Chapter 9
Accurate Pricing of Swaptions via Lower Bound

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Abstract We propose a new lower bound for pricing European-style swaptions for a wide class of interest rate models. This method is applicable whenever the joint characteristic function of the state variables is either known in closed form or can be obtained numerically via some efficient procedure. Our lower bound involves the computation of a one dimensional Fourier transform independently of the underlying swap length. Finally the bound can be used as a control variate to reduce the confidence interval in the Monte Carlo simulation. We test our bound on different affine models, also allowing for jumps. The lower bound is found to be accurate and computationally efficient.

Keywords Pricing • Swaptions • Characteristic function • Fourier transform • Lower bound

9.1   Introduction

Libor based derivatives (swaps, caps, swaptions) are the most liquid derivatives traded in financial markets. In particular a European swaption is a contract that gives the right to its owner to enter into an underlying interest rate swap, i.e. it is an European option on a swap rate. It can be equivalently interpreted as an option on a portfolio of zero coupon bonds.

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Let t be the current date, T be the option expiration date, $T_1, \ldots, T_n$ be the underlying swap payment dates (by construction $t < T < T_1 < \ldots < T_n$) and R the fixed rate of the swap. The payoff of a receiver swaption is

$$payoff = \left(\sum_{h=1}^{n} w_h P(T, T_h) - 1 \right)^+,$$

where $w_h = R \cdot (T_h - T_{h-1})$ for $h = 1, \ldots, n-1$ and $w_n = R \cdot (T_n - T_{n-1}) + 1$, $P(T, T_h)$ is the price at time T of a zero coupon bond expiring at time T_h. The no-arbitrage fair price at time t is the discounted risk neutral expected value of the payoff

$$C(t, T, \{T_h\}_{h=1}^{n}, R) = \mathbb{E}_t \left[e^{-\int_t^T r(\mathbf{X}(s))ds} \left(\sum_{h=1}^{n} w_h P(T, T_h) - 1 \right)^+ \right], \qquad (9.1)$$

where $r(\mathbf{X}(s))$ is the short rate at time s, and $\mathbf{X}(s)$ denotes the state vector at time s of a multi-factor stochastic model.

Due to their importance and popularity, swaption market quotations are often used for calibration of interest rate models. However calibration procedure involves the pricing of a large number of swaptions (different option maturity, swap tenors and strikes), so the availability of an efficient pricing algorithm is required. Moreover Basel III accords introduced the Credit Value Adjustment (CVA) charge for over the counter contracts.[1] It is interesting to note that for the most simple and popular kind of interest rate derivative, i.e. interest rate swap, the (unilateral) C.V.A. can be estimated as a portfolio of forward start European swaptions.[2]

Since a closed-form formula of swaption price does not exist for many popular interest rate models, then several approximated pricing method have been developed in literature specially for affine interest rate models. The most important are those of Munk (1999), Collin-Dufresne and Goldstein (2002), Singleton and Umantsev (2002), and Schrager and Pelsser (2006).

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[1]CVA of a contract price C(t) is the risk neutral expectation of the loss

$$CVA(t) := LDG \int_0^T \mathbb{E}[e^{-\int_0^t r(s)ds} max(C(t), 0)] \, d\mathbb{P}_D(t),$$

where LGD is the Loss Given Default and $\mathbb{P}_D(t)$ is the default probability in the interval $(t, t + dt)$. Risk adjusted price is: $C(t) - CVA(t)$.

[2]C.V.A. of an interest rate swap with payment dates $T_1, \ldots, T_n$ and fixed rate R, can be approximated by the following portfolio of swaptions:

$$CVA(t) \simeq LGD \sum_{i=1}^{n} \mathbb{P}_D(T_{i-1}, T_i) \cdot SWO(t, \{T_h\}_{i-1}^{n}, R).$$

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Munk approximates the price of an option on a coupon bond by a multiple of the price of an option on a zero-coupon bond with time to maturity equal to the stochastic duration of the coupon bond. This method is fast but not very accurate for out of the money options.

The method of Collin-Dufresne and Goldstein is based on a Edgeworth expansion of the density of the swap rate and requires the calculation of the moments of the coupon bond. This procedure can be very time consuming.

Singleton and Umantsev (2002) introduce the idea of approximating the exercise region in the space of the state variables. This method has the advantage of producing precise results for a wide range of strikes (in particular for out of the money swaptions); however, it does not admit a simple extension to general affine interest rate models, because it requires the knowledge in closed form of the joint probability density function of the state variables. Moreover the Singleton and Umantsev method requires the calculus of as many Fourier inversion as the number of payment dates of the underlying swap contract. Hence the run time algorithm increases with the swap length.

Similarly to Singleton and Umantsev (2002), we propose a lower bound which is based on an approximation of the exercise region via an event set defined through a function of the model factors

$$\mathcal{G} = \{\omega \in \Omega : g(\mathbf{X}(\omega), T) \geq k\}, \tag{9.2}$$

where $\mathbf{X}$ is the vector of the model factors, Ω is the state space and g is a suitably chosen function approximating the exercise boundary.

Our pricing formula consists in the valuation of the option on the approximate exercise region and requires a single Fourier transform, performed with respect to the parameter k in formula (9.2).

The approximation we propose has several advantages. First of all it is a lower bound, so the direction of the error is known a priori, further it is very general as it can be applied to a wide class of models, provided that the characteristic function of the state variates is known (explicitly or numerically). It involves the computation of only one Fourier inversion, independently of the number of cash flows of the underlying swap. Finally, it can be used as a control variate to improve the accuracy of the Monte Carlo simulation method.

The paper is organized as follows. Section 9.2 introduces a general formula for lower bound on swaption prices, based on an approximation of the exercise region. Then we apply the general lower bound formula to the case of affine interest rate models and we find an efficient algorithm to calculate analytically the approximated swaption price. Section 9.3 describes the approximate exercise set defined by the logarithm of the ZCBs portfolio geometric mean. Section 9.4 shows the results of numerical tests. Conclusive remarks are presented in last section.

9.2 A Lower Bound on Swaption Prices

In this section, we derive a general formula for lower bound on swaption prices. The price formula (9.1), once a change of measure to the T-forward measure is used, becomes

$$
C(t, T, \{T_h\}_{h=1}^n, R) = P(t, T) \cdot \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h P(T, T_h) - 1 \right)^+ \right]
$$

$$
= P(t, T) \cdot \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h P(T, T_h) - 1 \right) I(\mathcal{A}) \right], \qquad (9.3)
$$

where I is the indicator function and $\mathcal{A}$ is the exercise region seen as a subset of the space events Ω

$$
\mathcal{A} = \{ \omega \in \Omega : \sum_{h=1}^n w_h P(T, T_h) \geq 1 \}.
$$

Indeed, we observe that for any event set $\mathcal{G} \subset \Omega$

$$
\mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h P(T, T_h) - 1 \right)^+ \right] \geq \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h P(T, T_h) - 1 \right)^+ I(\mathcal{G}) \right]
$$

$$
\geq \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h P(T, T_h) - 1 \right) I(\mathcal{G}) \right].
$$

Then by discounting we obtain

$$
C(t, T, \{T_h\}_{h=1}^n, R) \geq LB(\mathcal{G}) := P(t, T) \cdot \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h P(T, T_h) - 1 \right) I(\mathcal{G}) \right], \quad (9.4)
$$

i.e. $LB(\mathcal{G})$ is a lower bound to the swaption price for all possible sets $\mathcal{G}$.

9.2.1 Affine Models

For affine interest rate models the price at T of a zero coupon bond with expiration date T_h can be written as the exponential of a linear combination of the state variables

$$
P(T, T_h) = e^{\sum_{j=1}^d b_{h,j} X_j(T) + a_h} = e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h}, \qquad (9.5)
$$

where $a_h = A(T_h - T)$ and $\mathbf{b}_h = \mathbf{B}(T_h - T)$ are functions of the payment date T_h and are typical of each model. We know from Duffie and Kan (1996) and Duffie et al. (2000) that under certain regularity conditions, the functions $A(\tau)$ and $\mathbf{B}(\tau)$ are the solution of a system of $d + 1$ ordinary differential equations that are completely determined by the specification of the risk-neutral dynamics of the short rate. These equations can be solved through numerical integration starting from the initial conditions $A(0) = 0, \mathbf{B}(0) = \underline{0}$ and the solutions are known in closed form for most common models.

