence on the use of precise definitions as the basis for reasoning is not meant to be, literally, applicable to all of K–12 but only to roughly *grade* 5 *and up*.[11] These are the grades where reasoning begins to assume a critical role and the non-learning of mathematics starts to become most pronounced. We hasten to add that we do not by any means imply that definitions and reasoning do not matter in grades K–4; emphatically they do. After all, the foundation of learning how to reason from precise definitions must be laid in those grades. However, at least in K–3, the pedagogical and psychological components of teaching may be even more important than the content component. Therefore, a discussion of definitions and reasoning in the early grades will have to be more nuanced than is possible in the limited space we have here.

A second point we should make is that the use of definitions and the presentation of proofs in grades 5–12 must respect the reality of the school classroom. It is time to recall requirements (1) and (2) at the beginning of Sect. 4.2: we want mathematics that is both correct *and* usable in the school classroom. We therefore expect definitions to be introduced with motivation and background information, in ways that are grade-level appropriate.

We can illustrate with the teaching of fractions. By no later than the fifth grade, we expect a fraction to be defined as a point on the number line constructed in a prescribed way (see Wu, 1998, 2011a, Part 2, and CCSSM, 5.NF).[12] But does this

[11] In making this assertion, I am trying to be as conservative as possible. Larry Francis pointed out to me, for example, that the definition of a fraction as a certain point on the number line is essentially given in the third grade of the CCSSM: 3.NF.2.

[12] Also see the preceding footnote.

mean a fifth grade teacher should ram this definition down students' throats on day one of a fifth grade class? Not at all. We would expect something more persuasive to precede it. For example, when a textbook *for teachers* introduces this definition, it devotes six-and-a-half pages to explain the genesis and the need for such a definition (see Wu, 2011a, pp. 177–183). In fact, by the time this book gets to fractions, it has already spent a chapter explaining the virtues of the number line as a tool for codifying the mathematics of whole numbers (Wu, 2011a, Chapter 8). For another example, when the same book for teachers defines what fraction division means, it spends four pages reviewing the relevant definitions of subtraction and division for whole numbers and giving an intuitive meaning to the division of "simple" fractions (Wu, 2011a, pp. 283–286).

A final example is about the definition of a genuinely abstract concept, that of the **probability of an event.** This is without a doubt a difficult concept for middle school students. Therefore in a book for teachers (Wu, 2016a, pp. 121–141), no *general* definition of probability is given in the first 12 pages of the exposition on this topic. Instead, these 12 pages are devoted entirely to examples of coin tossing and dice throwing, and the probability *of each example* is defined specifically for that example; these definitions are relatively easy to accept because experiments can be performed to test the plausibility of each of these definitions. When the general definition is finally given at the end of these 12 pages of examples, the abstract pattern of the earlier definitions of the probability for each individual example is already in clear evidence, and the general definition becomes nothing but a summary of the earlier ones.

It remains to point out that the motivation for definitions in *student* textbooks will have to be even more expansive and more considerate. While one would not expect such elaborate preparation for the introduction of each and every definition, these three examples do serve the purpose of clarifying the recommendation that precise definitions be given in grades 5–12.

4.2.3 Textbook School Mathematics (TSM)

School textbooks are a powerful force in teachers' lives because teachers' lessons usually follow the textbooks. It is unfortunately the case that the mathematics encoded in the school textbooks of roughly the four decades from 1970 to 2010 is a very defective version of mathematics. Let us call it **Textbook School Mathematics (TSM)** (Wu, 2014a, Introduction; Wu, 2015). Because colleges and universities—as pointed out on page 47—make scant effort to help preservice teachers revisit and revamp their knowledge of TSM, what teachers know about school mathematics generally consists of nothing more than TSM. Consequently, teachers have no choice but to teach their students what they themselves were taught as school students so that they too imprint TSM on their own students. It therefore comes to pass that this body of defective mathematics knowledge gets recycled in schools from generation to generation.

In order for teachers to acquire a content knowledge base that respects mathematical integrity, i.e., satisfies condition (2), we must begin by helping them to recognize and replace their knowledge of TSM.

It is a legitimate question whether the concept of TSM has any validity. Does it exist? This question becomes all the more pressing when one realizes that the mathematics education reform of the 1990s (National Council of Teachers of Mathematics, 1989, 2000) took place within the last four decades and the reform was a revolt against the school mathematics of the 1970s and 1980s. How can TSM possibly span both eras, pre-reform and post-reform? We will leave a more detailed answer to these questions to Appendix 2: The Existence of TSM (p. 86) so as not to interrupt the present discussion of teachers' content knowledge. However, a little reflection will immediately reveal that the following features are equally common in pre-reform or post-reform texts: lack of *precise definitions* (e.g., fractions, negative numbers, the meaning of division of fractions, decimals, constant rate, percent, slope, etc.), the absence of *precise reasoning* for major skills (e.g., how to add or multiply fractions, how to multiply or divide decimals, why negative times negative is positive, how to write down the equation of a line passing two given points, how to locate the maximum or minimum of a quadratic function, etc.), and the failure to explain the *purpose* of studying major topics such as the standard algorithms, rounding off whole numbers or decimals, functions, exponential notation of numbers (why write $\sqrt{b}$ as $b^{1/2}$?), trigonometric functions (are right triangles *that* important?), etc. (Also see Wu, 2014a.)

The most egregious errors of TSM lie in rational numbers (especially in fractions), linear equations of two variables and linear functions of one variable, and middle school and high school geometry. Since these topics will be discussed at some length in the next section, what we are going to do here is describe how TSM, in its treatment of the laws of exponents in high school algebra, manages to violate all five fundamental principles of mathematics.

The *laws of exponents* in question state that for all $a, b > 0$ and for all *real numbers* s and t, we have:

(E1) $a^s \cdot a^t = a^{s+t}$
(E2) $(a^s)^t = a^{st}$
(E3) $(a \cdot b)^s = a^s \cdot b^s$

The starting point is of course the easily verified simpler versions of (E1)–(E3) for all $a, b > 0$ and for all *positive integers* m and n,

(E1′) $a^m \cdot a^n = a^{m+n}$
(E2′) $(a^m)^n = a^{mn}$
(E3′) $(a \cdot b)^n = a^n \cdot b^n$

The first order of business in generalizing $(E1')$–$(E3')$ to $(E1)$–$(E3)$ is to define a^0 and a^{-n} for any positive integer n. The way TSM tries to motivate the definition $a^0 = 1$ is by either asking students to believe that the validity of patterns ($\ldots a^3 = a^4/a$, $a^2 = a^3/a$, $a = a^2/a$) also validates $a^0 = a^1/a = 1$ or by claiming that since $(E1')$ holds, we must have $a^2a^0 = a^{2+0} = a^2$ so that by dividing both sides of $a^2a^0 = a^2$ by a^2, we get $a^0 = 1$. This kind of *speculative reasoning* is of course an integral part of doing mathematics provided it is clearly understood to be *speculative*. However, precision not being a main concern of TSM, this motivation for the definition of a^0 is presented—informally to be sure—as "reasoning," and the result is that this motivation for a definition is commonly misconstrued as a proof of the theorem that *for any* $a > 0$, $a^0 = 1$. The same comment applies to the definitions of $a^{-n} = 1/a^n$ and $a^{1/n} = \sqrt[n]{a}$. Such imprecision contributes to teachers' confusion between what a *definition* is and what a *theorem* is.[13]

Once the concept of a^r has been defined for all rational numbers, the next step is to explain, to the extent possible, why ($E1$)–($E3$) are valid for all rational numbers s and t. Unfortunately, TSM simply dumps these laws of exponents for rational exponents on students with nary a word of explanation. Let us be clear: we do not want these laws for rational exponents to be completely proved in a high school classroom either, because these proofs are long and tedious (see, e.g., Wu, 2010b, pp. 183–191). Yet some special cases are so important that they deserve to be proved in full, e.g., the following special case of ($E3$):

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab} \quad \text{for all positive integers } n \tag{4.3}$$

This equality, especially the case $n = 2$, is almost ubiquitous in the middle and high school mathematics curriculum, but it seems to be the case that either TSM assumes (4.3) without comment or, if a proof is attempted, it is not correct.[14]

Now the laws of exponents are taken up in textbooks long after the concept of a function has been taught. Therefore, there is no excuse for not pointing out, emphatically, that these laws are in fact remarkable properties of the exponential functions. Yet TSM introduces these laws almost always as "number facts," and even when it gets around to discussing exponential functions, no special effort is made to finally establish the relation of these so-called number facts with the exponential functions. Thus the real *purpose* of studying these laws of exponents (i.e., they are characteristic properties of exponential functions) goes by the wayside, and students are likely to lose sight of the fact that it is automatic in mathematics to isolate the properties common to a given class of functions. In this light, the laws of exponents are to exponential functions as the addition theorems (of sine and cosine) are to the trigonometric functions.[15] This is mathematical *coherence* in action. But, instead, TSM makes students believe that the exponential notation is a just a game we play in order to rewrite $\sqrt[n]{a}$ in the fancy notation $a^{1/n}$. Without any exposure to the reasoning behind the laws of exponents, students end up seeing these laws as undecipherable statements about a quaint notation that they must commit to memory.

[13] I have personally witnessed this confusion not just in the USA but also in Australia and China.

[14] Part of the difficulty of obtaining a correct proof of (4.3) is that the *uniqueness* of the positive n-th root of a is part of the definition of $\sqrt[n]{a}$, but TSM seems unwilling to confront the concept of uniqueness.

[15] Or, more generally, as the addition theorems are to complex exponential functions.

There is an additional flaw in TSM in its failure to at least comment on the meaning of a^s when s is an irrational number such as π or $\sqrt{3}$. See the discussion in Chapter 9 of Wu (2016b) that presents a more reasonable way to address the laws of exponents overall.

4.2.4 The Data

Since the two requirements (1) and (2) on page 56 for the content knowledge that teachers need pull in opposite directions, it is by no means obvious how to provide teachers with this knowledge. Following Poon (2014), let us call content knowledge that satisfies both requirements of *principle-based mathematics*. TSM certainly satisfies requirement (1) of principle-based mathematics, but it fails requirement (2) in spectacular ways as we have just seen. Conversely, one can easily cobble together a coherent exposition of all the standard topics in school mathematics by making a judicious selection of various pieces from the required courses of a university math major, but the result will not come close to resembling school mathematics, i.e., it cannot satisfy requirement (1). For example, to college math majors, a rational number—in particular a fraction—is just an equivalence class of ordered pairs of integers, but that is not something we would try to teach to fourth or fifth graders. Similarly, to these majors, the maximum of a quadratic function can be simply obtained by differentiating the function and setting the derivative equal to zero to obtain the point at which the function achieves the maximum. However, 10th or 11th graders have to learn how to locate this maximum point without the benefit of calculus and so on. Incidentally, these examples also give an indication of why school mathematics cannot be a proper subset of college mathematics (see page 48).

To the extent that the goal of school math education is to teach students *mathematics*, teachers cannot afford to teach them TSM, period. TSM is *incorrect* mathematics. *The need to replace teachers' knowledge of TSM by principle-based mathematics is therefore absolute.* Beyond such theoretical considerations, it would also be reassuring if we could get some indication from another source that principle-based mathematics is beneficial to mathematics learning. There *is* an indirect reassurance from the CCSSM. These standards have taken a major step in moving away from TSM to principle-based mathematics. One look at the standards on fractions (grade 3 to grade 6), rational numbers (grade 6 to grade 7), and geometry in grade 8 and high school will be enough to convince a reader of this fact. The belief in principle-based mathematics is therefore at least shared by some reasonable people. Beyond that, one would like to have some large-scale data for this purpose. Thus far, there is little or no such data for the obvious reason that principle-based mathematics has not yet been available on a reasonable scale either in professional development for teachers or in the K–12 classroom. Perhaps more telling is the fact that, with rare exceptions (e.g., Hill, Rowan, and Ball, 2005; Ball, Hill, and Bass, 2005), the education research community has traditionally neglected content and its role in instruction (see the reference to the "missing paradigm" on page 6 of Shulman, 1986). What data we have is so meager that it borders on the anecdotal.

In her Berkeley dissertation (Poon, 2014), Rebecca Poon explored the impact of content knowledge training on student learning. She did a case study of four teachers (three in fourth grade and one in sixth grade) who received (to varying degrees) training in principle-based mathematics. Three were on the West Coast (but not in California) and one on the East Coast. Through personal interviews and teachers' notes, she studied how these teachers taught one topic: the division interpretation of a fraction. This allowed her to sample the teachers' ability to faithfully implement the basic message of principle-based mathematics, especially definitions, precision, and reasoning. Then she looked at their students' state test scores and compared them to the scores of other *comparably matched*[16] students who were taught by teachers without any training in principle-based mathematics. Her conclusion is that "the average effect of PBM (principle-based mathematics) training on student achievement was significant and substantial" (*ibid.*, p. 63), but there are uncertainties about whether the positive effect on student achievement can be attributed exclusively to the training in principle-based mathematics.

The article Alm and Jones (2015) would seem to be the only relevant published article we can cite. The authors reported a success story about students in remedial courses in a small liberal arts college when principle-based mathematics (based on Wu, 2011a) was taught. They attribute the success to the emphasis on the use of precise definitions (particularly in fractions) and coherence (of fractions and algebra). The authors added:

> The a priori case that students are better off learning better mathematics is clear enough. The a posteriori case that student learning in the classroom is actually improved is more complicated (but anecdotal evidence and our observations certainly support it). In particular, small sample sizes are a major issue. We are currently working on constructing a multiyear study over several cohorts to measure the practical effectiveness of the approach described here. (*Ibid.*, footnote on page 1364.)

My own summer institutes from 2000 to 2013 were devoted to principle-based mathematics. Over the years, teachers from those institutes have let me know how the institutes had impacted their students' learning, but none—with two exceptions—provided me with usable data. I will now briefly mention the results from those two exceptions. I will also mention the data from another teacher at the end.

Kyle Kirkman (kirkmanks1@gmail.com) was a first-year K–6 Math Interventionist in 2015–2016 at the Pan-American Charter School of Phoenix, AZ. The school uses the *Galileo K–12* Online Assessment System from Assessment Technology Incorporated (ATI). His charge was to work with *RTI* (Response to Intervention) students to bring them up to grade level. Students' progress is monitored by the "growth" of their test scores, measured in the following way. For

[16] This is a long story. Please see Sections 4.3, 6.1–6.3 of Poon (2014).

each quarter (of the school year), students take a Galileo K–12 test at the beginning and another one at the end, and the score of the latter minus the score of the former is by definition their *growth* in the quarter. (The Galileo K–12 test at the end of the first quarter doubles as the test at the beginning of the second quarter, the test at the end of the second quarter doubles as the test at the beginning of the third quarter, and so on.) The following tables (numbers are rounded to the nearest one) summarize the comparison of the **average growth** of Kirkman's RTI students with that of the non-RTI students. Some comments will also be found after the tables.

Fall-QT 1, 2015:

	Gr K	Gr 1	Gr 2	Gr 3	Gr 4	Gr 5	Gr 6	**K–6 Av**
Non-RTI students	126	−31	15	49	30	37	8	33
RTI students	236	50	55	74	59	139	68	97
RTI student growth minus non-RTI student growth	**110**	**80**	**40**	**25**	**29**	**102**	**59**	**64**

Fall-QT 2, 2015:

	Gr K	Gr 1	Gr 2	Gr 3	Gr 4	Gr 5	Gr 6	**K–6 Av**
Non-RTI students	−13	63	48	41	98	15	63	45
RTI students	19	119	30	36	95	−10	64	50
RTI student growth minus non-RTI student growth	**32**	**56**	**−19**	**−5**	**−13**	**−25**	**2**	**6**

Spring, 2016:

	Gr K	Gr 1	Gr 2	Gr 3	Gr 4	Gr 5	Gr 6	**K–6 Av**
Non-RTI students	108	97	63	51	85	58	1	66
RTI students	188	177	78	96	142	100	52	119
RTI student growth minus non-RTI student growth	**80**	**80**	**15**	**45**	**57**	**42**	**51**	**53**

The average growth of the RTI students obviously far exceeds that of the non-RTI students except in the second table. Kirkman explained that in the second quarter, he stopped working with his students of the first quarter and got a new group of students. Moreover, in an effort to work with more students, he moved students in and out of his class in shorter intervals than a quarter. The strategy backfired, as the table shows. In the Spring, he worked with the same group of students all through the semester, and the Galileo K–12 test at the end of the third quarter was not administered.

He described how his knowledge of principle-based mathematics helped him:

> I have learned that precise mathematical definitions are critical. If precision is lacking, students will fill in any missing or vague elements of the definition with whatever happens to be present in their paradigm that seems to fit the idea. Not all of mathematics is intuitive in nature, so this can definitely lead to erroneous conclusions.

Larry Francis (larrydotfrancis@gmail.com) taught Title 1 math intervention groups, in 2014–2015, in grades 1–5 at Helman Elementary School of Ashland, OR. Grouped by grades, students came to his classroom for 30 min four times each week. Below is a comparison of the average grade-by-grade gains in percentile scores on the 2014–2015 fall-spring easyCBM™ CCSS benchmark tests of his Title 1 students compared with those of their classmates in their home classrooms (classroom A and classroom B). In nine out of ten classrooms, these previously underperforming Title 1 students outperformed their classmates, sometimes dramatically. Title 1 students' scores have been removed from their respective classrooms' scores for this comparison. Furthermore, the fall-spring numbers are the nationally normed percentile scores according to easyCBM™.

	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5
Title 1 math	19	10	11	2	16
Classroom A	2	−12	7	−8	1
Classroom B	−3	−8	3	4	12

According to Francis, "Precise definitions were crucial. Helping first and second graders with counting doesn't mean you need to tell them a bunch of definitions, but you need to make it clear that fundamentally a number is a thing you count with." What he learned from the summer institutes is "to reorganize my knowledge of arithmetic into a much more mathematical form. I continued to 'know' almost all the old things I used to know, but your [institutes] got me to reorganize that knowledge. …I am sure that reorganizing my knowledge contributed to my students' successes."

Finally, I have some data from a teacher who was not at any of my summer institutes. Katie Bunsey (kate.bunsey@lakewoodcityschools.org) teaches fourth and fifth grades at Hayes Elementary School of Lakewood, OH. I happen to have been mentoring her, long distance, for the past three years on whole numbers and fraction using Wu (2011a). She has just reported to me her fifth grade students' 2016 math scores on the Ohio State Assessment (administered by AIR):

- Seventy-seven percent of her students scored proficient or above, whereas only 62% of Ohio's fifth graders scored proficient or above, and only 63% of her school district's fifth graders were proficient or above.
- Among those students who had her for 2 years (in their fourth and fifth grades), 84% were proficient or above, but among those who had her for only 1 year (in fifth grade), only 70% were proficient or above.

- Her students comprised only 16% of the district's fifth grade population (65 out of 397), but 27% (respectively, 21%) of the district's students who scored proficient (respectively, accelerated) were her students.

For the record, let it be mentioned that, together with an evaluation specialist, I did apply for grants (to NSF-EHR in 2010 and to IES in 2013) to evaluate the effectiveness of principle-based mathematics in the classroom. They were not funded.

4.3 TSM Confronts Mathematical Integrity

We will discuss in this section, in considerable detail, the chasm that separates TSM from principle-based mathematics. It reveals the vast distance we will have to travel if we want to provide mathematics teachers with the content knowledge they need in order to competently discharge their basic obligation as teachers. What should stand out in the following discussion is the damage TSM has inflicted on mathematics learning. TSM is not learnable except by rote, as all irrational ideas are not learnable except by rote. If nothing else, this recognition should be incentive enough for us to join forces to undo this damage by eradicating TSM.

4.3.1 The Importance of Definitions: The Case of Fractions

Consider the teaching of fractions *in grade 5 and up*. In TSM, a fraction is not given a precise definition *that can be used as the starting point for logical reasoning*. The resulting absence of reasoning in the teaching of fractions therefore opens the floodgates to mathematics-with-no-reasoning, a.k.a. *rote-learning,* regardless of all the hands-on activities, analogies, and metaphors that rush in to fill this vacuum (cf. Wu, 2010c). Although such a statement about the teaching of fractions is generally accepted by most as a given, it may nevertheless strike others as being too harsh. Let us therefore back it up by giving a more detailed analysis.

In TSM, fractions are usually introduced with pictures galore and fascinating stories about the different ways fractions are used in everyday life, together with the statement that a fraction can be interpreted as at least one of three things: parts-of-a-whole, quotient (division), and ratio. Here is one example:

> A fraction has three distinct meanings.
>
> Part-whole. The part-whole interpretation of a fraction such as $\frac{2}{3}$ indicates that a whole has been partitioned into three equal parts and two of those parts are being considered.
>
> Quotient. The fraction $\frac{2}{3}$ may also be considered as a quotient, $2 \div 3$. This interpretation also arises from a partitioning situation. Suppose you have some big cookies to give to three people. ... [If] you only have two cookies, one way to solve the problem is to divide each cookie into three equal parts and give each person $\frac{1}{3}$ of each cookie so that at the end, each person gets $\frac{1}{3} + \frac{1}{3}$ or $\frac{2}{3}$ cookies. So $2 \div 3 = \frac{2}{3}$.

> Ratio. The fraction $\frac{2}{3}$ may also represent a ratio situation, such as there are two boys for every three girls. (Reys, Lindquist, Lambdin, and Smith, 2009, p. 266.)

The same viewpoint persists in the research literature. The usual introduction of the concept of a fraction is by way of the same multiple representation approach:

> Rational numbers[17] can be interpreted in at least these six ways (referred to as subconstructs): a part-to-whole comparison, a decimal, a ratio, an indicated division (quotient), an operator, and a measure of continuous or discrete quantities. Kieren (1976) contends that a complete understanding of rational numbers requires not only an understanding of each of these separate subconstructs but also of how they interrelate. (Behr, Lesh, Post, and Silver, 1983, p. 92.)

The mathematical flaws of these "multiple interpretations" of fractions are analyzed in Wu (2011a, p. 178; 2016a, pp. 5–6), respectively, but we are here to focus on the impact of such teaching on student learning. The overriding fact is that none of this information answers students' burning question about *what a fraction is*. To ask students to accept a fraction as part-whole, quotient, and ratio all at once is pedagogically untenable. First of all, the part-whole interpretation involves two whole numbers: the number of equal parts the whole has been divided and also the number of parts that are taken, so are we trying to tell them that a fraction is two numbers? The same is true for the ratio interpretation: the fraction $\frac{2}{3}$ is the number 2 (the number of boys) *and* the number 3 (the number of girls), two numbers again.[18] The pedagogical issue with the "quotient" interpretation is far more subtle and therefore far more insidious in the long run. Students' knowledge of "quotient" (division) is based entirely on their experience with whole numbers, where it is always about $24 \div 6$, $72 \div 4$, or $45 \div 15$. In other words, the dividend is known ahead of time to be a multiple of the divisor so that the "equal group" interpretation of division can make sense. Now teaching is generally about building on students' prior knowledge, and this time the prior knowledge is about "quotient." Keeping this in mind, can we as competent teachers ask students to divide two cookies into three equal groups? Six cookies or nine cookies, that is for sure. But *2* cookies? This is pedagogically unsound to say the least, because students' prior knowledge would not allow them to absorb this information. But since TSM insists on ramming this unnatural demand down their throats, right here TSM is pulling the rug from under their feet. Indeed, if they had any illusion at all about mathematics being *learnable* in the sense of a careful scaffolding of its steps with reasons given for the progression from one step to the next, it has been wiped out in one fell swoop. The formidable task they face is to try to understand a new gadget called a *fraction* by first submitting themselves to the uncomfortable proposition that whatever they have strived to learn about "quotient" is simply not good enough. Now they must ask themselves: what else must they unlearn before they can climb the mathematical ladder? Such thoughts cannot be an auspicious beginning for the arduous journey ahead.

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[17]This term is being used erroneously for *fractions.*

[18]Teachers that I have worked with told me consistently that students have difficulty conceptualizing a fraction as a single number.

It may be thought that the preceding analysis of "quotient" is not accurate because students do know about dividing an arbitrary whole number by a nonzero whole number before coming to fractions. For example, $5 \div 3$ is so-called 1 $R2$,[19] i.e., quotient 1 and remainder 2. In this light, $2 \div 3$ would be the two numbers 0 and 2, as in 0 $R2$. This then leads back to an earlier impasse about the meaning of the fraction $\frac{2}{3}$: this meaning of the fraction is *two numbers* 0 and 2. It is an insurmountable task to relate "0 and 2" to the concept of part-whole (i.e., partition the whole into three equal parts and consider two of them), and failing to do that, we are guaranteeing non-learning again.

If we may use an analogy, asking students to believe at the outset that a fraction is three dissimilar things all at once is akin to asking them to look at a picture of a house obtained by superimposing three different views of the same house on each other. *Students get no clarity.* Such multiple representations of a fraction also beg the question: In a given situation, which representation should students use? Or should they use all three to make sure? Teaching based on TSM cannot provide answers to these natural questions.

Leaving students in this state of puzzlement, TSM nevertheless asks them to freely compute with fractions and use them to solve word problems. Can competent teaching afford to make students do things by rote for 6 or 7 years (from grade 5 to grade 12) without informing them what they are doing? To make matters worse—or perhaps because of the lack of definition of a fraction—definitions for all concepts related to fractions seem to be completely missing as well. For example, students never get a precise definition for the intuitive and basic concept of "one fraction being bigger than another." Instead, they are taught that *if* they change both fractions to fractions with the same denominator, and then they can see which is bigger. Now the reason this is worth pointing out is that it exemplifies a recurrent theme in TSM: Never mind whether you know what you are doing or not, because we are going to tell you *what to do*, and then you will get the right answer. As for the arithmetic operations on fractions, the plaintive refrain of "Ours is not to reason why, just invert and multiply" says it all: in TSM, one does not teach the definition for the division of fractions. The case of fraction addition, however, deserves a closer look (Wu, 1998, p. 24; 2011a, p. 228), and we will do just that.

In place of a precise definition of the addition of two fractions, TSM usually provides profuse verbal descriptions and pictorial illustrations of putting parts-of-a-whole together. Competent teaching on the most basic level however demands that, at this juncture of students' mathematics learning trajectory, they be exposed to a clear and logical argument that *leads from the definition of the sum of two fractions to an explicit formula for the sum.* Unhappily, without a precise definition of a fraction and a precise definition of the addition of fractions in TSM, such a demand cannot be met. What students get in place of reasoning is a formula for the sum involving LCD (*least common denominator*) that has to be memorized by

[19]The multiple errors inherent in this notation 1 $R2$ should be better known. See, e.g., Wu (2014b, p. 6).

rote. This is where fraction phobia seems to begin. Being cognizant of this fact, some have gone so far as to advocate de-emphasizing the addition of fractions, perhaps with a view toward reducing students' anxiety (e.g., National Council of Teachers of Mathematics, 1989, p. 96). In a climate of no definitions and therefore no reasoning, any attempt at *teaching fractions for understanding*—no matter how well-intentioned—becomes an oxymoron.

It remains to explain why we believe students in grade 5 and up deserve to learn about the reasoning that leads from the *definition* of the sum of two fractions to the explicit formula for the sum. In a nutshell, this is the basic survival skill in navigating the mathematical waters of roughly grades 6–12. It therefore behooves students to begin acquiring this skill through the study of fractions. We have to recognize that the mathematics in grade 5 and beyond will be increasingly abstract and will be increasingly dependent on having precise definitions and logical deductions therefrom to make sense of the abstractions. The concept of fractions is the first genuine abstraction students face in school mathematics because fractions do not show up *naturally* in the real world (think of $\frac{7}{13}$ or $\frac{21}{11}$); if we want students to learn what a fraction is, it is incumbent on us to tell them, *precisely, what we want them to know about fractions.* This is what precise definitions can accomplish. If our goal is to nurture students' mastery of abstractions, then we can do no better than employ precise definitions in the teaching of fractions. Indeed, once students enter the world of fractions around the fourth or fifth grade, the march toward abstraction in the school curriculum becomes inexorable. Fractions are followed by negative numbers (particularly the multiplication and division of negative numbers), the use of "variables"[20] and the concept of *generality*, transformations of the plane and basic isometries, congruence and especially similarity, functions and their graphs, principle of mathematical induction, complex numbers, etc. The learning of each and every one of these concepts will require extra effort on the part of students—in the same way that the learning of fractions requires extra effort—because of the elevation in the level of abstraction. Competent teaching must therefore take note of students' battles ahead and prepare them accordingly.

Let it be known in no uncertain terms that we do not argue against appropriate use of stories, hand-on activities, and multiple representations to round off the intuitive picture of a concept *if a precise definition is part of the presentation and the primacy of the definition is understood* (see the protracted discussion of the definition of a fraction in Wu, 2011a, pp. 173–182, or Wu, 2016a, pp. 2–10). However, TSM promotes the idea that students can learn what an abstract concept such as fraction is, *without a definition,* solely by being exposed to a multitude of stories and activities to illustrate these multiple "meanings." This idea is predicated on the assumption that mathematics can be learned by what we call **inductive guessing**. This is the process of letting students work informally with a given concept to guess the properties this concept *might possess* and allowing their guesses to coalesce, over time, to form a complete picture of the concept. *But no precise*

[20]Please see the discussion of "variables" on page 67.

definitions. The fact that mathematics learning largely fails to materialize when fractions are taught exclusively by inductive guessing is by now beyond dispute. For example, fraction phobia has become almost a national pastime (there are numerous strips in the Peanuts and FoxTrot comic strips on fraction phobia). This failure has dramatically crystallized in a TIMSS fraction item for eighth grade, as pointed out in Askey (2013). To my knowledge, there is no data to establish a causal relationship between inductive guessing and students' non-learning, but the ongoing school mathematics education crisis (cf. National Academy of Sciences, National Academy of Engineering, and Institute of Medicine, 2010; National Mathematics Advisory Panel, 2008) would seem to *strongly suggest* that such a causal relationship does exist.

The case against the sole reliance on *inductive guessing* in mathematics learning is rooted in the fact that correct reasoning requires a precise hypothesis as the starting point and a precise conclusion as the endpoint. If an abstract concept is nothing but the amalgamation of impressions accrued from inductive guessing, then it would be, by its very nature, imprecise because impressions vary from person to person. Consequently it cannot be *reliably* used in either the hypothesis or the conclusion of any reasoning and, without reasoning, there would be no mathematics. The virtue of a precise definition for an abstract concept is therefore that it "tames" the abstractness by providing precise information about *what the concept is*, no more and no less. Moreover, it is in the nature of mathematics that, once a definition is given, it will not change with time. If a fraction is defined in grade 4 to be a point on the number line constructed in a specific manner, then students can count on its being the same in every grade thereafter. This property of permanence makes the concept learnable because it allows students to stop wasting time trying to guess what a fraction might be in another situation but concentrate instead on getting to know fractions by using them in logical reasoning. In this way, students will get to derive *all* the known properties of fractions, including, in particular, what it means to add fractions and why the formula for adding fractions without using LCD is correct (cf. Wu, 2011a, Part 2, especially Section 14.1). No guesses, and no *deus ex machina*. Such an experience will give students the confidence that mathematics is learnable, and this confidence will in turn empower them to conquer the many more abstractions to come.

It remains to make a comment about the definition of a fraction as a point on the number line constructed in a prescribed manner (cf. Jensen, 2003; Wu, 1998, 2011a). In the event that such a definition is adopted, it is imperative to use the *same* definition throughout the whole development of fractions, including multiplication, division, ratio, and percent. If we abandon this definition at any point and choose, for example, to represent a fraction as a rectangle to discuss multiplication (as some have done), then we would be sending the erroneous signal that a definition is something we use when it is convenient but, otherwise, it is not to be taken seriously. Worse, we will be showing clearly that mathematics has no coherence, because it does not always tell the same story about a concept (fraction) but changes its story line at will. This will wreak havoc with student learning.

4.3.2 Other Garbled Definitions in TSM

There are more garbled definitions in TSM than we can count that have a profound effect on teaching and learning, but we will limit ourselves to four of them: decimals, constant rate, variable, and slope.

First, decimals. A finite decimal such as 0.2037 is defined in TSM as "2 tenths, 3 thousandths and 7 ten-thousandths." In terms of student learning, this causes damage in at least two different ways. The first is that it leads to students' misconception of a decimal as a fragmented collection of little bits of 2 tenths, 3 thousandths, and 7 ten-thousandths when they should be learning that a decimal is a single number. This is because *thousandths, ten-thousandths,* etc. are almost invisible quantities to students in elementary school, and they don't know how to integrate these new tidbits into a single number. Could such a fragmented conception of a decimal be a factor in students' difficulty in comparing decimals and computing with decimals? This would make for an interesting research project in cognition. Secondly, if the vague statement "2 tenths, 3 thousandths and 7 ten-thousandths" is phrased in precise language, then it will state clearly that 0.2037 is *the sum of the following fractions:*

$$0.2037 = \frac{2}{10} + \frac{0}{100} + \frac{3}{1000} + \frac{7}{10{,}000} \tag{4.4}$$

Unfortunately, TSM teaches decimals and fractions separately, making believe that they are different kinds of numbers (this may be the reason TSM uses the imprecise language "2 tenths, 3 thousandths and 7 ten-thousandths" to hide the fact that a finite decimal is a fraction). Since there is no attempt in TSM to ensure that the arithmetic of decimals is taught only after fraction addition has been introduced, the teaching of finite decimals in TSM is *mathematically* unlearnable.

A correct definition of 0.2037—historically as well as mathematically—is that it is the fraction

$$\frac{2037}{10{,}000}$$

which is of course equal to the sum of four fractions on the right side of (4.4). Likewise, all finite decimals are nothing but fractions whose denominators are powers of 10 (see Wu, 2011a, p. 187; 2016a, p. 17; CCSSM, 4.NF.5 and 4.NF.6). We may summarize the need of a correct definition for finite decimals as follows. On the one hand, it restores the *coherence* of mathematics by showing that, instead of three kinds of numbers—whole numbers, decimals, and fractions—there is only one kind of numbers, namely, fractions. On the other hand, the correct definition allows for simple (and correct) explanations of the addition and multiplication algorithms for finite decimals. For the multiplication algorithm, TSM has forced teachers to teach *by rote* the correct placement of the decimal point in the product, whereas it is a simple consequence of the product formula for fractions (see Wu, 2011a, p. 269; 2016a, pp. 68–69).

We will next look at the absence of any definition for *constant speed* or, more generally, for *constant rate* in TSM. We will show that this absence has very serious consequences because it spawns the bogus concept of *proportional reasoning*. We can begin the discussion with a typical rate problem:

(P1). David drove 936 miles in 13 hours. At the same rate, how long will it take him to drive 576 miles?

According to TSM, we teach students that the "rate" of 936 miles in 13 hours should immediately suggest that we look for the "unit rate," which is $936/13 = 72$ mph. Therefore, proportional reasoning tells us that the answer is $576/72 = 8$ hours. Very simple. But is it?

If the problem had asked instead,

(P2). David drove 936 miles in 13 hours. At the same rate, how long will it take him to drive 2808 miles?

then this would be a reasonable problem for the following reason. No matter how one interprets "at the same rate," one would agree that it carries the information that, in every 13 hours, David covers 936 miles. So in 26 hours, he would cover 1872 miles ($1872 = 2 \times 936$), and every 39 hours he would cover 2808 miles ($2808 = 3 \times 936$). The answer to (P2) is therefore 39 hours.

But to ask how long it would take David to drive 576 miles? This adds complexity to the problem that makes (P1) unsolvable. Indeed, suppose David cruised for the first 10 hours at 70 mph, so that at the end of 10 hours, he had driven 700 miles. Knowing that he should get to his destination in 13 hours, he sped up and managed to cover the remaining 236 miles in 3 hours.[21] That was how he drove the 936 miles in 13 hours. Now if you want to know *at the same rate*, how long it would take him to drive 576 miles, he will have to ask you whether it is the rate in the first 576 miles of his trip or the last 576 miles or somewhere in between. If the first 576 miles, for example, then at 70 mph, it will take him $576/70 = 8\frac{8}{35}$ hours. *Not* 8 hours as claimed. Can anyone dispute that $8\frac{8}{35}$ is as good an answer as 8 to (P1)? Moreover, if we consider his rate in the last 576 miles of his trip, then it will take him $7\frac{6}{7}$ hours to cover 576 miles. Obviously there are other possibilities. Therefore, as is, (P1) is a problem that admits many correct solutions, and as such, it is not an acceptable mathematics problem.

What happens is that the given data that David drives 936 miles in 13 hours is not precise enough to yield a definitive answer to (P1). The *implicit assumption* in all such problems in TSM that would render a definitive answer possible is that David drives at the same *constant speed* throughout. By bringing out this implicit assumption, we reformulate (P1) to read:

(P3). David drives at a constant speed and he drove 936 miles in 13 hours. At the same constant speed, how long will it take him to drive 576 miles?

[21] He drove the last stretch of 236 miles in New Mexico where the freeway speed limit is 75 mph most of the time.

In TSM, the assumption of constant speed in such problems is usually missing, and even when it is mentioned, the concept of "constant speed" is understood *intuitively* without a precise definition. The idea seems to be that if the *words* themselves sound familiar, then definitions will be superfluous. Since an expression such as "driving at 70 miles an hour" is part of everyday language and people already have a *vague* understanding of it, a precise definition would be considered unnecessary in TSM. The fact is that the definition of "constant speed" is quite subtle, but when it is defined precisely and *is put to use in the solution of* (P3),[22] the solution turns out to be very simple. In particular, the correct solution does not make use of "proportional reasoning" in any shape or form (Wu, 2016a, pp. 108–115, and especially Section 7.2 of Wu, 2016b), and instead of the mysterious invocation of "unit rate," it shows how the concept of "unit rate" follows naturally from the definition of constant speed. Most importantly, the correct solution of such rate problems restores *reasoning* to teaching and mathematics education.

The cavalier attitude that TSM takes toward definitions also materializes in another form. Freed of the responsibility to provide definitions, TSM is at liberty to create fictitious mathematical concepts, the most notorious among these being that of a **variable**, "a quantity that varies." The inability to master this concept, according to an informal survey of the teachers that I have come in contact with, has been a real stumbling block for teachers and students alike in the learning of algebra. Yet, they feel compelled to grapple with this concept because:

> Understanding the concept of *variable* is crucial to the study of algebra; a major problem in students' efforts to understand and do algebra results from their narrow interpretation of the term. (National Council of Teachers of Mathematics, 1989, p. 102.)

I believe it is time for mathematics education to face the reality that "variable" is not a mathematical concept but is a cultural vestige of the way mathematicians in the eighteenth and nineteenth centuries referred to elements in the domain of definition of a function. If the function $f(t)$ describes the location of a moving particle in 3-space at time t, then as *t changes its value*, so does $f(t)$. So it is suggestive to think of t as a "variable." However, it is wrong to believe that learning must always be built on students' prior existing knowledge. Sometimes learning requires a revision, or at least some form of modification, of this knowledge. For example, we routinely speak about *sunrise* and *sunset* in everyday life, which suggests unmistakably that the sun revolves around the earth. Few would object to these expressions. But it will not do—purely for the sake of building on this prior misperception—to tell students in a science class that indeed the sun rises and sets because it revolves around the earth. At that point of students' education, it is time for them to recognize the limitations of the commonly used suggestive language and embrace the correct scientific information that it is the rotation of the earth that causes the illusion that the sun revolves around the earth. The truth is that the earth revolves around the sun.

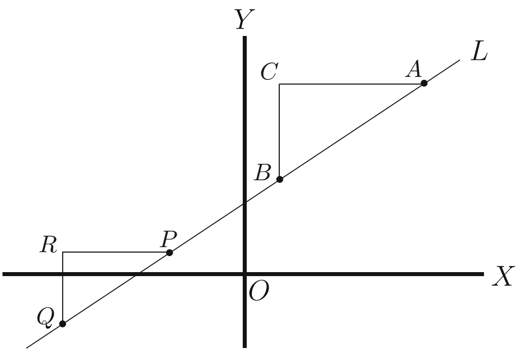
[22] We cannot overemphasize the fact that we need definitions in mathematics because they furnish the foundation for logical reasoning.

If we feel scandalized by a science class that does not clear up "sunrise" as a *human* misconception, then why do we complacently accept the teaching of "variable" in a mathematics class as a valid mathematical concept, or worse, that it is a "concept crucial to the study of algebra"? We do not advocate that we banish the word "variable" from mathematics because,

> ... the word *variable* has been in use for more than three centuries and, sooner or later, you will run across it in the mathematics literature. The point is not to pretend that this word doesn't exist but, rather, to understand enough about the use of symbols to put so-called *"variables"* in the proper perspective. Think of the analogy with the concept of *alchemy* in chemistry; this word has been in use longer than *variable*. On the one hand, we do not want alchemy to be a basic building block of school chemistry, and, on the other hand, we want every school student to acquire enough knowledge about the structure of molecules to know why alchemy is an absurd idea. In a similar vein, while we do not make the concept of "variable" a basic building block of algebra, we want students to be so at ease with the use of symbols that they are not fazed by the abuse of the word "variable" because they know how to interpret it correctly. (Wu, 2016b, p. 3.)

This discussion points to the need for school mathematics to move away from concepts without definitions—"variable" in this case—and engage students instead in the far more important issue of the correct use of symbols. When symbols are used correctly in school mathematics, "variable" as a *mathematical concept* will disappear from the school curriculum (cf. Wu, 2010b, Section 1; 2016b, Chapter 1).

Our final example of the mishandling of definitions in TSM is the concept of the *slope* of a line in the coordinate plane. Students' difficulty with slope is well documented (cf. Postelnicu, 2011), but it does not seem that the education research that looks into this difficulty has taken note of a serious mathematical flaw in the usual definition given in TSM (the two papers of Newton & Poon, 2015a,b are among the exceptions). The TSM definition states that the *slope* of a (nonvertical) line L in the coordinate plane is the following "rise over run": let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two distinct points on L, and then the "rise over run" is $\frac{RQ}{RP}$ (the "rise" RQ and the "run" RP), where R is the point of intersection of the vertical line through Q and the horizontal line through P, and it would be minus this quantity if the line slants the other way. In a more compact form, the slope is the following ratio: $\dfrac{p_2 - q_2}{p_1 - q_1}$.



What is obviously missing in this definition is the assurance that this ratio is the same regardless of which two points on L are chosen. In other words, suppose we take two other points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ on L instead of P and Q, then the ratio computed with A and B is the same as the one above computed with P and Q. More precisely, we should have

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{a_2 - b_2}{a_1 - b_1} \tag{4.5}$$

This equality is important because if the slope of L is really a property of *the line L itself,* then it has to be the same number regardless of which two points on L are chosen. Fortunately, Eq. (4.5) is indeed correct (see Section 4.3 of Wu, 2016b), but its proof requires some knowledge of similar triangles. The latter fact is not mentioned in TSM.

The reason a correct definition of slope, in the sense of making explicit Eq. (4.5), is important for mathematics learning is twofold. The first is that the general confusion about slope appears to include the misconception that it is a pair of numbers, "rise" and "run," but not *a single number* attached to the line itself.[23] In this light, one virtue of providing a proof of Eq. (4.5) is to reinforce the message that these are *numbers* that we are trying to prove to be equal. Such a proof may help to dislodge those students from this misconception.[24] A second reason is that it is difficult to solve problems related to *slope* without the explicit knowledge that slope can be computed by *choosing any two points on the line* that suit one's purpose. (Compare Wu, 2016b, pp. 72–76 on the proof of the graph of $ax + by = c$ being a line.) The lack of this knowledge is the cause of students' well-known difficulty with learning all aspects of the graphs of linear equations. For example, they are forced to memorize by brute force—often without success—the four forms of the equation of a line (two-point, point-slope, slope-intercept, and standard) because they are not taught any reasoning in connection with any part of the concept of slope. According to a recent survey (Postelnicu & Greenes, 2012) of students' understanding of (straight) lines in introductory algebra, the most difficult problems for students are those requiring the *identification of the slope of a line from its graph*. That these research findings could actually be correct is almost unfathomable. Think about this for a moment: to compute the slope of a line, all you have to do is grab *any* two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on the line and form the ratio $\frac{p_2 - q_2}{p_1 - q_1}$. This is trivial, but only if you happen to know, emphatically, that you can take *any two points* on the line for this purpose.

The correct use of definitions in school mathematics does matter after all.

[23] This echoes the phenomenon mentioned in Sect. 4.3.1 about students' confusion over a fraction also being a pair of numbers.

[24] It may be mentioned that the particular definition of slope in Section 4.3 of Wu (2016b), brings out the fact from the beginning that the slope is a single number.

4.3.3 Geometry in Middle School and High School

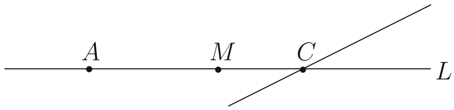
The non-learning that has been taking place in the high school geometry course of TSM is perhaps too well known to require comments (see, e.g., Schoenfeld, 1988). Incidentally, there may never be a better argument for the importance of teachers' content knowledge than Schoenfeld's account of what passes for "geometry teaching" in a TSM classroom. This kind of non-learning actually has its roots in the middle school curriculum and beyond. In this subsection, we will briefly summarize the three main issues and leave the more extended discussion to Section 4.1 of Wu (2016a).

(A) In TSM, the high school geometry course sticks out like a sore thumb among other courses in school mathematics. In the latter, reasoning is lacking and the opportunity to write a proof is nearly nonexistent, but in the former, literally *everything* demands a proof. This incongruity breeds non-learning.
(B) The discord between what is taught in middle school geometry regarding congruence, similarity, and scale drawing and what is taught about the same topics in the high school geometry course is too great for an average student to overcome.
(C) The high school geometry course is taught in isolation, as if it were unrelated to the rest of the school curriculum. In reality, certain geometric tools are critically needed for the teaching of slope of a line and the graphs of quadratic functions. The failure of the typical course to meet this need is an unfortunate missed opportunity to broaden its appeal and make it relevant to school mathematics.

We will add a few comments to round off the picture. Regarding (A), it has been a recurrent theme of this article to emphasize the overall lack of reasoning in TSM. Therefore students' transition into the high school geometry course may be likened to a nonswimmer being thrown into a lake in icy January and told to sink or swim. Trauma and bad results are preordained. To make matters worse, the TSM high school geometry course also insists on starting with axioms and proving a series of boring and geometrically obvious theorems at the beginning.[25] For illustration, we will make use of the well-known text of Moise and Downs (1964). We hasten to add that the text of Moise and Downs is on a higher plane than TSM, but it does following the tradition of "trying to prove everything." It was written in response to the call of the New Math of the 1960s (see Raimi, 2005; Wikipedia, New Math). It purports to use a modified version of Hilbert's axioms of 1899 (cf. Hilbert, 1950) to prove *every* theorem in plane geometry. With this in mind, we find on page 177 the following theorem.

Theorem 6-5. *If M is a point between points A and C on a line L, then M and A are on the same side of any other line that contains C.*

[25]For the lack of space, we will not take up the opposite kind of TSM geometry course which is all hands-on activities without a single proof. See, for example, Serra (1997).



One can imagine that not much of the discussion in the first 176 pages can be stimulating to the average beginner.

To give a little context to this discussion, let me relate a personal experiment. I taught the mathematics of the secondary curriculum (see Wu, 2011c, pp. 44–54) to preservice high school teachers many times in 2006–2010. In these courses, proofs are provided for all the theorems, including all the major geometry theorems to be found in a high school course and beyond (e.g., the nine-point circle). One day I suddenly popped the following question to a class of about 20 preservice teachers: "You know that proofs in the high school geometry course are considered very difficult. Now that you have proved many geometric theorems much harder than those in your high school course, can you tell me whether you still find the proofs of these geometric theorems to be too hard?" It took them a few seconds to even understand my question, because (they later told me) having been with me for almost a year up to that point and having been conditioned to proving everything, they had ceased to differentiate between a geometric theorem and a nongeometric one. That was the reason they didn't understand what I was referring to. Naturally, their answer was *no*. The geometric proofs were not harder.

If principle-based mathematics is taught in K–12, the overall situation regarding (A) will improve considerably because students would be already accustomed to reasoning and proofs *before* they take the high school geometry course. The course itself can be improved too. One proposal of a new foundation for the course is to use the **basic isometries** (rotations, reflections, and translations) to define congruence and use congruence and dilation to define similarity. Congruence and similarity now become tactile concepts rather than abstract inscrutable ones, and the classical criteria for triangle congruence (SAS, ASA, SSS) can now be proved as theorems. In addition, by assuming sufficiently many facts to get the geometric development started, we also avoid having to prove many uninteresting and possibly subtle theorems at the beginning, such as Theorem 6-5 in Moise and Downs (1964). (For more details, see Wu, 2013b, 2016a, Chapters 4 and 5.) It is easy to believe that such a new foundation will provide an easier access to geometry for students, but obtaining data to verify this fact may be less easy. It will have to be large scale, long term, and therefore expensive. However, the fact that the CCSSM also came to the same conclusion regarding such a new foundation gives us hope that there will be ample data on this issue in the years ahead.

Congruence and similarity provide a natural segue to (B) above on the discontinuity between middle school geometry and high school geometry in TSM.

There are two major disruptions in the transition from middle school geometry to high school geometry. First, TSM defines *congruence* as same size and same shape and *similarity* as same shape but not necessarily the same size. These statements are intuitive and attractive, but they are comically inadequate as *mathematical definitions* because they lack *precision*. For example, if we draw the acute triangle with three sides of lengths 20, 67.1, and 70 with respect to any unit of measurement (see page 50), then it will appear to have the same size and same shape as the right triangle of sides 20, $30\sqrt{5}$, and 70.[26] But these two triangles are not congruent. Of course, such a "definition" of congruence or similarity has the virtue that it is applicable to any shape, curved or otherwise. But high school congruence and similarity are suddenly defined precisely for polygons in terms of corresponding angles and corresponding sides *and for nothing else.* Does this mean that one can only reason about polygons when it comes to the concepts of congruence and similarity but that there is no way to express whether two parabolas, for example, are congruent or similar? This jarring discrepancy does no service to the *coherence* of mathematics or to mathematics learning.

It goes without saying that, with such inadequate definitions, the middle school geometry of TSM cannot sustain any *reasoning* about congruence or similarity. And there is none.

A second major disruption in the teaching of congruence and similarity lies in the way TSM treats the basic isometries in middle school and high school. In middle school, basic isometries are taught only for the purpose of fun activities and art appreciation, e.g., the sometimes subtle symmetries exhibited in Escher's prints and how the beauty of tessellations in church windows and Islamic mosaic art is enhanced by the different kinds of symmetries. Nothing is about the *purposefulness* of the basic isometries in school mathematics. Consequently, teachers who are immersed in TSM get the mistaken idea that the basic isometries are valuable only for so-called *transformational geometry*, which is roughly about doing the fun activities of moving geometric figures around the plane—using a coordinate system if necessary—and identifying symmetries in art works. To these teachers, the basic isometries are not about mathematics at all because the isometries appear to have nothing to do with the proofs of theorems in the high school course. While there are references to basic isometries near the end of some high school textbooks, they are mostly ornamental. In TSM, the basic isometries are long forgotten by the time of the high school geometry course. In this climate, it is therefore not surprising that, when in 2012 the Department of Education of a state on the East Coast produced a document on CCSSM geometry for its high school teachers, all 80 pages of it were devoted to transformational geometry but not a word about the serious business of using the basic isometries to understand congruence and proofs in high school geometry.

It should be quite clear that teachers' knowledge of TSM geometry will not enable them to teach the geometry of middle school or high school in any sensible

[26]Note that $30\sqrt{5} = 67.082\ldots$

way. We must help them to revamp their knowledge base. This is another reminder that teachers' content knowledge does matter.

Finally we briefly discuss the critical role of geometry in making sense of the algebra of linear and quadratic functions. The need of similar triangles for an understanding of the slope of a line has already been brought out in Sect. 4.3.2. The CCSSM have already asked for a reshuffling of our middle school curriculum so that eighth graders are at ease with the AA criterion for triangle similarity when they take up the graph of linear equations in two variables (CCSSM, 8.G.5). As for quadratic functions, the long and short of it is that the graph of $f(x) = ax^2 + bx + c$ is a translation (in the sense of basic isometries) of $F_a(x) = ax^2$, and furthermore, the graphs of $F_a(x) = ax^2$ (where $a > 0$) are similar to each other under a dilation with center at the origin O (Sections 10.2 and 10.3 in Wu, 2016b). These two facts together clarify the structure of quadratic functions: at least conceptually, every quadratic function is qualitatively the same as the function $F_1(x) = x^2$.

At the moment, the above approach to quadratic functions is inaccessible to students because translations are not precisely defined in the usual high school geometry course, and similarity between graphs of quadratic function does not make mathematical sense because similarity applies only to polygons. The chasm between TSM and principle-based mathematics is real indeed.

4.3.4 How Coherence and Purposefulness Impact Learning

It is easy to explain, in theory, the reason that mathematics developed *coherently* and *purposefully* will improve student learning. Obviously, when some events are told as a coherent story and the narrative is propelled forward with a purpose, they will be more memorable to readers than if the same events are presented as a laundry list. This is why even a rushed reading of *Don Quixote*—all 1000 pages of it—will leave readers with vivid memories of the Don's amazing exploits, whereas reading pages of a phone book, no matter how conscientiously done, will leave the readers with no memorable highlights whatsoever. We will present two examples that are consistent with such a narrative. The first one shows how *in*coherent mathematics can impede mathematics learning, and the other suggests that, by infusing the teaching of a seemingly boring topic with *purposefulness*, one can make it more learnable.

The first example is the way TSM teaches equivalent fractions, the basic tool students need to put any two fractions on a common footing (Wu, 2011a, Section 13.4). To see, for example, why $\frac{7}{3} = \frac{14}{6}$, TSM provides the following explanation:

$$\frac{7}{3} = 1 \times \frac{7}{3} = \boxed{\frac{2}{2} \times \frac{7}{3} = \frac{2 \times 7}{2 \times 3}} = \frac{14}{6} \tag{4.6}$$

The problem with such an "explanation" lies in the step enclosed in the box: It assumes, inexplicably, that before students know what it means to *add* two fractions with unequal denominators, they already know how to *multiply* them $\left(\frac{2}{2} \text{ and } \frac{7}{3}\right)$ by the so-called *product formula*, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. In greater detail, a coherent mathematical progression through fractions could reach the product formula by at least one of two ways: either

> definition of fraction ⟶ equivalent fractions ⟶ definition of fraction multiplication using "fraction of a fraction" ⟶ the product formula

(Jensen, 2003, Section 7.1; Wu, 2016a, pp. 60 ff.; CCSSM, 5.NF.4) or

> definition of fraction ⟶ definition of fraction multiplication using *area* of a rectangle ⟶ the product formula

(Wu, 1998, p. 25, 2011a, pp. 263 ff.). In either case, the proof of the product formula is a difficult one for fifth graders and should by no means be used to explain something as basic as equivalent fractions. What TSM has done in (4.6) is to shred the basic structure of mathematics for the expediency of making equivalent fractions *look* easy. However, there is a price to pay: Students get the idea that, although they don't know what "$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$" means, they are supposed to believe it because it *looks right*. This naturally suggests to them that, in mathematics, *if it looks right, it must be true.* So why not $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$? It is a popular sport to lament that even some freshmen in prestigious universities make such a ghastly mistake, but this is not students' fall from grace. This is our collective handiwork!

Before giving the second example, we first make a few observations about purposefulness in the context of mathematics learning. The first is that mathematical research is overwhelmingly about investigations with a purpose. The purpose of an investigation is always front-and-center because it provides a focal point for the researcher's thinking. Serious mathematical work is rarely the result of a random walk through the mathematical jungle to pick up low-hanging fruits. The school curriculum, being the distillation of serious mathematical work through the ages, should reflect as much as possible the purposefulness of such investigations. There is an additional connection between *research* and *learning*: they are fundamentally two sides of the same coin. They are both driven by curiosity, and researchers and learners alike try to peer into the (to them) unknown.[27] For this reason, learners will benefit from knowing the purpose of learning a new concept or a new skill because the purpose helps them to focus their own thinking too. Teachers should be aware of this aspect of mathematics learning and, for this reason, should get to know the purpose behind every concept and every theorem.[28]

Now consider the teaching of the skill of *rounding* (to the nearest hundred, nearest thousand, etc.) in TSM. Personally I have never come across a teacher who

[27] Of course there is also a big difference: whereas a helping hand is usually available if the learner fails to learn, there is no such helping hand when a researcher gets stuck!

[28] In this day and age when inquiry-based learning is encouraged, we hope that such learning can be conducted by emphasizing the *purpose* of any inquiry.

is not bored by this skill as presented in TSM; it seems to be totally pointless and mechanical, but it doesn't have to be that way. Imagine a teacher engaging students in a discussion of what they hear from TV or radio about the temperature of the day. Ask students why they often hear things like "today is a mild day in the 70s." Why not say, "today's temperature will be 74"? Make them realize that such precision is both *unattainable* and *unnecessary*. Ask them if they would change the way they dress if the temperature were 72 instead of 74, but also point out that they would likely change if the temperature were "in the 60s" rather than "in the 70s." Next, ask them in case the temperature is 68° whether they would describe it as "about 60 degrees" or "about 70 degrees"? Pursuing this line of give-and-take, a teacher can lead students to naturally *round to the nearest ten* without using any jargon or imposing any rigid rules. Then tell them after the fact that what they did is what is known as "rounding to the nearest ten" and that they did so because they were in fact trying to strike a happy medium between being informative and being sensible. In any case, by the time the teacher gets around to summarizing their findings of "rounding to the nearest ten" into some simple rules ("34 will be rounded down to 30, but 78 will be rounded up to 80"), the rules will sound neither boring nor pointless. They have a *purpose.*

The teacher can likewise talk about the population of a city like Berkeley. In Wikipedia, the estimated 2014 population is 118,853. Ask the students how much faith they have in this estimate. Consider the daily births and deaths, the expected influx and outflow of people, and other issues such as the homeless population and undocumented immigrants. Ask them whether they think it is appropriate to list the estimate as 118,853. Do they think the last three digits, 853, mean anything? If not, how do they want to list it? 118,000 or 119,000? In fact, for the purpose of general information, wouldn't an estimate of 120,000 make more sense? Now let them know they are *learning to make a decision* about whether to round to the nearest thousand or nearest ten thousand. In this case, the skill of rounding serves the *purpose* of making sense of the world around us. It doesn't have to be a fossilized skill from TSM at all.

For more details on rounding and estimation from this perspective, see Wu (2011a, Chapter 10).

4.4 What Does It Mean to *Know* a Fact in Mathematics

We started off this article by asking what mathematics teachers need to know in order to achieve basic teaching competence. Having described in some detail the nature of this content knowledge, we now bring closure by addressing what it means to *know* a fact mathematically (this discussion should be compared with Ball et al., 2005). In mathematics education, *knowing* a fact commonly means *knowing it by heart* (having memorized it). In mathematics, however, the same word means much more. To say you **know a fact** in mathematics means you know:

(a) what it says *precisely*,
(b) what it says *intuitively*,
(c) why it is true,
(d) why it is worth knowing,
(e) in what ways it can be put to use,
(f) how to put it in the proper perspective.

See Wu (2013a, p. 11). As in the case of "mathematical integrity," there is no pretense that such a characterization of "know" will be accepted by all mathematicians, but undoubtedly most would find it acceptable. The idea is that *knowing a fact* means being able to tell the *whole story about this fact* rather than just a few sound bites. It should also be said that there may not always be a good answer to each and every one of (d)–(f) in every situation. Moreover, since we are asking for content knowledge to ensure teaching competence *at the most basic level,* we will ignore (f) in subsequent discussions as its answer tends to be more sophisticated (see, e.g., pp. 25 and 38 of Wu, 2013a). That said, I believe a teacher should certainly make an effort to raise these questions all the time and try to get as many of them answered as possible. (Incidentally, the ability to answer most of these questions most of the time is intimately related to the coherence of mathematics.)

Again, it should come as no surprise that these are questions all mathematics researchers ask themselves again and again throughout the course of their work. Recalling once more the kinship between research and learning, we recognize that many students will be pondering the same questions (regardless of whether or not they can explicitly articulate them) when they are confronted by a new concept or a new theorem. A teacher must come prepared for these questions.

We give an example: what should a teacher know about the theorem on equivalent fractions? We will answer the preceding questions (a)–(e) in the same order:

(a) Given a fraction $\frac{m}{n}$, then for any nonzero whole number c, $\frac{m}{n} = \frac{cm}{cn}$. (In a fifth grade classroom, one will have to begin by using concrete numbers rather than symbols.)
(b) Don't get hung-up on the fraction symbol, e.g., $\frac{2}{3}$; it is the corresponding *point* on the number line that counts. A fraction is *a certain point on the number line,* and the symbol is nothing more than a representation of the point. Also get used to recognizing $\frac{2}{3}$ as $\frac{24}{36}$ or $\frac{18}{27}$. The moral is: nether the numerator nor the denominator in a fraction symbol means all that much by itself; it is the relative size of the numerator *and* the denominator that matters. For example, if we ask for *half* of $\frac{2}{3}$ of an apple pie, it is obvious: $\frac{1}{3}$ of the pie. Now if we ask for *a fifth* of the same amount of apple pie, is it any harder? Not much, because $\frac{2}{3} = \frac{10}{15}$, so a fifth of $\frac{10}{15}$ of the apple pie is $\frac{2}{15}$ of the pie.
(c) To prove $\frac{2}{3} = \frac{5\times 2}{5\times 3}$, for example, we ask whether the 2nd point in the sequence of thirds on the number line is the *same* point as the 10th point in the sequence of fifteenths. If we divide each segment between consecutive points in the sequence of thirds into 5 segments of equal length, the unit [0, 1] is immediately divided into 15 equal parts and we get the sequence of fifteenths. Now count carefully, and the truth of the assertion is obvious.
The general proof is no different.
(d) If you work with fractions at all, you will be seeing equivalent fractions all day long. This theorem figures prominently in every discussion of fractions, including the hows and whys of the arithmetic operations on fractions: $+, -, \times, \div$.
(e) Each time you get stuck on a problem involving fractions, your conditioned reflex ought to be: can I use equivalent fractions to get me out of this jam? More often than you can imagine, this strategy will work. For example, see (b) above.

4.5 Professional Development

It may be self-evident at this point, but we will nevertheless demonstrate presently, that any professional development (PD) that manages to pry open the grip of TSM on teachers and introduce them to principle-based mathematics will not be easy to come by. Let us consider two examples.

First, suppose some high school teachers want you to help them learn about quadratic functions. TSM being what it is, you know they are likely to have been misinformed about the need to understand the graphs of quadratic functions for the purpose of understanding the functions themselves. Equally likely, they may not realize that the study of quadratic *equations* is a very small part of the study of quadratic functions. They may also have been misinformed about the technique of completing the square, the fact that it is just as important for the study of *functions* as for the derivation of the quadratic formula. There will be a lot to talk about, but you feel comfortable telling the teachers that 2 days of PD should be enough.

Next, suppose some elementary teachers come to you and ask for PD that explains why definitions are important. You probably do a double take before replying because that is a big job! In your thinking, you may likely begin with the definition of fractions. Considering how much misinformation about fractions has been handed out in TSM, you figure that 1 day may not be enough to explain why the various TSM "definitions" of a fraction are not mathematically acceptable and why the definition in terms of the number line will promote better learning. Because the teachers are likely not to have come across any definitions for the addition, subtraction, multiplication, and division of fractions either, you want to take this opportunity to explain how this absence has led to "Ours is not to reason why, just invert and multiply," among other things. You want to convince them that having definitions for these operations is as important as getting the computational formulas because the definitions will make it possible to *explain* these formulas. This will take more time, because you cannot just *tell* them what the definitions of the operations are and move on; you must also explain the associated reasoning in detail because they have never seen it before. There is another reason you cannot rush them: they have been living with mathematics-without-definitions all through K–16 as well as all through their professional lives. You cannot change someone's habits of 20-some years overnight. You will need even more time.

