A Study on Optimal Preventive Maintenance Policies for Cumulative Damage Models

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1 Introduction

The preventive maintenance policy is one of the most important problems in the reliability theory and the maintenance one. The preventive maintenance models are classified into several categories, and many kinds of preventive maintenance policies have been discussed (e.g., [1–6, 10, 12–14]). Especially, there exists a preventive maintenance policy taking account of damage by shocks as one of them (e.g., [7, 9, 11]).

In this chapter, we consider the extended preventive maintenance policies for cumulative damage models with stochastic failure levels. That is, we discuss the optimal preventive maintenance policies for the system that fails when the cumulated amount of damage by shocks exceeds a stochastic failure level, assuming a continuous distribution and a discrete one, respectively. We apply the expected costs per unit time in the steady state as criteria of optimality and seek the optimal policies minimizing these expected costs. We show that there exists a unique optimal policy under certain conditions, respectively. Furthermore, we refer to the modified models where the shock does not always give the damage to the system.

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2 Preventive Maintenance Policies for a Cumulative Damage Model with a Continuous Distribution

2.1 Model and Assumptions

1. Consider a one–unit system.
2. System failure is revealed at failure immediately and each failed unit is scrapped without repair.
3. An unlimited number of spare units are immediately available when they are needed.
4. The original unit begins operating at time 0 with the cumulated amount of damage 0.
5. The planning horizon is infinite.
6. The damage occurred by shocks is additive.
7. The unit fails only when the cumulated amount of damage by shocks exceeds the stochastic failure label r.v. (random variable) W ($W \geq 0$). The r.v. W obeys a cdf (cumulative distribution function) $D(w)$ ($w \geq 0$) with a pdf (probability density function) $d(w)$.
8. When the cumulated amount of damage exceeds the predetermined exchange level w_0 ($0 \leq w_0 < \infty$) by any shock, the unit is exchanged if it has not failed, by the spare one immediately (i.e., the preventive maintenance). On the other hand, the unit is replaced if it has failed by any shock in a similar fashion (i.e., the corrective one).
9. The exchange and replacement actions are executed instantaneously. The new exchanged and replaced units take over the operation immediately.
10. The similar cycles are repeated from time to time. That is, an interval from the start of the original unit (the exchange or the replacement) to the following exchange or replacement is defined as one cycle, and the cycle repeats itself again and again.
11. The time interval between $j-1$st shock and jth one is r.v. T_j ($j = 1, 2, 3, \ldots$; $T_j \geq 0$ and r.v. T_1 is the time interval between the time 0 and the first shock), and the amount of damage by jth shock is r.v. X_j ($X_j \geq 0$), where r.v. X_i is independent of r.v. T_j ($i \neq j$).
12. There exist n types of shock modes and the shock mode i occurs with probability a_i ($i = 1, 2, \ldots, n$; $\sum_{i=1}^{n} a_i = 1$; $a_i \geq 0$). Under the condition that the mode of jth shock is the ith type, r.v. T_j obeys a cdf $F_i(t)$ ($t \geq 0$), and r.v. X_j obeys a cdf $G_i(x)$ ($x \geq 0$), that is, to say r.v. T_j obeys the cdf $F(t) = \sum_{i=1}^{n} a_i F_i(t)$, and r.v. X_j obeys the cdf $G(x) = \sum_{i=1}^{n} a_i G_i(x)$. Furthermore, we put $\int_0^\infty t \mathrm{d}F_i(t) = 1/\lambda_i$ and $\int_0^\infty t \mathrm{d}F(t) = 1/\lambda$ and this implies the relation $1/\lambda = \sum_{i=1}^{n} a_i/\lambda_i$. The amount of damage by jth shock, r.v. X_j has a renewal function $M(x)$ and a renewal density $m(x)$.

13. The costs considered are a cost c_0 suffered for each exchange before a failure (each preventive maintenance) and a cost c_i suffered for each replacement after a failure (each corrective one) due to shock mode i, where $c_i > c_0$ since the corrective maintenance is more expensive than the preventive one.

