Chapter 1
A Maximal Regularity Approach to the Analysis of Some Particulate Flows

D. Maity and M. Tucsnak

Abstract This work presents some recent advances in the mathematical analysis of particulate flows. The main idea we want to emphasize is that, for a variety of fluid models the corresponding coupled systems have a common structure, at least in the linearized case. Within this framework, several model problems are considered and studied in detail. This includes a simple toy model, motion of a piston in a heat conducting gas, motion of a rigid body in a viscous incompressible fluid and motion of a solid in a compressible fluid.

Keywords Compressible fluid • Existence • Fluid-structure interactions • Global well-posedness • Heat conducting gas • Navier-Stokes • Uniqueness • Viscous incompressible fluid

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1.1 Introduction and Hilbert Space Analysis of a Toy Model

1.1.1 The General Context

In the remaining part of this work the term *particulate flows* designs the coupled motion of a collection of rigid bodies and of a fluid surrounding them. Such systems occur, for instance, in aerodynamics (flow around an aircraft), medicine (blood flow in vessels), zoology (swimming of aquatic animals). The mathematical study of

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these problems rises several challenges, the main one being due to the fact that the domain filled by the fluid is one of the unknowns of the problem. Another difficulty which has to be tackled is that the dynamics of the system couples equations of different nature: ordinary differential or partial differential equations modeling the solid with the partial differential equations (compressible or incompressible Navier-Stokes) modeling the fluid.

A first important idea we want to develop in this work is that such a system can be mathematically tackled as a perturbation (in an appropriate sense) of the equations describing the fluid alone. More precisely, we see the coupled linearized fluid-structure system like a boundary controlled fluid system, with the boundary control given by an appropriate dynamic feedback which satisfies a "smallness" condition. For the considered applications, this smallness condition follows from a compactness type property of the operator describing the dynamic feedback. We first apply this methodology to a toy problem and then to systems describing particulate flows in a viscous compressible fluid. The incompressible case, apriori simpler, seems more difficult to be included in the general framework we have constructed. For this case we refer to the rich existing literature (see, for instance, Geissert et al. [17] or Martín and Tucsnak [27] and references therein).

A second important idea is that we study the wellposedness of the considered initial and boundary values problems in spaces of functions which are L^p with respect to time and L^q with respect to the space variable, with arbitrary $p,\ q > 1$. Most of the existing literature on the mathematical analysis of particulate flows consider the Hilbert space setting, corresponding to $p = q = 2$. (The only exceptions we are aware of are Geissert et al. [17] Hieber and Murata [19].) Quitting the Hilbert space setting clearly complicates the analysis. This is essentially due to the fact that the maximal regularity of the solutions of the linearized problems is no longer implied by the analytic character of the associated semigroup. Instead, a more sophisticated property of the generators, called $\mathcal{R}$-sectoriality, has to be investigated. One of the advantages of this approach is that the extra integrability properties obtained by taking $p,\ q > 2$ allow us to avoid estimates on higher order derivatives and also to correctly define the changes of variables which naturally occur in the study of particulate flows (such as the equivalence of Eulerian and Lagrangian formulations for compressible flows).

Let us first describe those basic equations which are independent of the properties of the fluid. The domain occupied by the fluid and the particles is $\Omega \subset \mathbb{R}^3$, a connected open bounded set with C^2 boundary. Let $m \in \mathbb{N}$ be the number of particles let $h_1, h_2, \ldots, h_m$ be the (variable) positions of their centers of mass. For every $k \in \{1, 2 \ldots, m\}$ we denote by R_k the proper orthogonal matrix (also a variable one) giving the orientation of the kth particle, whose position is thus given by

$$S(h_j, R_j) = h_j + R_j(S_{0,j} - h_{0,j}) \qquad (j \in \{1, \ldots, m\}),$$

where $S_{0,j}$ and $h_{0,j}$ stand, for each $j \in \{1, \ldots, m\}$ for the set occupied by the jth solid, respectively the position of its center of mass, at $t = 0$. The fluid is supposed to be incompressible, homogeneous with *density* $\rho > 0$ and it occupies the domain

$$F(h_1, R_1 \dots h_m, R_m) := \Omega \setminus \bigcup_{k=1}^{m} S(h_k, R_k).$$

Regardless the considered type of fluid, we know that the *Cauchy equations* hold in fluid domain. More precisely, we have

$$\rho\,[\dot{v} + (v \cdot \nabla)v] - \operatorname{div}\sigma = \rho b \qquad (t \geqslant 0,\ \ x \in F(h_1(t), R_1(t), \dots, h_m(t), R_m(t)), \tag{1.1}$$

v is the Eulerian velocity field of the fluid, σ is its Cauchy stress field and b is the density of exterior forces (supposed to be known). The equations of motion of the solids are given by Newton's laws and they can be written

$$M_j \ddot{h}_j = -\int_{\partial S(h_j(t),R_j(t))} \sigma n \,\mathrm{d}\Gamma + \int_{S(h_j(t),R_j(t))} \rho_j b \,\mathrm{d}x, \quad t \geqslant 0, j = 1, \dots, m, \tag{1.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(J_j\omega_j) = -\int_{\partial S(h_j(t),R_j(t))} (x - h_j) \times \sigma n \,\mathrm{d}\Gamma$$
$$+ \int_{S(h_j(t),R_j(t))} (x - h_j) \times \ \rho_j b \,\mathrm{d}x, \quad t \geqslant 0, j = 1, \dots, m, \tag{1.3}$$

$$\frac{\mathrm{d}R_j}{\mathrm{d}t}(t) = A(\omega_j(t))R_j(t) \quad t \geqslant 0, j = 1, \dots, m, \tag{1.4}$$

where ρ_j is the density of the solid $S(h_j(t), R_j(t))$ (supposed to be a known constant), $\omega_j(t)$ is its angular velocity, the notation $\times$ stands for the usual vector product in $\mathbb{R}^3$, whereas n denotes the unitary normal vector field to $\partial S(h_j(t), R_j(t))$ oriented towards the interior of each solid. The skew symmetric matrix $A(\omega)$ is defined by

$$A(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \text{for all} \quad \omega \in \mathbb{R}^3. \tag{1.5}$$

Moreover, for every $j \in \{1, \dots, m\}$, M_j stands for the mass of $S(h_j(t), R_j(t))$ and $J(h_j(t), R_j(t))$ denotes the inertia matrix of $S(h_j(t), R_j(t))$ defined by

$$J(h_j(t), R_j(t))a \cdot b$$
$$= \rho_j \int_{S(h_j(t),R_j(t))} \left[a \times (x - h_j(t))\right] \cdot \left[b \times (x - h_j(t))\right] \mathrm{d}x \qquad \text{for all} \quad a, b \in \mathbb{R}^3. \tag{1.6}$$

In order to close the system, Eqs. (1.1)–(1.4) have to be supplemented with a constitutive law for the fluid, with appropriate boundary conditions and with the initial conditions, valid jour $j \in \{1, \dots, m\}$,

$$v(x, 0) = v_0(x) \qquad (x \in F(h_{0,j}, R_{0,j})), \tag{1.7}$$

$$h_j(0) = h_{0,j},\ \dot{h}_j(0) = g_{0,j},\ R_j(0) = R_{0,j},\ \omega_j(0) = \omega_{0,j} \tag{1.8}$$

The constitutive law and the boundary conditions in the case of a viscous incompressible fluid will be introduced in the following sections.

The outline of this work is as follows. In the next subsection, in order to give a flavour of the results to come, we introduce a toy model in one space dimension, in which the Navier-Stokes system is replaced by the viscous Burgers equations. The solid is replaced by a mass-point evolving under the action of the surrounding "Burgers" fluid. In the remaining part of the first section we develop the existence and uniqueness theory for the corresponding coupled PDE system, in a Hilbert space framework. The second section is devoted to the introduction of several more realistic models of fluid-structure interactions. More precisely we consider the systems modelling the motion of a piston in a 1D viscous heat conducting gas, then of a rigid body in a viscous incompressible fluid and finally the motion of a rigid body in a three dimensional viscous compressible fluid filling a bounded domain. Section 1.3 contains an introduction to the theory of maximal regularity for evolution equations, namely those which are associated to $\mathcal{R}$-sectorial operators. Moreover, we make precise here the common structure of the linearized problems for various particulate flow systems, and we prove a useful perturbation result. Section 1.4 first revisits the analysis of the toy problem introduced in Chap. 1, this time in an $L^p - L^q$ setting. The last part of this chapter is devoted to local in time existence results, still in an $L^p - L^q$ setting, for the two other systems introduced in Sect. 1.2.