But it is not just fractions that need definitions; whole numbers do too. In TSM, even concepts in the whole numbers do not have definitions. Few teachers will remember the definition of adding or multiplying two whole numbers (see, e.g., Case 12 of Schifter, Bastable, and Russell, 1999), much less why these definitions are relevant. After all, can the algorithms not be taught simply by rote? Therefore few will be able to explain the virtues of the standard algorithms for addition and multiplication, among other things. In fact, even fewer will be able to give a precise definition of the long division *algorithm*. Recall: to define an algorithm one must state the *precise procedure* as well as the *desired* outcome in a general context. To the extent that neither appears to have been done in the education literature, you begin to realize that explaining the significance of definitions is much more than changing the teachers' perception about "definitions" per se. You are in fact called upon to revamp their mathematical knowledge base—which is steeped in TSM—into principle-based mathematics. You have to change their belief system *and* rebuild their content knowledge from the ground up. Clearly *two weeks* will not be enough.

These examples hint at the difference between the run-of-the-mill kind of PD and the kind that aims at providing teachers with principle-based mathematics. The fundamental difficulty with the latter is the stranglehold that TSM has had on teachers for such a long time; if we want to enable teachers to teach correct mathematics, we will have to retrofit their knowledge base. This is hard, unpleasant work.[29] Let us start with preservice teachers. They have had 13 years of TSM by the time they get to college. Even if their undergraduate program offers courses on K–12 mathematics, these courses will have the burden of convincing them, point-counter-point, that the TSM they are familiar with is not correct and therefore not learnable by students, so that they had better replace it with something that is logical and coherent. This is a hard sell because, in all the years preservice teachers were in school, they saw with their own eyes that "mathematics" (i.e., TSM) was nothing more than a bag of tricks to memorize in order to score well on standardized tests and move on to the next class. They had no conception of the *logical and coherent progression of ideas* in principle-based mathematics. For example, they have all been taught that it is legitimate to "prove" equivalent fractions, i.e., $\frac{m}{n} = \frac{mc}{nc}$, by the following string of equalities (see Eq. (4.6) on page 73),

$$\frac{m}{n} = \frac{m}{n} \times 1 = \frac{m}{n} \times \frac{c}{c} = \frac{mc}{nc}.$$

Now imagine the hard work that is necessary to retroactively explain to them the fatal mathematical *incoherence* in this one line.

If we try to teach fractions without *directly* confronting preservice teachers with such fatal errors but only tell them what the correct reasoning is, will they realize on their own that *what they think they know* is wrong? If not, how then can we expect them to turn around and be advocates for principle-based mathematics? Changing teachers' minds about the precision, reasoning, and coherence of mathematics is clearly more than making a few tweaks here and there in the TSM they know. We will have to retrace essentially all the mathematics they have ever learned in school and *revamp it systematically* before any new ideas of principle-based mathematics can hope to sink in. At this point, perhaps what was said in Sect. 4.1 about the need for *long-term* PD will begin to make sense.

The last I heard, the pervasive dominance of TSM in school mathematics is largely unknown and unmentioned in the education literature and in Schools of Education, and the need for content-based professional development is widely ignored. Certainly the urgent need of professional development to explicitly undo

[29]For an example of why there can be no shortcuts in this kind of professional development, see the analysis of Garet et al. (2011), in Wu (2011c, pp. 20–31).

the ills of TSM is unheard of. In addition, there is as yet no awareness in most mathematics departments that the standard math majors do not necessarily make good high school teachers. If there are still any doubts about this fact, the recent study of Newton and Poon (2015a), should lay them to rest. Beyond this awareness, there is the obstacle of finding the right personnel to do this kind of PD. We have a long way to go.

The issues facing the PD of *inservice* teachers are even more dire. Districts do not invest (or do not have the funds to invest) in long-term professional development, and the teachers in the trenches do not have the time and energy to make the intensive effort to relearn the content during the regular school year. Unless something extraordinary happens soon, TSM will continue to be the default content in teaching and learning in schools for the foreseeable future.

Finally, we should address the naive question of why not just expunge TSM by the most direct method possible, namely, by rewriting school textbooks? To properly answer this question will take a separate article, but the short answer is that textbook publishers worry about their bottom line but not necessarily about good education. For slightly more details, see Keeghan (2012), and pp. 84 ff. of Wu (2015).

In summary, we have isolated the singularly destructive presence of TSM in school mathematics—especially its wanton disregard of definitions and reasoning—as a target for the mathematical reconstruction of the average teacher's knowledge base. Some may question whether this critique of TSM and the advocacy of its obliteration are necessary or appropriate. Our answer is affirmative, very much so. The school curriculum is a vast terrain, and teachers' misconceptions from TSM in this terrain are not confined to a few spots or a few chosen pathways; they are minefields that lay waste to the entire territory. Any attempt at professional development without *confronting and removing TSM*, such as the above "proof" of equivalent fractions or the pseudo-definition of the slope of a line mentioned in Sect. 4.3.2, runs the danger of "floating down a smooth-flowing river, so broad that you can seldom see either bank; but, when from time to time a promontory comes into view, you are surprised that it is a new one, as you have been unconscious of movement."[30] It would be irresponsible of us to usher complacent teachers through a tour of the K–12 landscape that they *think* they recognize through the lens of their TSM-infused misconceptions *without* explicitly making them realize that they must now leave these misconceptions behind. We want teachers and teacher-educators to become aware of the pressing need to eradicate TSM.

Having said that, I am compelled to point out in the spirit of full disclosure that the emphasis on the need to replace TSM—especially the malpractice of pretending to do mathematics-without-definitions and reasoning—and the urgency of the need to implement (content-based) PD to help teachers dislodge TSM are strictly my personal conviction thus far. These issues are not to be found in other recent discussions of teachers' content knowledge, e.g., Common Core (2012), Conference Board of the Mathematical Sciences (2001, 2010), National Council of Teachers of Mathematics (2014), and Zimba (2016). *Caveat emptor.*

[30]Bertrand Russell's critique of George Santayana's literary style; see Russell (1956, p. 96).

4.6 Pedagogical Content Knowledge (PCK)

So far, we have focused our attention on the reality of teaching and learning in the classroom. However, the question about what mathematical content knowledge teachers need has theoretical implications as well. In his well-known address (Shulman, 1986), Shulman initiated an inquiry into the kind of content knowledge that all teachers need for teaching. He introduced the concept of **pedagogical content knowledge (PCK)**, which is roughly the bridge that leads from content expertise to the process of teaching. The starting point is thus subject matter content knowledge. According to Shulman:

> We assume that most teachers begin with some expertise in the content they teach. . . . Our central question concerns the transition from expert student to novice teacher. . . . How does the novice teacher (or even the seasoned veteran) draw on expertise in the subject matter in the process of teaching? (*ibid.*, p. 8).

The precise nature of this "expertise in the content" is therefore foundational to his work on teacher education. This naturally leads us to ask what "some expertise in the content they teach" might mean and what it entails.

To the extent that Shulman was looking into all content disciplines all at once, a precise definition of this content expertise in general is out of the question because such a definition would have to be specific to each discipline. However, since we are now only considering the teaching of *mathematics*, it is incumbent on us to be as precise as possible about what constitutes *mathematical content expertise.* At this point, the picture can get murky. Since the content knowledge that an overwhelming majority of teachers possess is TSM, can mathematics teacher education be built on a foundation of TSM? Obviously not. So how then should the discussion of PCK *in mathematics* proceed? Can we assume that this requisite content knowledge is *principle-based mathematics?* An affirmative answer will bring clarification to the concept of PCK in mathematics and clear the way for us to get to work on providing the minimum content knowledge for PCK. Unfortunately, this remains very much an open question at the moment.

We can perhaps more deeply appreciate the preceding concerns if we take up the refinement of PCK in mathematics teaching proposed in Ball et al. (2008). These authors isolated what they called **subject matter knowledge for teaching** (*ibid.*, p. 402) as the *content* foundation of PCK. In their work, this knowledge is further subdivided into three categories. In our effort to understand what this subject matter knowledge for teaching consists of *in mathematical terms*, however, we find it more revealing to turn to a series of questions posed on page 402 of their article.

> Where, for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Where do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime or how and why the long division algorithm works? (Ball et al., 2008, p. 402)

In the view of Ball et al., the *subject matter knowledge for teaching* that they have in mind is the home for answers to questions such as these. Let us therefore try to answer them one by one.

• "Where, for example, do teachers develop explicit and fluent use of mathematical notation?"

The authors have put their fingers on a key issue in the school curriculum: how to properly use mathematical symbols. Since TSM is cavalier with the symbolic notation—lack of *precision*—it ends up with the bogus concept of a "variable" (see page 67).[31] Clearly TSM is very far from being the requisite *subject matter knowledge for teaching*, at least in this instance. The need to address the use of mathematical notation naturally comes with the requirement of *precision* in principle-based mathematics. In fact, precision suggests that symbols be used, albeit gently, in the elementary classroom in the statements of the commutative laws and associative laws for whole numbers and fractions (cf. Wu, 2011a, p. 42; also see Section 1.3). When in the middle grades the use of symbols becomes both necessary and intensive, teachers must come to terms with a fundamental fact regarding the use of symbols:

> Each time one uses a symbol, one must specify precisely what the symbol stands for.

This is given the name *the basic protocol in the use of symbols* in Wu (2016b, p. 4) (also see Wu, 2010b, Section 1). When such precision is duly observed, the usual symbolic computations in school mathematics are demystified as nothing more than computations with numbers. The whole of Chapters 1 and 3 and Section 2.1 of Wu (2016b) are devoted to an explanation of this fact from different angles. To the extent that this aspect of principle-based mathematics seems to be neglected in the mathematics literature—not to mention the education literature—the concerns of Ball et al. are entirely justified. We must teach teachers more than TSM.

• "Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept?"

This question does not even make sense in TSM because TSM has shown no appetite for definitions. So once again, teachers who know only TSM will not possess the *subject matter knowledge for teaching*.

In professional development materials, establishing the equivalence of definitions is a very rare occurrence even in principle-based mathematics because such an occasion is not commonly called for. For example, because there is as yet no usable definition of a fraction in school mathematics other than that using the number line (see the discussion of the following question), we are not in a position to compare the pedagogical pros and cons of different definitions or prove their equivalence, no matter how desirable such a discussion may be. A slight exception is the equivalence

[31] Some go even further and define a "variable" as a symbol without qualification, and sentences involving symbols-without-qualification are then called **open sentences** (e.g., UCSMP, 1990, p. 4). But the concept of "open sentence" is not needed for doing mathematics.

of the two definitions of fraction multiplication that is implicit in the discussion on page 74 of the product formula, i.e., the definition using "fraction of a fraction" (Wu, 2016a, p. 58; CCSSM, 5.NF.4) and the definition using the area of a rectangle (Wu, 2011a, p. 263). This equivalence is mentioned on page 262 of Wu (2011a), but no proof was offered. The equivalence is implicitly proved by combining Section 17.3 of Wu (2011a) and Theorem 1.6 on page 65 of Wu (2016a). Indeed, the former proves that the area definition implies the "fraction of a fraction" definition, while the latter proves the converse. In any case, this kind of knowledge is beyond principle-based mathematics even if it is compatible with it.

• "Where do they learn definitions for fractions and compare their utility?"

Again, not in TSM, because there is no definition for a fraction in TSM. See the discussion in Sect. 4.3.1. The first part of this question implicitly assumes that there are usable definitions of a fraction in school mathematics. As of 2016, the assumption is correct, but unfortunately there is only one such definition at the moment, which was the one was put forth in Wu (1998), and subsequently put to use in Jensen (2003) and Wu (2011a, 2016a), and put to partial use in Siegler et al. (2010). It would also appear to be the one in CCSSM, 3.NF.2. So as far as mathematics is concerned, a comparison of different definitions of a fraction is not yet a reality in 2016.

The second part of this question suggests that, perhaps, the authors meant to ask whether any of the existing TSM "interpretations" of a fraction (see Sect. 4.3.1) can be used as a *definition* of a fraction and, if so, how do they compare? Let us first consider this question in the context of advanced mathematics. Then one of them—the quotient interpretation—can indeed serve as a definition, but perhaps not others. It is known (in advanced mathematics) that a fraction $\frac{m}{n}$ can be defined as a division, $m \div n$, but this has to be done with great care. For example, $m \div n$ cannot be recklessly tossed around as in TSM (see page 61), but has to be defined abstractly as the solution of $nx = m$. Then this solution can be proved to be equal to the fraction $\frac{m}{n}$, which is understood to be the equivalence class of the ordered pair (m, n). However, even this brief description is enough to reveal that such a discussion is way beyond the level of school mathematics and is therefore inappropriate for the consumption by teachers. In summary then, the answer as of 2016 is that there is only one usable definition of a fraction in school mathematics.

• "Where do they learn what constitutes a good mathematical explanation?"

A "mathematical explanation" is of course just a "proof." Given the paucity of reasoning in TSM, one does not look for proofs in TSM. So emphatically TSM does not provide the *subject matter knowledge for teaching* that Ball et al. are looking for. If I understand the question, Ball and her co-authors are asking how teachers can learn to decide whether a proof is *correct* or not and, if correct, how to present it in an accessible way to students. Let us start with the former.

The ability to reason is not an instinctive one and has to be carefully nurtured. My own observation is that among teachers, especially elementary teachers, their prolonged immersion in TSM has often rendered them incapable of routinely asking *why*, much less looking for the answer. In the mathematical education of teachers, I believe we have to help teachers regain their reasoning faculty in at least two ways. First, they have to get used to the *mechanics* of proofs by a process of total immersion: learn the proof of every assertion in the mathematics they teach. Second, they have to get a feel for the overall architecture of mathematics by working through a systematic logical development of school mathematics.

To illustrate the first point, consider the assertion that, even without a definition of fraction division, one can derive the invert-and-multiply rule (see page 44):

$$\boxed{\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times (3 \times 5)}{\frac{4}{5} \times (3 \times 5)}} = \frac{2 \times 5}{4 \times 3} = \frac{2}{3} \times \frac{5}{4}$$

If teachers' sensibilities in reasoning have been heightened by a prolonged exposure to proofs, their conditioned reflex would be immediately alarmed by the fact that, in the first equality above, the left side of the equality, $\frac{2}{3}/\frac{4}{5}$, is as yet undefined. They know that to say $A = B$ is to say they already know what each of A and B is before asserting that they are equal. Therefore they would see right away that there is no way the equality can make sense. Next, let us see why they need to have an overview of the hierarchical structure of school mathematics. Look at the TSM proof of a special case of equivalent fractions, $\frac{7}{3} = \frac{14}{6}$, quoted on page 73:

$$\frac{7}{3} = 1 \times \frac{7}{3} = \boxed{\frac{2}{2} \times \frac{7}{3} = \frac{2 \times 7}{2 \times 3}} = \frac{14}{6}$$

Now equivalent fractions come near the beginning of every discussion of fractions, but this proof makes use the product formula for the multiplication of fractions. That should be enough to raise a red flag to teachers reading this proof, because they should have an overall understanding of the logical structure of fractions: no matter how fractions are developed, multiplication is never easy, and the product formula requires hard work. They should suspect right away that it is probably wrong to make use of a result that only appears down the road to prove something that is foundational.

In 2016, most teachers only know TSM but not principle-based mathematics. If we expect them to know the *subject matter knowledge for teaching*, we must begin by helping them go through an immersion in proofs and a point-by-point systematic development of the mathematics they teach. It was exactly the lack of any systematic exposition of principle-based mathematics that provided the initial impetus for the writing of the six-volume work: Wu (2011a, 2016a,b, to appear).

• "Do they learn why 1 is not considered prime or how and why the long division algorithm works?"

As usual, this question has no answer in TSM. The fact that 1 is not defined to be a prime has to do with the uniqueness of prime factorization (see Section 3.1 of Wu, 2016a), but TSM skirts any explicit mention of either *existence* or *uniqueness* (e.g., what $\sqrt{2}$ means is never seriously discussed in TSM). Next, if we reformulate the second part of this question about the long division algorithm in mathematical terms, then what it asks for is a formal statement of the algorithm as a *theorem* (i.e., with a hypothesis and a conclusion) as well as a *proof* of this theorem. This by itself is a most remarkable question because it seems not to have been previously raised in the education literature (and therefore never answered either). At its best, TSM provides heuristic arguments for the "division house" by using analogies or metaphors, but nothing remotely resembling a *proof*, i.e., a sequence of precise steps that progresses logically from hypothesis to conclusion. One of the difficulties is that, in TSM, the hypothesis of such a theorem has never been clearly stated. Moreover, the conclusion in TSM of the division-with-remainder of 125 by 4 is "31 $R1$," but there is no known mathematical reasoning that has the nonsensical statement "31 $R1$" as a conclusion. Now, it is possible in principle-based mathematics to explicitly describe the algorithm and—following the description—to systematically present a sequence of *simpler* divisions-with-remainders that ends with the equality $125 = (31 \times 4) + 1$. This may be the proof that Ball et al. are looking for. See Section 7.3 of Wu (2011a) for a formulation of such a theorem, and *ibid.*, Section 7.5 for its proof. The recognition of the long division algorithm as a theorem and a knowledge of its proof should without a doubt be part of every elementary teacher's minimal content knowledge.

In summary: If we are interested in the kind of content knowledge that can provide answers to the preceding five questions from Ball et al. (2008), then we must abandon TSM and look for something even more comprehensive than principle-based mathematics. Our conclusion is therefore that the subject matter knowledge for teaching that Ball et al. assumed to be foundational for PCK is a bit beyond principle-based mathematics.

In Sect. 4.5, we expressed the pessimism that we may not have a system in place, nor the requisite personnel, to provide mathematics teachers with the content knowledge for achieving basic teaching competence. If the preceding analysis of Ball et al. (2008), is correct, then fundamental to both Shulman's theory of PCK in mathematics and its refinement in Ball et al., is a content expertise for teaching that exceeds principle-based mathematics. We must therefore pool our resources together to *try to* provide this basic content knowledge for teachers before we can seriously contemplate tackling PCK. Let us begin by teaching them principle-based mathematics.

In a 2005 article (Shulman, 2005), Shulman said tongue-in-cheek that "Teacher education does not exist" because educators had failed to converge on a set of "signature pedagogies" that characterize all of teacher education. In the same vein, we can say that *teacher education in mathematics does not exist* because we haven't found (yet) a way to give teachers the content knowledge they need to achieve a basic level of competence in mathematics teaching.

Appendix 1: Applied Mathematics

The five principles (A)–(E) of Sect. 4.2.1 may be said to be foundational to the integrity of *pure mathematics*, which is the discipline that is driven principally by its internal logic and its internal imperatives (the sense of beauty, the sense of structure, etc.). However, its allied discipline of *applied mathematics*, which mediates between pure mathematics on the one hand and science and technology on the other, is never far from school mathematics. Consider, for example, the following problem:

> Two shuttle trains traveling at constant speed go between cities A and B which are 15 miles apart. It takes the first train 10 hours to make the trip, but it takes the second train 12 hours. Suppose now the first train is at city A and the second train is at city B and they take off at the same time on parallel tracks. How long will it be before they meet?

Notice that, as stated, this problem cannot be solved because we don't know precisely what "distance between cities" means, and we are also not given the lengths of the trains. Does the "distance between cities" mean the distance between city centers or the shortest distance between the outskirts of the cities, or is it the distance between the train stations? Let us assume that it is the latter. Now suppose the first train is 528 ft long (= 1/10 miles). Then the train doesn't travel 15 miles in going from city A to city B; it only travels $(15 - \frac{1}{10})$ miles and the given data actually means this train travels $(15 - \frac{1}{10})$ miles in 10 hours. Similarly, suppose the second train has length 264 ft (= 1/20 miles). Then this train travels $(15 - \frac{1}{20})$ miles in 12 hours. Now a little reflection will reveal that if "meeting" of the trains means the meeting of the *fronts* of the trains, then they will meet after they have traveled a combined distance of $(15 - \frac{1}{10} - \frac{1}{20})$ miles. Without proceeding further with this analysis, it is quite clear that a seventh grade *school mathematics problem* cannot afford to be this unwieldy.

In order to make the problem manageable to a middle school classroom, the standard simplification is to imagine that both trains are *points* without length. By further assuming the precise distance between the train stations of the two cities to be 15 miles, we are now given that these trains will travel 15 miles in 10 and 12 hours, respectively. With these simplifications understood, then this problem can be solved in the usual way as a typical *mathematics problem*.

This process of translating a word problem into a "doable" *mathematics problem* by making "reasonable simplification" is what is formally called "modeling." Applied mathematics may be said to be the study of mathematical problems whose solutions require modeling. The train problem is a rather trivial example of problems in applied mathematics. Most such problems arise from science or technology, and the modeling that is required for their solutions usually requires a heavy dose of *scientific* knowledge. To solve these problems, we will have to deal with concepts whose primary definitions lie "in the real world," so to speak, outside mathematics. For example, in dealing with the electric field in Newtonian physics, mathematicians may believe that the electric field is precisely defined by the gradient of the solution of Poisson's equation. But in physics, what truly matters is the force exerted on a test charge by the field. Therefore if the same problem is taken up in nineteenth century electrodynamics, the modeling of the field changes. The mathematical definition of the field will now involve Maxwell's equation and the combined electric plus magnetic force on a test charge. There will be other variations if the context changes to non-quantum special relativity or special relativistic quantum field theory. All the while, the electric field "out there" remains the same. Moreover, the reasoning used will involve a substantial amount of science in addition to mathematics, and the purpose behind the problem would likely lie more in science rather than in pure mathematics. The fundamental principles that characterize the integrity of applied mathematics will therefore be a slightly modified version of each of (A)–(E), as we have just indicated.

However, given the lack of coordination between the teaching of school science and school mathematics as of 2016, the chances of being able to do substantive applied *school* mathematics are essentially nonexistent, because such problems inevitably involve serious science. Problems like the train problem above typically constitute the only kind of applied mathematics that can be taught in K–12, and the modeling that is required for their solution is no more than certain *formal* conventions that—like the modeling of a train by a point—once set up can be learned quickly.[32] As the preceding solution of the train problem shows, once these conventions are understood, the usual applied problems in school mathematics quickly become part of pure mathematics again.

It is for this reason that we believe that the fundamental principles (A)–(E) are sufficient to characterize the integrity of the mathematics of K–12 in year 2016.

Appendix 2: The Existence of TSM

To people not directly involved with the professional development for mathematics teachers or the evaluation of school mathematics textbooks, TSM is an unbelievable concept: could a *nation's* textbooks be so bad for so long? Could it be that someone is taking poetic license to create this concept for purposes that are not entirely intellectual? This appendix addresses these doubts and suggests projects for research to confirm or refute the validity of this concept.

The most reliable way to identify TSM is to read, in succession, several textbooks for the same grade from major publishers. Using this article as a guide, the reader will not fail to notice the many similarities—and the anti-mathematical qualities—among these books. In order to generate data for research, however, we will have to suggest a far cruder methodology. We are going to write down a small list of observable characteristics to be used for detecting the presence of TSM. Note that it is easy to expand this list. For middle school mathematics, simply look up "TSM"

[32]This may explain why it is almost impossible to find sensible assessment items on modeling.

in the indices[33] of Wu (2016a,b). For high school mathematics, the volumes of Wu (to appear) will serve the same purpose when they are finally published. Because the volume Wu (2011a) was published before the term TSM was coined, it is slightly more difficult to come up with a list for elementary school mathematics. Nevertheless, it is not difficult to single out the many implicit references to TSM in Wu (e.g., pp. 106, 177–178, 206, 228, 332, 335, etc.).

What we suggest is to use the items on the following list to check the school mathematics textbooks from the major publishers. If over 75% of these books (in a fixed grade band) contain the error described by each item on this list (i.e., relevant to the grade band), then the validity of TSM would be beyond doubt. Moreover, one can get further confirmation by a survey of teachers using these items. Again if over 75% of the teachers confirm that these errors were exactly what they were taught when they were students, that would be a double confirmation of the validity of the concept of TSM. For this kind of research, the participation of a very competent mathematician will be crucial.

It should also be pointed out that many of the errors in the following list are recorded in the lessons of the teachers in the casebooks of Barnett, Goldstein, and Jackson (1994), Merseth (2003), Schifter et al. (1999), and Stein, Smith, Henningsen, and Silver (2000).

Here is the list:

(I) Missing or garbled basic definitions. (By "definition," we mean as in Sect. 4.2.1 a precise and mathematically correct statement about a concept that is put to use in the textbook for reasoning.)

Number; division-with-remainder; fraction; decimal; one fraction *bigger than* another; addition, subtraction, multiplication, and division of fractions; ratio; percent; constant speed; negative fraction; addition, subtraction, multiplication, and division of rational numbers; variable; expression; equation; polynomial; length of curve, area of region in a plane, and volume of solid in 3-space; scale drawing; slope of a line; half-plane of a line in the plane; the graph of an inequality, equation, or function.

(II) Wrong instructions.

(*a*) Writing a division-with-remainder, e.g., 17 by 5, as $17 \div 5 = 3\,R2$.
(*b*) Add two fractions by the use of the least common denominator of the fractions.
(*c*) Introduce mixed numbers before fraction addition.
(*d*) Expanding the product of two linear polynomials by the mnemonic device of FOIL.
(*e*) Teach *order of operations* as a major skill by the mnemonic device of PEMDAS.
(*f*) Define slope of a line as *rise-over-run* without emphasizing that it is *a single number* attached to the line.
(*g*) *Define* in a high school algebra text that two lines in the plane are perpendicular if and only if the product of their slopes is -1.

[33] These indices are not in those volumes but are obtainable from http://www.ams.org/publications/authors/books/postpub/mbk-98 and http://www.ams.org/publications/authors/books/postpub/mbk-99.

(III) Lack of reasoning (proof) for any of the following basic facts:

(a) The long division algorithm for whole numbers.
(b) The product formula of fractions: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.
(c) The invert-and-multiply rule for the division of fractions.
(d) The multiplication algorithm for the product of two finite decimals.
(e) The theorem $(-x)(-y) = xy$ for rational numbers x and y.
(f) The theorem $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$ for all rational numbers a and b ($b \neq 0$).
(g) The theorem that the graph of $ax + by = c$ is a line.
(h) The theorem that the graph of a linear inequality is a half-plane.
(i) The theorem that the solution of a system of two linear equations in two variables is the point of intersection of the two lines defined by the linear system.
(j) The theorem that a linear function attains its maximum or minimum at a vertex of the feasibility region in linear programming.
(k) The formula for the vertex of the graph of a quadratic function.
(ℓ) For any positive a and b and any positive integer n, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.
(m) The Factor Theorem for polynomials of one variable.
(n) The addition formulas for sine and cosine for *all* angles (i.e., not just acute angles).

(IV) Lack of purpose for basic skills or concepts.

(a) Why round off whole numbers or decimals?
(b) Why do we need negative numbers?
(c) Why do we need absolute values?
(d) Why teach rotations, translations, and reflections in middle school if they seem to be useful only for art appreciation?
(e) Why do we need to know the *slope* of a line?
(f) Why change the notation of $\sqrt[n]{a}$ to $a^{1/n}$ and $\frac{1}{a}$ to a^{-1}?

(V) Incoherence in the teaching of geometry.

Congruence is defined to be same size and same shape in middle school, but in the high school geometry course, it is *redefined* as equal sides and equal angles for polygons but nothing else. There is no explanation as to why once students are in high school, they will no longer be concerned about the congruence of curved figures. Similarly, *similarity* is defined to be same shape but not necessarily the same size in middle school, but in the high school geometry course, it is *redefined* as proportional sides and equal angles for polygons but nothing else. (See Sect. 4.3.3 for a more nuanced discussion.)

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Chapter 5
Knowing Ratio and Proportion for Teaching

James J. Madden

Abstract Ratio and proportion have been part of school mathematics since the earliest manifestations of anything like school math in the Middle Ages. In this paper, I compile and comment on statements from primary sources of the last 2300 years to exhibit ideas that appear to have influenced the treatment of these topics in schoolbooks today. Historical sources clarify many points about the contemporary curriculum, supporting the contention that an understanding of history of ideas concerning ratio and proportion is an important component of knowledge of mathematics for teaching.

5.1 Introduction

Some of the occasionally puzzling things that we read in school mathematics textbooks, or find in discussions about standards or in commentaries about school math, can best be explained by reference to the long, complicated history of the curriculum. When we read that the quantities used in forming a ratio must be of the same kind, we are catching an echo of Definition 3 of Book V of Euclid's *Elements*: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind." Similarly, the statement that a proportion is an equality between two ratios refers back to Definition 6 of the same book: "Let magnitudes which have the same ratio be called proportional." Euclid, and two millennia of scholarly writings on Euclid, have influenced the way we speak about proportion today.

Another powerful influence, largely independent of the classical tradition, developed with the emergence of mercantilism in Europe in the Middle Ages. The rule of three is a method for solving the proportions that arise in trade, such as deducing the cost of one amount of a commodity from the cost of another amount, assuming that the conditions of the sale remain the same. The rule was known in antiquity and was described in texts such as al-Khwārizmī's *Algebra* (c. 820 CE) and Fibonacci's *Liber Abaci* (1202 CE). It was always closely associated with numerical computations and the use of units. From the thirteenth to the sixteenth century, the

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method was taught in the so-called abbaco schools that sprang up in northern Italy to provide the training in calculation required for the trades. As the Renaissance spread northward from Italy across Europe, the curriculum of these schools spread with it (Bjarnadóttir, 2014; Heeffer, 2009), eventually making its way to the New World and helping to shape the American mathematics curriculum in the eighteenth and nineteenth centuries. Abraham Lincoln wrote of learning the rule of three in the brief autobiography he prepared for his 1860 presidential campaign. Even today, one finds on the Internet many problems that are modern versions of schoolbook exercises from this tradition.

Ideas and habits that shape teaching practice tend to persist from generation to generation by a kind of cultural replication (Stigler & Hiebert, 2009). With respect to ratio and proportion, it is not merely that artifacts of the past survive. An examination of history shows continuity over more than a millennium. Some traditions have imprinted the practices of the present to such an extent that it can be difficult to make sense of the latter without reference to the former. A deep, mathematically informed understanding of the history of mathematics—not at the higher levels of scholarship, which is where most histories tend to focus, but at the level of the classroom—has practical relevance for teachers, teacher-educators, textbook authors, curriculum specialists, and anyone else who might influence mathematics education.

This essay assembles evidence in support of this thesis. In Sect. 5.2, I collect numerous statements of the rule of three from the seventh century up to the present. As we shall see, the formulations of this rule remained stable over more than ten centuries. Clearly, there has been a tendency for teachers (or at least textbooks) to perpetuate certain ways of thinking and behaving mathematically.

In Sect. 5.3, I return to Book V of the *Elements*. This work influenced scholarly writing about ratio and proportion from the Middle Ages onward. Wallis and Newton referred to Euclidean proportion in explaining the new conception of the number system that was beginning to emerge in the seventeenth century. Wallis discussed explicitly the reasons why ratios (as comparisons of magnitudes) could only be formed between magnitudes of the same kind. Galileo used patterns of reasoning borrowed from Euclid in considering the proportional relationships that arise in physics. Thinking in terms of magnitudes and their classical ratios—rather than numbers and operations on them—is characteristic of physics from the early Enlightenment up till today. In addition, important influences of Book V on contemporary mathematics occur in the theory of measurement (see Hölder, 1901 and Michell, 1999) and in the mathematics of ordered algebraic structures (Bigard, Keimel, & Wolfenstein, 1977). The Common Core Standards for grades K-5 treat measurement in a manner that is consistent with the Euclidean approach. However Book V shows a profound connection between the topic of ratio and proportion and the topic of measurement that seems to be overlooked in contemporary school math.

In Sect. 5.4, I examine some comments by mathematicians and mathematics educators about ratio and proportion in the contemporary curriculum. I believe that they sacrifice some aspects of the Euclidean theory that are meaningful in the sciences. In Sect. 5.5, I present a modern interpretation of a central idea of *Elements*, Book V. The point here is to show that the contents of this book are neither quaint nor outdated. It contains ideas that are relevant to how we think about the modern curriculum. I have tried to present the most important ideas in a way that will be accessible to a broad audience. Finally, in the last section, I gather together the conclusions that I think can be gleaned from this historical sight-seeing tour, and I share some final thoughts.

This essay is not intended to be a contribution to scholarship on the history of mathematics in the usual sense. I wish to suggest the relevance of the history of the mathematics curriculum to modern problems of teaching and instructional design. I hope that the reader will conclude that significant "discursive formations" (to borrow a phrase from M. Foucault) can be identified in the history of mathematics teaching and that they illuminate the structure of the modern curriculum. This paper will have achieved its goal if readers come away convinced that there is something to be gained by studying what one might call the "archeology of the mathematics curriculum." The task for the future is to pursue this in a disciplined and systematic way, with the aim of contributing to knowledge of mathematics for teaching.

5.2 The Rule of Three

Early in 1859, Abraham Lincoln's friend, Jesse Fell, asked Lincoln to prepare an autobiography, hoping to use it to help generate publicity for the potential presidential nominee. Lincoln's response, a letter of four paragraphs,[1] was used as a basis for an article that appeared in the *Chester County Times*, February 11, 1860, the day before Lincoln's fifty-first birthday. In the second paragraph, Lincoln described his boyhood in Spencer County, Indiana, where his family moved in 1816:

> . . . It was a wild region, with many bears and other wild animals still in the woods. There I grew up. There were some schools, so called; but no qualification was ever required of a teacher, beyond "readin, writin, and cipherin" to the Rule of Three. If a straggler supposed to understand latin, happened to sojourn in the neighborhood, he was looked upon as a. . . wizzard. There was absolutely nothing to excite ambition for education. Of course when I came of age I did not know much. Still somehow, I could read, write, and cipher to the Rule of Three, but that was all.

In pioneer communities like Lincoln's, families contracted a schoolmaster and paid a fee for each child. When he was around eleven, Lincoln attended Azel Dorsey's school in Little Pigeon Creek, Indiana, a mile and a half from the Lincoln cabin. "It was built of unhewn logs, and had holes for windows, in which greased paper served for glass. The roof was just high enough for a man to stand erect" (Lamon, 1872, p. 33). At the schools he attended during the next several years, he kept a "ciphering book" much of which his stepmother preserved. After Lincoln's

[1]The Library of Congress holds authenticated reproductions of the original letter, which may be viewed on the internet.

death, his law partner, William Herndon, acquired it. Ten of the surviving pages are reproduced in the first volume of the Rutgers edition of Lincoln's collected works (1953). One page bears the title "The Single Rule of Three."

Besides the so-called subscription schools like those that Lincoln attended, there were other forms of basic education at that time. Many boys in their early teens entered indentures, or contracts of apprenticeship, which obligated them to work for a master and reciprocally obligated the master to care for and educate the young person. Often, these contracts required the master to teach his charge to read, write, and "cipher to [or as far as, or through] the rule of three." This meant learning arithmetic with whole numbers and fractions, conversions of units of measure, weight and currency, and techniques for basic proportions. Numerous examples of such contracts can be found in genealogical databases, accessible via the Internet. One web site, for example, contains a copy of the Apprentice Bonds from Cumberland County, North Carolina. There we read that on December 6, 1823, the orphan Leonord Cason, about 14 years of age (thus sharing his birth year with Lincoln), was bound to a certain David D. Salmon, "To learn the art and trade of a saddler and harness maker and to be taught to read, write and cypher through the rule of three." Other boys were bound in apprenticeships as carpenters, bricklayers, coopers, blacksmiths, chair-makers, cabinetmakers, hatters, shoemakers, tailors, carriage-makers, farmers, millwrights, clerks, accountants, printers, bookbinders, and so on—all of them to be trained to "read, write and cipher through the rule of three." Lincoln's words echo the formulaic language of the learning standards of his day. In writing what he did in his autobiography, Lincoln was saying that his education met the typical requirements for the education of young man preparing for a trade.

What, then, is the rule of three? In modern algebraic notation, the rule is expressed as follows:

$$\text{for any positive numbers } a, b, c, x \text{ if } \quad \frac{a}{b} = \frac{c}{x} \quad \text{then} \quad x = \frac{b\,c}{a}\,.$$

However, the algebraic formulation tells us little about the teaching and use of the rule. Every application of the rule requires recognizing the roles of the numbers involved, including the units of measure in which the problem is stated and the units required for the answer. This is the pragmatic dimension of the rule, distinct from the symbolic structure by which the rule is expressed and the idea it encapsulates, but essential in teaching, learning, and using the rule.

In early sources, the rule was presented as a procedure for finding the value of x, given a, b, and c and an understanding of the roles they play in a transaction. For example, given that b shillings are paid for a ounces, if you want to find what must be paid for c ounces, multiply b and c and divide this number by a. This is a number of shillings, which must be expressed as a number of pounds, a number of shillings, and a number of pence before the solution is complete.

Smith (1958, p. 483) states that the name for the rule originated in India, though similar rules were stated in older materials from other places. Brahmagupta, who wrote around 630 CE, says (in translation):

> In the rule of three, argument, fruit and requisition [are names of the terms]: the first and last terms must be similar. Requisition, multiplied by the fruit, and divided by the argument is the produce. (Colebrooke, 1817, p. 287)

The procedure was described by al-Khwārizmī in his *Treatise on Algebra*, written early in the ninth century in Baghdad. The brief *Chapter on Transactions* (which is more or less independent of the other major chapters of the work) contains the following statement:

> Know that all transactions between people, be they sales, purchases, exchange, hire, or any others, take place according to two modes, and according to four numbers pronounced by the enquirer: the evaluative quantity, the rate, the price, and the evaluated quantity. . . . [A]mong these four numbers, three are always obvious and known, and one of them is unknown . . . You examine the three obvious numbers. Among them it is necessary that there be two, of which each is not proportional to its associate. You multiply [them] and divide the product by the other obvious number . . .; what you get is the unknown number sought . . . . (Rasheed, 2009, p. 196)

The words *evaluative quantity*, *rate*, *price*, and *evaluated quantity* are translations of Arabic words that were used to differentiate the roles of the numbers. In modern notation, according to the translator, these numbers stand in the following relationship:

$$\frac{\text{evaluative quantity}}{\text{rate}} = \frac{\text{evaluated quantity}}{\text{price}}.$$

The chapter contains only a few, trivial examples to illustrate the application of the rule. It seems to be a report on widely used commercial practices, rather than guide for teaching them.

Now we jump ahead several centuries and shift focus from the East to pre-Renaissance Italy. Here, between 1200 and 1300 CE, as the mercantile revolution gathered momentum, communal and independent schools grew up to meet the needs of increasing numbers of young men headed for commercial and civic careers. The mathematics required for commerce was taught in the abbaco schools, which first appeared in northern Italy after 1250 (Goldthwaite, 1972). Historian Jens Høyrup has argued that the curriculum of the abbaco schools was derived from a culture of practical mathematics based on the Hindu-Arabic system that was well established in northern Africa, Spain, and southern France by the twelfth century (Høyrup, 2005). Presumably, this had been carried from the East by the expansion of Islamic civilization.

Late in the twelfth century, the young Fibonacci traveled through northern Africa, absorbing the mathematics used there and recording it in his *Liber Abaci* (1202). Fibonacci's book is often cited as a source for the emerging abbaco curriculum, but Fibonacci clearly was not the only conduit (Høyrup, 2005). In the abbaco schools, boys (roughly) between the ages of 10 and 13 learned how to write numbers with Hindu-Arabic numerals; how to perform the basic algorithms for whole-number addition, subtraction, multiplication, and division; and how to calculate with fractions. After this, they studied "commercial mathematics (in varying order): the rule of three, monetary and metrological conversions, simple and composite interest and reduction to interest per day, partnership, simple and composite discounting, alloying, the technique of a single false position and area measurement" (Høyrup, 2014, page 120). In short, boys in the abbaco schools learned to cipher to the rule of three—plus some.

Chapter 8 of the *Liber Abaci* opens with a paragraph that echoes al-Khwārizmī:

> In all commercial transactions, four proportional numbers are always found, of which three are known, but the remaining one is unknown. The first of the three known numbers is the number of units sold, be they bundles, or weights, or measures. A bundle might be, for example, a hundred hides or a hundred goatskins, or similar things: a weight might be a *cantarum*, or a *centum*, or a *libra* or an *uncia*, or something similar. A measure might be a *metra* of oil, or a *sestario* of corn, or a *canne* of cloth. The second number is the price of the sale to which the first number refers, and it may be a quantity of denari, or of bezants, or of tareni or some other monetary unit. The third is another quantity of the same merchandise as in the sale, and the fourth is the unknown price [to be determined]. (Boncompagni, 1857, page 83; free translation by JJM, aided by Sigler, 2012)

The first example in the chapter asks, *"If 100 rolls cost 40 pounds, how many rolls can I buy for 2 pounds?"* (a roll is a unit of weight.) Solutions were found by means of what Fibonacci called the "Principal Method," which goes as follows. Write the number of items of the first sale in the upper right of a square and in the upper left write the price paid; in the lower left, write the price in the second sale and to the right, leave a blank:

40	100
2	?

Multiply the two numbers that lie in the ascending diagonal and then divide by the number in the upper left. The result is the price to be paid.

The simple problem in the previous paragraph illustrates the procedure, but it is not at all representative of the kinds of problems that Fibonacci discusses. In the third problem, for example, the price of 27 rolls is sought, given that 100 rolls sells for 13 pounds. This is found to be $3 + {}^{51}/_{100}$ pounds, by the Principal Method. Fibonacci goes on to express the result in the form that would be needed in an actual transaction, as a number of pounds plus a number of soldi plus a number of denari. There being 20 soldi in a pound and 12 denari in a soldo, the result is 3 pounds, 10 soldi, and $2 + {}^{4}/_{10}$ denari. Chapter 8 contains nearly 150 examples illustrating the rule and, in virtually all cases, the units of measure, and the monetary denominations require attention. Eventually, in some problems, four different units are used in stating the problem, and much of the effort in finding a solution goes into making the required conversions. The problems, we can assume, are typical of those that merchants encountered in an age when different regions had their own systems of weights and measures and their own coinage.

During the 1300s, the abbaco curriculum acquired a stable, durable form. From the earliest times, many abbaco masters prepared handwritten manuscripts recording problems and solutions, as they may have been used in instruction. About 250 of these survive in libraries all over the world (van Egmond, 1981). As the new technology for printing spread across Europe after 1450, printed textbooks in practical mathematics began to appear. It is said that the first of these was the *Treviso Arithmetic*, of 1478. (A translation of this anonymous work is in Swetz, 1987.) A large part of this text is devoted to examples of applications of the rule of three. As in Fibonacci's work, there is much attention to units, unit conversions, and expressing monetary amounts in mixed denominations.

As Renaissance culture spread north through Europe, the mathematical culture of the abbaco schools spread with it. The *Bamberger Rechenbuch* by Ulrich Wagner (1483) contains a section on the rule, though here it is called the golden rule. Robert Recorde's *The Ground of Artes* (1543), one of the earliest printed books on arithmetic in the English language, has a chapter entitled "The golden rule, and the backer rule with divers questions therto belongynge." Recorde's book does not appear to me to be a manual for instruction, but more an exposition for a literate audience. Only a few illustrative examples of the rule are provided, but as in the works already mentioned, they require careful attention to the roles of the numbers involved and to the required unit conversions.

The rule was featured in Cocker's famous *Arithmetick*, which first appeared in 1677. (Several editions can be viewed complete on Google Books.) In the 48th edition (1736), Chapter 10 is entitled *The Single Rule of Three Direct*. It begins on page 87 as follows:

> 1. The Rule of Three (not undeservedly called the Golden Rule) is, that by which we find out a fourth Number, in Proportion unto three given Numbers, so as this fourth Number that is sought may bear the same Rate, Reason, or Proportion to the third (given) Number, as the second doth to the first, from whence it is also called the Rule of Proportion.

A few paragraphs later, we read:

> 6. In the Rule of Three, the greatest Difficulty is to discover the Order of the 3 Terms of the Question propounded, *viz.*, which is the first, second and third; which that you may understand; observe that of the Three given Numbers, two always are of one Kind, and the other [is] of the same Kind, with the proportional Number that is sought . . .
>
> 7. . . . to find out the fourth number . . ., multiply the second Number by the third, and divide the Product thereof by the first . . .

Following a page of explanation, there are 15 examples worked in detail, each filling about a page. These are very much like the problems in the *Liber Abaci*, in that they require discerning the roles of the quantities, converting units, and expressing the answer in a form appropriate for trade. In the following example, *C.* stands for a hundredweight, which consists of 4 quarters (*qrs.*), each being 28 pounds (*l.*) in weight. A pound (money) (*l.*) is 20 shillings (*s.*).

> *Quest. 10.* If 3 *C.* 1 *qr.* 14 *l.* of Raisins cost 9 *l.* 9 *s.* what will 6 *C.* 3 *qrs.* 14 *l.* cost?

Thomas Dilworth's *Schoolmaster's Assistant*, first published in London in 1743 and surely influenced by Cocker, became one of the most popular early arithmetic texts in the United States, with numerous North American printings between 1770 and 1820. Dilworth begins his presentation of the rule of three with the following catechism:

> Q. *By what is the* Single Rule *of* Three *known?*
> A. By *three Terms*, which are always given in the Question to find a *Fourth*.
> ...
> Q. *What do you observe concerning the* first *and* third Terms?
> A. They must be of the same Name and Kind.
> Q. *What do you observe concerning the* fourth Term?
> A. It must be of the same Name and Kind with the *Second*.
> ...
> Q. *How is the fourth* Term *in* Direct Proportion *found?*
> A. By multiplying the second and third Terms together and dividing that Product by the first Term.

The first problem following the instructional part reads, "If 3 *Oz*. of silver cost 17*s*. what will 48 *Oz*. cost?" The answer is worked out by multiplying 48 and 17 to get 816 and then dividing by 3 to get 272. This number of shillings is then converted to pounds and shillings by dividing by 20 (the number of shillings in a pound) to get the final answer: 13*l*. and 12*s*. The questions and answers from Dilworth quoted above are written out verbatim in Lincoln's ciphering book. Here we also read in Lincoln's hand the statement of the problem of the 3 *Oz*. of silver and its solution, as well as several other problems from Dilworth. Lincoln's teacher must have been using a copy of the *Schoolmaster's Assistant*. (Much more information about Lincoln's mathematics education, and his ciphering book in particular, can be found in Ellerton and Clements, 2014.)

In the late nineteenth century, schoolbooks began incorporating modern algebraic notation; see White (1870), for example. Rather than a cipher with four numbers, students would write an equation between two ratios, e.g., $12/30 = 42/x$. In a 1921 manual for teachers (Klapper, 1921), we read:

> A proportion is merely one method of writing a simple equation, and with the use of the letter x allowed, the equation form is likely to replace that of proportion. ... For example, consider this problem: If a shrub 4 ft. high casts a shadow 6 ft. long at a time that a tree casts one 54 ft. long, how high is the tree? Here we may write a proportion in the form
>
> $$6\text{ ft.} : 4\text{ ft.} = 54\text{ ft.} : (?),$$
>
> not attempting to explain it, but applying only an arbitrary rule. This is the old plan. Or we may put the work into equation form,
>
> $$\frac{x}{54} = \frac{4}{6},$$
>
> and deduce the rule for dividing the product of the means by the given extreme ... (page 183)

No less than before, students need to discern the roles of the various numbers in order to place them in the appropriate graphical schema. Operationally, the method remains as it has been all along. The connection to algebra clarifies some points that would not have been evident from the rule for the manipulation of numbers. For example, various cancelations that are justified by the algebraic content can be used to simplify the calculation.

We have seen the use of the word "proportion" in the historical sources. The quotes from Cocker (and other remarks made by Cocker in the chapter we looked at) give the impression that he regarded proportion as a more theoretical topic that provided the justification for the rule of three. By the time of Cocker, Euclid was widely studied and written about by English scholars, and the connections to practical mathematics were probably recognized. The language in the teachers' manual suggests that the author understood the phrase "a proportion" to refer a problem of the type that the rule of three was meant to dispatch and that he expected students to deal with such problems by well-practiced but poorly grasped routines.

Looking back, we see that the rule of three has been a robust schema that for hundreds of years has been a stable part of the school mathematics experience. There have been changes in appearance, including more prominent reference to proportion and attempts to link the notation to algebra. Yet even today, if one searches the web for problems on ratio and proportion, much of what one finds reflects the ancient patterns with modern adaptations. The following comes from the Khan Academy web site: "Pamela drove her car 99 kilometers and used 9 liters of fuel. She wants to know how many kilometers (k) she can drive with 12 liters of fuel."

In the Common Core Standards for Mathematics, the rule of three is not mentioned, nor are the manipulations associated with it. In grade 6, students represent and reason about ratios and collections of equivalent ratios, and in grade 7 they learn to recognize proportional relationships between varying quantities and to represent them with an equation of the form $y = kx$, where k is a constant. The standards shift away from setting up and solving *proportions*, i.e., equations of the form $\frac{A}{B} = \frac{C}{x}$ with A, B, and C constants, and focus on *proportional relationships*, i.e., the relationships between variables x and y that are expressed by $y = kx$. It is not my intention to describe the vision of the new standards here, but I would like to draw attention to the fact that some observers of teachers have noted that the rule of three schema seems to have a powerful hold on the thinking of many, to the extent of creating the suspicion that it hinders the understanding of *proportional relationships* in the manner suggested by the standards; see (Stanley, 2014).

5.3 Euclidean Ratio and Proportion

A completely different tradition concerning proportion springs from Greek mathematics. This is described in Book V of Euclid's *Elements*. The terms A and B in the Euclidean ratio of A to B are not numbers but things, classically called "magnitudes." Lengths, areas, weights, and temporal durations are kinds of magnitudes. When forming a Euclidean ratio, the magnitudes must be of the same kind; otherwise, the sort of direct comparison required to make a ratio is not possible. According to Plato, understanding this was basic knowledge; those who believed that a line might measure a surface, or a surface a volume, exhibited an ignorance "more worthy of a stupid beast like the hog than of a human being" (*Laws*, 819d, A. E. Taylor, trans.).

The ancient Greek concept of number was fundamentally different from ours. In Greek mathematics, *number—ἀριθμός (arithmos)*—referred to a multitude composed of units. This idea encompassed the counting numbers 2, 3, 4, ... but none of the other things that today we call number. On the other hand, the ancient Greeks understood there to be nonnumerical quantities of many kinds: line segments, areas, volumes, weights, etc. The generic term for these was *μεγέθη (megethe)*, typically translated as *magnitudes*. These were not numbers and were not associated with numbers, but nonetheless one could operate upon magnitudes of a given kind by doing some (but not all) of the things we do with numbers. For example, given two different magnitudes of the same kind, one could determine which is larger by direct juxtaposition, or one could add them together by manipulations specific to the kind: in the case of lengths, by placing them end to end in a straight line; for polygonal areas, by cutting along lines and joining along edges; and for volumes of a liquid, by putting them in a single container. Most importantly (for us), given two magnitudes of the same kind, one could form a *ratio—λόγος (logos)*—between them. Magnitudes forming equivalent ratios were said to be *in proportion—ἀνάλογον (analogon)*. Because Greek ratios are not formed from numbers, but from magnitudes, the meaning of ratio and proportion in ancient Greek thought was different from present-day schoolbook notions, but it is nonetheless relevant to the modern curriculum in some unexpected ways, especially in measurement and in understanding quantity concepts.

Euclid's *Elements* was influential in European mathematics from the late Middle Ages. Translations into the vernacular languages of Europe were made in the sixteenth century, and in the seventeenth century, the study of Euclid was basic to a scientific education. During the seventeenth century, some English mathematicians strove to blend the Euclidean framework with the more modern number concepts that were then emerging. John Wallis (1685, page 79) described the idea of a Euclidean ratio as follows (with italics as in the original):

> [The] whole definition of λόγος (*Ratio, Rate*, or *Proportion*) ... [is] *that Relation of two Homogeneous Magnitudes* (or Magnitudes of the same kind,) *how the one stands related to the other, as to the* (Quotient, or) *Quantuplicity*: That is, *How many times*, (or How much of a time, or times,) *one of them contains the other*. The *English* word *How-many-fold*, doth in part answer it, ... but because beside these which are properly called *Multiple* or *Many-fold*, (such as the *Double, Treble*, &c. which are denominated by whole Numbers,) there be many others to be denominated by Fractions, (proper or improper,) or Surds, or otherwise; ... to which would answer (in *English*,) *How-much-fold*, (if we had such a word) ...

Ratio in the sense described here is not a relationship between numbers but is the means by which we pass from magnitudes of a nonnumerical kind to numbers.

Wallis was explicit on this point: “When a comparison in terms of ratio is made, the resultant ratio often (namely with the exception of the ‘numerical genus’ itself) leaves the genus of quantities compared, and passes into the numerical genus, whatever the genus of quantities compared may have been” (Wallis, 1968). This, of course, is not something that Euclid said; it is a new spin on Euclid, made possible by the new conception of number. The closest analog we have in modern thought to the Euclidean ratio of *A* to *B* is the *measure of A by B*. Indeed, Otto Hölder referred to Euclid in proposing an axiomatization for measurement in his fundamental paper of 1901 (Hölder, 1901). We shall return to Hölder’s work in Sect. 5.5.

One of the most important applications of ratio in the *Elements* occurs at the beginning of Book VI, where Euclid shows that the ratio of the areas of two triangles with the same altitude is equal to the ratio of their bases. Although Euclidean ratios are relationships between magnitudes of the same kind, Euclid can compare a ratio between things of one kind to a ratio between things of another. The famous criterion for sameness of ratio is given in Definition 5 of Book V and is recognized as a precursor of the modern definition of real number. (We will say more about Definition 5 in §5, below.) As we have already mentioned, the term that Euclid used to describe equal ratios is ἀνάλογον, which is translated into English as “in proportion.” The term is introduced in Definition 6: “Magnitudes which have the same ratio are said to be in proportion.” That is to say, when the ratio of *A* to *B* is the same as the ratio of *C* to *D*, we say that the magnitudes form a *proportion*. (As we can see in the first quote from Wallis, the word “proportion” has in addition been used to refer to ratios.)

There is a powerful tradition related to Euclid’s definitions. If one looks for “ratio and proportion” on the Internet, one finds numerous statements along the lines of the following:

> The ratio of two quantities of the same kind is the quotient of their measures. . . . An equality of two ratios is called a proportion. (1977, p. 38)

The influence of the Euclidean paradigm is evident, but the use of measurement to pass to numbers *before* taking ratios is a modern twist and a very peculiar—if not incoherent—one if viewed from a Euclidean perspective. Measurement itself is the formation of a ratio between nonnumerical inputs. Therefore, we cannot explain what a ratio is by reference to measurement. To do so would be circular. An orthodox modern Euclidean would explain the meaning of “in proportion” in the following way: “If the measure of *A* by *B* is the same as the measure of *C* by *D*, we say that the four quantities are in proportion.” To repeat, the quantities *A*, *B*, *C*, and *D* are *not* themselves numbers, and no one of them is naturally associated with any number. It is only the measure of *A* by *B* and the measure of *C* by *D* that can be thought of as numbers.

At this point, some of the statements made in this section may seem obscure. We will elaborate and clarify in the following sections.

5.4 Mathematicians on School Math

In recent years, many mathematicians have commented on the meaning of ratio in school mathematics. They seldom mention the Euclidean conception or take it seriously. The following passages are from texts or papers by mathematicians. Because I am quoting out of context, these snippets may not communicate accurately the intent of the author. Therefore I make no attributions. Regardless of the author's intent, these passages contain ideas about ratio that I think we can recognize and identify in many discussions of school math.

> Ratios are essentially just fractions, and understanding and working with ratios and proportions really just involves understanding and working with multiplication, division, and fractions. ... To say that two quantities are in a ratio A to B means that for every A units of the first quantity there are B units of the second quantity.

> By definition, given two ... [numbers] A and B, where $B \neq 0$ and both refer to the same unit (i.e., they are points on the same number line), the ratio of A to B, sometimes denoted by $A{:}B$, is the ... [number] A/B.

> We say that the ratio between two quantities is $A : B$ if there is a unit so that the first quantity measures A units and the second measures B units. ... Two ratios are equivalent if one is obtained from the other by multiplying or dividing all the measurements by the same nonzero number. ... A proportion is a statement that two ratios are equal.

In one way or another, the authors of these passages all say that we form a ratio out of a pair of numbers or that a ratio is nothing but a pair of numbers. Notice that in all three statements, A and B stand for *numbers*. The things themselves—what I have been calling the *magnitudes*—are mentioned but never named. If we take these statements seriously, the term "ratio" is not essential part of mathematical vocabulary, but rather it is a word used to signal that the numbers that are involved originate as the measures of two things whose relationship is of concern. The words quoted above are suggestive of the notion that the vocabulary of mathematics includes words for numbers, for sets of numbers, for arrays of numbers, for relationships between numbers, and for operations on numbers but does *not* include words that refer to things in the world.

The sciences other than mathematics take a different view. The quantities of physics are not labeled numbers but magnitudes much as conceptualized by Euclid. The basic magnitudes are length, mass, and time, and other magnitudes are composites of the basic magnitudes, e.g., velocity is length/time, acceleration is velocity/time, force is mass · acceleration, energy is length · force, and power is energy/time. If a unit is chosen for each basic magnitude, then each instance of each magnitude has an associated number. But in physics, it is more productive to reason with the magnitudes than with the numbers assigned to them through a choice of unit. This is the position advocated in many physics textbooks. Physicist Sanjoy Mahajan explains as follows; see Mahajan (2010, page 4). The inclusion of units, such as feet or feet per second in a problem about a falling body, he says, "creates a significant problem. Because [if we are given that] the height is h feet, the variable h does not contain the units of height: h is therefore dimensionless." If the other variables in the given problem are also numerical, then they are also dimensionless, and likewise any combination of them is dimensionless. Consequently, no combination is favored. However, the kinds of the given quantities can guide us—and indeed they *will* guide us—if we use variables to stand directly for magnitudes. We should not pose the problem of a falling body by asking for "the number v of feet per second that the body is moving after falling h feet, given the acceleration a in feet per s^2." Instead, we should understand each variable to stand for a quantity with a kind (or "dimension"), and we should recognize that we may only combine and compare magnitudes in a manner that is consistent with their kinds. We benefit thereby, because the physical meaning is built in to the terms with which we reason. If we ask, "What is the velocity v after falling a distance h, given the acceleration a," then evidently, the only magnitude we can compound from h and a that has the same genus as v is the square root of $h \cdot a$, and so we can expect v to be proportional to the square root of $h \cdot a$. We double the velocity by increasing the height by a factor of 4. Readers of Newton's *Principia* will find it filled with passages where the reasoning is of this kind but far more sophisticated. Newton's arguments about complex proportional relationships typically do not mention units or numbers. He uses units and numbers only when presenting experimental data.

5.5 Euclidean Magnitudes and Measurement

Above, we quoted Wallis saying that when we take the ratio of two magnitudes (which need not be numbers) we create a number. Two centuries after Wallis, Hölder provided a systematic elaboration of this idea. In this section, I will give a simplified account of what Hölder said. This is based on a set of basic ideas about magnitudes that are implicit in Book V of the *Elements*, together with some modern ideas about number.

From the presentation in Book V, we can infer that Euclid assumed several things about the members of each specific kind of magnitude. Hölder carefully disentangled these assumptions and called them the axioms for measurement. We can express them as follows:

(1) *Compare.* Given two objects of one kind, either they are equivalent (as members of their kind), or if they are not equivalent, then one is larger than the other. Moreover, if A is larger than B and B is larger than C, then A is larger than C.
(2) *Add and subtract.* Given two objects of a kind, we can add them to make a larger thing of the same kind. For example, we can put things with length end to end, or we can bind two masses together, etc. (We are not adding numbers! We are operating directly on things, much as first-graders do in some curricula before they ever learn to make measurements and represent the sizes of things with numbers; see the contribution of H. Bass to the present volume.) A smaller

magnitude may be removed from a larger one of the same kind. Moreover, addition and subtraction of magnitudes have the following properties:

(a) Addition is not sensitive to the order in which the parts are joined or assembled (i.e., it is associative and commutative).
(b) Subtraction is the inverse of addition; that is to say, if we add B to A and then subtract it, then we get back to A. And if we subtract B from A and then add it back, we get back to A.
(c) Adding the same magnitude to two others preserves their order. In other words, if A is less than B, then $A + C$ is less than $B + C$. The same is true of subtraction; if A is less than B, then $A - C$ is less than $B - C$. If A is equivalent to B and the same magnitude is added to—or subtracted from—both, then the resulting magnitudes are equivalent.

(3) *Duplicate and form integer multiples.* We can make copies of a magnitude—as many as we like. We may add two, three, or any number of duplicates of a given magnitude to itself and thus double, or triple, or form any multiple we please of the given magnitude.

Remark Let us stop for a moment to note some important consequences of the first three items. If A is a magnitude and we add together m copies of A, we call the result mA. The properties of addition imply that *if* $A < B$ (respectively, $A = B$, $A > B$), *then* $mA < mB$ (respectively, $mA = mB$, $mA > mB$) for all m. By assumption (1), for any A and B, exactly one of the conditions $A < B$, $A = B$, $A > B$ holds. Therefore, *if* $mA < mB$ (respectively, $mA = mB$, $mA > mB$) for any particular m, *then* $A < B$ (respectively, $A = B$, $A > B$).

Infinitesimal magnitudes had no role in the Euclidean theory of ratio. On this point, Euclid was explicit. Definition 4 states, “Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.” Accordingly, we add the Archimedean axiom to the list of properties that the magnitudes of a given kind must possess:

(4) Given a lesser and a greater magnitude, some multiple of the lesser exceeds the greater.

If we take conditions (1)–(4) together, they form a system of axioms. As we have said, they were first isolated by Hölder in (1901). Today, mathematicians will recognize them as an informal statement of the axioms for the positive part of an Archimedean totally ordered group. Math educators, on the other hand, will see here a collection of ideas that are closely related to the sequence of developmental benchmarks that children attain in mastering measurement. By age 5, children are able to identify measurable attributes, such as length and weight, to compare things with respect to length or weight and to use representations to make comparisons between objects that cannot be compared directly. After this, they acquire the ability to put several things in order with respect to a measurable attribute that they all share and to build up varying lengths by laying units end to end (or varying weights by combining weights in an appropriate way). Following this, the ability to compare and add are elaborated and refined, while the idea of using a number of identical units to represent an arbitrary length (or weight) develops (Sarama & Clements, 2009, pp. 289–292). The Common Core proposes standards for measurement in grades K-5 that reflect these stages. In grades K-2, children work with materials that directly mirror the abstract attributes of magnitudes that we listed.

At this point, we can continue the exposition in two ways. One way will be agreeable to mathematicians. It is brief and it states the mathematical content with great efficiency, but it is likely to be meaningless to many readers. The other way will be accessible to patient readers who have a modest mathematical background. It reveals historical connections and elaborates notions in the present-day curriculum concerning measurement, ratio, and proportion in interesting ways. We will go quickly through the first way and go carefully through the second.

For mathematicians, the heart of the matter is Hölder's theorem, which says the following. Suppose $\mathcal{G}$ is an Archimedean totally ordered group. (We will write $\mathcal{G}$ in additive notation.) Let $0 < B \in \mathcal{G}$. For each $0 < A \in \mathcal{G}$, define the real number $[A{:}B]$ by the following rule:

$$[A{:}B] := \sup\left\{ \frac{n}{m} \;\middle|\; mA \geq nB\,;\ m, n \in \mathbb{N} \setminus \{0\} \right\} \in \mathbb{R},$$

where sup means supremum, i.e., least upper bound. Evidently $[B:B] = 1$. It can be shown that for all $A, C \in \mathcal{G}_{>0}$, the following are true:

(i) $A \leq C \;\Rightarrow\; [A{:}B] \leq [C{:}B]$;
(ii) $[A + C{:}B] = [A{:}B] + [C{:}B]$.

