Mathematics for Makers and Mathematics for Users

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To Reuben Hersh who posed an incisive problem:
What is Mathematics, Really?

Introduction

As I argue in my paper Borovik (2016), the current crisis in school-level mathematics education is a sign that it reaches a bifurcation point and is likely to split into two streams:

- education for a selected minority of children / young people who, in their adult lives, will be filling an increasingly small share of jobs which really require mathematical competence (I call them *mathematical makers*); and
- basic numeracy and mathematics awareness classes for the rest of population, *end users* of technology saturated by mathematics – which, however, will remain invisible to them.

In this paper, I discuss challenges arising in mathematics education for *makers* of mathematics. This is a theme which is rarely discussed in the mathematics education literature. It demands re-thinking of basic assumptions underpinning the mainstream mathematics education.

I invite the reader to discard taboos and start a frank and open discussion of the difficult problem:

> What is Mathematics Education, Really?

In the changing socioeconomic environment of mathematics, it needs to be addressed from the first principles.

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A Thought Experiment: Replicators

I suggest a simple thought experiment. Science fiction books occasionally mention an imaginary device: a *universal replicator*. It consists of two boxes; you put an object in a box, close the lid, and instantly get its undistinguishable fully functional copy in the second box. In particular, a replicator can replicate smaller replicators.

Now imagine an economy based on replicators. It needs two groups of producers: a small group of engineers who build and maintain the biggest replicator and a diverse, but still small, group of artisans, designers, and scientists who produce a single original prototype of each object. Let us enhance the functionality of replicators and assume that they can store originals in memory and share them with other replicators.

Then the world needs only one baker who has produced, once, a flavorsome and precisely textured original of a loaf, baguette, etc. of every kind of bread.

This hypothetical economy also needs a service sector, mostly waste disposal.

Next, try, if you can, to imagine a sustainable, stable, equal, and democratic model of education that supports this lopsided economy.

A wild and irrelevant fantasy? Alas, it is not. This apocalyptic future is already upon us—in the information sector of economy, where computers act as replicators of information. Mathematics, due to its special role in information technology, is the most affected part of human culture. The new patterns of division of labor split *mathematics for makers* of mathematics from *mathematics for end users* of mathematical technology and trigger a crisis of mathematics education. The latter increasingly focuses on mathematics for *users* and undermines itself because a sustainable reproduction of mathematics requires teachers educated as *makers*.

The Ultimate Replicating Machines

I borrowed the title of this section from a chapter in my book Borovik (2010). In the book, I followed Davis and Hersh (1981) who defined mathematics as

> *a study of mental constructs with reproducible properties*,

and I argued that the essence of mathematics was in its precise replicability which imitated the rigid stability of the laws of the physical universe. In the domain of technology,

> *mathematics is the ultimate in the technology transfer.* (Stewart 1990)

A mathematical theorem needs to be proved only once—and then it can be used for centuries. An algorithm needs to be developed only once—and then it can serve, as the Google Ranking Algorithm does, as a kingpin of a global information system.

In previous historic epochs, every use of a mathematical result required the participation of a human, who, depending on the level of his or her tasks, had to be trained in arithmetic (e.g., a bank clerk) or, in addition, had to learn elementary algebra, logarithms, and trigonometry (e.g., an artillery officer.[1]) The development and teaching of mathematics was shaped by a natural requirement: mathematics had to be understood and used by humans. In particular, this technological imperative defined the role of proofs at higher stages of mathematics education: the ability to derive formulae was used as a criterion of understanding.

Nowadays, mathematics works mostly inside of computers, and its applications are developed for being fed directly into computers, with humans excluded from this process and reduced to the role of uninformed *end users*.

Instant replicability of mathematics and the economy of scale, combined together, create a peculiar singularity:

> in mass produced devices such as smartphones, the per unit cost of mathematics encoded and hardwired within the device converges to zero.

For the end user, the mathematical component of technology frequently remains invisible. Even if the user is aware of the presence of a particular mathematical tool in the product used, it is not accessible for maintenance or repairs (try to reprogram a microchip in your credit card!). But this does not actually matter: even if the mathematical core is reached, it remains incomprehensible for anyone with the exception of a very narrow circle of experts.