1.1.2 Introduction of a Toy Model

The viscous Burgers equation is often used as a toy model for the Navier-Stokes equations. In this section we consider a similar simplification for the system describing the motion of a rigid body in a viscous fluid. Assuming that, instead of the Navier-Stokes equations, the fluid is described by the one dimensional viscous Burgers equation, the system writes

$$\begin{cases} \dot{v}(t, y) - v_{yy}(t, y) + v(t, y)v_y(t, y) = 0 & t \geqslant 0,\ y \in (-1, 1),\ y \neq h(t), \\ v(t, -1) = v(t, 1) = 0 & t \geqslant 0, \\ \dot{h}(t) = v(t, h(t)) & t \geqslant 0, \\ \ddot{h}(t) = [v_y](t, h(t)) & t \geqslant 0, \\ v(0, y) = v_0(y) & y \in (-1, 1), \\ h(0) = h_0, \qquad \dot{h}(0) = g_0. \end{cases} \tag{1.9}$$

In (1.9), $v = v(t, y)$ denotes the Eulerian velocity field of the fluid filling the interval $(-1, 1)$, whereas $h = h(t)$ indicates the position of the point mass and the derivative with respect to time is denoted by a dot. Moreover, the force exerted by the fluid on the mass is given by the jump of the derivative of v when crossing the mass, denoted by $[v_y](t, h(t))$. For the sake of simplicity, we have assumed that the mass of the body, the viscosity and the density of the fluid are equal to one.

The main result of this chapter reads as follows:

Theorem 1.1 *Assume that* $v_0 \in H_0^1(-1, 1)$, $h_0 \in (-1, 1)$ *and* $g_0 \in \mathbb{R}$ *are such that* $v_0(h_0) = g_0$. *Then the system* (1.9) *admits a unique solution* $\begin{bmatrix} v \\ h \end{bmatrix}$ *with*

$$v \in C([0, \infty); H_0^1(-1, 1)) \cap H_{\text{loc}}^1((0, \infty); L^2(-1, 1)), \qquad h \in H^2\left((0, T), (-1, 1)\right),$$

with the restriction of v *to* $x \in (-1, h_0)$ *(respectively to* $(h_0, 1)$*) in* $L_{\text{loc}}^2((0, \infty); H^2(-1, h_0))$ *(respectively in* $L_{\text{loc}}^2((0, \infty); H^2(h_0, 1))$*).*

Note that the global character of the wellposedness result above implies that the mass point does not reach the extremities of the interval, i.e. the solid will not touch the boundary. The methodology used in next section extends to the case of several point-masses and in this case we can show that the point-masses do not collide in finite time.

1.1.3 Change of Variables

An important step in proving our wellposedness results is to use a change of variables mapping the time dependent interval $[-1, h(t)]$ (respectively $[h(t), 1]$) on the fixed one $[-1, h_0]$ (respectively $[h_0, 1]$). More precisely, we set $z(t, x) = v\,(t, y)$, where

$$x = \begin{cases} \frac{(h_0+1)y+h_0-h(t)}{h(t)+1} & (y \in [-1, h(t)]), \\ \frac{(h_0-1)y+h(t)-h_0}{h(t)-1} & (y \in [h(t), 1]). \end{cases} \tag{1.10}$$

It is easily checked that (1.10) can be rewritten as

$$y = \frac{(1 - kh(t))x - h_0 + h(t)}{1 - kh_0}, \qquad k = \operatorname{sgn}(x - h_0). \tag{1.11}$$

The following proposition shows that by using the change of variable (1.10) the system (1.9) is equivalent with a system written in a fixed spatial domain.

Proposition 1.2 *Let* $T > 0$, $v_0 \in L^2[-1, 1]$, $h_0 \in (-1, 1)$, $g_0 \in \mathbb{R}$, *and assume that*

$$v \in C([0, T]; H_0^1(-1, 1)) \cap H^1((0, T); L^2(-1, 1)), \qquad h \in H^2\left((0, T), (-1, 1)\right),$$

Then $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ *is a solution of* (1.9) *on* $[0,T]$ *if and only if, the triplet* $\begin{bmatrix} z \\ g \\ h \end{bmatrix}$*, where* $z(t,x) = v\,(t,y)$*, with* x *given by* (1.10)*, satisfies, for every* $t \in [0,T]$*,*

$$\begin{cases} \dot{z} - z_{xx} = \frac{k(h-h_0)}{1-kh}\left[2 + \frac{k(h-h_0)}{1-kh}\right] z_{xx} + \frac{1-kx}{1-kh} g z_x - \frac{1-kh_0}{1-kh} z z_x, \; x \in (-1,1)\setminus h_0 \\ z(t,-1) = z(t,1) = 0 \\ z(t,h_0) = g(t) \\ \dot{g} - [z_x]\,(t,h_0) = (h-h_0)\left[\frac{kz_x}{1-kh}\right](t,h_0) & t \in (0,T) \\ \dot{h}(t) = g(t) \\ z(0,x) = z_0(x) & x \in (-1,1) \\ h(0) = h_0, \qquad g(0) = g_0. \end{cases} \tag{1.12}$$

Proof Using the change of variables (1.10)–(1.11), simple calculations show that (1.9) can be rewritten, for $t \in [0,T]$:

$$\begin{cases} (1-kh)\dot{z} - \frac{(1-kh_0)^2}{1-kh} z_{xx} - (1-kx) g z_x + (1-kh_0) z z_x = 0, \; x \in (-1,1)\setminus h_0 \\ z(t,-1) = z(t,1) = 0 \\ z(t,h_0) = g(t) \\ m\dot{g}(t) = \left[\frac{1-kh_0}{1-kh} z_x\right](t,h_1) \\ \dot{h}(t) = g(t) \\ z(0,x) = z_0(x), & x \in (-1,1), \\ h(0) = h_0, \qquad g(0) = g_0. \end{cases} \tag{1.13}$$

After some simple calculations we see that the above equations are equivalent to the system (1.12). □

1.1.4 Local in Time Existence and Uniqueness of Solutions

The main result in this section states as follows.

Proposition 1.3 *Assume that* $v_0 \in H_0^1(-1,1)$*,* $h_0 \in (-1,1)$ *and* $g_0 \in \mathbb{R}$ *are such that* $v_0(h_0) = g_0$*. Then there exists* $T_{max} > 0$ *such that for every* $T \in (0, T_{max})$*, the system* (1.9) *admits a unique solution*

$$v \in C([0,T]; H_0^1(-1,1)) \cap H^1((0,T); L^2(-1,1)), \qquad h \in H^2\left((0,T),(-1,1)\right), \tag{1.14}$$

with the restriction of v *to* $x \in (-1,h_0)$ *(respectively to* $(h_0,1)$*) in* $L^2((0,T); H^2(-1,h_0))$ *(respectively in* $L^2((0,T); H^2(h_0,1))$*). Moreover, for every* $t \in [0,T_{\max})$

we have

$$\frac{1}{2}\int_{-1}^{1} v^2(t,y)\mathrm{d}y + \frac{1}{2}(\dot{h}(t))^2 = -\int_0^t \int_{-1}^{1} v_y(\sigma,y)^2\,\mathrm{d}y\,\mathrm{d}\sigma - \int_0^t \dot{h}^2(\sigma)\,\mathrm{d}\sigma. \tag{1.15}$$

Finally, only one of the alternatives holds true

1. *The solution is global, i.e.* $T_{max} = \infty$.
2. *We have either that* $\inf_{t\in[0,T_{max})}(1-h(t)) = 0$ *(which means that the mass touches the boundary) or that* $\sup_{t\in[0,T_{max})} \|v(t,\cdot)\|_{H_0^1(-1,1)} = \infty$.