Moreover from Duffie et al. (2000), we know that for affine models the risk neutral expected value of an exponential payoff has the form

$$\mathbb{E}_t\left[e^{-\int_t^T r(\mathbf{X}(u))du}e^{i\boldsymbol{\lambda}^\top \mathbf{X}(s)}\right] = e^{\tilde{A}(s-t,T-s,\boldsymbol{\lambda})+\tilde{\mathbf{B}}(s-t,T-s,\boldsymbol{\lambda})^\top \mathbf{X}(t)},$$

where $\mathbf{X}$ and $\boldsymbol{\lambda}$ are in $\mathbb{R}^d$ and the functions $\tilde{A}(\tau, T-s, \boldsymbol{\lambda})$ and $\tilde{\mathbf{B}}(\tau, T-s, \boldsymbol{\lambda})$ are solutions of the same ODE system of the zero coupon bond case, but with different initial conditions ($\tilde{A}(0, T-s, \boldsymbol{\lambda}) = A(T-s)$ and $\tilde{\mathbf{B}}(0, T-s, \boldsymbol{\lambda}) = i\boldsymbol{\lambda} + \mathbf{B}(T-s)$). Then the T-forward characteristic function of the model factors $\mathbf{X}$ can be obtained by

$$\mathbb{E}_t^T\left[e^{i\boldsymbol{\lambda}^\top \mathbf{X}(s)}\right] = \frac{1}{P(t,T)}\mathbb{E}_t\left[e^{-\int_t^T r(\mathbf{X}(u))du}e^{i\boldsymbol{\lambda}^\top \mathbf{X}(s)}\right]$$

$$= e^{\tilde{A}(s-t,T-s,\boldsymbol{\lambda})-A(T-t)+(\tilde{\mathbf{B}}(s-t,T-t,\boldsymbol{\lambda})^\top -\mathbf{B}(T-t)^\top)\mathbf{X}(t)}.$$

Since we are interested in the case $s = T$ (forward measure at expiry date of the option), then we define the function

$$\Phi(\boldsymbol{\lambda}) = \mathbb{E}_t^T\left[e^{i\boldsymbol{\lambda}^\top \mathbf{X}(T)}\right] = e^{\tilde{A}(T-t,\boldsymbol{\lambda})-A(T-t)+(\tilde{\mathbf{B}}(T-t,\boldsymbol{\lambda})^\top -\mathbf{B}(T-t)^\top)\mathbf{X}(t)}, \quad (9.6)$$

where $\tilde{A}(\tau, \boldsymbol{\lambda}) = \tilde{A}(\tau, 0, \boldsymbol{\lambda})$ and $\tilde{\mathbf{B}}(\tau, \boldsymbol{\lambda}) = \tilde{\mathbf{B}}(\tau, 0, \boldsymbol{\lambda})$.

We define the set $\mathcal{G}$ using a linear function of the state variates

$$\mathcal{G} = \{\omega \in \Omega : g(\mathbf{X}(T)) \geq k\} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) + \alpha \geq k\},$$

where $\boldsymbol{\beta}$ is a constant vector of length d, $\alpha \in \mathbb{R}$ and k is a free parameter. k can be chosen such that it optimizes the value of the lower bound.

Proposition 1 *The lower bound to the European swaption price, for affine interest rate models, is given by the following formula*

$$\widehat{LB}(t, T, \{T_h\}_{h=1}^n, R) = \max_{k \in \mathbb{R}} LB(k; t, T, \{T_h\}_{h=1}^n, R), \quad (9.7)$$

where

$$LB(k; t, T, \{T_h\}_{h=1}^n, R) = P(t, T) \frac{e^{-\delta k}}{\pi} \int_0^{+\infty} e^{-i\gamma k} \psi_\delta(\gamma) d\gamma, \qquad (9.8)$$

and

$$\psi_\delta(\gamma) = \left(\sum_{h=1}^n w_h e^{a_h} \Phi\left(-i\mathbf{b}_h + (\gamma - i\delta)\boldsymbol{\beta}\right) - \Phi\left((\gamma - i\delta)\boldsymbol{\beta}\right) \right) \frac{e^{(i\gamma+\delta)\alpha}}{i\gamma + \delta},$$

$$(9.9)$$

where δ is a positive constant.

Proof See Appendix A.1.

9.3 The Geometric Average Approximate Exercise Region

The approximate exercise set is defined through the logarithm of the geometric average of the portfolio of zero coupon bonds

$$\mathcal{G} = \{\omega \in \Omega : g(\mathbf{X}(T)) \geq k\},$$

$$G(\mathbf{X}(T)) = \prod_{h=1}^n P(T, T_h)^{w_h},$$

$$g(\mathbf{X}(T)) = \ln\left(G(\mathbf{X}(T))\right) = \sum_{h=1}^n w_h \ln(P(T, T_h)).$$

In particular $\mathcal{G}$ and $g(\mathbf{X})$ are given by

$$\mathcal{G} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) + \alpha \geq k\},$$

where $P(T, T_h) = e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h}$, $\boldsymbol{\beta} = \sum_{h=1}^n w_h \mathbf{b}_h$ and $\alpha = \sum_{h=1}^n w_h a_h$.

Since we don't know the optimum value of the parameter k, then the pricing method requires the maximization of the lower bound, $LB(k; t, T, \{T_h\}_{h=1}^n, R)$, seen as a function of k.

The optimization can be accelerated looking for a good starting point. We suggest the following

$$\tilde{k} = \log\left(\frac{1}{\sum_{h=1}^n w_h} \right) = -\log\left(\sum_{h=1}^n R(T_h - T_{h-1}) \right).$$

According to this choice $\mathcal{G}_{\tilde{k}} = \{\omega \in \Omega : g(\mathbf{X}(T)) \geq \tilde{k}\}$ is the greatest possible subset of the true exercise region, $\mathcal{A}$.

In fact normalizing the weights, the expression of the true exercise region can be rewritten as

$$\mathcal{A} = \{\omega \in \Omega : \sum_{h=1}^{n} w_h P(T, T_h) \geq 1\} = \{\omega \in \Omega : \sum_{h=1}^{n} \tilde{w}_h P(T, T_h) \geq e^{\tilde{k}}\}$$

$$= \{\omega \in \Omega : A(\mathbf{X}) \geq e^{\tilde{k}}\},$$

where $A(\mathbf{X})$ is the arithmetic mean of the ZCBs portfolio, $\tilde{w}_h = \frac{w_h}{\sum_{h=1}^{n} w_h}$ and so $\sum_{h=1}^{n} \tilde{w}_h = 1$.

By the arithmetic-geometric inequality we know that $A(\mathbf{X}) \geq G(\mathbf{X}) \ \forall \mathbf{X}$, then $\forall k > \tilde{k}$

$$\mathcal{A} \supseteq \mathcal{G}_{\tilde{k}} \supseteq \mathcal{G}_k.$$

Instead if $k < \tilde{k}$ then it is no more guaranteed that $\mathcal{G}_k$ is a subset of the true exercise region.

9.4 Models and Numerical Results

This section presents the examined models and discuss the numerical results.

9.4.1 Affine Gaussian Models

Affine Gaussian models assign the following stochastic differential equation (S.D.E.) to the state variable $\mathbf{X}$

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} - \mathbf{X}(t)) \, dt + \Sigma \, d\mathbf{W}(t),$$

$$\mathbf{X}(0) = \mathbf{x}_0,$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, K is a $d \times d$ diagonal matrix and Σ is a $d \times d$ triangular matrix. The short rate is obtained as a linear combination of the state vector $\mathbf{X}$; it is always possible to rescale the components $X_i(t)$ and assume that

$$r(t) = \phi + \sum_{i=1}^{d} X_i(t)$$

without loss of generality ($\phi \in \mathbb{R}$).

The T-forward characteristic function of $\mathbf{X}$ is

$$\Phi(\boldsymbol{\lambda}) = \mathbb{E}_t^T \left[e^{i\boldsymbol{\lambda}^\top \mathbf{X}(T)} \right] = e^{i\boldsymbol{\lambda}^\top \mu(t,T) - \frac{1}{2}\boldsymbol{\lambda}^\top V(t,T)\boldsymbol{\lambda}},$$

where μ is the T-forward expected value and V is the covariance matrix (assuming K is diagonal):

$$\mu(t,s) = \mathbb{E}_t^T[\mathbf{X}(s)] = \mathbb{E}_t[\mathbf{X}(s)] - (I_{n\times n} - e^{-K(s-t)})K^{-1}\Sigma\Sigma^\top(K^{-1})^\top$$

$$\mathbf{g} - Ve^{-K^\top(T-s)}(K^{-1})^\top\mathbf{g}$$

$$\mathbb{E}_t[\mathbf{X}(s)] = e^{-K(s-t)}(X(t) - \boldsymbol{\theta}) + \boldsymbol{\theta}$$

$$V_{ij}(t,s) = (\Sigma\Sigma^\top)_{ij} \left(\frac{1 - e^{-(K_{ii}+K_{jj})(s-t)}}{K_{ii} + K_{jj}} \right).$$

where $\mathbf{g} = [1, 1, 1, \ldots, 1]^\top$ is a column vector of length d.

For this type of process, the lower bound can be calculated analytically

$$LB(k; t, T, \{T_h\}_{h=1}^n, R) = P(t,T) \left(\sum_{h=1}^n w_h e^{a_h + \mathbf{b}_h^\top \mu + \frac{1}{2}V_h + \frac{1}{2}d_h^2} N(d_h - d) - N(-d) \right),$$

(9.10)

where

$$d = \frac{k - \boldsymbol{\beta}^\top \mu - \alpha}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}},$$

$$d_h = \mathbf{b}_h^\top \mathbf{v},$$

$$V_h = \mathbf{b}_h^\top (V - \mathbf{v}\mathbf{v}^\top)\mathbf{b}_h,$$

$$\mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}},$$

and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$ is the cumulative distribution function of a standard normal.