Indeed, mathematical results and concepts involved in practical applications are much deeper and more abstract and difficult than ever before. And we have to accept that they are beyond the reach of the vast majority of graduates from mathematics departments in our universities.

The cutting edge of mathematical research moves further and further away from the stagnating mathematics education. From the point of view of an aspiring PhD student, mathematics looks like New York in the Čapek Brothers' book *A Long Cat Tale*:

> *And New York – well, houses there are so tall that they can't even finish building them. Before the bricklayers and tilers climb up them on their ladders, it is noon, so they eat their lunches and start climbing down again to be in their beds by bedtime. And so it goes on day after day.* (Čapek and Čapek 1927, reprinted 1996, p. 44).

If you think that these words are too jocular for a discussion of a matter so serious as the fate of mathematics as a cultural system, please take into consideration that Joseph and Karel Čapek were the people who coined the word *robot* for a specific

[1]It is worth to remember that in the first half of the 20th century, school mathematics curricula in many nations were dictated by the Armed Forces' General Staffs—this is why trigonometry was the focal point and apex of school mathematics: in the era of mass conscription armies, it was all about preparation for training, in case of war, of a sufficient number of artillery and Navy officers and aircraft pilots. With this legacy, we still cannot make transition to a more human mathematics.

socioeconomic phenomenon: a device or machine whose purpose is to replace a human worker.[2] Almost a century ago, they were seriously thinking about the social impact of technological and scientific progress.

Investment cycles and research and development cycles in many modern industries are just two years long. On the other hand, proper mathematics education still takes at least 15 years from the age of 5 to the age of 20—or even 20 years if postgraduate studies are needed.

As I argue in Borovik (2016), mathematics education is being torn apart by this tension between the deepening specialization of labor and increasing length of specialized education and training required for jobs at the increasingly thin cutting edge of technology. Public discourse on education policy is in a mess.[3] Key issues are ignored; this is not surprising—when the society does not know the answer, the question does not exist. Nothing said by politicians, by industry, by experts in mathematics education, etc. can be taken at face value.

For example, if banks and insurance companies were interested in having numerate customers—as they occasionally claim—we would witness the golden age of school mathematics: fully funded, enjoying cross-party political support, promoted and popularized by the best advertising companies in all forms of mass and social media. But they are not; banks and insurance companies need a numerate workforce – but even more so they need innumerate customers. 25 years ago in the West, the benchmark of arithmetic competence at a consumer level was the ability to balance a checkbook. Nowadays, bank customers can instantly get full information about the state of their accounts from an app on a mobile phone together with timely advice tailored to individual circumstances on the range of recommended financial products. This kind of service can be characterized in alternative terms: a bank can instantly exploit the customer's vulnerability.

Mathematics for Makers and Mathematics for Users

Makers and Users

The new patterns of division of labor split mathematics for makers from mathematics for end users of mathematical technology. How to describe the two? The replicability of mathematics mirrors the stability of laws of the physical universe, which is captured by an apocryphal formula:

[2]The root of the word "robot" is Slavic and means "work."

[3]Perhaps I would never write this paper and its predecessor (Borovik 2016) if I had not had a chance to observe, at a close range, the recent National Curriculum reform in England.

> *Mathematics is the language of contracts with Nature which Nature accepts as binding.*

Therefore, in my understanding, mathematics for makers is mathematics for those whose duties include writing contracts with nature and inventing, in the process, new mathematics and new ways to apply mathematics—they could be mathematicians, engineers, and scientists. In terms of the "universal replicator" simile from section "A Thought Experiment: Replicators," it is mathematics for those who produce the originals for subsequent replication.

In particular, it is mathematics that cannot be entrusted to computers.

We need to remember that

> mathematicians and physicists are stem cells of a technologically advanced society.

They are re-educable, able to change their role and metamorphose[4]—and, as I argue in Borovik (2016), this is made possible by frequent changes of the mode of their mathematical thinking in the process of their learning of mathematics.