An important role in the proof of the above proposition is played by a self-adjoint operator which we introduce below. Consider the Hilbert space

$$H = L^2(-1,1)\times\mathbb{R},$$

endowed with the inner product

$$\left\langle \begin{bmatrix}\varphi_1\\ p_1\end{bmatrix}, \begin{bmatrix}\varphi_2\\ p_2\end{bmatrix}\right\rangle = \int_{-1}^{1}\varphi_1(x)\varphi_2(x)\,\mathrm{d}x + p_1p_2. \tag{1.16}$$

We define the unbounded operator $A_0 : \mathcal{D}(A_0) \to H$,

$$\mathcal{D}(A_0) = \left\{ \begin{bmatrix}\varphi\\ p\end{bmatrix} \in \begin{matrix} H_0^1(-1,1)\\ \times\\ \mathbb{R}\end{matrix} \;\middle|\; \begin{matrix}\varphi|_{(-1,h_0)} \in H^2(-1,h_0),\\ \varphi|_{(h_0,1)} \in H^2(h_0,1),\\ \varphi(h_0) = p\end{matrix} \right\}. \tag{1.17}$$

$$A_0\begin{bmatrix}\varphi\\ p\end{bmatrix} = \begin{bmatrix}-\varphi_{xx}\\ -[\varphi_x](h_0)\end{bmatrix} \qquad \left(\begin{bmatrix}\varphi\\ p\end{bmatrix} \in \mathcal{D}(A_0)\right). \tag{1.18}$$

Proposition 1.4 *The operator* A_0 *is positive in* H. *Moreover, the corresponding space* $H_{\frac{1}{2}}$ *(i.e.,* $\mathcal{D}(A_0^{\frac{1}{2}})$ *endowed with the graph norm of* $A_0^{\frac{1}{2}}$*) is*

$$H_{\frac{1}{2}} = \left\{ \begin{bmatrix}\varphi\\ p\end{bmatrix} \in H_0^1(-1,1)\times\mathbb{R} \;\middle|\; \varphi(h_1) = p \right\}, \tag{1.19}$$

endowed with the inner product

$$\left\langle \begin{bmatrix}\varphi_1\\ p_1\end{bmatrix}, \begin{bmatrix}\varphi_2\\ p_2\end{bmatrix}\right\rangle_{\frac{1}{2}} = \int_{-1}^{1}\varphi_{1,x}(x)\varphi_{2,x}(x)\,\mathrm{d}x. \tag{1.20}$$

Proof We first check that A_0 is symmetric. Indeed, for any $\Phi_i = \begin{bmatrix}\varphi_i\\ p_i\end{bmatrix} \in D(A_0)$, $i = 1, 2$, we have that

$$\begin{aligned}\langle A_0\Phi_1, \Phi_2\rangle &= -\int_{-1}^{h_1} \varphi_{1,xx}(x)\varphi_2(x)\,\mathrm{d}x\\ &\quad -\int_{h_1}^{1} \varphi_{1,xx}(x)\varphi_2(x)\,\mathrm{d}x - [\varphi_{1,x}](h_1)p_2 - [\varphi_{1,x}](h_1)r_2\\ &= \int_{-1}^{1} \varphi_{1x}(x)\varphi_{2x}(x)\,\mathrm{d}x = \langle \Phi_1, A_0\Phi_2\rangle. \end{aligned} \tag{1.21}$$

We next check that A_0 is onto. For $F = \begin{bmatrix}f\\ g\end{bmatrix} \in H$, the equation $A_0\Phi = F$, of unknown $\Phi = \begin{bmatrix}\varphi\\ p\end{bmatrix} \in \mathcal{D}(A_0)$ writes

$$\begin{cases} -\varphi_{xx}(x) = f(x) & x \in (-1, h_1) \cup (h_1, 1)\\ \varphi(a) = p\\ -[\varphi_x](h_1) = g. \end{cases}$$

Elementary considerations on the differential equation $-\varphi_{xx} = f$ show that the above system has a unique solution $\begin{bmatrix}\varphi\\ p\end{bmatrix} \in \mathcal{D}(A_0)$ so that A_0 is onto. Since we have already shown that A_0 is symmetric, a classical result (see, for instance, [34, Proposition 3.2.4]) implies that A_0 is self-adjoint.

On the other hand, taking $\Phi_1 = \Phi_2 = \Phi = \begin{bmatrix}\varphi\\ p\end{bmatrix}$ in (1.21) we see that,

$$\langle A_0\Phi, \Phi\rangle = \int_{-1}^{1} \varphi_x^2(x)\,\mathrm{d}x,$$

which implies (1.20). □

As a consequence of the positivity A_0 and of a classical result (see, for instance, Lemma 3.3 and Theorem 3.1 of [4]),we obtain:

Corollary 1.5 *For every $t_0, t_1 > 0$, $Y_0 \in H$ and $f \in L^2([t_0, t_1], H)$ there exists a unique $Y \in C\left([t_0, t_1], H_{\frac{1}{2}}\right) \cap L^2([t_0, t_1], H_1)$ such that*

$$\begin{cases} \dot{Y}(t) + A_0Y(t) = f(t) \ t \in (t_0, t_1)\\ Y(t_0) = Y_0. \end{cases} \tag{1.22}$$

Moreover, there exists an absolute positive constant K such that, for every $Y_0 \in H_{\frac{1}{2}}$ and $f \in L^2([t_1, t_2], H)$, we have

$$\|Y\|^2_{\mathcal{C}\left([t_1,t_2],H_{1/2}\right)} + \|A_0 Y\|^2_{L^2([t_1,t_2],H)} \leqslant \|A_0^{\frac{1}{2}} Y_0\|^2_H + K\|f\|^2_{L^2([t_1,t_2],H)} \qquad (Y_0 \in H_{1/2},\ f \in L^2\left([t_1, t_2], H\right)). \tag{1.23}$$

Remark 1.6 In PDE terms the above corollary says that if $T > 0$, $z_0 \in H_0^1(-1, 1)$, $g_0 \in \mathbb{R}$, $f_1 \in L^2([0, T], L^2(-1, 1))$ and $f_2 \in L^2[0, T]$, are such that $v_0(h_0) = g_0$ then then the solution $\begin{bmatrix} z \\ g \end{bmatrix}$ of the system

$$\begin{cases} \dot{z}(t, x) - z_{xx}(t, x) = f_1(t, x), \ x \in (-1, h_0) \cup (h_0, 1), & t \in (0, T), \\ z(t, -1) = z(t, 1) = 0, & t \in (0, T), \\ z(t, h_0) = g(t), & t \in (0, T), \\ \dot{g}(t) - [z_x]\,(t, h_0) = f_2(t), & t \in (0, T), \\ z(0, x) = z_0(x), & x \in (-1, 1), \\ g(0) = g_0, \end{cases} \tag{1.24}$$

satisfies

$$\|z\|^2_{C([0,T],H_0^1(-1,1))} + \|g\|^2_{C[0,T]} + \|z\|^2_{L^2[0,T],H^2(-1,h_0))} + \|z\|^2_{L^2[0,T],H^2(h_0,1))} \leqslant \|z_0\|^2_{H_0^1(-1,1)} + |g_0|^2 + K\left(\|f_1\|^2_{L^2([0,T],L^2(-1,1))} + \|f_2\|^2_{L^2[0,T]}\right). \tag{1.25}$$

Another important ingredient are the properties of the operators $(\mathcal{G}_k)_{k=1}^4$ which are defined (as suggested by the right hand side of (1.12)) by

$$\mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t, x) = \frac{k(h(t) - h_0)}{1 - kh(t)} \left[2 + \frac{k(h(t) - h_0)}{1 - kh(t)}\right] z_{xx}(t, x), \tag{1.26}$$

$$\mathcal{G}_2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t, x) = \frac{1 - kx}{1 - kh(t)} g(t) z_x(t, x), \tag{1.27}$$

$$\mathcal{G}_3 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t, x) = -\frac{1 - kh_0}{1 - kh(t)} z(x, t) z_x(t, x), \tag{1.28}$$

$$\mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t) = (h(t) - h_0) \left[\frac{kz_x}{1 - kh}\right](t, h_0), \tag{1.29}$$

where z, g satisfy (1.24) and

$$h(t) = h_0 + \int_0^t g(\sigma)\, \mathrm{d}\sigma. \tag{1.30}$$

We give below some of the properties of these operators.