See details in Appendix A.2.

9.4.2 Multi-factor Cox-Ingersoll-Ross (CIR) Model

The state vector of the model evolves according to the following system of SDE

$$dX_i(t) = a_i(\theta_i - X_i(t))dt + \sigma_i \sqrt{X_i(t)}dW^i(t),$$

$$\mathbf{X}(0) = \mathbf{x}_0,$$

where $i = 1, \ldots, d$, $W^i(t)$ are independent standard Brownian motions, a_i, θ_i and σ_i are positive constants. The short rate is obtained by

$$r(t) = \phi + \sum_{i=1}^{d} X_i(t),$$

where $\phi \in \mathbb{R}$.

Using the results in Collin-Dufresne and Goldstein (2002), we can calculate the zero coupon bond price and the T-forward characteristic function. In particular we deduce the functions $A(\tau)$, $\mathbf{B}(\tau)$, $\tilde{A}(\tau, \boldsymbol{\lambda})$ and $\tilde{\mathbf{B}}(\tau, \boldsymbol{\lambda})$ in formula (9.6),

$$A(\tau) = -\phi \, \tau + \sum_{j=1}^{d} \left[\frac{2a_j\theta_i}{h_j - a_j}\tau - \frac{2a_j\theta_j}{\sigma_j^2} \ln\left(\frac{(a_j + h_j)(e^{h_j\tau} - 1) + 2h_j}{2h_j} \right) \right],$$

$$B_j(\tau) = \frac{-2\,(e^{h_j\tau} - 1)}{(a_j + h_j)(e^{h_j\tau} - 1) + 2h_j},$$

$$\tilde{A}(\tau, \boldsymbol{\lambda}) = -\phi \, \tau + \sum_{j=1}^{d} \left[\frac{2a_j\theta_i}{h_j - a_j}\tau - \frac{2a_j\theta_j}{\sigma_j^2} \ln\left(\frac{\sigma_j^2((i\lambda_j + \mu_j^+) - (i\lambda + \mu_j^-)e^{h_j\tau})}{2h_j} \right) \right],$$

$$\tilde{B}_j(\tau, \boldsymbol{\lambda}) = \frac{i\lambda_j(\mu_j^- - \mu_j^+ e^{h_j\tau}) + \frac{2(e^{h_j\tau} - 1)}{\sigma_j^2}}{i\lambda_j(e^{h_j\tau} - 1) + (\mu_j^- e^{h_j\tau} - \mu_j^+)},$$

where $h_j = \sqrt{a_j^2 + 2\sigma_j^2}$ and $\mu_j^{\pm} = \frac{-a_j \pm h_j}{\sigma_j^2}$.

9.4.3   Gaussian Model with Double Exponential Jumps

The vector of model factors evolves according to the following S.D.E.

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} \, - \mathbf{X}(t))\, dt + \Sigma \, d\mathbf{W}(t) + d\mathbf{Z}^+(t) - d\mathbf{Z}^-(t),$$

$$\mathbf{X}(0) = \mathbf{x}_0,$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, K is a $d \times d$ diagonal matrix, Σ is a $d \times d$ triangular matrix and $\mathbf{Z}^{\pm}$ are pure jumps processes whose jumps have fixed probability distribution ν on $\mathbb{R}^d$ and constant intensity $\mu^{\pm}$. The short rate is obtained as a linear combination of the state vector $\mathbf{X}$. In particular $\mathbf{Z}^{\pm}$ are compound Poisson processes with jump size having exponential distribution

$$Z_l^{\pm} = \sum_{j=1}^{N^{\pm}(t)} Y_{j,l}^{\pm},$$

where $l = 1, \ldots, d$ is the factor index, $N^{\pm}(t)$ are Poisson process with intensity $\frac{\mu^{\pm}}{d}$ [3] and $Y_{j,l}^{\pm}$, for a fixed l, are independent identically distributed exponential random variables of mean parameters $m_l^{\pm}$:

$$Y_{j,l}^{\pm} \sim \nu(m_l^{\pm}) = \frac{1}{m_l^{\pm}} \exp\left(\frac{y}{m_l^{\pm}}\right).$$

Since $\mu^{\pm}$ do not depend on $\mathbf{X}$, we know from Duffie et al. (2000) that: (a) the functions $\mathbf{B}(\tau)$ and $\tilde{\mathbf{B}}(\tau, \boldsymbol{\lambda})$ in formula (9.6) are independent of jumps and they are calculated as in affine Gaussian model case; (b) the functions $A(\tau)$ and $\tilde{A}(\tau, \boldsymbol{\lambda})$ in formula (9.6) are sum of two components, diffusive and jump: $A(\tau) = A^D(\tau) + A^J(\tau)$ and $\tilde{A}(\tau, \boldsymbol{\lambda}) = \tilde{A}^D(\tau, \boldsymbol{\lambda}) + \tilde{A}^J(\tau, \boldsymbol{\lambda})$. The diffusive part can be obtained as in affine Gaussian model case.

Then we obtain that

$$\Phi(\boldsymbol{\lambda}) = \mathbb{E}_t^T\left[e^{i\boldsymbol{\lambda}^\top \mathbf{X}(T)}\right] = \Phi^D(\boldsymbol{\lambda})\, e^{\tilde{A}^J(T-t,\boldsymbol{\lambda}) - A^J(T-t)}, \tag{9.11}$$

where $\Phi^D(\boldsymbol{\lambda})$ is the T-forward characteristic function of affine Gaussian model. The jumps component is calculated using the characteristic function of the jumps size distribution ν

$$\theta^{\pm}(\mathbf{c}) = \int_{\mathbb{R}^d} e^{\mathbf{c}^\top \mathbf{y}} d\nu(\mathbf{y}) = \frac{1}{d}\sum_{j=1}^d \frac{1}{1 - m_j^{\pm}c_j} = \frac{1}{d}\sum_{j=1}^d \frac{\eta_j^{\pm}}{\eta_j^{\pm} - c_j},$$

$$\tilde{A}^J(T-t,\boldsymbol{\lambda}) = \mu^+ \int_t^T ds\, \left(\theta^+\left(\tilde{\mathbf{B}}(s,\boldsymbol{\lambda})\right) - 1\right) + \mu^- \int_t^T ds\, \left(\theta^-\left(\tilde{\mathbf{B}}(s,\boldsymbol{\lambda})\right) - 1\right),$$

$$A^J(T-t) = \tilde{A}^J(T-t,\mathbf{0}),$$

where $c \in \mathbb{C}^d$ and $\eta_j^{\pm} = \frac{1}{m_j^{\pm}}$. The function $\tilde{A}^J(\tau, \boldsymbol{\lambda})$ is available in closed form

$$\tilde{A}^J(\tau, \boldsymbol{\lambda}) = \frac{\mu^+}{d} \sum_{j=1}^d \frac{-\tau}{1 + \eta_j^+ K_{jj}} + \frac{\eta_j^+}{1 + \eta_j^+ K_{jj}} \log\left(\frac{(1 + i\lambda_j K_{jj})e^{-K_{jj}\tau} - 1 - \eta_j^+ K_{jj}}{K_{jj}(i\lambda_j - \eta_j^+)}\right)$$

$$+ \frac{\mu^-}{d} \sum_{j=1}^d \frac{-\tau}{1 - \eta_j^- K_{jj}} - \frac{\eta_j^-}{1 - \eta_j^- K_{jj}} \log\left(\frac{(1 + i\lambda_j K_{jj})e^{-K_{jj}\tau} - 1 + \eta_j^- K_{jj}}{K_{jj}(i\lambda_j + \eta_j^-)}\right).$$

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[3] $N^{\pm}(t)$ represent the number of positive or negative jumps before the time t.