On the other hand, mainstream mathematics education gradually sheds its content and loses its meaning: in its present form, it is not actually needed in the world of end users (Simeonov 2016).

Essence and Phenomenon

At a bit more philosophical level, the issue boils down to the difference between *essence* and *phenomenon*.

I was lucky that my philosophy lecturer at my university was one of the prominent Russian philosophers of that time, Mikhail Rozov. In the oppressive ideological environment of Soviet Russia, he was a quiet nonconformist. In one of the areas of his professional work, he dared to develop Niels Bohr's *complementarity principle* as a general principle of epistemology and then applied it to the humanities—this

[4]While Brexit is still in the news, it is worth to mention that Dominic Cummings, the Director of the *Vote Leave* campaign, explains in his blog http://bit.ly/2ePmyA2 that he hired, instead of professional pollsters and public relations people, some physicists for analysis of voters' intentions. He writes: *Physicists and mathematicians regularly invade other fields* but other fields do not invade theirs *so we can see which fields are hardest for very talented people. It is no surprise that they can successfully invade politics and devise things that rout those who wrongly think they know what they are doing.*

required some courage. In his lectures, he found a clever way to circumvent the official dogma by announcing to me and my peers that all of us, by default, were instinctive dialectic materialists and that he did not hope to advance us any further because of our general ignorance. This allowed him to teach us an honest history of philosophy instead of the official course of dialectic materialism. However, he took care to demystify some sacred dogmas of the official philosophy, in particular the essence/phenomenon double act of the Hegelian dialectics. He taught us:

> *You can describe a table knife in two ways.*
>
> (1) *It is a long narrow flat piece of steel slightly sharpened at one edge, with a handle attached.*
> (2) A thing for spreading butter on bread.
>
> (1) *describes how a knife is made; it is its* essence. (2) d*escribes how a knife is used; it is its* phenomenon.

For me, mathematics is all about how mathematics is made; I am a maker, not a user. Steven Strogatz published in *The New Yorker* blog[5] a brilliant popular article about how the number π is *used*; for me, it is more important to understand how the number π is *made* (or discovered). I work with the *essence*.

Pattern Matching

You have perhaps heard an expression popular in the mathematics education community:

> "*mathematics is the science of patterns*."

This is mathematics for users. Mathematics for makers can be described as

> "the science of structures behind patterns."

Meanwhile, the "mathematics is the science of patterns" approach (I suggest to call it *patternism*) is becoming popular within some parts of the applied/industrial mathematics community. I heard talks with suggestions to abandon the formulation of mathematical models of real-world objects and processes, as well as their subsequent analysis by mathematical means. Instead of that, it was suggested to run pattern-matching algorithms over large data sets.

This technology has every right to exist; it could be quite useful, especially when you are interested in "bulk" solutions which work correctly with a sufficiently high probability, securing, on average, acceptable profit margins. But this is not mathematics as we know it.

[5] S. Strogatz, *Why Pi Matters*, 15 March 2015. http://www.newyorker.com/tech/elements/pi-day-why-pi-matters?intcid=mod-most-popular.

There is an intrinsic danger in the patternalist technology: there is a possibility that very soon it will be monopolized by a few algorithms/systems, the same way as social media are dominated by likes of FACEBOOK and TWITTER. Verification of results could become a problem. There is a danger that the flaws of social media which resulted in the present "fake news" scandal might be reproduced, on a grander scale, in the "big data" technology—with computers spreading "fake data" among themselves.

Besides writing contracts with nature, the professional competences of the new generation of makers should include the ability to *control* computers. Mathematicians must remain prepared to face intellectual challenges so important and critical that they cannot be entrusted to computers.

As a corollary, serious mathematics education has to remerge with computer science.

Interestingly, some trends in mainstream education appear to lead in the exactly opposite direction.

Our new information environment becomes more and more saturated by pattern recognition and pattern matching (predictive typing is a prominent example).[6] There are some first signs that it starts poisoning teaching and learning mathematics. As it is argued in a deep and revealing paper by Yagmur Denizhan, this has already triggered changes in students' approach to learning: they started to imitate generic statistically shaped parse/substitute/append computer algorithms akin to that of pattern-matching Google Translate or predictive typing.