Lemma 1.7 *Let $T > 0$ and let $\mathcal{G}_k$, with $k \in \{1,2,3,4\}$, be the operators defined in (1.26)–(1.29). Then, for every $k \in \{1,2,3\}$, the operator $\mathcal{G}_k$ maps $L^2([0,T], L^2[-1,1])$ to $L^2([0,T], L^2[-1,1])$, whereas $\mathcal{G}_4$ maps $L^2[0,T]$ to $L^2[0,T]$. Moreover, assume that*

$$\|f_1\|^2_{L^2([0,T],L^2[-1,1])} + \|f_2\|^2_{L^2([0,T]} \leqslant R^2, \quad \|v_0\|^2_{H^1_0(-1,1)} + |g_0|^2 \leqslant M^2, \quad 1-|h_0| \geqslant \varepsilon, \tag{1.31}$$

for some $R,\ M,\ \varepsilon > 0$. Then there exists a constant $C = C(\varepsilon) > 0$ such that for every $T \leqslant \frac{\varepsilon}{2\sqrt{M^2+KR^2}}$ (with K being the constant in (1.25)) we have

$$\left\| \mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2([0,T],L^2[-1,1])} \leqslant TC(\varepsilon)\,(M^2 + KR^2) \qquad (k \in \{1,2,3\}), \tag{1.32}$$

$$\left\| \mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2[0,T]} \leqslant \sqrt{T} C(\varepsilon) \left(M^2 + KR^2\right). \tag{1.33}$$

Finally, if h is defined by (1.30) we have that

$$|h(t)| \leqslant 1 - \frac{\varepsilon}{2} \qquad (t \in [0,T]). \tag{1.34}$$

Proof In order to prove (1.34) it suffices to note that, using (1.25), we have

$$|h(t)| \leqslant |h_0| + \int_0^T |g(\sigma)|\, \mathrm{d}\sigma \leqslant 1 - \varepsilon + T\sqrt{M^2 + KR^2} \leqslant 1 - \frac{\varepsilon}{2} \qquad (t \in [0,T]).$$

The facts that $\mathcal{G}_1,\ \mathcal{G}_2,\ \mathcal{G}_3$ map $L^2([0,T], L^2[-1,1])$ to $L^2([0,T], L^2[-1,1])$ and that $\mathcal{G}_4$ maps $L^2[0,T]$ to $L^2[0,T]$ follow from (1.25) and from simple Sobolev embeddings.

In the remaining part of this proof we denote by $\widetilde{C}(\varepsilon)$ a generic positive constant depending only on ε.

In order to prove (1.32) we first note that (1.34) implies that

$$\frac{1}{1-kh(t)} \leqslant \frac{2}{2-\varepsilon} \qquad (t \in [0,T]). \tag{1.35}$$

By combining (1.26) and (1.35) it follows that

$$\left\| \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2([0,T],L^2[-1,1])} \leqslant T\widetilde{C}(\varepsilon)\, \|g\|_{C([0,T],L^2[-1,1])}\, \|z_{xx}\|_{L^2(0,T,L^2(-1,1))}.$$

Combining the last estimate with (1.25) we obtain that (1.32) holds for $k = 1$.

In order to prove (1.32) holds for $k = 2$, we note that the definition of $\mathcal{G}_2$, combined with (1.35) and the Cauchy-Schwarz inequality, imply that

$$\left\| \mathcal{G}_2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} (t,x) \right\|_{L^2[0,T],L^2(-1,1))} \leqslant \widetilde{C}(\varepsilon) \|g\|_{L^2[0,T]} \|z_x\|_{L^2(0,T,L^2[-1,1])}$$

$$\leqslant T\widetilde{C}(\varepsilon)\|g\|_{C[0,T]}\|z_x\|_{C([0,T],L^2[-1,1])} \leqslant T\widetilde{C}(\varepsilon)\left(\|g\|^2_{C[0,T]} + \|z_x\|^2_{C([0,T],L^2[-1,1])}\right).$$

The last estimate and (1.25) imply that (1.32) holds for $k = 2$.

The fact that (1.32) holds for $k = 3$ can be proved in a completely similar manner, so we omit the details.

In order to prove (1.33) we note that the definition (1.29) of $\mathcal{G}_4$, estimate (1.35) and a classical trace theorem imply that

$$\left| (h(t) - h_0) \left[\frac{kz_x}{1 - kh} \right] (t, h_0) \right|$$

$$\leqslant \widetilde{C}(\varepsilon) \int_0^T |g(t)|\,\mathrm{d}t \left(\|z(t,\cdot)\|_{H^2(-1,h_0))} + \|z(t,\cdot)\|_{H^2(h_0,1))} \right).$$

The above estimate and (1.25) imply that

$$\left\| (h(t) - h_0) \left[\frac{kz_x}{1 - kh} \right] (t, h_0) \right\|_{L^2[0,T]}$$

$$\leqslant \sqrt{T}\widetilde{C}(\varepsilon) \left(\|g\|^2_{C[0,T]} + \|z(t,\cdot)\|^2_{H^2(-1,h_0))} + \|z(t,\cdot)\|^2_{H^2(h_0,1))} \right),$$

which, combined with (1.25), yields (1.33). □

Lemma 1.8 *With the notation and assumptions in Lemma 1.8, suppose that* $\widetilde{f}_1, \widetilde{f}_2 \in L^2([0,T], L^2[-1,1])$ *satisfy*

$$\|\widetilde{f}_1\|^2_{L^2([0,T],L^2[-1,1])} + \|\widetilde{f}_2\|^2_{L^2([0,T]} \leqslant R^2, \tag{1.36}$$

Then there exists a constant $C = C(\varepsilon) > 0$ *such that for every* $T \leqslant \frac{\varepsilon}{2\sqrt{M^2+KR^2}}$ *(with* K *being the constant in* (1.25)*) we have*

$$\left\| \mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_k \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{L^2([0,T],L^2[-1,1])}$$

$$\leqslant TC(\varepsilon)\sqrt{M^2 + KR^2}\left(\|f_1 - \widetilde{f}_1\|_{L^2([0,T],L^2[-1,1])} + \|f_2 - \widetilde{f}_2\|_{L^2[0,T]}\right) \qquad (k \in \{1,2,3\}), \tag{1.37}$$

$$\left\| \mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_4 \begin{bmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{bmatrix} \right\|_{L^2[0,T]}$$
$$\leqslant \sqrt{T} C(\varepsilon) \sqrt{M^2 + KR^2} \left(\|f_1 - \widetilde{f_1}\|_{L^2([0,T],L^2[-1,1])} + \|f_2 - \widetilde{f_2}\|_{L^2[0,T]} \right). \tag{1.38}$$

Proof The proof is based on estimates which are very close to those used in proving Lemma 1.7. More precisely, we first note that, estimate (1.35) for h and for $\widetilde{h}$ implies that

$$\left\| \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_1 \begin{bmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{bmatrix} \right\|_{L^2([0,T],L^2[-1,1]} \leqslant \widetilde{C}(\varepsilon) \|h - h_0\|_{L^\infty[0,T]} \|z_{xx} - \widetilde{z}_{xx}\|_{L^2([0,T],L^2[-1,1])}$$
$$+ \widetilde{C}(\varepsilon) \left\| (h - h_0) z_{xx} (1 - k\widetilde{h}) - (\widetilde{h} - h_0) \widetilde{z}_{xx} (1 - kh) \right\|_{L^2([0,T],L^2[-1,1])}. \tag{1.39}$$

Using the inequality

$$\|h - h_0\|_{C[0,T]} \leqslant T \|g\|_{C[0,T]}, \tag{1.40}$$

together with (1.25) it follows that the first term in the right hand side of (1.39) satisfies

$$\|h - h_0\|_{L^\infty[0,T]} \|z_{xx} - \widetilde{z}_{xx}\|_{L^2([0,T],L^2[-1,1])}$$
$$\leqslant T \sqrt{M^2 + KR^2} \left(\|f_1 - \widetilde{f_1}\|_{L^2([0,T],L^2[-1,1])} + \|f_2 - \widetilde{f_2}\|_{L^2[0,T]} \right). \tag{1.41}$$

