Chapter 11
Extended Representations of Belief States

The AGM model is a simple and elegant representation of quite complex phenomena. Obviously, the trade-off between simplicity and relevance can be made differently. Many of the modifications of the framework that have been proposed consist in extensions of the belief state representation so that it contains more information than what is contained in a belief set or belief base. We have already studied one such extension in Sections 7.1 and 10.2. This chapter is devoted to additional extensions.

11.1 Probability and Plausibility

The AGM model and other logical approaches to belief revision represent features of doxastic behaviour that differ from those represented by probabilistic models. The degrees of belief represented for instance by entrenchment relations do not coincide with probabilities [298]. It seems difficult to construct a reasonably manageable model that covers both the logic-related and the probabilistic properties of belief change. (Problems connected with the lottery and preface paradoxes have a major role in creating this difficulty [217, 237].)

However, some authors have explored the interrelations between the two types of models. Lindström and Rabinowicz showed how belief revision can be connected with accounts of conditional probability that allow the condition to have probability zero [230]. Makinson further investigated this and other connections between the two frameworks [244]. Insights from the AGM model can be used as an impetus for considering "revisionary" accounts of conditional probability, i.e., accounts in which $\mathfrak{p}(q,r)$, the probability of q given r, is not defined in the standard way. (According to the standard definition, $\mathfrak{p}(q,r)$ is equal to $\mathfrak{p}(q\&r)/\mathfrak{p}(r)$ when $\mathfrak{p}(r) \neq 0$, and otherwise undefined.) Furthermore, the notion of non-prioritized revision that has been developed in the AGM tradition (revision not satisfying $p \in K * p$ for all p; see Chapter 8) can be usefully transferred to a probabilistic context. There it corresponds to "vacuous" conditionalizing when the condition is too unbelievable

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to be taken seriously, i.e., $\mathfrak{p}(q,r) = \mathfrak{p}(q)$ when r is highly unlikely. Makinson also discussed "hyper-revisionary" probabilistic conditionalization, in which the fact that something believed to be very improbable actually happens is taken as a reason to believe that the probability was underestimated. There is an analogy between hyper-revisionary conditionalization and belief revision that violates the AGM property that if $K + p$ is logically consistent, then $K * p = K + p$. Such violations would be justified if K and p are epistemically but not logically incompatible.

In order to investigate the relationship between AGM and Bayesian conditionalization, Bonanno introduced what he called the qualitative Bayes rule, namely that

> "... if at a state the information received is consistent with the initial beliefs–in the sense that there are states that were considered possible initially and are compatible with the information–then the states that are considered possible according to the revised beliefs are precisely those states." [39]

Bonanno constructed and characterized a model of belief revision that satisfies this condition. It complies with the AGM postulates for partial meet revision.

Friedman and Halpern have developed a model based on a notion of plausibility that is a generalization of probability. Instead of assigning to each set A of sentences a number $\mathfrak{p}(A)$ in $[0,1]$, representing its probability, they assign to it an element $Pl(A)$ of a partially ordered set. $Pl(A)$ is called the "plausiblity" of A. If $Pl(A) \leq Pl(B)$ then B is at least as plausible as A. A sentence p is believed if and only if p is more plausible than $\neg p$. Changes in belief take the form of changes in the plausibility ordering. Conditions on such changes have been identified that produce a revision operation that is essentially equivalent with partial meet revision [111, 113].

Several other authors have presented probability-based and plausibility-based belief revision models that have close connections with the AGM model [20, 82, 13, 16].

11.2 Ranking Models

In Wolfgang Spohn's ranking theory of belief change, a belief state is represented by a ranking function κ that assigns a non-negative real number to each possible world W, representing the agent's degree of disbelief in W [314, 315, 316]. A sentence p is assigned the value $\kappa(p) = min\{\kappa(W) \mid p \in W\}$. Furthermore, p is believed if and only if $\kappa(\neg p) > 0$, i.e., if and only if every $\neg p$-world is disbelieved to a non-zero degree. The conditional rank of q given p is $\kappa(q \mid p) = \kappa(p\&q) - \kappa(p)$. For any sentence p and number x, the $p \mapsto x$-conditionalization of κ is defined by:

$$\kappa_{p\mapsto x}(q) = min\{\kappa(q \mid p), \kappa(q \mid \neg p) + x\}$$

Contractions, expansions and revisions can all be represented as conditionalizations. The outcome depends on the numerical values involved. In addition, other operations such as the strengthening or weakening of beliefs already held are straightforwardly representable in this framework. Important results on belief revision based on ranking functions, including an axiomatic representation that clarifies their relationship to AGM operations, have been reported by Hild and Spohn [198]. A generalization of Spohn's ranking functions has been proposed by Weydert [330].

11.3 Conditionals and the Ramsey Test

Belief revision theory has primarily been concerned with belief states and inputs expressed in terms of classical sentential (truth-functional) logic. The inclusion of non-truth-functional expressions into the language has interesting and often surprisingly drastic effects.