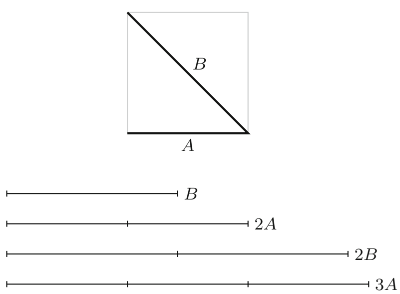
Furthermore, $A \mapsto [A:B]$ has a unique extension to an injective order-preserving group homomorphism from $\mathcal{G}$ to the additive real numbers. (Interestingly, Hölder's theorem does not require the hypothesis that $\mathcal{G}$ be commutative—the commutative property for $\mathcal{G}$ is implied by the other hypotheses; for background and a complete proof, the reader may consult (Bigard et al., 1977, pp. 48–50); see also Madden (2008) for elaborations relevant to measurement.) Notice that if $mA = nB$ for some positive integers m, n, then $[A : B] = n/m$. If there are no positive integers m, n such that $mA = nB$, then $[A : B]$ is an irrational number. We never assumed that a magnitude could be divided into equal parts—that is to say, we did not assume $\mathcal{G}$ to be divisible. If $\mathcal{G}$ *is* divisible, then $[A : B]$ is simply the supremum of the set of positive rational numbers q such that $qB \leq A$. In view of this, $[A : B]$ may reasonably be called "the ratio of A to B" or "the measure of A by B," because it has the properties that we expect of these things. In particular, to recall the words of Wallis, $[A{:}B]$ answers *How-much-fold* of B there is in A.

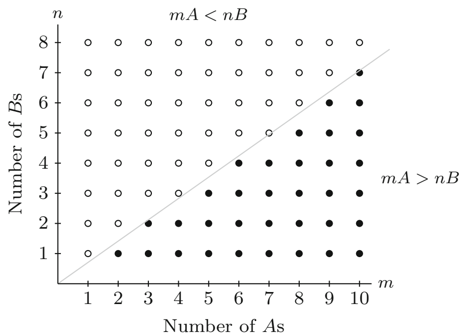
Now, let us examine the same material from a less technical vantage point. For concreteness, we will concentrate on lengths. Recall that we can (in principle) compare *any* two lengths by putting them side by side, lining them up at one end, and observing which goes further. If your pencil and mine line up at both ends, then *as lengths* they are the same. (Of course, the two pencils are different physical objects, but when we are concerned with them as lengths, this difference makes no difference; a philosopher might say that lengths are "equivalence classes" of objects of experience.) We can add any two lengths by putting them end to end. We can form multiples by duplicating and adding repeatedly. We can do all this with no need to measure or to assign numbers to the lengths, and these operations are well-behaved in the sense that the assumptions above are true of them. It is precisely because we can do these things, and because the outcomes are so governed, that we can form and compare ratios. How so? This will take some space to explain.

Suppose A and B are lengths. If we form a multiple of A and a multiple of B, then we can compare those multiples. Either they will be the same (with respect to length) or one will be larger than the other. Further, we need not stop with a single pair of multiples. We may consider *all* pairs, mA and nB, where m and n are allowed to range over all whole numbers. As we shall see, when we reason about the ratio of A to B, we must consider all such pairs of multiples.

Imagine all possible number pairs arrayed as a grid in the first quadrant of the coordinate plane, where (m, n) is the point lying m steps to the right of the origin and n steps above it. For each pair (m, n), just one of the following is true: $mA < nB$ or $mA = nB$ or $mA > nB$. Let us decorate the grid points according to which of the options we find. If $mA < nB$, we draw an open circle at (m, n). If $mA = nB$, we draw a red dot at (m, n). If $mA > nB$, we draw a black dot at (m, n). Note that we use a *square* grid—the horizontal steps are the same size as the vertical ones. Our grid is being used to record and label the number pairs only. We do not use the multiples of A and B in laying out the grid. We refer to the magnitudes only in deciding how to decorate the points with circles or dots.

The picture below shows the result of following this rule when A is the side of a square and B is its diagonal. The point $(2, 1)$ in the grid is black, because two As placed end-to-end exceed one B. Similarly, the point $(3, 2)$ is black, because three As exceed two Bs. We have drawn a light gray line through the origin in such a manner that it separates the black dots from the open circles. The point at $(7, 5)$ is above the line because $7A < 5B$ (though the circle around it happens to touch the line). There are no red dots in this picture, nor will there be any if the picture is extended, because we can never find a multiple of A that is equal to a multiple of B.





According to Euclid, the classification of the number pairs (m, n), which is illustrated in the grid diagram, tells us all there is to know about the ratio of A to B. That is, *if* we know *for which* (m, n) *it is true that* $mA < nB$, *and* we know *for which* (m, n) *it is true that* $mA = nB$, *and* we know *for which* (m, n) *it is true that* $mA > nB$, *then* we know *everything* about the ratio of A to B. All this is recorded in the diagram of dots and circles, since every (m, n) eventually gets marked with a circle or a red dot or a black one as the diagram is extended. Euclid says that if A and B are magnitudes of the same kind and C and D are magnitudes of the same kind as one another (but possibly of a different kind than A and B), then the ratio of A to B is the same as the ratio of C to D if and only if the diagram for A and B is the same as the diagram of C and D. This explains the meaning of the famous (and famously obscure) Definition 5 of Book V:

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Returning to our exposition of Hölder's ideas, we will use diagrams in place of the formal reasoning he employed. We will show that *every ratio of magnitudes has a real number associated with it in a canonical way*. We begin by listing some things that follow from conditions (1)–(4) concerning the diagram for a pair of magnitudes A, B. These are all things that Euclid would have understood, though of course he did not use dot diagrams.

First, if we draw a line through the origin and a point (m, n), then all the grid points on that line will be decorated in the same way—if one is circled, then they all are; if one is colored black (respectively, red), then they all are. This follows from the remarks after (3).

Second, in any diagram, there will be some circles and some black dots. This follows from (4), since given any m, there will be some n such that $mA > nB$. Thus, every column will have some circles. Symmetrically, given any n, there will be some m such that $nB < mA$. Thus, every row will have some black dots.

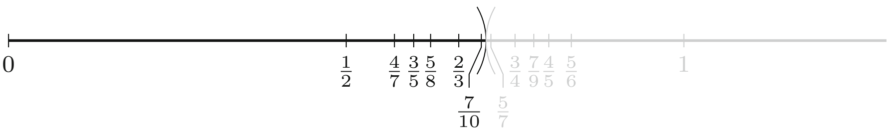
Third, if a line through the origin passes through a black grid point, then all the grid points below this line are black. Similarly, if a line through the origin passes through a circled grid point, then all the grid points above this line are circled. To see this, suppose (m, n) is black and ℓ is the line through $(0, 0)$ and (m, n). Suppose (m', n') lies below ℓ. Then $(m\,n', n\,n')$ lies on ℓ, and $(m'\,n, n'\,n)$ lies below $(m\,n', n\,n')$ on the same vertical line, so it's black. Since (m', n') and $(m'\,n, n'\,n)$ lie on the same line through the origin, the former is also black. The claim about circled points is seen by a similar argument.

Fourth, if a line through the origin passes through a red grid point, then all the grid points below that line are black and all the grid points above that line are circled. This can be seen by the same reasoning used in the previous observation.

Let us consider what happens if $mA = nB$ for some (m, n). In this case, the line through $(0, 0)$ and (m, n) marks the boundary between the black points and the circled points. Of course, all the grid points that lie on this line are red. According to

Euclid's Definition 5, the ratio of the numerical quantity n to the numerical quantity m is the same as the ratio of A to B. The reason for this is that the diagram for A and B is the same as the diagram for n and m since both diagrams have a red dot at (m,n). In this case, we associate the rational number n/m with the ratio of A to B. This takes care of the ratios that, as Wallis said, are "denominated by whole Numbers ... [or] by Fractions, (proper or improper)."

Now, let us consider what happens if $mA \neq nB$ for all (m,n). In this case, we need a different approach. With each grid point (m,n), we can associate the fractional number n/m. We may color the points on the fraction line in conformity with the decorations on the grid: color n/m black if the grid point (m,n) is black and color n/m gray if the grid point (m,n) is circled. Since $mA \neq nB$ for all (m,n), no points will be red. The first observation above shows that there is no ambiguity in the way we assign colorings. Condition (1) assures that every grid point is decorated, and therefore every positive fraction will be either black or gray. The second observation shows that some positive fractions will be colored black and some will be gray. Finally, the third observation shows that if n/m is colored black, then every positive fraction less than n/m will also be black, while if n/m is colored gray, then every positive fraction greater than n/m will also be gray. In particular every black number is less than every gray number. Hence, the decorated fraction line will appear as in the picture below. Here, we have marked the fractions corresponding to the grid points closest to the line we drew in the previous diagram.



The last step appeals to the modern definition of the real number system. A partition of the (positive) rational numbers into two sets with the properties of the black and gray sets above is called a Dedekind cut. To be precise, a Dedekind cut is a coloring of the rational numbers by two colors—black and gray, say—in such a manner that every rational number is colored, some rational numbers are black and some are gray and every black number is less than every gray number. The real number system has the property that for any Dedekind cut, there is a unique real number that is greater than or equal to each rational number colored black and less than or equal to each rational number colored gray. This is the number we associate with the ratio of A to B.

This has been rather long-winded, and essentially it has brought us to the definition of the function $A \mapsto [A : B]$ that we made above in a single line. On the other hand, the ideas and imagery that have entered this discussion might conceivably be incorporated in actual curriculum materials. For example, imagine a lab experiment where we attempt to measure the weight of a 10d common nail using a (lighter) 6d common nail as a unit. We might set up a balance and place various numbers of 10d nails in one pan and then add 6d nails to the other pan until the balance tips. We could record the data on a chart like what we made above, coloring the point (m, n) black if m 10d nails weigh more than n 6d nails. Specifically, beginning with a square grid of open circles in rows and columns labeled 1 through 20 (say), we could start by placing a 10*d* nail in one pan and then adding 6d nails to the other pan until the balance tips. Then we blacken the circles in the first column, up to the last one before the tipping point. Next, add a 10d nail to the first pan, and then add 6d nails to the other pan until tipping, and blacken dots in the second column by the same rule as the first. After moving through several columns, draw a line separating the black dots from the open ones. The slope is a good approximation of the measure of a 10d nail by a 6d one. The more columns we mark before making the line, the better the approximation, and since when there are m 10d nails in a pan, we are in essence measuring *one* 10d nail by mth parts of a 6d nail.

5.6 Conclusions

As we survey ideas related to ratio and proportion, a couple of things stand out. The first has to do with the meaning of the symbols that are used in reasoning about quantities. Even in the simplest of situations, such as the problem with the rolls, there is a problem in the world (How much should I receive for 2 pounds?), and there are symbols that we use to represent the problem and that we manipulate to find the answer. In several different contexts, we raised the question of whether the symbols refer to numbers or to objects in the world (or possibly to abstractions intermediate between the things we experience and the objects of the orthodox modern mathematical universe). In some respects, this does not seem to matter. The question might be dismissed as a philosophical concern with no implications for teaching, since it really makes no difference what the symbols mean in a metaphysical sense, but only what students do with them. But this assumes that the question makes no difference to the learners themselves. It very well might! The symbols that we use are present in our experience alongside everything else that we experience. That is, we are aware of the symbols themselves and are instinctively interested in how they work. When a child draws a picture, the picture itself becomes part of the world, and the child will speak about the picture, explain its parts, and develop and modify its meaning by talking about it (Woleck, 2001). In a manner that is not entirely different, learners are concerned about how meanings work in the symbol systems they use: "What refers to what? How do I recognize the connections? Why do I say or write this or that, and what does the result mean?" A good account of the meanings of things is not a philosophical indulgence but a solid support for student learning.

Laying out what the terms in a domain of knowledge refer to is a basic task of artificial intelligence. In order to develop a system for recording, filing, and systematically searching and retrieving medical information, for example, information engineers need a representation of the kinds of things that might be mentioned in a medical record and the kinds of relationships each might have to every other thing. A patient has a name, a date of birth, a weight, a pancreas, a prescription for eyeglasses, and innumerable other things. These things fall into classes and are related (or not) in ways dictated by the classes. Some things may change, some not. The weight may cause concern for the pancreas, but not for the eyeglasses. To sort these things out, the engineer will create what is called an ontology: a set of specifications about what there *is* in this knowledge domain, what the terminology refers to, and what properties and relations the objects may have to one another. The word "ontology" also has philosophical connections, but here we understand an ontology simply as a *very* explicit, practical specification of what a domain of discourse is about.

Our historical review of ratio and proportion has demonstrated that there are several competing ontologies for proportional reasoning. Up till now, no one has attempted to make the different competing ontologies fully explicit or to compare how the different alternatives might work out in a curriculum. The first step, clearly, should be to find an appropriate framework for sketching out the ontological alternatives. How to do this and how to put the final results to use are topics for future research.

The second thing that stands out is the intimate connection between measurement and proportional reasoning. It is interesting that in the Common Core Standards, the measurement and data domain spans kindergarten through grade 5, whence in sixth grade this domain vanishes and the ratio and proportional relationships domain appears. Reasoning with rates and proportions, I suggest, is more dependent upon the ability to understand the measurement process than widely acknowledged. The history of ratio and proportion bears this out. Of course, the ability to take measurements, calculate rates, put measure numbers into formulae, and "cancel units" at the appropriate times is important. But we need to attend to more than the mechanics. What is the *explanation* for a cancelation such as the following?

$$10{,}000\,\cancel{\text{feet}} \times \frac{0.3048\,\text{m}}{1\,\cancel{\text{foot}}} = 3048\,\text{m}$$

Perhaps you think that the words are just decorations to remind us that the 10,000 refers to feet, and the 0.3048 refers to meters. Or perhaps you prefer to think of the words as symbols for magnitudes that are here being multiplied by numbers. In either case, why is it that this cancelation procedure, which we have validated previously for numbers, can be carried over to this nonnumerical context? I cannot provide a complete rigorous answer that could be grasped in any seventh-grade classroom. I challenge readers to propose one. Most interesting proportional relationships involve heterogeneous quantities and a rate that relates the amount of one quantity in given units to the amount of the other, in other units. How do we change units—convert the driver's miles per hour to the runner's minutes per mile? We need a deep grasp of measurement in order to do this. It seems to me that the opportunity to produce a curriculum that ties measurement more closely to ratio and proportion is wide open and that the work to be done is great but has great potential.

I would like to close with some remarks of a broader nature. What teachers know and the knowledge that they value depends upon the knowledge and the values that are distributed throughout the systems that support teachers and teaching. The authors of textbooks; the people who train, observe, and evaluate math teachers; the people who develop and promote school policies; the people who compile standards; and the people who design and evaluate tests—all of them use special forms of mathematical knowledge and have their own mathematical priorities. At the system level, as opposed to the level of the individual teacher, "mathematical knowledge for teaching" becomes a matrix of meanings, understandings, habits of mind, and values that circulate among individuals in different roles in the organizations, agencies, and institutions that impact teaching. At this level, "mathematical knowledge for teaching" is more of a cultural entity than the set of understandings and abilities that we might find, or fail to find, in an individual. Culture is an emergent social phenomenon, not what is in someone's head.

Until the end of the twentieth century, the most powerful influencers of this culture were probably the traditions within the teaching community, the textbook writers and publishers, the professional organizations for teachers, and the university programs that prepared teachers. Textbooks, as concrete records of practice, were surely very influential. In the past several decades, new forces have come on the scene: the various systems of standards (created by the NCTM, the states themselves, and now the producers of the Common Core), the massive high-stake testing programs resulting from federal legislation, and increased use of test data in teacher evaluation. Within the last few years, there has been an explosion in the availability of curriculum materials on the Internet.

The scholarly discipline of mathematics has always been about creating and describing efficient, coherent systems of ideas. The same mindset ought to be applicable to school mathematics. It should be possible to lay out the content of school mathematics in good mathematical style, with rigorous definitions, clear logic, and appropriate, unambiguous symbolism. Roger Howe's essays on topics in school mathematics are examples of this. Historically, however, mathematicians have not been the chief architects of school mathematics—it has had no chief architects. It has developed like the ancient cities that Descartes contemplated in the *Discourse on Method*, which "from being at first only villages, have become, in course of time, large towns" and which, as a consequence, are "usually but ill-laid out compared with the regularly constructed towns which a professional architect has freely planned on an open plain." He added that "it is not customary to pull down all the houses of a town with the single design of rebuilding them differently, and thereby rendering the streets more handsome . . .," and similarly, it would be "preposterous for a private individual to think of reforming a state by fundamentally changing it throughout, and overturning it in order to set it up amended . . . [or to contemplate a] similar project for reforming the body of the Sciences, or the order of teaching them established in the Schools . . .."

The culture of the curriculum, as sustained by the institutions described above, is traditional and syncretic. For whatever reasons and by whatever mechanisms, this culture preserves patterns of expression and habits of thought, meeting pressure for change by absorbing and transforming what is newly thrust upon it, forcing new things into the spaces between old structures, or on top of them, or within them.

It mixes and juxtaposes ideas, in much the same way popular culture samples and remixes styles, cuisines, icons, and beliefs. Knowledge for teaching *as it is* at the present time, rather than *as we might wish it to be*, resembles what knowledge for healing was 150 years ago: a mixture of folkways, craft wisdom, and science, shaped as much by social influence as by reason. To change it one would need to change the institutional conditions around teaching, ... but all this is something to take up at another time. My main thesis here is that the knowledge for teaching that we have at present—no matter what anyone might envision as a replacement—is the result of cultural process spanning centuries. In many cases, the intellectual sources have been reasonable and coherent, though this is not always evident in the resulting hodgepodge. If we wish to replace what we presently have with something better, the first step should be to understand truly what we have.

I would like to end on an inspirational note that Dick Stanley mentioned to me. Descartes may have been quite right that it would be preposterous for an individual to undertake the rebuilding of a city. But the preposterous and the impossible are two different things. At the behest of Napoleon III, between 1853 and 1870, Georges-Eugène Haussmann led a massive renovation of Paris, tearing down vast tracts of ancient buildings and laying out the majestic city we know today. Though he was forced from his position as Prefect of the Seine in 1870 by political opponents, the project continued, reaching completion in 1927 with the opening of the Boulevard Haussmann. Might we, in the present century, achieve something analogous for the mathematics curriculum?

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Chapter 6
How Future Teachers Reasoned with Variable Parts and Strip Diagrams to Develop Equations for Proportional Relationships and Lines

Sybilla Beckmann and Torrey K. Kulow

Abstract Findings are presented from an analysis of how six future middle-grade teachers reasoned with strip diagrams and a variable parts perspective on proportional relationships to develop and explain equations in two variables. One equation was for two quantities varying together and one was for a line through the origin in a coordinate plane. Both equations involved a constant of proportionality that was not a whole number. The future teachers' arguments were mathematically valid and relied on reasoning quantitatively about strip diagrams. The arguments also treated variables as quantities but rarely described the variables as numbers of units. Some arguments combined an interpretation of fractions as multiples (or iterates) of unit fractions with an interpretation of multiplication as a whole number of equal groups. In contrast, most arguments involving a fractional multiplier interpreted multiplication as "of." All the points of tension that the future teachers encountered while developing their equations concerned referent units for quantities. The points of tension were resolved by focusing on referent units or on equality. Based on the data, extensions to current theories on reasoning quantitatively and with variables are proposed.

6.1 Background

Students use equations to model situations throughout their schooling, but it is only in the middle-school grades (grades 6, 7, 8) that they begin to use equations in two variables to model situations of quantities varying together. One especially important class of examples is proportional relationships, in which quantities vary

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together in a fixed ratio and therefore have a fixed multiplicative relationship. Two quantities that vary together in a proportional relationship can be modeled by a graph that is a line through the origin in a coordinate plane and by an equation of the form $y = \mathrm{m}x$, where m is a constant of proportionality and x and y are variables.

In this introduction, we first discuss how we see the study of proportional relationships as part of a bigger landscape of mathematical ideas and practices. We then discuss central definitions and framing ideas that we use in our courses for future teachers. Among these, one particular approach to proportional relationships—variable parts—has only recently been recognized in mathematics education research (Beckmann & Izsák, 2015). Using the variable parts perspective to generate and explain equations in two variables is the focus of the study we report on in this chapter.

6.1.1 Equations for Proportional Relationships as Part of a Multiplicative Conceptual Field

The domain of ratio and proportional relationships is known to be essential yet one of the most challenging to learn (e.g., Kilpatrick, Swafford, & Findell, 2001; Lamon, 2007). The National Mathematics Advisory Panel (2008) noted that the interrelated topics of fractions, decimals, percent, ratios, and proportional relationships provide a critical foundation for algebra (p. 18). The National Research Council (2012) listed "Scale, proportion, and quantity" as a crosscutting concept for science (p. 3) and stated that ratio relationships are key to forming mathematical models that interpret scientific data (p. 90). Yet, the National Center on Education and the Economy (2013) identified weak conceptual understanding of middle-school mathematics—especially arithmetic, ratios and proportions, and simple equations—as the most important obstacle to readiness for community college, where most vocational and technical education takes place and where many students begin 4-year college degrees.

We view proportional relationships, lines in the plane through the origin, and their equations as part of a *multiplicative conceptual field* (Vergnaud, 1983, 1988, 1994)—a web of interrelated ideas that also include whole-number multiplication and division, fractions, ratio, and rate. With such a view, we want students to connect and build on multiplicative ideas as their education progresses. For example, reasoning to develop equations in two variables for proportional relationships and lines could build on reasoning for solving missing-value proportion problems, which in turn could build on reasoning about multiplication and division with quantities. Consonant with this view, current recommendations include that students in kindergarten through grade 8 should develop their conceptual understanding for solving ratio, rate, and proportion problems before being exposed to cross multiplication as a procedure to use to solve such problems (Siegler et al., 2010) and that students in grade 6 should use ratio and rate reasoning to solve problems, for example, by reasoning with strip (tape) diagrams and double number lines (Common Core State Standards Initiative [CCSS], 2010).

More generally, the CCSS (2010) describe the kinds of mathematical reasoning students should engage in and what kinds of mathematical arguments students should develop. The CCSS Standard for Mathematical Practice 2, "Reason abstractly and quantitatively" (p. 6), describes mathematically proficient students as able to make sense of quantities and their relationships. The CCSS Standard for Mathematical Practice 3, "Construct viable arguments and critique the reasoning of others" (p. 6), describes mathematically proficient students as able to use stated assumptions, definitions, and previously established results in constructing arguments. Given the current recommendations and expectations for middle-grade students, their teachers also need opportunities to reason quantitatively and construct viable arguments that connect and explain ideas in the multiplicative conceptual field.

How then might middle-grade students and their teachers reason about ideas in the multiplicative conceptual field to develop equations in two variables for proportional relationships, including cases of lines through the origin? Such reasoning will draw on ideas about multiplication, division, fractions, variables, and equations, among others. Large bodies of research in mathematics education have investigated students' and, to a lesser extent, teachers' reasoning about these topics. We will not survey that research here, but in the following discussion, we provide the references upon which we draw directly.

6.1.2 Reasoning with Quantities

The CCSS Standard for Mathematical Practice 2, "Reason abstractly and quantitatively" (CCSS, 2010, p. 6), describes quantitative reasoning as creating coherent representations and attending to the meaning of quantities, not just how to compute them, among other attributes. In discussing the high school standards, the CCSS describe quantities as "numbers with units, which involves measurement" (p. 58), which is consistent with some other authors (e.g., Schwartz, 1988).

We take a slightly different view of quantity, close to Thompson's (1994), by not requiring any measurement unit to be specified or in mind. So we consider a person to be treating an entity as a quantity if they state or imply that the entity has a quality that is or could be described as some number of units. But we also consider measurement units themselves to be quantities. So if a person treats an entity as a unit with which to describe another entity numerically, for example, by taking a fraction of it or considering some number of copies of it, we consider a person to be treating both entities as quantities. For example, if a person describes and accurately explains one strip as 2/5 of another strip, then we consider the person to be treating the two strips as quantities. If the person treats x and y as standing for those strips and describes y as 2/5 of x, we consider the person to be treating x and y as quantities.

With our view of quantity, for a person to be treating a letter such as x as a quantity, they *need not* be viewing x as standing for some number (an unknown number or a number from a specified set). We believe this view of quantity is consistent with the definition of physical quantity given by the National Institute of Standards and Technology (NIST, n.d.). For example, NIST uses h_W to stand for the physical quantity "height of the Washington Monument," but the numerical value of h_W depends on a choice of unit. When expressed in meters, its numerical value is 169; when expressed in feet, its numerical value is 555.

6.1.3 Reasoning with Variables

Providing an adequate definition of "variable" is known to be difficult (Schoenfeld & Arcavi, 1988), but a variable is often understood as representing "an unknown number, or, depending on the purpose at hand, any number in a specified set" (CCSS, 2010, p. 44, Standard 6.EE.6). Mathematics education researchers have found that students conceive of variables in a variety of ways (for a recent summary, see Lucariello, Tine, & Ganley, 2014). Among them is the well-known (mis)conception of treating a variable as a shorthand label for an object or unit, for example, treating S as shorthand for *students* rather than as a number of students (see McNeil et al., 2010). This (mis)conception has been described as one source of errors in writing equations in two variables. For example, many college science students wrote the eq. $6S = P$ when given the information that there were six times as many students as professors at a university (Clement, 1982; Clement, Lochhead, & Monk, 1981).

Although some interpretations of variables as labels do lead to statements that are mathematically incorrect, we think this topic needs further examination. In particular, we claim it is possible to treat a variable as a label but in such a way that it also functions as a quantity, as we described above. We think that such uses of variables are not misconceptions and in fact may be a productive part of quantitative reasoning. Later in this chapter, we present examples showing how future teachers treated variables in that way and developed correct equations based on sound quantitative reasoning.

6.1.4 Interpreting Fractions

The CCSS (2010) define fractions in the grade 3 standard 3.NF.1 as follows, which we will refer to as "the CCSS definition of fraction":

> Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand the fraction a/b as the quantity formed by a parts of size 1/b. (p. 24)

In grade 4 standards 4.NF.3 and 4.NF.4, the CCSS ask students to understand fractions as sums or multiples of unit fractions (fractions with numerator 1). We therefore view the two-part CCSS fraction definition as describing fractions as iterates, sums, or multiples of unit fractions.

In our courses for future teachers, we use the CCSS definition of fraction but with an elaboration to draw attention to the fraction's referent whole. For example, we might describe 8/3 as the quantity formed by 8 parts, each of size 1/3 of the unit amount. There is some ambiguity in the language. Does it mean 8 parts, each of size (1/3 of the unit amount), or does it mean 8 parts, each of size 1/3, of the unit amount? One might think of the latter expression as highlighting that 8/3 stands for some number of units and the former expression as elaborating what exactly that number is. We want our future teachers to be able to think of 8/3 in both of those ways.

We note that there are other ways to define or think about fractions (for a summary, see Lamon, 2007). For example, one can define the fraction A/B with division, by thinking of it as the quantity in one share when A units are distributed equally among B shares. One might also think of the fraction A/B not as an expression for a single number but as a pair of numbers, i.e., the ratio A to B. When students use language such as "A out of B," it could be that they have a kind of ratio idea in mind.

6.1.5 Interpreting Multiplication

In our courses for future middle-grade teachers, we use a quantitative "equal groups" definition for multiplication that is consistent with the grade 3 CCSS Standard 3.OA.1 about whole-number multiplication (CCSS, 2010, p. 23). In a situation involving quantities, we say that

$$M \bullet N = P$$

if M is a number of equal groups, N is a number of units in one whole group (or each group), and P is the number of units in M groups in the situation. We call M the multiplier, N the multiplicand, and P the product (e.g., Greer, 1992). We define division as multiplication with an unknown multiplier or multiplicand.

Our quantitative equal groups definition applies not only to whole numbers but also to (nonnegative) fractions and decimals (e.g., Beckmann, 2014). We think it is important to have such a definition in our courses for future teachers so that (1) we can make the case that multiplication is a coherent operation across different kinds of numbers and (2) we have a tool for determining whether a problem is solved by multiplication.

We note that it is possible to use an equal groups *interpretation* of multiplication without explicitly invoking the equal groups *definition* of multiplication. For example, a person might reason that $2^2/_3 \bullet A = B$ because it takes $2^2/_3$ copies of A to make B, but they might not describe $2^2/_3$ as a number of groups or they might not describe A as the number of units in one group and B as the number of units in $2^2/_3$ groups.

Another way to think about multiplication when the multiplier is a fraction is as "of" (e.g., Boulet, 1998). For example, we might think of 1/2 • 1/3 as "1/2 of 1/3." Some authors treat an "of" interpretation (or sense) of multiplication as distinct from an equal groups interpretation (e.g., Greer, 1992, distinguished equal groups situations from part/whole situations, which are worded with "of"), and students and teachers may see them as separate as well. We consider an "of" interpretation of multiplication to follow from an equal groups interpretation because a shorthand way to describe M • N is "M groups of N," which we interpret as "M of N" when M is a fraction.

We acknowledge that it is possible to use the word "of" for multiplication in a mechanical, unthinking way, for example, by automatically inferring multiplication from the word "of," regardless of context. We do not agree with such a use of keywords in mathematics. For this reason it might seem better to avoid an "of" interpretation of multiplication altogether. However, as we will show, future middle-grade teachers were able to make sound use of an "of" interpretation of multiplication and use it to develop equations by reasoning about relationships between quantities.

6.1.6 The Variable Parts Perspective for Reasoning About Proportional Relationships

As Beckmann and Izsák (2015) discussed, the quantitative equal groups definition of multiplication can unify multiplication, division, and proportional and inversely proportional relationships. Furthermore, the definition leads to two ways of viewing proportional relationships: a *multiple batches* perspective, which has been widely studied, and a *variable parts* perspective, which had been largely overlooked in mathematics education research. Although the variable parts perspective is new in mathematics education research, it is implicit in some East Asian curricula (e.g., Fujii & Iitaka, 2012) and fits with the model method used in Singapore (e.g., Kaur, 2015; Ng & Lee, 2009).

Beckmann and Izsák (2014, 2015) have argued that a variable parts perspective may be especially valuable for developing equations for proportional relationships. We illustrate with the following fertilizer scenario:

> *Fertilizer scenario*: A type of fertilizer is made by mixing nitrogen and phosphate in an 8-to-3 ratio. Suppose you will use N kilograms of nitrogen and P kilograms of phosphate, where N and P are unspecified numbers of kilograms, which could vary.

From a variable parts perspective, we view the N kg of nitrogen as 8 parts and the P kg phosphate as 3 parts, where all the parts are the same size as

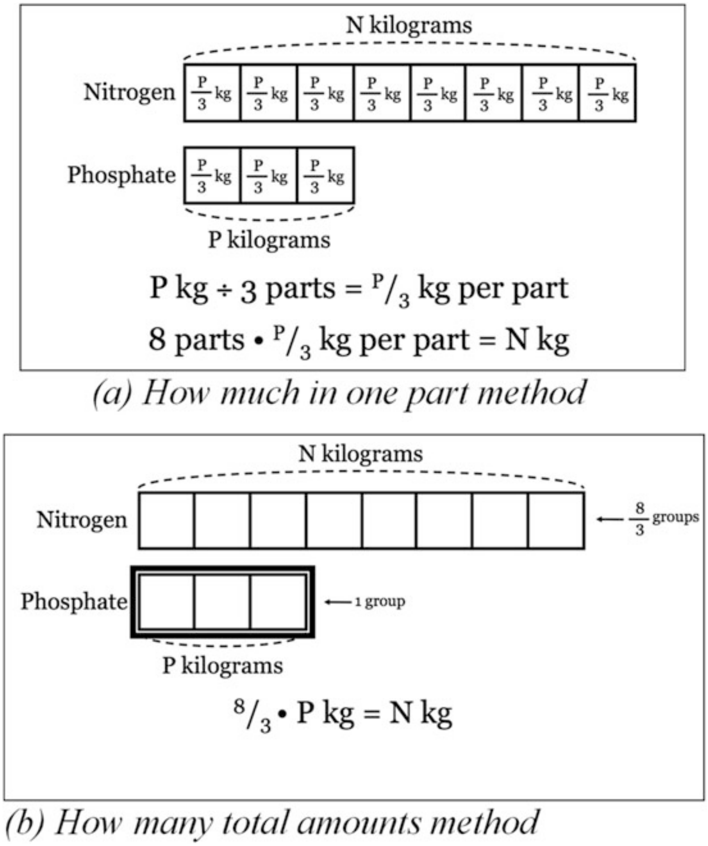


Fig. 6.1 Using the variable parts perspective to derive two equations relating *N* and *P* for the fertilizer scenario. (**a**) How much in one part method. (**b**) How many total amount method

each other, as indicated in the strip diagrams in Fig. 6.1. To develop equations relating *N* and *P*, we can reason about multiplication and division with a strip diagram in several different ways (Beckmann, Izsák, & Ölmez, 2015). With the "how much in one part" method (see Fig. 6.1a), we can first find that each part contains $^{P}/_{3}$ kilograms. There are 8 parts of nitrogen; therefore, according to the equal groups definition of multiplication, $8 \bullet {}^{P}/_{3} = N$. With the "how many total amounts" method (see Fig. 6.1b), we can treat the phosphate as 1 group of *P* kilograms. Because 3-part phosphate strip fits into the 8-part nitrogen strip $^{8}/_{3}$ times, the nitrogen is $^{8}/_{3}$ groups. Therefore, according to the equal groups definition of multiplication, $^{8}/_{3} \bullet \mathrm{P} = \mathrm{N}$. The strip diagram visually illustrates the constant relative magnitudes of the two quantities and makes the constant of proportionality, $^{8}/_{3}$, visually explicit.

6.1.7 The Variable Parts Perspective for Reasoning About Lines in a Coordinate Plane

Beckmann and Izsák (2014, 2015) have argued that a variable parts perspective may be especially valuable for situations involving geometric similarity, such as lines and their slope. For example, Fig. 6.2 shows two screenshots from a Geogebra sketch (see http://ggbm.at/UutLbwxl) that illustrates how we can view the points on the

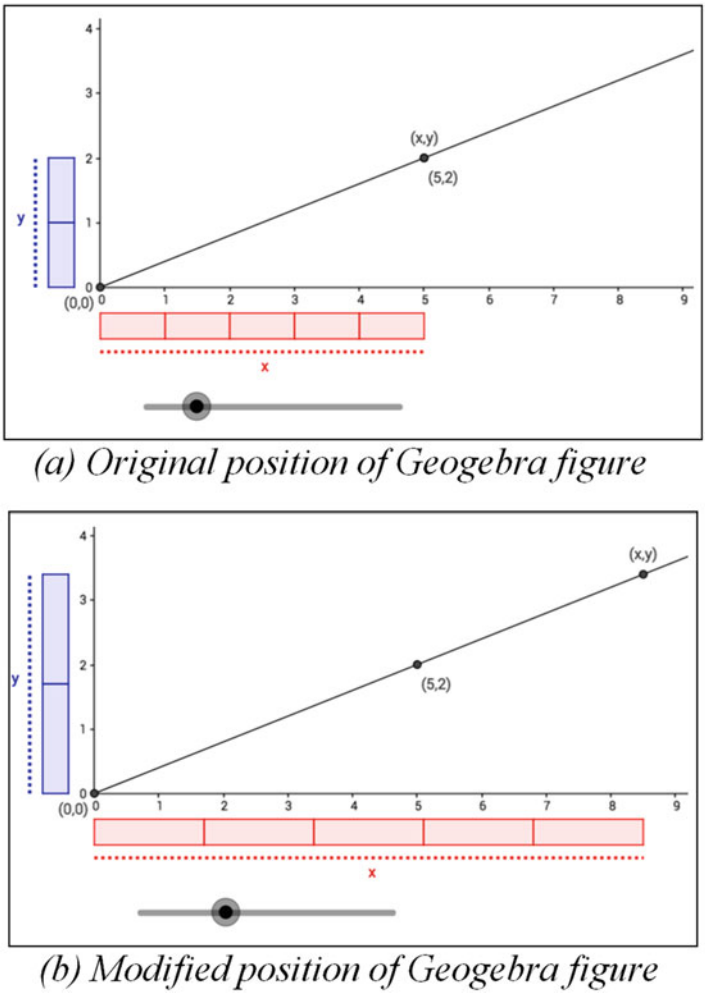


Fig. 6.2 A variable parts perspective on points on a line. (**a**) Original position of Geogebra figure. (**b**) Modified position of Geogebra figure

line through the origin and (5, 2) in terms of a variable parts perspective. The x- and y-coordinates of a point on the line are in the fixed 5-to-2 ratio. As a point (x, y) moves along the line, the 5 parts that make x and the 2 parts that make y all remain the same size as each other, but that size changes depending on the location of the point (which can be moved by moving the slider in the Geogebra sketch). The same reasoning as above, for the fertilizer scenario, applies for developing an equation for the line.

6.1.8 Research Questions

Given that the variable parts perspective has been proposed as potentially valuable for reasoning about proportional relationships, a first question is whether it is even a viable approach and, if so, what the reasoning looks like. We can ask this question for students as well as for teachers. In this report we consider the viability of a variable parts perspective for future middle-grade teachers. Can future middle-grade teachers learn to reason quantitatively with a variable parts perspective to generate equations in two variables? What are the characteristics and qualities of this reasoning? What ideas do the future teachers choose to use? What is hard or tricky? We report here on a small-scale investigation into these questions, as part of a larger, ongoing project.

In the rest of this chapter, we address the following research questions about six future middle-grade teachers who took courses that emphasized reasoning about multiplication and division with quantities and developed the variable parts (and multiple batches) perspectives on proportional relationships:

Research Question 1 What ideas, concepts, and ways of reasoning did the future teachers use as they reasoned from the variable parts perspective to develop and explain an equation the form constant • variable = variable for a proportional relationship?

Research Question 2 What ideas, concepts, and ways of reasoning did the future teachers use as they reasoned from the variable parts perspective to develop and explain an equation for a line through the origin in a coordinate plane?

Research Question 3 What points of tension did future teachers experience as they reasoned from a variable parts perspective to develop equations in two variables?

By ideas and concepts, we mean things such as the interpretation of multiplication as "of" or that fractions can tell us how much of something there is. We are interested not only in whether future teachers can produce and explain correct equations but also in what ideas and concepts they use to do so and how they assemble those ideas and concepts. It is only through a detailed, fine-grained understanding of reasoning that we can hope to design learning experiences that tap into ways of thinking that are available and productive.

6.2 Methods

Data for this report comes from an ongoing study of future teachers' reasoning about multiplication and division, fractions, and proportional relationships. As part of the broader study, the project team selected six individuals from a class of 22 future middle-grade mathematics teachers who took arithmetic content course offered in Fall 2014. These six individuals were selected to be mathematically diverse based on their performance on a fractions survey (Bradshaw, Izsák, Templin, & Jacobson, 2014). They were each interviewed two times during the arithmetic course in the Fall 2014 semester and four times during Spring 2015 while taking an algebra content course. Another project team member conducted all of the interviews.

Beckmann taught the arithmetic and the algebra course and used a textbook in both courses (Beckmann, 2014). Throughout both courses, Beckmann emphasized reasoning with the CCSS definition of fraction and the equal groups definition of multiplication (discussed above). In the arithmetic course, when the equal groups definition was extended from whole numbers to fractions, the class discussed the connection to the "of" interpretation of multiplication. The algebra course developed both the variable parts and multiple batches perspectives for solving proportion problems and for reasoning about proportional relationships. The algebra course included instruction in using a variable parts perspective to develop equations in two variables, including lines in a coordinate plane. Both courses emphasized developing mathematically valid arguments and explanations for solution methods, procedures, and equations.

The data for this study came from the fifth set of interviews, which was conducted at the end of February 2015. The data included a video recording and transcript of the interview with each participant and a scanned copy of each participant's written work on the tasks given during the interview.

For this report, we selected the two tasks from the fifth interview to analyze for each of the six participants. One task asked participants to use a variable parts perspective to develop an equation in two variables for a proportional relationship (see Fig. 6.3); we refer to this task as the "fertilizer task." The other task asked

Task 2: A type of fertilizer is made by mixing nitrogen and phosphate in an 8-to-3 ratio. Suppose you will use *N* kilograms of nitrogen and *P* kilograms of phosphate, where *N* and *P* are unspecified numbers of kilograms, which could vary.

Please explain how to solve the following problems.

Task 2A: Use the variable parts perspective and a math drawing to derive and explain an equation of the form (fraction) • $P = N$, where "fraction" is a suitable fraction or mixed number. Attend carefully to the definition of multiplication used in Dr. Beckmann's class when discussing (fraction) • P.

Fig. 6.3 Task 2A from Interview 5

Task 3A: Please see the geogebra sketch. Can you use features of the drawing to express a relationship between *x* and *y* that holds for points (*x*,*y*) on the line?

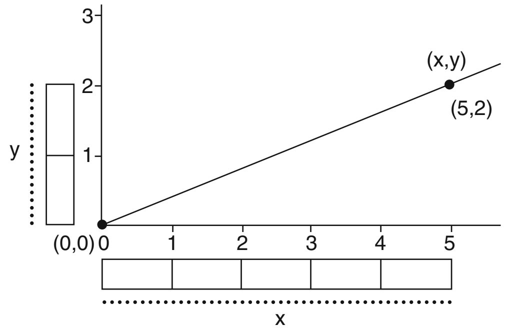


Fig. 6.4 Task 3A from Interview 5

participants to develop an equation for a line using features of the given diagram (see Fig. 6.4); we refer to this task as the "line task." The line task was presented together with a dynamic Geogebra sketch shown on an iPad (see Fig. 6.2 and http://ggbm.at/UutLbwxl). These two tasks were similar to tasks on an in-class test given about a 2 weeks earlier in the algebra course.

We analyzed the participants' initial responses to the tasks, before follow-up questions. Our thinking is that these initial responses are likely to use ideas and ways of reasoning that the participants felt most comfortable with and confident in.

To analyze the participants' responses, we first wrote "cognitive memos" summarizing and describing how each participant reasoned about each task. To find themes, we further condensed each cognitive memo by making a list describing the ideas, concepts, and ways of reasoning the participant used as they worked on the task. When making each list, we considered the participant's use of key class tools related to reasoning with variable parts: the CCSS definition of fraction, the equal groups definition of multiplication, and strip diagrams. We also considered the viability of the participant's argument. We discussed these cognitive memos and lists until we agreed upon a set of themes: main ideas, concepts, and ways of reasoning that we found across multiple participants.

To describe the ideas, concepts, and ways of reasoning that we identified, we found we needed a criterion for when a person is treating an entity as a quantity. This led us to develop the discussion about quantity we presented above. Based on that discussion, we take as evidence that a person is treating an entity as a quantity if they do any of the following: describe the entity as some number of another entity (thus using the other entity as a unit); use the entity to describe another entity as some number of the entity (thus using the entity as a unit); use any of the words size, amount, or quantity to describe or discuss the entity; and describe the entity as equal to some number of another entity.

To determine the points of tension that the future teachers experienced as they explained their equations, we examined the future teachers' arguments and considered them as sequences of assertions. Because the arguments were constructed in the moment, they involved false starts and questions about what to do next. By a "point of tension," we mean either a spot in the sequence where the future teacher considers competing ideas or a spot where the future teacher's argument takes a turn in direction.

6.3 Results and Discusson

This section discusses the results of our analysis of the ideas, concepts, and ways of reasoning that the future teachers used when developing an equation for a proportional relationship and for a line. We first present our analysis of the future teachers' reasoning about the fertilizer task (Fig. 6.3). We then present our analysis of the future teachers' reasoning about the line task (Fig. 6.4). Finally, we discuss some issues that cut across the two tasks.

6.3.1 How Future Teachers Developed and Explained an Equation for a Proportional Relationship Using the Variable Parts Perspective

In response to the fertilizer task (Fig. 6.3), all six of the future teachers developed an equation of the form (fraction) • $P = N$ and five of the six explained their equation by reasoning about quantities before the interviewer posed follow-up questions. (We do not consider clarifying questions to follow-up questions.) In this section, we first present the written work and verbal explanations given by three future teachers. Next, we discuss the ideas, concepts, and ways of reasoning used by all five of the future teachers who explained their equation before they were asked follow-up questions. Finally, we discuss points of tension that the future teachers encountered as they developed and explained their equations.

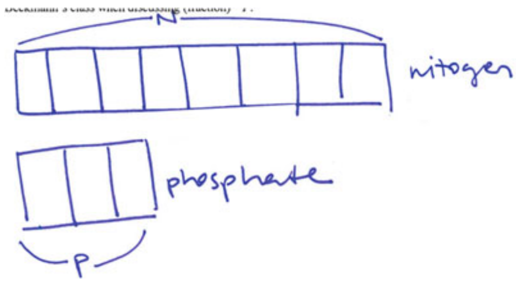
6.3.1.1 Diana, Kelly, and Jeff's Reasoning

We present the responses of Diana, Kelly, and Jeff to the fertilizer task—up to the point of follow-up questions—because these responses show a range of reasoning and provide examples of all of the themes we highlight in the next section. These responses also illustrate all the points of tension we found in the future teachers' arguments.

Throughout this chapter, when we present transcript excerpts from an interview, a future teacher's words are in italics and notes on their gestures or inscriptions are in brackets. We use ellipses to denote deletions from the excerpt, including interviewer comments or questions.

Diana's Reasoning

When working on the fertilizer task, Diana first draws the following:



While producing this strip diagram, Diana explains:

> *So I guess first I'd start off with just drawing out the parts, so if it's 8, it's 8 parts to 3 parts of nitrogen and phosphate, and then this* [draws a bracket with "N" above the nitrogen strip] *is some number of N. Or some yeah, N kilograms while this* [draws a bracket with "P" below the phosphate strip] *is P kilograms.*

Diana continues to explain:

> *I'm looking for a fraction times P equals N, we're looking at something of P* [points to phosphate strip] *is equal to N* [points to nitrogen strip]*, so we're relating them. So if you're, the equation is calling for N* [points to nitrogen strip] *in terms of P* [points to phosphate strip]. *So since P is going to be our whole* [circles P in the equation (fraction) • P = N in the problem statement] *or yeah reference whole or unit whole, then our whole is out of 3 parts* [points to 3-part phosphate strip]. *So since they're all the same size, since this has 1, 2, 3, 4, 5* [points to parts of the nitrogen strip] *the, since this* [points to the entire nitrogen strip] *has 8 parts, it would be 8 out of 3* [writes $^8/_3$] *or nitrogen would be 8/3 of phosphate or times phosphate would equal nitrogen* [completes writing 8/3 • P = N].

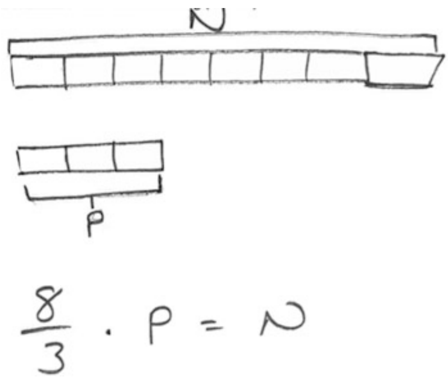
In her initial written work and verbal explanation, Diana draws and annotates a strip diagram to show the 8 parts nitrogen and the 3 parts phosphate. In doing this, she uses the strip diagram as an initial means for orienting or organizing her thinking and work. Diana also uses her strip diagram to develop an equation through a logical argument. She reasons that because the equation is calling for *N in terms of P*, P will serve as the *reference whole or unit whole*, so the 3 parts of the phosphate strip make that reference whole. She then infers that the 8 parts comprising the nitrogen strip are 8/3 of the phosphate strip. Thus, she develops her logical argument by reasoning about her strip diagram.

Diana's argument relies on interpreting multiplication as "of." She derives her equation by first translating the multiplication dot in "(fraction) • P = N" to "of." She indicates this when stating, *I'm looking for a fraction times P equals N, we're looking at something of P is equal to N.* This "of" language leads her to determine that *P* is the reference whole for the fractional multiplier she is looking for (*So since P is going to be our or yeah reference whole or unit whole*). Once she has established the reference whole, she quickly reasons that the nitrogen is 8/3 of that whole and therefore that the multiplier is 8/3.

Throughout her argument, Diana treats the nitrogen and phosphate strips and all their parts as quantities. She explicitly refers to the phosphate strip as *3 parts, P kilograms,* and a *reference whole or unit whole*. She explicitly refers to the nitrogen strip as *8 parts* and *N kilograms.* She also states that the parts are *all the same size* and that *nitrogen would be 8/3 of phosphate.* Diana's argument also relies on treating the letters *N* and *P* as quantities. For example, she says, *we're looking at something of P is equal to N,* and *P is going to be our whole or yeah reference whole or unit whole* while circling the letter P in the equation "(fraction) • P = N" in the problem statement. Although she refers to N and P in terms of kilograms while producing her strip diagram (*Or some yeah, N kilograms while this is P kilograms*), she does not refer to kilograms again as she develops her equation. Thus, Diana's argument primarily relies on treating N and P as quantities based on a number of parts as opposed to a number of kilograms.

Jeff's Reasoning

Jeff starts the fertilizer task by writing the following:

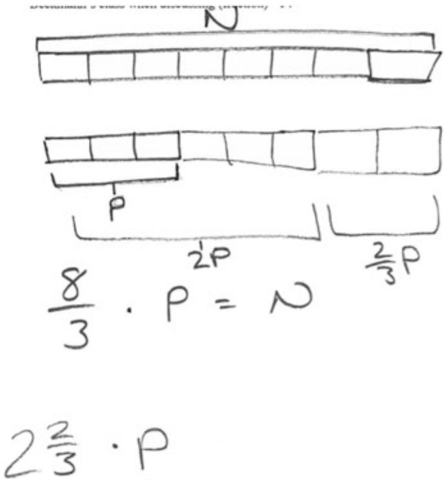


After writing this, Jeff explains:

> *So I'm seeing the relationship eight part*s [points to the entire N-strip] *to three parts* [points to the entire P-strip] ... *Eight parts for nitrogen, or eight kilograms of nitrogen, for three kilograms of phosphate. So now I, I wrote the equation, but I'm thinking that if this is my*

> *equation* [i.e., $^{8}/_{3} \bullet P = N$], *each part should be P and not the whole thing. But I think I'm correct in this thinking ... I'm debating whether the whole thing should be P* [i.e., entire P-strip] *or each part should be P, but I'm pretty sure the whole thing should be P ... I'm thinking that, in seeing the eight to three ratio, I'm thinking each part should be P and then multiplied by eight thirds, but that doesn't make sense which is why I'm pretty confident that this* [i.e., the entire P-strip] *does.*

Jeff then continues to add to his written work:



While doing this, Jeff explains:

> *Because another way of, for me to write it* [i.e., the equation] *would be two and two thirds times P* [writes "$2^{2}/_{3} \bullet P$"], *which in essence would be, this would be one P* [points to entire P-strip] *if I added another right here, one two three* [adds 3 more parts to the P-strip], *this in turn would be two P* [writes bracket labeled "2P" under the 6 parts of the 2P-strip] *and then these last two would be two thirds P* [adds 2 more parts with a bracket under them labeled "$^{2}/_{3}$P"], *which would give you the total N.*

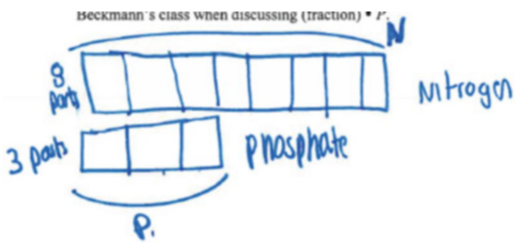
In his initial written work and verbal explanation, Jeff draws and annotates a strip diagram to show the 8 parts nitrogen and the 3 parts phosphate. In doing this, he uses the strip diagrams as a means for orienting or organizing his thinking and work. Jeff also uses his strip diagram to verify his equation through a logical argument. Initially he is confused about whether the referent unit for P in his equation ("$^{8}/_{3} \bullet P = N$") is the entire phosphate strip or one part of the phosphate strip (*I'm debating whether the whole thing should be P or each part should be P).* He decides that it would not make sense for each part to be P because it would not make sense to multiply each part by $^{8}/_{3}$ (*each part should be P and then multiplied by eight thirds, but that doesn't make sense*). He determines that the entire strip should be P because the total N, which is made of 8 parts, is $2^{2}/_{3}$ copies of the 3-part phosphate strip. This means that $2^{2}/_{3} \bullet P$ *would give you the total N* which is consistent with his original equation $^{8}/_{3} \bullet P = N$. Thus Jeff develops a logical argument by reasoning about his strip diagram.

Jeff's argument relies on an equal groups interpretation of multiplication, which is facilitated by interpreting the multiplier $^8/_3$ as the mixed number $2^2/_3$. In explaining $2^2/_3 \bullet P$, he considers copies of P: *if I added another right here, one two three* [adds 3 more parts to the P-strip]*, this in turn would be two P* [writes bracket labeled "2P" under the 6 parts of the P-strip] *and then these last two would be two thirds P* [adds 2 more parts to the P-strip with a bracket under them labeled $^2/_3$P] *which would give you the total N*. Thus Jeff considers groups of P: one P, two P, and 2/3 P. In a later follow-up question when he is asked if it is easier for him to think about the situation from a mixed number point of view as opposed to an improper fraction point of view, he responds, *It clarifies it for me. I can see the, I can see the whole two parts, if that makes sense, and then the additional two thirds.* Thus the mixed number $2^2/_3$ seems to help Jeff see the relationship between the amount of nitrogen and phosphate in the equation $^8/_3 \bullet P = N$.

Throughout his argument, Jeff treats the nitrogen and phosphate strips and all their parts as quantities. He explicitly refers to the phosphate strip as *three parts* and *three kilograms* and uses the quantity of 3 parts for P to describe 6 parts as "2P," 2 parts as "$^2/_3$P," and 8 parts as "$2^2/_3$P." He explicitly refers to the nitrogen strip as "eight parts" and "eight kilograms." Jeff's argument also relies on treating the letters *N* and *P* as quantities. For example, he says, *I'm debating whether the whole thing should be P* [i.e., entire P-strip] *or each part should be P* and *this in turn would be two P*, *and then these last two would be two thirds P which would give you the total N.* Although he refers to N and P in terms of kilograms while producing his strip diagram and initial equation (*Eight parts for nitrogen, or eight kilograms of nitrogen, for three kilograms of phosphate*), he does not refer to kilograms again as he explains why the equation appropriately represents the situation. Thus, Jeff's argument primarily relies on treating N and P as quantities based on a number of parts as opposed to a number of kilograms.

Kelly's Reasoning

Kelly starts the fertilizer task by drawing the following:

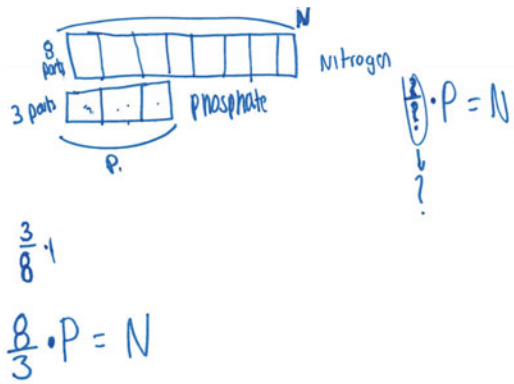


While writing this, Kelly explains:

> *This is going to be my nitrogen, or N is what I'll use. It's 8 parts. And this is my phosphate, P, 3 parts. And these parts are the same size. All of them are. Well they're supposed to be.*

> *And I'm trying to make the equation fraction times P equals N. So I have to figure out what my fraction's going to be to make this equation true* [writes "$^{?}/_{?}$ • P = N" on the paper]. *That's supposed to be a fraction* [points to "$^{?}/_{?}$"]... *So the way I first think of this* [points to "$^{?}/_{?}$ • P = N"] *is what of phosphate is going to equal nitrogen? So ...* [writes "$^{3}/_{8}$" and "$^{8}/_{3}$ • P = N"] *I'm just thinking ... Whether I need $^{3}/_{8}$ of P equals N or $^{8}/_{3}$ of P equals N. And I think it's going to be $^{8}/_{3}$* [points to "$^{8}/_{3}$" in "$^{8}/_{3}$ • P = N"] *of phosphate* [colors over the "•" in "$^{8}/_{3}$ • P = N"] *is equal to nitrogen. Because if you only have 3 parts* [points to each part in the phosphate strip] *of size 1, $^{1}/_{8}$ of phosphate, then you're only going to have $^{3}/_{8}$* [points to the first three parts of the nitrogen strip] *of the nitrogen, but we want to have all of N, and not just a part of it. So I need $^{8}/_{3}$* [points to "$^{8}/_{3}$" in "$^{8}/_{3}$ • P = N"] *of P* [points to the entire phosphate strip], *right? Each of P* [points to each part of the phosphate strip] *is size $^{1}/_{3}$, and I need 8 of these $^{8}/_{3}$ to be equal to N* [points to "N" in "$^{8}/_{3}$ • P = N"], *because there are 8 parts in N. So, like this is $^{1}/_{3}$* [points to the first part of the phosphate strip], *$^{1}/_{3}$* [points to the second part of the phosphate strip], *$^{1}/_{3}$* [points to the last part of the phosphate strip]. *I would need 8 of these $^{1}/_{3}$ parts of P to be equal to N.*

Prior to being asked follow-up questions, Kelly's written work is as follows:



In her initial written work and verbal explanation, Kelly draws and annotates a strip diagram to show the 8 parts nitrogen and the 3 parts phosphate. She uses the strip diagram together with her equation $^{?}/_{?}$ • P = N to orient and organize her thinking and work. Kelly also uses the strip diagram together with her equation $^{?}/_{?}$ • P = N to develop her final equation $^{8}/_{3}$ • P = N through a logical argument. When deciding whether the equation is "$^{3}/_{8}$ • P = N" or "$^{8}/_{3}$ • P = N," she uses the strip diagram to reason that she wants all of N, not just 3/8 of the nitrogen strip (*if you only have 3 parts of size 1, $^{1}/_{8}$ of phosphate, then you're only going to have $^{3}/_{8}$ of the nitrogen, but we want to have all of N and not just a part of it*). She then concludes that the equation is "$^{8}/_{3}$ • P = N," since *I need $^{8}/_{3}$ of P, right? Each of P is size $^{1}/_{3}$, and I need 8 of these $^{1}/_{3}$ to be equal to N.* Thus, she develops her logical argument by reasoning about her strip diagram.

Kelly's argument relies on interpreting multiplication as "of." She derives her equation by first translating the multiplication dot in "(fraction) • P = N" to "of" in stating, *So the way I first think of this* is *what of phosphate is going to equal nitrogen.* She then uses this "of" language to interpret the equation "$^{8}/_{3}$ • P = N" as $^{8}/_{3}$ *of P equals N.* She even colors over the multiplication dot as she makes this interpretation. However, Kelly's "of" interpretation of multiplication does not seem completely solid because when she considers the possibility $^{3}/_{8}$ *of P equals N* she rejects it based on interpreting $^{3}/_{8}$ as *3 parts of size 1,* $^{1}/_{8}$ *of phosphate.* In doing so, she is not considering the 3/8 as taking a portion of the phosphate because she points to the 3 parts that make up the entire phosphate strip. When she refers to the 3/8 as 3 parts of size 1/8, the referent should be nitrogen, not phosphate.

Kelly's argument also relies on interpreting the equal sign as indicating, in a sense, that the stuff on the left side of the equation is equal to the same amount of stuff on the right side of the equation. Throughout her explanation, she uses the language of "[some amount] of P equals N. Kelly reasons that the equation is not "$^{3}/_{8} \bullet P = N$" because *you're only going to have* $^{3}/_{8}$ *of the nitrogen, but we want to have all of N, and not just a part of it*. Instead, she explains, *I need* $^{8}/_{3}$ *of P* because *I would need 8 of these* $^{1}/_{3}$ *parts of P to be equal to N.* Kelly's argument also relies on the CCSS definition of fractions. Kelly uses this definition when interpreting $^{8}/_{3}$ as *8 of these* $^{1}/_{3}$ *parts of P*.

Throughout her argument, Kelly treats the nitrogen and phosphate strips and all their parts as quantities. She explicitly refers to the phosphate strip as *3 parts* and the nitrogen strip as *8 parts.* She additionally states that *these parts all the same size* and refers to the size of one part as $^{1}/_{3}$ and $^{1}/_{3}$ *parts of P.* Kelly's argument also relies on treating the letters *N* and *P* as quantities. For example, she says, *I would need 8 of these 1/3 parts of P to be equal to N.* Kelly does not refer to the N and P or the strips in terms of kilograms at any point during her argument. Thus, Kelly's argument relies on treating N and P and the strips as quantities based on a number of parts as opposed to a number of kilograms.

6.3.1.2 Ideas, Concepts, and Ways of Reasoning the Future Teachers Used as they Developed an Equation for the Fertilizer Task

From our analysis of the future teachers' reasoning as they worked on the fertilizer task (Fig. 6.3), we found several themes in the ideas, concepts, and ways of reasoning they used. These future teachers (1) used strip diagrams as an organizing and thinking tool for developing equations; (2) developed mathematically valid arguments based on reasoning about quantities; (3) used strips and letters to represent quantities, although not necessarily numbers of kilograms; and (4) mostly used an "of" interpretation of multiplication. We discuss each of these themes below.

Strip Diagrams as Organizing and Thinking Tools for Developing Equations

Before writing or discussing equations, all six of the future teachers drew and annotated a strip diagram consisting of 8 parts for the nitrogen and 3 parts for the phosphate. Thus, the future teachers initially used the strip diagrams as organizers and as a means for representing and relating the quantities of nitrogen and phosphate provided in the fertilizer task. We claim that the strip diagrams served more than just an organizing function for the future teachers because five of the six future teachers (Alice, Claire, Diana, Jeff, and Kelly) made extensive references to their strip diagrams, in gestures and words, while developing their equations. These five future teachers repeatedly connected specific components of their strip diagrams with specific components of their equations as they reasoned to develop their equations.

Linda was the only future teacher who did not discuss her strip diagram while developing her equations. Immediately after drawing her strip diagram, she wrote and stated two equations, $^8/_3 \bullet P = N$ ("eight-thirds of P equals N") and $^3/_8 \bullet N = P$ ("three eighths times N equals P"), but did not explain how she had formulated them. Linda later shared her reasoning about the equation $^8/_3 \bullet P = N$ when the interviewer asked her follow-up questions about how she interpreted the meaning of equal sign in her equation and how she interpreted the equation in terms of the definition of multiplication used in class. Since Linda did not explain her initial reasoning about the fertilizer task during the interview, we do not include an analysis of her work in the remainder of this section.

Mathematically Valid Arguments Based on Reasoning about Quantities

All five of the future teachers who explained the equation $^8/_3 \bullet P = N$ without prompting provided arguments that were mathematically valid, logical, and complete. Four of the five future teachers considered alternative avenues before arriving at such an argument, which shows that the derivation of such an equation was not necessarily easy or automatic for them.

None of the future teachers based their arguments on mnemonics or other devices that are not logic-based. For example, one could make the following argument:

> N is to P as 8 is to 3, so $^N/_P = {}^8/_3$. By multiplying both sides by P, we have $N = {}^8/_3 \bullet P$.

This argument relies on turning an analogy into an equation and then using algebra to derive an equivalent equation. While such an argument might be acceptable in some circumstances, this type of argument does not rely on reasoning quantitatively about the strips, their parts, or the letters N and P. None of the five future teachers provided an argument like this one; they all formulated their arguments by reasoning about relationships between quantities.

Strips and Letters Represent Quantities, Although Not Necessarily Numbers of Kilograms

All five of the future teachers treated their nitrogen strip, their phosphate strip, and the parts of those strips as quantities and engaged in quantitative reasoning as they developed or explained their equations. All of the future teachers related their nitrogen and phosphate strips to each other and described that relationship between the quantities in terms of numbers of parts or numbers of entire strips. Four of the five future teachers (all but Jeff) explicitly referred to the size of parts or strips at some points during their explanations.