Concerning the second term in the right hand side of (1.39) we note that

$$\left\| (h - h_0) z_{xx} (1 - k\widetilde{h}) - (\widetilde{h} - h_0) \widetilde{z}_{xx} (1 - kh) \right\|_{L^2([0,T],L^2[-1,1])}$$
$$\leqslant \|(h - h_0) z_{xx}\|_{L^2([0,T],L^2[-1,1])} \; \|h - \widetilde{h}\|_{L^\infty[0,T]}$$
$$+ \|(h - h_0)(1 - kh)\|_{L^\infty[0,T]} \; \|z_{xx} - \widetilde{z}_{xx}\|_{L^2([0,T],L^2[-1,1])}$$
$$+ \|\widetilde{z}_{xx}(1 - kh)\|_{L^2([0,T],L^2[-1,1])} \; \|h - \widetilde{h}\|_{L^\infty[0,T]}.$$

Using in the above inequality the fact that

$$\|h - \widetilde{h}\|_{L^\infty[0,T]} \leqslant T \|g - \widetilde{g}\|_{L^\infty[0,T]}, \tag{1.42}$$

together with (1.42) and (1.25), we obtain that

$$\left\| (h - h_0) z_{xx} (1 - k\widetilde{h}) - (\widetilde{h} - h_0) \widetilde{z}_{xx} (1 - kh) \right\|_{L^2([0,T],L^2[-1,1])}$$
$$\leqslant T \widetilde{C}(\varepsilon) \sqrt{M^2 + KR^2} \left(\|f_1 - \widetilde{f_1}\|_{L^2([0,T],L^2[-1,1])} + \|f_2 - \widetilde{f_2}\|_{L^2[0,T]} \right). \tag{1.43}$$

By combining (1.41) and (1.43) we obtain that (1.37) holds for $k = 1$. The proof of (1.37) for $k \in \{2, 3\}$ is very similar (but quite tedious) so we omit it here.

In order to prove (1.38) we note that from the definition (1.29) of $\mathcal{G}_4$ and from (1.35) it follows that

$$\left\| \mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_4 \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{L^2[0,T]} \leqslant \|h(t) - h_0\|_{C[0,T]} \left\| \left[\frac{k(z_x - \widetilde{z}_x)}{1 - kh} \right](t, h_0) \right\|_{L^2[0,T]}$$

$$+ \|h - \widetilde{h}\|_{C[0,T]} \left\| \left[\frac{\widetilde{z}_x)}{1 - kh} \right](t, h_0) \right\|_{L^2[0,T]} + \gamma\sqrt{T}\|g - \widetilde{g}\|_{C[0,T]} + \sqrt{T}\|h - \widetilde{h}\|_{C[0,T]}.$$

The above estimate, combined with (1.35), (1.40), (1.42) and (1.25), implies the conclusion (1.38). □

We are now in a proposition to prove the main result in this section.

Proof of Proposition 1.3 Let

$$\mathcal{X} = L^2([0, T], L^2[-1, 1]) \times L^2[0, T],$$

and let $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ be defined by

$$\mathcal{N} = \begin{bmatrix} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 \\ \mathcal{G}_4 \end{bmatrix},$$

where $(\mathcal{G}_k)_{1 \leqslant k \leqslant 4}$ have been defined in (1.26)–(1.29).

Let $M > 0$ be such that

$$\|z_0\|^2_{H^1_0(-1,1)} + |g_0|^2 \leqslant M^2, \tag{1.44}$$

and let $\varepsilon > 0$ such that

$$|h_0| \leqslant 1 - \varepsilon. \tag{1.45}$$

We denote by $\mathcal{B}_M$ the ball in $\mathcal{X}$ of radius M. From Lemma 1.7 it follows that

$$\left\| \mathcal{N} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{X}} \leqslant (T + \sqrt{T})C(\varepsilon)(M^2 + M + 1) \qquad \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right) \in \mathcal{B}_M).$$

The last estimate implies that $\mathcal{N}$ maps $\mathcal{B}_M$ into $\mathcal{B}_M$ if

$$T \leqslant \left[C(\varepsilon) M^2 \right]^{-1}. \tag{1.46}$$

By applying Lemma 1.8 it follows that

$$\left\| \mathcal{N}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{N}\begin{bmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{bmatrix} \right\|_{\mathcal{X}} \leqslant \sqrt{T}\frac{C(\varepsilon)(M^2+M+1)}{M} \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{bmatrix} \right\|_{\mathcal{X}} \qquad \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ \begin{bmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{bmatrix} \in \mathcal{B}_M \right).$$

The last estimate implies that the restriction of $\mathcal{N}$ to $\mathcal{B}_M$ is a strict contraction provided that

$$T \leqslant \frac{1}{2}\left[\frac{C(\varepsilon)(M^2+M+1)}{M}\right]^{-1}. \tag{1.47}$$

Consequently, for every T satisfying (1.46) and (1.47) we have that $\mathcal{N}$ has a unique fixed point $\begin{bmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{bmatrix} \in \mathcal{B}_M$. Denoting by $\begin{bmatrix} \widehat{z} \\ \widehat{g} \end{bmatrix}$ the solution of (1.24) with $f_1 = \widehat{f}_1$ and $f_2 = \widehat{f}_2$ we clearly have that $\begin{bmatrix} v \\ h \end{bmatrix}$ with

$$v(t,\cdot) = \widehat{z}(t,\cdot), \quad h(t) = h_0 + \int_0^t g(\sigma)\,\mathrm{d}\sigma \qquad (t \in [0,T]), \tag{1.48}$$

satisfy all the equations in (1.9), with the restriction of v to $x \in (-1, h_0)$ (respectively to $(h_0, 1)$) in $L^2([0,T]; H^2(-1,h_0))$ (respectively in $L^2([0,T]; H^2(h_0,1))$). Moreover, according to (1.34) we have that $h(t) \in (-1,1)$ for every $t \in [0,T]$, so that $\begin{bmatrix} v \\ h \end{bmatrix}$ is indeed the desired local in time solution of (1.9).

According to classical arguments, this solution can be extended to a solution defined on $[0, T_{max})$.

Finally, assume that both assertions in the second alternative in Proposition 1.3 are false. Denoting

$$M = \sup_{t\in[0,T_{max})} \|v(t,\cdot\|_{H_0^1(-1,1)}, \quad \varepsilon = \inf_{t\in[0,T_{max})} (1 - |h(t)|),$$

the first part of the proof shows that there exists $\delta = \delta(\varepsilon, M) > 0$ such that for every $t \in [0, T_{max})$ the solution can be extended on $[t, T+\delta]$. This clearly implies that $T_{max} = \infty$, i.e., that the solution is global. □

1.1.5 Proof of the Global Well-Posedness Result

The key estimates used to prove the above theorem are given in the result below.

Proposition 1.9 *With the notation and assumptions in Proposition 1.3, let* $T \in [0, T_{max})$ *and let* $\begin{bmatrix} v \\ h \end{bmatrix} : [0,T] \to H_0^1(-1,1) \times \mathbb{R}$ *be the local in time solution of* (1.9) *constructed in Proposition 1.3. Moreover, assume that the initial data* v_0 *satisfies* $\|v_0\|_{H^1(-1,1)} \leqslant M$*, for some* $M > 0$*. Then there exists a constant* $K = K(M,T)$ *such that*

$$\|v(t,\cdot)\|_{H_0^1(-1,1)} \leqslant K(M,T) \qquad (t \in [0;T]). \tag{1.49}$$

$$\int_0^T \left[\int_{-1}^{h(t)} v_{yy}^2(t,y)\,\mathrm{d}y + \int_{h(t)}^1 v_{yy}^2(t,y)\,\mathrm{d}y + |\ddot{h}(t)|^2 \right] \mathrm{d}t \leqslant K(M,T). \tag{1.50}$$

Proof We follow step by step the method used in [36].