9.4.4 Balduzzi, Das, Foresi and Sundaram Model

In the model proposed in Balduzzi et al. (1996), the interest rate follows the same stochastic process as in CIR model, but the long-mean $\theta(t)$ and the variance $V(t)$ are stochastic, according to the following system of SDEs

$$\begin{cases} dr(t) = k\,(\theta(t) - r(t))\,dt + \sqrt{V(t)}\,dW(t) \\ d\theta(t) = \alpha\,(\beta - \theta(t))\,dt + \eta\,dZ(t) \\ dV(t) = a\,(b - V(t))\,dt + \phi\,\sqrt{V(t)}\,dY(t) \\ dW(t)dY(t) = \rho\,dt \\ r(0) = r_0,\ \theta(0) = \theta_0 \text{ and } V(0) = v_0 \end{cases}$$

k, α, β, η, a, b and ϕ are positive constants and $\rho \in [-1, 1]$. In order to align the notation with previous sections we denote $\mathbf{X}(t) = [r(t), \theta(t), V(t)]^\top$. In this model the characteristic function of the state vector $\mathbf{X}$ can not be obtained in closed form,[4] but can be calculated numerically, solving the following system of ordinary differential equations

$$\frac{d\tilde{A}(\tau, \boldsymbol{\lambda})}{d\tau} = -\alpha\beta\tilde{B}_2(\tau, \boldsymbol{\lambda}) + \frac{1}{2}\eta^2\tilde{B}_2(\tau, \boldsymbol{\lambda})^2 - ab\tilde{B}_3(\tau, \boldsymbol{\lambda}),$$

$$\tilde{B}_1(\tau, \boldsymbol{\lambda}) = \lambda_1 e^{-k\tau} - \frac{1 - e^{-k\tau}}{k},$$

$$\tilde{B}_2(\tau, \boldsymbol{\lambda}) = (k\lambda_1 + 1)\frac{e^{-k\tau} - e^{-\alpha\tau}}{\alpha - k} + \lambda_2 e^{-\alpha\tau} - \frac{1 - e^{-\alpha\tau}}{\alpha},$$

$$\frac{d\tilde{B}_3(\tau, \boldsymbol{\lambda})}{d\tau} = \frac{1}{2}\tilde{B}_1(\tau, \boldsymbol{\lambda})^2 + a\tilde{B}_3(\tau, \boldsymbol{\lambda}) + \frac{1}{2}\phi^2\tilde{B}_3(\tau, \boldsymbol{\lambda})^2 + \rho\phi\tilde{B}_1(\tau, \boldsymbol{\lambda})\tilde{B}_3(\tau, \boldsymbol{\lambda}).$$

The functions $A(\tau)$ and $\mathbf{B}(\tau)$ (i.e. the zero coupon bond price) can be obtained solving the previous system and setting $\boldsymbol{\lambda} = \underline{0}$.

9.4.5 Numerical Results

Apart from the Vasicek model for which a simple closed form solution is available, Monte Carlo is used as benchmark for the computation of the true swaption price. The 97.5% mean-centered Monte Carlo Confidence Interval[5] is used as measure of

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[4]A semi-analytical solution for functions $\tilde{A}(\tau, \boldsymbol{\lambda})$ and $\tilde{B}_3(\tau, \boldsymbol{\lambda})$ is available but it requires the evaluation of Kummer's functions of the first and second kind. Kummer's functions are not analytic but have series and integral representation. However we find that the numerical solution of the ODE system is much more efficient than the evaluation of the semi analytical form.

[5]Note that we use the quantile function of a Student's t distribution.

the accuracy of the Monte Carlo method. Simulation of the affine 3-factor Gaussian and 2-factor CIR models is implemented by sampling from the exact distribution. The Gaussian with jumps model and the BDFS model are simulated using an Euler-Maruyama scheme with a time step of 0.0005. The number of simulations is chosen depending on the complexity of the model and it is specified in each table caption. Antithetic variates technique is also used for the affine 3-factor Gaussian model and the BDFS model.

For the Vasicek model and the 3-factor Gaussian model, lower bound is obtained via the closed formula described in Sect. 9.4.1. For the 2-factor CIR model and the Gaussian with jumps model, the integrals involved in the lower bound are evaluated by a Gauss-Kronrod quadrature rule, using Matlab's built-in function **quadgk**. For the BDFS model the system of ordinary differential equations is solved numerically using the Matlab function **ode45** based on the Dormand–Prince method. Due to the complexity of the problem we adopt a Gauss-Legendre quadrature rule. The calculus of the lower bound with a geometric mean region $\mathcal{G}$ involves the optimization of the function $LB(k; t, T, \{T_h\}_{h=1}^n, R)$ with respect to the parameter k. The optimization is performed via Matlab functions **fminunc**.

Another important fact is that our lower bound formula is very suitable to be used as a control variate to reduce Monte Carlo error. The approximated formula is easy to implement in a Monte Carlo scheme and turns out to be very effective (see Caldana et al. (2014) for details).

Swaption prices for different tenor and maturities are reported in Tables 9.1, 9.2, 9.3, 9.4, and 9.5 with the relative overall computing time for each pricing method.

Moreover, Figs. 9.1, 9.2, 9.3, 9.4, and 9.5 compare graphically the relative error of the three proposed pricing methods.

9.4.5.1 Vasicek Model, Three-Factors Gaussian Model and Cox-Ingersoll and Ross Model

We verify the accuracy of our new lower bound using models and parameter values already examined in literature[6]

- Vasicek model: $K = 0.05$, $\theta = 0.05$, $\Sigma = 0.01$, $x_0 = 0.05$ and $\phi = 0$;

- 3-factors Gaussian model: $K = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $\theta = [0, 0, 0]^\top$, $\sigma = [0.01, 0.005, 0.002]^\top$, $\rho = \begin{bmatrix} 1 & -0.2 & -0.1 \\ -0.2 & 1 & 0.3 \\ -0.1 & 0.3 & 1 \end{bmatrix}$, $\Sigma = \text{diag}(\sigma) \cdot \text{chol}(\rho)$,[7] $x_0 = [0.01, 0.005, -0.02]$ and $\phi = 0.06$;

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[6]Schrager and Pelsser (2006) and Duffie and Singleton (1997) for the 2-factors C.I.R. model.

[7]diag(σ) means the diagonalization of the vector σ and chol(ρ) means the Cholesky decomposition of the correlation matrix ρ, where σ and ρ are the volatility vector and the correlation matrix, respectively, of the original paper.

Table 9.1 The three tables represent matrices of swaption prices at three different strikes for the Vasicek model. For each swaption we report on the first line the price in basis points and on the second line the error in basis points. The error of the lower bound is the difference from the exact Jamshidian method (Jam.)

Vasicek model						
Payer swaptions (ATMF)						
Option mat.	1		2		5	
Swap length	LB	Jam.	LB	Jam.	LB	Jam.
1	35.670 10^{-4}	35.670	46.836 10^{-4}	46.836	59.501 10^{-4}	59.501
2	67.953 10^{-4}	67.953	89.234 10^{-4}	89.234	113.412 10^{-4}	113.412
5	147.645 10^{-4}	147.645	193.957 10^{-4}	193.957	246.875 10^{-4}	246.875
10	238.273 10^{-4}	238.273	313.243 10^{-4}	313.243	399.674 10^{-4}	399.674
Payer swaptions (ITMF: $0.85 \times$ ATMF)						
Option mat.	1		2		5	
Swap length	LB	Jam.	LB	Jam.	LB	Jam.
1	80.591 10^{-4}	80.591	86.861 10^{-4}	86.861	91.405 10^{-4}	91.405
2	155.872 10^{-4}	155.872	167.452 10^{-4}	167.452	175.622 10^{-4}	175.622
5	353.282 10^{-4}	353.282	376.199 10^{-4}	376.199	391.032 10^{-4}	391.032
10	605.661 10^{-4}	605.661	637.301 10^{-4}	637.301	654.439 10^{-4}	654.439
Payer swaptions (OTMF: $1.15 \times$ ATMF)						
Option mat.	1		2		5	
Swap length	LB	Jam.	LB	Jam.	LB	Jam.
1	11.247 10^{-4}	11.247	21.169 10^{-4}	21.169	35.825 10^{-4}	35.825
2	20.781 10^{-4}	20.781	39.551 10^{-4}	39.551	67.525 10^{-4}	67.525
5	41.394 10^{-4}	41.394	81.330 10^{-4}	81.330	142.400 10^{-4}	142.400
10	58.744 10^{-4}	58.744	121.059 10^{-4}	121.059	220.008 10^{-4}	220.008

- 2-factors Cox-Ingersoll and Ross model: $\mathbf{a} = [0.5080, \ -0.0010]^{\top}$, $\boldsymbol{\theta} = [0.4005, \ -0.7740]^{\top}$, $\boldsymbol{\sigma} = [0.023, \ 0.019]^{\top}$, $x_0 = [0.374, \ 0.258]$ and $\phi = -0.58$.

  Numerical results for these models are shown in Tables 9.1, 9.2 and 9.3.