Among the several formal interpretations of non-truth-functional *conditionals*, such as counterfactuals, one is particularly well suited to the formal framework of belief revision, namely the so-called Ramsey test. It is based on a suggestion by Frank Ramsey (1903–1930) that has been further developed by Robert Stalnaker and others [318, pp. 98–112]. The basic idea is that "if p then q" is taken to be believed if and only if q would be believed after revising the present belief state by p. Let $p \mathrel{\Box\!\!\rightarrow} q$ denote "if p then q", or more precisely: "if p were the case, then q would be the case". The Ramsey test says:

$$p \mathrel{\Box\!\!\rightarrow} q \text{ holds if and only if } q \in K * p.$$

If we wish to treat conditional statements like $p \mathrel{\Box\!\!\rightarrow} q$ on par with statements about actual facts, then they will have to be included in the belief set when they are assented to by the agent; thus:

$$p \mathrel{\Box\!\!\rightarrow} q \in K \text{ if and only if } q \in K * p.$$

However, inclusion in the belief set of conditionals that satisfy the Ramsey test will require radical changes in the logic of belief change. As one example of this, contraction cannot then satisfy the inclusion postulate ($K - p \subseteq K$). The reason for this is that contraction typically provides support for conditional sentences that were not supported by the original belief state. Hence, if I give up my belief that John is severely mentally retarded, then I gain support for the conditional sentence "If John has lived 30 years in London, then he understands the English language" [151].

A famous impossibility theorem by Gärdenfors shows that the Ramsey test is incompatible with a set of plausible postulates for revision [132]. The crucial part of the proof consists in showing that the Ramsey test implies the following monotonicity condition:[1]

$$\text{If } K \subseteq K', \text{ then } K * p \subseteq K' * p.$$

[1] The proof is straightforward: Let $K \subseteq K'$ and $q \in K * p$. The Ramsey test yields $p \mathrel{\Box\!\!\rightarrow} q \in K$, then $K \subseteq K'$ yields $p \mathrel{\Box\!\!\rightarrow} q \in K'$, and finally one more application of the Ramsey test yields $q \in K' * p$.

This condition is incompatible with the AGM postulates for revision, and it is also easily shown to be implausible. Let K be a belief set in which you know nothing specific about Ellen and K' one in which you know that she is a lesbian. Let p denote that she is married and q that she has a husband. Then we can have $K \subseteq K'$ but $q \in K * p$ and $q \notin K' * p$.

Several solutions to the impossibility theorem have been put forward. One option investigated by Rott and others is to reject the Ramsey test as a criterion for the validity of conditional sentences [186, 285]. Levi accepts the test as a criterion of validity but denies that such conditional sentences should be included in the belief set when they are valid. In his view, they lack truth values and should therefore not be included in belief sets [221]. This, of course, blocks the impossibility result. Levi and Arló-Costa have investigated a weaker version of the Ramsey test that is not blocked by Gärdenfors's result and is also compatible with the AGM model [12, 15].

In a somewhat similar vein, Lindström and Rabinowicz have proposed that a conditional sentence expresses a determinate proposition about the world only relative to the subject's belief state. Given a conditional statement $p \mathrel{\Box\!\!\rightarrow} q$ and a belief set K, there is some sentence $r^K_{p \Box\!\!\rightarrow q}$ such that $p \mathrel{\Box\!\!\rightarrow} q$ holds in the belief state represented by K if and only if $r^K_{p \Box\!\!\rightarrow q} \in K$. In this way we can have the Ramsey test in the form

$$r^K_{p \Box\!\!\rightarrow q} \in K \text{ if and only if } q \in K * p,$$

which is not blocked by the impossibility result [233, 232].

Yet another option is to accept both the Ramsey test and the inclusion of conditional sentences into the belief set. Then belief sets containing $\Box\!\!\rightarrow$ will behave very differently under operations of change than the common AGM belief sets, and the standard AGM postulates will not hold [151, 286]. Not even the simple operation of expansion can be retained. Reusing an example from Section 5.4, we can suppose that you have no idea about John's profession, but then "expand" your belief set by the belief that he is a taxi driver. You will then lose the conditional belief that if John goes home by taxi every day, then he is a rich man–hence this is not an expansion after all [151]. As was noted by Rott, "[e]xpansions are not the right method to 'add' new sentences if the underlying language contains conditionals which are interpreted by the Ramsey test" [286].

Ryan and Schobbens have related the Ramsey test to update rather than revision (cf. Section 10.1) and found the test to be compatible and indeed closely connected with update operations [303].

Kern-Isberner has proposed a framework for revision that is based on a conditional valuation function that assigns (numerical) values to both non-conditional and conditional sentences. In this framework–which differs from AGM in important respects–conditional sentences can be elements of belief sets, and revisions can be performed with conditional sentences as inputs [209]. A partly similar approach has been developed by Weydert [331].