All five of the future teachers also treated the letters N and P as quantities. They related N and P to their respective 8-part nitrogen strip and 3-part phosphate strip, which they treated as quantities. They also used language relating N and P to each other or to the parts of a strip diagram. Two of the future teachers (Alice and Diana) referred explicitly to P as either a unit amount or a reference whole and three (Alice, Jeff, and Kelly) referred to "the total N" or indicated they saw N as all of the nitrogen.

Although all five of the future teachers treated the nitrogen and phosphate strips and the letters N and P as quantities, Claire was the only person to describe strips and letters in terms of kilograms throughout her explanation. She always referred to the letters N and P as "N kilograms" and "P kilograms." Her explanation also focused on the size of each part. None of the other future teachers used kilograms while they were developing and explaining their equations, although Diana and Jeff did refer to kilograms at the start of the fertilizer task.

The "of" Interpretation of Multiplication

Although the fertilizer task explicitly asked the future teachers to "attend to the definition of multiplication used in Beckmann's class" (the equal groups definition), none of the five future teachers referred to this definition as they initially produced and explained their equations. This was not because the future teachers did not remember the class definition. Later, when the interviewer asked them to interpret their equations using that definition, all of the future teachers were able to do so.

Instead of using the class (equal groups) definition of multiplication to explain their equations, all but Jeff interpreted multiplication as "of," and Claire used a combination of an "of" interpretation and an equal groups interpretation of multiplication. Therefore, it seems the future teachers were initially inclined to reason using an "of" meaning of multiplication as opposed to using the equal groups definition of multiplication.

We claim that the future teachers in this study used an "of" sense of multiplication as a thinking tool, not as a mechanical crutch to avoid thinking. Their interpretation of multiplication as "of" helped them translate the relationships they saw in their strip diagrams into equations. Alice, Claire, and Diana consistently used the "of" interpretation to connect a fractional multiplier to its referent whole.

Jeff was the only one of the five future teachers who did not explain his equation using an "of" sense of multiplication. Instead, he used an equal groups interpretation of multiplication with the multiplier $2^2/_3$. To do so, he used the idea that the multiplier indicates how much or how many of the multiplicand it takes to make the product. Specifically, he interpreted the mixed number multiplier $2^2/_3$—but not the improper fraction $^8/_3$—as how many Ps it takes to make the same amount as N.

6.3.1.3 Points of Tension and their Resolution: Referent Unit and Equality

Out of the six future teachers, Linda was the only one to produce the equation $^8/_3 \bullet P = N$ immediately after drawing a strip diagram. The other five explained their reasoning as they developed their equation. As they did so, the five future teachers encountered various points of tension. All these points of tension, or their resolution, involved some aspect of referent unit. Two of the points of tension were resolved by focusing on equality.

Alice and Claire encountered the same point of tension. They both started their arguments by selecting the nitrogen as the referent unit and, as a result, described the phosphate as $^3/_8$ of the nitrogen. They then changed direction by taking the phosphate as referent.

A second point of tension, experienced by Kelly, occurred when she debated between competing ideas. She wondered whether the multiplier for her equation should be "$^3/_8$" or "$^8/_3$." She explained, *I'm just thinking ... Whether I need $^3/_8$ of P equals N, or $^8/_3$ of P equals N.* To resolve this point of tension, Kelly focused on identifying which multiplier would give her *all of N* and *equal to N.* She explained, *because if you only have 3 parts of size 1, $^1/_8$ of phosphate, then you're only going to have $^3/_8$ of the nitrogen, but we want to have all of N, and not just a part of it. So I need $^8/_3$ of P, right? Each of P is size $^1/_3$, and I need 8 of these $^1/_3$ to be equal to N.* Thus, she determined that the multiplier should be $^8/_3$.

A third point of tension, experienced by Jeff, occurred as he tried to verify his equation by interpreting it using his strip diagram. When interpreting his equation using his strip diagram, Jeff's point of tension came in deciding whether the "P" in his equation referred to each part of his 3-part phosphate strip or the entire phosphate strip. This point of tension seemed tied to the way in which Jeff viewed quantities described in terms of ratios since he explained, *in seeing the 8 to 3 ratio* [gestures to the 8-part strip and the 3-part strip], *I'm thinking each part should be P.* To resolve the tension, Jeff selected the entire phosphate strip to be P, rewrote the multiplier as a mixed number, and then showed how 2 Ps and another $^2/_3$ P *would give you the total N.* Thus, he determined that the entire phosphate strip should be the referent unit for his equation.

All of the above points of tension involved some aspect of referent unit, and two of three focused on equality. When Alice and Claire took the nitrogen as the referent unit, they were perhaps not anticipating that the desired equation of the form (fraction) $\bullet P = N$ implies that P will be the referent unit for the fractional multiplier. When Kelly debated between $^3/_8$ of P and $^8/_3$ of P, she seemed not to be attending to the referent unit P. In fact her language *3 parts of size 1, $^1/_8$ of phosphate* used the incorrect referent for 1/8. She referred to the 3 parts that are inside the phosphate strip, which might be why she said *of phosphate*. But to describe those parts as having size 1/8, the referent for the 1/8 should be nitrogen, not phosphate. Instead of resolving her point of tension by focusing on what the referent unit signified, she resolved it by focusing on equality with all of N. When Jeff resolved his debate between each part being P or the whole strip being P, he used the entire phosphate strip as a referent for the mixed number $2^2/_3$.

6.3.2 How Future Teachers Developed and Explained an Equation for a Line in a Plane Using the Variable Parts Perspective

In response to the line task (see Fig. 6.4), all six of the future teachers produced and explained the equation $^2/_5\ x = y$ or $y = {}^2/_5\ x$ before they were asked follow-up questions. Several of the future teachers also produced and explained other equations, including $^5/_2\ y = x$, $x = {}^5/_2\ y$, and $y = 2(^1/_5 x)$. Perhaps because the line task was the last planned task of the interview, the future teachers' explanations were not as elaborate as their explanations on the fertilizer task (only one of the future teachers started the task before 50 min into the interview and two started it after an hour). In this section, we first present the written work and verbal explanations given by three future teachers. Next, we discuss the ideas, concepts, and ways of reasoning used by all six future teachers before they were asked follow-up questions.

Unlike the fertilizer task, we found that the future teachers did not encounter points of tension as they developed and explained their equations. Thus we do not include a section on this for the line task.

6.3.2.1 Alice, Kelly, Claire, and Diana's Reasoning

We present the responses of Alice, Kelly, Claire, and Diana to the line task—up to the point of follow-up questions—because these responses show a range of reasoning and provide examples of all of the themes we highlight in the next section.

Alice's Reasoning

Alice starts the line task by briefly moving the slider on the Geogebra sketch and indicating (by nodding) that she is familiar with this kind of Geogebra sketch from class. Alice then writes the equation "$^2/_5 \bullet x = y$" while explaining:

> *So y, where like two fifths of x equals y ... Because you have two parts here, and five here, and they're the same size, so two fifths of x would be two of those* [points to first two parts of the x-strip]*, because you have five total, would equal the same size as y.*

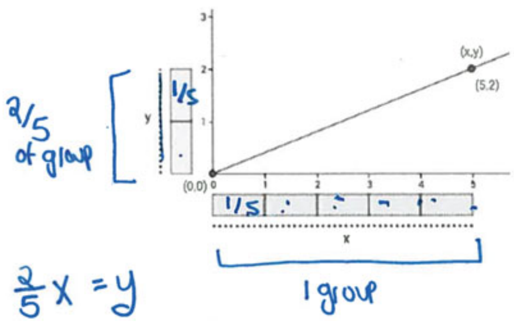
In her written work and verbal explanation for the line task, Alice provides a correct equation and a viable argument for her equation. Her explanation relies on reasoning about the strips and letters as quantities because she refers to the strips as 2 parts and 5 parts, connects them to the letters X and Y, and refers to sizes of the parts and the Y strip. She interprets multiplication as "of" when translating the statement *two fifths of x equals y* into the equation "$^2/_5 \bullet x = y$." Her explanation relies on interpreting the "$^2/_5$" as *two parts*, although she does not explicitly describe the parts as having a size of $^1/_5$ even though she states that the two parts and 5 parts are the same size (*two parts here, and five here, and they're the same size*). Alice did not discuss a connection between the strips and points on the line.

Kelly's Reasoning

Kelly starts the line task by moving the slider on the Geogebra sketch and indicating (by nodding) that she is familiar with this kind of Geogebra sketch from class. Kelly then explains:

> *Well each of these parts* [points to parts of the x-strip and the y-strip] *is going to be the same size, there are 2 of them in Y* [points to the y-strip] *and there are 5 of them in X* [points to the x-strip], *and we'll just say for this that X is 1 total group* [draws a bracket under the x-strip and labels it "1 group"]. *Which means if there's 2 parts – so each of these is $^1/_5$* [writes "$^1/_5$" in a part of the x-strip], *and that's the same with the Y parts* [writes "$^1/_5$" in a part of the y-strip], *but there's 2 of them* [draws a bracket to the left of the entire y-strip]. *So this is $^2/_5$ of group* [writes "$^2/_5$ of group" to the left of the bracket by the y-strip] *because there are 2 parts* [points to both parts of the y-strip] *of size $^1/_5$ of the total $^5/_5$* [points to each part of the x-strip]. *So $^2/_5$ X equals Y* [writes "$^2/_5$x = y"]*?*

Prior to being asked follow-up questions, Kelly's written work is as follows:



In her written work and verbal explanation, Kelly develops a correct equation, $^2/_5x = y$, and provides a viable argument for it. Her explanation relies on reasoning about the strips and letters as quantities because she refers to the numbers and sizes of the strips' parts, notes that the parts are the same size, and connects the strips to the letters X and Y. She uses the class (equal groups) definition of multiplication to formulate her equation when stating *we'll just say for this that X is 1 total group* and then describing the y-strip as *$^2/_5$ of group*. She also interprets the fraction "$^2/_5$" using the CCSS definition of fraction when making statements such as *there are 2 parts of size $^1/_5$ of the total $^5/_5$*. Kelly did not discuss a connection between the strips and points on the line.

Claire's Reasoning

When Claire is given the dynamic Geogebra sketch, she nods to indicate she has seen one like this. She moves the slider on the Geogebra sketch briefly and then explains:

> *So like I interpret this with a variable parts perspective since they're two parts versus 4, 5 parts which are each the same size. So Y has 2 parts, and X has 5 parts. And they're each the same size. So that means 1 of 2 parts well, let's see I'll start with X, okay. So that means 1 of 5 parts of X is* $^1/_5$ *X. And . . . Y has 2 parts so since 1 of 5 parts of X is* $^1/_5$ *of X, each of these 2 parts of Y is going to be* $^1/_5$ *of X. So that means Y has 2 parts, each size* $^1/_5$ *X. So that means Y equals 2 times* $^1/_5$ *X* [writes "y = 2(1/5x)" and "y = 2/5x"]. *Or Y equals* $^2/_5$ *X. I mean it's the same as saying 2 times* $^1/_5$ *X, or* $^1/_5$ *X plus* $^1/_5$ *X. Because Y has those 2 parts they're each* $^1/_5$ *of X, since X has 5 parts, each part* $^1/_5$ *X. So whichever numbers we put in for example X and Y is 5, 2 in one coordinate. So if we put 5 in for X,* $^2/_5$ *times X which is* $^2/_5$ *times 5, is 10 over 5, which means that when X equals 5, Y equals 2.*

In her written work and verbal explanation for the line task, Claire develops two correct equations, $y = 2(^1/_5\ x)$ and $y = {}^2/_5\ x$, and provides a viable argument for them.

Her explanation relies on reasoning about the strips and letters as quantities because she refers to X and Y in terms of the strips and their numbers of parts and notes that the parts are the same size. In her argument, Claire uses both an "of" interpretation and an equal groups interpretation of multiplication. She uses an "of" interpretation of multiplication when she refers to $^1/_5$ *of X*, which she writes as $^1/_5$ x. She uses an equal groups view of multiplication when explaining how y is made of 2 parts. She explains, *so that means Y has 2 parts, each size* $^1/_5$ *X . So that means Y equals 2 times* $^1/_5$ *X,"* and writes this as "y = 2($^1/_5$ x)." She does not refer to the fractions $^1/_5$ or $^2/_5$ separately from x in her explanation. Her wording (e.g., *Y has those 2 parts they're each* $^1/_5$ *of X, since X has 5 parts, each part* $^1/_5$ *X*) seems like a blend of the CCSS definition of fraction with an equal groups interpretation of multiplication.

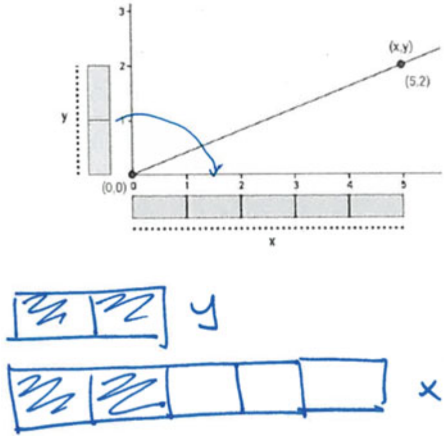
Although Claire does not describe it this way, we can interpret her use of the equation $y = 2(^1/_5\ x)$ to obtain the equation $y = {}^2/_5\ x$ as an application of the associative property of multiplication.

Claire indicates that her equations hold generally for points on the line with the brief mention *whichever numbers we put in* although she only illustrates this with a single point (5,2).

Diana's Reasoning

When Diana is given the dynamic Geogebra sketch, she nods slightly to indicate this is similar to something they have seen in class. The interviewer explains she can move the slider to see the blue parts and red parts expanding or contracting and that we want her to have this in mind. She briefly moves the slider on the Geogebra sketch.

Diana then makes the following drawing:



While making this drawing, she explains:

> *Oh so for relating X and Y, if we were to rotate Y horizontally* [draws curved arrow indicating rotating the y-strip 90° clockwise] *you would get that 2 parts, that there's 2 parts of Y for 5 parts X* [draws a 2 part y-strip and 5 part x-strip parallel to each other]. *So essentially Y is 2 parts* [shades the two parts of the y-strip and the first two parts of the x-strip] *of the X 5 parts. So $^{2}/_{5}$ of X. So you can say Y is equal to 2 parts* [points to the first two parts of the x-strip] *out of 5 parts* [points to the entire x-strip] *times X* [writes "y = $^{2}/_{5}$ *x"]. *So for slope you could say for every time it goes up 2* [points to the "2" in "y = $^{2}/_{5}$ * x" then each part of the y-strip shown in the printout of the Geogebra sketch], *it goes over 5. And no matter the size, or it works for all points X and Y just because the number of parts will never change* [points to the two parts in the y-strip she drew and some parts in the x-strip she drew] *even though the size will, the proportions will stay the same. So it will always be, Y would always be $^{2}/_{5}$ of X.*

In her written work and verbal explanation for the line task, Diana develops the equation $y = {}^{2}/_{5} * x$ and provides a viable argument for it. She starts her explanation by rotating the y-strip to make it parallel to the x-strip, so that the strips are oriented in the way that strip diagrams are typically drawn. Her explanation relies on reasoning about the strips and letters as quantities because she refers to the strips and X and Y as numbers of parts and describes Y as $^{2}/_{5}$ of X.

When stating, *So essentially Y is 2 parts of the X 5 parts. So $^{2}/_{5}$ of X. So you can say Y is equal to 2 parts out of 5 parts times X*, Diana uses both an "out of" interpretation of the fraction $^{2}/_{5}$ and an "of" interpretation of multiplication to formulate the equation "y = $^{2}/_{5}$ * x." There is no indication that she is using the CCSS definition of fraction to interpret $^{2}/_{5}$, although it is possible she is thinking of the fraction that way.

In her explanation for the line task, Diana implicitly connects the strip diagram to the line when discussing slope, which she interprets in terms of the parts of the strips. When stating, *And no matter the size, or it works for all points X and Y* *just because the number of parts will never change even though the size will, the proportions will stay the same*, Diana explains that the equation works for all points and that, since the numbers of parts stay the same while the parts can change in size, Y will always be $^2/_5$ of X.

6.3.2.2 Ideas, Concepts, and Ways of Reasoning the Future Teachers Used as they Developed an Equation for the Line Task

From our analysis of the future teachers' reasoning as they worked on the line task (Fig. 6.4), we found several themes in the ideas, concepts, and ways of reasoning they used. These future teachers (1) used strip diagrams as an organizing and thinking tool for developing equations; (2) developed mathematically valid arguments based on reasoning about quantities; (3) used strips and letters to represent quantities, although not necessarily quantities that vary; and (4) used "of" and equal groups interpretations of multiplication. We discuss each of these themes below.

Strip Diagrams as Organizing and Thinking Tools for Developing Equations

All the future teachers used strip diagrams to develop and explain their equations. Unlike the fertilizer task, the line task provided a static image of a strip diagram in the task statement (Fig. 6.4) as well as a dynamic strip diagram in a Geogebra sketch shown on an iPad. While working on the line task, each participant had the opportunity to manipulate the dynamic strip diagram in the Geogebra sketch. All parts in the initial strip diagram provided in the Geogebra sketch were 1 unit long (as shown in Fig. 6.2a), and each participant could change the size of all of the parts by moving a slider in the bottom of the sketch. Figure 6.2b shows an example of the Geogebra sketch when the size of each part is larger than 1 unit. The future teachers spent little time manipulating the dynamic sketch, and most indicated that they were familiar with this kind of sketch from class.

Two of the future teachers, Diana and Linda, discussed moving the two strips to make them parallel. Diana made a new drawing showing the strips parallel to one another while Linda verbally described flipping the y-strip over so that it would be parallel to the x-strip. In doing this, Diana and Linda oriented the strips in the way that was consistent with how the strip diagrams are usually drawn.

None of the future teachers explicitly discussed the connection between the strips and the points on the line. This might be because they simply took this connection for granted or were used to focusing on strips to explain equations. Diana did discuss the slope of the line in terms of the strips; however she did not make an explicit reference to a plotted point.

Two of the future teachers, Jeff and Linda, were asked about the connection between the strips and a point on the line in a follow-up question. Jeff explained that they were talking in another class about the relationship between the distances on the axes and the values of coordinates. Linda said that she was not thinking about this connection while she was developing her equations but that it would be reasonable and that she would draw a rectangle.

Mathematically Valid Arguments Based on Reasoning About Quantities

All the future teachers provided viable arguments explaining how at least one of their equations represented the relationship between the x-strip and y-strip. Although Jeff's explanation of his equation $^2/_5\ x = y$ lacked detail, he gave an adequate explanation for the equation $^5/_2\ y = x$, using a method like Claire's.

Strips and Letters Represent Quantities, Although Not Necessarily Quantities that Vary

In their explanations, all the future teachers used words and gestures relating the strips, their parts, and the letters x and y to each other and treated these entities as quantities. However, we cannot tell from their arguments whether they viewed x and y as numbers. For example, it is possible that they viewed x and y as quantities but not as x units and y units.

Only Claire and Diana discussed that their equations hold generally, giving some sense that x and y could vary, and only Diana explained that the parts could change in size but that the numbers of parts stay the same, so that y will always be $^2/_5$ of x. It is possible that the future teachers took for granted that x and y would vary, given that the Geogebra sketch shows this, and therefore didn't say anything about it. In any case, most of them did not highlight the idea of variation.

The "Of" and Equal Groups Interpretations of Multiplication

The future teachers' use of multiplication was more varied on the line task than it was on the fertilizer task. Alice and Diana used only the "of" interpretation of multiplication, Jeff and Kelly used only the equal groups interpretation of multiplication, and Claire and Linda used a combination of both the "of" and equal groups interpretations of multiplication.

Of the future teachers using an equal groups interpretation of multiplication, Kelly was the only one who gave an argument using a fractional multiplier. When Claire, Jeff, and Linda used an equal groups interpretation of multiplication, they did so with a whole-number multiplier. For example, to explain his equation $^5/_2\ y = x$, Jeff essentially explained the eq. $5 \bullet (^1/_2\ y) = x$ and implicitly used the associative property of multiplication. Jeff's wording is not as clear and explicit as Claire's, but he implies that x is made of 5 of the $^1/_2\ y$ parts:

> *So five halves Y* [points to $^5/_2$ in $^5/_2$ y = x], *each one of these is a half* [points to the 2 parts in the y-bar]. *Multiply it five times will give you X* [gestures across the entire 5-part x-bar].

6.3.3 Further Discussion of the Future Teachers' Reasoning on the Two Tasks

Looking at the future teachers' responses across both tasks, we find the diversity of their arguments notable. They used only a small collection of mathematical ideas (e.g., the CCSS definition of fraction, equal groups, and "of" interpretations of multiplication, strips and letters representing quantities), yet the ways in which they used these ideas to formulate arguments varied. They all started with the same strip diagrams and ended with the same equations (more or less), yet they varied in how they strung together ideas from the collection of ideas commonly used in class. In seeing this variation, we wonder whether it might reflect differences in the future teachers' ecologies of mathematical ideas. This is a topic for further study.

Given that developing algebraic equations is difficult, we find it noteworthy that all six of our future teachers developed and explained correct equations on both tasks. We expect future middle-grade teachers to have no trouble producing correct equations for lines. But in our experience, although they know that lines have certain kinds of equations, they do not know why that is so before our instruction. (We note that our line task assumes consequences of geometric similarity, which was not discussed in our algebra course.)

On the fertilizer task, none of our future teachers produced an incorrect equation with a "reversal error" (Clement, Lochhead, & Monk, 1981) by arguing as follows:

> The ratio of nitrogen to phosphate is 8 to 3 so 8N = 3P. Dividing both sides by 8, we have N = $^8/_3$ P.

After reading the statement of the fertilizer task, Linda wrote the notation "8 N: 3P" at the top of her page, which is similar to the reversal error equation, but after drawing a strip diagram, she immediately wrote correct equations and she explained them in follow-up questions.

We also did not see a "testing values" argument for the equation $^8/_3 \bullet P = N$ on the fertilizer task. In the past, we have seen some students debate between two constants of proportionality, just as Kelly debated between $^8/_3$ and $^3/_8$. Other future teachers resolved their debate by plugging in values (e.g., plugging in 3 for P and 8 for N). This method of plugging in values can help determine which coefficient is plausible, but it doesn't prove that the equation holds in general, for all coordinated values of the variables.

In contrast with a "testing values" argument, the arguments Kelly and others produced explain why the equation $^8/_3 \bullet P = N$ holds in general. However, other than Claire, we do not know if the future teachers saw that P and N in the equation $^8/_3 \bullet P = N$ could stand for numbers of *kilograms* that vary together. Their explanations were in terms of numbers of parts. From an expert point of view, an explanation derived from the parts of the strip diagram is general—it does not matter whether we view the nitrogen and phosphate in terms of numbers of parts or in terms of kilograms, the amount of nitrogen is $^8/_3$ of the amount of phosphate. Said another way, the number of groups of phosphate it takes to make the same amount as the nitrogen is $^8/_3$, whether we view the quantities in terms of parts or in terms of kilograms. But we do not know if the future teachers thought about their equations that way. Future research should investigate this issue.

Across the two tasks, the future teachers used a mixture of CCSS fraction language and "out of" language for fractions. Some arguments used a blend of the CCSS definition of fraction and the equal groups definition of multiplication. For example, on the line task Claire and Jeff did not describe the fraction $^2/_5$ separately from multiplication. When writing and describing her equation $y = 2(^1/_5 x)$, Claire explained, *Y equals 2 times* $^1/_5$ *X* and *Y has those 2 parts they're each* $^1/_5$ *of X, since X has 5 parts, each part* $^1/_5$ *X*. Claire also used similar reasoning on the fertilizer task. This way of reasoning allowed for using a whole-number multiplier instead of a fractional multiplier. In fact, across the two tasks, almost all of the arguments that used a fractional multiplier used an "of" interpretation of multiplication. There were only two exceptions: Jeff used an equal groups interpretation with the multiplier $2^2/_3$ on the fertilizer task and Kelly used an equal groups interpretation with the multiplier $^2/_5$ on the line task.

From an expert perspective, the following three ideas may seem to be closely related: (1) the equal groups definition of multiplication, (2) an "of" interpretation of multiplication, and (3) the idea that a multiplier tells us how many or how much of the multiplicand it takes to make the product amount. All three ideas were useful for the future teachers in this study. The data we examined for this study are not enough to determine whether our future teachers saw these ideas as connected and whether they could move fluidly from one to another. In any case, these connections may not be evident for future teachers, and we think that instruction should aim to help future teachers make such connections. We note that the connection between the equal groups definition and the "of" interpretation of multiplication is facilitated by putting the multiplier first and the multiplicand second in the equal groups definition of multiplication. (The order of the multiplier and the multiplicand is simply a convention, which is different in different countries.)

6.4 Conclusion

This chapter explored the viability of using strip diagrams and a variable parts perspective to generate equations in two variables for proportional relationships and for lines. We studied six future middle-grade teachers who were in our courses and were selected for diversity in their reasoning about fractions. We found that these participants reasoned successfully with a variable parts perspective and strip diagrams to develop and explain equations in two variables for (1) quantities varying together in a proportional relationship and (2) a line through the origin in a coordinate plane. Their reasoning relied on relating strip diagrams to letters (variables) and treating strips and letters as quantities. Their arguments were mathematically viable.

Our study was limited both in numbers of participants that we studied (only six), in the tasks we investigated (only two), and the fact that the course instructor is also the first author of this report. The purpose of the study was not to make general claims about all students or all future middle-grade teachers but rather to look in detail into the nature of the reasoning that our participants used. This reasoning was surely shaped by the experiences the future teachers had in their courses; however we cannot claim that other future teachers in other such courses would reason this way. Further research is needed to examine the viability and utility of a variable parts perspective more generally, for both teachers and students.

Despite its limitations, our study adds to a body of work that points to the utility of reasoning with strip (tape) diagrams (e.g., Beckmann, 2004; Englard, 2010; Kaur, 2015; Murata, 2008; Ng & Lee, 2009). Given that elementary school students are able to use strip diagrams to solve algebra word problems (e.g., start unknown problems) without letter-symbolic algebra (Ng & Lee, 2009), studies investigating how middle-grade students might reason with the combination of strip diagrams and algebraic equations should be worthwhile. In particular, students who have learned from curricula that use strip diagrams may be in an especially good position to learn to develop equations from a variable parts perspective.

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Chapter 7
Giving Reason and Giving Purpose

Yvonne Lai, Mary Alice Carlson, and Ruth M. Heaton

Abstract Skillful teaching involves seeing mathematics in ways that are coherent and making decisions during planning and teaching. In particular, teaching requires making decisions about what connections to make, when to make them, and how students might make them. We posit that it is important for teachers and teacher educators to understand the pedagogical work of making connections, as this pedagogical work positions students to access ideas later in the curriculum. We analyzed the teaching and planning sessions of a first grade teacher to examine the question: What are the characteristics of planning that make it possible for students to connect mathematics in ways that are productive in the short and long term? We frame this work in terms of connections that "give reason" (Duckworth, *The having of wonderful ideas and other essays on teaching and learning*. New York: Teachers College Press, 1996) and "give purpose" – making sense of mathematical representations and arguments and increasing students' access to content and practices valued by the discipline. We provide a concrete decomposition of the pedagogical work of planning for connections that give reason and give purpose. To illustrate the components we identify, we use the example of a first grade lesson whose goal was to help students transition from counting one by one when adding or subtracting to using the base ten system more intentionally. We close by describing possible future work in two arenas: designing opportunities to learn teaching that makes connections well and identifying learning opportunities made possible by such teaching.

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7.1 Giving Reason and Giving Purpose

Mrs. Reynolds was preparing a lesson for her first grade mathematics class. It was late February, and her students have made a lot of progress since September when many counted one by one to add numbers. To find that 4 + 5 equals 9, students wrote "1, 2, 3, 4" and counted on five more as they wrote, "5, 6, 7, 8, 9." Now the children were using more efficient strategies. They were "doubling": finding 4 + 4 (doubling) then adding 1 to obtain 9. On larger sums, a few students were using "easy tens." To find 7 + 5 equals 12, students noticed that 5 could be decomposed into 3 and 2, that 3 and 7 are 10, and that adding the remaining 2 gives a sum of 12. Students represented easy tens with "breaking down" (Fig. 7.1c) and "big hops" (Fig. 7.1d). However, some students were still using counting on, shown with "little hops" (Fig. 7.1a); and some relied on using ten frames (Fig. 7.1b). These strategies are less efficient.

Mrs. Reynolds decided that the lesson would begin with a "number talk" (Parrish, 2010): an activity where students talk through mental strategies for finding sums. Later, students would record their own solution to two story problems. Consider the lesson details Mrs. Reynolds needed to address as she planned the sums to assign for the number talk and story problems, which strategies to emphasize, how to use

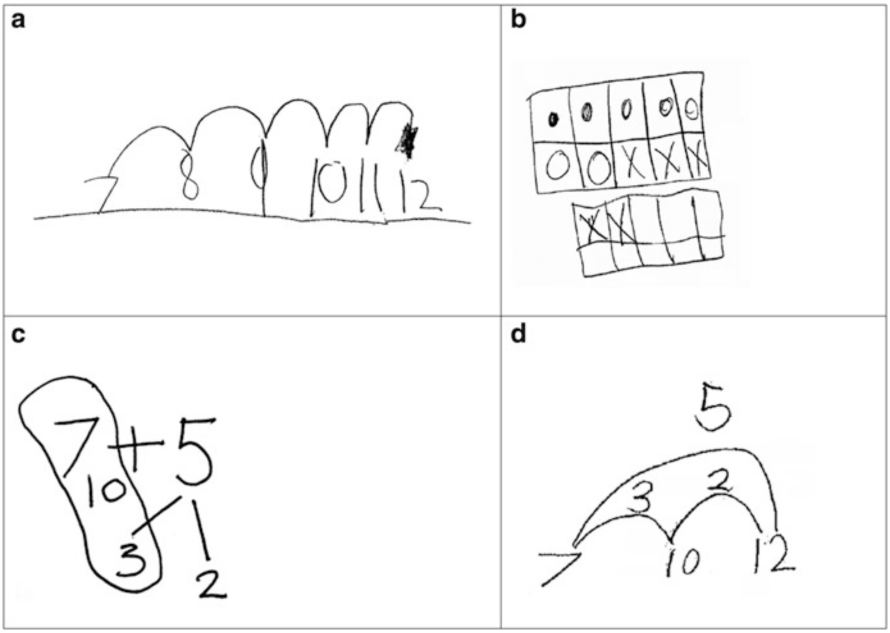


Fig. 7.1 (**a**) Little hops on the number line. (**b**) Ten frames. (**c**) Breaking down. (**d**) Big hops on the number line

student work to guide all students toward the main point of the day's lesson, and how today's lesson might impact students' understanding of mathematics encountered in later grades.

The way teachers address the details like the ones listed above makes or breaks a lesson's coherence. In the case of Mrs. Reynolds, on the one hand, the number talk could be used to see what strategies students are likely to use and encourage the strategies most productive for students to practice. The story problems could then reinforce these strategies. On the other hand, if the sums in the number talks and story problems have no potential to build on each other or if the efficient strategies are not much more efficient than counting on, or cannot easily be applied, there is little reason for students to practice them. Mrs. Reynolds also had to consider the mathematics happening. How do the different strategies help children access ideas later in the curriculum? Connecting ideas to build across time, using student work, is challenging.

This chapter examines the way a teacher addressed these details in one particular lesson, how planning shaped the lesson, and what the lesson highlights about teaching more generally. We identify useful characteristics of connections, and then we examine the question:

> *What are characteristics of planning that make it possible for students to connect mathematics in ways that are productive in the short- and long-term?*

We identify components of such lesson planning. We begin the chapter by describing our research setting, a Math Studio, and explain why the Math Studio project was a particularly fitting site for investigation into a teacher's planning practices. After describing the lesson, we comment on the nature of the connections the teacher made. Finally, we discuss the components of planning for connections.

7.2 Math Studio

This study focuses on Mrs. Reynolds and her first grade students. Mrs. Reynolds and Miss Curtis were two teachers in a mid-sized school district in the Midwestern United States whose classrooms were the site of a 2011–2012 professional development project called Math Studio.[1] Math Studio is a variant of Lesson Study (Lewis, 2002). Teachers, along with school and district leaders, come together to observe mathematics lessons and to discuss the teaching in planning and debriefing sessions. A facilitator (in this study, the third author) works with the teachers so instructional decisions are visible to the observers. In our study, the live facilitation was particularly important during the teaching. Short conferences between the

[1]The original design and implementation of Math Studio is led by Linda Foreman of the Teachers Development Group, West Linn, OR.

facilitator and the studio teachers, which took place during lessons and which observers could hear, helped capture the real-time decisions of Mrs. Reynolds and Miss Curtis, as well as their pedagogical reasoning.

Math Studio design includes an emphasis on enhancing teachers' mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) and increasing students' metacognitive development by attending carefully to their own mathematical ideas in relation to others' ideas (Franke, Carpenter, Levi, & Fennema, 2001). Its design also intentionally makes the subtle but critical nuances of teaching and teacher decision-making and pedagogical reasoning visible. The observers hear what classroom teachers anticipate about students' engagement with mathematics content and see how and why teaching decisions are carried out. Math Studio is public in the sense that there are observers in the classroom, but the public nature is more than just live observation. In Math Studio, teachers communicate their thoughts to a wide audience. Teachers and coaches from multiple grade levels, including university level, share their responses to planning and debriefing. These conversations are public in the sense that they respond explicitly to perspectives beyond those of the grade taught. The teachers' reasoning and the purpose for their instructional decisions are meant to be accessible to those in K-12 and beyond.

7.3 Lesson

We now return to Mrs. Reynolds' lesson.[2] We describe the number talk and the story problem. Mrs. Reynolds' purposes were to increase students' inclination toward an understanding of the big jump on the number line strategies, especially when working with easy tens. When it comes to making connections, three aspects of her approach are noteworthy. First, Mrs. Reynolds used the number talk to foreshadow the next part of the lesson and orient students toward using specific strategies. Second, Mrs. Reynolds' approach connects arithmetic and place value – the second of Howe's (2011) "Three Pillars of First Grade Mathematics." In the lesson, Mrs. Reynolds' students work on problems that involve higher addition facts (addition facts whose sum is within 20). The emphasis of students' work is not on memorizing these facts "but understanding how to produce them, and their connection to place value notation" (Howe, 2011, p. 4). Finally, Mrs. Reynolds made extensive and intentional use of the number line throughout the lesson, an approach that is, in our experience, uncommon. Moreover, carefully planned teaching moves, including appropriately sequenced questions, are needed to help students move productively between number line and symbolic representations of addition and subtraction expressions (Clements & Sarama, 2009). Recommendations for helping elementary

[2]The narrative has been edited for grammar and coherence. It skips over some short exchanges between Mrs. Reynolds and her students, so it is easier to keep track of how the lesson progresses.

students with arithmetic emphasize extensive use of the number line (e.g., Gersten et al., 2009), in part because of its critical role in understanding non-integer numbers (Siegler et al., 2010).

7.3.1 Using a Number Talk to Orient and Foreshadow

The students were seated in a semicircle in front of a small whiteboard. Math Studio observers were seated around the edges of the classroom, with the Math Studio facilitator standing nearby. To open the lesson, Mrs. Reynolds assigned the string of sums below.

$$5 + 5 + 8 =$$
$$3 + 4 + 6 =$$
$$4 + 5 + 6 + 5 =$$

Writing them on the board, one at a time, she asked the children how they found each sum. For the first, a student answered, “I was thinking 5 and 5 make 10.” Mrs. Reynolds asked the student to show this on the board. The student walked to the front of the room and faced the class. He connected the two 5s with a vee and said, “I knew there were 8 left so that makes 18.”

Cody raised his hand. He erased the vee, extended lines from the 8, wrote 4 and 4, and said, “I combined the 4 with the 5,” as he circled the two pairs of 4 and 5. “And then I added the 9 and the 9 together. And I know that 9 plus 9 equals 18 because 8 plus 8 equals 16, so 2 more would be 18.” After another student asked for and obtained clarification on Cody’s strategy, Mrs. Reynolds moved the class to the second sum.

Bradley offered the following explanation. “I think it’s 13. Because when you combine these [points to 4 and 6] you get 10 and then you get 13.”

Peyton raised her hand next. She drew one ten frame and then a second ten frame. Mrs. Reynolds addressed the class, “As she’s solving it, watch how she solves it. See if you see a connection between how she and Bradley solved it.” Mrs. Reynolds watched the students. Turning back to Peyton, she asked, “Why did you make another ten frame, Peyton?”

Peyton explained, “When I make my ten frames, I know that that’s a whole 10 and then I added 3 more. And then I said, 10, 11, 12, 13.”

Mrs. Reynolds asked, “How did you know – and listen carefully here – to fill the first ten frame all the way?”

Peyton looked at her work. She began, “Because it’s, it’s …,” and then circled the 4 and 6.

Mrs. Reynolds responded, “So you saw a full 10?”

Peyton nodded, “Yeah. And then I saw a 3.”

Finally, for the third sum, Karen said, "I think that it's 20 because 6 and 4 equals 10 and 5 and 5 is 10, and I know two 10s make 20."

Autumn raised her hand. She said, "I combined the 4 and 5 to make 9 and 6 and 5 to make 11. And then if you don't have 10 over here and you have one more than 10 over here, then you can give one away, and then you have 9 and 11, and then you have 20."

Mrs. Reynolds asked other students to retell Autumn's solution. When Caleb repeated Autumn's first sentence but not the second, Mrs. Reynolds prompted, "And then there was something else. What was Autumn telling us?" She helped Caleb recreate Autumn's solution and then reiterated to the class the logic of creating two 10s from 9 and 11.

7.3.2 *Using Story Problems to Represent Strategies on a Number Line*

After the number talk, students returned to their seats. Mrs. Reynolds asked the students to work on the following problems at their desks, individually, and to demonstrate their thinking as clearly as possible:

I baked 7 cookies. The next day I baked 5 more. How many cookies did I make?

I went to a party. I was holding 15 balloons. Then 8 of them popped. How many balloons are left?

As the Math Studio community anticipated during the planning session the day before, for the first problem, the students produced solutions typified by the four shown in Fig. 7.1a–d. Now Mrs. Reynolds faced the question of how to use these solutions to orient students toward the goal of showing big hops on the number line. Mrs. Reynolds, Miss Curtis, and the Math Studio facilitator conferred. They discussed the strengths and limitations of showing the following solutions:

- Ten frames (Fig. 7.1b) and then big hops on the number line (Fig. 7.1d)
- Ten frames (Fig. 7.1b) and then breaking down (Fig. 7.1c)
- Little hops on the number line (Fig. 7.1a) and then big hops on the number line (Fig. 7.1d)
- Breaking down (Fig. 7.1c) and then big hops on the number line (Fig. 7.1d)

Ultimately, Mrs. Reynolds opened the discussion by asking Katie to explain her breaking down solution. She then constructed a big hop on the number line solution and at the same time interpreted the solution in terms of little hops. Figure 7.2 shows the board as the first grade class saw it, after the discussion. We now describe how Mrs. Reynolds orchestrated the three solutions, in parallel. This narrative continues just after Katie presented her work.

Mrs. Reynolds asked the class, "Let's take a look at using Katie's strategy and how we could do that on a number line. What number do we want to start with? Peyton?"

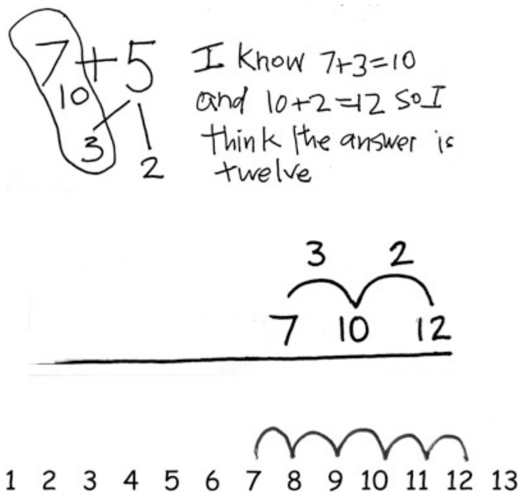


Fig. 7.2 Katie's solution and Mrs. Reynolds' use of big hops and little hops on the number line, as they appeared to the class after a discussion

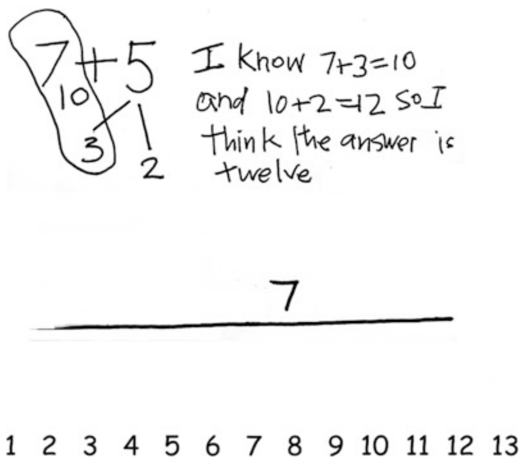


Fig. 7.3 Katie's solution and Mrs. Reynolds' introduction to the big hop on the number line solution

Peyton supplied the starting number 7.

Mrs. Reynolds repeated, "Seven. And I know that 7 cookies is what I started with the first day. So I'm going to put that up here. First graders, what happened that next day when I made cookies?" The board displayed the number line as shown in Fig. 7.3.

The first graders chorused, "You made 5 more." Mrs. Reynolds then demonstrated how to continue mapping Katie's solution onto the number line (Figs. 7.4 and 7.5).

Mrs. Reynolds: But Katie didn't just add 5 cookies. What did Katie do? She broke it down. Could we break numbers down on the number line, too? So if Katie took her 5 and she broke it down to make an easy 10, I could make my first hop [draws an arc from 7 to 10 on the number line] how many?

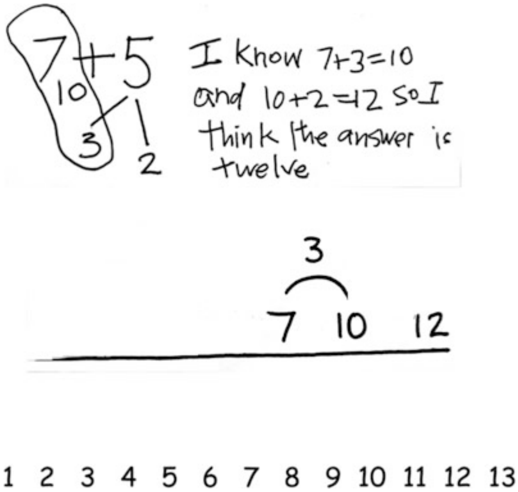


Fig. 7.4 Mrs. Reynolds' work, continued

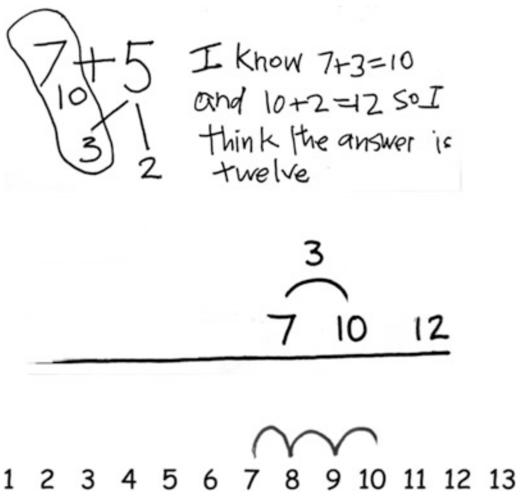


Fig. 7.5 Mrs. Reynolds' work, continued

Students: Three.

Mrs. Reynolds: I can break down the 5 with a 3. And that goes to 10. Let's check that. [Mrs. Reynolds references a number line on bottom of the board.] I start at 10 and I do one hop, two hops, and three hops.

Mrs. Reynolds: But did I just make 3 cookies?

Students: No.

Mrs. Reynolds: How many more cookies did I make?

Student: Two more because I know that 2 plus 3 equals 5.

Mrs. Reynolds: Two more. So I do another hop of 2. [Mrs. Reynolds draws an arc and labels it "2."] So how many cookies did I make?

The board now appeared to the first graders as in Fig. 7.1.

Mrs. Reynolds used the arcs to confirm that the total number of cookies is 12 and that the total number of hops is equal to 5. She selected another student, Thomas, to explain his breaking down solution. She then asked the students to work with a partner to represent Thomas's strategy using big hops on the number line. While the students worked, Mrs. Reynolds planned the subtraction problem discussion with the third author. Students had produced breaking down solutions with doubling (splitting the 8 into 4 and 4), little hop on the number line solutions, and ten-frame solutions. They discussed sharing:

- Breaking down and then asking students to draw analogous big hops on the number line
- Little hops on the number line and then asking students to draw analogous big hops on the number line

After reviewing the big hops representation of Thomas's solution, Mrs. Reynolds turned to the subtraction problem. After Mandy shared a little hops strategy, Mrs. Reynolds asked the class how to represent this with big hops. One student suggested breaking down the 8 into 5 and 3. After two students described the big hops, Mrs. Reynolds depicted the big hops in parallel with little hops as she had with the addition problem, showing a big hop from 15 to 10 and then from 10 to 7.

Finally, to end the lesson, she asked the students to find $8 + 6$ using a big hops strategy on the number line. One student, who had previously used ten frames even when asked to use other strategies, used big hops for this last problem. After circulating the class, the facilitator and Mrs. Reynolds agreed to go over the students' solutions during the debriefing to determine what to do in the next lesson.

7.3.3 Observations About the Lesson

From the numbers selected to comments she made within each part of the lesson and the solutions she highlighted, Mrs. Reynolds consistently moved the students toward using big hops on the number line. She also intentionally emphasized the importance of tens and the number line. When the first student used the fact that $5 + 5 = 10$ in the number talks, Mrs. Reynolds made a point of asking the student to show this visually to the rest of the students. When Cody used a doubling strategy, Mrs. Reynolds did not comment on it, but she highlighted Peyton's use of ten frames and asked students to pay particular attention to the parallels between Bradley's and Peyton's solutions. In the final number talk, Mrs. Reynolds again emphasized the use of tens as well as the importance of being able to recreate each other's solutions. Mrs. Reynolds' emphases during the number talk directly supported the instruction of the story problem, where many students used easy tens. As students shared solutions to the story problem, Mrs. Reynolds deftly moved from one representation to the next, using the breaking down representation as a basis for the big hops on the number line and using the little hops to make sense of both the breaking down and the big hops. Throughout this lesson, Mrs. Reynolds connected mathematical ideas while moving toward a larger goal.

Mrs. Reynolds' instructional decisions emphasized important concepts while helping her students develop procedural skills. These decisions may be unusual in US mathematics classrooms. US children have been documented to continue to rely on using known additive relationships such as doubles to find unknown sums rather than using tens and therefore missing opportunities to build a foundation for understanding place value (Clements & Sarama, 2009; Fuson & Kwon, 1992; Murata, 2004). Selecting 5 and 7 for the story problem steered students away from using known double facts to find an unknown total. The expressions Mrs. Reynolds selected for the number talk implicitly suggested the use of the associative and commutative properties to make tens. Peyton's use of ten frames emphasized the total as "ten and some more." When Bradley worked with the first sum, he showed some understanding that the order of numbers added does not matter.[3] Throughout this lesson, the way that Mrs. Reynolds set up connections oriented students toward larger goals and significant mathematical ideas.

In the next section, we address why the connections in the lesson are noteworthy. We highlight two essential characteristics to use in considering connections more generally. Based on the analysis of Math Studio Data, we then describe components of planning that make it more likely that students make connections that serve the short and long term. Then, to help the reader understand where our decomposition (Grossman et al., 2009) comes from, we precede it with a description of the data collected from Math Studio, how we worked with the data, and the role of the three authors. Finally, we illustrate the components using the planning for the introductory episode as an example.

7.4 Giving Reason and Giving Purpose

We highlight two characteristics of connections in teaching, which we term "giving reason" and "giving purpose." We illustrate that teaching can accomplish these simultaneously, using the connections between Peyton's and Bradley's solutions and between Katie's solution and the big and little hops solutions. The phrase "giving reason" is from Duckworth (1996), who used it to describe a stance of reading students' work with the aim of understanding the reasoning the students may have used.

We use "giving reason" to refer to connections that help students make sense of mathematical representations and arguments. These connections show that one can reason through solutions different from one's own and understand why the solution can be analogous to one's own. Bradley's and Peyton's solutions both combined

[3] Howe and Epp (2008) named this the "Any-Which-Way Rule."

numbers to make ten, but Peyton represented her thinking using a ten frame. Ms. Reynolds' questioning helped her students see the parallels between Peyton's and Bradley's solutions. Both focused on adding 4 and 6 first, resulting in one 10 to which the students added 3. By reinforcing to students that filling a ten frame is a way to represent known facts about 10, students who rely on the concreteness of a ten frame may begin to move toward more abstract representations and do so knowing that the mathematics still holds. Similarly, Mrs. Reynolds talked the students through why Katie's breaking down solution was equivalent to the big hops solution and used little hops as a basis for why the representations are equivalent. The connections from concrete to abstract may help students move toward more efficient representations on firm footing.

We use "giving purpose" to describe connections that increase students' access to content and practices valued by the discipline of mathematics. The connections in Mrs. Reynolds' teaching were selected with the lesson purpose in mind: to move the students toward using efficient strategies with more abstract representations. In the case of this lesson, the efficient strategy and abstract representation emphasized were easy tens and big hops on the number line. This purpose aligns with recommendations for school mathematics standards (National Governors' Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010), and it supports learning mathematics in grade school and beyond. Viewing the sums between 10 and 20 as "ten and some more" emphasizes place value and plays a critical role in understanding algorithms for multi-digit addition. Finally, decomposing and representing big hops to ten on the number line introduced a strategy that students continue to use throughout elementary school, particularly when adding and subtracting multi-digit numbers, as well as a representation they would use in high school and beyond. Beyond the potential to help students understand content encountered in the future, the connections in Mrs. Reynolds' teaching also modeled mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996) including describing solutions completely and precisely and constructing viable arguments. These are practices that are common across mathematics. Moreover, insofar as coherence is an intrinsic part of mathematics, teaching mathematics must include supporting students in experiencing mathematics as coherent.

The characteristics of connections that we refer to as "giving reason" and "giving purpose" are not new. They are often treated separately in teacher education literature (though there are exceptions, such as the *Everyday Mathematics* materials from the University of Chicago). For instance, there is some consensus among educators that learners benefit from connecting related ideas and representations and that teaching should create opportunities for learners to identify and explain connections (e.g., Carpenter et al., 1999; Stein, Grover, & Henningsen, 1996; Stein, Smith, Engle, & Hughes, 2008). These connections, which "give reason," are compelling because they show that even reasoning done in a relatively short time span, such as a number talk discussion, is worth attending to. In the literature, on one hand, connections that "give reason" often take place at the level of a particular idea, task, or representation. Implications of the particular situation for mathematics more generally, especially connections to later content, are often not addressed. On the other hand, connections that “give purpose” emphasize similarities across mathematics. Consider treatments of elementary mathematics from an advanced standpoint (e.g., Klein, 1908; Moise, 1963; Usiskin, Peressini, Marchisotto, & Stanley, 2003). By building from number and operation on integers to arithmetic on complex numbers and then quaternions, Klein (1908) underscored structural commonality in number systems encountered in elementary, secondary, and disciplinary mathematics. However, connections that give purpose may not motivate the scope and sequence of a particular unit of mathematics, even if it does motivate the goals of the unit. For instance, knowing that quaternions are not commutative might illustrate that commutativity is not to be taken for granted, but it does not give insight into how to show that fraction multiplication commutes.

What makes the connections special in Mrs. Reynolds’ teaching is that they simultaneously give reason to the children and have the potential to give purpose to the mathematics. In the remainder of this chapter, we examine the role of lesson planning in making these connections possible. We describe the data that we used, the Math Studio planning session for the lesson discussed above, and essential features of the planning.

7.5 Data and Analysis

Data for this study include five videotaped planning, teaching, and debriefing sessions, as well as associated lesson plans and student work. Through data from the Math Studio project, we observed how Mrs. Reynolds and Miss Curtis planned and taught five lessons, reflected about teaching and learning, and weighed major decisions while teaching. To respond to the question of interest (“What are characteristics of planning that make it possible for students to connect mathematics in ways that are productive in the short and long term?”), we iteratively selected and analyzed video segments. Each segment consisted of instruction or planning that addressed a particular mathematical or pedagogical task within the lesson. Instructional segments were segments of videos of the lessons. Planning segments came from two sources: the planning session itself and the in-the-moment conferences between the Math Studio teacher and the facilitator. These conferences represent a kind of planning done within instruction, during which the teacher decides what should be done next, based on what students know and the lesson’s goals.

For each instructional segment, we asked: What ideas were connected and how? We then examined planning segments, asking: Were the connections made intentional? What were the intentions, and how did they shape the way in which the connections were made? Through this process, we selected instructional segments from our larger data set for closer analysis. We reexamined this set of segments, asking: What pedagogical problem does the teacher attend to? We then analyzed the planning segments for decision points that shaped the problem solving. Finally, we reviewed the debriefing sessions for consistency. If our conclusions about how planning had shaped the lesson were contradicted by comments in the debriefing, then we revised our conclusions.

The first and second authors led the analysis. The organizers and facilitators of the Math Studio project studied here were the second and third authors. It is often productive for the research team analyzing mathematics teaching to have complementary backgrounds (Thames, 2009). The first author is a mathematician by training. The second and third authors have taught elementary school and are educators by training. Together, the three authors have taught elementary mathematics, led professional development of elementary teachers, and conducted mathematics education research and mathematics research. The analysis was informed by their experiences in teaching and mathematics.

7.6 Planning

We highlight three segments from the planning session. In the first, the teachers discuss students' readiness for the lesson goal of using more efficient strategies. In the second, the teachers plan the story problems. In the third, they discuss overall considerations for the lesson. Prior to the planning session, the teachers had emailed the second author to express an interest in planning story problems. The main facilitator for this planning session was the third author.

7.6.1 Appraising Students' Past and Current Mathematical Work

In the library, 18 educators from Mrs. Reynolds' and Miss Curtis's school district were seated. They included teachers, who taught different grade levels from K-12, and coaches (mathematics teaching specialists who worked directly with teachers to improve instruction). The educators were arranged around Mrs. Reynolds, Miss Curtis, the second and third authors, and the two math coaches who worked with Mrs. Reynolds and Miss Curtis.

Addition and subtraction within 20 is a critical area in first grade mathematics, so students' and teachers' work in a school year involves developing increasingly sophisticated and efficient strategies for combining numbers. Developing fluency with numbers within 20 is important as a foundation for arithmetic with larger numbers. Mrs. Reynolds wanted more students to use more efficient strategies. In recent lessons, there were hints that students may be ready to move away from counting on strategies.

"Readiness" became an important topic for the planning session. Prompted to describe how she knew that students were ready for more efficient strategies, Mrs. Reynolds described looking not only at what representations students used but also how students used those representations. She observed that students no longer filled in multiple ten frames one box at a time, without identifying each frame as a" ten," as they had earlier in the year. They now recognized when a sum consisted of a "whole ten" and "some extras." The students' comfort with decomposing numbers into tens and ones indicated an opportunity to make a connection to the number line. For example, suppose students used ten frames to work on 7 + 5. The observation that "I have a whole ten and some extras" can be used to model working with a number line, pointing out that 3 more than 7 is 10 and two more will make 12. The teachers identified using "big hops" to 10 as appropriate territory for advancing students' mathematical work. The coaches concurred with this choice and noted that using 10 as a "landmark number" now could help students be ready to use 100 as a landmark number in third grade.

The Math Studio facilitator commented that some students could be uncomfortable with the more sophisticated strategies if they are "daunted by the idea of trying something new." She suggested that when students struggle with more efficient strategies, "it may be helpful for them to see the connection between a way they know and are comfortable with and a new way" (Math Studio Planning Session, February 21, 2012, p. 5). Mrs. Reynolds and Miss Curtis discussed ways that they had themselves pointed out similarities between strategies – but had not given students the responsibility to make connections. Following a brief pause, the facilitator moved the discussion to planning story problems.

7.6.2 Planning the Sum Featured in the Story Problem

Mrs. Reynolds and Miss Curtis expressed concern that their students did not understand all types of addition and subtraction problems. They considered the possibility of working on addition and subtraction problem types that challenged the students, such as compare problems (e.g., I have 2 nephews and 5 nieces. How many more nieces than nephews do I have?). Miss Curtis suggested focusing on either strategies or problem types, but not both. Although they did not explicitly state agreement on Miss Curtis's point, both teachers primarily discussed strategies from this moment onward.

The teachers discussed number combinations most likely to elicit the decomposition of addends to reach a "whole ten" and "some extras" and the number line representation. Mrs. Reynolds suggested avoiding "doubles plus one" and "doubles minus one"; in her experience, students "would only see those strategies" with sums such as 7 + 6 or 8 + 7, whereas students were more likely to use easy tens with sums such as 8 + 5. The teachers then considered what they could do if students did not readily select to use the number line.

7.6.3 Responding to Anticipated Student Work

Mrs. Reynolds said, "We hope that the students do use the number line. But if they don't, we can take a student's breaking down solution and ask, 'How could you do this on a number line?' We could pose this question to pull students to the number line."

Once Mrs. Reynolds articulated that connections to the number line could be made even if the number line did not arise from student work, the facilitator focused on making the connections possible.

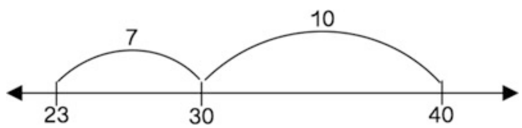
Facilitator: This seems like a good opportunity. The idea of easy ten and moving toward an open number line might work together with the idea of big hops. So when you're comparing strategies, you can be intentional about moving toward an open number line.[4]

Implicit in the facilitator's suggestion that the teachers "be intentional" was the idea that much of the work involved in creating opportunities for comparisons between strategies and movement to the number line would have to be done during the lesson. In other words, the teachers would be particularly attentive for opportunities to relate student strategies to one another and opportunities to represent strategies as "big jumps" on the number line.

7.6.4 Observations About Planning and Its Impact on Instruction

We note that the planning discussions are implicitly framed by two questions: How are students progressing relative to what they have learned and what they will encounter? What is the significance of students' and teachers' actions relative to the connections they make available? The answers to these questions shape the lesson. Mrs. Reynolds noticed that her students had begun recognizing when a sum consisted of a "whole ten" and "some extras." However, they were still using ten frames. Both teachers also realized that they had not given students as much responsibility for connecting representations as they might have. Articulating students'

[4]Open number lines do not contain any predetermined markers. Instead, numbers and markers are added to create records of students' mental computation strategies. For example, 23 + 17 could be recorded as follows.



past and current mathematical work and assessing their "readiness" to move forward created a foundation upon which Mrs. Reynolds could make instructional decisions for the lesson. We interpret this realization as having impacted the lesson in several instances, including when Mrs. Reynolds asked Peyton to explain the connection between her strategy and tens and when she asked students to participate in an explanation of why the little hops, big hops, and breaking down solutions were equivalent.

Furthermore, in our analysis across all planning segments, the work of planning opportunities to make connections has three components, which are exemplified by the segments described above. First, the teachers identified goals for the class based on their knowledge of the students and curriculum. Then, the teachers set the goal of moving students toward easy tens and solutions with big hops on a number line. Finally, the teachers designed questions to elicit number line strategies, anticipated student thinking for different sums, and prepared for how to respond if students did not use the number line. This discussion played an important role in identifying opportunities to make mathematical connections. In fact, many students did not use the number line, and those that did may have counted one by one (see Fig. 7.1a–d). In all three components, the teachers' discussion responded to the two framing questions. The planning was key to the instruction's success, particularly the attention to how students may or may not connect ideas and what the connections might look like. We now describe the three components in more detail.

7.7 Decomposition of Planning for Connections that Give Reason and Give Purpose

7.7.1 Components

We identify the components as (1) identifying current and new mathematical ideas and evaluating them, (2) choosing a new mathematical idea to focus on, and (3) designing work that elicits current ideas, new ideas, and the connections between them. Each component requires substantial mathematical knowledge for teaching (Ball et al., 2008).

7.7.2 Identifying Current and New Mathematical Ideas and Evaluating Them

This first component brings together teachers' knowledge of the mathematics content and their students. It requires teachers to draw on their knowledge of the ways students initially encounter a mathematical idea as well as the ways those ideas have developed over time. Mrs. Reynolds and Miss Curtis identified current and new mathematical ideas when they articulated counting on strategies for addition that some students relied on and the newer, more efficient strategies that other students were beginning to use. What is important to notice is that Mrs. Reynolds and Miss Curtis identified strategies in the context of their own students' mathematical development. By drawing on general understanding of students' mathematical development to interpret the thinking of a specific group of students, Mrs. Reynolds and Miss Curtis were able to evaluate students' readiness for particular strategies and anticipate opportunities for connecting. In this way, Mrs. Reynolds and Miss Curtis considered ways to give purpose to the students' mathematical work. The less efficient strategies that some students were likely to use could set up opportunities for meaningful connections to new strategies and help students develop a more integrated understanding of the mathematical ideas at hand. Evaluating new strategies as potentially worthwhile for students involved considering the strengths and limitations of those strategies generally, as well as the potential for students to meaningfully use those strategies.

7.7.3 Choosing a New Mathematical Idea to Focus on

Choosing a new mathematical idea to focus on integrates teachers' knowledge of content, students, and curriculum. Having sketched the potential terrain of the lesson, Mrs. Reynolds and Miss Curtis then could decide which paths to make most visible to students. They stepped back and considered how addition strategies would develop over time. Some strategies, such as using ten frames, would eventually be less powerful as students worked with larger numbers. Other ideas, such as the open number line, would continue to be developed over time within the context of performing and visualizing mental computations. Still other ideas, such as place value and placing emphasis on the base ten structure of numbers, were foundational to work with number and operations. Moreover, these latter strategies engaged students in mathematical practices such as looking for and making use of structure (NGACBP & CCSSO, 2010), worthwhile endeavors for students regardless of the specific content at hand. In this way, carefully choosing a mathematical idea to focus on gave purpose to the mathematics within the lesson. The work that the students did on that day increased their access to ideas they would need well beyond elementary school.

Part of choosing a new mathematical idea to focus on is to recognize when competing agendas arise. For instance, when discussing subtraction problems, Miss Curtis advised that their lessons focus either on subtraction problem types or strategies, but not both. At this point, they might have chosen to change the focus of the lesson, which would likely have had implications for the rest of the lesson. They chose to focus on strategies, which influenced the types of problems they designed.

7.7.4 Designing Work that Elicits Current Ideas, New Ideas, and the Connections Between Them

The third component of planning for connections that give reason and give purpose involves design work that integrates the prior two. In this component, teachers develop plans that elicit students' current ideas, encourage them to explore new ideas, and create opportunities for making substantive connections between them. In the case of Mrs. Reynolds and Miss Curtis, this work involved planning both prior to and during teaching. Prior to the lesson, Mrs. Reynolds and Miss Curtis designed a sequence of sums for the number talk and selected numbers for the story problems that encouraged students to make tens. During the lesson, through her conversation with the Math Studio facilitator, we saw that Mrs. Reynolds' and Miss Curtis's story problems elicited both less efficient and more efficient strategies. Mrs. Reynolds then planned while teaching to make use of the strategies that actually emerged. It should be noted that this kind of design work did not require students to use the number line strategy, which Mrs. Reynolds and Miss Curtis had hoped to see. Instead, the design gave students reasons to choose the strategy themselves. Mrs. Reynolds and Miss Curtis could have designed a lesson wherein the teacher demonstrated decomposing numbers to make a ten and then require students to use the same strategy. Such an approach is deceptively straightforward and may seem to accomplish the goals at hand. However, it falls short of giving reason and giving purpose because the teacher, not the students, does the work of connecting. Mrs. Reynolds and Miss Curtis designed a lesson that facilitated *students* making connections between and among strategies.