Table 9.2 The three tables represent matrices of swaption prices at three different strikes for the 3-factors Gaussian model. The bottom line of each table provides the overall computation time for the different pricing methods. For each swaption we report the price in basis points estimated with the Monte Carlo method, MC, the Lower Bound approximation (LB) and the Monte Carlo method with the control variable technique, MC (CV). Monte Carlo prices without and with control variable method are estimated using 10^7 and respectively, 10^5 simulations, antithetic variates method and the exact probability distribution of the state variables at the maturity date of the swaption. Below each Monte Carlo price, the confidence interval at 97.5% is reported in basis point

Three-factor Gaussian model									
Payer swaptions (ATMF)									
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	20.814	20.817	20.817	23.554	23.554	23.554	23.217	23.207	23.207
	0.010		10^{-4}	0.011		10^{-4}	0.011		10^{-4}
2	33.114	33.119	33.119	38.430	38.434	38.434	38.728	38.722	38.722
	0.015		10^{-4}	0.018		10^{-4}	0.018		10^{-4}
5	53.325	53.312	53.312	63.676	63.686	63.686	65.673	65.683	65.683
	0.025		10^{-4}	0.029		10^{-4}	0.030		10^{-4}
10	65.583	65.579	65.583	79.090	79.062	79.067	82.164	82.156	82.159
	0.030		0.001	0.036		0.001	0.037		10^{-4}
Overall	MC	LB	MC (CV)						
time (sec)	32.045	0.122	3.657						
Payer swaptions (ITMF: 0.85 × ATMF)									
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	79.446	79.445	79.445	78.404	78.404	78.404	69.446	69.442	69.442
	0.003		10^{-4}	0.005		10^{-4}	0.005		10^{-4}
2	154.563	154.563	154.563	150.909	150.911	150.911	131.95	131.949	131.949
	0.003		10^{-4}	0.005		10^{-4}	0.007		10^{-4}
5	361.470	361.469	361.469	346.275	346.275	346.275	295.162	295.162	295.162
	0.001		10^{-4}	0.003		10^{-4}	0.006		10^{-4}
10	636.982	636.982	636.982	604.809	604.81	604.81	508.838	508.840	508.840
	0.001		10^{-4}	0.002		10^{-4}	0.003		10^{-4}
Overall	MC	LB	MC (CV)						
time (sec)	32.023	0.117	3.543						
Payer swaptions (OTMF: 1.15 × ATMF)									
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	1.571	1.570	1.570	2.823	2.824	2.824	3.798	3.794	3.794
	0.003		10^{-4}	0.005		10^{-4}	0.006		10^{-4}
2	1.065	1.065	1.065	2.611	2.612	2.612	4.324	4.322	4.322
	0.003		10^{-4}	0.006		10^{-4}	0.008		10^{-4}
5	0.150	0.150	0.150	0.904	0.905	0.905	2.569	2.569	2.57
	0.001		10^{-4}	0.004		10^{-4}	0.007		10^{-4}
10	0.003	0.003	0.003	0.076	0.076	0.076	0.517	0.516	0.517
	10^{-4}		10^{-4}	0.001		10^{-4}	0.003		10^{-4}
Overall	MC	LB	MC (CV)						
time (sec)	32.024	0.113	3.541						

Table 9.3 The three tables represent matrices of swaption prices at three different strikes for the 2-factor C.I.R. model. The bottom line of each table provides the overall computation time for the different pricing methods. For each swaption we report the price in basis points estimated with the Monte Carlo method, MC, the Lower Bound approximation (LB) and the Monte Carlo method with the control variable technique, MC (CV). Monte Carlo prices without and with control variable method are estimated using 10^7 and respectively, 10^5 simulations and the exact probability distribution of the state variables at the maturity date of the swaption. Below each Monte Carlo price, the confidence interval at 97.5% is reported in basis point

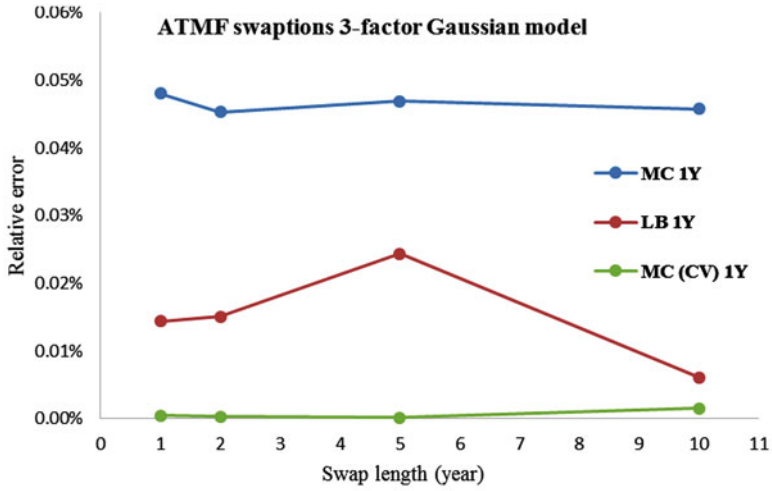| Two-factor Cox-Ingersoll-Ross model								
Payer swaptions (ATMF)								
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	48.490	48.466	48.466	59.386	59.361	59.361	67.006	66.970	66.970
	0.044		10^{-4}	0.054		10^{-4}	0.060		10^{-4}
2	85.908	85.871	85.871	106.938	106.890	106.890	123.894	123.830	123.830
	0.078		10^{-4}	0.098		10^{-4}	0.113		10^{-4}
5	169.474	169.427	169.428	216.993	216.879	216.881	261.486	261.366	261.368
	0.157		10^{-4}	0.202		10^{-4}	0.246		10^{-4}
10	265.862	265.779	265.818	345.184	344.949	344.992	423.059	422.857	422.887
	0.251		0.004	0.331		0.005	0.418		0.004
Overall	MC	LB	MC (CV)						
time (sec)	23.118	1.456	1.924						
Payer swaptions (ITMF: $0.85 \times$ ATMF)								
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	107.584	107.577	107.577	116.846	116.839	116.839	114.947	114.930	114.930
	0.025		10^{-4}	0.034		10^{-4}	0.043		10^{-4}
2	208.047	208.037	208.037	222.373	222.363	222.363	217.236	217.208	217.208
	0.040		10^{-4}	0.058		10^{-4}	0.078		10^{-4}
5	475.686	475.668	475.669	493.334	493.301	493.301	473.362	473.330	473.331
	0.065		10^{-4}	0.109		10^{-4}	0.165		10^{-4}
10	812.501	812.470	812.482	825.291	825.202	825.219	778.599	778.559	778.573
	0.092		0.002	0.168		0.003	0.275		0.002
Overall	MC	LB	MC (CV)						
time (sec)	23.121	1.199	1.667						
Payer swaptions (OTMF: $1.15 \times$ ATMF)								
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	16.004	15.973	15.973	24.476	24.446	24.446	34.601	34.546	34.546
	0.062		10^{-4}	0.072		10^{-4}	0.077		10^{-4}
2	23.777	23.724	23.724	40.022	39.964	39.964	61.943	61.838	61.838
	0.113		10^{-4}	0.134		10^{-4}	0.146		10^{-4}
5	33.668	33.565	33.567	68.868	68.740	68.742	124.627	124.396	124.399
	0.239		10^{-4}	0.288		0.001	0.323		0.001
10	42.602	42.425	42.459	99.260	99.004	99.045	196.625	196.230	196.266
	0.394		0.005	0.483		0.005	0.555		0.005
Overall	MC	LB	MC (CV)						
time (sec)	23.121	1.314	1.782						

Table 9.4 The three tables represent matrices of swaption prices at three different strikes for the 2-factor Gaussian model with Double Exponential Jumps. The bottom line of each table provides the overall computation time for the different pricing methods. For each swaption we report the price in basis points estimated with the Monte Carlo method, MC, the Lower Bound approximation (LB) and the Monte Carlo method with the control variable technique, MC (CV). Monte Carlo (MC) price is estimated using four millions simulations, an Euler scheme with a time step equal to 0.0025 and the antithetic variates technique. Monte Carlo is also performed using 10^5 simulations and control variates method. Below each Monte Carlo price, the confidence interval at 97.5% is reported in basis point