Multiplying the first equation in (1.9) v_{yy} and integrating on $(-1, h_0)$ and $(h_0, 1)$, we obtain that for every $t \in [0,T]$ we have

$$\int_{-1}^{h(t)} v_{yy}^2\,\mathrm{d}y = -\frac{1}{2}\int_{-1}^{h(t)} \frac{\partial}{\partial t}(v_y^2)\,\mathrm{d}y + \dot{v}(t,h(t)-0)v_y(t,h(t)-0) + \int_{-1}^{h(t)} v\,v_y v_{yy}\,\mathrm{d}y, \tag{1.51}$$

$$\int_{h(t)}^1 v_{yy}^2\,\mathrm{d}y = -\frac{1}{2}\int_{h(t)}^1 \frac{\partial}{\partial t}(v_y^2)\,\mathrm{d}y - \dot{v}(t,h(t)+0)v_y(t,h(t)+0) + \int_{h(t)}^1 v\,v_y v_{yy}. \tag{1.52}$$

On the other hand, differentiating the third equation in (1.9) it follows that

$$\dot{v}(t,h(t)\pm 0) = \ddot{h}(t) - \dot{h}(t)v_y(t,h(t)\pm 0) \qquad (t \geqslant 0),$$

so that

$$\dot{v}(t,h(t)\pm 0)v_y(t,h(t)\pm 0) = \ddot{h}(t)v_y(t,h(t)\pm 0) - \dot{h}(t)v_y^2(t,h(t)\pm 0). \tag{1.53}$$

On the other hand

$$\int_{-1}^{h(t)} \frac{\partial}{\partial t}(v_y^2)\,\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{h(t)} v_y^2\,\mathrm{d}y - \dot{h}(t)v_y(t,h(t)-0),$$

$$\int_{h(t)}^1 \frac{\partial}{\partial t}(v_y^2)\,\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{h(t)} v_y^2\,\mathrm{d}y + \dot{h}(t)v_y(t,h(t)+0),$$

so that

$$-\frac{1}{2}\int_{-1}^{h(t)}\frac{\partial}{\partial t}(v_y^2)\,\mathrm{d}y-\frac{1}{2}\int_{h(t)}^{1}\frac{\partial}{\partial t}(v_y^2)\,\mathrm{d}y=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}v_y^2\,\mathrm{d}y-\frac{1}{2}\dot{h}(t)[v_y](t,h(t)).$$

By combining the last formula with (1.51), (1.52) and (1.53) it follows that

$$\begin{aligned}\int_{-1}^{h(t)}v_{yy}^2\,\mathrm{d}y+\int_{h(t)}^{1}v_{yy}^2\,\mathrm{d}y&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}v_y^2\,\mathrm{d}y-\frac{1}{2}\dot{h}(t)[v_y](t,h(t))\\&\quad-\ddot{h}(t)[v_y](t,h(t))+\dot{h}(t)[v_y^2](t,h(t))+\int_{-1}^{h(t)}v\,v_yv_{yy}\,\mathrm{d}y+\int_{h(t)}^{1}v\,v_yv_{yy}\,\mathrm{d}y.\end{aligned}$$

In the second term of the right hand side of the above formula we use the fact that

$$[v_y](t,h(t))=\ddot{h}(t), \tag{1.54}$$

and we obtain that

$$\begin{aligned}\int_{-1}^{h(t)}v_{yy}^2\,\mathrm{d}y+\int_{h(t)}^{1}v_{yy}^2\,\mathrm{d}y&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}v_y^2\,\mathrm{d}y-\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\dot{h}^2(t)\\&\quad-\ddot{h}^2(t)+\dot{h}(t)[v_y^2](t,h(t))+\int_{-1}^{h(t)}v\,v_yv_{yy}\,\mathrm{d}y+\int_{h(t)}^{1}v\,v_yv_{yy}\,\mathrm{d}y.\end{aligned} \tag{1.55}$$

The last two terms in the right hand side of the above formula can be estimated, using the Cauchy-Schwarz inequality, to give

$$\begin{aligned}&\left|\int_{-1}^{h(t)}v\,v_yv_{yy}\,\mathrm{d}y+\int_{h(t)}^{1}v\,v_yv_{yy}\,\mathrm{d}y\right|\\&\quad\leqslant\|v(t,\cdot)\,v_y(t,\cdot)\|_{L^2[-1,1]}\left(\|v_{yy}(t,\cdot)\|_{L^2[-1,h(t)]}+\|v_{yy}(t,\cdot)\|_{L^2[h(t),1]}\right).\end{aligned} \tag{1.56}$$

Using the classical interpolation inequality

$$\|\psi\|_{C[-1,1]}\leqslant\|\psi'\|_{L^2[-1,1]}^{\frac{1}{2}}\|\psi\|_{L^2[-1,1]}^{\frac{1}{2}}\qquad(\psi\in H_0^1(-1,1)),$$

together with (1.15) it follows that

$$\|v(t,\cdot)\,v_y(t,\cdot)\|_{L^2[-1,1]}\leqslant M\|v_y(t,\cdot)\|_{L^2[-1,1]}^{\frac{3}{2}}.$$

Inserting the last inequality into (1.56) we obtain that

$$\left|\int_{-1}^{h(t)} v v_y v_{yy}\,\mathrm{d}y + \int_{h(t)}^{1} v v_y v_{yy}\,\mathrm{d}y\right| \leqslant \frac{M^2}{2}\|v_y(t,\cdot)\|_{L^2[-1,1]}^3 + \frac{1}{2}\left(\|v_{yy}(t,\cdot)\|_{L^2[-1,h(t)]}^2 + \|v_{yy}(t,\cdot)\|_{L^2[h(t),1]}^2\right) \qquad (t \in [0,T]). \tag{1.57}$$

By combining the last inequality and (1.55), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v_y^2\,\mathrm{d}y + \int_{-1}^{h(t)} v_{yy}^2\,\mathrm{d}y + \int_{h(t)}^{1} v_{yy}^2\,\mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}t}|\dot{h}(t)|^2 + |\ddot{h}(t)|^2 + \leqslant K_1\left(1 + \|v_y(t,\cdot)\|_{L^2[-1,1]}^3 + \dot{h}(t)[v_y^2](t,h(t))\right) \qquad (t \in [0,T]), \tag{1.58}$$

with K_1 depending only on M. In order to estimate the last term in the right hand side of (1.58) we note that, for almost every $t \in [0,T]$, we have

$$\begin{aligned}\dot{h}(t)[v_y^2](h(t),t) &= v(t,h(t))\left[v_y^2(t,h(t)+0) - v_y^2(t,h(t)-0)\right] \\ &= \int_{-1}^{h(t)} \left(v(t,y)v_y^2(t,y)\right)_y\,\mathrm{d}y + \int_{h(t)}^{1} \left(v(t,y)v_y^2(t,y)\right)_y\,\mathrm{d}y = \int_{-1}^{1} v_y^3(t,y)\,\mathrm{d}y \\ &\quad + 2\int_{-1}^{h(t)} v(t,y)v_y(t,y)v_{yy}(t,y)\,\mathrm{d}y + 2\int_{h(t)}^{1} v(t,y)v_y(t,y)v_{yy}\,\mathrm{d}y\end{aligned}$$

Using (1.57) in the last inequality we deduce that

$$\dot{h}(t)[v_y^2](h(t),t) \leqslant \int_{-1}^{1} v_y^3(t,y)\,\mathrm{d}y + M^2\|v_y(t,\cdot)\|_{L^2[-1,1]}^3 + \|v_{yy}(t,\cdot)\|_{L^2[-1,h(t)]}^2 + \|v_{yy}(t,\cdot)\|_{L^2[h(t),1]}^2.$$

Inserting the last inequality in (1.58) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v_y^2\,\mathrm{d}y + \int_{-1}^{h(t)} v_{yy}^2\,\mathrm{d}y + \int_{h(t)}^{1} v_{yy}^2\,\mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}t}|\dot{h}(t)|^2 + |\ddot{h}(t)|^2 \leqslant K_2\left(1 + \|v_y(t,\cdot)\|_{L^2[-1,1]}^3 + \int_{-1}^{1} v_y^3\,\mathrm{d}y\right) \qquad (t \in [0,T]), \tag{1.59}$$

with K_2 depending only on M. To estimate the last integral in the right-hand side of the above formula we use the interpolation inequality (4.13) from [36] which asserts the existence of a universal constant $K > 0$ that

$$\|v_y\|_{L^\infty[-1,1])} \leqslant K\left[\|v_y\|^{\frac{1}{2}}_{L^2[-1,1]}\left(\|v_{yy}\|^{\frac{1}{2}}_{L^2[-1,h(t)]} + \|v_{yy}\|^{\frac{1}{2}}_{L^2[h(t),1]}\right) + |\ddot{h}(t)|\right].$$