7.7.5 Interaction Among Components and Knowledge Used

The three components interact: identifying influences choosing; designing can generate new potential goals and also influence identifying and choosing. When the goal for the lesson is stable, more attention can turn to interactions between designing how to elicit current ideas, connections to the new goal, and the new goal itself. Thus, the components shape each other over time. How one lesson is designed influences the choices available for the next lesson, and lessons in the past and future constrain the current lesson: what is possible to identify and select comes from the connections and new images previously elicited, and what has been connected and elicited was influenced by prior choosing and identifying.

Table 7.1 illustrates the work of "setting up" or planning based on the narrative presented earlier in this chapter based on our data. The questions support teachers in reflecting on and planning for how students might respond, what strategies they might use, what to monitor students' work for, how to select students to present ideas, and connecting ideas to deepen understanding (Stein, Engle, Smith, & Hughes, 2008).

Table 7.1 Decomposition of the work of setting up opportunities for connecting

<table>
<tr><th colspan="2">Tasks of setting up</th><th>Setting up prior to and while teaching</th><th>Questions to support teacher reflection and planning</th></tr>
<tr><td colspan="2">Identifying current and new mathematical ideas</td><td>Current ideas: less efficient strategies for adding (e.g., counting on using ten frames and number lines, doubling strategies)
New ideas: representing numbers on an open number line; breaking numbers down and making easy tens</td><td>What current understandings do students hold?
What may be new and worthwhile for students to move toward in upcoming lessons?</td></tr>
<tr><td colspan="2">Choosing a new, focal mathematical idea</td><td>Transition from less efficient to more efficient addition and subtraction strategies
Using “big jumps” on the number line</td><td>What is mathematically worthwhile about the named new ideas?
What ideas would the curriculum support, and what ideas would support using the curriculum in future units?
What would help students make mathematical progress in the longer term?</td></tr>
<tr><td rowspan="2">Designing Work that Facilitates Connecting</td><td>Eliciting public, individual representations of current ideas</td><td>Designing problems on which students are unlikely to use doubling
Designing problems for which breaking down with easy tens would use jumps of distance more than one
Asking students to “show a strategy,” not only to find an answer</td><td>What representations do students find available?
What representations help students grasp connections to new ideas?
How might challenges in the task be leveraged to give reason or purpose to the connection?
Which representations will be the focus of discourse about current ideas?</td></tr>
<tr><td>Eliciting public connections between current and new ideas</td><td>Mapping shared strategies to the open number line and to easy tens throughout the lesson
Selecting strategies that use easy tens to share in whole class discussion
Showing “big” and “little” jumps on the number line result in the same sum or difference</td><td>How do current ideas connect to new ideas?
What current and new ideas do students use in their mathematical work?
Which new ideas do students make connections to?</td></tr>
</table>

(continued)

Table 7.1 (continued)

Tasks of setting up		Setting up prior to and while teaching	Questions to support teacher reflection and planning
	Eliciting public, individual representations of the new idea	Asking a student to show their strategy to the class, and asking the other students to articulate the key points of the presenting student's explanation Asking students to individually show a breaking down strategy on an open number line for a new problem	What are the opportunities for students to represent new ideas? Are those opportunities accessible to students? How can student thinking about new ideas be made visible to the teacher?

7.8 Cultivating Teaching that Gives Reason and Gives Purpose

> We became convinced that no amount of theory can affect children in schools except as it becomes a fundamental part of a teacher's thinking.
> – Duckworth, *The Having of Wonderful Ideas*, 3rd ed., p. 86

Teaching mathematics means coordinating the demands of responding to children and conveying mathematics with integrity. Scholars have named these dual demands as "twin imperatives of responsibility and responsiveness" (Ball, 1993, p. 374) and "being authentic (that is, meaningful and important) to the immediate participants and authentic in its reflection of a wider mathematical culture" (Lampert, 1992, p. 310). We pair giving reason with giving purpose to situate these demands in the work of making connections.

Our work builds on firsthand accounts of the complexities of teaching (e.g. Heaton, 2000; Lampert, 2001) and calls to "build knowledge of and for teaching practice" (Ball, Sleep, Boerst, & Bass, 2009, p. 459). By examining the planning, teaching, and debriefing of actual lessons and by taking advantage of the insights afforded by the Math Studio structure, we offer a decomposition of practice that takes seriously the affordances and challenges that arise when designing and enacting lessons that give reason and purpose. Facilitating mathematical connections meets Ball et al.'s criteria for "high-leverage practices for beginning teaching of mathematics" (p. 461) that is generalizable, useful, and teachable. If children are going to come to see mathematics as an integrated, coherent set of ideas, then teachers need to facilitate connections that give reason and give purpose, regardless of the grade being taught, content under discussion, curricula being used, or instructional style of the teacher. Our "Tasks of Setting Up" and "Design Work that Facilitates Connecting" are an effort to make the pedagogical work of connecting "teachable." We capture critical decision points teachers may face and describe the practice of connecting around those decision points so that it can be integrated into preservice teacher education courses and professional development opportunities.

We suggest future work in two arenas: designing opportunities to learn teaching that gives reason and purpose through the pedagogical work of connecting and identifying learning opportunities made possible by such teaching. Participants of the Math Studio project leveraged 2 h planning sessions per Math Studio lesson in the presence of a large community and external expertise – time and effort that is likely impossible to carry out every day, no matter the benefits. Moreover, the learning that Math Studio offered its participants accumulated over a long period of time and potentially as “unconscious competence” rather than explicit teachable knowledge. However, if the decomposition laid out here does capture consequential decision points, it provides scaffolding for deliberate individual and community practice as well as in teacher education and professional development. Designing opportunities to learn teaching would also entail designing opportunities to learn when and how to embrace decision points.

We see significant learning opportunities for students and teachers in this intentional teaching. In our theory, giving reason and giving purpose are qualities of teaching that position students to experience mathematics as meaningful and important. Examining this theory would involve studying how students themselves make connections across ideas and use connections to reason through ideas and how the ways that teaching gave reason and gave purpose may or may not have shaped student activity. These studies would be informed at least in part by analyses of elementary school mathematics (e.g., Howe, 2011; Howe & Epp, 2008), which discuss the structure of the mathematics across the elementary grades and how elementary mathematics can foreshadow more advanced mathematics. Understanding the learning of teachers would also involve studying what learning opportunities are afforded to teachers through planning for lessons that give reason and give purpose. Through planning, teachers may come to connect mathematics and student thinking in new ways which they can bring with them into their teaching for that lesson as well as subsequent lessons and to communities to which they belong.

The design of teaching the work of teaching might involve collaboration among teachers, teacher educators, and mathematicians. The connections that we are saying are important in K-12 teaching might not show up in any context but K-12 mathematics. Since the warrants for connections are mathematical and the context for connection is teaching and learning, connections may be a promising boundary object for mathematicians and teacher educators to learn to work across disciplines.

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Chapter 8
Who Are the Experts?

John Ewing

Abstract K-12 mathematics education has undergone many changes over the past decades, sometimes driven forward by mathematicians, sometimes by mathematics educators. Those two groups have often disagreed, but we are now in a more cooperative period in which experts on both sides seem to agree more and argue less. But in reaching this rapprochement, we have left out another group of experts who offer valuable perspectives and fresh ideas—the K-12 mathematics teachers themselves. In doing so, we not only miss the opportunity to draw on different expertise but also inadvertently demean the teaching profession itself.

There is nothing so stupid as an educated man,
if you get him off the thing he was educated in.

Will Rogers

Education has many experts.

At a fundraising event not long ago, the event chair got up to welcome everyone. In his opening remarks, he critiqued his fifth grader's mathematics class. He was earnest, urbane, and a successful businessman, but he has no experience in teaching (or mathematics). Nonetheless, he gave a short précis on his theory of skills versus understanding for mathematics instruction.

When I advertised recently for an executive position in a program for teachers, typical résumés looked something like this: at most 2 years of teaching, usually in Teach for America, followed by a position coaching teachers, moving on to head an organization that promotes some particular education policy. In their cover letters, candidates described themselves as "education experts."

Then there are the economists. Economists of every ilk are newly focused on education, analyzing massive data-sets, employing powerful mathematical models,

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and citing arcane statistics.[1] They are especially enamored with value-added models, which purport to linearly order teachers according to their ability to increase student achievement. They draw extravagant inferences about causality, making claims about effects on students decades in the future.[2] They create institutes of education research and publish technical studies that titillate reporters, who uncritically describe them as "education experts"—in spite of the fact that most have never actually studied education or taught in K-12 classrooms.[3]

Who gets counted as expert makes a difference. Experts have credibility; they have authority; they have respect. Many of the recent problems in education have arisen because everyone counts him or herself as expert[4] and, of course, when

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[1]Perhaps the most famous and frequently cited economist is Eric Hanushek at the Hoover Institute. Here is a typical paper from Hanushek (Hanushek, 2011):

> "Most analyses of teacher quality end without any assessment of the economic value of altered teacher quality. This paper begins with an overview of what is known about the relationship between teacher quality and student achievement. Alternative valuation methods are based on the impact of increased achievement on individual earnings and on the impact of low teacher effectiveness on economic growth through aggregate achievement. A teacher one standard deviation above the mean effectiveness annually generates marginal gains of over $400,000 in present value of student future earnings with a class size of 20 and proportionately higher with larger class sizes. Replacing the bottom 5–8% of teachers with average teachers could move the U.S. near the top of international math and science rankings with a present value of $100 trillion."

[2]The most notable recent such paper was by three Harvard economists: Chetty, Friedman, and Rockoff (2014). An excerpt from the abstract is illustrative:

> "Using school district and tax records for more than one million children, we find that students assigned to high-[Value Added] teachers are more likely to attend college, earn higher salaries, and are less likely to have children as teenagers. Replacing a teacher whose VA is in the bottom 5% with an average teacher would increase the present value of students' lifetime income by approximately $250,000 per classroom."

[3]For example, the distinguished Hoover Institution at Stanford University publishes research that is frequently cited. The Hoover Institution currently lists eight distinguished fellows with "expertise in education" (http://www.hoover.org/fellows?expertise=638). Half of those eight are economists; two others are political scientists.

[4]Much of the popular reaction to the Common Core State Standards illustrate this point. In dozens of postings, parents complain about the new mathematics standards, often unaware of what the standards actually say. Frequently, they cite their credentials as engineer or business executive, giving them expert status. ("Her husband, who is a pipe designer for petroleum products at an engineering firm, once had to watch a YouTube video before he could help their fifth-grade son with his division homework.") This was described by Motoko Rich in a *New York Times* article on June 29, 2014, "Math Under the Common Core has even Parents Stumbling." Similarly, on July 29, 2012, the *New York Times* published a prominent essay "Is Algebra Necessary?" by Andrew Hacker (http://www.nytimes.com/2012/07/29/opinion/sunday/is-algebra-necessary.html). The author is a professor political science who other than twice teaching an experimental course in mathematical literacy has no credentials in mathematics education whatsoever. The article drew national attention.

everyone is expert, no one is. This is a general problem in education, and the causes are complex and varied. Today I am thinking specifically about *mathematics* and the groups that lay claim to expertise in *mathematics* education.

I realize the word "expert" is not precisely defined. It comes from the Latin, *experitus*. In its original English meaning, the noun meant "a person wise through experience." But through its use in legal proceedings, "expert" also now refers to someone whose authority derives from some special acquired knowledge.

Whatever the precise definition, two groups are clearly counted among the experts—mathematicians and mathematics educators.

Mathematicians, at least some of them, have always been counted as experts in teaching and learning their own discipline. We have many examples of mathematicians who have become experts in education, including of course the honoree of this conference, Roger Howe. This is not a new phenomenon; its roots go back to antiquity. Examples include mathematicians like George Pólya and Ralph Boas, and before them Hans Freudenthal and Felix Klein, and before that Weierstrass and Cauchy, and so on.[5] Not new and not surprising. Is there any field in which its practitioners don't think about transmitting their knowledge to the next generation? Isn't this what culture is about?

The contentious issue is *not* whether some mathematicians are experts but rather to what extent mathematical expertise is *necessary* for educational expertise. Pólya gave one answer when he was advising prospective mathematics teachers:

> No amount of courses in teaching methods will enable you to explain understandably a point that you do not understand yourself. *Know your subject.*[6]

This simple maxim has been contorted into ridiculous propositions—for example, that expertise in mathematics is *sufficient* for expertise in education. I guess there are people who believe this, but surely Pólya was not among them. Also, while Pólya believed mathematical knowledge was necessary, he knew you can't learn just any mathematics in order to be an expert teacher; you need to master particular mathematics, in the same way that you need to master particular mathematics if you are going to do research in a specific field. Whatever the relationship of mathematical and educational expertise, some mathematicians must surely be counted as experts.

In the past two centuries, *mathematics educators* have also risen to expert status. Mathematics educators focus on the aspects of education that are particular to mathematics. Beginning in the early nineteenth century with Colburn's system[7]

[5]The International Commission on Mathematical Instruction (ICMI) was established at the Fourth International Congress of Mathematicians in 1908. The original aim was to support the "widespread interest among mathematicians in school education" (http://www.mathunion.org/icmi/icmi/a-historical-sketch-of-icmi/). The eminent mathematician Felix Klein was its first president; Hans Freudenthal was the eighth.

[6]Pólya (1958)

[7]Coburn (1821) (An 1884 reprint can be found at https://archive.org/details/intellectualari00colbgoog).

of "discovery learning" (perhaps the first "New Math"), mathematics educators proposed new methods for instruction and debated education's goals. For decades, educators emphasized utility[8]; for a time they promoted the idea of "transference[9]" (that learning mathematics teaches one how to think), then moved on to concentrate only on "meaningful" mathematics, and so on. Mathematics educators played a prominent role in the debates about the place of mathematics in school education.[10] This was especially true during the late nineteenth century and the first few decades of the twentieth, when both the MAA and NCTM were founded.[11] Later, their influence grew during and after the New Math, Back-to-Basics, and NCTM Reform.

But experts can disagree, and disagreements can be ugly. The Math Wars[12] had many complicated causes, but one was simple: Mathematicians and mathematics educators each made claims to superior expertise.

Mathematics educators pointed out the mathematicians were amateurs in K-12 education and had no *evidence* for their pronouncements because they didn't do education research.[13] In return, mathematicians pointed out the educators sometimes made incorrect or misleading mathematical statements.[14] In any case, education research wasn't the same as mathematics research. The research was frequently inconsistent or inconclusive; problems in education tended to be "divergent" (in the sense of E. F. Schumacher[15]) with two studies sometimes leading to opposite recommendations. Both sides traded accusations, sometimes about substance and more often about form.

Of course, neither group was entirely right nor entirely wrong. Mathematicians sometimes proposed nutty ideas that were wholly inappropriate for K-12 students.[16] Mathematics educators sometimes made statements that spawned nutty movements, with various groups insisting that children rediscover all mathematics or dismissing fractions because calculators made them irrelevant.[17] And while we can't blame

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[8]Grouws (1992).

[9]Willoughby (1967).

[10]Ibid. p 7.

[11]Ibid. p 11.

[12]For a perceptive view of the "Math Wars," see the two articles by Allyn Jackson: Jackson (1997a) and Jackson (1997b).

[13]An illustration of this point of view can be found in Mathews (1996).

[14]An illustration of this point of view can be found in Andrews (2001).

[15]Schumacher (1978).

[16]Tom Lehrer: http://curvebank.calstatela.edu/newmath/newmath.htm.

[17]In his 1997 State of the Union address, President Clinton proposed a voluntary national mathematics test to be given to students in the eighth grade. The Department of Education formed a Mathematics Committee in order to gauge the feasibility of such a test, and the Committee held hearings at various locations around the country. Individuals and groups were invited to testify at those hearings, including representatives from both the mathematics and the mathematics education communities. The ideas mentioned here were presented at those hearings—along with many others. The dramatic divergence of views about mathematics education was sobering. The test was eventually abandoned. See Bass (1998).

either side for Wu's School Mathematics,[18] we can blame both for fiddling, like Nero, while others created dreadful textbooks, mediocre standards, confused curricula, and insane assessments—much of it mathematically flawed and pedagogically barren. Like all wars, the Math Wars had collateral damage.

In recent years, there has been a truce as both mathematicians and mathematics educators accepted the other's expert status.[19] That's good for everyone and certainly good for education.

But in arranging the truce, and acknowledging their mutual expertise, mathematicians and mathematics educators have excluded a third group of experts—the practicing teachers.

This has become painfully evident to me in the past several years as I've worked with many accomplished teachers.[20] Workshops and conferences on K-12 mathematics education often contain no teachers at all.[21] Keynote talks are uniformly from distinguished university professors or a CEO of some large company. Discussions about curriculum and policy take place without any input from those who are responsible for implementation. Presentations make sweeping generalizations about teachers, without any check on reality. And when teachers are present, they tend to be invited as audience participants or token panelists.

This is not a mere oversight, and the exclusion of classroom teachers from expert status has been conscious and systematic.

When mathematicians talk about teachers, they often focus on elementary school teachers, who are generalists, in spite of the fact that by any reasonable measure the elementary grades represent less than half of K-12 mathematics. Mathematicians make categorical statements about *all* teachers. They refer to the teachers' content knowledge deficit, their poor training, and their desperate need for professional development, as though teachers were some monolithic collection, all with identical flaws.[22] If similar remarks were made about an ethnic or religious group, they would be condemned as shameful stereotyping.

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[18]Wu (2014) Also see Wu's "Mathematical Education of Teachers, Part I: What is Textbook School Mathematics?" Posted on February 20, 2015 on the AMS Blog (http://blogs.ams.org/matheducation/2015/02/20/mathematical-education-of-teachers-part-i-what-is-textbook-school-mathematics/)

[19]Ball, Ferrini-Mundy, Kilpatrick, Milgram, Schmid, and Schaar et al. (2005)

[20]Math for America is a program that offers 4-year fellowships to outstanding math and science teachers, who come together in a scholarly community to work on both content and pedagogy. In 2016, the program in New York City has approximately 1000 teachers; another 900 are in a similar program in the rest of New York State, with another 300 in other cities.

[21]For example, *US News* holds an annual conference on STEM education (http://usnewsstemsolutions.com/). In 2015 the conference had more than 50 speakers—CEOs from major corporations, university deans, heads of nonprofits, and a single classroom teacher. It also included many panels, and of all the panelists there was a single teacher among them. This is typical of major national education conferences.

[22]Wu (2011) A perceptive and engaging article that makes many valid points but containing a number of unsubstantiated generalizations about teachers. For example:

Mathematics educators argue that teachers have a narrow view of education.[23] What would a teacher know about teacher training? What would a teacher know about professional development? What would a teacher know about teacher evaluation? Teachers are trained; teachers are developed; teachers are repaired.

There are some reasons for these attitudes. Mathematicians are status conscious (like everyone else), and they value mathematical knowledge above all else. Even the most accomplished teachers are unlikely to be doing mathematical research. Moreover, we have spent the past three decades proclaiming a crisis in education, math and science in particular, and for much of that time, the blame has fallen on teachers.[24] After several decades of incessant teacher bashing, the status of teachers is at a low ebb. The *public* views teachers as anything but expert—mathematicians echo the public attitude.

The situation for mathematics educators is more complicated. First, if you are in the business of training new teachers, you get to see teachers in the making—not a pretty sight. Almost every teacher struggles for a time before mastering the craft. Brand new teachers are hardly expert at anything.

But there is another reason, more subtle than the first and indirectly a consequence of the Math Wars. Increasingly, mathematics educators have sought validation through the evidence of "research"—and frequently research that is considered "scientific," using large collections of data. This is supposed to make

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> "At the moment, most of our teachers do not know the materials of the three grades above and below what they teach, because our education system has not seen to it that they do." (p 381)

[23]Teacher educators sometimes intentionally downplay the expertise of practicing teachers. For example, the following passage appears in Thames and Bal (2013):

> "Another impediment to progress is the inclination to persist with outdated and refuted ideas about teacher quality, especially with respect to content knowledge .... The focus tends to be on teacher quality, particularly when it comes to teachers' inadequate content knowledge. However, the issue is not teacher quality, but teaching quality." (p 34)

[24]Here is a small sample of major reports, all of which are unabashedly critical of US education and US teachers:

A Nation at Risk. 1983. Report from the National Commission on Excellence in Education. http://www2.ed.gov/pubs/NatAtRisk/index.html

Rising Above the Gathering Storm. 2007. Report from Committee on Science, Engineering, and Public Policy; National Academy of Sciences and National Academy of Engineering. http://www.nap.edu/catalog/11463/rising-above-the-gathering-storm-energizing-and-employing-america-for

Rising Above the Gathering Storm, Revisited: Approaching Category 5. 2010. Report from Members of the "Rising Above the Gathering Storm" Committee; National Academy of Sciences; National Academy of Engineering; Institute of Medicine. http://www.nap.edu/catalog/12999/rising-above-the-gathering-storm-revisited-rapidly-approaching-category-5

U.S. Education Reform and National Security. 2012. Report from the Independent Task Force on U.S. Education Reform and National Security chaired by Joel Klein and Condoleezza Rice, http://www.cfr.org/united-states/us-education-reform-national-security/p27618 education respectable like medicine. Randomized controlled trials are the gold standard. Elaborate statistical analyses become standard tools. As a consequence, education research often focuses on test scores ("student achievement"). This is why research is now so often carried out by economists, rather than by philosophers like Dewey or psychologists like Piaget. Teachers' thoughtful discourse about teaching, based on experience, of the sort done in Finland and other high-performing countries, is not counted for much in America today. In the era of "evidence-based research," teachers don't earn much respect.

I do not dispute for a moment that mathematicians, at least some, are experts in mathematics education. I do not dispute that mathematics educators, at least some, are experts as well. Their combined expertise is essential to K-12 education.

But classroom teachers, at least some, are experts too! They have a different kind of expertise that derives from experience. The most expert have a special grasp of content that comes from the day-to-day struggle to unpack what they know in a variety of ways, for different students or sometimes for a single student who fails to understand. The most expert possess a deep, complex understanding of education's goals, not as the number who are counted proficient but as the number who are curious, creative, and intellectually passionate. The most expert understand their students and student thinking in ways that neither mathematicians nor mathematics educators are able to do.[25]

[25]The assertion of this paragraph—that practicing teachers have expertise that both overlaps and compliments that of mathematicians and mathematics educators—is regularly challenged by (some) mathematicians and educators. This itself illustrates the teacher's dilemma: When they are not viewed as experts, they are marginalized in policy making, both education and professional. This makes it hard to find examples of teachers' expert influence on policy. Nonetheless, the participation of teachers in discussions about major recent initiatives to expand computer science, extend the introduction of algebra in earlier grades, and build coherent curricula based on the Common Core amply illustrate their actual and potential expertise.

One recent example stands out. In a recent op-ed ("Wrong Way to Teach Mathematics" Feb 27, 2016) in the *New York Times*, the political scientist Andrew Hacker wrote to denounce the usual mathematics requirements for high school graduation (http://www.nytimes.com/2016/02/28/opinion/sunday/the-wrong-way-to-teach-math.html). He urged eliminating almost all mathematics requirements and eliminating a common curriculum. He promoted a light-weight quantitative literacy, criticizing more standard (high school) courses in statistics. Hacker elaborated on these ideas in his book, *The Math Myth and Other STEM Delusions,* 2016, New York: The New Press. While these ideas have been debated in many venues, the mathematics and mathematics education community has been relatively ineffective in addressing them. Many prominent educators and writers have sympathized with Hacker, dismissing the reactions of the mathematics community as self-interested. Teachers themselves have addressed the issues most effectively, both the details and the tenor of his proposals. For example, see the blog posts of two highly accomplished mathematics teachers:

Patrick Honner at http://www.mathforamerica.org/news/when-it-comes-math-teaching-let%E2%80%99s-listen-math-teachers

Amy Hogan at http://alittlestats.blogspot.com/2016/05/the-wrong-way-to-target-math-part-iii.html

These are experts through experience, and they can be valuable colleagues—not junior colleagues to be mentored, developed, or repaired but colleagues from whom we can learn. They are every bit as expert as that event chair who never taught, or the *Teach for America* Corps member who taught for a single year, or the economist who works on data rather than ideas.

Who gets counted as expert has consequences. By reinforcing the public's low respect for teachers, we further diminish the prestige of the teaching profession. Is it any wonder that fewer and fewer talented young people show an interest in becoming teachers? And prominent public discussions about education policy, such as the one taking place about the Common Core, shape public opinion in profound ways. Teachers have played virtually no role[26] in the recent Common Core discussions except as pathetic pawns who are not up to the task of implementing the higher, more challenging standards.[27] Is it any wonder that many teachers haven't embraced the Common Core? It is amazing that, despite the way the standards have been used to bash teachers, some have stepped up to do the job—the *hard* job—of actually implementing a coherent curriculum in a system that makes it very difficult to build coherence.

Not all teachers are experts, just like not all mathematicians are. But mathematicians and mathematics educators should recognize and respect those who are because they can be valuable partners *and* because public respect will follow from theirs. This is the surest way to make teaching attractive, to improve the profession, and ultimately to make *more* teachers who are experts.

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[26]The development process for the Common Core State Standards is described in detail at http://www.corestandards.org/about-the-standards/development-process/. This describes teacher involvement as follows:

> "Teachers played a critical role in development
>
> The Common Core State Standards drafting process relied on teachers and standards experts from across the country. Teachers were involved in the development process in four ways:
>
> 1. They served on the Work Groups and Feedback Groups for the ELA and math standards.
> 2. The National Education Association (NEA), American Federation of Teachers (AFT), National Council of Teachers of Mathematics (NCTM), and National Council of Teachers of English (NCTE), among other organizations were instrumental in bringing together teachers to provide specific, constructive feedback on the standards
> 3. Teachers were members of teams states convened to provide regular feedback on drafts of the standards.
> 4. Teachers provided input on the Common Core State Standards during the two public comment periods."

For mathematics, the Work Group consisted of 51 individuals: exactly 2 of them were practicing teachers. The Feedback Group consisted of 22 individuals: exactly 1 of them was a practicing teacher.

[27]Wingert, Pat. 2014. "When Teachers Need Help in Math." Atlantic, October 2 http://www.theatlantic.com/education/archive/2014/10/when-teachers-need-help-in-math/381022/

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Part III
Identifying and Structuring Core Ideas and Practices in K-12 Mathematics Curriculum

Chapter 9
Building on Howe's Three Pillars in Kindergarten to Grade 6 Classrooms

Karen C. Fuson

Abstract Howe (2014, Three pillars of first grade mathematics, and beyond. In: Li Y. & Lappan G. (eds), Mathematics curriculum in school education, Springer, Dordrecht, pp 183–207) identified three pillars of first grade mathematics and beyond that described central mathematical and sense-making aspects of major Common Core State Standards Math (National Governors Association Center for Best Practices, Council of Chief State School Officers. 2010) domains. This chapter builds on each pillar by sharing visual models that have been powerful in helping students learn the aspects identified by Howe. Visual models are central core ideas and practices in the CCSS–M and deserve attention and discussion. The research-based examples discussed here are simple math drawings that students can make and use in their own ways in problem solving and explaining of thinking. Such drawings support the math talk discussions that are at the heart of the CCSS–M and of the mathematical practices. They enable (Howe's, 2014, Three pillars of first grade mathematics, and beyond. In: Li Y. & Lappan G. (eds), Mathematics curriculum in school education, Springer, Dordrecht, pp 183–207) three pillars to come to life in the classroom. Teachers and students can come to appreciate all of these pillars: Pillar I, the power of robust understanding of the operations of addition and subtraction including situations that give meaning to the operations and levels of single-digit addition and subtraction; Pillar II, an approach to arithmetic computation that intertwines place value with the addition/subtraction facts; and Pillar III, making connections between counting number and measurement number.

Howe (2014) identified three pillars of first grade mathematics and beyond that describe central mathematical and sense-making aspects of major Common Core State Standards Math (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) domains. The Howe paper circulated before it was posted on the website in 2010, and it was influential in the design of the Common Core State Standards Math. This chapter builds on each pillar by sharing visual models that have been powerful in helping students learn the aspects

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identified by Howe. I draw on 35 years of research in classrooms to discuss strengths and limitations of various visual models. This research has shown me the power of research-based math drawings that students make. This research was carried out in independent research studies and in funded research as part of the Children's Math Worlds Project that led to the publication of these models in the K to grade 6 math program *Math Expressions*. The research studies and experience with classrooms using the *Math Expressions* programs have provided extensive teacher data about the effectiveness of these visual models. Visual models are central core ideas and practices in the CCSS–M and deserve attention and discussion. But which visual models should we be using and why? We need discussion of this issue for various math domains. This chapter is a contribution to such discussion.

9.1 Pillar I: A Robust Understanding of the Operations of Addition and Subtraction

9.1.1 Situations that Give Meaning to the Operations

The major real-world situations that give meaning to addition/subtraction and to multiplication/division have been the focus of much research (e.g., see National Research Council, 2001, 2009). Problem classifications of these situations drawn from research are given in the Common Core State Standards on pages 88 and 89 (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). I spent many years trying different diagrams students could use especially with larger numbers, fractions, and decimals (Fuson, 1988; Fuson, Carroll, & Landis, 1996; Fuson & Smith, 2016; Fuson & Willis, 1989; Willis & Fuson, 1988). Students can make their own drawings. But specially designed diagrams provide a common visual language to support discussion, and they provide consistency across kinds of numbers.

The final set of research-based diagrams that were successful in hundreds of classrooms is shown in Fig. 9.1. Students must learn meanings of equations, so equations were chosen as the visual support for the simplest kind of problems add to/take from. These situations show action over time, so it is natural for students to write each step of an equation as a step in the problem situation over time. Put together/take apart diagrams with the total on the top and two legs for the addends were found to help students understand these situation actions. Comparison bars show the two compared quantities in an additive comparison situation, and the difference quantity created by information about the situation is shown as an oval that makes the smaller quantity as long as the bigger quantity. The equal groups multiplication/division situations use the put together/take apart drawing repeatedly, and the multiplicative comparison situations draw the repeated quantity bar repeatedly. The array/area situations begin by drawing all of the objects or squares but quickly get abbreviated to a drawn rectangle in which the factors are along the sides and the product is inside. This model also reflects the traditional long division format.

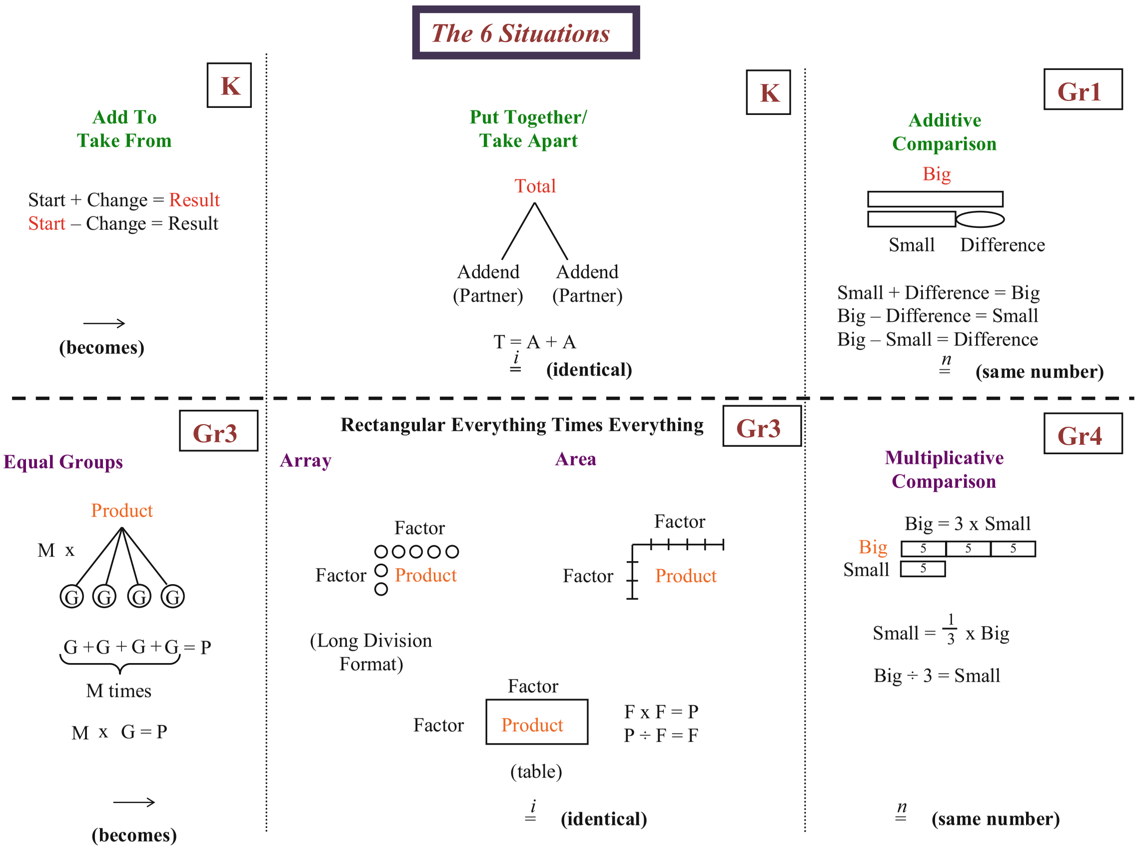


Fig. 9.1 CCSS addition (*top row*) and multiplication (*bottom row*) word problem situations and *Math Expressions* diagrams

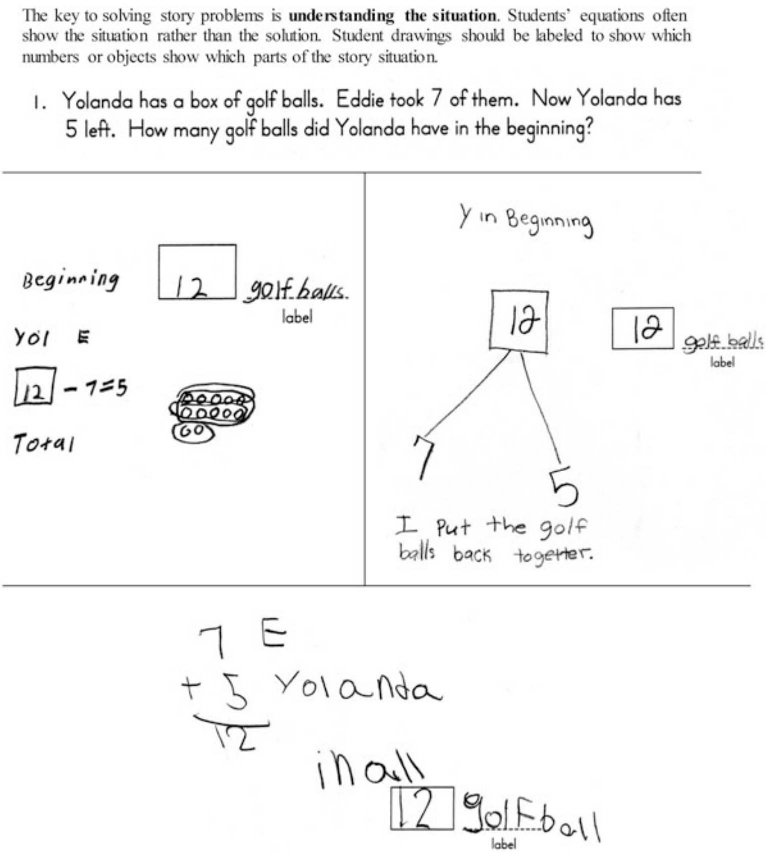


Fig. 9.2 Labeled math drawings for an unknown start problem

Each type of situation has three quantities, and each quantity can be the unknown. Some unknowns are more difficult than other unknowns. These differences create the learning path of difficulty across addition/subtraction situations that extend from kindergarten to grade 2. The key to solving word problems is understanding the situation and then making a labeled drawing if needed. Students' equations often show the situation rather than the solution. They then think about their drawing or equation to solve the problem. A difficult take from:start unknown problem is shown in Fig. 9.2. At the top left, the equation shows the situation, and the student then draws quantities to show the adding of 5 and 7 to make 12. Students often represent and solve in different ways. Two other approaches are shown in Fig. 9.2. Older students can use the same diagrams to support varied approaches for problems with multi-digit numbers and fractions.

For more information about the learning path of difficulty of the problem types and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Operations and Algebraic Thinking (OA) in the CCSS: Part 1 Problem Situations and Problem Solving at http://www.karenfusonmath.com.

9.1.2 Levels in Adding and Subtracting Single-Digit Numbers

Students worldwide go through three levels of conceptualizing and carrying out adding and subtracting (e.g., Fuson, 1986, 1992; Fuson & Fuson, 1992; Fuson & Willis, 1988). At level 1 they can only think of one number at a time. So adding is a three-step adding-to process in which they focus on the first addend, then on the addend added to the first addend, and then on the total of both addends. Subtracting is the reverse taking-from process that begins with the total, takes one addend from that total, and then focuses on the remaining other addend. At level 2, students can conceptually embed both addends within the total so that they can begin the counting of the total with the counting of the second addend: to add, they count on from the first addend to find the total, keeping track of how many are counted on and saying the last counted word as the total. To subtract, they count on from the first addend to find the unknown second addend, keeping track of and stopping as they say the total, and then seeing how many they counted on. At level 3, students can decompose and recompose addends within the total. So, for example, they can carry out the general method for single-digit adding and subtracting in which one addend is decomposed to make a ten with the other addend: for example, $8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14$. Methods from all three levels are in the CCSS–M, level 1 at kindergarten and levels 2 and 3 at grade 1.

Howe (2014) discussed in his Pillar II the importance of decomposing a number into two addends in different ways. In the CCSS–M, such decompositions are a kindergarten standard:

K.OA.A.3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2 + 3$ and $5 = 4 + 1$).

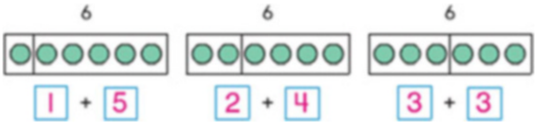
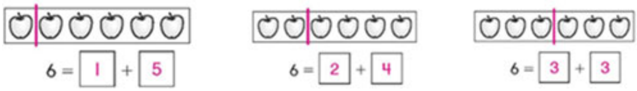
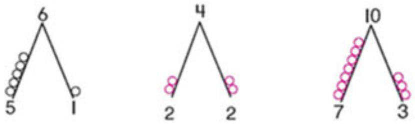
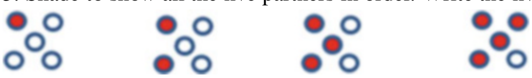
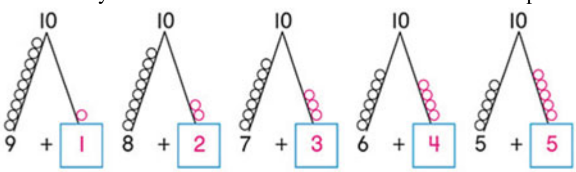
Notice that the equations to record these decompositions have the total alone on the left and the addends are added on the right side. This reflects the taking apart action in the situation and is helpful in overcoming the prevalent view by older students that an equation must have two numbers on the left and one number on the right. It is helpful for kindergarten and grade 1 children to see equations of this form that show the meaning of the situation.

In my own research, I have found that decomposing numbers into two addends helps children move to the level 2 methods of single-digit adding and subtracting that require the addends to be embedded within the total: counting on to find a total or to find an unknown addend. Such level 2 embedding of the addends within the total also allows children to solve the more difficult problem subtypes like add to or take from change unknown and start unknown problems (Fuson & Smith, 2016). Students can represent change unknown situations by a situation equation such as

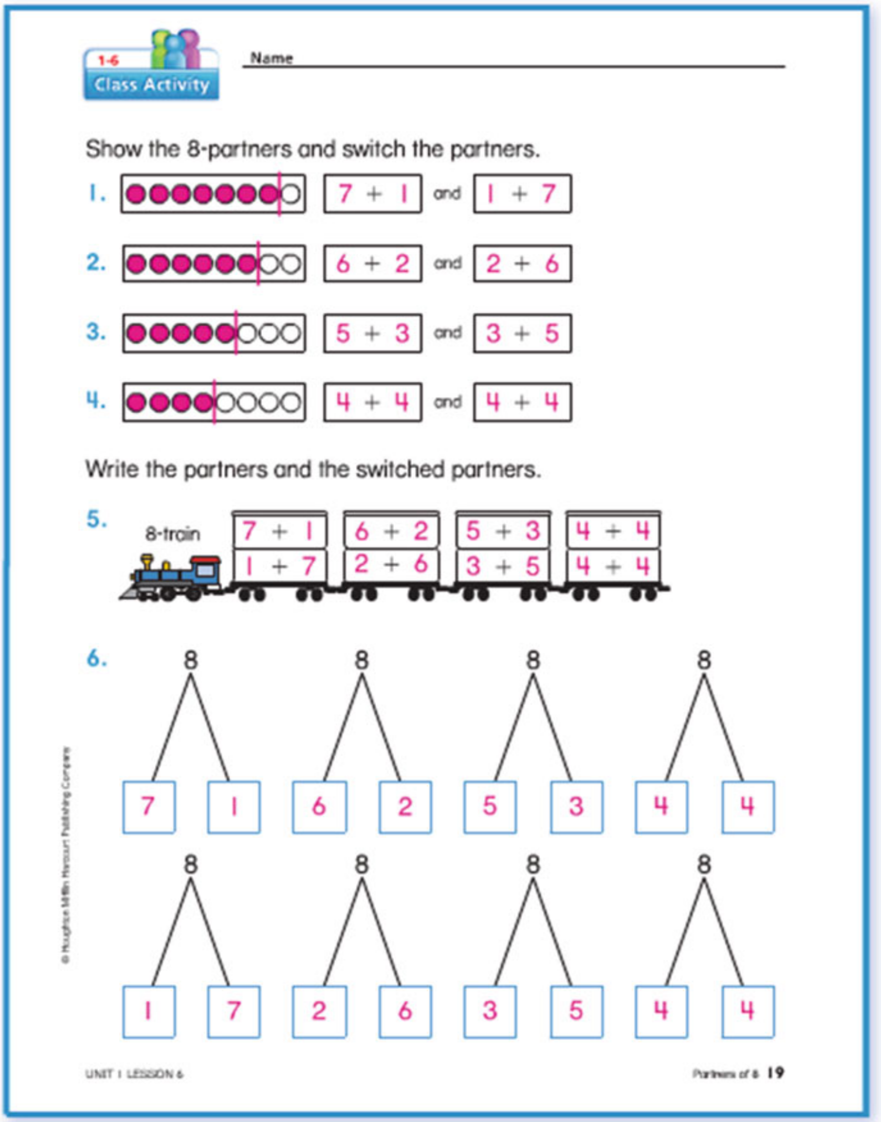
8 +? = 14 or 14 −? = 8 and can solve them by level 2 counting on from 8 to 14 to find the unknown addend. Start unknown problems such as the problem in Fig. 9.2 also require an understanding of where totals and addends are in equations or diagrams and how to relate these three quantities to find the unknown.

Visual supports for decomposing that I have found to be effective in kindergarten are shown in Table 9.1. Students count out things to make a given number and partition these in various ways with a break-apart stick. Later, as in the tasks in Table 9.1, the partitioning is shown in drawings on paper by a break-apart stick and by shading. Students write the two addends that are created. These addends are called *partners* because this word was found to help students relate these two numbers. In grade 1 (see Fig. 9.3), students move on to using these visual supports to decompose larger numbers and to relate decompositions that reverse the order of the addends (using the commutative principle). The decompositions also become

Table 9.1 Percentage correct on partner (addend) tasks for kindergarten children

Unit	%	Task
3	90	1. Write the partners. 
4	92	2. Draw a line to show the partners. Write the partners. 
4	92	3. Draw tiny tumblers on the math mountain. 
4	85	4. Write the partner equation. 
5	88	5. Shade to show all the five partners in order. Write the five partners. 
5	83	6. Draw tiny tumblers on the math mountain and write the partner. 

Note. These tasks fall centrally within the following CCSS: K.OA.3



Student Activity Book page 19

Fig. 9.3 Grade 1 partner switches

mostly numerical for first graders, as these small numbers take on quantitative meanings from extensive work in kindergarten. Such decompositions appear again in grade 4 as CCSS–M standard 4.NF.B.3b. This work helps students understand that unit fractions obey the same principles as whole numbers, reduces the common

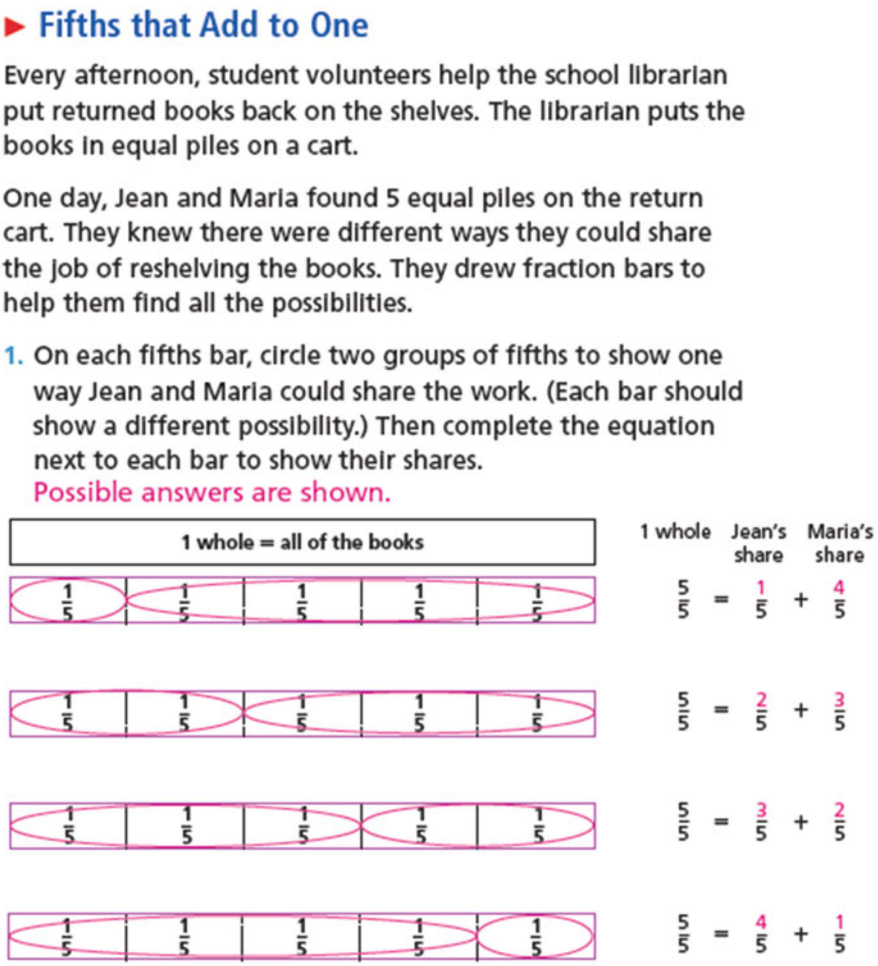


Fig. 9.4 Decomposing fractions into addends/partners

error of adding tops and bottoms when adding fractions (because students see that only the top numbers are added and that the unit fraction number does not change), and generalizes decomposing a number into addends (see Fig. 9.4).

Decomposing a number into addends is the second step in doing the general level 3 make-a-ten method: $8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14$. In the first step $(8 + ?)$, one must know the number that makes ten with the first addend. In the second step, one decomposes the second addend into the number added to ten and the rest of the second addend: $8 + (2 + ?)$, where $2 + ? = 6$. In the third step, one must know $10 + 4$, a total made with ten. All three prerequisites for the make-a-ten method are kindergarten CCSS–M:

K.OA.A.4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

K.OA.A.3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2 + 3$ and $5 = 4 + 1$).

K.NBT.A.1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., $18 = 10 + 8$); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

This method and these prerequisites are emphasized in East Asian countries but have not been emphasized in this country, especially the second step of decomposing a number discussed by Howe (2014). As kindergarteners have time to learn these prerequisites, understanding and carrying out the make-a-ten method will become easier.

However, this method is more difficult in English than in East Asian languages based on Chinese that say 14 as *ten four*. Saying a number between ten and twenty as *a ten* and *some ones* helps with all three steps in the make-a-ten method. In contrast, an English word such as *fourteen* has a reversal in the ten and the ones that complicates the relationship with the written numeral 14. *Ten* is not said clearly (how many adults know that teen means ten?). And the number of ones is not said clearly in *eleven*, *twelve*, *thirteen*, and *fifteen*. For these reasons, the level 2 counting on methods may be enough for CCSS–M OA problem solving in grades 1 and 2. But make-a-ten methods can be helpful in CCSS–M NBT multi-digit adding and subtracting, as is discussed in the next section. For more information about the learning path of three levels of adding/subtracting and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Operations and Algebraic Thinking (OA) in the CCSS: Part 2 The K, 1, 2 Learning Paths for OA + and – (at http://www.karenfusonmath.com).

9.2 Pillar II. An Approach to Arithmetic Computation that Intertwines Place Value with the Addition/Subtraction Facts

Howe's (2014) Pillar II involves two major conceptions:

(a) Understanding that a two-digit number is made of some tens and some ones.
(b) In adding or subtracting, you work separately with the tens and the ones, *except* when regrouping is needed.

Both of these concepts extend to larger numbers with more places. Howe pointed out that it would be useful to have a term for the numbers created by a decomposition into place value numbers, for example, in 243 = 200 + 40 + 3. He suggested that such numbers be termed *single-place numbers*; the 200, 40, and 3 would be called single-place numbers. This is a helpful observation and might make it easier for students to conceptualize and discuss such parts. But I suggest instead the term *place value parts* for such numbers because they are parts and they explicitly name place values. Howe's two concepts above form the basis for general methods of adding and subtracting for any number of places. Students need to be able to add and subtract the single-digit addends discussed in Pillar I. And they need to understand how to think about and have a written method to record grouping when adding and ungrouping when subtracting. For two-digit numbers, students will group ten ones to make one ten whenever their total for the place value parts in a given place is ten or more. In general, they will group ten of one kind of place value parts to make one of the place value parts in the next-left column. And they will ungroup one of one kind of place value parts to make ten of the place value parts in the next-right column.

I did research for many years to ascertain what visual supports would help students understand these two vital conceptions that underlie multi-digit adding/subtracting and what general written methods were easy for students to understand and explain and relate to visual supports (Fuson, 1998, 2003; Fuson & Li, 2009; Fuson & Smith, 1997; Fuson, Smith, & Lo Cicero, 1997; Fuson et al., 1997). Students need to see and understand the quantities that make the place value parts for any number. Secret-code cards that can be layered to show place value parts and math drawings that students can make to show the quantities for each place value are both very helpful to students. The fronts and the backs of the secret-code cards are shown in Fig. 9.5. Unlayered, the cards show the place value parts (400 and 80 and 6). When layered on top of each other, the cards show the usual single-digit form of our base-ten numerals (486). But the little numbers on the top left of each card remind students of the place values for each part and of the zeroes that are hiding under the other digits. These cards are called secret-code cards because they show the secret code of our numbers, and students love the term. The quantities named by the place value parts are shown on the back of the cards: 4 hundreds, 8 tens, and 6 ones. Students learn how to draw these quantities by drawing on columns of ten dots to make a ten-stick and making a box around ten such columns of ten dots (or ten such ten-sticks) to make a hundred-box. Soon students make quick-hundred and quick-ten drawings that are just a hundred-box and a ten-stick,

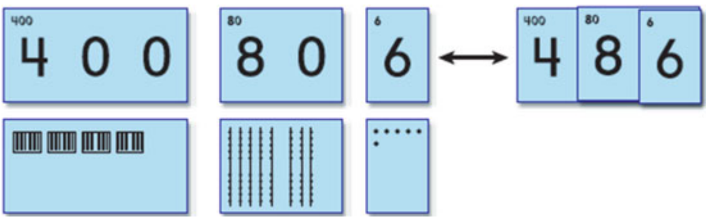


Fig. 9.5 Secret-code cards for 486

but they understand the quantities involved. Secret-code cards can be used on a millions frame to show the groups of three numbers in millions, thousands, and ones. Secret-code cards can also be extended in the opposite direction to show decimal place value parts.

These visual models support working separately with the place value parts, as described in Pillar IIb and in the CCSS–M. The CCSS–M critical areas for each grade at which new multi-digit computation is introduced specify that students are to "develop, discuss, and use efficient, accurate, and generalizable methods" for that computation. They further specify that students are to understand that adding and subtracting involve adding or subtracting place value parts, composing or decomposing these parts as needed. Importantly, the CCSS–M also specify that students use concrete models or drawings, relate strategies based on place value to a written method, and explain why the methods work. For example, in grade 2, NBT standards 7 and 9 state:

2.NBT.B.7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, and ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

2.NBT.B.9. Explain why addition and subtraction strategies work, using place value and the properties of operations.[3] [[3]Explanations may be supported by drawings or objects.]

There are different ways to write generalizable methods that meet the above specifications. There is no such thing as "a standard algorithm" in spite of the widespread use of this term. Many different methods have been used historically in this country and in other countries, often several at the same time. The National Research Council report adding it up made this point and showed and discussed many methods (National Research Council, 2001). Fuson and Beckmann (2012) followed the lead of the NBT Progression document (the Common Core Writing Team, 7 April 2011) and summarized that *the standard algorithm* for an operation implements the following mathematical approach with minor variations in how the algorithm is written:

- Decomposing numbers into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number
- Using the one-to-ten uniformity of the base-ten structure of the number system to generalize to large whole numbers and to decimals

Fuson and Beckmann then identified variations in written methods for recording the standard algorithm for each operation, showed visual models that supported understanding of the written methods, and discussed criteria for evaluating which variations might be used productively in classrooms. A similar discussion for teachers of advantages and disadvantages of various written methods for addition and subtraction is given in National Council of Teachers of Mathematics (NCTM) (2011). Fuson and Li (2009) identified and analyzed a number of variations of written methods for multi-digit addition and subtraction found in textbooks in China, Japan, and Korea.

These analyses converge on one method of addition and one method of subtraction that are superior to others. The addition method is shown in Fig. 9.6, where drawings and a student explanation are shown for each step in adding using place value parts. Questions by other students follow at the bottom. This classroom example implements the CCSS–M and Pillar II. Notice, as you read, the example of how the drawings can support listeners' understanding of the explanation and of the questions by other students and can clarify both aspects of multi-digit adding identified above.

This method, often called New Groups Below, has several conceptual and procedural advantages compared to the current common method in which the new groups (the little 1 s) are written above the columns. It supports place value understanding by:

- Making it easier to see the teen sums for the ones (16 ones) and for the tens (14 tens), rather than separating these teen sums in the space above and below the problem so that it is difficult to see the 16 or the 14.
- Allowing students to write the teen numbers in the usual order as 1 then 6 (or 1 then 4) instead of writing the 6 and then "carrying" or grouping the 1 above.
- Making it easier to see where to write the new 1 ten or 1 hundred in the next left place instead of above the left-most place (a well-documented error that arises more with problems of 3 or more digits and is easier to make when one is separating the teen number below and above the problem).
- Making it easier to carry out the single-digit additions because you add the two larger numbers you see and then increase that total by 1, which is waiting below. When the 1 is written above the column, students who add the two numbers in the original problem often forget to add the 1 on the top. Many teachers emphasize that they should add the 1 to the top number, remember that number and ignore the number they just used, and add the mental number to the other number they see. This is more difficult than adding the two numbers you see and then adding 1.

Notice in Fig. 9.6 how the drawings use five groups to support the level 3 make-a-ten methods. When adding 9 ones and 7 ones, you can see that the 9 needs one more to make ten; this one ten can be written below in the tens column waiting for it to be added. The 7 has been decomposed into 1 to make ten and 6 left, so the 6 ones can be written below as the total number of ones. Similarly for the tens, 8 tens

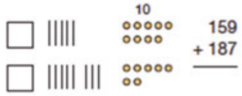
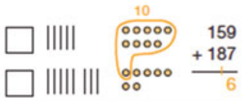
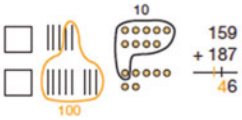
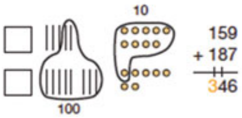
Math Drawing and Problem	Explanation Using Place-Value Language About Hundreds, Tens, and Ones
a. 	I drew one hundred, five tens, and nine ones to show one hundred fifty nine, and here below it I drew one hundred, eight tens, and seven ones for one hundred eighty seven. I put the ones below the ones, the tens below the tens, and the hundreds below the hundreds so I could add them easily.
b. 	See here in my drawing, nine ones need one more one from the seven to make ten ones that I circled here, and I wrote 10. That leaves six ones here. With the numbers the seven gives one to the nine to make ten that I write over here in the tens column, see one ten. And I write six ones here in the ones column.
c. 	With the tens, I start with eight because it is more than five so it is easier. I get two tens from five tens to make ten tens, see here, and I write one hundred here to remind me that the ten tens make one hundred. There are three tens left in the five tens and I have one more ten from my ones (see here in my drawing and the one ten at the bottom of the tens column). That makes four tens and the one hundred. So in my problem I write the one hundred below in the hundreds column and the four tens in the tens column.
d. 	There are three hundreds, two in the original numbers I'm adding and one new hundred from the ten tens. I write three hundreds here in the hundreds column. Are there any questions? Yes, Stephanie.
Student Question	**Explainer Answer**
Stephanie: For the tens, you never said fourteen tens as the total of the tens. Why not?	Because when I'm making ten tens, I just can write that one hundred over here with the hundreds and just think about how many tens I need to write. But I can think eight tens and five tens is thirteen tens and one more ten is fourteen tens, so that is one hundred and four tens. You can do it either way. (Aki)
Aki: Do you still need to make the drawings or did you just make them so you could explain better?	I don't have to make the drawings, but I can explain better with a drawing because you can see the hundreds, tens, and ones so well. (Jorge)
Jorge: Do you do make-a-ten in your head or just know those answers?	I just know all of the nine totals because of the pattern: the ones number in the teen number is one less that the number added to nine because it has to give one to nine to make ten. So nine plus seven is sixteen. I just know that pattern super fast. For eight plus five, I do make-a-ten fast, sort of just thinking five minus two is three, so thirteen. (Sam)
Sam: I know five and eight is thirteen, so why did you write a four in the tens column, Karen?	Because I had one more ten from the ones. See here in the drawing: nine ones and one one from the seven ones make ten ones. I wrote 10 here to remind me, and here in the problem I wrote the new one ten below where I can add it in after I find thirteen. You have to write your new one ten big enough to be sure you see it.
Sam: Oh yes, I see it now. I can see the new one ten when I write it, but I couldn't see yours.	OK, thanks. I'll write it bigger next time so everyone can see it.

Fig. 9.6 Three-digit addition using New Groups Below with student drawings, explaining, and questioning. The explainer stands to the side and points with a pointer to parts of the math drawing or to parts of the problem as they are mentioned. Pointing is a crucial part of the explanation. Reprinted with permission from Focus in Grade 2: Teaching with Curriculum Focal Points, copyright 2011, by the National Council of Teachers of Mathematics. All rights reserved

can be seen to need 2 more to make ten tens; this new one hundred can be written below the hundreds column. The 5 tens are decomposed into 2 to make ten with 8 tens and 3 tens left; the 3 can be added to the 1 ten waiting below and then the 4 tens written below the tens column. With experience, the make-a-ten method can be done mentally in this multi-digit adding context and then perhaps in other contexts. Such five-group visual models are used widely in East Asian classrooms. They can be used from the first day of kindergarten displayed on a poster with numerals to help children build understanding of single-digit numbers. These five groups are used on the backs of the secret-code cards shown in Fig. 9.5 to help children see how many hundreds, tens, and ones more easily.

Two written methods for subtracting after decomposing into place value parts are shown in Fig. 9.7. The better method is shown first. Before you subtract a given kind of place value part (a given column), you need to check if you can subtract the bottom number from the top number: *Is the top number greater than or equal to the bottom number?* If not, you need to get more of those units in the top number by ungrouping one unit from the left to make ten more of the units in the target column. All of these "checking and ungrouping if needed" steps can be done first, either from the left or from the right. Then all of the subtracting can be completed either from the left or from the right. These subtractions can actually be completed in any order, but going in one direction systematically creates fewer errors. This taking care of all needed ungrouping first is shown in Fig. 9.7 as method A with math drawings for a three-digit example and then without drawings for a six-digit number at the bottom to show how the method generalizes. Students can stop making drawings as soon as they understand and can explain the steps.

Ungroupings from the left and from the right are shown for the six-digit example. You can see how these ungroupings differ by looking at the ungroupings in the second and fifth columns. In ungrouping from the left, the 6 hundred thousands give 10 ten thousands to the 2 ten thousands, making 12 ten thousands and leaving 5 hundred thousands. Then the 3 thousands need more thousands (to subtract the 6 thousands), so the 12 ten thousands give 10 thousands to the right making 13 thousands and leaving 11 ten thousands. In ungrouping from the right, you ungroup moving to the left, and when you get to the ten thousands place, you have taken 1 ten thousands to give 10 thousands to the 3 thousands to make 13 thousands. The steps of ungrouping involve the same quantities, but they are done in different orders as shown by the ungrouped numbers above the problem.

Separating the two major kinds of steps involved in multi-digit subtracting as in this method is conceptually clear and makes it easier to understand that you are not changing the total value of the top number when you ungroup. You are just moving units around to different columns. Many students prefer to move from left to right, as they do in reading, and productive mathematical discussions can take place as students explain why they can go in either direction and still get the same answer.

The method B variation in Fig. 9.7 involves following the same steps but alternating between ungrouping and subtracting. Alternating steps is more difficult for students, and this method sets up the common subtraction error of subtracting the top from bottom number when it is smaller (e.g., for 94–36, get 62). Even when you know you should check and ungroup if needed, alternating steps prompts errors. For example, in the three-digit number in step 2, you have just subtracted 6 ones from 13 ones to get 7 ones. You look at the next column and see 1 and 5, and 4 pops into your head (if you are only in second grade). You write 4 and move left. In the

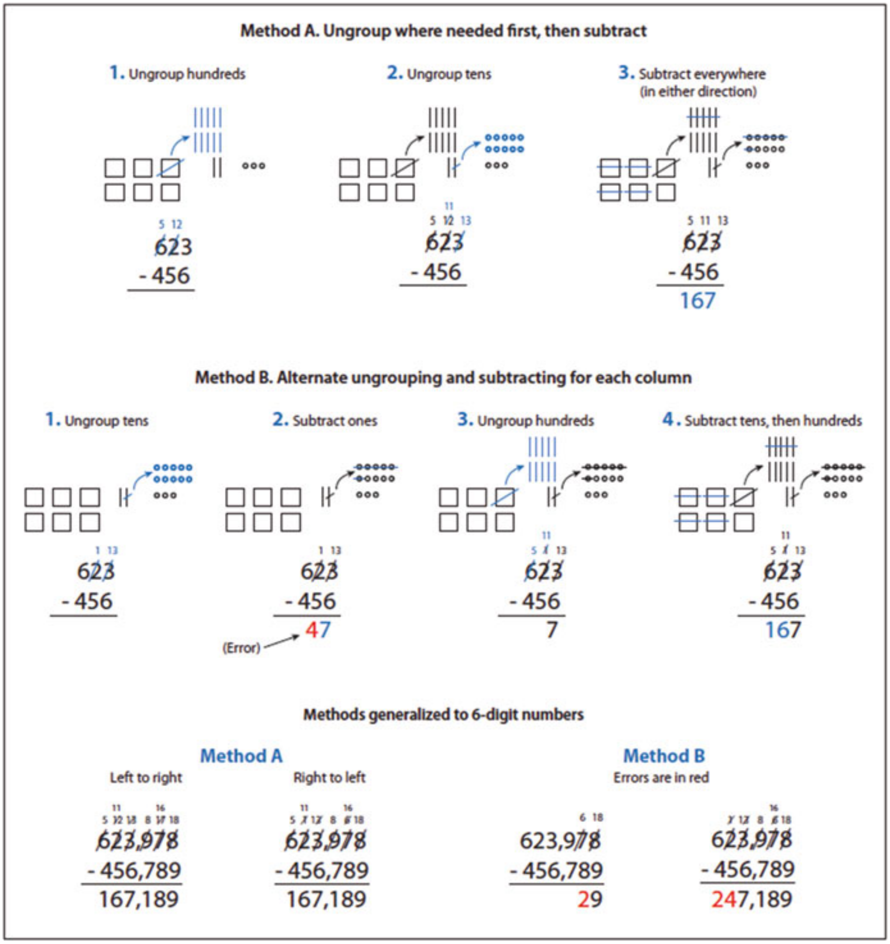


Fig. 9.7 Multi-digit subtraction methods

six-digit problem, the three errors that can be created by alternating ungrouping and subtracting in method B are in red. Although this alternating method can be used for numbers of any size, it is not as easy or conceptually clear as method A. For two-digit numbers, the alternating method B and non-alternating method A are the same because there is no iteration of the steps.