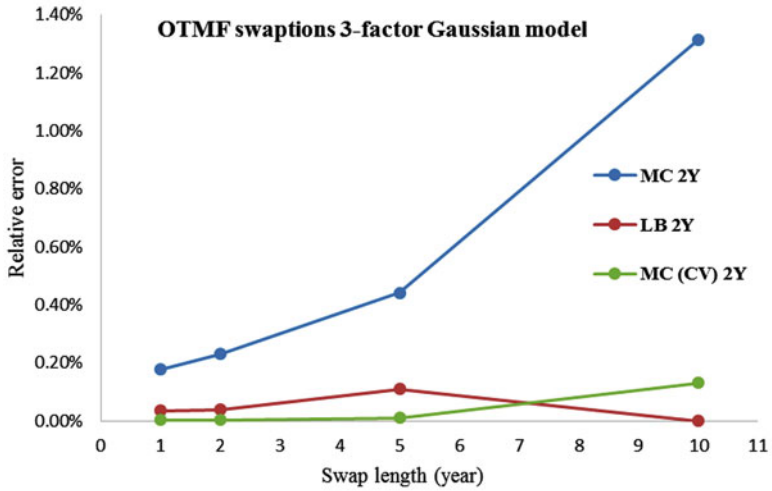| Two-factor Gaussian model with Double Exponential Jumps								
Payer swaptions (ATMF)								
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	44.013	44.046	44.046	52.773	52.766	52.766	59.145	59.204	59.204
	0.063		10^{-4}	0.074		10^{-4}	0.082		10^{-4}
2	82.968	83.042	83.042	102.255	102.242	102.242	119.461	119.629	119.629
	0.117		10^{-4}	0.143		10^{-4}	0.165		10^{-4}
5	96.950	97.030	97.031	120.664	120.652	120.652	142.807	143.021	143.021
	0.137		10^{-4}	0.169		10^{-4}	0.196		10^{-4}
10	97.542	97.623	97.623	121.396	121.384	121.384	143.643	143.858	143.858
	0.138		10^{-4}	0.170		10^{-4}	0.198		10^{-4}
Overall	MC	LB	MC (CV)						
time (sec)	5.396×10^3	2.643	84.485						
Payer swaptions (ITMF: 0.85 × ATMF)								
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	63.821	63.878	63.878	68.253	68.267	68.267	69.103	69.164	69.164
	0.074		10^{-4}	0.084		10^{-4}	0.088		10^{-4}
2	138.588	138.729	138.729	148.623	148.667	148.667	153.982	154.172	154.172
	0.146		10^{-4}	0.169		10^{-4}	0.184		10^{-4}
5	349.841	350.150	350.150	347.712	347.888	347.888	340.988	341.320	341.320
	0.214		10^{-4}	0.250		10^{-4}	0.275		10^{-4}
10	904.579	904.933	904.934	887.761	887.995	887.995	856.180	856.559	856.559
	0.215		10^{-4}	0.264		10^{-4}	0.305		10^{-4}
Overall	MC	LB	MC (CV)						
time (sec)	5.396×10^3	2.643	84.057						
Payer swaptions (OTMF: 1.15 × ATMF)								
Opt. mat.	1			2			5		
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	28.669	28.679	28.679	39.757	39.729	39.729	50.158	50.215	50.215
	0.051		10^{-4}	0.065		10^{-4}	0.076		10^{-4}
2	44.305	44.319	44.319	66.449	66.396	66.396	90.386	90.535	90.535
	0.087		10^{-4}	0.117		10^{-4}	0.144		10^{-4}
5	9.461	9.449	9.449	22.651	22.573	22.573	41.176	41.275	41.275
	0.041		10^{-4}	0.072		10^{-4}	0.105		10^{-4}
10	0.051	0.048	0.048	0.299	0.304	0.304	1.563	1.564	1.564
	0.005		10^{-4}	0.009		10^{-4}	0.019		10^{-4}
Overall	MC	LB	MC (CV)						
time (sec)	5.396×10^3	2.643	84.070						

Table 9.5 The three tables represent matrices of swaption prices at three different strikes for the BDFS. The bottom line of each table provides the overall computation time for the different pricing methods. For each swaption we report the price in basis points estimated with the Monte Carlo method, MC, the Lower Bound approximation (LB) and the Monte Carlo method with the control variable technique, MC (CV). Monte Carlo (MC) price is estimated using four millions simulations, an Euler scheme with a time step equal to 0.0005 and the antithetic variates technique. Monte Carlo is also performed using 10^5 simulations and control variates method. Below each Monte Carlo price, the confidence interval at 97.5% is reported in basis point

| Balduzzi, Das, Foresi and Sundaram model								
Payer swaptions (ATMF)								
Opt. mat.	1			2			5	
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	85.08	85.00	85.00	102.24	102.17	102.16	97.45	97.45	97.45
	0.06		10^{-3}	0.07		10^{-3}	0.06		10^{-3}
2	148.39	148.27	148.26	177.40	177.28	177.27	167.37	167.38	167.38
	0.10		10^{-3}	0.11		10^{-3}	0.09		10^{-3}
5	250.22	250.01	250.01	296.89	296.68	296.68	276.29	276.28	276.28
	0.17		10^{-3}	0.18		10^{-3}	0.15		10^{-3}
10	290.96	290.71	290.72	345.34	345.10	345.10	321.69	321.68	321.68
	0.19		10^{-3}	0.21		10^{-3}	0.17		10^{-3}
Overall	MC	LB	MC (CV)					
time (sec)	12.256×10^3	894	1.139×10^3					
Payer swaptions (ITMF: 0.85 × ATMF)								
Opt. mat.	1			2			5	
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	122.53	122.44	122.44	148.64	148.57	148.57	152.26	152.25	152.25
	0.05		10^{-3}	0.06		10^{-3}	0.04		10^{-3}
2	234.64	234.51	234.50	277.64	277.52	277.51	277.32	277.31	277.31
	0.09		10^{-3}	0.1		10^{-3}	0.07		10^{-3}
5	521.94	521.72	521.72	578.32	578.11	578.11	543.71	543.68	543.68
	0.11		10^{-3}	0.12		10^{-3}	0.08	0	0
10	881.46	881.21	881.21	908.49	908.26	908.26	799.72	799.70	799.70
	0.07		10^{-3}	0.08		10^{-3}	0.07		10^{-3}
Overall	MC	LB	MC (CV)					
time (sec)	12.256×10^3	930	1.176×10^3					
Payer swaptions (OTMF: 1.15 × ATMF)								
Opt. mat.	1			2			5	
Swap length	MC	LB	MC (CV)	MC	LB	MC (CV)	MC	LB	MC (CV)
1	55.91	55.84	55.84	66.35	66.29	66.28	57.30	57.30	57.30
	0.06		10^{-3}	0.07		10^{-3}	0.06		10^{-3}
2	85.74	85.65	85.64	104.05	103.95	103.94	90.40	90.40	90.40
	0.09		10^{-3}	0.11		10^{-3}	0.09		10^{-3}
5	92.5	92.38	92.38	124.12	123.99	123.98	113.7	113.69	113.69
	0.13		10^{-3}	0.16		10^{-3}	0.13		10^{-3}
10	48.09	48.00	48.01	81.30	81.18	81.17	86.16	86.15	86.15
	0.10		10^{-3}	0.14		10^{-3}	0.13		10^{-3}
Overall	MC	LB	MC (CV)					
time (sec)	12.256×10^3	928	1.172×10^3					