11.4 Modal, Doxastic, and Temporal Sentences

The inclusion of *modal sentences* in belief sets has been investigated by Fuhrmann. Let $\Diamond p$ denote that p is possible. The following, seemingly reasonable definition:

$$\Diamond p \in K \text{ if and only if } \neg p \notin K,$$

gives rise to problems similar to those exhibited in Gärdenfors's theorem, and essentially the same types of solutions have been discussed [115].

Lindström and Rabinowicz have investigated the inclusion into a belief revision framework of *introspective beliefs*, i.e., allowing for $\mathbb{B}p \in K$, where $\mathbb{B}p$ denotes "I believe p". Paradoxical results not unsimilar to those for conditionals are obtained in this case as well [234]. Similar results were obtained by Friedman and Halpern [112].

Dupin de Saint-Cyr and Lang introduced *temporally labelled sentences* into belief revision and proposed a belief change operation, called belief extrapolation, in which predictions are based on initial observations and a principle of minimal change [84]. Bonanno has developed logics that contain both a next-time temporal operation and a belief operation. The basic postulates of AGM revision are satisfied, and a strong version of the logic also satisfies the supplementary postulates [40, 41].

Booth and Richter have developed a model of fuzzy revision on belief bases. In this model, both the elements of the belief base and the input formulas come attached with a numerical degree (whose precise interpretation is left open). They showed that partial meet operations on belief bases can be faithfully extended to this fuzzy framework [52].

Finally, Fuhrmann has generalized partial meet operations to *arbitrary collections of (not necessarily linguistic) items* that have a dependency structure satisfying the Armstrong axioms for dependency structures in database relationships [117, 120].

11.5 Changes in Norms, Preferences, Goals, and Desires

Norms: The AGM model was partly the outcome of attempts to formalize changes in norms [5]. In spite of this, authors who tried to apply the AGM model to normative change have found it to be in need of rather extensive modifications to make it suitable for that purpose.

Boella, Pigozzi, and van der Torre analyzed normative change in a framework where a norm system is represented by a set of pairs of formulas. The pair $\langle p,q \rangle$ should be read "if p, then it is obligatory that q". In this framework, however, postulates for norm contraction and revision that are closely analogous to the AGM postulates give rise to inconsistency [38].

Governatori and Rotolo proposed a model for changes in legislation that, among several other aspects, also includes an explicit representation of time. Such a model can account for phenomena such as retroactivity that are difficult to deal with in an input-assimilating framework such as AGM [142].

Hansson and Makinson investigated the relationship between changes and applications of a norm system. In order to apply a norm system with conflicting norms to a particular situation, some of the norms may have to be ignored. Although these norms will remain intact for future situations, the problem of how to prioritize among conflicting items is similar to selecting sentences for removal in belief contraction [192].

Common law systems (case law systems), such as those of the United Kingdom, the United States, Canada, Australia, and New Zealand, have a structure that differs significantly from the civil law systems (Roman law systems) that dominate on the European continent and in Latin America. Most studies of changes in legal systems are primarily applicable to civil law systems in which the legal system changes through the modification of statutes. In common law systems, legal change takes place to a large extent through court decisions that are constrained by previous decisions made in other courts. John Horty [200] has investigated the nature of norm change in common law systems. It turns out to have interesting logical properties that differ from those of changes in statutory codes.

Preferences: A model of changes in preferences can be obtained by replacing the standard propositional language in AGM by a language consisting of sentences of the form $p \geq q$ ("p is at least as good as q") and their truth-functional combinations. The acquisition of a new preference takes the form of revision by such a preference sentence. The adjustments of the original preference state that are needed to maintain consistency in such revisions can be modelled by partial meet contraction. However, some modifications of the AGM model seem to be necessary in order to obtain a realistic model of preference change [144, 156, 218].

Goals and desires: Desires are often allowed to be contradictory, since they need not be actively pursued by the agent. Likewise, we often have goals that are difficult or impossible to combine. Intention selection is a process aimed at removing such contradictions, and ending up with a consistent set of intentions [265]. Paglieri and Castelfranchi have proposed a model of Data-oriented Belief Revision (DBR) in which significant attention is paid to the mutual influences between beliefs and goals [264]. In one direction, beliefs support and regulate goal processing and the transition from desires to intentions [62, 60, 61, 63]. In the other direction, goals can affect belief revision by determining the relevance (usefulness to the agent's current purposes) and likability (capability of fulfilling the agent's goal) of potential beliefs. Relevance increases the likelihood that a potential belief will be considered a candidate for belief, whereas likeability increases its chances of being actually believed, once considered. Boella, da Costa Pereira, Pigozzi, Tettamanzi, and van der Torre have also analyzed the role of goals in belief revision [37]. Their model is similar to DBR in its selection criteria, but it puts more emphasis on preventing wishful thinking from having any influence on the agent's beliefs.