For more information about how to support students through the learning path of understanding place value parts and making drawings to show them and use them in explaining multi-digit addition and subtraction, see the Teaching Progression on *Math Expressions* and Number and Operations in Base Ten (NBT) in the CCSS: Part 2 Place Value and Multi-digit Addition and Subtraction in K to G4 (at http://www.karenfusonmath.com).

9.3 Pillar III. Making Connections Between Counting Number and Measurement Number

9.3.1 Limitations of Length for Showing Place Value and Addition and Subtraction

Physical and Practical Issues Howe (2014) suggested that students use trains of 100-rods, 10-rods, and 1-cubes to show place value. Length does show how the place value parts get big quickly. But length is not practical for use in a classroom. Length is too long for students to use to show or add or subtract even two-digit numbers. Base-ten blocks have ten-sticks 10 cm long and 1-cubes 1 cm long. Length trains of ten-sticks and 1-cubes do not fit across most student desks, and most rooms do not have enough tables on which all students can work. Base-ten blocks use a 10 cm by 10 cm square for hundreds rather than length; this is more practical. But the blocks present other difficulties. They are expensive, leave no record of the steps in using them, cannot be used for homework, are difficult to show the whole class, and are cumbersome to relate to written methods. The drawings shown in Figs. 9.6 and 9.7 have none of these disadvantages.

Drawings that just use length are also problematic. The CCSS–M 2.MD.B.5 and CCSS–M 2.MD.B.6 specify that students should relate addition and subtraction to length by solving word problems involving lengths and by representing whole numbers and whole number sums and differences within 100 as lengths from 0 on a number-line diagram. However, to get 100 units across a page even horizontally, each unit is about 1.6 mm long. This is small. Consequently, a number-line diagram to 100 is too short to see numbers clearly and is too complex for students to draw even semi-accurately. So, students can work with a few examples already drawn on a page to see that their count models do extend to length models. They can use meter sticks marked into centimeters and decimeters in demonstrations for the whole class of these lengths related to their place value parts. But the tools for adding and subtracting that can actually be used by each student are drawings of hundreds, tens, and ones related to written methods as shown in Figs. 9.6 and 9.7.

Length Models Constrain the Addition and Subtraction Methods Students Can Easily Use Length models do not support Pillar II or general CCSS–M methods that compose separate place value parts because they keep one multi-digit number together and add to or take from that number. Such methods require advanced sequence counting skills as students add on or take from hundreds or tens or ones from a whole two-digit or three-digit number. I tried these methods in classrooms for several years, but I found that it was difficult for less-advanced or non-native English speakers to learn these sequence counting skills. These methods can be done using drawn place value parts as in Figs. 9.6 and 9.7 instead of length models. They still require the same sequence counting skills, but they do not require learning to use a different visual model. Further problems with these length model methods are discussed in Fuson and Beckmann (2012) and NCTM (2011). Among other issues, they are not generalizable to larger numbers.

9.3.2 Counting Number and Measurement Number Do Relate Well to Show Multi-digit Multiplication and Division

Count models of drawn place value parts as used for addition and subtraction lead into the array models (count models using things as units) and area models (measure models using units of measure) commonly used to visualize multiplication and division. For example, the known factors are the numbers of rows and of columns in an array or the lengths of the sides, and the product is the number of total things in the array or the number of unit squares in the area model. I have found with many classrooms that students can make such array or area drawings for small numbers on a dry-erase Math Board that shows 100 by 50 dots, each 4 mm apart. Students can draw around the dots to make arrays, or they can draw on the lengths between the dots to show area. Such drawings (e.g., for 24 × 37) show all of the drawn place value parts accurately to scale. Then students can move to drawing sketches and relate them to a written method. Eventually students drop the sketches and just do a written method.

There are written variations for multiplication and division that record the place value parts in somewhat different ways. Advantages and disadvantages of many of these are discussed in Fuson and Beckmann (2012). Approaches that I have found to be understandable by many students are shown in Fig. 9.8. The area model is shown on the left. For multiplication, students know and draw the lengths of the two sides, separating the place value units tens and ones. They draw line segments inside the rectangle to make subareas for the products of the place value parts and fill in the products for each subarea.

The expanded notation method shown in the top middle for multiplication is a common approach. But there are tricky parts of this method, so students in one classroom added the blue steps to help all of them see what was happening in each step and avoid their errors, and the multiplying was written for the largest place value unit product (the tens × the tens) first so that the other products could be aligned underneath. The blue steps can drop out when they are not needed. This fuller method is helpful to many students initially.

But I found in many classrooms that some students had difficulty with this method: they could not see what to multiply by what. The area model was clearer about what to multiply by what, so they would draw a little rectangle, record the products inside the subareas, and add them up on the right as shown in the place value sections method.

The one-row method shown on the top right is a common embedded method that alternates multiplying and adding and that writes the added-in value for the tens × ones step in the wrong place: 60 × 3 is 180, but the 1 hundred is written above the tens column (above the 4 and the 6). Better methods are discussed in Fuson and Beckmann (2012).

The rectangle sections method, on the bottom left for division, helps students relate multiplication to division as they see how the same area model can be used for both. Students first draw a length 40 for the tens part of the unknown factor

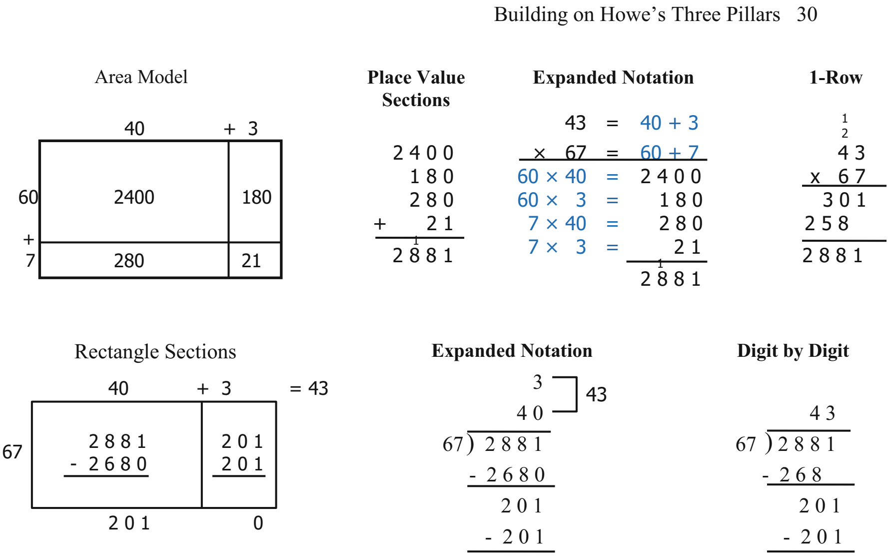


Fig. 9.8 Drawings and written variations of standard algorithms for multiplication and division

and multiply 67 by that number 40. They subtract the resulting 2680 from the total product to find the area of the subarea for the ones unit, getting 201. They draw the ones length 3, multiply 3 × 67, and subtract that from the area of the ones subarea. This problem has no remainder, but many problems do have a remainder. The other methods in the bottom row of Fig. 9.8 show the same steps of finding the tens and then the ones values of the unknown factor. These methods can be related to the area model so that students understand what they are doing, and students can discuss how all three methods relate to each other.

To return to the issue of length models with which this section began, alternating square and long shapes shows place values more easily than do just length models. The hundred square discussed earlier is the new larger square unit, ten of which can be composed in a tall column to make a thousand. Ten of these tall columns can be composed to make the new large square unit of ten thousand. Ten of these ten thousand units can be composed in a tall column to make a hundred thousand. Finally, ten of these hundred thousand long shapes can be composed to make a huge million square. The units for these models can be count numbers (dots) or measure numbers (tiny unit squares). Making such a display in the hallway has been a productive activity for many classrooms.

For more information about the learning path of multi-digit multiplication/division and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Number and Operations in Base Ten (NBT) in the CCSS: Part 3 Place Value and Multi-digit Multiplication and Division in G3 to G6 (at http://www.karenfusonmath.com).

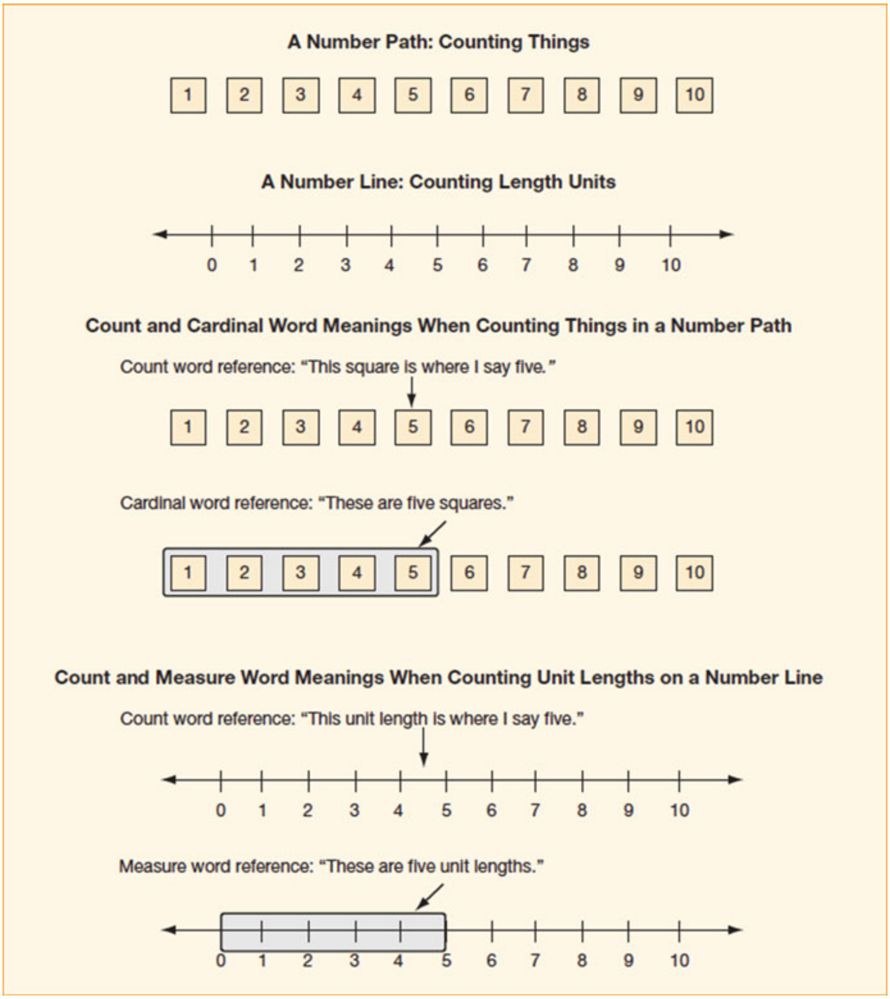
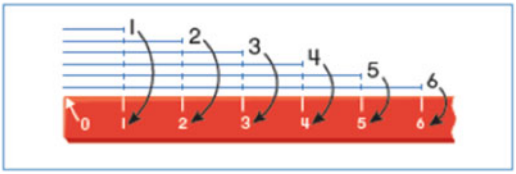


Fig. 9.9 Relationships between counting number, cardinal number, and measure number

9.3.3 Numbers on the Number Ray Tell Distances from the Endpoint/Origin

Howe's (2014) final major point concerning counting number and measurement number is the understanding that the numbers on the number ray tell *distances* from the endpoint/origin. This is a crucial understanding that provides a sound basis for placing whole numbers and fractions on the number-line diagram. Some people refer to whole numbers and fractions on the number line as points on the number line. Thinking only about points does not provide meanings for adding

Fig. 9.10 Seeing length units on a ruler by drawing successive lengths



and subtracting. How can you add one point and another point and get a third point? This is only possible if the points are actually endpoints of distances from zero created by length units. The CCSS–M 3.NF.A.2a and CCSS–M 3.NF.A.b use this relationship between interval/distance/length units and the endpoint of the interval/distance/length from zero to describe representing fractions on a number-line diagram.

Seeing the length/distance units on a number line is difficult because our brains are wired to see things not lengths. In Fig. 9.9, counting numbers are shown at the top, each within a square to make it easy to count them. Below that is a number line where the numbers represent the number of length units from 0. Notice how your eye is drawn by the numbers below the line and the little vertical marks for the ends of each unit. It is difficult to see the unit lengths on the line that lie between the numbers. In the middle are shown the relationships between count and cardinal meanings of number described in K.CC.B.4b: the last counted word tells how many things there are. Below that are shown the similar relationships between count and measure meanings of number: the last counted word tells how many unit lengths there are. Because of the visual difficulty and the off-by-one errors induced by number lines, the National Research Council reports (2001, 2009) conclude that number lines are not appropriate for PK, K, or grade 1 children. Visual count models like the number path shown at the top of Fig. 9.9 are appropriate. The CCSS–M is consistent with these recommendations, first introducing number lines at grade 2.

Rulers and bar graph scales have the same structure as a number line. Figure 9.10 shows a ruler. Notice how the eye is drawn by the points marked by the short vertical segments and by the numbers below these. We have to work hard to help students see and use the distances/lengths in rulers, bar graph scales, and number lines. One way is shown in Fig. 9.10. Students can draw one length unit and write a 1 after it, then very close below they draw two length units with a 2 after it, then three length units followed by a 3, etc. They then can think of a ruler as all of these lengths pushed together to make a single line with all of these lengths on it; the number of lengths so far is written at the endpoint of each of the lengths. They also can make little vertical segments as they measure lengths initially and then count those lengths to emphasize that they are measuring length units.

In Fig. 9.11, we can see three other ways to see the lengths on fraction number lines. First, at the top, a fraction bar in which one can see lengths is drawn above a number line in which the eye is drawn to points instead of lengths. The lengths in the fraction bar help one see the lengths in the number line. At the bottom, the number of unit fractions (seven) is shaded in the number bar and encircled in the

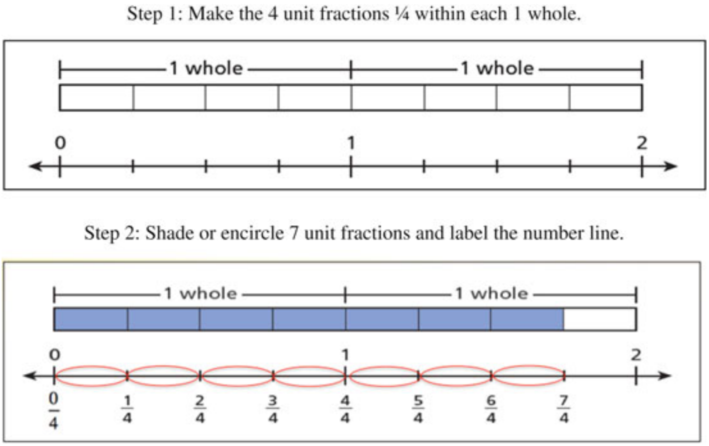


Fig. 9.11 Seeing the unit fraction lengths by shading or encircling

number line. This helps the viewer see the lengths. Students can also be asked to slide their finger along each length as they count the seven unit fraction lengths.

Making unit fraction drawings in these two steps also helps students make sense of unit fractions. Usually students just see the second step with some of the unit fractions shaded or otherwise marked. But then they do not see the total number of unit fractions, here four in one whole. They just see the two parts of the fraction embedded inside the whole. If only the second whole had been shown, students would see three parts shaded and one part not shaded in that second whole. Many students then say that the fraction is 1/3 because they see the parts 1 and 3 but not the total four parts. But in the top drawing in which four unit fractions are made in one whole, students can see the four unit fractions. So the right-hand bottom half of the drawing shows three parts shaded of the total four parts, so ¾. Here, to see that the bottom shows 7/4 and not 7/8, the top unit fractions could each have been labeled ¼.

Without consistent support to see the lengths in number lines, students make errors when drawing or labeling number lines for whole numbers or fractions. They may count the points beginning with the first point as 1 instead of as 0 and get one too few unit lengths. If they have a number line with the starting and end marks already made, they may make as many new marks as unit fractions, resulting in one too many unit lengths.

Number lines are an important mathematical tool, and students must come to understand the relationships between the distance/length units and the endpoints of these units that are labeled on the number line. For more information about the learning path of fraction conceptions and computation and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Number and Operations—Fractions (NF) in the CCSS–M (at http://www.karenfusonmath.com).

9.4 Visual Models Are Central Core Ideas and Practices in the CCSS–M and Deserve Attention and Discussion

We close by summarizing the importance of visual models for building understanding and explaining in classrooms. As the research-based examples here have shown, models can be simple math drawings that students can make and use in their own ways in problem solving and explaining of thinking. They support the math talk discussions that are at the heart of the CCSS–M. The CCSS–M specify eight mathematical practices that are to be implemented with the standards. These eight can be formed into four pairs (practices 1 and 6, practices 7 and 8, practices 4 and 5, practices 2 and 3) and given names to support their use in the classroom. A teacher can ask every day: "Did I support students to focus on math sense-making about math structure using math drawings (visual models) to support math explaining? And can I do this better tomorrow?" These mathematical practices, and the visual models that support their implementation, can help Howe's (2014) three pillars come to life in the classroom. Teachers and students can come to appreciate the power of robust understanding of the operations of addition and subtraction including situations that give meaning to the operations and levels of single-digit addition and subtraction (Pillar I), an approach to arithmetic computation that intertwines place value with the addition/subtraction facts (Pillar II), and making connections between counting number and measurement number (Pillar III). These are crucial aspects of CCSS–M OA, NBT, and NF standards.

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Chapter 10
Is the Real Number Line Something to Be Built or Occupied?

Hyman Bass

Abstract Number and operations form the backbone of the school mathematics curriculum. A high school graduate should comfortably and capably meet an expression like, "Let f(x) be a function of a real variable *x*," implying that the student has a robust sense of the real number continuum. This understanding is a central objective of the school mathematics curriculum, taken as a whole. Yet there are reasons to doubt whether typical US high school graduates fully achieve this understanding. Why? And what can be done about this? I argue that there are obstacles already at the very foundations of number in the first grade. The *construction narrative* of the number line, characteristic of the prevailing curriculum, starts with cardinal counting and whole numbers and then *builds* the real number line through successive enlargements of the number systems studied. An alternative proposed by V. Davydov, the *occupation narrative*, begins with *pre-numerical* ideas of quantity and measurement, from which the *geometric* (number) line, as the environment of linear measure, can be made present from the beginning and wherein new numbers progressively take up residence. I will compare these two approaches, including their cognitive premises, and suggest some advantages of the occupation narrative.

10.1 Two Story Lines of the Number Line

> *… we assumed that the students' creation of a detailed and thorough conception of a real number, underlying which is the concept of quantity, is the purpose of this entire subject, from grade 1 to 10 … the teacher, relying on the knowledge previously acquired by the children, introduces number as a … representation of a general relationship of quantities, where one of the quantities is taken as a measure and is computing the other.*
>
> - Vasily Davydov (1990)

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> *The device beyond praise that visualises magnitudes, and at the same time the natural numbers articulating them, is the number line, where initially only the natural numbers are individualised and named. In the didactics of secondary instruction the number line has been accepted, though it is often still imperfectly and inexpertly exploited*
>
> - Hans Freudenthal (1983)

Teaching is not only teaching in the moment. It must build on what students bring from their past. And it must prepare them for what is to come in their future. This paper examines the early foundation of number in the school curriculum and how this foundation can best support the development of an ultimate robust understanding of the continuous real number line. At play in this discussion is a fundamental mathematical duality:

The discrete-continuous duality	
Discrete	Continuous
Discrete counting	Continuous measurement
Whole numbers/integers	Real numbers
Cardinal/ordinal world: “the number queue” . . . • • • • • • • • • . . .	Measurement world: the number line . . .−−−+−−+−−+−−+−−+−−+−−+−. . .

The current number curriculum is a multi-year progression, which I call the *construction narrative*, from the left side of this picture to the right, with the number line growing as new numbers are successively filled in. The right side also makes an early curricular appearance, in the context of measurement, but without this being integral to the development of number. I will contrast this construction narrative with an alternative, based on ideas of V. Davydov, the *occupation narrative*, which begins with *pre-numerical* notions of quantity and (continuous) measurement. In this approach, the *geometric* (number) line can be made present from the beginning, as an environment in which numbers progressively take up residence and are named.

10.2 The Construction Narrative

The construction narrative begins with whole numbers and counting and progressively introduces new number systems. In all but one case (from rational to real numbers), the new system is created to enable solutions to equations formulated, but not solvable, in the previous system. In each extension, it is tacitly presumed, but not generally proved, that the arithmetic operations extend and that basic rules of arithmetic (commutativity, associativity, etc.) continue to hold.

The construction narrative of the real number line Cognitive premise: Children's early sense of small cardinals and large differences		
Number systems	Models	Conceptual frame
Whole numbers	(Finite) sets; disjoint union	Cardinal/ordinal
Fractions ($\geq$ 0)	Part-whole images *Solve*: a • x = b	Part-whole ratio measure; whole as unit
Integers*	Diverse, including the number queue Solve: a + x = b	Mirror reflection through 0 on the number line
Rational numbers	Formal combination of fractions $\geq$0 and negative numbers	Mirror reflection of fractions $\geq$0
Irrational numbers	Miscellaneous natural examples: $\sqrt{2}$, π, e	Incommensurability with 1
Real numbers	"Everything else." Infinite decimals. A significant conceptual gap	All points on the (continuous) number line
Complex numbers	"The complex plane" Solve: $x^2 + 1 = 0$	

(*) The curricular order of "integers" and "fractions ($\geq$ 0)" is sometimes reversed

10.2.1 Affordances of the Construction Narrative

This line of development fits with Leopold Kronecker's famous declaration "*God made the integers, all else is the work of man.*" Moreover, the cognitive premise is well founded in research (e.g., NRC (2009), or Butterworth, (2015)):

> *It is now widely acknowledged that the typical human brain is endowed by evolution with a mechanism for representing and discriminating numbers … when I talk about numbers I do not mean just our familiar symbols – counting words and 'Arabic' numerals, I include any representation of the number of items in a collection, more formally the cardinality of the set, including unnamed mental representations. Evidence comes from a variety of sources.*
>
> Brian Butterworth (2015)

10.2.2 Difficulties with the Construction Narrative

10.2.2.1 The Whole Number/Fraction Divide

Whole numbers are conceived as cardinalities of (discrete) sets, while fractions are conceived as relative measures of two (continuous) quantities, and so they seem to be different a species of numbers. The whole number 7 is thought of as a noun, whereas, when thinking of 3/4, it is hard to resist adding the word "of." A fraction is conceptually an *operator* on quantities, not a conceptually freestanding mathematical object, since, unlike cardinality, the unit of measure is unspecified and not implicit. This difference makes it difficult to arrange for these two number populations, and their interactions harmoniously cohabit the same (real) number universe. Of course, cardinality is appropriately viewed as a special (discrete) regime of measurement, but this perspective is not initially needed and so not made explicit.

Whole numbers ——>	Fractions
A whole number is, conceptually, a mathematical object."7"	A fraction is, conceptually, a mathematical operator. "3/4 of . . . "
A whole number is *the (discrete) measure* (cardinality) of a set	A fraction is *the relative measure* of two quantities
Addition/subtraction corresponds to composition/decomposition (set union)	Addition/subtraction corresponds to composition/decomposition of quantities
Multiplication corresponds to repeated addition (or whole number rescaling) or to Cartesian arrays	Multiplication corresponds to composition of operators or to rectangular area (in which case the product is a different species of quantity)
Whole numbers are denoted with base-10 positional notation	Fractions are denoted with the fraction bar notation
Computational algorithms are anchored in this notation	Computational algorithms are anchored in this notation
Whole numbers are born in the cardinal/ordinal world	Fractions are born in the worlds of (possibly continuous) measure (The cardinal world is one of these, though it is not typically seen this way)

10.2.2.2 The Continuum Gap

The passages from rational numbers to irrationals and then to real numbers are fragmentary and pretty much clouded in mystery in the school curriculum. The student knows little more than "some numbers are irrational." To build the real numbers with analytic rigor would exceed the resources of the school curriculum, but as a result students are left with a weakly developed concept image of real numbers. How would a high school student explain the meaning of $\sqrt{2} + \pi$ or $\sqrt{2} \bullet \pi$? Or 2^{π}, "the product of π copies of 2?" Our base-10 algorithms act first on the rightmost digits and so could not be applied to infinite decimal expansions.

10.3 The Occupation Narrative: Cognitive Premise

Here I report on ideas mainly due to Vasily Davydov (1975). The approach is founded on a complementary cognitive premise:

> *Children's understanding of measurement has its roots in the preschool years.* *Preschool children know that continuous attributes such as mass, length, and weight exist, although they can not quantify or measure them accurately. Even 3-year-olds know that if they have some clay and then are given more clay, they have more than they did before. Preschoolers*

cannot reliably make judgments about which of two amounts of clay is more; they use perceptual cues such as which is longer. At age 4-5 years, however, most children can learn to overcome perceptual cues and make progress in reasoning about and measuring quantities. Measurement is defined as assigning a number to a continuous quantity.

- Doug Clements and Michelle Stephan (2001)

Davydov's approach:

- Young children have a primordial sense of *quantity*, an attribute of physical objects (not only cardinalities): length, area, volume, weight, etc., without numerical associations.
- And of *addition* (composing and decomposing quantities of the *same species*), they can make rough comparisons of size ("Which is more?"), which Davydov has them express symbolically, as "B > T" and then infer that "B = T + C" for the "quantity difference" C. Venenciano and Dougherty (2014) describe this as "concurrent representation used to model change from a statement of inequality to a statement of equality."

Using two unequal areas of paper, the papers can be stacked such that the area of the larger piece that is not covered by the smaller piece can be cut off. The piece that is removed is defined as the difference. Similarly, beginning with the unequal areas of paper, by taping the precise amount of area to the smaller area to create a combined area equal to the larger area, defines the difference.

Given quantity B > quantity T: If B – C = T and B = T + C, then B = T by C.

The last statement is read "Quantity B is equal to quantity T by the difference, quantity C."

- Davydov develops in children such *algebraic relations*, involving "pre-numerical" quantities and hence *involving no numerical calculation*.
- This practice functions as a precursor of algebraic thinking.
- It imparts the correct sense of the meaning of the "=" sign that the (eventually numerical) value of the two sides is the same. This meaning is sometimes distorted when equations are used primarily in the context of numerical computation: "the right side is the computation of the left side."
- Davydov develops these ideas in first grade, *prior* to the introduction of whole numbers, in a measurement context. Whole numbers appear only late in the first term of first grade.

10.4 Occupation Narrative: Quantity, Unit, Measure, and Number

A quantity has no intrinsically attached number. Rather, given two quantities, A and U, then, taking U as a “unit,” the number we attach to A is “How much (or many) of U is needed to constitute A?” Thus, a number is a ratio of two quantities.

To understand a numerical quantity, it is necessary to specify, or know, the unit. And, for a given species of quantity, different units may be chosen: feet, inches, meters – for length; quarts, pints, and liters – for liquid volume; etc. *To numerically simplify a sum* of two numerical quantities, they must be of the *same species* and expressed with the *same unit*. (“Can’t add apples and oranges”) That is why, in place of value algorithms for addition, we vertically align the digits with the same place value, i.e., with the same base-10 units. That is why, in adding fractions, we seek common denominators (“unit fractions”).

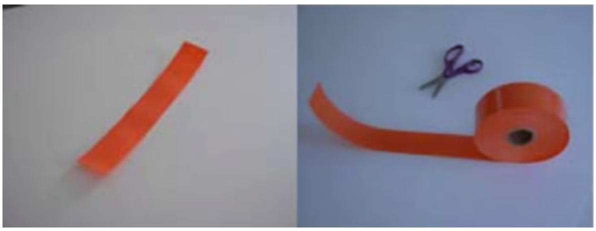
In principle, numerical quantities manifest *the full continuum of (positive) real numbers. Whole numbers arise, in every measure regime*, when a quantity is composed exactly of a set of copies of the unit. *This is how to comprehend whole numbers in the general measure context, not simply cardinal counting.* (In the cardinal world, the default unit is the one element set, and each set is composed of a set of copies of this unit.)

Of course, cardinal can be viewed as a (discrete) measurement context. However, since it is natural to choose the one-element set as unit, there is no a priori need to even introduce the concept of unit. Thus, in the cardinal introduction of whole number, the very concepts of unit and of measure relative to a unit do not immediately rise to conscious consideration. Later, when introducing multiplication and place value, other sets are taken to function as units, but, again, this typically is not explicitly linked conceptually to the domain of continuous measure. This is related to the “whole number/fraction divide” discussed above.

This notion of number as a ratio of quantities may seem somewhat sophisticated and not appropriate for very young children. Davydov argues the contrary, as demonstrated by the following activity design, to solve what he calls the “*fundamental problem of measure”*: *given a quantity A, reproduce A in a different place and time.*

Here is how he enacts this with children (see Moxhay 2008).

1. A strip of tape, A, is on a table. In the next room is a roll of tape.
2. *Task*: Cut off a piece of that roll of tape exactly the length of A. But you are not allowed to move A.



3. *Different approaches*:
   - Make a guess, from a remembered image. This is very inexact.
   - If given a spool of string, cut off a piece of string the length of A. This is exact but *needs a mediating equivalent quantity*, the string.)
   - Suppose you are given a stick of wood, longer than A. Mark it at the length of A, and use this to measure off the tape.
   - Suppose you have a piece of wood *shorter* than A, then you can count off lengths of the piece to *measure* A. In this case, *the child actually constructs the idea of measurement and engages the concept of unit.*

This activity design, which leads the learner to the concepts of measure and of unit, creates what Harel (2003) calls *intellectual necessity* and exemplifies a *didactical situation*, in the sense of Brousseau (1997). If we imagine this experiment with cardinal instead of linear measure, several conceptual and cognitive steps would be missing, and the first approach would suffice.

10.5 Some History

Vasily Davydov (1930–1988) was a Vygotskian psychologist and educator in the Soviet Union. With colleagues, in the 1960s, he developed a curriculum starting with quantity (of real objects) and measure. Adaptations of the Davydov early grades curriculum have been implemented in the USA, with some claims of success. See, for example, Dougherty and Slovin (2004); Schmittau (2005); and Moxhay (2008). Many of these ideas are present in the NCTM and Common Core State Standards Initiatives, in the context of measurement, but not integrated with the development of number.

In Bass (1998), I speculated about the possibility of an early introduction of the continuous number line in the school curriculum without, then, being aware of Davydov's work.

10.6 Coordinatizing the Geometric Line ("Descartes in Dimension One")

Length is a one-dimensional quantity and may be applied to one-dimensional attributes of objects in general. We typically measure length with a measuring stick or a tape measure. The latter is more adaptable if, for example, we want to measure a hat size or the circumference of a tree trunk. The essential feature of the tape is that it is flexible but inelastic (it doesn't stretch). Notice that no unit of measurement has yet been specified. On the other hand, the tape is two-dimensional, not one-dimensional. From the point of view of linear quantity, a better metaphorical model might be a string, eventually indefinitely long, that we can take to be a heuristic model of the geometric line. On this "line," two points have a well-defined distance represented by the length of string (mathematically, the interval) between them. This "geometric line" is the environment of linear quantity and measure.

The geometric line is *coordinatized* with numbers by *choice of an ordered pair of points that we call 0 and 1*. Then we take the interval [0, 1] as the unit of linear measure. Moreover, the direction from 0 to 1 is taken as the positive orientation of the line, the direction of numerical increase. Note that the line has an intrinsic "linear structure" arising from "betweenness": Given three points, one will lie between the other two. This does not yet specify which one is largest. There are *two possible "linear orders*" on the geometric line. In choosing the ordered pair (0, 1), we specify not only the unit of measure, [0, 1], but also the order (orientation) of the line by declaring that 1 is greater than 0, so 0 to 1 is the positive direction on the line. Our general convention is to depict the line horizontally and to take (left –> right) as the positive direction.

A whole number N is then placed on the line by concatenating, to the right, N copies of the unit, starting at 0 and placing N at the final right endpoint. Note that this placement is essentially measure theoretically, not based on cardinal. Children are sometimes confused by counting hash marks, where the copies of the unit meet, instead of counting intervals. Fractions are similarly placed on the line using a subunit, $[0, 1/d]$, where d is the denominator of the fraction.

This, in fact, foreshadows the general geometric concept of number on the (coordinatized) number line: A point, a, on the number line represents the number that is the measure of the *oriented interval* from 0 to a. (This will be the negative of a if 0 lies between a and 1.) In fact, one may reasonably think of the oriented interval $[0, a]$ as a one-dimensional vector. From this point of view, adding a to a general number, x, can be geometrically viewed as translation of the line by the vector $[0, a]$, a given distance in a given direction.

10.7 Conclusion: What Is Achieved by This Occupation Narrative of the Number Line?

- As mentioned earlier, Davydov's early introduction of pre-numerical quantities provides an introduction to algebraic notation and relations and to a robust sense of the meaning of the "=" sign.
- The whole number/fraction divide is bridged: The part-whole introduction of fractions is inherently a measurement approach, the *whole* being the unit of measure. Though cardinal counting is also a measurement context, that point of view is not emphasized, because there is a natural default choice of unit (the one-element set), and so the very concept of unit, and its possible variability, need not enter conscious reflection or discussion. Here it is proposed that one emphasizes the appearance of whole numbers in every measurement context. In fact, placement of whole numbers on the number line already requires appeal to (continuous) linear measure.
- The main point is that *the geometric line, anchored in the context of linear measure, is present almost from the beginning. The progressive enlargements of the number world simply supplies numerical names to more and more of the (already present) points on the line*.
- While a few irrational numbers can be identified and located on the number line, it can be pointed out that many (even most) numbers are irrational and that even though we have not named them, they are there as points on the geometric number line and leaving no "holes."
- The number line makes it possible, from the beginning, to geometrically interpret the operations of adding, or multiplying by, a real number. (See I and Dougherty (2015) for a measurement treatment of multiplication.)

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Chapter 11
What Content Knowledge Should We Expect in Mathematics Education?

Richard Askey

Abstract Two content topics will be described. One deals with geometric measurement: length, area, and volume. These are important topics and they have not been learned very well. In particular, what types of relations can one consider when dealing with these three aspects of geometric measurement? The second topic is fractions, and different views of what students should be expected to learn. A third topic will be briefly discussed. This is a new book which has many mathematical topics which arose in secondary classes, with comments on the content both as it relates to student learning and sometimes how the mathematics fits into what had previously been studied in earlier grades and also how the ideas fit into later material. The aim of the work was to set up a framework on mathematical content knowledge for teaching mathematics in secondary school. The authors asked for comments from readers, so some will be given.

11.1 Introduction

Roger Howe has not only done very important work in mathematics, but he has taken on the task of trying to do important work in mathematics education at two ends, early elementary school and college level. This is rare and even rarer to succeed at both levels. His elementary school level work includes the following (Howe, 2012): *Three Pillars of First Grade Mathematics*. This illustrates very important aspects of whole numbers and addition and subtraction. Here is a comment following this article:

> *I have been an elementary math coach for 10 years and an elementary teacher for 29 years before that. We are always working on these concepts, but I have never read such an excellent, explicit article that pushes all the important understandings for the primary students.*

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At the college level, Barker and Howe wrote Barker & Howe (2007), which can be used as a text for a college course in geometry. Parts of this book can influence what teachers should know in detail about geometry, and some other parts contain material secondary level mathematics teachers should be familiar with.

11.2 Concepts and Skills

One of the recommendations from the National Mathematics Advisory Panel (2008, p. xix) is:

> *To prepare students for Algebra, the curriculum must simultaneously develop conceptual understanding, computational fluency, and problem solving skills. Debates regarding the relative importance of these aspects of mathematical knowledge are misguided. These capabilities are mutually supportive, each facilitating learning of the others.*

Here is another recommendation (National Mathematics Advisory Panel, 2008, p. xxi):

> *The mathematics preparation of elementary and middle school teachers must be strengthened as one means for improving teachers' effectiveness in the classroom. This includes preservice teacher education, early career support, and professional development programs. A critical component of this recommendation is that teachers be given ample opportunities to learn mathematics for teaching. That is, teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach.*

Let me add a little to the last. In addition to mathematical knowledge for teachers, there are others who need more mathematical knowledge. Mathematics support people, both those in schools and in many other places, need deeper knowledge and a more comprehensive view of the curriculum, how it developed in earlier grade bands, and where it is going in later years. Mathematics educators have an important role to play in the preservice education of teachers, in their research, and in knowing more about the topics Lee Shulman mentioned teachers need to know (Shulman, 1986). One of these is knowledge of curriculum. With respect to this topic, let me just mention one example which is common to many textbooks on algebra and geometry.

Definition *Two nonvertical lines are parallel if and only if their slopes are equal. Two nonvertical lines are perpendicular if and only if the product of their slopes is* -1.

All of the groups mentioned above should know the difference between a theorem and a definition. Since parallel and perpendicular have geometric definitions, this should be a theorem, not a definition. Some people from the groups listed above should have written publishers and requested an appropriate change. As someone who has done this in person at displays at meetings and in letters to publishers, this seems to be something which surprises people. Some staff agree that a change should be made, but some defend what was written. Finally, with the Common Core (CCSSO and NGA, G-GPE.5), this is now listed as a theorem, as it should have been all along. The lead author of the geometry book containing the above *Definition* has a Ph.D. in mathematics from a major US university and clearly should know the difference between a definition and a theorem, so the problem is not just content knowledge, but expectations and taste. I am not mentioning which book this is since many books have this error.

This paper will point out some more examples where both conceptual understanding and computational skills are far too weak.

11.3 Geometric Measurements

In the book *Accessible Mathematics* (Leinwand, 2009, p. 92), Steven Leinwand wrote the following:

> *For this reason, effective instruction balances a focus on conceptual understanding (such as the meaning of area and perimeter and how they are related) with a focus on procedural skill (such as how to find the area and perimeter of plane figures).*

This is a book written for elementary and middle school teachers. Let us first address the issue of area and perimeter, a common topic in primary school. A problem devised by Deborah Ball and used later by Liping Ma needs to be mentioned about teacher knowledge concerning a possible connection between area and perimeter. The story is told that a student comes to class very excited. She has figured out a new property that the teacher has never told the class. She said that she has discovered that as the perimeter of a rectangle increases, the area also increases, and illustrates this with two examples, a square of side 4 cm and a rectangle with sides 4 cm and 8 cm. The question for teachers is: How do you respond?

If you have not read Ma's description of the results when this question was asked of some teachers in the USA and in China, let me strongly suggest you read Chap. 4 in Ma (2010). There were 23 US teachers, 12 of them had 1-year experience of teaching, and the other 11 averaged 11 years of teaching. Two of these teachers accepted the student's claim, three investigated the claim, and one was able to show that the claim was false. The rest said they would look it up, often because they did not remember formulas for the perimeter and/or the area of a rectangle. Ma interviewed 72 Chinese elementary school teachers. About the same percent accepted the claim. The rest worked on the problem, and about 70% of them were able to show that the student's claim was false. One gave a very cogent answer (Ma, 2010, p. 97):

> *The area of a rectangle is determined by two things, its perimeter and its shape. The problem of the student was that she only saw the first one. Theoretically, with the same perimeter, let's say 20 cm, we can have infinite numbers of rectangles as long as the sum of their lengths and widths is 10 cm. For example, we can have $5 + 5 = 10$, $3 + 7 = 10$, $0.5 + 9.5 = 10$ even $0.01 + 9.99 = 10$,* etc., etc. *Each pair of addends can be the two sides of a rectangle. As we can imagine, the area of these rectangles will fall into a big range. The square with*

sides of 5 cm will have the biggest area, 25 square cm, while the one with a length of 9.99 and a width of 0.01 will have almost no area. Because in all the pairs of numbers with the same sum, the closer the two numbers are, the bigger the product they will produce.

A number of years ago, I read a manuscript of a book which had the following problem:

Find rectangles whose area is equal to its perimeter.

To show that this problem made no sense, I suggested considering a rectangle with sides 3 and 6. The numbers one gets for perimeter and area are both 18. Consider the unit length to be 1 foot to begin with, and then change to yards. The dimensions of the rectangle are now 1 yard and 2 yards so that the perimeter is 6 and the area is 2. Then change to inches and one gets $2(36 + 72) = 216$ for the perimeter and $36 \times 72 = 2592$ for the area. So, are the perimeter and area the same or is one larger? One now sees that none of these is true; one cannot compare lengths with areas since their units of measurement are different. I omitted feet and square feet which suggests that these cannot be compared. However, one can compare the square of length with area, and for rectangles this is a problem which can be given and solved in late primary school or middle school. It is closely related to the claim made by the Chinese teacher of the square having the largest area among all rectangles with the same perimeter. The usual way this isoperimetric (same perimeter) problem is used in the USA is when the area of a rectangle has only been discussed when the side lengths are positive integers. Then, for many specific examples, there are only finitely many cases to consider and these can be treated by computation.

There are a number of different ways to treat the general case. One is to take a rectangle with sides $a > b$ and use it to construct much, but not all, of the square with side $(a + b)/2$ by removing a smaller rectangle with sides $(a - b)/2$ and b and placing it on top of the remaining rectangle. A small square of side $(a - b)/2$ is missing, so the area has decreased by this much. What has been done here is to give a geometric proof of the identity:

$$((a+b)/2)^2 - ab = ((a-b)/2)^2.$$

Since the right-hand side is positive when $a \neq b$, we have shown that $((a + b)/2)^2 \geq ab$. Divide by $(2(a + b))^2$ the square of the perimeter for the original rectangle and the square to get

$$ab/(2(a+b))^2 \leq 1/16.$$

This shows that the area A of the original rectangle divided by the square of the perimeter P satisfies

$$A/P^2 \leq 1/16 \qquad (11.1)$$

and equality holds only when the rectangle is a square.

This is a direct relationship between area and perimeter for a rectangle but clearly not what was being asked for. I think it is too much to expect many teachers to go this far, but the description Ma quoted from a Chinese teacher is something we could hope for. We are very far from this now. I would like math specialists and high school math teachers to know all of this. In a new book (Wu, 2016, p. 197), Wu includes the isoperimetric inequality (11.1) for rectangles as a problem. This is the first of two books he has written for middle school teachers.

Teachers in elementary and middle schools need to know some things about lengths and areas of figures that are not necessarily rectilinear. There is a difference between lengths and areas in a plane, which is important but almost completely ignored in school mathematics. When dealing with area, one can find, intuitively, both upper and lower approximations to the area, and, for standard figures showing up in school mathematics, these can be refined to get good approximations to the area of the figures. For arc length in the plane, if one connects successive points on a curve, one gets a lower approximation to the length of the curve, and for smooth curves, this is what will be used later in calculus to motivate the formula for the arc length of a curve. However, an upper bound on the arc length, even for a circle, requires work and is not appropriate for school mathematics.

How much of this do you think Leinwand had in mind, and how much do you think most readers of his book would know? The following will give some idea. Near the start of Chap. 9 in Leinwand (2009), there is the following:

> *What is the formula for the volume of a sphere? Really, do you know it? Have you forgotten it? Do you ever use it? Do you even care? ... But now return to our middle school and high school classes where memorizing and regurgitating the formula* $V = (4/3)\pi r^3$ *is a perfect way to sort students out on the basis of memorization criteria that have little relation to understanding and actually using the formula.*

Here is what he seems to think *understanding* this formula is:

> *... knowing how much 4/3 is, what* π *is equal to, what the r represents, and what that little elevated 3 means – that is how to use the formula once it is presented.*

He could at least have mentioned why the exponent is 3, since volumes of similar figures change as the cube of the factor of dilation. He could also have mentioned the fact that the constant π is the same constant which appears in the formulas for the area and circumference of a circle. Mathematics has many miracles, some minor and some major, and these should be celebrated. The area of a circle is a constant times r^2 and the circumference is a constant times r. The fact that these constants differ only by a factor of 2 is a miracle which can be motivated relatively easily. The fact that the same constants in the volume of a sphere and its surface area are also rational numbers times the same constant for a circle is a bit deeper. It is possible to define a four-dimensional sphere, and here the relevant constant is a rational multiple of π^2. I do not expect most mathematics educators to know this last part, but the two- and three-dimensional results are miracles and should be appreciated by people doing mathematics education. The factor of 1/3 is also interesting; it comes from a cone which, in school mathematics, can be motivated from a pyramid, but that is too far of a stretch to mention in a book like the one Leinwand wrote. However, it should be much more relevant in a high school class than to suggest that one needs to ask students "how much 4/3 is."

Let us return to the problem of perimeter and area of rectangles. Here is another way to treat this problem at an older age. It is often useful to revisit a problem after new ideas have been introduced. The following is an example of this. If the sides of a rectangle have lengths a and b, then the area is ab and the perimeter is $2(a + b)$. Let us simplify the last to $a + b$ and form a function as follows:

$$f(x) = x^2 + (a + b)\,x + ab = (x + a)\,(x + b)\,.$$

Setting this equal to 0 gives a quadratic equation with real roots. We know a necessary and sufficient condition for this, which for this equation gives

$$(a + b)^2 – 4ab \geq 0 \text{ and equal to 0 if and only if } a = b.$$

Recall that the inequality we just proved is the one obtained before when considering the isoperimetric inequality for rectangles. One aspect of this argument is an example of an important fact, the connections between the coefficients in a quadratic polynomial and the zeros of this polynomial. This connection also holds for polynomials of higher degrees. This suggests the question: Is there is a similar theorem in three dimensions? It is much more likely that an algebraic method would generalize easily than that a cutting and pasting argument like the one in two dimensions would. The argument below contains aspects which all high school teachers would benefit from, and some middle school teachers would also.

The natural analogue of a rectangle is a rectangular prism, or a box to give it a shorter name. If the edge lengths are a, b, c, then the volume is abc, the surface area is $2(ab + ac + bc)$, and the sum of the edge lengths is $4(a + b + c)$. Again, we will form a function using the simplified products; that is, the coefficients 2 and 4 will be dropped. Set

$$g(x) = (x + a)\,(x + b)\,(x + c) = x^3 + (a + b + c)\,x^2 + (ab + ac + bc)x + abc.$$

The zeros of this function are real, and the last time we had that we used the criteria for a quadratic equation to have real zeros. We can do the same thing by reducing the cubic to a quadratic by a method which gives real roots to the quadratic. Fortunately there is a result like this in calculus. A real-valued differentiable function which has two zeros has a zero of the derivative which lies between the two zeros of the function. This is Rolle's theorem. Next,

$$g'(x) = 3x^2 + 2\,(a + b + c)\,x + (ab + ac + bc)\,.$$

This function has real zeros if and only if $4(a + b + c)^2 – 4 \times 3(ab + ac + bc) \geq 0$. As before, this inequality can be rewritten as

$$2\,(ab + ac + bc)\,/16(a + b + c)^2 \leq 1/24. \tag{11.2}$$

The left-hand side is the surface area of the box divided by the square of the sum of the edge lengths, and the constant on the other side is the same ratio for all cubes, so in particular for the cube which has the same sum of the edge lengths. Also, there is equality only if the given box is a cube. Thus for a box with fixed sum of the edge lengths, the largest surface area this box could have is when it is a cube. There should two more theorems of this type. If the sum of the edge lengths is given, the largest volume a box can have is when it is a cube, and if the surface area of the box is given, the largest the volume can be is when the box is a cube.

We would like to use a similar proof, but to do that, we need an operator which reduces the degree of a polynomial by one and does not remove the coefficient *abc*. Here are two ways this can be done. One is to reverse the coefficients in $g(x)$ by setting $h(x) = x^3 g(1/x)$. The other way is to make $g(x)$ homogeneous by introducing a new variable y:

$$h(x, y) = x^3 + (a + b + c)\, x^2 y + (ab + ac + bc)\, xy^2 + abcy^3.$$

Then take a derivative with respect to y. I will leave this problem now, so that the reader can have some fun with it. The basic idea used here is implicit in the work of Newton (1972) and explicit in a paper by Maclaurin (1729). Some of the connections with isoperimetric inequalities were stated by Hardy, Littlewood, and Polya (1952, p. 36), as are readable treatments of the inequalities of Newton and Maclaurin. The full story works in n-dimensions. The three-dimensional theorems are interesting for another reason. All that is needed to prove them is Rolle's theorem and a little algebra. When Rolle's theorem is done in calculus, it is just used as a step to getting the mean value theorem, which is then used in various ways. It is nice to have an application of Rolle's theorem which students can appreciate for its own sake. A paper (Askey et al., 2015) on the three-variable inequalities has appeared in Mathematics Teacher.

11.4 Fractions

Leinwand's book deals mostly with suggestions for teaching. His *Instructional Shift 8* is:

> *Minimize what is no longer important, and teach what is important when it is appropriate to do so.*

Here is what he wrote about fractions (Leinwand, 2009, p. 56):

> *Sevenths and ninths. When was the last time you encountered a seventh or a ninth in everyday life? Because nearly all encounters with fractions are limited to ruler fractions such as 1/2, 1/4, 1/8, and 1/16, thirds and sixths, and fifths and tenths, one has to question the need to find a common denominator for fifths and elevenths. Only in a textbook in a math class do we impose the lunacy of 3/13 + 4/7!*

For someone like me who read the original NCTM Standards back in the early 1990s, this might bring back memories of a longer paragraph which includes not only *small denominators* but the following (National Council of Teachers of Mathematics, 1989, p. 96):

This is not to suggest, however, that valuable instruction time should be devoted to exercises like 17/24 + 5/18 or 5 3/4 × 4 1/4, which are much harder to visualize and unlikely to occur in real-life situations.

As bad as the NCTM paragraph is, what Leinwand wrote is worse. What conceptual understanding could one have of fractions if one cannot find a common denominator for fifths and elevenths? Is it really significantly harder to add *3/13 + 4/7* than to add *3/4 + 4/5*? One is *(3 × 7 + 13 × 4)/(13 × 7)* and the other is *(3 × 5 + 4 × 4)/(4 × 5)*. The first is then *(21 + 52)/91 = 73/91* and the second is *(15 + 16)/20 = 31/20*. All of the numerical computations can be done mentally. If one wants to go one step further and write both as an integer and a fraction between 0 and 1, the second has an extra step, which adds a bit to the complexity so the two computations have about the same complexity. At least the NCTM example of addition had denominators which had some common factors, so finding the least common divisor adds to the complexity. Fortunately, the Common Core does not suggest that least common denominators be used when starting to add fractions with different denominators. One reason for not mentioning least common denominators when starting to add fractions with different denominators is that doing that means having to introduce two new ideas at the same time. It is usually much harder to learn two new things at the same time than to learn them separately.

One referee of this paper suggested dropping the section on fractions. Let me add a bit to help explain why I think it is necessary to focus on this topic. First, here are some results on an eighth grade TIMSS fraction problem from 2011.

Which shows a correct method for finding 1/3–1/4?

A. (1 – 1)/(4 – 3)
B. 1/(4 – 3)
C. (3 – 4)/(3 × 4)
D. (4 – 3)/(3 × 4)

Here are a few of the results on this question. The numbers are percents:

	Correct	A	B	C	D
Average	37.1	25.4	26.0	9.4	37.1
Korea	86.0	2.7	6.9	4.2	86.0
USA	29.1	32.5	26.1	10.7	29.1
Finland	16.1	42.3	29.5	8.7	16.1

You can read more results at http://tinyurl.com/z118a7u or (Askey, 2015) and links provided there.

I hope most readers are as concerned about this as I am. After the first few NAEP results, there were articles written about how poorly our students did on fractions. One is on the web and it contains other references. See Post (1981). One item was 1/2 + 1/3 and 33% of the students got it right. Unfortunately the alternate possible responses were not included, and I have been unable to find them on the NAEP website. There are other results mentioned in this paper, and the highest score was 74% of 8th grade students correctly picking the answer to 4/12 + 3/12.

There are some other fraction problems given in TIMSS which set up a quandary both with the question about 1/3–1/4 and among the results there. Here are two questions and a few results. These are from TIMSS 1995 and were given to students in grades 7 and 8.

K9 $3/4 + 8/3 + 11/8 =$

A 22/15 B 43/24 C 91/24 D 115/24

The eighth grade international average was 49% correct. The international average was 35% for answer A. For the USA it was 42% for A and 45% for D. Notice that both A and B have answers which are smaller than 2, and the sum is clearly larger than 2 since the second fraction is 2 2/3. 11/8 = 1 3/8 > 1 3/9 = 1 1/3, so the sum of the second and third fractions is larger than 4, but 91/24 < 4 since 24*4 = 96. Those picking A clearly did not realize this type of argument, but it is likely that some of the others did since the percent of students picking B or C was small.

Here is another problem which involves subtraction.

L17 What is the value of 2/3–1/4–1/12?

A 1/6 B 1/3 C 3/8 D 5/12 E 1/2

For both the USA and internationally, the most common answer was the correct one, B: 39% USA and 42% internationally. The second most popular response was D, with 25% USA and 26% internationally. My guess as to why this was the second most popular answer is the denominator. Contrast this problem with the 1/3–1/4 one. It is much harder to do the calculations in an exam setting, and there were five possible answers rather than 4 so one might expect a larger percent of students to be able to answer the 1/3–1/4 problem correctly. Yet 39% of the students did the more complicated calculation correctly, while only 29% gave the correct answer for 1/3–1/4. When I wrote the comments on the 2011 TIMSS question which was linked earlier, I knew that two of the wrong answers were picked because their forms were somewhat like what one would do for a whole number subtraction problem. I wrote that the other incorrect answer might have some reason for picking it, but did not mention what I now think is a reason, and this reason also helps explain why all three wrong answers might be picked. There is a relatively new book (Kahneman, 2011) by Daniel Kahneman, *Thinking, Fast and Slow*. Kahneman argues that humans have two levels of thinking, which he calls System 1 and System 2. The first is what is used initially, and if it comes up with an answer which seems reasonable, that is usually as far as the thinking goes. If not, the second system is used. The first might come up with any of the four answers. The two with answers equivalent to 0 and 1 can be explained by analogy with whole numbers. The other two could occur in the following ways. Suppose a student knows that there is a formula for how to find the difference between two fractions and it is a somewhat messy formula. These two answers are somewhat messy so a student who is unsure what to do might pick one of these two answers at random. If 10% did this and got a negative number, I would expect that about 10% would have gotten the correct answer with no more sure knowledge, so the 29% correct could be replaced by about 19% who knew enough to do the calculation correctly via a rapid system in their brain. Here I am assuming that the problem of 1/3–1/4 seems so simple that few students will have to think hard about how to come up with an answer that requires the slower part of the brain. That would not be the case for 2/3–1/4–1/12. There is a way to do this rapidly, $1/4 + 1/12 = 3/12 + 1/12 = 4/12 = 1/3$, and $2/3–1/3 = 1/3$, but few of our students will do this. For the 1995 problems, the popular incorrect answer for the addition problem clearly comes in a way similar to what Kahneman describes, and I suspect that the most common mistake for the subtraction problem came in a similar way for some students, but most will have had to do some thinking about how the subtraction is actually done. Of course, just some thinking might not be enough for students to do the calculation correctly.

11.5 Mathematical Understanding for Secondary Teaching

There is a recent book with the title of this section (Heid et al., 2015). This is part of a long-term project of mathematics educators at the Pennsylvania State University and the University of Georgia. Their goal was to start to map out for secondary mathematics what has been done for primary school mathematics under the name of Mathematical Understanding for Secondary Teaching, or as they summarize, MUST. This book (Heid et al., 2015) contains 43 situations which arose in school classes and comments on the mathematics involved or, in some cases, that could have been involved in the lesson or as important background information for teachers and supervisors. These examples and general knowledge were used to develop the MUST framework. There are three general categories: Mathematical Proficiency, Mathematical Activity, and Mathematical Context of Teaching. The first has five parts carried over from *Adding It Up* (National Research Council, 2001), Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, and Productive Disposition, and a sixth has been added, Historical and Cultural Knowledge. The first chapter, written by Jeremy Kilpatrick, ends with:

> *Just as we have sought the input of many mathematicians, mathematics teachers, and teacher educators during construction of this framework, we welcome comments on our final product from those in the field.*

There are many comments which could be given, but here only one situation will be discussed. This is Chap. 41, Situation 35, *Calculation of Sine*.

The prompt is: *After completing a discussion on special right triangles (30°-60°-90° and 45°-45°-90°), the teacher showed students how to calculate the sine of various angles using a calculator. A student then asked*: *“How could I calculate sin(32°) if I do not have a calculator.”*

Various methods are described: the definition of sine for a right triangle with the use of a protractor and ruler or with dynamical geometry software, a secant line using 30° and 45°, a tangent line from 30°, and a Taylor polynomial of degree 7 of *sin(x)* about 0. There is a very important problem where this question arose and was solved in a different way by Ptolemy. Greek trigonometry dealt with chords in circles rather than triangles because this was the setting for uses in astronomy. Tables needed to be computed, and two things were available: chords associated with angles of 36° as well as the angles mentioned previously and how to find the lengths of a chord of half the length of a given chord in terms of the length of the given chord. Ptolemy’s theorem could be stated and a little information given about how he used it to get good approximations for that early time. In the course of outlining this, it would be clear that Ptolemy’s theorem was used as we would now use the easier addition formulas for sine and cosine. Teachers and some students would then see two important aspects of the history of mathematics. Very good work was done long ago, and when learning about it, one can often see how problems led to development of new mathematical results. One can also frequently see how it is now easier to do what had been done because of later work. Twenty years ago I did not know Ptolemy’s theorem nor how useful it could be as a source of different proofs which use material high school teachers should know well, but based on experience in a course on proofs at the post calculus level, few good college students knew this material. Here is an illustration of how to complicate what should be a very simple proof.

From (Wikipedia, 2017), the proof of Ptolemy’s theorem is easily reduced to proving the following trigonometric identity:

$$\sin(a+b)\sin(b+c) = \sin(a)\sin(c) + \sin(b)\sin(a+b+c).$$

This is followed with: “Now by using the sum formulae, $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, it is trivial to show that both sides are equal to” [and then a complicated expression which is the sum of four terms each of which is the product of four factors is given]. Look it up to see if you think this is trivial. However, if one uses a simple corollary of the cosine formula, $\cos(x-y) - \cos(x+y) = 2\sin(x)\sin(y)$, one easily sees that both sides are equal to $[\cos(a-c) - \cos(a+2b+c)]/2$. When teaching this proof of Ptolemy’s theorem, it was started in class and given as a homework problem to complete. A simple proof of the addition formulas had been given, but nothing had been said about how the product formulas followed from the full addition formulas. All of the students had taken the full calculus sequence and this included moderately complicated integrals, so they will have had problems where the product formulas would have been natural tools to use. A few had been able to work out a proof directly using the addition formulas, but none of them did it as slickly as the writer of the Wikipedia article did, and I do not consider that argument as trivial. It is tedious and ugly. None of them used the product formula to give a very nice proof.

I was very pleased that a sense of history was added to the MUST framework but a bit disappointed that no one thought of or knew enough to add Ptolemy's work as a good example of how history could be used to illustrate some important points.

11.6 Conclusions

I wish this could have been a more positive paper. The report (National Mathematics Advisory Panel, 2008) from the National Mathematics Panel Advisory group was very good, but it has basically been ignored almost since it was published. The final version of the Common Core Mathematics Standards (CCSS-M, 2010) is much better than I thought it would be, and parts of it could make a big difference if a few other things happen. One is adequate professional development. By and large that has not happened. Textbooks need to be improved. As Wu has remarked, it will take a lot of work to replace TSM [textbook school mathematics] by something much closer to mathematics. See (Wu 2017) for this and a much broader treatment of needed content knowledge than has been given here. The goal of the present paper is to illustrate to some extent the depth of the problems. The problems go well beyond content knowledge for teachers, it exists at all levels. Roger Howe has spent a lot of time and energy on whole numbers, and this has been appreciated. As school mathematics becomes more abstract, which it does with fractions, the problems become harder. Wu has spent a lot of time and energy on fractions, and the revised treatment of fractions in the Common Core is very similar to what he has been writing for about 15 years. However, the comments in Section 3 from Leinwand's book show that much more education needs to be done in the general mathematics education community. There is a review of Leinwand and Mathematics (2009) in the Spring 2015 issue of the *NCSM Newsletter*. The review includes the following:

> *The concluding chapters focus on accountability and provide a variety of practical expectations that rely on commitment from all levels to effect change. The book includes a lesson plan template, a crib sheet for raising mathematics achievement for all, and a research-based vision of teaching and learning with 12 interrelated characteristics of effective instruction in mathematics. This has become part of the introduction in my methods course for beginning middle level and high school mathematics teachers.*

There were no comments on the statements: *question the need to find a common denominator for fifths and elevenths* or *the lunacy of 3/13 + 4/7*. One wonders why such bad advice is almost never called out in the mathematics education literature. Leinwand's book (Howe) was published in 2009, and the first draft of these standards below high school was only released in late 2009, so Leinwand's book reflects his views before the Common Core Standards were available. One hopes it is somewhat different now.

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Chapter 12
Approaching Euclidean Geometry Through Transformations

Zalman Usiskin

Abstract Almost 50 years ago, Arthur Coxford and I developed a high school geometry course in which geometric transformations were fundamental to the mathematical development. Properties of reflections and size changes (dilations) were taken as postulates and used to deduce properties of symmetric figures, the traditional theorems of triangle congruence and similarity, and develop the relationships among the various types of isometries. In *Continuous Symmetry: From Euclid to Klein*, Roger Howe and William Barker also approach Euclidean geometry through transformations. The similarities and differences in these approaches are examined in this paper.

Mathematics and mathematics education are quite different fields. Even the most casual look at the journals in our fields can attest to this fact. We come closest to each other when we are speaking about curriculum and instruction, that is, about the scope, sequence, timing, presentation, and audience for mathematics. But even then there are significant differences. As one goes down the grades, greater and greater attention needs to be given to the student and the pedagogy rather than to the mathematics itself. The challenge for all of us is to be true to the mathematics and to mathematical truth, but we cannot expect materials written for all or virtually all students to have the same amount of rigor and detail that is necessary in materials written for serious students of mathematics, and we strive to make materials accessible, exciting, and worthwhile for students so as to maximize their interest in learning our subject.

Most mathematicians and most mathematics educators do not venture into each other's fields, but mathematicians have long assisted mathematics education without leaving the stark side.[1] We are here because Roger has been one of those

[1]Earlier in the conference, Sybilla Beckmann had humorously spoken of mathematicians' work in mathematics education as going over to the "dark side."

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mathematicians who have ventured into mathematics education. My wife is a managing editor in mathematics at Pearson and knows Roger because he has been involved with their elementary program *enVision Mathematics* as a subject-matter expert. In my remarks today I wish to speak about a major mathematical effort of Roger's that no one has yet mentioned that I consider to be a wonderful contribution he has made to mathematics education as a mathematician, and it connects his work with mine in a way quite interesting to me and I hope interesting to you as well. It is his work in college-level geometry.