Fig. 9.1 Three-factor Gaussian model: relative error in percentage. The error of the lower bound is the difference from Monte Carlo value. (**a**) ATMF-1 year maturity. (**b**) OTMF-2 years maturity
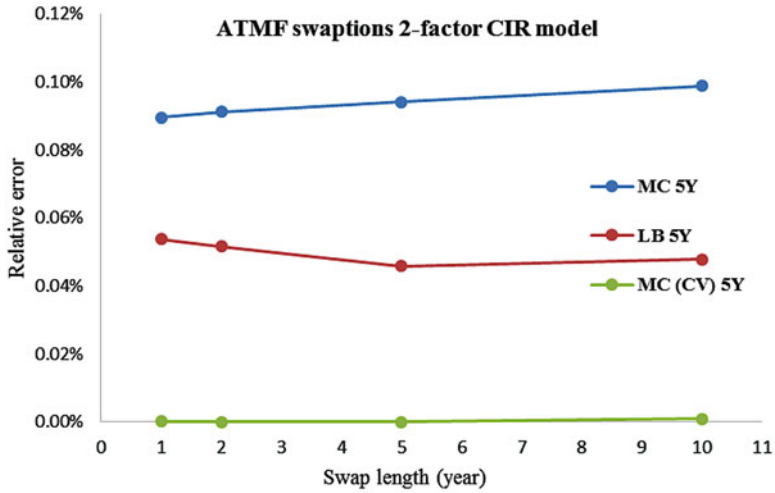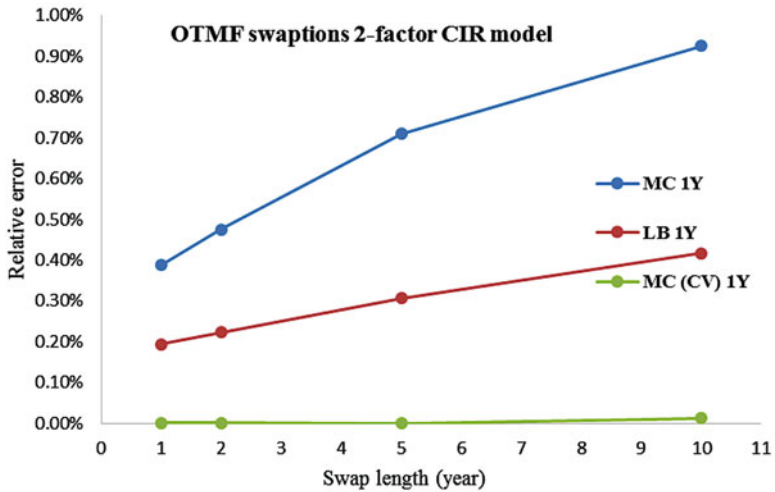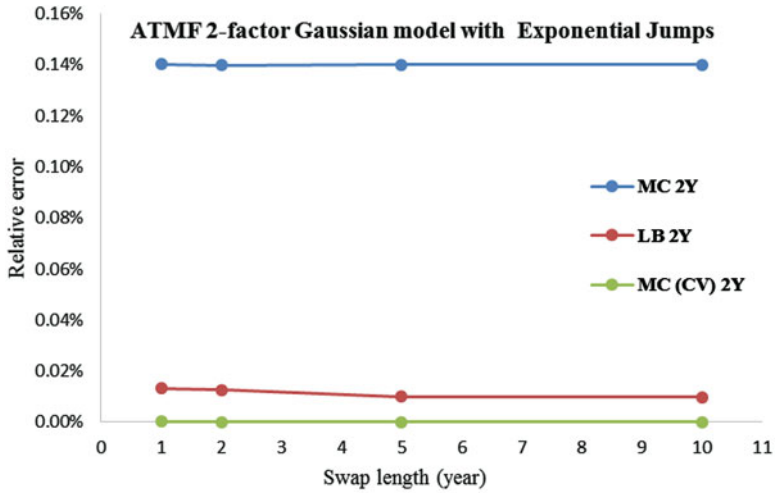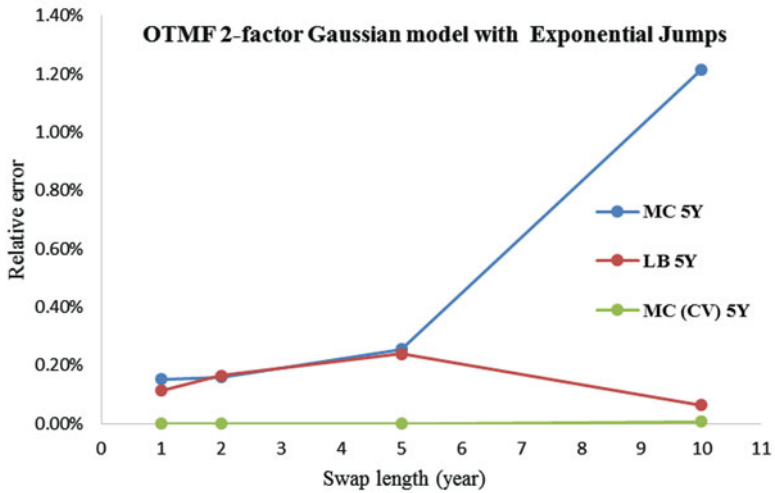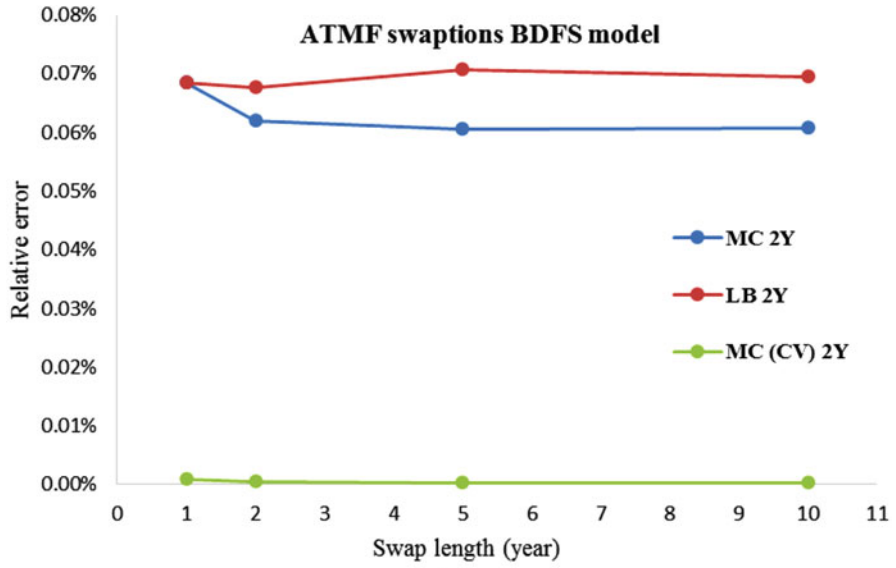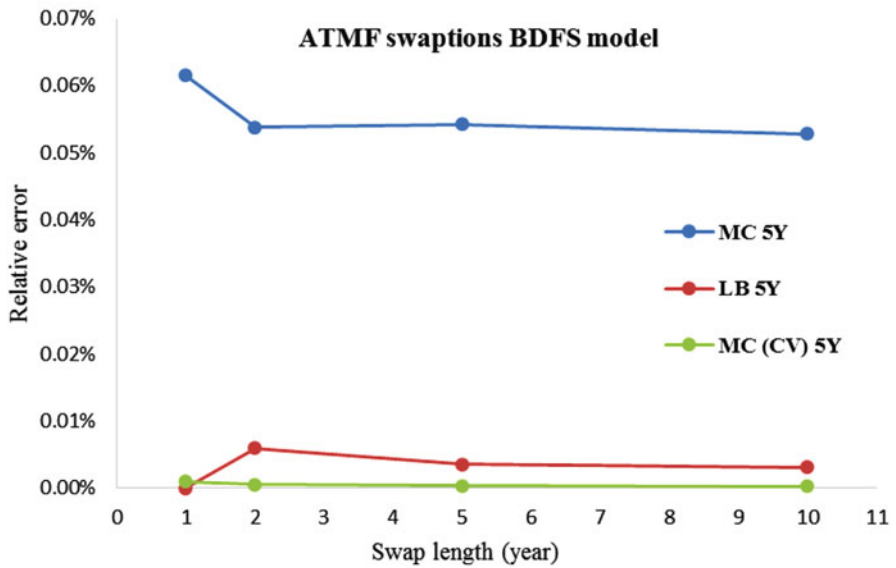
Fig. 9.2 Two-factor Cox-Ingersoll-Ross model: relative error in percentage. The error of the lower bound is the difference from Monte Carlo value. (**a**) ATMF-5 year maturity. (**b**) OTMF-1 years maturity
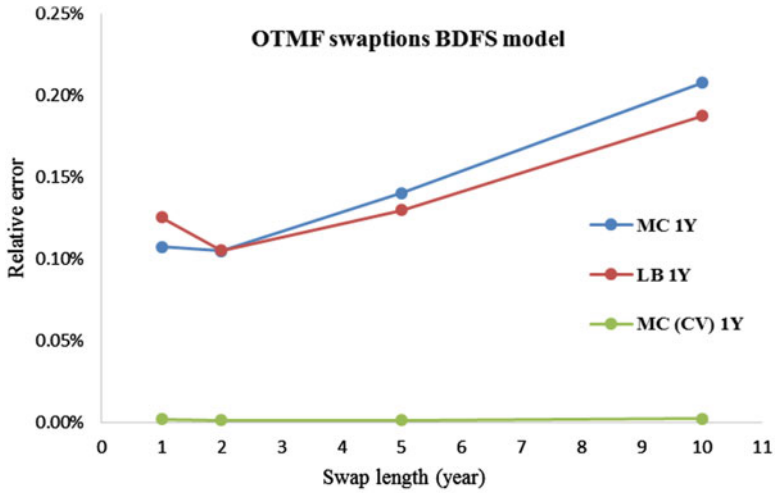
Fig. 9.3 Two-factor Gaussian model with Double Exponential Jumps: relative error in percentage. The error of the lower bound is the difference from Monte Carlo value. (**a**) ATMF-2 year maturity. (**b**) OTMF-5 years maturity
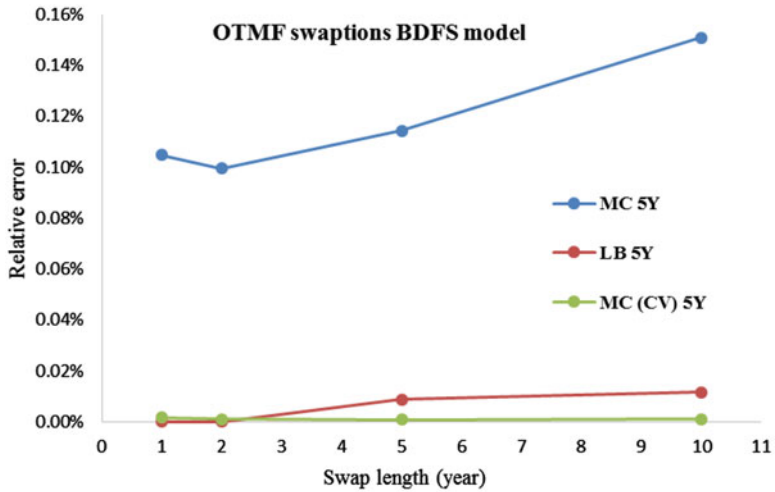
Fig. 9.4 Balduzzi, Das, Foresi and Sundaram model: relative error in percentage. The error of the lower bound is the difference from Monte Carlo value. (**a**) ATMF-2 year maturity. (**b**) ATMF-5 years maturity

Fig. 9.5 Balduzzi, Das, Foresi and Sundaram model: relative error in percentage. The error of the lower bound is the difference from Monte Carlo value. (**a**) OTMF-2 year maturity. (**b**) OTMF-5 years maturity

9.4.5.2   Two-Factor Gaussian Model with Double Exponential Jumps

We test the affine Gaussian with Jumps interest rate model using the following parameter values

- Gaussian parameters: $K = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}$, $\theta = [0,\ 0]^\top$, $\boldsymbol{\sigma} = [0.01,\ 0.005]^\top$,

  $\rho = \begin{bmatrix} 1 & -0.2 \\ -0.2 & 1 \end{bmatrix}$, $\Sigma = \text{diag}(\boldsymbol{\sigma}) \cdot \text{chol}(\rho)$, $x_0 = [0.01,\ 0.005]$ and $\phi = 0.005$;
- Jumps parameters: $\boldsymbol{\mu}^+ = 0.001$, $\mathbf{m}^+ = [0.01,\ 0.01]$, $\boldsymbol{\mu}^- = 0.001$, $\mathbf{m}^- = [0.01,\ 0.01]$.