> *What led me to the line of thought underlying this article was a strange situation I encountered sometime in 2007 or 2008. It was a new attitude in my sophomore class that I never observed before during my (by then) 18 years' career. During the lectures whenever I asked some conceptual question in order to check the state of comprehension of the class, many students were returning rather incomprehensible bulks of concepts, not even in the form of a proper sentence; a behaviour one could expect from an inattentive school child who is all of a sudden asked to summarise what the teacher was talking about, but with the important difference that – as I could clearly see – my students were listening to me and I was not even forcing them to answer. After observing several examples of such responses I deciphered the underlying algorithm. Instead of trying to understand the meaning of my question, searching for a proper answer within their newly acquired body of knowledge and then expressing the outcome in a grammatically correct sentence, they were identifying some concepts in my question as keywords, scanning my sentences within the last few minutes for other concepts with high statistical correlation with these keywords, and then throwing the outcome back at me in a rather unordered form: a rather poorly packaged piece of Artificial Intelligence.*
> *It was a strange experience to witness my students as the embodied proof of the hypothesis of cognitive reductionism that "thinking is a form of computation". Stranger, though, was the question why all of a sudden half a century after the prime years of cybernetic reductionism we were seemingly having its central thesis actualised.* (Denizhan 2014)

Alas, I am in agreement with Yagmur Denizhan—I observe this behavior in my own students.

[6] I attempted to write some notes that became a fragment of this paper, on a tablet with predictive typing. A remarkable experience—predictive typing does not help to formulate any new thoughts but speeds up composing routine emails.

I wish to mention, in passing, another cultural phenomenon that I call *cartoon physics menace*[7]: a systematic suppression of laws of mechanics (and physics) in the virtual worlds of CGI movies, cartoons, and computer games, where our (grand)children spend an ever-increasing part of their lives and where everything can happen—pigs may fly. I am afraid it kills the all important "physical" intuition of the real world—the same way as an out-of-tune music toy can damage a child's sense of tone pitch.

What Will Replace the Present System of Mathematics Education?

As I argue in Borovik (2016), the present model of "mathematics education for all" is unsustainable, and, not surprisingly, the first cracks have started to appear. I concluded my paper with a warning that I wish to repeat here.

> Democratic nations, if they are sufficiently wealthy, have three options:
>
> A. Avoid limiting children's future choices of profession, teach rich mathematics to every child—and invest serious money into thorough professional education and development of teachers.
> B. Teach proper mathematics, and from an early age, but only to a selected minority of children. This is a much cheaper option, and it still meets the requirements of industry, defence and security sectors, etc.
> C. Do not teach proper mathematics at all and depend on other countries for the supply of technology and military protection.
>
> Which of these options are realistic in a particular country at a given time, and what the choice should be, is for others to decide.

My own instincts make me to go for option A, but it could happen to be unrealistically expensive—and unlikely to have support of every parent and every teacher.

Meanwhile, there are signs of option B emerging as the preferred one—at least in some countries.

In England, the recent green paper *Building Our Industrial Strategy*[8] sets the aim of

> "*expanding the number of specialist maths schools across the country*" (p. 16),

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[7]You can watch on YOUTUBE a useful compilation of relevant episodes from *The Looney Tunes* (the classics of the genre): Zac Snively, *Wile E. Coyote vs. The Road Runner Physics*, https://www.youtube.com/watch?v=EdGxf5sYdsU, 26 March 2015.

[8]Her Majesty's Government, *Building Our Industrial Strategy*. Green Paper, January 2017. http://bit.ly/2kh3roa.

and, which is much more telling, signals a shift of the preferred, from the government's point of view, career destination of

> *today's PhD students [who] are often tomorrow's research leaders, entrepreneurs and industrial researchers* (p. 29)

from academia to the industry.

Option B means separation of mathematics education for makers from education for end users.

But what is mathematics education for makers?

This question has never been seriously discussed. To answer it, Reuben Hersh's famous question (Hersh 1999):

> *What Is Mathematics, Really?*

needs to be recast as

> What Is Mathematics Education, Really?