Roger and I both received our doctorates in 1969. His doctorate was in mathematics and mine in mathematics education, but there was something in common with our dissertations. Both had to do with groups – his with nilpotent groups and mine with transformation groups. My remarks today traverse about four decades and are about a more specific idea, namely, our common interest in approaching geometry through transformations – mine as expressed in my textbook writing (Coxford & Usiskin, 1971; Coxford, Hirschhorn, & Usiskin, 1991; Usiskin et al., 1997; Benson et al., 2009) and Roger's as expressed in his work with William Barker in their book *Continuous Symmetry: From Euclid to Klein* (2007) and in a revision of Chap. 1 available online on the AMS website. Even though Roger's work is not directly in mathematics education, it is valuable to mathematics teaching at all levels because it provides a mathematical grounding for approaching Euclidean geometry via transformations that is sorely needed in today's environment.

Allow me to relate how I fell into this curricular area. I had the opportunity to do my bachelor's work in mathematics at the University of Illinois, the home of the first of the new math projects, UICSM. And because I had always wanted to be a mathematics teacher, my undergraduate work included mathematics education courses on the side in which UICSM was discussed, and I learned that it is possible to approach the same school mathematics in qualitatively different ways. I went to Harvard for my master's degree because I wanted to learn about SMSG, the biggest of the new math projects, from Ed Moise, who had written the SMSG Geometry text. I studied more geometry there with his college text *Elementary Geometry from an Advanced Standpoint*.[2] I loved the content, but I thought it was rather unimportant in anyone's mathematics education. I argued publicly then that, while a little bit of geometry needed to be known by all students, the one-year geometry course should be abolished from the high school curriculum. My argument was quite simple: In college as a mathematics major, there was little use for most of the geometry I learned in high school. The area and volume formulas I used in calculus could be learned before high school. Abstract algebra, real analysis, complex variables, etc., did not use any of that geometry. My favorite one-liner was: "There are few functions whose graph is a triangle."

[2] I took two courses with Moise's book as the text, one taught by Moise himself and the other taught by Thomas A. Lehrer, who at the time was writing material for the television program That Was The Week That Was (TW)[3]. The only reference Tom made to his other life in this course was to mention, when non-Euclidean geometry first became the subject of discussion, that "regardless of what you might have heard, there is no evidence that Lobachevsky ever plagiarized."

But the next couple of years, when I taught full time in a high school, I found that it was only in geometry that I was teaching mathematics! In algebra I was spending almost all classroom time on algorithms – given this equation, here is what you can do to solve it; given this expression, here is how you might simplify it; and given this word problem, here is a trick to translate the information into mathematics. But in geometry I was teaching the essence of mathematical reasoning, how to get from some given information to a desired conclusion, how to show that the truth of one statement follows from others, and where results fit into a mathematical system.

When, as a first-year doctoral student at the University of Michigan, my advisor Joe Payne suggested looking into the idea of approaching geometry through transformations, I immediately responded. I learned that it was Art Coxford, then a young professor in mathematics education, who had wanted to try this approach, and, after only a couple of meetings, we decided to write a full course using transformations. If we could develop a postulational system, here was a way to approach geometry that would enable the teaching of deduction and proof, that would give students mathematics that would be useful in their later study of linear and abstract algebra, and that would elegantly deal with the all-important concepts of congruence and similarity. We arranged that we would each teach a geometry class the next school year at the University High School.

Here is a quote from Roger's book.

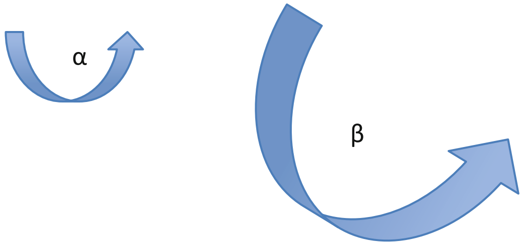
> We have tried to write a book that honors the Greek tradition of synthetic geometry and at the same time takes Felix Klein's Erlanger Programm seriously. (p. ix)

This is just like what Art Coxford and I tried to do – to honor our tradition of a single-year synthetic geometry course in high school that would follow the Erlanger Programm.

What does it mean to "follow the Erlanger Programm"? To me the key is found in general definitions of congruence and similarity that apply to any kinds of figures and involve transformations: Two figures α and β are **congruent** if and only if there is a distance-*preserving* transformation that maps α onto β.[3]

Two figures α and β are **similar** if and only if there is a distance-*multiplying* transformation that maps α onto β.

[3] The figures α and β here have purposely been chosen not to have any angles or segments to show the generality of the definitions of congruence and similarity.



"Following tradition" means having a development of the theorems of Euclidean geometry from postulates and careful definitions. Barker and Howe base their postulational development largely on that of Moise (1963). Moise himself combined the work of David Hilbert at the turn of the twentieth century (Hilbert, 1902) with that of George David Birkhoff in the 1930s (Birkhoff, 1932).

The result, for Barker and Howe, is the following structure of postulates (Table 12.1). The real numbers and their properties are assumed. There are incidence and plane separation postulates; the ruler and protractor postulates come basically from Birkhoff[4]; SAS congruence comes from Hilbert; and we know we need Playfair's parallel postulate or an equivalent somewhere.

It is notable here that the SAS congruence proposition is a postulate, whereas it was a theorem for Euclid. Euclid had tacitly assumed that figures could be moved without changing their measures. This tacit assumption is significant; it means that the lengths of segments and measures of angles are consistent throughout the plane. By assuming SAS congruence, this consistency is made explicit.

The postulate set Art Coxford and I used is almost identical to that of Barker and Howe (Table 12.2). After all, I was a student of Moise. But we did not use SAS congruence. Instead, we introduced reflections and assumed that every reflection was a transformation that preserved distance, collinearity, betweenness, and angle measure. This is obviously quite a powerful postulate, but it is easier to understand than SAS congruence, and it serves the same purpose.

We made this choice for mathematical and pedagogical reasons. Pedagogically, one of the attributes of transformations that attracted Art Coxford to the approach was that students could get their hands into the geometry – they could draw images of figures. Our course would contain much that is hands-on. Mathematically, as soon as we had students drawing, they were drawing images of images, and so they were finding images under composites of reflections. We defined rotations and translations as composites of reflections, so our reflection postulate enabled us to conclude immediately that rotations and translations preserve distance and angle measure. Then we could define congruence for any figures in terms of reflections and composites of reflections.

[4]Birkhoff's postulate set is notable for including the SAS similarity proposition rather than SAS congruence as a postulate. A few years after publishing his postulate set, Birkhoff used this set for a high school geometry text he wrote with Ralph Beatley (Birkhoff & Beatley, 1941).

Table 12.1 Postulates for Euclidean geometry (Barker & Howe, p. 120 and Revision 2.0)

0. We assume the existence of the complete ordered field R of real numbers
1. (Incidence) (a) The plane contains at least three non-collinear points
(b) Given two points, there is exactly one line containing them.
2. (Ruler) There is a 1–1 mapping from the points of a line onto R and a distance function applicable to any pair of points on that line satisfying the rest of these axioms. (The length of a segment is defined as the distance between its endpoints.)
3. (Plane separation) For any line L, the points not on L consist of a union of two disjoint nonempty convex sets H1 and H2 (half planes) such that every segment joining a point of H1 to a point of H2 intersects L.
4. (Protractor)
(a) Every angle has a measure between 0 and 180.
(b) If ray AB is on line L, then for every number r with $0 < r < 180$, there exists a unique ray AC in each half plane of L such that $m\angle CAB = r$.
(c) (Angle addition) If ray AD is in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.
(d) If two angles form a linear pair, then they are supplementary.
5. (SAS congruence) If there is a correspondence between two triangles such that two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle (i.e., have the same lengths and angle measures), then the triangles are congruent.
6. Parallel postulate (Playfair) For any line L and point P not on L, there exists a unique line L' parallel to L containing P.

Table 12.2 Some postulates for Euclidean geometry (from Coxford and Usiskin, pp. 599–601)

0–4 and 6. (Essentially as in Barker and Howe)
5. (Reflection) (replacing SAS congruence)
(a) Given a reflecting line, every point has exactly one reflection image.
(b) Reflections preserve collinearity.
(c) Reflections preserve betweenness.
(d) Reflections preserve distance.
(e) Reflections preserve angle measure.
7. (Orientation) A convex polygon $A_1A_2 \ldots A_n$ is either clockwise oriented or counterclockwise oriented but not both. Reflections switch orientation.
8. (Size change) Every size transformation preserves collinearity, betweenness, and distance. Under a size transformation of magnitude k, the distance between any two images is k times the distance between their preimages.

We used the language and notation of function composition and called the result of following one transformation by another the composite, because we wanted to take advantage of this language in the introduction of functions, which at that time in the United States was standard fare in the course following geometry. We felt that introducing the standard $T(P)$ function notation for the image of P under the transformation T made it much easier for students to understand $f(x)$ notation

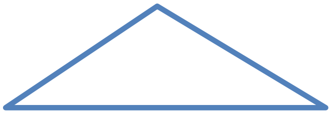


Fig. 12.1 An isosceles triangle

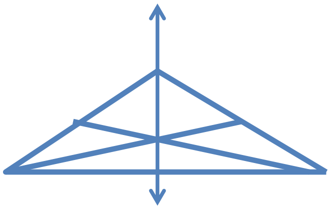


Fig. 12.2 An isosceles triangle with its symmetry line and two medians

for functions of real numbers when that was introduced.[5] A dissertation of Andy Kort (1971) involving students who had studied from our text verified our informal finding.

An aspect of approaching geometry through transformations that is seldom mentioned is that they lead students to view figures as a whole rather than as assemblages of sides and angles. Consider an isosceles triangle (Fig. 12.1).

Why do we *think* the base angles of this triangle are congruent? I don't think it is because we split the triangle into two right triangles and use SAS to prove the base angles are corresponding parts of congruent triangles. I think we view the base angles as congruent because we see the symmetry of the triangle. In fact, every geometry text that I have ever seen pictures the isosceles triangle like the one in Fig. 12.1, with a horizontal base so that the line of symmetry is vertical, in essence recognizing that it is the symmetry of the figure that we want to have as a picture in our head. The traditional proof only establishes what we already *know* due to the symmetry. So another reason for starting with reflections was to use properties of reflections to prove that certain figures are reflection-symmetric, and from that symmetry comes all sorts of properties.

For instance, we can immediately deduce that two medians of the isosceles triangle are the same length, because they are reflection images of each other over the isosceles triangle's symmetry line; so are two of the angle bisectors and two of the altitudes (Fig. 12.2). And these pairs of segments all intersect on the reflecting line because a point on the reflecting line coincides with its reflection image over that line. And from this we can quickly extend the idea to demonstrate the symmetry and get properties of other reflection-symmetric figures, including equilateral triangles, isosceles trapezoids, kites, and rhombuses.

[5]An advantage of using T(P) before f(x) is that "T(P)" is a natural abbreviation for a particular image of a particular point P under a particular transformation T. The substitution of r for T when T is a reflection, and R for T when T is a rotation has no counterpart with the first functions students encounter. For instance, we have never seen q substituted for f when f is a quadratic function, or c substituted for f when f is a cubic function. Later, of course, students do have names such as "sin" and "log" for those functions, but these do not typically occur in a student's early experience.

Perhaps the trickiest aspect of dealing with transformations in a rigorous or quasi-rigorous manner has to do with directionality. This manifests itself in the concepts of orientation and of directed angle. We want to talk about reflections switching orientation, but a figure does not have an orientation a priori. We can say we go around a triangle in the plane in a clockwise direction, but if we look at the plane from the bottom rather than the top, it will be counterclockwise. Here, in developing the mathematics, I was afraid of the technicalities. This is why we punted by introducing a postulate that a polygon, named by its vertices in order, has either clockwise or counterclockwise orientation and its reflection image has the other orientation.

Barker and Howe treat this elegantly by working in the concept of *parity*. They define (p. 194) an isometry to be orientation-preserving if it is the product of an even number of reflections and orientation-reversing if it is the product of an odd number of reflections. This requires prior knowledge that no isometry can be both the product of an odd and an even number of reflections. Discussing the algebra of transformations and the classification of isometries is something we left to the last chapter in our book, where we studied groups of transformations. Barker and Howe use the algebra of transformations to handle this idea in a very nice way.

We did not discuss directed angles. We hand-waved clockwise and counterclockwise rotations. With respect to what are usually today called dilations but what we called size transformations, we introduced a postulate because it is reasonably difficult to prove that a dilation of magnitude *k* multiplies all distances by *k*. Barker and Howe treat this in detail.

Art Coxford and I titled our book *Geometry – A Transformation Approach* (Coxford & Usiskin, 1971). We did not title it "Geometry – The Transformation Approach" because there are so many ways to use transformations to develop geometry in a mathematical way just as there are so many ways to avoid their use. Sequence matters. Notation matters. The level of rigor matters. Applications matter. The most elegant proofs possible of some theorems may be in "The Book," as Erdos described it (Schechter, 1998), but there is no section of "God's book" for the learning of a mathematical concept. That depends too much on one's experience and on the quality of the exposition of the concept.

In the Common Core, 4 of the 10 middle school geometry standards and 10 of the 43 high school geometry standards involve transformations, material that is foreign to many if not most of today's mathematics teachers. In the past year, I completed a study of textbooks published in the United States since 1960 regarding their treatment of geometric transformations (see Usiskin, 2014). My analysis indicates that even in those school geometry texts that give some attention to transformations, the treatment is often confused and incoherent and unrelated to the rest of the topics in the geometry course. And, sadly, the Common Core standards do not take advantage of the knowledge about transformations that students are expected to amass in their study of geometry.

As an example of not taking advantage, here is the first of three Common Core standards dealing with building new functions from existing functions.

Standard F-BF.3

Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given in the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Even if students have only been introduced to reflections, rotations, and translations, as in the eighth grade standards – even if they have not yet studied those transformations again as dictated in the high school geometry standards – they could use the language to see that replacing f(x) by f(x) + k results in the graph of the function being translated k units up; replacing f(x) by f(x + k) results in the graph of the function being translated k units to the right; a function is even if its graph is symmetric to the y-axis; a function is odd if its graph is point-symmetric to the origin.

But perhaps this would be expecting too much of the Common Core writers. Witness this quote:

> We have been surprised and pleased at how far this idea can be taken. (Barker and Howe, p. ix)

Art Coxford and I were likewise surprised. From the start, we knew of the connections with matrices and linear algebra and with groups and abstract algebra. And we knew we could treat congruence and similarity within geometry quite nicely. Through the volumes of Yaglom (1962, 1968) in the *New Mathematical Library* of the Mathematical Association of America, we learned of nontrivial geometry propositions with elegant proofs using transformations. Slightly after our book appeared, Escher drawings became known and popular, adding the word "tessellation" to our vocabulary. But more immediately applicable to high school mathematics were some applications to algebra and trigonometry that surprised us.

> Our text is, in fact, the first of a projected two-volume work. The second volume will go beyond traditional Euclidean geometry by introducing coordinates, discussing different geometries – affine and non-Euclidean (hyperbolic and spherical/elliptical) – in a projective setting, and ending with an interpretation of Einstein's *Special Theory of Relativity* as an analog in higher dimensions of hyperbolic plane geometry. (Barker and Howe, p. x)

I don't know if this text has been completed.[6] I do know that I am looking forward to seeing it, because I am certain it will be a significant contribution to the mathematics for those mathematics educators who are interested in the theory behind school mathematics. In particular, the introduction of coordinates overlaps with what is today standard high school mathematics.

[6] At the symposium, Roger responded that the second volume is in progress.

A transformation background provides many opportunities for the application of geometrical concepts and language in the algebra and pre-calculus experiences of students.

Here are some of the most important applications.

Group 1: *Using geometric symmetry*

1. From their synthetic definitions, proving that all ellipses and all hyperbolas have two perpendicular symmetry lines (and thus are both reflection- and rotation-symmetric).
2. Proving that the graphs of all odd functions are point-symmetric to the origin; all even functions are reflection-symmetric to the y-axis.

Group 2: *The Graph Translation Theorem and its consequences*

Graph Translation Theorem: In a relation described by a sentence in x and y, the following two processes yield the same graph:

1. Replacing x by $x - h$ and y by $y - k$;
2. Applying the translation $T(x, y) = (x + h, y + k)$ to the graph of the original relation, the graphs are congruent in the strict geometric sense. [Proof: Let the graph of the original relation be the set of points R $= \{(x,y)$: f$(x, y) = 0\}$ and let $x' = x + h$ and $y' = y + k$. We wish to describe the set of points R$' = \{(x',y')$: f$(x, y) = 0\}$. Notice how the dummy variables differ from the variables in the equation. Since $x = x' - h$ and $y = y' - k$, by substitution, R$' = \{(x',y')$: f$(x' - h, y' - k) = 0\}$. That is, the description of R$'$ is found by replacing x and y in the original relation by $x' - h$ and $y' - k$, respectively. Since we traditionally use x and y both for the variables in the preimage and the image, the description of R$'$ is found by replacing x by $x - h$ and y by $y - k$.]

There are many corollaries to this theorem.

Corollary 1 From the Pythagorean theorem, an equation for the circle with center at (0,0) and radius r is $x^2 + y^2 = r^2$, so an equation for the circle with center at (h,k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$ (similar for other graphs).

Corollary 2 Point slope: Since the line through (0,0) with slope m has equation $y = mx$, parallel lines have the same slope, and a line is parallel to its translation image, the translation that maps (0,0) to (x_1, y_1) shows that the line through (x_1, y_1) with slope m has equation $y - y_1 = m(x - x_1)$.

Corollary 3 Phase shift: The graph of $y = \sin(x - b)$ is the translation image of the graph of $y = \sin x$ under the horizontal translation $T(x, y) = (x + b, y)$. As a special case, since $\cos x = \sin(x + \pi/2)$, the graphs of the cosine and sine functions are translation images of each other and therefore congruent.

Corollary 4 The graphs of $y = b^x$ and $y = ab^x$, $a > 0$, are congruent. [Proof: Let $d = \log_b a$. Then, $b^d = a$, so $y = ab^x = b^d b^x = b^{x+d}$, so the graph of $y = ab^x$ is the image of $y = b^x$ under the translation $T(x,y) = (x - d,y)$, and the graphs are congruent.]

A continuing theme throughout elementary mathematics is the existence of analogies between addition and multiplication. The multiplicative analogue of the Graph Translation Theorem is the Graph Scale-Change Theorem.

Group 3: *The Graph Scale-Change Theorem and its consequences*

Graph Scale-Change Theorem: In a relation described by a sentence in x and y, the following two processes yield the same graph:

1. Replacing x by x/a and y by y/b;
2. Applying the scale-change transformation $S(x,y) = (ax, by)$ to the graph of the original relation; and if $a = b$, then the two graphs are similar in the strict geometric sense. [The proof is the multiplicative analogue to the proof of the Graph Translation Theorem.]

Corollary 1 The graph of $y = A\sin x$ has amplitude A; the graph of $y = \sin(Bx)$ has period $2\pi/B$.

Corollary 2 The graphs of $y = x^2$ and $y = ax^2$ are similar; thus, all parabolas are similar. [Proof: Apply the dilation with center (0,0) and magnitude $1/a$ to the graph of $y = x^2$. The image has equation $ay = (ax)^2$, equivalent to $y = ax^2$. More generally, define a parabola synthetically as the set of points that are equidistant from a given point (the focus) and a given line (the directrix). Let P and Q be any two parabolas in a plane. Then there exists a similarity transformation S that maps the focus and directrix of P onto the focus and directrix of Q, with the magnitude of S being the ratio of the distances between the foci and directrices of the two parabolas. Since each parabola is determined by its focus and directrix, the parabolas are similar.]

Corollary 3 More generally, two conics are similar if and only if they have the same eccentricity. [Proof: Define a conic section as the set of points whose distance from a given point F (a focus) is k (its eccentricity) times its distance from a given line d (its directrix). Then, since a similarity transformation applied to the conic preserves k, then by the same argument as for the parabola, two conics with the same k can be mapped onto each other by a similarity transformation, so they are similar.]

Corollary 4 The graphs of $y = b^x$ and $y = c^x$ ($bc \neq 0$, $b \neq 1$, $c \neq 1$) are similar; that is, the graphs of all exponential functions are similar.

Group 4: *Elegant proofs using rotations*

1. The product of slopes of non-horizontal/non-vertical perpendicular lines is -1. [Proof: We use the fact that the composite of two reflections over intersecting lines is a rotation whose center is the intersection of the lines and whose magnitude is twice the measure of the angle between the lines measured from the first reflecting line to the second. Thus the composite of reflections over the x-axis and the line $y = x$ is a counterclockwise rotation of 90°, and so the image of (x, y) under this rotation is $(-y, x)$. Let a line L contain the origin and the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_2 \neq x_1$. The slope of L is $(y_2 - y_1)/(x_2 - x_1)$, and under a rotation of 90°, the images $P' = (-y_1, x_1)$ and

$Q' = (-y_2, x_2)$, so the rotation image L′ of L has slope $(x_2 - x_1)/(-y_2 + y_1)$, and the products of the slopes of L and L′ is -1. Since translating a line does not affect its slope, we have shown that the product of slopes of two oblique lines is -1 whether or not they contain the origin. Another way to think about this is that a 90° counterclockwise rotation switches coordinates and changes the first coordinate to its opposite. (A 90° clockwise rotation likewise switches coordinate but changes the second coordinate to its opposite.) In the slope formula, these switches interchange numerator and denominator and multiply one of them by -1. The product of the slopes is then -1.]

2. Formulas for $\cos(\Theta + \Phi)$ and $\sin(\Theta + \Phi)$. [Proof: Define cos x and sin x as the first and second coordinates of R_x (1,0), the rotation of magnitude x. Then R_x (0,1) = (−sin x, cos x). By the Matrix Basis Theorem, a 2×2 matrix for R_Θ, is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Multiply the matrices for R_Θ and R_Φ to obtain the matrix for $R_{\Theta+\Phi}$, and notice that the elements of this matrix give (twice each!) the sum formulas for $\cos(\Theta + \Phi)$ and $\sin(\Theta + \Phi)$.]

I expect Barker and Howe to look at the applications of transformations on the coordinate plane differently than I have. There are so many ways to look at the same mathematics. While it is natural that mathematicians and mathematics educators would see things differently, it is always nice to have people with different backgrounds look at the same subject. We enrich each other by our discussions of how to approach mathematics as long as we do not become so enamored with our favorite approach that we cannot tolerate any other.

Thanks to Roger for his many contributions to mathematics and mathematics education, thanks to the organizers for inviting me, and thanks to you for listening.

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Chapter 13
Curricular Coherence in Mathematics

Al Cuoco and William McCallum

Abstract Building on the work of Schmidt et al., we propose a definition of curricular coherence for K–12 mathematics that encompasses both the arrangement of topics, which we call coherence of content, and the habits of mind the curriculum fosters in students, which we call coherence of practice. We give examples to illustrate each.

13.1 Introduction

> Coherence: the quality or state of cohering: such as
>
> (a) systematic or logical connection or consistency
> (b) integration of diverse elements, relationships, or values
>
> —Merriam-Webster.com. Accessed March 4, 2017.

What does it mean for a curriculum to be coherent? Schmidt, Wang, and McKnight (2005) offer a definition of coherence applied to mathematics content standards in terms of the logical progression of and deep structures in the discipline (see pp. 527–529 for a discussion of other notions of curricular coherence). Our goal in this paper is to elaborate this definition in two directions. First, we distinguish between standards and curriculum and flesh out the definition of content coherence for curriculum. Second, we pursue the idea of "integration of values" to propose a second aspect of curricular coherence: a coherence of practice that guides the way students do mathematics, following a consistent set of principles.

Dedicated to Roger Howe on the occasion of his 70th birthday.

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There are many definitions of curriculum and many qualifiers for different types of curriculum (intended, enacted, achieved, hidden). In this paper, we are interested in definitions that emphasize the temporal, sequential nature of curriculum, curriculum as a sequence of learning experiences rather than curriculum as a set of knowledge and skills. Standards set expectations for the knowledge and skills; curriculum is how we help students meet those expectations.

While standards might remain fixed—a mountain we aim to help our students climb—different curricula designed to achieve those standards might make different choices about how to accommodate variation in the abilities and preparation of students. Some curricula might allow students to move at different rates or work in different ways, while others might insist that all students share the same experience. Verbal wars have raged about such decisions, but whatever the choices made, they are choices about curriculum, not standards; they are how you get there, not where you are heading.

For example, consider the fact $9+6 = 15$. The Common Core State Standards in Mathematics (CCSSM) (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010) set the expectation that students know this fact from memory. How are students to achieve this goal? One might adopt a traditional method and have student chant the math facts in chorus, repeating day after day, until they are fixed in memory. One might adopt a progressive method which has students experimenting with cubes or rods, putting them together and counting, slowly acquiring the addition facts through repeated exposure to experiences that illustrate them. Or one might adopt an extensible method. If you understand $9 + 6 = 15$ as a result of making a ten, as in $9 + 6 = 9 + (1 + 5) = (9 + 1) + 5 = 10 + 5 = 15$, then not only do you have $9 + 6 = 15$, but you have also opened the door to $9 + 7 = 16$, $9 + 8 = 17$, $9 + 16 = 25$, $29 + 6 = 35$, and so on. No matter which path is taken, adherents of different methods can embrace the standard itself. Indeed, having common standards allows one to make scientific comparisons of different methods.

While standards by themselves are nothing without a viable way of achieving them, standards can help or hinder the writing of coherent curriculum. In that regard, CCSSM made some important shifts from previous state standards, which were often simple bulleted lists of performance objectives, all of equal importance. This made it easy to write assessments; just write ten questions for each standard.

But you wouldn't describe the journey up a mountainside in steps of equal size. If you were thinking about how to get your students up the mountain, you would make sense of things: identify key landmarks and stretches of trail to single out—a long path through the woods, a steep climb up a ridge.

By the same token, mathematics has its landscape. CCSSM pays attention to this landscape by laying out pathways, or progressions, that span across grade levels and between topics, so that a third grade teacher understands why she is teaching a particular topic, because it will help students with some other topic in the next grade and build on what they already know. The standards were built around coherent progressions, developed in collaboration with teachers, education researchers, and mathematicians. As a result, they are coherent in the sense of Schmidt and Houang (2012).

For this reason, we will base the discussion in this paper on CCSSM. However, the same points could be made with respect to any coherent set of standards.

13.2 Coherence of Content

By the *content* of a curriculum, we mean the collection of mathematical concepts and skills it aims to teach. That content can be arranged coherently or not. Schmidt et al. (2005) define a coherent arrangement as one that is:

> articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. That is, what and how students are taught should reflect not only the topics that fall within a certain academic discipline, but also the key ideas that determine how knowledge is organized and generated within that discipline. This implies that to be coherent, a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. These deeper structures then serve as a means for connecting the particulars (such as an understanding of the rational number system and its properties).

This definition suggests three specific ways in which a coherent arrangement of topics might be achieved: through logical sequencing, through evolution from particulars to deep structures, and through using deep structures to make connections.

13.2.1 Logical Sequencing

The first property of a coherent curriculum is that it makes clear the logical sequence of mathematical concepts.

Consider, for example, the concepts of similarity and congruence. It is quite common in school curricula for similarity to be introduced before congruence. This comes out of an informal notion of similarity as meaning "same shape" and congruence as meaning "same shape and same size." However, the fact that the informal phrase for similarity is a part of the informal phrase for congruence is deceptive about the mathematical precedence of the concepts. For what does it mean for two shapes to be the same shape (that is to be similar)? It means that you can scale one of them so that the resulting shape is both the same size and the same shape as the other (that is congruent). Thus, the concept of similarity depends on the concept of congruence, not the other way around. This suggests that the latter should be introduced first.

This is not to say you can never teach topics out of order; after all, it is a common narrative device to start a story at the end and then go back to the beginning, and it is reasonable to suppose that a corresponding pedagogical device might be useful in certain situations. But the curriculum should be designed so that the learner is made aware of the prolepsis.

Making clear the logical dependence of mathematical concepts does not mean that the entire curriculum should be built up from axiomatic foundations. It is a common criticism of the New Math, deserved or not, that it tried to do this. Wu (1996) has proposed the weaker notion of local axiomatics as an organizing principle:

> before the proof of a theorem, make clear what statements are assumed to be true and proceed to show how to use them in the proof. This shows students how to demonstrate the truth of a statement on the basis of explicit hypotheses. A reasonable mathematics education should aim for at least this much.

For example, in geometry, CCSSM recommends defining congruence and similarity in terms of rigid transformations and dilations and taking for granted their basic properties (that rigid transformations preserve distance, angle, and parallelism and that dilations preserve all these except distance). From this basis, one can prove the basic criteria for congruence and similarity of triangles and get quickly to interesting theorems, rather than making the long march from the Euclidean axioms.

Although the examples we give here are from secondary school, the principle applies at elementary grades as well. Logical sequence is established by reasoning and proof, and these should be present at all grade levels, in grade-appropriate forms. (See Ball and Bass (2003) for a discussion of reasoning and proof in a third grade classroom.)

13.2.2 Evolution from Particulars to Deep Structures

The principle of logical sequencing can determine the ordering of a set of topics. Since time is one-dimensional, and curriculum occurs over time, some principle for ordering is necessary. However, mathematics is not a linearly ordered set of topics; it is better viewed as a network. A deep structure is, roughly speaking, a node in that network with many connections. Of course, this is not a precise definition; the organization of the subject into a network is to a certain extent a matter of judgment and preference, although some connections are dictated by the principle of logical sequencing. However, this will serve for a start in describing the principle of evolution from particulars to deep structures.

We focus on two ways in which the process of evolution occurs: *extension* and *encapsulation*. Extension is a process by which a particular principle is repeatedly applied to ever-broader systems, thus revealing its nature as a deep structure. Encapsulation is a process by which a related array of concepts and skills becomes encapsulated into a single compound concept or skill.

Extension is exemplified in the way that arithmetic with whole numbers is extended to fractions, integers, and rational numbers through a program of preserving the properties of operations. The fact that $(-3) \times (-5)$ is 15 is a definition, rather than a theorem—it has to be that way if we want arithmetic with integers to obey the distributive property. The properties of operations start from observation of particular instances and evolve into powerful deeper structures undergirding the number system.

In a similar way, the meaning of rational exponents is determined by a desire to extend the properties of exponents for whole number expressions.

Encapsulating processes and viewing them as single objects is a theme that runs throughout the history of mathematics—the methods of one generation often become the objects of study for the next. A coherent curriculum acknowledges this phenomenon by introducing certain ideas as methods and then gradually providing experiences that allow students to encapsulate these methods and work with them as elements of a new system.

An example is the development of fractions. CCSSM in Grade 3 has a standard:

> Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

Later students are expected to place fractions on the number line and reason about equivalent fractions using this representation. A fraction as a number on the number line encapsulates many prior ideas and activities: dividing a physical object into halves or thirds; recognizing a geometric figure as a fraction of a larger figure representing the whole; moving from area representations to linear measurement representations; understanding the number line as marked off in unit lengths; subdividing those lengths into n equal parts and thinking of those parts as a new sort of unit, an nth, and measuring out distances in those new units; and correlating all these activities to the numerator and the denominator of the fraction.

This is a powerful encapsulation and therefore one that needs to be approached gradually, from concrete models (e.g., sharing brownies) to area models (subdividing geometric figures) to tape diagrams (which are transitional between area models and linear models) to number lines as abstractions of tape diagrams.

Encapsulation builds coherence by tying what where previously disparate ideas and actions into a tightly connected structured bundle which becomes viewed as an object in its own right.

An important type of encapsulation is the evolution of representations. In advanced mathematics, a representation is a mapping from an abstract algebraic system to one that is more concrete, in a way that preserves operations and makes them more amenable to calculation. This is similar to representations in school mathematics: they are often a way of making a concept more concrete, and they should preserve structure or information. For example, the representation of whole numbers in base ten makes it possible to grasp both the magnitude and the details about them in a very compact form and makes it possible to perform operations on them. (For an account of the extent and power of this encapsulation, see Howe & Epp, 2008.) The representation of rational numbers on the number line makes

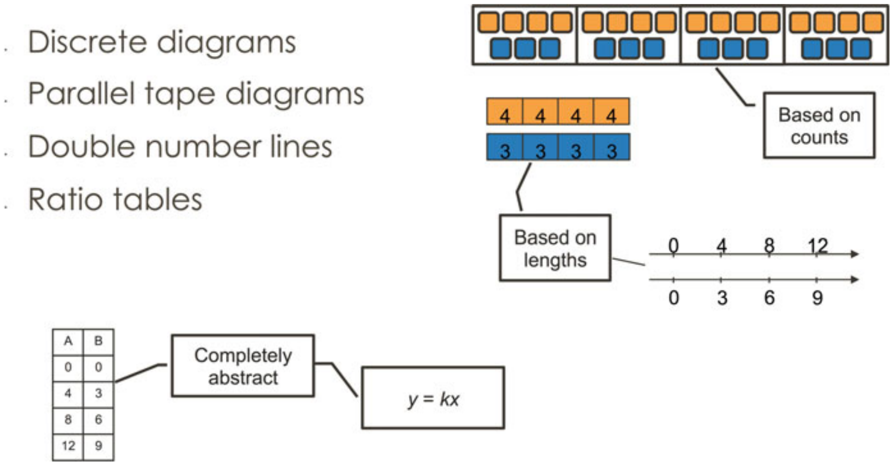

Fig. 13.1 Evolution of representations for proportional relationships

it possible to relate operations on them to concatenation or scaling of lengths. The representation of geometric figures and transformations using coordinates makes it possible to prove theorems by algebraic calculations.

Mature representations in school mathematics are a form of encapsulation and should be developed through a sequence of representations that, over time, reveal the structural features of the representation and the ways in which they preserve information about the object being represented. In early grades, students might start with pictorial representations; but even then the picture should be more than a picture, it should carry information about the situation. Over time, such pictures evolve into more abstract diagrammatic representations. Figure 13.1 shows such an evolution for representations of proportional relationships in middle school.

13.2.3 Using Deep Structures to Make Connections

A difficult question in designing a curriculum is to decide which topics go together. The logical and evolutionary considerations described above help, in that they provide guidance on the ordering of topics. But that still leaves many decisions to be made. Our goal in this final section about coherence of content is to show some examples of how deep structures can guide these decisions.

CCSSM in sixth grade has the following standard about percents in the Ratio and Proportional Reasoning domain:

> 6.RP.A.3c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

One approach to implementing this standard in a curriculum would be to have a section on percents that covers everything in this standard. But there is another possibility which attends the difference between the two parts of this sentence before and after the semicolon. The first part introduces the concept of percent. The second half involves solving problems that are tantamount to solving the equation $px = q$, where p and q are constants. This is related to a standard in the Expressions and Equations domain:

> 6.EE.B.7. Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all non-negative rational numbers.

Thus, another possibility might be to split this standard into two places in the curriculum, with the introduction to percents occurring as a type of rate, in the section where ratios and rates are studied, and percent problems occurring in the section where solving equations is studied. If percents are regarded as a deep structure, one might choose the first arrangement; if rates and equations are regarded as deep structures, then one might choose the second.

Another example of using a deep structure to make connections is the profound connection between geometry and algebra. In this case, the deep structure is the use of coordinates to represent geometric objects. For example, it is common in algebra classes to use quadratic functions to show that, among all rectangles of a given perimeter, the square maximizes area. Here, one uses algebra to model the area as a function of a side length, transforms the algebra to reveal the maximum value of the function, and then translates back to the geometric context. In this approach, algebra informs geometry.

Another method shows how geometry can inform algebra; it reasons directly about area. Imagine a 12 by 16 rectangle. Experiments with geometry software suggest that a square of side 14 maximizes area for this perimeter. If this is so, it should be possible to dissect the rectangle and fit the pieces into the square with something left over. One such dissection is shown in Fig. 13.2. Trying several other rectangles of perimeter 56, a regularity emerges. Expressing this regularity in

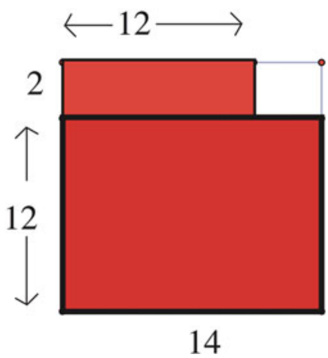

Fig. 13.2 Dissecting a 12×16 rectangle into a 14×14 square

precise language leads to an algebraic identity that captures the dissection. Using an $a \times b$ rectangle, one has

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab \tag{13.1}$$

This identity, inspired by geometric reasoning, can, of course, be verified in an algebra course. But its roots in geometry give it some extra meaning. And, it can be used to show how far off the rectangle is from the square.

There's another connection: Eq. 13.1 can be used to establish a strong algebraic-geometric mean inequality:

$$\frac{a+b}{2} \geq \sqrt{ab}, \text{ with equality } \Leftrightarrow a = b \tag{13.2}$$

Rather than separating the parts of this connection into two chapters or lessons, a coherent curriculum could use *one* story to develop both the necessary algebra and geometry, making it explicit that the main point is the connectivity of the ideas.

13.3 Coherence of Practice

We value coherence of content because we believe a coherently arranged curriculum makes it possible for a student to see the subject as a whole, to understand the logical connections and deep structures, and to use that understanding for more efficient problem-solving and better retention of knowledge and procedures. But making it possible does not make it probable. Therefore, we propose another aspect of coherence, coherence of practice. The way students do mathematics, their mathematical practice, may have an effect on their ability to take advantage of a coherent curriculum. CCSSM, influenced by the work of Cuoco, Goldenberg, and Mark (1996) and Harel (1998), attempts to describe the characteristic features of the practice of mathematics in the eight Standards for Mathematical Practice. Although these are often referred to as "the practices," they are better viewed as eight angles on a single complex construct, mathematical practice. We propose that certain aspects of that construct can promote a coherent understanding of mathematics. Here we focus on two aspects, using structure and abstraction.

13.3.1 Using Structure

Standard for Mathematical Practice no. 7 (MP7) in CCSSM is "Look for and make use of structure." In a sense, all of mathematics is about finding, building, and using structure. So a description of the many faces of how structure is central to mathematics would fill volumes. Here we discuss three faces of structure: structure in mathematical objects (expressions and geometric figures), mathematical systems, and systems for notation.

Structure in arithmetic and algebraic expressions reveals what might be called "hidden meaning." For example, writing $x^2 - 6x - 7$ as $(x - 3)^2 - 16$ reveals that, for real values of x, the expression assumes values greater than or equal to -16 (and it assumes that value only when $x = 3$). Writing it as $(x - 7)(x + 1)$ highlights the values of x that make the expression 0.

Treating pieces of expressions as a single "chunk" can simplify calculations; seeing that

$$4x^2 - 8x + 3$$

can be written as

$$(2x)^2 - 4(2x) + 3$$

helps one obtain the factorization from the (easier) factorization of

$$z^2 - 4z + 3.$$

This example can be generalized to encompass all polynomial expressions, providing students with a general purpose tool that can be used to transform a general polynomial into one with leading coefficient 1. It amounts to a change of variable in order to hide complexity, a practice that is useful all over mathematics.

Hidden meaning in geometric figures often involves the creation of auxiliary lines not originally part of a given figure. Two classic examples are the construction of a line through a vertex of a triangle parallel to the opposite side as a way to see that the angle measures of a triangle add to 180° and the introduction of a symmetry line in an isosceles triangle to see that the base angles are congruent. Another kind of hidden structure makes use of the invariance of area when it is calculated in more than one way—finding the length of the altitude to the hypotenuse of a right triangle, given the lengths of its legs, for example.

On a larger scale than individual expressions or figures, students throughout their K–12 career study systems that have underlying structural similarities. In elementary and middle school, students study properties of integers; they perform arithmetic with integers and rational numbers and their properties as a system, including the properties of operations on them. The same program takes place in elementary algebra, this time with polynomials in one variable (usually with rational coefficients). Using the same vocabulary and pointing out the structural similarities between $\mathbb{Z}$ and $\mathbb{Q}[x]$ can bring coherence between two main algebraic structures in school mathematics.

A final example of using structure, or not, is in the view that students form of the base ten notational system. The compactness and regularity of this system make it useful for efficient computation and estimation. But in that compactness, there is also the danger of superficial, and therefore fragile, grasp of procedures. The Number and Operations in Base Ten domain in CCSSM lays out a progression designed to help students learn to see the decimal expansion of a rational number as, in advanced language, a linear combination of powers of 10 with coefficients taken from integers between 0 and 9 helps. Similarly, viewing a polynomial in x as a linear combination of powers of x can lead to an understanding of polynomial algebra as a system in its own right. Writing $3x^2 - 7x + 5$ "in base $(x - 2)$" as

$$3(x - 2)^2 + 5(x - 2) + 3$$

reveals another kind of hidden meaning in the expression.

Emphasis on structure is a distinguishing feature of how topics and methods are developed in a coherent curriculum. The following quote from Conference Board of the Mathematical Sciences (2012) concerns high school mathematics but applies more generally to the whole of K–12 mathematics:

> "the mathematics of high school" does not mean simply the syllabus of high school mathematics, the list of topics in a typical high school text. Rather it is the structure of mathematical ideas from which that syllabus is derived.

13.3.2 Abstraction

A theme that runs throughout a coherent curriculum is a cross-grade emphasis that helps students develop and use the many faces of abstraction. One of the most important uses of abstraction is captured in the CCSSM Standard for Mathematical Practice no. 8 (MP8), "Look for and express regularity in repeated reasoning." It asks students to abstract a process from several instances of that process in a way that doesn't refer to the inputs to any particular instance. Describing that process in precise algebraic language allows one to create general algorithms, equations, expressions, and functions. This practice can bring coherence to many seemingly different areas of the curriculum that often cause students difficulty.

The description of MP8 in National Governors Association Center for Best Practices and Council of Chief State School Officers (2010) gives the following example:

> By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $\frac{y-2}{x-1} = 3$.

Helping students develop the habit of testing several numerical points to see if they are on the line and then looking for and expressing the "rhythm" in their calculations gives them a way to find the equation of a line between two points without leaning on formulas (e.g., "point slope form"), and, more importantly, it gives them a general purpose tool for finding Cartesian equations of geometric objects, given some defining geometric conditions.

As another example, consider the task of building an equation. Teachers know that building is much harder for students than checking. The same practice of abstracting from numerical examples is useful here, too. For example, consider the stylized story problem:

> Emilio drives from Tucson to Phoenix at an an average speed of 60MPH and returns at an average speed of 50MPH. If the total time on the road is 4.4 hours, how far is Tucson from Phoenix?

The practice of abstracting regularity from repeated actions can be used to build an equation whose solution is the answer to the problem: One takes several guesses (for the distance) and checks them, focusing on the steps that are common to each of the checks. The goal isn't to stumble on (or approximate) an answer by "guess and check"; the goal is to come up with a general "guess checker" expressed as an algebraic equation:

$$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} = 4.4$$

These two examples seem quite different, but coherence comes from the fact that exactly the same mathematical practice is used to find an algebraic equation whose solution solves the problem.

13.4 How Do We Achieve Curricular Coherence?

It is fitting to conclude a paper in honor of Roger Howe with a call to action to the community of mathematician educators, to use Hyman Bass's term (Bass, 1977). Hung-Hsi Wu has written eloquently about that strange subject called Textbook School Mathematics: an arcane collection of tricks, topics, and mindless mnemonics (Wu, 2015). There are many reasons for its existence, some having nothing to do with curriculum. But mathematicians can bring fresh air to this subject. They can communicate a sense of what the subject itself is all about, a sense of excitement and power and a coherent view that makes it make sense. We call on mathematicians to find opportunities for partnership with educators and teachers and ways in which they can contribute to building a curriculum that is mathematically coherent, works in the classroom, and inspires the teaching profession to do its best and to help build a discerning professional community that owns the curriculum and can use it skillfully in the classroom and that has a sense of the craft and knowledge not held by a guild of experts but shared broadly in the profession.

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Part IV
Supporting and Engaging Mathematicians in K-12 Education

Chapter 14
Attracting and Supporting Mathematicians for the Mathematical Education of Teachers

Amy Cohen

Abstract Education is a complex process involving teachers and learners within surrounding institutional and social environments over which neither group has extensive control. However, complexity is not an excuse for despair or inactivity. We will not make progress unless each of us contributes what we can and does so without waiting for someone else to make some other part of the problem go away first. Mathematicians choose to work on the mathematical education of teachers for many reasons. While money, promotion, and reputation are important to some mathematicians at some times in their careers, this paper concentrates on other rewards and other forms of support. In particular, there is an attractive intellectual challenge involved in finding accessible and engaging ways to help teachers understand the mathematics itself. While much of the recommended intellectual and pedagogical support is relevant to all who teach, it is particularly relevant for mathematicians working with teachers who have been worn down by a litany of disrespect for K-12 education.

14.1 Formative Experiences of the Author

The author was an undergraduate at Harvard in the early 1960s and a graduate student at the University of California at Berkeley during the late 1960s. As a math major, the primary attractions and satisfactions of mathematics arose from solving problems and constructing efficient and sufficient justifications for those solutions. As a graduate student (supported variously by research assistantships, fellowships, and teaching assistantships), I learned that social and communication skills were important for most (admittedly, not for all) successful career paths in mathematics.

My most important preparation for teaching came in 1970, just as I was completing my PhD, during training to become a Math Specialist in Project SEED (Davis, 1960; Heaton & Lewis, 2011; Phillips & Ebrahimi, 1993; Wilson, 2003).

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We learned that a teacher must listen to what the students are saying! An unexpected answer to a question need not be "wrong." The student may have heard a question other than the one the teacher just asked. The teacher may have asked a different question than the one intended. The student may finally be answering a question asked 5 min ago after pondering over it in the meantime and getting the nerve to speak up. The student may indeed have made an error. The teacher needs to be agile enough to turn the student's comment into a productive occasion for teaching and a safe opportunity for learning.

Indeed, working mathematicians know that many plausible attacks on a problem turn out to be wrong without proving that the researcher is irredeemably stupid. John Tate once said something like the following to my honors calculus class: "I spend 50% of my time working hard and not getting anywhere; but maybe 10% of my time making some progress; and 40% of my time wondering how I can be so unproductive 90% of the time."[1] John Tate, a very distinguished mathematician, was making the point that even smart people struggle and make mistakes and waste time worrying about it. Learning, just like research, can involve struggle (Mizer, Howe, & Blosser, 1990; National Research Council, 2001). Professional mathematicians early in their education and career probably associated with people who were also mathematically talented and engaged. We sometimes forget that not all bright people like math enough to learn it – and not all decent human beings turn out to be mathematicians. Some intellectual humility is essential when working with students who may become teachers and with teachers who are trying to deepen their mathematical understanding.

In the late 1960s, the Berkeley School District was trying to overcome the effects of residential segregation by busing kids from various neighborhoods to integrated school buildings. Project SEED, based on work of Robert Davis and others in the Madison Project, engaged kids in guided mathematical discovery on topics that none had seen before – so no one would have an "advantage" (Davis, 1960; Phillips & Ebrahimi, 1993; Wilson, 2003). My work in two mixed second and third grade classrooms (supervised by certified teachers) was the most challenging teaching I have ever done! When it went well, it was very satisfying. When it didn't, I encountered many of the issues we still discuss, e.g., engagement, motivation, precision of language, and apparent preferences (by many children as well as by their parents) for rote learning over independent thinking.

My greatest failure came in trying to retain the participation of Eddie. He had been the brightest kid in either class, also the most cooperative and the most eager to offer ideas and suggest reasons. One day, he hit another kid and was immediately suspended. On his return to class, he never misbehaved again, but he never spoke up again either. We later learned that Eddie needed to be home with his preschool sibling while his single mother went to Mississippi to care for her dying mother. Eddie knew that if he hit a kid, he'd be suspended and could be at home. The family fell into the hands of Youth and Family Services. This is tough stuff for a new PhD to handle, especially one who grew up in a neighborhood with "intact" families who had lots of resources and knew how to use them. In my 44 years in

[1] This quotation is based on my memory of a comment made in a calculus course in AY1960–1961

university math departments since my time in Project SEED, I have had to learn much from experience. I am still trying to make up for my inability to reengage Eddie in learning math.

Between graduate school and promotion to full professor, I concentrated on navigating among the sometimes conflicting demands of research and teaching. Since that promotion,

I have served in an unusually large variety of educational roles at Rutgers:

- Director of the undergraduate program in mathematics
- Dean of a unit serving adults returning to college to start or complete BA or BS degrees
- Member of a team of colleagues introducing active learning activities into a number of undergraduate math courses at both freshman and junior levels
- Co-PI on a VIGRE (Vertical Integration in Graduate Research and Education) grant addressing (among other things) professional communication skills of graduate students and postdocs
- Most recently, PI on two Math Science Partnership (MSP) grants
  - One from NSF to work with mid-career math teachers in grades 6–9
  - One from the Department of Education for generalist teachers in grades 3–6
- Coordinator of my department's Math Outreach to Teachers program
- Organizer of Math Teacher Circles (Taton, 2015)

14.2 Major Lessons Learned in Math Science Partnerships

Since most of my experience in working with mid-career teachers has come from the two MSP grants mentioned above, it would help the reader to know how these were organized. The bulk of the continuing mathematical education was presented in 2-week summer institutes carrying graduate credit. The bulk of the professional development addressing mathematical pedagogy was presented during the school year. A typical day in a summer institute began with reports from two groups of teachers, each presenting and discussing several approaches to an assigned workshop problem from the previous day. There followed exposition and exploration of the new day's topic, a classroom-connections session, and a wrap-up period for teachers to work on their presentations and/or their "homework" for the next day. During the workshop period, teachers worked in groups on a set of mathematically rich problems with coaching from the instructional staff. The goal was to present engaging problems closely aligned with material the teachers actually taught, made slightly challenging in order to require the input of the full group but not so challenging as to be daunting. The coaches suggested questions the group could discuss and sometimes hints of scaffolding – but never just showed the teachers "what to do next." Teachers were generally amazed that they naturally found different approaches to a problem and that they learned as much from trying to understand their colleagues' approaches as they did from trying to motivate and justify their own approaches. It was often hard to bring teachers back from engagement in the workshop to discuss classroom connections explicitly. On the other hand, teachers often discussed their classroom experiences spontaneously during workshop sessions.

It is intellectually challenging for senior mathematicians to design workshop problems that are accessible and engaging for teachers and then to coach group work on these problems in ways that lead to mathematically honest learning. But good mathematicians appreciate a good challenge. We may not always realize what needs motivation or explanation. Indeed, we may not know how the terms "motivation" and "explanation" are understood. Teachers in K-8 often think of motivation as a way to obtain compliance "in the moment." Mathematics educators at institutions of higher education (IHEs) often think of motivation as a way to encourage commitment to a longer-term project like "preparing for further education and employment." When mathematicians describe their motivation for formulating a definition or theorem in some particular way, they often talk about making choices to facilitate understanding or proof (Vatuk, 2011). Teachers may see an "explanation" as a clear listing of steps to be followed; mathematicians may see "explanation" as a synonym for "proof" or "justification." Mathematicians can benefit from working with colleagues from mathematics education to obtain guidance on what is accessible, what is engaging, and what is intelligible to various audiences. Attention to what is engaging and accessible is valuable for teaching post-secondary students as well as for working with teachers.

University mathematicians are often accustomed to giving uninterrupted lectures lasting 50–80 min. The teachers in my MSP projects become impatient if a piece of exposition lasts longer than 15–20 min. They then want to explore the idea actively alone or with a partner before returning to listening. This gives the teachers a way to engage actively with the idea, to test their understanding, to ask questions, and to articulate concerns. This also gives the instructional team a chance to circulate, listen to discussions, and provide nearly instant feedback to the lead mathematician on what might need clarification or extension.

Mathematicians quickly adapted to this style once they understood the dynamic involved and provided "exploratory exercises" to break up their presentations. The participating teachers were amazed that their daily feedback forms could elicit prompt accommodations to their suggestions. Mathematicians tend to see abstraction and generalization as making it possible for learners with varying interests to apply general results in many different fields. Teachers, like many learners in mathematics-intensive fields, do not see the applicability without explicit guidance for making the connections. Teachers are very happy to see workshop problem sets that address the mathematics that they and their students find problematic.

It is possible (and essential) to offer courses in which mathematicians and math teachers interact as mutually respectful professionals and learn from each other. This work can be joyful and satisfying – just as it can sometimes be a struggle for all concerned. It takes time to build a coherent cohort of teachers, faculty, and graduate students – but the effort can pay off in activities like Math Teacher Circles (MTCs) that sustain the connection, bring in new participants, and don't cost as much to run as more intensive projects like Math and Science Partnerships, (Taton, 2015). IHE (short for institution of higher education) participants in MTCs benefit by gaining ideas to improve courses, develop new courses, and modify classroom practice. Teacher participants found that they modified their classroom practice – in particular by eliciting more open answers to the question "why?" Universities and mathematics departments benefit from opportunities to publicize how research-active faculty members also engage in valued service to STEM education.

Finding funds and institutional support for outreach to teachers is hard. Federal grants may not be available to sustain proven initiatives. Corporate and private foundations may not solicit proposals for these activities. "Political issues" are inappropriate for this paper.

14.3 Attracting Mathematicians to the Mathematical Education of Teachers

Here are the primary reasons given by colleagues in the mathematical education of teachers when asked what attracted them to begin or continue with this work. (At the risk of being tedious, I will spell out the phrase "mathematical education of teachers" to avoid confusion with publications with titles commonly abbreviated MET I or II.)

1. Meeting the intellectual challenge of constructing engaging and accessible problems from which teachers could acquire mathematical ideas that were correct and teachable in their classrooms.
2. Concern about what and how mathematics was being taught to their children or grandchildren in school. Concern that some teachers felt pressured to value speed, tricks that may be generally useful but not always valid, and rote memory over mathematical reasoning. Many mathematicians associate the word "rote" when used to modify "learning" or "teaching" as indicating an undue preference for memorization through repetition over development of understanding. Nonetheless, mathematicians realize that learners need to develop both automaticity and understanding in order to use mathematics in employment and in civic life.
3. Desire to learn more about working with teachers in order to make career plans for what types of jobs to seek after completing a PhD, a postdoc, or a short-term nontenure-track appointment.
4. Desire to learn more about methods of eliciting "active learning" in order to modify their own teaching practices.

5. Desire to learn more about teachers and teaching at K-8 or K-12 levels in order to design new courses for prospective teachers, enhance existing courses, and teach them.

Based on this admittedly small sample of views of senior faculty, junior faculty, and graduate students at a major public university, here are some suggestions about how to recruit mathematicians to participate in the mathematical educations of teachers.

Necessarily, one must either discover a pre-existing interest in the mathematical education of teachers or find ways to generate such interest. Discussion of issues of teaching and learning should be an acceptable form of conversation in a math department – this includes hallways and seminar rooms as well as private offices with the doors closed. I do not suggest that discussion of education should replace discussion about research, but it should not be taken as somehow unprofessional. The opinion leaders of a department can contribute to this atmosphere by encouraging discussion of courses, textbooks, teaching methods, and so on – even if they do not themselves focus primarily on the mathematical education of teachers.

TA training programs usually offer a basic introduction to teaching for graduate students. In departments where TAs are primarily assigned to lead discussion sessions, TA training may not cover all the issues of teaching at colleges and universities. For example, TA training may not discuss courses for prospective teachers. Furthermore, the connotation of "training" suggests that the trainees should simply do what they are told without necessarily understanding of the reasons for various policies and practices. Not all junior faculty members have had even rudimentary preparation for the full range of an academic teaching career and the special concerns of working with prospective teachers and mid-career teachers. Department leaders can urge inclusion of speakers on mathematics education at all levels in department colloquia and/or encourage a regular seminar on teaching mathematics.

Once one knows who is interested in improving the mathematical education of teachers, one needs to find out what their particular interests and skills are and what the opportunities are on campus and off campus. National, regional, and local meetings of the AMS and the MAA include sessions on K-12 education, undergraduate education of future teachers, and the continuing mathematical education of certified teachers in the public sector and teachers not needing certification in the private sector. There are specialty conferences and organizations focused on these topics as well. Explicit invitations to colleagues to engage in the improvement of teaching, whether at K-12 or at post-secondary level, should be made privately. Whoever makes such invitations should offer a small menu of reasonable and concrete suggestions of how to become active and should promise collegial support. One suggestion I once heard, namely, that someone interested in the mathematical education of teachers must change professions from mathematics to mathematics education research, seems to me quite unreasonable.

My colleagues were pleased to be asked to take part in my grant projects. They were surprised to benefit from discussion of mathematical materials and methods with other members of their instructional teams. They later recognized the irony of this surprise since they were accustomed to benefit from discussing ideas and methods with their research colleagues

Interest in teaching well is not incompatible with interest in research – but one cannot pursue both with full devotion at the same time throughout a career. Flexibility during careers is important. So are leadership and mentorship to encourage the right people to take an interest in the mathematical education of teachers at the right times in their professional lives. Faculty might prefer to teach a small class for future teachers than to teach a large lecture on calculus.

Don't demand unanimous consent in a department or faculty. Leon Henkin, formerly professor and chair of the department of mathematics at UC Berkeley, told me that change in academia is best accomplished by a "small cadre swimming in a sea of indifference."[2] He said that asking everyone to agree is a recipe for organized opposition. John Kelley, another Berkeley mathematics professor, said that one person trying to change a department was like a prisoner trying to escape from a steel cell by scratching with his fingernails. Both Henkin and Kelley did make changes; they worked assiduously with quiet tact.

Like many of us at this conference, I have always liked the security of a well-solved problem. We have not yet solved the problems of preservice, early-career, and mid-career mathematical education of teachers. These problems are not amenable to a single doctrinaire approach. Given a choice between "either/or" and "both/and," I recommend the latter.

Separating content and pedagogy is not, I believe, a productive path. The articulation of the mathematical practices in the Common Core and related work on "mathematical habits of mind" present challenges to those of us who teach in IHEs – as well as those who teach in K-12 (Cuoco, Goldenberg, & Mark, 1996; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Teachers must use these practices and habits explicitly so that students will appreciate their value and internalize them in order to benefit from them. As Jim Lewis has suggested, find ways to use the pedagogy courses to enhance mathematical understanding, and use math classes to model a variety of effective teaching methods. If possible, schedule a math course and a math pedagogy course in a pair of consecutive class periods so that they can be co-taught in a mutually supportive fashion ((Jim) Lewis, 2015; Heaton & Lewis, 2011).

Recognize the importance of good mathematical communication with students and the public, as well as with colleagues. Our profession will not prosper if many of us take the view that our natural audience contains only those who already know what we want to say. A VIGRE postdoc had gotten that impression in his graduate-student years. He said with frank surprise "You mean that people are supposed to understand a seminar talk?"[3] If we cannot make our own results intelligible and interesting even to ourselves, how can we expect the public to continue to fund our

[2]Recalled from a discussion at UC Berkeley in the 1970s

[3]Recalled from a seminar discussion at Rutgers in the early 2000s

search for results – and how can we expect the future students in our classes to make their teaching intelligible and engaging? Once upon a time, it was enough to teach those who could learn virtually without our help. Now the demand for STEM workforce at all levels means that we also need to educate also those who will only learn with our help.

14.4 Supporting Mathematicians in Work with Teachers

This section is long, because I believe it is important. Discussion of financial support for this work is discussed in the following section. Experienced mentors can provide important personal support. This section presents topics which can be very helpful.

14.4.1 General Remarks

Effective teaching requires agility in the classroom to provide multiple representations of the mathematics, multiple forms of motivation for the content, multiple strategies in problem-solving, multiple connections to realistic grade-appropriate "real-world" applications, and multiple ways of recognizing and addressing gaps in learning and understanding. It is an art to help students gain insight about how to select from these multiplicities.

Teachers also need to know the mathematics taught in earlier and following grades. They cannot be sure at what level they will be assigned to teach. They need to know what their students are bringing into their classes from earlier work. They need to know what they are preparing their students to do in the next grade. This kind of understanding of mathematics for teaching is hard to disentangle from the mathematical pedagogy by which teachers engage their students in the work of studying, learning, and applying the mathematics that is both grade-appropriate and developmentally appropriate for their students.

As mathematical preparation has become a prerequisite for many employment paths, mathematics teachers are increasingly asked to ensure that each student learns as much mathematics as possible in order to broaden options for "college and career." As women have won access to new employment opportunities, teaching is no longer their only option or even their best option. For these reasons, mathematicians working with teachers need to nurture mathematical talent and growth at whatever level they find it. A sensitive combination of tact and honesty is called for in cases in which individuals will not engage in learning broadly enough or deeply enough to become successful teachers.

The K-12 sector in the USA addresses a large and heterogeneous population of students with widely varying degrees of access to educational opportunities at home. Some have highly educated families who provide extensive advice and set high educational goals. Others come from families which may not speak

English at home, have very limited formal education, or have limited vocational aspirations. Teachers have had widely varying educational backgrounds in math and mathematical pedagogy.

Mathematicians interested in working with teachers should choose settings compatible with their preferences. Some may want to work with programs for the gifted and talented. Others may want to work in the schools their children attend. Yet others may feel called to work in districts with great challenges. In most settings, teachers usually want to learn material and methods that are directly relevant to their own teaching. "Elementary mathematics from an advanced point of view" may not be appreciated by teachers unless the instruction and discussion make explicit connections between that advanced point of view and classroom realities.

14.4.2 The Common Core State Standards in Mathematics (CCSS-M) (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and Related Documents

The CCSS-M articulate aspirations for what school children should be able to do and to understand. The proficiency standards in the CCSS-M mark benchmarks along learning trajectories for strands of content. Some readers infer incorrectly that instruction toward any particular proficiency for Grade N should only be provided only in Grade N. It is important to distinguish aspirational goals from curriculum development and assessment instruments (Partnership for Assessment of Readiness for College and Career (PARCC); Smarter Balanced Assessment Consortium)

The impetus for the CCSS-M document came from organizations of state governors and of state superintendents of schools (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These organizations mentioned several serious concerns including (1) that math curricula in the K-12 sector had become "a mile wide and an inch deep" (Li, 2007, and references therein) and (2) that children whose parents moved from place to place often saw some topics twice and others not at all. The CCSS-M phrase its standards not just in terms of computational fluency but also in terms of proficiency in explaining and justifying. Mathematicians working with preservice teachers and in-service teachers may need to wean undergraduate students and certified teachers alike from overreliance on rote memory and mnemonic tricks. For many mathematicians, the terms "rote teaching" and "rote learning" suggest overreliance on memorization based on frequent repetition with little or no explicit attention to understanding.

Mathematicians often learned school mathematics more quickly and much earlier than many other children. As a result, they may have inaccurate ideas about what is taught and how. The CCSS-M provide a good way for mathematicians to find out approximately what topics are taught when. Much has changed in the last three or four decades! It is not helpful to say anything that sounds like "As you have known since 3rd grade" to teachers who didn't learn it in 3rd grade, don't teach it in 3rd grade, and may have learned it by rote and without lasting results. Similarly, mathematicians' memories of HS or undergraduate courses that prepared them for college or doctoral work are not reliable evidence of what all HS teachers really know.

The standards of mathematical practice describe ways of thinking and working that facilitate the learning and application of mathematics both in school and in later life. Contrary to the views of some mathematicians and some professional development providers, the standards of practice cannot be sensibly separated from mathematical work. If teachers do not understand and employ these practices themselves, it is unlikely that children will appreciate their value and employ them for their own benefit. These practices are best understood by seeing examples of them in action. Thus, it is important that mathematicians use these practices and point them out explicitly so that teachers can "teach as they were taught." The primary mathematical practice is to make the math make sense. Since the concepts of "mathematical structure" and "repeated patterns of reasoning" often seem difficult for teachers to acquire, faculty should make a special effort to point out examples when they occur naturally

Mathematicians should also gain some familiarity with essays on "mathematical habits of mind" (Cuoco et al., 1996). One can have amusing discussions about whether a "mathematical practice" follows from a "habit of mind" or vice versa.