Under these model and assumptions, we derive the expected cost per unit time in the steady state and discuss the optimal preventive maintenance policies minimizing that expected cost.

2.2 Analysis and Theorems

The expected cost per one cycle $A_c(w_0)$ is given by

$$A_c(w_0) = c_0 \int_{w_0}^{\infty} [G(w) - \int_0^{w_0} \bar{G}(w-u)\mathrm{d}M(u)]dD(w) + \sum_{i=1}^{n} a_i c_i \left[\int_0^{w_0} \left\{ \bar{G}_i(w) + \int_0^{w} \bar{G}_i(w-u)\mathrm{d}M(u) \right\} \mathrm{d}D(w) + \int_{w_0}^{\infty} \left\{ \bar{G}_i(w) + \int_0^{w_0} \bar{G}_i(w-u)\mathrm{d}M(u) \right\} \mathrm{d}D(w) \right], \tag{1}$$

where $\bar{\psi}(\cdot) = 1 - \psi(\cdot)$, in general.

The mean time of one cycle $B_c(w_0)$ is

$$B_c(w_0) = (1/\lambda) \left[\int_0^{w_0} \{1 + M(w)\}\mathrm{d}D(w) + \{1 + M(w_0)\}\bar{D}(w_0) \right]. \tag{2}$$

We obtain the following when $w_0 = 0$ and $w_0 \to \infty$.

$$A_c(0) = c_0 \int_0^{\infty} G(w)\mathrm{d}D(w) + \sum_{i=1}^{n} a_i c_i \int_0^{\infty} \bar{G}_i(w)\mathrm{d}D(w) \ , \tag{3}$$

$$B_c(0) = 1/\lambda \ , \tag{4}$$

$$A_c(\infty) = \sum_{i=1}^{n} a_i c_i \int_0^{\infty} \left[\bar{G}_i(w) + \int_0^{w} \bar{G}_i(w-u)\mathrm{d}M(u) \right] \mathrm{d}D(w) \ , \tag{5}$$

and

$$B_c(\infty) = (1/\lambda) \int_0^\infty [1 + M(w)]\mathrm{d}D(w) \ . \tag{6}$$

Thus, the expected cost per unit time in the steady state is given by

$$C_c(w_0) = \frac{A_c(w_0)}{B_c(w_0)} \tag{7}$$

(see [15], p. 52).

Define the numerator divided by $m(w_0)\bar{D}(w_0)$ of the derivative of the right-hand side in Eq. (7) as

$$\begin{aligned} q_c(w_0) = [1/\bar{D}(w_0)] &\left[-c_0 \int_{w_0}^\infty \bar{G}(w - w_0)\mathrm{d}D(w) \right. \\ &\left. + \sum_{i=1}^n a_i c_i \int_{w_0}^\infty \bar{G}_i(w - w_0)\mathrm{d}D(w) \right] B_c(w_0) \\ &- A_c(w_0)(1/\lambda), \end{aligned} \tag{8}$$

where $q_c(0) = -c_0/\lambda < 0$.

We obtain the following theorems with respect to the optimal exchange level w_0^* minimizing the expected cost per unit time in the steady state $C_c(w_0)$ in Eq. (7).

Theorem 1 *There exists at least one positive optimal exchange level* w_0^* $(0 < w_0^* \leq \infty)$. *If* $q_c(\infty) > 0$, *then there exists at least one positive and finite optimal exchange level* w_0^* $(0 < w_0^* < \infty)$.