This question is especially important in the context of education for makers. This is what I focus on in the rest of the paper.

I am not discussing the mathematics education of users—it is where the present model of mathematics education is moving to, in chaotic jerky moves, like a caterpillar pulled by ants in the general direction of their anthill. Some rather extreme suggestions have been made— for example, Emil Simeonov made a case for

> *drastically reducing mathematics teaching in schools to the level of music teaching, and introducing specialized schools* (i) *to prepare future engineers and scientists,* (ii) *to prepare for all other professions who need mathematics and* (iii) *where all those children who are just interested in mathematics can go deeper into the subject.* (Simeonov 2016)

Educating Makers

As I write in my paper Borovik (2017a),[9] advanced specialist mathematics schools such as Kolmogorov School in Moscow, Fazekas,[10] or Lycée Louis-le-Grand[11] accumulated a considerable experience of advanced mathematics education at the secondary school level. It remains mostly undocumented, unpublished, and not

[9]Some of the material in this text is built on observations made in that paper.

[10]A good description of Fazekas can be found in Juhász (2012).

[11]Lemme (2012) contains a fascinating analysis of Lycée Louis-le-Grand. M. Lemme said in a private communication: *It should be borne in mind that the system of* classes préparatoires *never was meant to train mathematicians On the other hand, and however immodest it will sound, the Institute I went to,* Lycée Louis-le-Grand, *always made a specialty of training the best and in particular the few who would become professional mathematicians.*

properly analyzed. However, a blog of the London Mathematical Society contains a collection of papers Borovik (2012) on advanced-level specialist mathematics schools in various countries. Even a brief look shows that these schools are all different. Also it is immediately clear that they all provide *mathematics for the makers*.

They nurture in their students specific mental traits which are almost never discussed in the literature on mathematics education or mentioned in education policy discourse:

- the ability to engage the subconscious when doing mathematics;
- the ability to communicate intuition;
- the ability to learn by absorption;
- the ability to compress mathematical knowledge;
- capacity for abstract thinking;
- being in control of their mathematics.

I'll try to explain these objectives point by point. I will also argue that development of these mental traits is the essence of mathematics education for makers.

I have to make an important disclaimer: I am not proposing to impose the model of mathematics education as practiced in the best (one might wish to say: elite) specialist mathematics schools on the rest of the world: I only wish to discuss lessons that can be learnt from their experiences.

Engaging the Subconscious

This is an aspect of mathematical practice that is mostly unknown outside the professional community of mathematicians.

In humans, the speed of totally controlled mental operations is at most 16 bits per second. In activities related to mathematics, this miserable bit rate is further reduced to 12 bits per second in addition of decimal numbers and to 3 bits in counting individual objects. Standard school mathematics education trains children to work at that speed, controlling and verbalizing each step: "left foot, right foot ...." Perhaps they can learn to walk slowly—but not many of them will ever be able to run.[12]

By comparison, the visual processing module in the brain crunches 10,000,000 bits per second (Nørretranders 1998, pp. 138 and 143).

I offer a simple thought experiment to the readers who have some knowledge of school level geometry.

[12] Here I borrow some details from my book Borovik (2010).

> Imagine that you are given a triangle; mentally rotate it about the longest side. What is the resulting solid of revolution? Describe it. [Answer is in footnote.[13]] And then try to reflect: where has the answer come from?

The answer comes from your subconscious. This is the best kept secret of mathematics: it is done by the subconscious. Moreover,

> Mathematics, in one of its many facets, is a language for communication with the subconscious.

If you were able to answer the question about the rotating triangle, then you were able to pass your commands to the visual processing centers of your brain, which then managed to unambiguously interpret them and return you the result in a form ready for verbalization and communicating back to me.

It is like training a dog.

Dogs have many faculties which we, humans, are lacking, for example, a fantastic sense of smell. To exploit these faculties, we have to send our commands to the dog and interpret its reactions. A learner of mathematics is a dog trainer; his subconscious is his/her "inner dog" (or a puppy), a wordless creature with fantastic abilities, for example, for image processing or for parsing of symbolic input. The subconscious has to be trained to react to commands *triangle!*, *side!*, and *rotate!* in a way similar to a dog reacting to *sit!*, *bite!*, and *fetch!*.