The above estimate, combined with Young's inequality, implies that for every $\varepsilon > 0$ there exists a constant $c > 0$ with

$$\int_{-1}^{1} v_y^3\,\mathrm{d}y \leqslant \varepsilon|\ddot{h}(t)|^2 + \varepsilon\left(\int_{-1}^{h(t)} v_{yy}^2\,\mathrm{d}y + \int_{h(t)}^{1} v_{yy}^2\,\mathrm{d}y\right) + c\left(\int_{-1}^{1} v_y^2\,\mathrm{d}y\right)^{\frac{5}{3}} + c\left(\int_{-1}^{1} v_y^2\,\mathrm{d}y\right)^{2} \qquad (t \in [0,T]).$$

Choosing ε small enough and inserting the last inequality in (1.59) we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v_y^2\,\mathrm{d}y + \int_{-1}^{h(t)} v_{yy}^2\,\mathrm{d}y + \int_{h(t)}^{1} v_{yy}^2\,\mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}t}|\dot{h}(t)|^2 + |\ddot{h}(t)|^2 \\ \leqslant K_3\left[1 + \left(\int_{-1}^{1} v_y^2\,\mathrm{d}y\right)^{\frac{3}{2}} + \left(\int_{-1}^{1} v_y^2\,\mathrm{d}y\right)^{\frac{5}{3}} + \left(\int_{-1}^{1} v_y^2\,\mathrm{d}y\right)^{2}\right] \\ \leqslant K_4\left[1 + \int_{-1}^{1} v_y^2\,\mathrm{d}y + \left(\int_{-1}^{1} v_y^2\,\mathrm{d}y\right)^{2}\right],\end{aligned} \tag{1.60}$$

with K_3 and K_4 depending only on M. Integrating the above formula on $[0,\tau]$, with $\tau \in [0,T]$, it follows that

$$\int_{-1}^{1} v_y^2(\tau,y)\,\mathrm{d}y \leqslant \int_{-1}^{1} v_{0,y}^2(y)\,\mathrm{d}y + K_5\left[T + \int_0^\tau \left(1 + \int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y\right)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y\,\mathrm{d}t\right].$$

Using in the last estimate the fact, resulting from (1.15), that

$$\int_0^\tau \int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y\,\mathrm{d}t \leqslant K_6,$$

the conclusion (1.49) follows by applying Gronwall's inequality.

In order to prove (1.50) it suffices to integrate (1.60) with respect to time and to use (1.49). □

We are now in a position to prove the main result of this section.

Proof of Theorem 1.1 It suffices to prove that both assertions in the second alternative of Proposition 1.3 are false. The fact that the assertion

$$\lim_{t\to T_{max}} \|v(t,\cdot)\|_{H_0^1(-1,1)} = \infty,$$

is false for every $T_{max} \in [0,\infty)$ is a direct consequence of Proposition 1.9. We show below that the assertion saying that

$$\lim_{t\to T_{max}} |h(t)| = 1,$$

is false for every $T_{max} \in [0,\infty)$. To accomplish this goal, we first note that from (1.49) and (1.50) it follows that v can be extended to a function, still denoted by v, such that

$$v \in C([0,T_{\max}], H_0^1(-1,1)),$$

and v is Lipschitz with respect to x, uniformly with respect to $t \in [0,T_{\max}]$. We use now a contradiction argument . Indeed, assume that

$$\lim_{t\to T_{max}} h(t) = 1.$$

This means that h can be extended to a function in $C^1[0,T_{\max}]$ such that

$$\dot h(t) = v(t,h(t)) \qquad (t \in [0,T_{\max}]), \qquad h(T_{\max}) = 1.$$

On the other hand the function $\widetilde{h}(t) = 1$ for every $t \in \mathbb{R}$ is also a solution of the above initial value problem. By the Cauchy-Lipschitz theorem it follows that $h(t) = \widetilde{h}(t) = 1$ for every $t \in [0,T_{\max}]$, which is clearly a contradiction. □

In order to study the concept of weak solution of (1.9) it is useful to note that that the distance from the mass point to the boundary is bounded from below by a function depending only on the initial kinetic energy of the fluid-mass particle system and of the initial position of the particle.

Theorem 1.10 *Let $M > 0$. We assume that v_0, h_0 and g_0 satisfy the assumptions in Theorem 1.1 and that*

$$\int_{-1}^{1} v_0^2(y)\,\mathrm{d}y + g_0^2 + |h_0 - h_1|^2 \leqslant M^2.$$

Let $\begin{bmatrix} v \\ h \end{bmatrix}$ *be the corresponding solution of* (1.9)*. Then there exist the positive constants* K_0*, which depends only on* M*, and* K_1*, depending only on* M *and on* h_0*, such that*

$$1 + h(t) \geqslant K_1 \mathrm{e}^{-tK_0} \qquad (t \geqslant 0). \tag{1.61}$$

$$1 - h(t) \geqslant K_1 \mathrm{e}^{-tK_0} \qquad (t \geqslant 0), \tag{1.62}$$

Proof We give below only the detailed proof of (1.62), since the proof of (1.61) can be obtained with obvious adaptations. Moreover, we note that it suffices to prove (1.61) only for the values of t for which $h(t) \geqslant \frac{1}{2}$, i.e. for values of t such that

$$h(t) \geqslant \frac{3}{2}. \tag{1.63}$$

Consider the function φ defined by

$$\varphi(t,y) = \begin{cases} \frac{1+y}{1+h(t)} & \text{if } y \in [-1, h(t)], \\ \frac{1-y}{1-h(t)} & \text{if } y \in [h(t), 1]. \end{cases}$$

Then

$$\int_{-1}^{h(t)} \dot{v}(t,y)\varphi(t,y)\,\mathrm{d}y = \int_{-1}^{h(t)} \frac{\partial}{\partial t}\left(v(t,y)\varphi(t,y)\right)\,\mathrm{d}y - \int_{-1}^{h(t)} v(t,y)\dot{\varphi}(t,y)\,\mathrm{d}y$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{h(t)} v(t,y)\varphi(t,y)\,\mathrm{d}y - \dot{h}^2(t) + \frac{\dot{h}(t)}{(1+h(t))^2}\int_{-1}^{h(t)} (1+y)v(t,y)\,\mathrm{d}y \qquad (t \geqslant 0),$$

$$-\int_{-1}^{h(t)} v_{yy}(t,y)\varphi(t,y)\,\mathrm{d}y = \frac{\dot{h}(t)}{1+h(t)} - v_y(t, h(t)-0) \qquad (t \geqslant 0).$$

Summing up the two above formulae it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{h(t)} v(t,y)\varphi(t,y)\,\mathrm{d}y - \dot{h}^2(t) + \frac{\dot{h}(t)}{(1+h(t))^2}\int_{-1}^{h(t)} (1+y)v(t,y)\,\mathrm{d}y + \frac{\dot{h}(t)}{1+h(t)}$$
$$- v_y(t, h(t)-0) + \int_{-1}^{h(t)} v(t,y)v_y(t,y)\varphi(t,y)\,\mathrm{d}y = 0 \qquad (t \geqslant 0). \tag{1.64}$$

Similar calculations show that

$$\int_{h(t)}^{1} \dot{v}(t,y)\varphi(t,y)\,\mathrm{d}y$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_{h(t)}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y + \dot{h}^2(t) - \frac{\dot{h}(t)}{(1-h(t))^2}\int_{h(t)}^{1}(1-y)v(t,y)\,\mathrm{d}y \qquad (t \geqslant 0),$$

$$-\int_{h(t)}^{1} v_{yy}(t,y)\varphi(t,y)\,\mathrm{d}y = \frac{\dot{h}(t)}{1-h(t)} + v_y(t,h(t)+0) \qquad (t \geqslant 0).$$