Proof These results hold clearly since $q_c(0) = -c_0/\lambda < 0$. □

Theorem 2

1. *Suppose that* $q_c(w_0)$ *is strictly increasing.*

(i) *If* $q_c(\infty) > 0$, *then there exists a finite and unique optimal exchange level* w_0^* $(0 < w_0^* < \infty)$ *satisfying* $q_c(w_0) = 0$ *and the corresponding expected cost is*

$$\begin{aligned} C_c(w_0^*) = [\lambda/\bar{D}(w_0^*)] &\left[-c_0 \int_{w_0^*}^\infty \bar{G}(w - w_0^*)\mathrm{d}D(w) \right. \\ &\left. + \sum_{i=1}^n a_i c_i \int_{w_0^*}^\infty \bar{G}_i(w - w_0^*)\mathrm{d}D(w) \right]. \end{aligned} \tag{9}$$

(ii) *If* $q_c(\infty) \leq 0$, *then the optimal exchange level is* $w_0^* \to \infty$, *i.e., the unit continues operation until its failure and then it is replaced by the new one, and the corresponding expected cost is* $C_c(\infty) = A_c(\infty)/B_c(\infty)$.

2. *When* $q_c(w_0)$ *is decreasing, we have* $w_0^* \to \infty$.

Proof These results hold clearly from the monotone properties of $q_c(w_0)$ and Theorem 1. □

2.3 Remarks

In the present model, if we put $c_N = c_i$ $(i = 1, 2, \ldots, n)$, $D(w) = u(w - W_0)$ (unit function), and $w_0 < W_0$, or if we put $c_N = \sum_{i=1}^{n} a_i c_i$, $G(x) = G_i(x)$, $D(w) = u(w - W_0)$, and $w_0 < W_0$, we have

$$A_c(w_0) = c_0 + (c_N - c_0)\left[\bar{G}(W_0) + \int_0^{w_0} \bar{G}(W_0 - u)\mathrm{d}M(u)\right], \tag{10}$$

and

$$B_c(w_0) = [1 + M(w_0)]/\lambda. \tag{11}$$

This is equivalent to the result discussed by Nakagawa [9].

In the present model, we assume that the shock gives the damage to the system with probability 1. Next, we refer to the preventive maintenance policy for the modified continuous type cumulative damage model, where the shock does not always give the damage to the system (see [11]). That is, we consider the situation that the shock generates the damage to the system with probability p $(0 < p \leq 1)$, i.e., it does not with probability $1 - p$. Also for this situation, we can apply our results by using λp instead of λ.

3 Preventive Maintenance Policies for a Cumulative Damage Model with a Discrete Distribution

3.1 Model and Assumptions

We apply items 1–6, 9, 10, and 13 not only in Sect. 2.1 but also in this section, and we rewrite items 7, 8, 11, and 12 for the discrete distribution as follows.

7. The unit fails only when the cumulated amount of damage by shocks exceeds the stochastic failure label r.v. V ($V = 0, 1, 2, \ldots$). The r.v. V obeys a cdf $K(v)$ ($v = 0, 1, 2, \ldots$) with a pmf (probability mass function) $k(v)$ ($k(0) = 0$).
8. When the cumulated amount of damage exceeds the predetermined exchange level v_0 ($v_0 = 0, 1, 2, \ldots$) by any shock, the unit is exchanged if it has not failed, by the spare one immediately (the preventive maintenance). On the other hand, the unit is replaced if it has failed by any shock in a similar fashion (the corrective one).
11. The time interval between $k-1$st shock and kth one is r.v. B_k ($k = 1, 2, 3, \ldots$; $B_k = 0, 1, 2, \ldots$ and r.v. B_1 is the time interval between the time 0 and the first shock), and the amount of damage by kth shock is r.v. D_k ($D_k = 0, 1, 2, \ldots$), where r.v. D_i is independent of r.v. B_j ($i \neq j$).
12. There exist n types of shock modes and the shock mode i occurs with probability a_i ($i = 1, 2, \ldots, n; \sum_{i=1}^{n} a_i = 1; a_i \geq 0$). Under the condition that the mode of kth shock is the ith type, r.v. B_k obeys a cdf $F_i(b)$ ($b = 0, 1, 2, \ldots$) with a pmf $f_i(b)$ ($f_i(0) = 0$), and r.v. D_k obeys a cdf $G_i(d)$ ($d = 0, 1, 2, \ldots$), that is, r. v. B_k obeys the cdf $F(b) = \sum_{i=1}^{n} a_i F_i(b)$ with the pmf $f(b) = \sum_{i=1}^{n} a_i f_i(b)$, and r.v. D_k obeys the cdf $G(d) = \sum_{i=1}^{n} a_i G_i(d)$. Furthermore, we put $\sum_{b=0}^{\infty} b f_i(b) = 1/\lambda_i$ and $\sum_{b=0}^{\infty} b f(b) = 1/\lambda$ and this implies the relation $1/\lambda = \sum_{i=1}^{n} a_i/\lambda_i$. The amount of damage by kth shock, r.v. D_k has a renewal function $M(d)$ and a renewal probability mass function $m(d)$ (see [8]).