We need to look at that in more detail—so a further digression into the subconscious is needed.

Digression into the Subconscious

I share Paul Bloom's conjecture that the human brain contains the equivalent of two quite separate supporting structures for two different causality systems: one for the physical world another for the social world. As he metaphorically put it in Bloom (2004),

> We have two distinct ways of seeing the world: as containing bodies and containing souls (p. xii).

For some years, Bloom was trying to test his conjecture by psychological studies of infants:

> *We suggest that infants possess different systems – or modes of construal – for reasoning and learning about inanimate material objects versus reasoning about people (and possibly, all*

[13]Most people who I asked this question usually answered, after a few seconds of looking inside themselves, something like "*two circular cones glued at the shared base*".

intentional entities). This is supported by the present data, as well as by a body of research suggesting that infants interpret inanimate objects in terms of physics and not goals (e.g. Woodward, 1998; see also Kuhlmeier, Wynn, & Bloom, in preparation), and interpret people and other animate entities – but not inanimate objects – in terms of goals (e.g. Meltzoff, 1995; Shimizu & Johnson, in press; Woodward, 1998). We suggest that these systems are not the product of past learning. Instead, they provide the foundation for future learning. (Kuhlmeier et al 2004b; see also Kuhlmeier et al 2004a.)

If true, Bloom's conjecture could have some interesting consequences for the philosophy of mathematics because it allows us to modify the Davis-Hersh definition of mathematics mentioned in section "The ultimate replicating machines":

> Mathematics is a reproducible and verifiable modelling of the causality systems of the physical world in terms of causality relations of the social world.

Bloom's conjecture also allows us to describe mathematics as a language for communication between the two causality systems, thus creating a conceptual framework for my "inner dog" metaphor. Actually, it would be best to talk about "inner wolves"—all behavioral traits of dogs are present in wolves and only amplified or suppressed in dogs by breeding (Fogle 1990).

The "inner wolf" is the physical causality module of the mind; it lurks below the horizon of consciousness, and the key issue in learning mathematics is learning a language for communication with it.

Wolves (the real ones, not metaphorical) are remarkable for the apparent disconnection between their social and physical causality systems. They have a sophisticated signal system for social interaction. They also patiently observe and then can predict the behavior of their prey. But they do not communicate with each other about the prey!

Wolves can show social aggression toward other wolves, but it is very different from their "true predatory aggression"[14]: wolves do not feel any emotions toward their prey; emotions are reserved for other wolves.

In baboons, the disconnect between the perception of social and physical worlds is even more striking, and the book *Baboon Metaphysics* by Cheney and Seyfarth (2007) is quite revealing because, in evolutionarily terms, baboons are much closer to humans than wolves.

A society of baboons' male troop has a linear transitive hierarchy recalculated every day after each fight between adjacent members. Human boys in less humane places such as various kinds of borstals, reformatories, and juvenile prisons form a similar strict linear order hierarchy recalculated every day as a result of fights.

[14]The true predatory aggression is suppressed by breeding in most breeds of dogs.

But the social order of female baboons—with grandmothers[15] and even grand-grandmothers caring about their descendants—is very different. In the words of *The Baboon Metaphysics*, female baboons live in the *Jane Austen's World*. Female baboons also form a transitive linear hierarchy, stable – they do not fight for a higher place in the order – but which is recalculated by transitivity every time a daughter is born and is inserted into the linear order immediately after her mother and her older sisters. This is the reason why a book about baboons (Cheney and Seyfarth 2007) contains a definition of *transitive relation*—and explanations of its meaning are repeated in the text several times.

In short, baboons are users of transitivity and linear order—but they apply it only to the social world.[16] Their relations with the physical world are much more primitive than their social life.

It appears that the barrier for information exchange between the two causality systems has been broken only in humans—and only partially:

> fMRI reveals reciprocal inhibition between social and physical cognitive domains. (Jack et al 2013)

So I wish to reaffirm my conjecture:

> "the inner dog", as I described it in Section 6.1, is the physical causality module of our mind.