Educational psychologists tell us that it is easier to remember and apply information if it is familiar, makes sense, connects to other known information, and comes with a convincing basis for belief. The National Research Council report *Adding It Up* (National Research Council, 2001, page 115 et seq.) provides an extended discussion of strands of mathematical proficiency – no one of which alone suffices for learning and applying mathematics. These strands are (in alphabetical order) adaptive reasoning, conceptual understanding, procedural fluency, productive disposition, and strategic competence. Research indicates that capacity to learn and to apply math is enhanced if lessons alternate between a focus on concepts and practice with procedures (Rittle-Johnson & Koedinger, 2009).

14.4.3 Benefits of Broadly Based and Supportive Teams Especially for Summer Institutes

Focus groups conducted by the external evaluators of my MSPs revealed that the teachers were very grateful to be treated as professionals who had something to offer to the math faculty and others on their instructional teams. Teachers have become anxious and defensive as a result of a litany of criticism from many directions. Teachers reported that the friendly and collegial respect we offered distinguished our program from some others they had attended and helped to alleviate their anxiety.

Ideally, the team should be composed of mathematicians, mathematics teacher educators, mathematics education researchers with a strong background in mathematical content as well as pedagogy, professional development facilitators, and teachers who can serve as "peer mentors." The overlapping experience and expertise of such a team provides benefits when the team plans its instructional activities, participates in those activities, and debriefs afterward. Varying points of view, sensitivities, and insights support all team members.

Faculty who participated in MSPs at Rutgers reported on anonymous surveys that they have enjoyed informal conversation with participating teachers and with all teammates. They say that they have been able to learn more about K-12 education in NJ and to learn more about teaching in general. They were particularly struck by the value of constructive feedback in each of the daily debriefing sessions. They were also struck by their early discomfort at being observed daily by other members of their instructional teams. This last comment generates empathy into K-12 teachers' discomfort with frequent observations by district staff who often evaluate without offering supportive feedback.

14.4.4 Cognitive and Linguistic Issues in Communicating Math

Professional mathematicians and mathematics graduate students often find it difficult to communicate easily and effectively in groups of nonmathematicians. We can work more productively with teachers at various career stages (pre-certification, early-career, mid-career) if we become more aware of the so-called didactic obstacles that may impede communication.

Mistakes, misconceptions, and confusions can arise in many ways. Research in mathematics education can be enlightening about such sources of error. Subsections below mention some examples worthy of attention. For deeper discussion, see Alcock (2014) and Tall (2013).

Solomon Friedberg's book (Friedberg, 2001) is an excellent source of examples to spur conversation and understanding about teaching content as well as teaching learners. There are many other books on related subjects, but few by well-regarded mathematicians who have both served as department chair and been actively engaged in enhancing the teaching of mathematics by early-career mathematicians. The June/July 2015 issue of the *Notices of the American Mathematical Society* includes additional stimulating material on this subject (Bass, 2015; Denschler, Hauk, & Speer, 2015).

14.4.4.1 Quantifier Errors and Omissions

Children can be very imaginative, but also very literal-minded. Even adults tend to assume that statements are universal unless there is an explicit restriction to some domain of discourse. A child who is told by one teacher that "we cannot subtract 7 from 3" quite reasonably loses confidence in teachers' authority when a later teacher tells them that we can do just that. Many future high school teachers are accustomed to the convention that unquantified statements are universal. This leads to confusion. For example, the statement "matrix multiplication is not commutative" does not mean that a matrix product AxB is never equal to BxA. Precision is important, but too much precision can be tedious. It is an art to learn how much to say and when to say it.

14.4.4.2 Tendencies to Overgeneralize or Oversimplify

In many elementary classes and in entry-level university courses, we provide evidence in support of general statements rather than proofs. This may be developmentally appropriate, but teachers need to increase their mathematical sophistication during their education and career.

Multiplication of fractions is "easier" than the addition of fractions. Why should we not "just add the tops and add the bottoms?" Some of my colleagues wonder who would ever do that. I have had a certain amount of mischievous fun with them by asking what percentage they would enter on the cover of a two-part exam if the student scored 30/40 on the first part and 40/60 on the second part. Their usual quick answer is "70% since the student earned (30 + 40) points out of (40 + 60)."

After students have done many drill exercises on the distributive law, it is not surprising that they mistake the meaning of parentheses in function notation as an instance of the distributive law. They then believe in a "law of universal additivity" for functions, namely,

$$f\,(r+s) \text{ is always equal to } f(r)+f(s).$$

Calculator syntax often allows the omission of a final parenthesis. This may explain the following otherwise inexplicable computation:

$$[F\,(x+h)–F(x)]\,/\,[\,(x+h)–(x)] = [\,F(x+h–F\,(x]\,/\,[h] = h/h = 1$$

that I have seen not only in precalculus courses but also at AP calculus readings. The law of unintended consequences applies even to the doctrine of calculator benefits.

14.4.4.3 Misleading Diagrams: One Example Only

Some classroom materials use an analogy with pan balances to explain the steps in solving simple linear equations. To save vertical space, the diagram may show a seesaw with a one-pound weight and two crates on one side of the fulcrum and three one-pound weights and one crate on the other side. The directions may make two assumptions: first that each crate contains an object of the same unknown weight, say W pounds, and second that the seesaw is initially balanced. This situation is represented by the equation $1 + 2\,W = 3 + W$. Students draw the result of removing a one-pound weight from each side of the beam and a crate from each side. Then they subtract 1 and W from each side of the equation. Their picture is supposed to support the conclusion $W = 2$.

Does this make sense? Students are unlikely to believe that each crate has zero weight. Further, students who have played on a seesaw know that a lighter child farther from the fulcrum can balance a larger adult nearer the fulcrum on the other side. Distance from the fulcrum matters in the real world, but didn't matter in the materials described. *The Next Generation Science Standards* (NGSS Lead States, 2013) call on K-12 students to reason from careful observation of the physical world. If they do so, they will object to unrealistic modeling. Math teachers and mathematicians need to become aware of both science standards and math standards.

14.4.4.4 Technical Terms that Are Suggested by, but Not Coextensive with, Ordinary Usage

The notion of continuous function is closely related to the notion of a continuous motion. Some texts try to make formal definitions more accessible by saying "a function is continuous if its graph can be drawn without picking up one's pencil" or that "a function is continuous if its graph has no holes or gaps." The functions defined by $f(x) = 1/x$ and $g(x) = \sqrt{(x^2-4)}$ have graphs with big gaps but are continuous, in the sense of continuous on their domains.

Other so-called didactic obstacles arise. Works by David Tall (2013) and by Lara Alcock, for example, in (2014), provide a more extensive and deeper discussion than I have room for here.

14.4.5 The Variety of National Systems of Education

Mathematicians coming to the USA from other countries may find our system confusing. The USA sends a very large percentage of 18-year-olds to institutions of higher education. Elsewhere, a smaller portion of teenagers who graduate from high school go on to higher education. In some systems (e.g., in the UK), "college" may come between secondary education and university work, and first year university students are expected to have mastered calculus. In some systems, a School of Mathematics is responsible for only two tasks – producing publishable mathematics and producing publishing mathematicians. Colleagues who come from such systems may well be surprised to have future doctors, engineers, and economists in their undergraduate courses – to say nothing of future teachers.

14.4.6 Connecting Instructors of "Math for Teachers" with the Rest of a Math Department

Some departments rarely assign senior faculty to courses for future teachers (both K-8 and 9–12), relying instead on "mathematics educators" nontenure-track instructional staff. This "division of labor" makes it harder for regular faculty to appreciate the needs of prospective teachers, early-career teachers, and mid-career teachers. Fostering communication between those who teach math majors and those who teach teachers can lessen unconscious bias against teaching. I have heard occasional statements of the form "Students X and Y say they want to teach math in HS. They are too good for that. Let's redirect them to a doctoral program in math." It is ironic when these statements sometimes come from mathematicians who complain about the weakness of American education.

14.4.7 "Elegant Exposition" Versus "Effective Exposition" for Teaching Mathematics

The following, from Gian-Carlo Rota (1997, pp. 14–15), describes the lectures of Alonzo Church at Princeton University in the 1950s:

> His lectures are best described as polished diamonds. ... He would give as few examples as he could get away with. ... Not more than [four or five] examples were given in the entire [point-set topology] course. ... His proofs [as opposed to older proofs] were perfect but not enlightening. He did not want to admit ... that his proofs would be best appreciated if he gave the class some inkling of what they were intended to improve upon. ... Anyone who wanted to understand had to figure out later 'what he really meant' .... His conversation [with colleagues] was in stark contrast to his lectures. He would give out plenty of relevant and enlightening examples, and freely reveal the hidden motivation of the material.

This description presents a style still employed by some mathematicians – but not a style suitable for future teachers or mid-career teachers.

The following from Mark Kac (1992, pp. 7–18) indicates that current conflicts between "instructional rigor" and "student engagement" are not new.

> To me and many of my colleagues, mathematics is not just an austere, logical structure of forbidding purity, but also a vital, vibrant instrument for understanding the world ... . Complete axiomatization, someone has rightly said, is an obituary of an idea. ... There are worse things than being wrong – being dull and pedantic are surely among them.

Both quotations can stimulate vigorous discussions of teaching at all levels.

14.5 Rewarding Mathematicians for Work with Teachers

For a careful and extensive discussion of tangible and intangible "rewards," see the recommendations in the AMS report, *Toward Excellence* (Ewing, 1999). I have been very lucky with my department chairs, my deans, my provosts, and my university presidents. By and large, they have respected me and my work and have provided bits of funding to expand its reach. Indeed, they greet me pleasantly on campus. I wish more members of our professions enjoyed similar encouragement from broad-minded administrators. Here I will only recount some of the rhetorical questions I find myself asking in conversations about this topic.

How do we balance the educational role of an institution of higher education (IHE) with the research role? Can our profession value not only review papers and expository talks on relatively new research results but also value the scholarship of teaching and learning at K-12, undergraduate, and graduate levels? Why do we sometimes claim we can't give promotion credit for textbooks because of the royalties when we have no difficulty giving promotion credit for NSF summer support, competitive fellowships, and prizes? Why do we sometimes lump course development with service rather than with teaching? How can we best recognize the scholarly work of papers in mathematics education, expository papers, and review articles in annual reports, CVs, and the like? Some universities separate them out; some do not; others exclude them altogether.

Is money the only measure of our worth as mathematicians and as human beings? Why do we talk about "teaching load" but not about "research load?" Why do we reward research production with less teaching but sometimes sneer at senior colleagues who are reducing research production to attend to a wider variety of professorial duties: governance, teaching, course development, mentoring, editing, etc.?

Why do universities permit, and even support, travel by mathematicians during the academic year to give talks and attend conferences and thereby raise the research reputation of the university, but much less commonly encourage travel for conferences on education? Why is it okay to give the impression that research reputation is more important to a university than teaching quality?

14.6 Conclusions

Many wise people have discussed the topic of enhancing the effectiveness of education in mathematics. There is growing agreement. However, I doubt there will ever be total agreement. What progress we have made has been based, at least recently, on a willingness to observe the educational process carefully, to look for robust evidence where it may be possible to find it, and to compare fervent belief, common knowledge, and common sense with results of careful observation, trustworthy evidence, and careful reasoning. Roger Howe has contributed to this effort by listening, by reading, by questioning, and by contributing to reports on education as well as textbooks and mathematical research. I wish that we all could have more of the serenity, insight, and elegance that characterize his work.

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Chapter 15
The Contributions of Mathematics Faculty to K-12 Education: A Department Chair's Perspective

Solomon Friedberg

Abstract Post-secondary mathematics faculty members have an important role to play in the preparation of future mathematics teachers at all levels and in the support and professional development of in-service mathematics teachers. This work can be demanding and time-consuming and constitute a significant professional contribution. Some departments now recognize, support and reward this work, while others do not. This article offers a view of this landscape from the perspective of a department chair and provides some suggestions for conversations that could take place within the department, for conversations of practitioners with their chairs, and for conversations across the broader university. In particular, if this work is to be properly valued by departments and institutions, it must be evaluated in a thorough and sustained way.

15.1 Introduction

Mathematics departments in colleges and universities have many responsibilities in the broad realms of teaching, scholarship, and service. These responsibilities include subject matter delivery at a time when mathematics is of increasing importance in many disciplines and for many jobs, the development of new mathematical knowledge and its use to address complex problems, and service both within a given institution and in the wider realm of the mathematical sciences. These three categories are not mutually exclusive. For example, in 2013, more than 120

Dedicated to Professor Roger Howe on his 70th Birthday.

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mathematics institutes worldwide participated in the UNESCO initiative entitled Mathematics of Planet Earth, an initiative that provided opportunities for each of teaching, scholarship, and professional service and one that illustrates the importance of mathematics for the wider world.

The goal of this article is to place the work of mathematicians who are involved in K-12 mathematics education in the context of a mathematics department. Many reading this article will immediately identify such efforts as vital to the future economic interest of the United States (or indeed, any other country). These readers will recognize the unique and critical role that well-informed university mathematicians play in precollegiate mathematics, as well as the commitment in time, thought, and energy that such a role requires. However, it would be optimistic indeed to believe that every university mathematician sees work with K-12 education as among the responsibilities of a math department, let alone a worthy departmental priority. In Sect. 15.2, we note the broad range of tasks that are part of a university mathematics department's mission, placing the topic of mathematics education in context of the wider departmental agenda. We also discuss how other tasks that fall to departmental faculty are evaluated. Then in Sect. 15.3, we discuss mathematicians' involvement in math education from an institutional perspective, pointing out the way that a university's Administration might view such efforts. Finally, in Sect. 15.4, we turn explicitly to evaluation. If substantial work on K-12 math education does belong in a mathematics department and is genuinely important, then it follows that it should be recognized and rewarded as such. We explain consequences of this assertion and in particular suggest that viewing the work of mathematicians in the domain of K-12 math education as important and worthy of recognition goes hand in hand with finding ways to systematically evaluate it. We argue that the standard methods used to evaluate research miss a substantial amount of meritorious work and suggest a specific metric for such an evaluation. In each section, we provide indications of conversations that we believe are needed if the work of mathematics faculty on K-12 mathematics education is to be accepted, supported, and valued.

This article is complemented by an article concerning the support of education and outreach in a math department by Jerry Dwyer and Lawrence Schovanec (2017, this volume). Prof. Dwyer is a mathematician sited in a math department where his primary efforts are in math education and K-12 outreach, and Provost Schovanec is a mathematician who has served for many years as a university administrator. By contrast, the author of this paper is primarily a research mathematician but one who engages in some outreach and policy work in math education as well and who has also recently completed a period of 9 years as department chair. Though we arrive at the topic with different perspectives, the two articles paint a coherent picture, and the chapter by Dwyer and Schovanec may be of use in furthering the conversations suggested here.

15.2 The Work of Mathematics Departments: An Internal Perspective

In this section, we survey the primary goals of mathematics departments,[1] as this will put the work of mathematicians concerned with math education in context. Of course, the primary goals of a math department vary from one institution to the next and moreover may be understood and formulated differently by different faculty within a given department. But the description we give here applies to most research-oriented doctoral-granting departments. With the obvious deletions, it applies to many teaching-oriented departments as well.

For a research department, a key goal is for individual faculty to carry out important scholarship. Faculty are judged on whether or not they publish papers in top journals, speak at prestigious conferences, receive external research support through a competitive grants process, and receive other professional recognition such as editorships, fellowships, or awards (Tucker, 1993). For work in the broader mathematical sciences, the influence on cognate areas is also of importance. Ultimately, the goal is to develop mathematical ideas that change a field.

Teaching is a secondary or primary responsibility depending on the department. Teaching includes service teaching, the undergraduate program for majors, the supervision of undergraduate research projects, teaching in the graduate program, and the supervision of doctoral students. The assessment of the teaching of undergraduate classes typically relies heavily on student evaluations, though these are not always directly correlated with increased student learning (Carrell and West, 2010). At the graduate level, success is measured by the influence of one's doctoral students, in the long run by having doctoral students who go on to successful academic careers and in the short term by students who get postdoctoral positions at strong research institutions or other good jobs. The teaching of future teachers may be both one component of service teaching (e.g., classes for future elementary teachers) and one component of the teaching of majors (classes taken by future secondary school math teachers). However, it is only a part of each of these categories, and often[2] a small part.

Service is the final category in most evaluations of faculty performance. This includes departmental and university service on the one hand and external professional service on the other. Departmental service related to teaching is common (e.g., the

[1]We use this term to include all departments of mathematical sciences in the broad sense. Minor modifications may be required in discussing different institutional configurations such as separate departments of pure and applied mathematics, or of mathematics and statistics, but the broad picture is the same.

[2]As an example, 843 teaching credentials—in all subjects—were awarded by the entire University of California system (with nine undergraduate campuses) in 2013–2014 (Purdue and Suckow, 2015, Table G), while the total undergraduate headcount in this system in Fall 2013 was 188,088 (UC System Infocenter website). Of course, the size of programs to prepare future teachers varies considerably from institution to institution, and in some it is a good bit larger.

supervision of large courses, the advising of majors, the mentoring of graduate (or undergraduate) teaching assistants), while university service is an aspect of shared governance. In contrast to these types of service, most external service by math faculty members is related to scholarship: refereeing a paper, editing a journal, organizing a conference, serving on a grants committee. Work with in-service teachers is another example of external professional service but differs from the prior examples in that it does not advance scholarship in mathematics in the short run. In most universities, service is generally not evaluated through a separate formal process.

In this context, work in the domain of K-12 mathematics education is often a relatively small part of a mathematics department's mission. After all, the department is charged with educating the next generation of mathematically literate citizens, the next generation of disciplinary specialists who use mathematics in an essential way, be they economists, physicists, or engineers, and the next generation of mathematicians. The department is the primary place where new mathematics is discovered, where it is applied to solve problems. And the department must run itself, contribute to the wider institution in governance, and be part of the international professional community of mathematicians.

The statement that K-12 math education should be part of a math department's responsibilities is supported by two kinds of arguments. First, there is national need: the nation will not get the job of K-12 math education done without the input of professional mathematicians. Thus, our collective future depends upon this. Second, these efforts are good for the math department itself. Indeed, such efforts can offer a new way for math faculty to contribute to the university's mission, earn professional recognition, and generate grants. Moreover, offering math classes for future teachers can provide increased student credit hours and a pathway to jobs for undergraduate math majors. This work may even have a positive effect upon the doctoral program, making doctoral students more employable by giving them additional teaching experiences and allowing them to demonstrate to prospective employers that they are able to teach a wider variety of classes.

If work with math education is to be included within the list of responsibilities of a math department, then it would be helpful for departments to *make explicit* that involvement with K-12 math education is part of the department's mission. For example, this involvement could be explicitly included in the department's mission statement. At the author's institution, the department's mission statement states:

> The mission of the Department of Mathematics at Boston College is the creation of new knowledge and ideas of mathematics through research, and excellence in the effective transmission of mathematics new and old: to our students at appropriate levels, to mathematicians and scientists world-wide, and to the general public. We give our students the mathematical tools and vision needed for their success in future endeavors, while supporting the university's commitment to a well-rounded education. We are a strong source of mathematical perspective to the university, the profession, and the community. We provide our expertise and experience to our students, to the university, and to our profession through mentoring and advising, committee duties, editing, reviewing, and research collaboration. We act as a resource for those concerned with mathematics education at all levels, as well as for government and industry.

Arriving at a departmental agreement that work in K-12 math education belongs in the math department, or changing a mission statement to reflect this agreement, may well require *conversations within the department*. Here, individuals who are not themselves interested in mathematics education but see it as useful to the department and to society may be key. Since such work is increasingly common, and distinguished mathematics researchers including members of the National Academy of Sciences such as our honoree Prof. Roger Howe engage in it and speak about it articulately, there is external support for this. Though such conversations may well expose a range of opinions within the department, including some less favorable, ultimately they are necessary for such work to thrive.

15.3 The Work of Mathematics Departments: An External Perspective

The math department has multiple roles from the perspective of the university's Administration as well. It is the primary location for teaching, scholarship, and service related to mathematics and the repository of resources devoted to advancing teaching and research in mathematics. In this context, the work of mathematicians in education has some significant *negative* aspects:

- It does not support the primary research mission of the math department, which is the development and application of new mathematics.
- It does not contribute to the rating or visibility of the department's doctoral program.
- It does not contribute to the department's support of students in areas such as physics or economics that use math heavily.
- It does not contribute to interdisciplinary scholarship involving math and science, such as work on bioinformatics or cryptography.
- It contributes to only one limited aspect of the department's undergraduate mission, which may be very broad.

It may also not be helpful politically in terms of relations with cognate departments, such as those within a common division of physical sciences. Indeed, scientists interested in working alongside a serious math department may not believe that K-12 education should be a priority for a math department. Naturally, this differs from institution to institution.

Besides this, there are broader organizational questions. It is generally not best practice to fund the same effort in two different institutional units. Many universities already have faculty in teacher education who are in charge specifically of K-12 education. Why should the university put resources into this area in two different departments? And why is math special?

Mathematicians who support work in K-12 in math departments will need to seriously consider these arguments and be prepared to answer them.[3] Moreover, the issue of rewards for mathematicians working in math education is tied to these questions and their answers. Indeed, one expects rewards to reflect the university's priorities for the math department.

The formulation of coherent answers to these questions is probably not needed for readers of this volume. A short list is as follows:

- The involvement of mathematicians is valuable for, even critical to, K-12 education. Indeed, without this work, our nation is likely to continue to significantly lag top achieving counties in K-12 math achievement in the future.
- This work helps address the nation's critical shortage of students prepared to study STEM. Training enough STEM graduates is expected to be a key factor in the nation's long-term economic health.
- Increasing the supply of well-qualified math teachers can address one factor in the nation's troubling socioeconomic inequality. Indeed, with a shortfall of such teachers, it is our poorest citizens who are most likely to have under-prepared teachers.
- As the controversies over the Common Core make clear, citizens care about math in the context of education. Work in math education contributes to the community in a demonstrable way. It also increases the university's visibility.
- While it is hard to explain abstract theorems and technical advances to a general audience, work in math education is easier. These efforts will be understood and appreciated by alumni.

The ways that these pros and cons are weighed will vary greatly from institution to institution.

Just as Sect. 15.2 proposed an internal discussion resulting in a formal recognition of involvement in K-12 math education as a departmental responsibility, in this section, we propose an external discussion of this matter within the university setting. Our thesis is simple*: Mathematicians have a unique and valuable perspective on K-12 math education. Their involvement is critical to the country's long-term economic and societal health. Consequently, this involvement should receive institutional support.* This thesis is best argued not by an individual practitioner, but by the department. Ideally, mathematicians working in math education and their department chairs will together be articulate about why the work of mathematicians in math education matters and about why it is necessary for mathematicians to be the ones to do this. They must actively engage in *conversations outside the department*, seeking to convince others both within and outside the university. In interacting with their colleagues in science, it is helpful to emphasize the potential impact of their work on the STEM pipeline.

[3] Even if they work with administrators who already view the preparation of K-12 math teachers as an appreciable part of a math department's mission, such considerations may arise in competing for scarce university resources.

There is an important additional aspect to making such arguments. At a competitive institution, resources follow successes. If mathematicians working in education wish to receive resources, they must then document at every turn that their work has made a difference. To be sure, some of the impact will only be clear years later, but this is true for other departmental efforts as well. Math educators must follow best practices in documenting the effectiveness and impact of their efforts.

15.4 Evaluating Mathematicians' Work in Math Education

In the first two sections, we have discussed the work of mathematicians in K-12 education in a rather undifferentiated way. Let us now look more closely. As readers of this volume are well aware, mathematicians do many different things concerning K-12 math education. These include work with preservice teachers, such as helping future elementary teachers in learning the pedagogical content knowledge they will need (Ma, 2010) and mentoring math majors as they carry out their student teaching; work with in-service teachers such as helping them to deepen their background in algebra, geometry, or statistics or supporting them to develop "mathematical habits of mind"; work with talented K-12 math students through devices such as math circles or summer programs; work on standard documents such as the Common Core State Standards for Mathematics; work on curriculum and on the implementation of a curriculum; and work on math education policy such as service on a statewide advisory committee or public advocacy of specific policy decisions. See McCallum (2003) for several specific examples, which illustrate the range of work that is possible, and Friedberg (2014) and Briars and Friedberg (2015) for examples of the public advocacy of math policy decisions that reflect the vantage of a university mathematician.

Each of these activities could be the focus of a substantial project, one that requires a great commitment of time and energy. Moreover, such efforts may take the form of a multi-year project. An example is the design of a systematic program to develop and support strong math teachers, the solicitation of external funding for such a program, and, upon receipt of funding, its implementation. To be sure, such work may also lead to research on some topic in math education or to advising doctoral students in carrying out research on some topic in math education, but that is often not the primary goal of such an effort. In a systematic program to develop and support strong math teachers, for example, the primary goal is visibly not such research.

Which of these activities are the most meritorious? Publications are valued in academia but may not be the primary goal of truly worthwhile projects. Also, even highly recognized scholarship in math education might well be outside a math department's mission, and surely every mathematician wishing to contribute to math education does not need to become a math education researcher. Moreover, it is possible to spend a great deal of time on many of the tasks mentioned above without generating publications. Simply applying the approach for mathematics itself of considering papers in top journals as a critical indicator of success misses a great deal of important and potentially valuable work.

In view of this, it is important that mathematics faculty members working in K-12 education *discuss with their department chairs* the range and demands of tasks in math education. And if this work is to be truly valued, they must then seek recognition for those tasks. However, the pact in academia is clear: if you seek recognition, then you must agree to be evaluated by clearly defined criteria.

Accordingly, we make the following suggestion: *Mathematicians working in K-12 mathematics education should be evaluated concerning the quality of this work. The metric for achievement should be overall impact, the same metric we use in evaluating scholarship in mathematics itself.*

To be clear, it is not common to evaluate service, and the bulk of work in math departments in the area of math education falls into this category. However, failing to carry out such evaluation ultimately denigrates these contributions. Accordingly, we should not let the challenges of carrying out such an evaluation stand in the way of adopting this as the metric.

The author does not wish to minimize the challenge of measuring overall impact. Gathering evidence that work with teachers has made a difference to their classroom students is not easy and may be costly. It also requires familiarity with research protocols in education. Multi-year projects, such as ones directed at building a community of highly trained math teachers in a given geographical area, may take a long time to bear fruit but may be very effective in the long run. Innovations that are worthy of dissemination are potentially very valuable, but once again it may take a period of time to reach this stage and then to carry out effective dissemination. But these are common issues to assessing any project involving a complicated system.[4] Moreover, measuring external support is certainly realistic, and letters from peers could be solicited. A combination of data, grants, and external letters from peers would be a natural beginning to documenting impact. When mathematicians working in education are able to demonstrate that their work is effective on a large scale, they will have a strong case for rewards and support.

15.5 Conclusions

Mathematicians carry out a great deal of work related to K-12 math education. The mathematical expertise that they bring to this work is critical, and indeed this work is important for the country. However, most of this work is not traditional scholarship. To move forward, mathematicians carrying out such work should argue

[4] Recent discussions of metrics for evaluating the impact of scholarship in mathematics suggest that even the evaluation of mathematics is not entirely straightforward; see, for example, the section on metrics in Andrews (2012).

for three changes. First, at the departmental level, work *as mathematicians* in K-12 math education should be recognized as belonging to the math department and included in the department's mission statement. Second, at the university level, the math department should argue that this work belongs in a mathematics department and deserves the commitment of institutional resources. Finally, if this work is to be perceived as truly valuable, it must be subject to evaluation. The mathematics profession and wider university should broadly agree upon *overall impact* as the metric for evaluating such work. It should then reward excellence accordingly.

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Chapter 16
Supporting Education and Outreach in a Research Mathematics Department

Jerry Dwyer and Lawrence Schovanec

Abstract Research mathematics departments promote scholarly activities primarily focused on traditional mathematical research. These departments also have a role to play in advancing high-quality teaching, student mentoring, K-12 outreach, and the mathematical education of preservice teachers. A significant number of faculty pursue these activities and enjoy varying levels of support. This chapter describes the benefits that accrue to a department that supports these activities while also recognizing the challenges that are often presented. The authors offer the perspective of an established outreach mathematician and that of a senior administrator who has actively supported outreach from his time as a mathematics department chair through his role as a college dean, provost, and now a university president. They argue that support can be provided at several levels from within the department and across the university, with a specific emphasis on setting clear guidelines for tenure and promotion based on nontraditional scholarly output in the areas of education and outreach.

16.1 Introduction

There is a significant cadre of faculty in research mathematics departments around the nation who devote considerable time to issues of mathematics education and outreach. The support of those departments for this type of work varies across the spectrum from enthusiasm to apathy to opposition. In this chapter, we describe the typical roles played by those faculty and how these roles may be supported by national organizations and universities and in particular within the departments themselves. In the course of addressing these issues, suggestions for supporting and enhancing education and outreach are provided.

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16.2 Background

Articles reflecting on these roles have appeared in the *Notices of the American Mathematical Society.* The first, describing early experiences of an outreach mathematician and the chairperson who advocated for such a role, appeared in 2001 (Conway, 2001; Dwyer, 2001), and an update appeared in 2013 (Dwyer and Schovanec, 2013). The first author (Dwyer) of this chapter is the outreach mathematician involved in the earlier articles. The second author (Schovanec) was the chair of the Department of Mathematics and Statistics (M&S) at Texas Tech University (TTU) at the time that Dwyer was hired in 2003. Schovanec now serves as the President of TTU, having previously served as dean of the College of Arts & Sciences and as Provost of TTU. In each of his administrative roles, he has promoted outreach, engagement, and the associated reward structures within the department, the college, and the university.

In addition to the above, there are several aspects of the interface of mathematics and education that have been discussed in other articles in the *Notices of the American Mathematical Society* over the past 15 years. A good example is the article by McCallum (McCallum, 2003), which describes several of the issues related to the role of mathematicians working on educational issues in research mathematics departments.

Some of the earlier articles focused on a specialist dedicated to outreach as a primary role. The reflections offered in this chapter will apply to a broader group of faculty and encompass any work undertaken by mathematicians in education. They may also be applied to faculty who continue to pursue traditional research but may devote a limited portion of their time to mathematics education and outreach issues. It also includes faculty whose focus is primarily in the area of traditional mathematics education research while working in a mathematics department. This chapter primarily addresses the issues of mathematicians engaging in activities outside of traditional research in mathematics or working on education issues and how contributions in these areas are recognized and credited for the purposes of tenure (if applicable) and promotion within a traditional research mathematics department. The focus of the chapter is on faculty in research mathematics departments. It is acknowledged that there are many faculty in departments primarily devoted to teaching and those who may not be tenure track who are engaged in these activities, whose contributions need to be supported. There are also many mathematicians working on educational projects in other capacities, and their value is recognized for the expertise they bring to the field. These include mathematicians working in educational testing companies, on state boards of education, or as dedicated clinical teaching faculty not seeking tenure in a research mathematics department.

This article is complemented by another article concerning the support of education and outreach in a mathematics department by Solomon Friedberg (2017). Professor Friedberg is primarily a research mathematician, but one who engages in outreach and policy work in math education as well and who is also completing a period of 9 years as department chair. His perspectives as a current chair complement those of the outreach mathematician and senior administrator offered here. He argues that work as mathematicians in K-12 math education should be recognized as belonging to the mathematics department and should be included in the department's mission statement. He also suggests that the mathematics department should argue at the university level for a commitment of institutional resources to support these activities. The outreach work should also be subject to evaluation, with overall impact as the metric for evaluating such work.

There is an interesting historical perspective related to these matters and a rich history of over a century of research mathematicians in the United States and elsewhere interacting positively with K-12 teachers. Most recent advances have been in areas such as the development of the recommendations of the panel on the Mathematical Education of Teachers (MET). That document draws heavily on the advice of research mathematicians who are also engaged in mathematics education. New mathematics standards, such as the Common Core, have also been developed with significant input from research mathematicians. In this volume, dedicated to Roger Howe, it is fitting to acknowledge his numerous contributions to the discussion of what is appropriate mathematics in the K-12 realm and how the understanding of such mathematics must inform the preparation of mathematics teachers at all levels from kindergarten to 12th grade. It is also important to note that many of the observations and suggestions in this chapter have resulted from the discussions and presentations at the workshop in March 2015 at Texas A&M University in honor of Professor Howe's 70th birthday.

16.3 Education and Outreach Roles and Recognition

Education and outreach faculty in research mathematics departments pursue a wide range of activities. These include traditional mathematics education researchers, while others are predominantly engaged in teacher education or outreach to the K-12 community. Typical activities include visiting K-12 school, organizing math clubs and summer programs, developing teacher workshops, and developing and teaching courses for preservice teachers. In a broader context, the roles of mentoring, supervising undergraduate research, educational grant writing, and scholarship focused on improving undergraduate teaching such as curriculum development and textbook writing may be adopted. Such faculty usually interact extensively with colleagues outside the department and serve as liaisons across college and departmental boundaries, promoting collaboration on multidisciplinary projects that may involve funded support and joint publications. This latter realm of activity brings a new set of challenges. Indeed, a major issue facing faculty in a nontraditional academic role is that of publication. There are a limited number of appropriate outlets for disseminating research, and there are reservations from colleagues in traditional research areas about the quality of scholarship that results from outreach activities and the relevance of this type of work to the mission of the department.

There are many benefits of having faculty dedicated at various levels to educational research and engagement work. This is an aspect of departmental work that is easier to explain to alumni and parents and offers potential for financial contributions from that donor base. This type of activity also resonates with foundations that support community engagement and outreach. It is an area of work that addresses the need for more STEM graduates that is commonly highlighted in the national press. It also offers the potential to attract new graduate students that might not have otherwise pursued a career in mathematics had their option been restricted to the traditional research areas of study. Mathematics education is also a vibrant area in many other countries and offers new opportunities for engagement at an international level.

It is important to stress here that the authors strongly believe that there is a definite role for mathematicians in educational work. In particular, mathematicians can provide mathematical knowledge and perspective that may not be available to those with a more pedagogical focus. Conveying a sense of what it is to do mathematics brings enthusiasm to the classroom that has the potential to motivate and foster a deeper conceptual understanding among future teachers and other students. This can be achieved in a way that may not happen with instructors who don't have the same level of mathematical background. We believe that the mathematical knowledge of teachers is important and that mathematics faculty can foster that deeper conceptual understanding among teachers in a unique way. Similarly, mathematicians can develop master teachers in ways that aren't always possible for educators who don't have the same mathematical background.

The mathematician working on educational issues is often viewed by colleagues as having an expertise in the area of undergraduate teaching. In that context, there is an opportunity to support the teaching mission of the department by initiating activities and discussions that support personal development of teaching proficiency, innovation in the classroom, and generally broader discussions of the importance of teaching excellence. This expertise is critical in addressing new models of instruction and efforts to incorporate more active and engaged teaching strategies. Several research studies show that active and engaged learning leads to greater conceptual understanding (Epstein, 2013). There is a major advantage to the department if such teaching can be adopted and if such an approach can lead to greater student success and decreased failure rates that continue to challenge many departments. This issue is also of concern to upper administration, as the high failure rates in lower-level STEM courses influence retention and graduation rates and potentially tuition revenues. Departmental seminars may help to educate the faculty on teaching issues and also encourage them to reflect on their own teaching. They may be motivated to consider the extent to which their teaching contributes to genuine learning among their students. Increased reflection on teaching can also lead to the development of topics in the realm of the scholarship of teaching and learning. This is also discussed later in the context of the development of different models of scholarship in general. Particular opportunities may exist to mentor junior faculty on teaching. At Texas Tech University, the authors have developed a series of seminars where issues in teaching at all levels are discussed. As a result, several faculty have reported that they have begun to reflect more on their teaching and have incorporated strategies that wouldn't otherwise have been apparent to them.

The development of educational and outreach projects often depends on grant funding. This is both a challenge and an opportunity. University administration typically supports the grant writing process by identifying this effort with scholarly activity. Indeed, there may be acknowledgment that a successful grant proposal is similar to that of a peer-reviewed publication. But in addition to the scholarly implications of this work, there is also a clear financial benefit to the department as well as increased visibility across campus and beyond the university. Conversely, the university needs to provide logistical support for grant writing in all disciplines, as many faculty are inexperienced in this area. The need and emphasis on funding do have a positive payoff in the possibility of attaining larger grants and the visibility and recognition that they bring.

The authors have encountered some negative feedback from faculty. This includes criticism that outreach or educational work does not contribute to the traditional mission of either the undergraduate or graduate program of the department. Another criticism is the possibility of a duplication of effort since faculty in traditional departments of education are already active in these areas. Both of these criticisms are addressed in this chapter.

16.4 Alternative Scholarship

It is useful here to describe a broader context of scholarship in general. These distinctions in scholarship have been defined and described by Boyer in (1990). Boyer argues for a consideration of scholarship beyond the narrower confines of traditional research, which often treats the only valid work as that of discovering new knowledge and disseminating it. A traditional research mathematician is trained in this mode, engaged in the development of "knowledge of knowledge for it's own sake" or expanding the knowledge base of a discipline of specialty. Boyer's model broadens the definition of scholarship to include work such as that of disseminating existing knowledge in new ways or in ways that are accessible to wider audiences or in connecting older knowledge in some new way. This definition allows recognition for alternative scholarship or creative activity that has long been recognized in some disciplines. For example, in music, the scholarly work may be a performance, which is certainly not a written work with literature reviews and appropriate methodologies. We can also mention the "Scholarship of Teaching and Learning," an area of increasing familiarity to mathematicians in which contributions meet the standard expectations of scholarship, including public dissemination and rigorous review.

Mathematics departments have a role to play in advancing this notion of scholarship by creating awareness among the faculty of this approach and how such work is valid. It is important that the faculty understand that this is not trivial work and is not seen as an easy option compared to traditional mathematics research. It is also important to clarify that this alternative scholarship is more than service. It is not sufficient to perform outreach activities and write about them. The activity must be planned with a research agenda, developed in a methodical manner and the results analyzed and described in a manner that allows replication by the research community. This type of scholarship has much in common of course with traditional research in education and in the social sciences.

An example of this type of scholarship may be informative. A mathematician could organize a summer program for a group of middle school students from underrepresented groups. The program is developed with specific objectives in mind. The participants are surveyed before and after the program, and the changes in content knowledge and attitude are analyzed. This should result in a scholarly publication that includes a description of the program and an analysis of the extent to which the objectives are achieved. The research methodology may lack some rigor, and the article may not have set the theoretical context for a traditional educational journal. However, the article will be of great interest to fellow practitioners and has a place in a practitioner journal or as a technical paper in a popular magazine.

Departments also need to recognize educational and outreach work that isn't accompanied by publication. This can be a valuable part of a regular faculty member's service role. It may also be of greater importance to faculty who are not tenure stream and who could be supported by course release time or increased salary where appropriate.

16.5 Training in Educational Research

Most mathematicians have not been trained in rigorous educational research methodology. As a result, they struggle to set the context in terms of literature models or to implement appropriate methodologies. The usual approach is to perform some educational work, but never prepare it for publication in order to receive scholarly credit. Another approach is to collaborate with educational faculty in order to produce papers based on the outreach work.

It may be productive for interested mathematicians to consider taking the time to formally learn the methods and standards of educational research. This gives the faculty member the option of independently designing research studies and writing about them in a scholarly manner. However, this option may be too time-consuming for faculty who wish to remain active in traditional mathematics research and who only wish to devote a limited amount of time to outreach and education.

To address this issue, departments in the STEM disciplines may consider assisting in the development of educational expertise by hosting a series of seminars and workshops designed to disseminate research expertise in the area. A number of departments could combine to jointly host these workshops so that a critical mass of mathematics education and science education faculty could join together to benefit from the presentations. These workshops would need facilitators from education colleges who could share their knowledge. Such joint workshops then have the added benefit of increasing interaction between education colleges and colleges of arts and sciences, with the potential for new collaborations.

16.6 Mathematics Teacher Education

One area where mathematicians may find increased reward is that of the mathematical education of teachers. The K-12 education system faces numerous challenges many of which are beyond the control of university faculty. However, we are the educators of preservice teachers, and for 3–6 h per week, we have their undivided attention in our classrooms. We can make an impact by teaching well and facilitating deep conceptual understanding that these teachers can take into their own classrooms in future years. There are opportunities for interested and qualified mathematics faculty to teach preservice teachers and to inspire and educate those who teach future generations.

Recent national meetings and feedback from practitioners suggest that there is some change in the landscape of mathematics teacher education. There is merit in a different approach to teaching preservice courses, with greater emphasis on partnerships between education and STEM faculty. An example is the impact described by Sultan and Artzt (2005) when a mathematician and mathematics educator co-teach a class for preservice teachers. These kinds of collaborations are areas where the outreach mathematician is in an ideal position to contribute positively to the new landscape.

Teacher education also provides opportunities to connect with supportive national organizations that are committed to research and development on learning and how programs can support teacher educators. These include the National Council for Teachers of Mathematics (NCTM) and the Association of Mathematics Teacher Educators (AMTE) who host national and regional meetings on an annual basis. These organizations also offer avenues and collaborators that assist mathematicians in learning the type of research approach needed to examine educational issues and to assess the impact of various outreach and K-12 classroom intervention programs. Teacher-faculty interaction in guided workshops or in events such as Math Teacher's Circles also offers the opportunity for faculty to understand the perspective of K-12 teachers and to see how teachers often bring to the table a surprisingly diverse and high level of mathematical knowledge and problem-solving experience.

16.7 Teacher Professional Development

College faculty can find great reward also in working with practicing teachers through professional development workshops. These programs can involve a deep analysis of the mathematics that is taught at the K-12 level. Faculty are often surprised at the level of this depth and how PhD mathematicians can respect and complement the work of their K-12 colleagues. This collaborative approach also helps in the development of the community of mathematics teachers. The college faculty realize that they share much in common with their K-12 colleagues and can offer each other a mutually supportive environment.

The outreach or educational specialist is also in a strong position to seek external funding for programs related to the professional development of K-12 teachers. These programs are often the focus of grant funding opportunities. Beyond the more traditional sources for mathematics research and education, such as the NSF and DOE, educational initiatives are often specially addressed by state-funded programs and foundations. An example is the funding from the Greater Texas Foundation to support the Middle School Math and Science $(MS)^2$: Understanding by Design master's degree program. This program is an example of a revenue source for the development of masters' courses for in-service teachers. These courses are popular and offer excellent opportunities to develop deeper analysis of the K-12 mathematics subject matter. They also provide an example of a way that outreach mathematicians can contribute positively to the revenue stream of the mathematics department. A final point on revenue streams may draw upon the potential of working with K-12 school districts to provide financing for some programs. There is a critical shortage of highly qualified math and science teachers, and mathematics departments may be able to work with school districts to find creative ways of funding new teacher training and professional development programs. The mathematician who is interested in educational issues is in a key position to lead such innovations.

16.8 Case Study of Successes and Challenges

This case study describes a particular model of a mathematician dedicated to outreach and describes some programs that have proven successful and can be offered as possible starting points for faculty who want to begin some educational outreach work. These include presentations in K-12 classrooms, after school math clubs, and summer programs for K-12 students. These programs provide opportunities to develop positive relationships with K-12 teachers and to gain classroom experience that gives credibility when we teach preservice teachers or offer professional development workshops for in-service teachers. It is the authors' experience that it is essential to develop these projects as genuine collaborations rather than any form of study or remediation exercises. Small projects of this nature are often funded by foundations collaborating with the Mathematical Association of America (MAA) and may be less competitive for funding than the large federally funded programs. As such, they have the potential to provide experience in managing funded educational projects. A related point refers to the building of collaborative teams across disciplines, which is often an important facet of outreach work.

Successful teams are formed in an atmosphere of mutual respect and a recognition that different cultures exist in colleges of education compared to that in mathematics or other STEM departments.

At Texas Tech University, the hiring of an outreach mathematician provided the catalyst and leadership for a number of campus-wide initiatives and collaborations. Over the last several years, significant external funding has dramatically affected STEM education and outreach projects at TTU. A major development in support of the scholarship of outreach at TTU occurred in 2012 when the university adopted a revised tenure and promotion policy that recognized outreach and community engagement as part of a faculty member's contributions in potentially each of the areas of teaching, research, or service. The College of Arts & Sciences followed suit, and most recently the mathematics department at TTU has adopted such a clause in its promotion and tenure documents. Dwyer has been featured as an "Integrated Scholar" (Smith, 2011, http://www.depts.ttu.edu/provost/scholars/jerrydwyer.php), a distinction that TTU has enlisted to recognize contributions to teaching, research, and service, where outreach is recognized as a component of all three areas. In 2006, TTU was one of the 76 universities and the first in Texas to be included in the "community engagement" classification of the Carnegie Foundation for the Advancement of Teaching and regularly recognized in the President's Higher Education Community Service Honor Roll. In 2015, TTU was again selected by the foundation for its 2015 Community Engagement Classification. This distinction is partially based on data reflective of TTU's strategic priority to expand community engagement and evidence of extensive faculty-led community collaborations.

The impact of outreach and engagement on various constituencies is described in two reports based on studies funded by an NSF grant awarded for the integration of outreach activities at TTU. The first report (Dwyer, Miorelli, & Moskal, 2015a) presents the results of surveys administered to incoming freshmen that describe the impact of their participation as K-12 students in TTU outreach programs. The results show a moderate impact on their decision to pursue a STEM degree. A corresponding survey to graduating seniors shows that their retention in a STEM discipline was also moderately impacted by their participation in K-12 outreach either as a facilitator or as a participant. Indeed, the effect was seen to be similar to that of participation in an REU (undergraduate research) experience, which has long been held as a positive influence on student retention in the STEM disciplines.

The second report (Dwyer, Miorelli, & Moskal, 2015b) describes the self-reporting by faculty of their participation in outreach activities and their perception of the merit associated by the institution to those activities. The results show a greater awareness of outreach and a higher level of faculty morale among newer faculty.

The authors have clearly seen the advantages of having faculty dedicated to education and outreach with various levels of commitment ranging from that of a specialist outreach mathematician to that of faculty who may make as little as one presentation per year to high school students. They have seen new opportunities, rewards, and frustrations related to outreach mathematics. However, a number of challenges remain as TTU continues to advance toward a campus-wide recognition of the role of outreach, engagement, and interdisciplinary educational projects. The nature and level of these challenges may vary from institution to institution. Some colleges remain on the periphery of these developments, and for some departments, there is still a singular focus on traditional research scholarship as the route toward tenure and promotion. Junior faculty may be discouraged from pursuing educational or outreach-related scholarship prior to obtaining tenure. In Dwyer's case, the department required about 8 years before finally achieving near-unanimous appreciation and recognition of his contributions and of the role of outreach and engagement.

These issues are more pronounced at many other institutions that have faculty dedicated to education and outreach. In many cases, these faculty were hired in nontenure track roles or pursued outreach activities only after the traditional requirements for tenure and promotion were met. Some tenure track colleagues at peer institutions have faced negative tenure votes and lengthy appeals before their work was recognized. Others have gained tenure but failed to find consistent support or successful collaborators.

16.9 Developing Support Structures

Many challenges still confront mathematicians who choose to work on educational issues, and those challenges must be addressed at a number of levels. Some of these strategies have already been mentioned in previous paragraphs. In addition, at a national level, the American Mathematical Society (AMS) could offer renewed support through the committee on education and perhaps through recognition of educational outreach as a separate subject classification within the AMS listing of topics. On a related note, the challenging issue of publication opportunities could be ameliorated through the creation of a new journal dedicated to outreach mathematics. The AMS or MAA could play a role in developing and promoting such a journal. Perhaps, the time has also arrived for outreach mathematicians/educators to join together to create a Special Interest Group or SIGMAA.

It is clear that outreach and engagement is recognized in several important ways at a national level, for instance, the WK Kellogg Foundation Community Engagement Scholarship Awards or the C. Peter Magrath Community Engagement Scholarship Award. The importance of this type of recognition is often evident at the university level through the institutional effort devoted to these types of recognition, as well as explicit mention in the university strategic plan. What is always not so transparent is the local significance of this activity at the departmental or college level. For one who aspires to work in the area of outreach, understanding the landscape for support of outreach is worth investigating when deciding where to pursue a career.

The major challenge often occurs within the department, where support is needed to pursue projects and ultimately in terms of votes for tenure and promotion. The typical reward system may not recognize that an outreach mathematician should be tenured and promoted based on the quality of his or her alternative scholarship as is the case in a traditional track of pure mathematical or educational research. The first step for a mathematics department should be an amendment to the reward and promotion structure to take into account the nontraditional role of an educational or outreach mathematician. This will also be achieved through a focused hiring letter that addresses the role of the scholarship of outreach and engagement. In terms of the overall mission of the mathematics department, the work of mathematicians in mathematics education should be explicitly stated in the mission statement or strategic plan of the department. Finally, a younger faculty member should find the support of a senior faculty member who can act as an advocate for the cause of outreach. A respected traditional researcher can be a powerful influence within the department even if that researcher is not personally involved in outreach or educational matters. The senior faculty advocate can also gauge and provide feedback on local attitudes in a way that a newer or more junior faculty member might not be able to accomplish.

In this volume dedicated to Roger Howe, it is again appropriate to mention the role that a senior and distinguished mathematician from outside the department can play in advocating for recognition for this work. It is the authors' experience that many of their faculty colleagues are convinced that mathematics education work is important when a distinguished mathematician such as Professor Howe advocates for this work. Other senior research mathematicians could adopt a similar role where their influence extends to the national level.

The education of faculty colleagues on these issues is critical. The chair has a very important role to play in educating faculty and in recruiting faculty for outreach and math education roles related to either K-12 or undergraduate learning issues. This includes recruiting faculty to teach math content courses for preservice teachers. It is not as difficult as some may suspect. First, we note that the mathematics of elementary and secondary school is deep and complex and rewarding for some mathematicians who choose to explore it in depth. A series of seminars in this area can be most helpful, especially by inviting K-12 teachers to attend these seminars and bring their perspective to the university faculty. These teachers can address issues of language and diversity and cognitive development that are not always known to mathematics faculty. This understanding should lead to a greater empathy with teachers and a greater appreciation of faculty colleagues who choose to work in these areas.

The education or outreach researcher also has a role to play in cultivating relationships within the department in a respectful manner. He or she has to recognize that some faculty aren't particularly interested in math education issues and shouldn't attempt to push a particular agenda on the department. It is the authors' experience that a positive and collaborative approach yields the greatest level of support.

General feedback from successful programs on all of these integration efforts is that the process must be gradual and change doesn't happen overnight. There is a long-standing tradition in most departments of what are appropriate roles and it will take time to alter this. This issue was captured nicely in a quotation from the March 2015 workshop: it was stated that it was “the work of a generation.”

Beyond the department level, various types of support should be introduced at the college and campus levels. The benefits of breaking down the siloes of traditional research factions can be promoted at the college and campus levels. Tenure and promotion guidelines at the college and institutional levels would certainly be of benefit. A specialized review process should be considered where the tenure or promotion vote could include colleagues in STEM education roles in other departments. These votes or recommendations need not replace the departmental vote but could be considered by the dean or provost in making final decisions. Campus-wide awards and small grants for outreach and engagement activities should be introduced to draw attention to such engagement and increase the visibility of the winners, many of whom remain unknown outside of their own departments. Solid levels of support from the upper administration should also be conveyed and implemented at the local level through effective support and promotion from the department chair. These levels of support should also be extended to faculty who are not tenure stream and who choose to pursue math education and/or outreach roles. Long-term contracts and course releases are examples of such possible administrative support. With such support it can be hoped that a greater number of mathematics faculty will pursue educational and outreach activities and reap the rewards that are becoming apparent in those departments that have successfully embraced this emerging role.

The challenges facing outreach and education-oriented mathematicians and the support structures to address those challenges have been described. There is, however, at least one area where the outreach mathematicians can support their own cause. This is in the area of assessment and evaluation. This aspect is also discussed as a major recommendation in the complementary article by Friedberg (2017). There is a critical need to document and quantify the impact of mathematics outreach and education activities. This leads to the question of how we might assess the impact of various projects. Assessment of this sort is not well developed, and it is often difficult to attach numbers to impact factors. Possible measures could include the number of new STEM degrees awarded, improved success rates in critical courses, the number of new teachers produced, the number of dollars generated, the number of newspaper and popular press articles written, as well as more formal research studies. An example of such a study could be one that measures the learning impacts of faculty intervention efforts at the K-12 level. However, it is recognized that measuring such metrics will be time-consuming and may require course release time or other support in order to facilitate the work. It could even be argued that the assessment of outreach itself may be a valuable contribution in its own right for a faculty member who has an interest in that area. If positive outcomes can be shown to accrue, then it is easier to advocate for this role with the upper administration of the university. It is also easier to convince faculty colleagues, university administrators, and funding agencies that the outreach work has significant merit in any department.

16.10 Conclusions

The previous paragraphs have described the roles of an outreach/educational researcher in a traditional research mathematics department. The benefits and challenges have been addressed, and several suggestions have been offered that might lead to greater recognition of these roles. The authors have often referenced their own particular department in illustrating specific examples that illustrate the benefits of embracing such a role. It should be acknowledged that these experiences are not unique to their institution. But as a result of their personal engagement in these endeavors, they have become strong advocates for the adoption of these roles in other departments. It is their hope that the reflections presented here will encourage others to consider the hiring of mathematicians wishing to pursue educational activities. Furthermore, it is hoped that the suggestions offered would be useful at several levels in preparing guidelines to navigate through the various obstacles that may present themselves along these relatively new pathways.

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Erratum to: Attracting and Supporting Mathematicians for the Mathematical Education of Teachers

Amy Cohen

Erratum to:
Chapter 14 in: Y. Li et al. (eds.), *Mathematics Matters in Education*, Advances in STEM Education, https://doi.org/10.1007/978-3-319-61434-2_14

In this chapter, the author name in the fifth reference was incorrect. It should have been "Deshler" and not "Denschler". The correct spelling in reference should be as follows:

Deshler, J., Hauk, S., & Speer, N. (2015). Professional development in teaching for mathematics graduate students. *Notices of the AMS, 62*(6), 638–643.

The original version of the chapter has also been revised.

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Index

A
Abbaco schools, 94, 97–99
Accomplishments, 10–12, 20, 31, 63, 158, 166, 177–179, 265, 297
Addition, 10, 17, 21–24, 28–32, 37, 50, 55, 62, 63, 65, 71, 75, 77, 79, 86–88, 94, 98, 103, 106, 152, 157, 159, 161, 162, 165, 167, 186–201, 206, 212–214, 219, 220, 226, 228, 229, 237, 242, 246, 270, 288, 291, 296
Areas, 23, 26, 74, 82, 87, 98, 101–103, 161, 186, 195, 201, 202, 212, 213, 221–225, 234, 249, 251, 253, 279, 281, 284, 288–293, 298
Attracting mathematicians, 263–266

C
CCSSM. *See* Common Core State Standards in Mathematics (CCSSM)
Coherence, 4, 6, 14, 20, 45, 49–51, 55–57, 64, 65, 72–76, 78, 113, 114, 119, 121, 151, 152, 159, 168, 179, 180, 245–255, 263, 278, 282
Common Core State Standards in Mathematics (CCSSM), 14, 16, 20, 27, 30, 31, 44, 46, 51, 52, 56, 65, 71–74, 82, 119, 186, 189, 191–193, 195, 196, 200, 204, 206, 246–250, 252–254, 267–268, 283
Congruent figures, 36, 72, 235, 237, 238
Contributions, 4, 5, 7, 9–17, 95, 105, 186, 234, 240, 243, 277–285, 288–291, 295, 296, 298
Counting, 6, 22, 31, 33, 59, 102, 150, 151, 161, 165, 167, 189, 190, 193, 200–206, 210, 211, 214, 216, 217, 246
Counting numbers, 6, 22, 102, 201–204, 206
Cubes, 29, 200, 223, 225, 246
Curriculum, 4, 6, 9, 13–17, 20, 27, 28, 38, 44, 45, 47, 49, 51, 55, 63, 68, 70, 71, 73, 74, 79, 81, 93–95, 97, 99, 102, 105, 107, 110, 112, 113, 122, 146, 164, 165, 167, 168, 177, 179, 188, 210, 212, 216, 220, 233, 234, 245–255, 267, 283, 289

D
Decomposition of teaching, 159, 162, 168, 169, 189, 190
Definitions, 6, 26, 29, 35, 44, 48–50, 52–55, 57–69, 72, 74, 77–83, 85–87, 102–104, 106, 109, 110, 113, 118–123, 126, 127, 134–136, 139–141, 144, 145, 175, 220, 221, 229, 235, 236, 240, 245–249, 262, 271, 291
Departmental and university perspective on math education, 282–285, 288, 289, 297, 298
Distance, 28, 29, 59, 60, 85, 105, 142, 167, 203–205, 216, 217, 236, 237, 239, 248, 249, 255, 271

E
Early childhood mathematics, 20, 106, 107, 111, 150, 160, 168, 176, 189, 213, 214, 263, 267
Elementary school mathematics, 5, 13–15, 87, 161, 169, 177, 219, 221
Engaging mathematicians, 4, 5, 7, 10, 262, 269, 278, 288–291

Y. Li et al. (eds.), *Mathematics Matters in Education*, Advances in STEM Education,
DOI 10.1007/978-3-319-61434-2

Equation in two variables, 54, 73, 88, 117–120, 125, 126, 145
Euclid, 23, 93, 94, 101–106, 109, 110, 236
Experts, 80, 144, 145, 173–180, 234, 255

F
Festschrift, v, 3
Fibonacci, 97–99
Fractions, 6, 14, 15, 23–30, 44, 46, 47, 49–54, 56, 57, 59–65, 69, 73, 74, 76–83, 87, 88, 96, 98, 102, 104, 110, 118–122, 125–130, 132–134, 136, 137, 139–141, 143–145, 160, 176, 186, 188, 191, 192, 203–205, 211–212, 214, 216, 217, 225–228, 230, 247, 249, 270
Fundamental principles of mathematics, 48–52, 54, 86

G
Geometric transformations, 37, 233–243, 250
Geometry, 6, 15, 23, 28, 31, 37, 46, 47, 51, 54, 56, 70–73, 88, 220, 221, 229, 233–243, 248, 251, 252, 283

H
Howe, Roger, 3–17, 19–38, 113, 152, 158, 169, 175, 185–206, 219, 220, 230, 236, 237, 239, 240, 243, 249, 255, 260, 273, 281, 289, 297

I
Inequalities, 50, 87, 88, 213, 223–225, 252, 282
Interdisciplinary field, 4
Isometries, 63, 71–73, 239

K
K-12 outreach, 278, 295

L
Length, 17, 23, 26, 28–30, 33, 37, 50, 54, 72, 76, 85, 87, 101, 102, 104–108, 200–202, 204, 205, 212–216, 221–225, 229, 236–238, 249–251, 253
Lines, 28–32, 34–37, 51–54, 102, 117–146, 150, 200, 205, 209–210, 220, 237, 242, 249

M
Magnitudes, 21–23, 93, 94, 101–112, 123, 210, 237, 239, 242, 249
Math drawings, 186, 188, 194, 197, 198, 206
Mathematical content knowledge, 5, 80
Mathematical knowledge for teaching, 4, 44, 46, 47, 113, 164, 220, 290, 293
Mathematical practice, 165, 206, 252, 255, 265, 268
Mathematicians, 4–7, 9–15, 17, 19, 28, 30, 46–48, 67, 85, 87, 94, 102, 104–107, 113, 161, 169, 175–180, 228, 233, 234, 243, 247, 255, 259–273, 278–285, 288–299
Mathematicians in k-12 education, 47, 278, 280, 282–285
Mathematics, 3–7, 9, 19–38, 43–88, 93, 118, 150, 173, 219–230, 233, 245–255, 259, 277–285, 287–299
Mathematics curriculum, 4, 6, 10, 13–16, 28, 55, 94, 95, 114
Mathematics education, 3–6, 9–17, 19, 21, 22, 44, 50, 54, 64, 67, 75, 94, 100, 119, 120, 122, 161, 174–176, 179, 219–230, 233–235, 248, 262, 264, 268, 269, 273, 278, 280, 281, 284, 287–290, 292, 297
Mathematics education of teachers, 15–17, 67, 94, 119, 161, 179, 220–224, 228–230, 234, 268
Mathematics educators, 4–6, 17, 28, 175–180, 220, 223, 228, 233, 240, 272
Mathematics matters, 3–7, 17
Mathematics matters in education, 3–7
Math Science Partnership (MSP), 261–263, 268, 269
Math talk, 206
Measurement, 6, 17, 25, 26, 28–30, 37, 50, 72, 94, 98, 102–112, 119, 200–206, 210, 212–217, 221–225, 249
Model method, 122, 200
MSP. *See* Math Science Partnership (MSP)
Multiplication, 20–22, 26, 27, 50, 63–65, 74, 77, 82, 83, 87, 88, 98, 104, 118, 119, 121–123, 125–127, 130, 132–136, 138–145, 160, 186, 187, 201–203, 214, 217, 242, 270

N
Number, 4, 11, 20, 44, 94, 118, 150, 179, 186, 210, 219, 236, 247, 261, 289
Number line, 6, 17, 28–30, 52, 53, 64, 76, 77, 81, 104, 119, 150, 152–159, 162–168, 200, 203–205, 209–217, 249

O
Ordered algebraic structures, 94

P
Pedagogical content knowledge (PCK), 46, 80–84
Place value, 14, 16, 20–22, 30, 37, 52, 152, 158, 193–202, 206, 214
Planning, 6, 12, 151, 152, 154, 158, 160–169
Possibilities for professional recognition, 279, 280
Preservice elementary teachers, 12, 15–17
Productive mathematical connections, 164, 168
Proportional relationship, 6, 23, 94, 101, 105, 112, 117–146, 250

Q
Quantity, 6, 22, 23, 25–27, 67, 68, 97, 98, 102, 104, 105, 110, 112, 118–121, 127, 132, 186, 188, 209, 210, 212–216, 249, 250

R
Ratio and proportion, 6, 93–114, 118
Real number line, 209–217
Reasoning, 6, 29, 43, 45, 47, 49–52, 55, 57, 60, 62–64, 66, 69–72, 77–79, 82–84, 87, 88, 94, 105, 109, 111, 112, 118–120, 122–135, 138–146, 152, 158, 159, 213, 248, 250, 252, 254, 263, 268, 273
Rectangles, 31–33, 35, 64, 74, 82, 143, 186, 201, 221–224, 251, 252
Rectangular prisms, 224
Rewarding mathematicians, 272–273, 282, 297
Rule of three, 93–101

S
Scholarly activities, 291
Sense-making, 46, 185, 206
Squares, 23, 31–33, 35, 77, 98, 105, 108, 111, 186, 200–202, 204, 221, 222, 225, 251, 252
Strip diagram, 117–146
Subtraction, 17, 28–30, 53, 77, 87, 98, 106, 152, 157, 159, 161, 162, 165, 167, 186–201, 204, 206, 212, 219, 227, 228
Supporting mathematicians, 5, 7, 259–273
Symmetry, 6, 10, 31–38, 72, 238–240, 253

T
Tape diagram, 119, 146, 249
Teacher education, 3, 5, 10, 13, 80, 84, 159, 168, 169, 220, 281, 289, 293
Teacher preparation in mathematics, 5, 9, 10, 13, 16, 25, 289
Tenure and promotion guidelines, 295
Textbook school mathematics (TSM), 5, 46, 51, 53–56, 60–75, 77–87, 230, 255

U
Units, 15, 20–27, 29, 30, 37, 50, 72, 93, 96, 98, 99, 102, 104–107, 110, 112, 119–122, 127, 129–131, 136–137, 142, 143, 160, 191, 192, 195, 198, 200–202, 204, 205, 211, 214–217, 222, 249, 261, 281

V
Variable parts, 117–146
Variables, 44, 54, 63, 65, 67, 68, 73, 81, 87, 88, 101, 105, 117–120, 126, 144, 145, 225, 234, 240, 241, 253
Visual models, 6, 15, 185, 186, 195, 196, 198, 200, 206
Volumes, 3, 5, 11, 13, 26, 87, 96, 102, 105, 213, 214, 223–225, 234, 240, 253, 278, 282, 283, 289, 297

W
Word problems, 62, 85, 146, 187, 188, 200, 235

Y
Yale Teachers Institute (YTI), 14–16, 30