Under these model and assumptions, we derive the expected cost per unit time in the steady state and discuss the optimal preventive maintenance policies minimizing that expected cost.

3.2 Analysis and Theorems

The expected cost per one cycle $A_d(v_0)$ is given by

$$\begin{aligned} A_d(v_0) = c_0 \sum_{v=v_0+1}^{\infty} [G(v) - \sum_{l=0}^{v_0} \bar{G}(v-l) m(l)] k(v) \\ + \sum_{i=1}^{n} a_i c_i [\sum_{v=0}^{v_0} \{\bar{G}_i(v) + \sum_{l=0}^{v} \bar{G}_i(v-l) m(l)\} k(v) \\ + \sum_{v=v_0+1}^{\infty} \{\bar{G}_i(v) + \sum_{l=0}^{v_0} \bar{G}_i(v-l) m(l)\} k(v)]. \end{aligned} \tag{12}$$

The mean time of one cycle $B_d(v_0)$ is

$$B_d(v_0) = (1/\lambda) \left[\sum_{v=0}^{v_0} \{1 + M(v)\} k(v) + \{1 + M(v_0)\} \bar{K}(v_0) \right]. \tag{13}$$

We obtain the following when $v_0 = 0$ and $v_0 \to \infty$.

$$A_d(0) = c_0 \sum_{v=1}^{\infty} G(v)k(v) + \sum_{i=1}^{n} a_i c_i \sum_{v=1}^{\infty} \bar{G}_i(v)k(v) , \tag{14}$$

$$B_d(0) = 1/\lambda , \tag{15}$$

$$A_d(\infty) = \sum_{i=1}^{n} a_i c_i \sum_{v=0}^{\infty} [\bar{G}_i(v) + \sum_{l=0}^{v} \bar{G}_i(v-l)m(l)]k(v) , \tag{16}$$

and

$$B_d(\infty) = (1/\lambda) \sum_{v=0}^{\infty} [1 + M(v)]k(v) . \tag{17}$$

Thus, the expected cost per unit time in the steady state is given by

$$C_d(v_0) = \frac{A_d(v_0)}{B_d(v_0)} \tag{18}$$

(see [15], p. 52).

Define the numerator divided by $m(v_0+1)\bar{K}(v_0)$ of the difference of $C_d(v_0)$ in Eq. (18) as

$$\begin{aligned} q_d(v_0) = [1/\bar{K}(v_0)] \Bigg[-c_0 \sum_{v=v_0+1}^{\infty} \bar{G}(v-v_0-1)k(v) \\ + \sum_{i=1}^{n} a_i c_i \sum_{v=v_0+1}^{\infty} \bar{G}_i(v-v_0-1)k(v) \Bigg] B_d(v_0) \\ - A_d(v_0)(1/\lambda). \end{aligned} \tag{19}$$

We obtain the following theorems with respect to the optimal exchange level v_0^* minimizing the expected cost per unit time in the steady state $C_d(v_0)$ in Eq. (18).

Theorem 3

1. *If* $q_d(\infty) > 0$, *then there exists at least one finite optimal exchange level* v_0^* $(0 \le v_0^* < \infty)$.
2. *If* $q_d(0) < 0$, *then there exists at least one positive optimal exchange level* v_0^* $(0 < v_0^* \le \infty)$.