Neurophysiology is still in infancy, but there is at least one example of a state-of-the-art experimental study:

> *Our work addresses the long-standing issue of the relationship between mathematics and language. By scanning professional mathematicians, we show that high-level mathematical reasoning rests on a set of brain areas that do not overlap with the classical left-hemisphere regions involved in language processing or verbal semantics. [...] Our results suggest that* ***high-level mathematical thinking makes minimal use of language areas and instead recruits circuits initially involved in space and number****.* (Emphasis is mine – AVB.) *This result may explain why knowledge of number and space, during early childhood, predicts mathematical achievement.* (Amalric and Dehaene 2016)

The physical causality module has immense raw processing power, but it is mute. The social causality module has access to language but otherwise is very slow. It has to train the physical module, much as people train dogs (i.e., domesticated wolves).

[15]There are no baboon granddads—males' lives are short and brutal, and they die young, mostly killed by other male baboons. From the day of reaching his position at the top, alpha male rarely stays alive for more than a year. Lower down the hierarchy, fights become less physically harmful and more ritualistic.

[16]I write more about baboons and their mathematics in Borovik (2017c).

I think it is obvious to every working research mathematician that, in their professional community, mathematicians are ranked by the size and strength of their inner dogs. When two mathematicians meet, their inner dogs start to sniff each other.

I dare to suggest that children who grew up to become mathematicians are to some degree aware of the existence of their dog (or puppy) and perhaps even love it and care about it.

I collected hundreds of mathematicians' testimonies about difficulties they experienced in their earliest encounters with mathematics (you may read some of them in Borovik (2017b)). A generic one was being misunderstood by adults. The most frequent specific difficulty was telling the left from the right—for lack of logical justification for the distinction between the two. A child can be told by adults "this is left and this is right," but his inner dog may tell him, using its posture and a skeptical position of its ears as means of communication "sorry, master, but they smell the same to me." For a child, to retain his/her mathematical ability means to retain the ability to listen to his subconscious and not to hurry to accept, as absolute truth, what he is told by adults.

How can a learner of mathematics start engaging his/her subconscious? Perhaps even without noticing it—in sharing his/her intuition with other likely minded young mathematicians. I say more on that in the next section "Sharing Intuition."

Sharing Intuition

There are four conversants in a conversation between two mathematicians: two people and their two "inner dogs."

When mathematicians talk about mathematics face-to-face, they frequently use language:

- which is very fluid and informal;
- is improvised on the spot;
- includes pauses (for a lay observer, very strange and awkwardly timed) for absorption of thought;
- has almost nothing in common with standardized mathematics "in print."

Mathematicians are trying to convey a message from their "inner dogs" directly to their colleagues' "inner dogs."

Alumni of high-level specialist mathematics schools are "birds of a feather" because they have been initiated into this mode of communication at the most susceptible age, as teenagers, at the peak of intensity of their socialization/shaping group identity stream of self-actualization. Learning to speak to a peer's "inner dog" is an efficient way to learn a language for communication with your own "inner dog."

This process is remarkably similar to the way toddlers learn to think by first directing at themselves the speech of their parents directed at them—and then interiorizing it.

Learners of mathematics need to talk to each other to develop this crucial interiorization of their outward-directed speech—and to talk informally—in a language they invent themselves.

In this context, the role of mathematics teachers goes beyond giving to students examples of "proper" mathematical language; teachers have to provide their students with a rich diet of challenging problems which go beyond the application of procedural recipes, stimulate mathematical thinking, and thus require the use of a deeper intuition and sharing of intuition.

In that respect, mathematics is not much different from arts. Part of the skills that children get in higher-level music schools, acting schools, ballet schools, and art schools is the ability to talk about music, acting, ballet, and art with intuitive, subconscious parts of their minds—and with their peers—in a semi-secret language which is not recognized (and perhaps not even registered) by the uninitiated.