  Numerical results for this model are shown in Table 9.4.

9.4.5.3   Balduzzi, Das, Foresi and Sundaram Model

In order to prove the accuracy of our bounds for a wider class of models, we consider a stochastic volatility (and long run mean) model. We use the following parameter values, proposed in Balduzzi et al. (1996) : $k = 0.25$, $\alpha = 0.76$, $\beta = 0.12$, $\eta = 0.02$, $a = 0.29$, $b = 0.0007$, $\phi = 0.003$ and $\rho = -0.12$.
Numerical results for this model are shown in Table 9.5.

9.5   Conclusions

This paper provides a new lower bound method for pricing of swaptions that is accurate, fast and applicable to a wide range of interest rate models. Our algorithm is particularly efficient because it requires the computation of only one Fourier inversion. Existing approximations (for instance Singleton and Umantsev method) require a number of Fourier inversions equal to the number of payment dates of the underlying swap. An approximate exercise region defined by the log-geometric mean of the ZCBs portfolio is tried out for the first time for swaption pricing. From the numerical tests we find that the approximation is much faster than Monte Carlo method and it is also very accurate across different maturities, tenors and strikes. Numerical results are presented across a wide class of models, including model with jumps and stochastic volatility. For all these models our lower bound is applicable and it is accurate, instead the Singleton and Umantsev method is not applicable if the density function is not known in analytical form. Moreover the lower bound is very effective as control variate to reduce the computation time and the error of the Monte Carlo. Hence, our model could be very suitable also for calibration purposes.

A   Appendix

A.1   Proof Proposition 1

We consider the lower bound to the swaption price as in formula (9.4) for affine models:

$$LB(k; t, T, \{T_h\}_{h=1}^n, R) = P(t, T) \cdot \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h \, e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1 \right) I(\mathcal{G}) \right]^+$$

where the set $\mathcal{G} = \{\omega \in \Omega : g(\mathbf{X}(T)) \geq k\} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) + \alpha \geq k\}$.

According to Carr and Madan (2000), we introduce the dampening factor $e^{\delta k}$, then we apply the Fourier Transform with respect to the variable k to the T-forward expected value and we obtain:

$$\psi_\delta(\gamma) = \int_{-\infty}^{+\infty} e^{i\gamma k + \delta k} \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h \, e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1 \right) I(g(\mathbf{X}(T)) \geq k) \right] dk$$

$$= \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h \, e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1 \right) \int_{-\infty}^{+\infty} e^{i\gamma k + \delta k} I(\boldsymbol{\beta}^\top \mathbf{X}(T) + \alpha \geq k) dk \right]$$

$$= \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h \, e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1 \right) \int_{-\infty}^{\boldsymbol{\beta}^\top \mathbf{X}(T) + \alpha} e^{i\gamma k + \delta k} dk \right]$$

Since the dampening factor δ is positive, then the module of the integrand function decays exponentially for $k \to -\infty$ and the Fourier Transform is well defined, so:

$$\psi_\delta(\gamma) = \mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1 \right) e^{(i\gamma + \delta)(\boldsymbol{\beta}^\top \mathbf{X}(T))} \right] \frac{e^{(i\gamma + \delta)\alpha}}{i\gamma + \delta}$$

Using the characteristic function of $\mathbf{X}$, calculated under the T-forward measure: $\Phi(\boldsymbol{\lambda}) = \mathbb{E}_t^T \left[e^{i\boldsymbol{\lambda}^\top \mathbf{X}} \right]$, the function $\psi_\delta(\gamma)$ can be written as:

$$\psi_\delta(\gamma) = \left(\sum_{h=1}^n w_h e^{a_h} \Phi \left(-i\mathbf{b}_h + (\gamma - i\delta)\boldsymbol{\beta} \right) - \Phi \left((\gamma - i\delta)\boldsymbol{\beta} \right) \right) \frac{e^{(i\gamma + \delta)\alpha}}{i\gamma + \delta}$$

Finally the lower bound is the maximum with respect to k of the inverse transform of $\psi_\delta(\gamma)$:

$$LB(k; t, T, \{T_h\}_{h=1}^n, R) = P(t, T) \frac{e^{-\delta k}}{\pi} \int_0^{+\infty} e^{-i\gamma k} \psi_\delta(\gamma) d\gamma$$

A.2 Proof of the Analytical Lower Bound for Gaussian Affine Models

$\mathbf{X} \sim N(\boldsymbol{\mu}, V)$ in T-forward measure and $g(\mathbf{X}(T)) = \boldsymbol{\beta}^\top \mathbf{X} + \alpha \sim N(\boldsymbol{\beta}^\top \boldsymbol{\mu} + \alpha, \boldsymbol{\beta}^\top V \boldsymbol{\beta})$

Then the approximate exercise region $\mathcal{G}$ becomes:

$$\mathcal{G} = \{\omega \in \Omega : g(\mathbf{X}(T)) > k\} = \{\omega \in \Omega : z > d\}$$

where z is a standard normal random variable and

$$d = \frac{k - \boldsymbol{\beta}^\top \boldsymbol{\mu} - \alpha}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}.$$

The lower bound expression can be written using the law of iterative expectation:

$$LB(k; t, T, \{T_h\}_{h=1}^n, R) = P(t, T) \, \mathbb{E}_t^T \left[\mathbb{E}_t^T \left[\left(\sum_{h=1}^n w_h e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1 \right) \Big| z \right] I(z > d) \right]$$

Conditionally to the random variable z, the variable $\mathbf{X}$ is distributed as a multivariate normal with mean and variance:

$$\mathbb{E}_t^T[\mathbf{X}|z] = \boldsymbol{\mu} + z \cdot \mathbf{v}$$
$$Var(\mathbf{X}|z) = V - \mathbf{v}\mathbf{v}^\top$$
$$\mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$$

We can now compute the inner expectation, using the normal distribution property:

$$LB(k; t, T, \{T_h\}_{h=1}^n, R) = P(t, T) \left(\sum_{h=1}^n w_h \, \mathbb{E}_t^T \left[e^{\mathbf{b}_h^\top \boldsymbol{\mu} + z\mathbf{b}_h^\top \mathbf{v} + \frac{1}{2} V_h} I(z > d) \right] - \mathbb{E}_t^T \left[I(z > d) \right] \right)$$

where $V_h = \mathbf{b}_h^\top (V - \mathbf{v}\mathbf{v}^\top) \mathbf{b}_h$.

Maximizing with respect to k, involved in the definition of d, we found the lower bound:

$$LB(t, T, \{T_h\}_{h=1}^n, R) = \max_{k \in \mathbb{R}} \left(\sum_{h=1}^n w_h e^{a_h + \mathbf{b}_h^\top \boldsymbol{\mu} + \frac{1}{2} V_h + \frac{1}{2} d_h^2} N(d_h - d) - N(-d) \right)$$

where $d_h = \mathbf{b}_h^\top \mathbf{v}$ and $N(x)$ is the cumulative distribution function of standard normal variable.

To make faster the optimization of the lower bound with respect to the parameter k, we compute the first order approximation of maximum point as a starting point. Equation for stationary points:

$$\frac{\partial LB(t,k)}{\partial k} = \frac{P(t,T)}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}} \left(\sum_{h=1}^{n} w_h e^{a_h + \mathbf{b}_h^\top \boldsymbol{\mu} + \frac{1}{2} V_h + d_h^2/2} N'(-(d-d_h)) - N'(-d) \right) = 0$$

Taylor first order expansion:

$$N'(-(d-d_h)) = N'(-d) + d_h N''(-d) + o(d_h) = N'(-d)(1 + d d_h) + o(d_h)$$

We substitute the first order expansion in the derivative expression and we obtain:

$$\sum_{h=1}^{n} w_h e^{a_h + \mathbf{b}_h^\top \boldsymbol{\mu} + \frac{1}{2} V_h + d_h^2/2}(1 + d d_h) - 1 = 0$$

So the first order guess of the maximum point is:

$$d_{guess} = \frac{1 - \sum_{h=1}^{n} w_h e^{a_h + \mathbf{b}_h^\top \boldsymbol{\mu} + \frac{1}{2} V_h + d_h^2/2}}{\sum_{h=1}^{n} w_h e^{a_h + \mathbf{b}_h^\top \boldsymbol{\mu} + \frac{1}{2} V_h + d_h^2/2} d_h}$$

$$k_{guess} = \sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}\, d_{guess} + \boldsymbol{\beta}^\top \boldsymbol{\mu} + \alpha$$

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