Proof These results hold clearly. □

Theorem 4

1. *Suppose that* $q_d(v_0)$ *is strictly increasing.*
   - (i) *If* $q_d(0) < 0$ *and* $q_d(\infty) > 0$, *then there exists a finite and unique optimal exchange level* v_0^* *(* $0 < v_0^* < \infty$*) satisfying* $q_d(v_0 - 1) < 0$ *and* $q_d(v_0) \geq 0$. *We have the following relationship with respect to the optimal expected cost.*

$$\begin{aligned}
&[\lambda/\bar{K}(v_0^* - 1)]\left[-c_0 \sum_{v=v_0^*}^{\infty} \bar{G}(v - v_0^*)k(v) \right.\\
&\left. + \sum_{i=1}^{n} a_i c_i \sum_{v=v_0^*}^{\infty} \bar{G}_i(v - v_0^*)k(v)\right]\\
&< C_d(v_0^*)\\
&\leq [\lambda/\bar{K}(v_0^*)]\left[-c_0 \sum_{v=v_0^*+1}^{\infty} \bar{G}(v - v_0^* - 1)k(v) \right.\\
&\left. + \sum_{i=1}^{n} a_i c_i \sum_{v=v_0^*+1}^{\infty} \bar{G}_i(v - v_0^* - 1)k(v)\right].
\end{aligned} \tag{20}$$

   - (ii) *If* $q_d(\infty) \leq 0$, *then the optimal exchange level is* $v_0^* \to \infty$, *and the corresponding expected cost is* $C_d(\infty) = A_d(\infty)/B_d(\infty)$.
   - (iii) *If* $q_d(0) \geq 0$, *then the optimal exchange level is* $v_0^* = 0$, *i.e., the unit is exchanged or replaced (failure) by the new one at the first shock. The corresponding expected cost is* $C_d(0) = A_d(0)/B_d(0)$.
2. *When* $q_d(v_0)$ *is decreasing, we have* $v_0^* \to \infty$ *or* $v_0^* = 0$.

Proof These results hold clearly from the monotone properties of $q_d(v_0)$ and Theorem 3. □

3.3 Remarks

We put

$$u(d) = \sum_{j=0}^{d} \delta(j) = 1, d = 0, 1, 2, \ldots, \tag{21}$$

where

$$\delta(d) = \begin{cases} 1, & d = 0, \\ 0, & d = 1, 2, 3, \ldots. \end{cases} \tag{22}$$

In the present model, if we put $c_N = c_i$ $(i = 1, 2, \ldots, n)$, $K(v) = u(v - V_0)$, and $v_0 < V_0$, or if we put $c_N = \sum_{i=1}^{n} a_i c_i$, $G(d) = G_i(d)$, $K(v) = u(v - V_0)$, and $v_0 < V_0$,

$$A_d(v_0) = c_0 + (c_N - c_0)[\bar{G}(V_0) + \sum_{l=0}^{v_0} \bar{G}(V_0 - l)m(l)], \tag{23}$$

and

$$B_d(v_0) = [1 + M(v_0)]/\lambda. \tag{24}$$

This is equivalent to the result discussed by Kaio and Osaki [7].

In the similar fashion of Sect. 2.3, when the shock generates the damage to the system with probability p $(0 < p \leq 1)$, i.e., it does not with probability $1 - p$, we can apply our results by using λp instead of λ.

4 Concluding Remarks

In this chapter, we have discussed the preventive maintenance policies for the extended cumulative damage model, in which the system fails when the cumulated amount of damage by shocks exceeds a stochastic failure level, assuming several shock modes, and a continuous distribution and a discrete one, respectively. We have applied the expected costs per unit time in the steady state as criteria of optimality and sought the optimal policies minimizing these expected costs. We also have shown the relationships between the results of this chapter and ones obtained in the earlier contributions.

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