Learning by Absorption

For talking to each other, the best option is to meet face-to-face, and specialist mathematics schools provide the best environment for that:

> "*students find their tribe and learn from each other*."[17]

This is an aspect of mathematics/physics education of "mathematically able" children which is almost never mentioned: "mathematically inclined" (my preferred term) children have a high capacity to learn by absorption. This trait remains dormant in the mainstream school environment but will be activated when kids find themselves surrounded by children *like them*. My university has a large and vibrant community of mathematics PhD students, and it is a place where learning by absorption can be observed "in the wild." It is less known that the same could happen with a certain kind of 13–16-year-old kids when they form a small learning community.

Indeed, who will teach them in their professional future? They will have to teach themselves and learn from each other. The key to the success of mathematics education for makers is the creation of a self-learning environment where students learn by absorption.

[17] A. Wolf, quoted in L. McClure, *All students should receive excellent math teaching not just those in specialist maths schools*, http://bit.ly/2k33Kj3, posted 3 February 2017, accessed 18 February 2017.

Compression and Abstraction

The specific modus of communication based on sharing intuition triggers the development of another mental skill specific for mathematics: compression of information. In the words of William Thurston, one of the greatest mathematicians of recent times,

> *Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.* (Thurston 1990)

In its turn, compression requires abstraction; I wrote in Borovik (2013) about the strange fate of abstract thinking and the paradoxical situation when computer science requires much higher levels of abstract thinking than is developed in recipients of the mainstream mathematics education in school and university.

We are talking about the next generation of mathematicians who, most likely, will routinely use automated proof checkers and engineers who will be using modelling and analytic software of a similar degree of sophistication. To be efficient and safe in their work, they will need a firm grounding in computer science and a sharpened ability for abstract thinking.

On the other hand, children in their early teens are quite open to absorption of abstract concepts; after all, they are grappling with other important abstract concepts in their lives, for example, “love.”

Being in Control

On this point, I refer the reader to my recent paper Borovik (2017b) where I discuss emotions related to a person’s control (or lack of control) of his/her mathematics:

> sense of danger; sense of security; confidence, feeling of strength; feeling of power;

which eventually lead to the ultimate emotion of mathematics:

> realisation that you know and understand something that no-one else in the world knows or understands – and that you can prove that.

These higher-level emotions are not frequently discussed in the context of mathematics education—but, remarkably, they are known not only to professional research mathematicians but also experienced by many children in their first encounters with mathematics.

I think it is self-evident that mathematics education for makers should nurture independence of their thinking and put them in control of their mathematics.

Conclusions

The social role of mathematics is changing. To save mathematics as a cultural system, we need to take special care of education of the next generation of mathematically competent *makers*, perhaps at the background of collapsing mass mathematics education.

I tried to argue that

mathematical intuition, ability to share intuition, compression, abstraction, and being in control

should be seen as the cornerstones of mathematics education for makers.

These key skills can be nurtured by uniting mathematically inclined students with their tribe, encouraging the communication of mathematics, and providing children with rich mathematics, gentle academic guidance, and a strong value system.

The aim of my paper is to start a discussion.

I understand that I pose more questions than give answers.

For example, I have not said a single word about what should be taught (perhaps I can only suggest that mathematics needs to be re-united with physics and computer science). I completely ignored all organizational, administrative, and political issues.

Instead, I tried to focus on methodological and pedagogical challenges highly relevant for a selective “deep” mathematics education which are ignored in the current model of mass education.

I have to warn that this is a political and ideological minefield. Academically selective education is a hot potato, at least in Britain.

The social/physical duality of causality modules of human mind is an even more difficult theme. Indeed, another obvious area of human activity affected by interaction between the two causality system is religion, myth, and magic—I mention them briefly in Borovik (2017b). The literature on this topic is already saturated by references to “body-soul duality.” Neurophysiology is still in its infancy, and identification of the two causality systems appears to be too subtle a problem for direct experimental study at the current level of research technology.[18]

Despite its first successes, experimental neuroscience does not yet provide us any certainty or protection from distracting ideological debates.

Meanwhile, let us abandon taboos and start a frank and open discussion of this difficult problem.

[18]See Yeo et al (2017) for a meta-analysis of some of findings – they are very interesting, but they are not enough.

Disclaimer

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