Summing up the last two formulae we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{h(t)}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y + \dot{h}^2(t) - \frac{\dot{h}(t)}{(1-h(t))^2}\int_{-1}^{h(t)}(1-y)v(t,y)\,\mathrm{d}y$$
$$+ \frac{\dot{h}(t)}{1-h(t)} + v_y(t,h(t)+0) + \int_{h(t)}^{1} v(t,y)v_y(t,y)\varphi(t,y)\,\mathrm{d}y = 0 \qquad (t \geqslant 0).$$

The above formula and (1.64) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y + \frac{\dot{h}(t)}{(1+h(t))^2}\int_{-1}^{h(t)}(1+y)v(t,y)\,\mathrm{d}y$$
$$-\frac{\dot{h}(t)}{(1-h(t))^2}\int_{h(t)}^{1}(1-y)v(t,y)\,\mathrm{d}y + \frac{2\dot{h}(t)}{1-h^2(t)}$$
$$+\ddot{h}(t) + \int_{-1}^{1} v(t,y)v_y(t,y)\varphi(t,y)\,\mathrm{d}y = 0 \qquad (t \geqslant 0).$$

The last formula implies that

$$\frac{2\dot{h}(t)}{1-h^2(t)} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y - \ddot{h}(t) - \frac{\dot{h}(t)}{(1+h(t))^2}\int_{-1}^{h(t)}(1+y)v(t,y)\,\mathrm{d}y$$
$$+\frac{\dot{h}(t)}{(1-h(t))^2}\int_{h(t)}^{1}(1-y)v(t,y)\,\mathrm{d}y - \int_{-1}^{1} v(t,y)v_y(t,y)\varphi(t,y)\,\mathrm{d}y \qquad (t \geqslant 0).$$

It follows that

$$-\frac{2\dot{h}(t)}{1-h^2(t)} \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y + \ddot{h}(t) + |\mathcal{A}_1(t)| + |\mathcal{A}_2(t)|, \qquad (1.65)$$

where

$$\mathcal{A}_1(t) = -\frac{\dot{h}(t)}{(1+h(t))^2}\int_{-1}^{h(t)}(1+y)v(t,y)\,\mathrm{d}y + \frac{\dot{h}(t)}{(1-h(t))^2}\int_{h(t)}^{1}(1-y)v(t,y)\,\mathrm{d}y \quad (t \geqslant 0),$$

$$\mathcal{A}_2(t) = -\int_{-1}^{1} v(t,y)v_y(t,y)\varphi(t,y)\,\mathrm{d}y \qquad (t \geqslant 0).$$

The expressions defined on the last two formulas can be estimated by

$$|\mathcal{A}_1(t)| \leqslant 2|\dot{h}(t)|\|v\|_{L^\infty([0,T],L^2[-1,1])} \leqslant 2M|\dot{h}(t)| \leqslant 2M^2 \qquad (t \geqslant 0),$$

$$|\mathcal{A}_2(t)| \leqslant \sqrt{2}\|v_y(t,\cdot)\|^2_{L^2[-1,1]}\|\varphi\|_{L^\infty([0,T]\times[-1,1])} \leqslant \sqrt{2}\|v_y(t,\cdot)\|^2_{L^2[-1,1]} \qquad (t \geqslant 0).$$

The last two estimates and (1.65) imply that

$$-\frac{2\dot{h}(t)}{1-h^2(t)} \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y + \ddot{h}(t) + 2M^2 + \sqrt{2}\|v_y(t\cdot)\|^2_{L^2[-1,1]} \qquad (t \geqslant 0). \tag{1.66}$$

Integrating (1.66) on $[0,t]$ it follows that for every $t \geqslant 0$ we have

$$\ln\left(\frac{1-h(t)}{1+h(t)}\right) - \ln\left(\frac{1-h_0}{1+h_0}\right) \leqslant \int_{-1}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y - \int_{-1}^{1} v(0,y)\varphi(0,y)\,\mathrm{d}y$$
$$+ \dot{h}(t) - g_0 + 2M^2 t + \sqrt{2}\int_0^t \|v_y(\sigma,\cdot)\|^2_{L^2[-1,1]}\,\mathrm{d}\sigma \leqslant t\widetilde{K}_0(M) + \widetilde{K}_1(M).$$

The last estimate, combined with (1.63), implies the conclusion (1.61). □

1.1.6 Bibliographical Notes

The first papers considering the coupling of viscous Burgers equation with Newton laws as a simplified fluid-structure interaction system are Vázquez and Zuazua [35, 36], where global wellposedness and long time behavior have been investigated. Similar models have been studied from a control theoretic viewpoint in Doubova and Fernández-Cara [15], Liu et al. [25] and Cîndea et al. [6]. Our presentation above follows [25] and [6].

1.2 Examples of Systems Modelling Fluid-Structure Interactions

In this chapter we introduce some systems modelling the motion of particles in a fluid, considering problems in one ore several space dimensions. We also describe some change of variables allowing to consider the governing equations in a fixed spatial domain and we postpone to the next chapters the study of the corresponding wellposedness results.

1.2.1 Motion of a Piston in a Heat Conducting Gas; a 1D Model

We consider a one dimensional model for the motion of a particle (piston) in a cylinder filled with a viscous compressible heat conducting gas. The extremities of the cylinder are fixed. The gas is modelled by the 1D compressible Navier-Stokes-Fourier equations, whereas the piston obeys Newton's second law. We assume that the piston is thermally conducting. More precisely, we consider the initial boundary value problem

$$
\begin{aligned}
&\partial_t\varrho + \partial_\xi(\varrho w) = 0, && (t \geqslant 0, \xi \in [-1,1] \setminus \{h(t)\})\\
&\varrho\left(\partial_t w + w\partial_\xi w\right) - \partial_{\xi\xi} w + \partial_\xi(\varrho\vartheta) = 0, && (t \geqslant 0, \xi \in [-1,1] \setminus \{h(t)\}),\\
&\varrho\left(\partial_t \vartheta + \partial_\xi\vartheta w\right) - \partial_{\xi\xi}\vartheta - \left(\partial_\xi w\right)^2 + \varrho\vartheta\partial_\xi w = 0, && (t \geqslant 0, \xi \in [-1,1] \setminus \{h(t)\}),\\
&w(t, h(t) \pm 0) = \dot{h}(t), \quad \vartheta(t, h(t) \pm 0) = Q(t) && (t \geqslant 0),\\
&m\ddot{h}(t) = [\partial_\xi w - \varrho\vartheta](t, h(t)), \quad \dot{Q}(t) = [\partial_\xi\vartheta](t, h(t)), && (t \geqslant 0),\\
&w(t,-1) = 0 = w(t,1), \quad \partial_\xi\vartheta(t,-1) = 0 = \partial_\xi\vartheta(t,1), && (t \geqslant 0),
\end{aligned}
\tag{1.67}
$$

with the initial conditions

$$
\begin{aligned}
&h(0) = h_0, \qquad \dot{h}(0) = g_0, \quad Q(0) = Q_0\\
&w(0,\xi) = w_0(\xi), \quad \varrho(0,\xi) = \rho_0(\xi), \quad \vartheta(0,\xi) = \vartheta_0(\xi) \qquad (\xi \in [-1,1] \setminus \{h_0\}).
\end{aligned}
\tag{1.68}
$$

In the above equations, $\varrho(t,\xi)$ is the density, $w(t,\xi)$ is velocity of the fluid, $\vartheta(t,\xi)$ is the temperature of the fluid (all in Eulerian coordinates), m is the mass of the particle and h is the trajectory of the mass point moving in the fluid. The symbol $[f](\xi)$ denotes the jump of f at ξ i.e.

$$[f](\xi) = f(\xi+) - f(\xi-).$$

We now rewrite the system (1.67)–(1.68) in Lagrangian mass coordinates. This change of variables has been widely used in the literature devoted to the study of one dimensional compressible flows (see, for instance, [3] and references therein). One of the advantages of this change of coordinates is that the positions of the piston becomes fixed. We begin by introducing the characteristic lines $\chi(t;\eta)$ defined by

$$\partial_t \chi(t,\eta) = w(t,\chi(t,\eta)), \qquad \chi(0,\eta) = \eta \qquad (\eta \in [-1,1]). \tag{1.69}$$

The first equation in (1.67) can be written

$$\rho_0(\eta) = \rho(t,\chi(t,\eta))\frac{\partial \chi}{\partial \eta}(t,\eta) \qquad (t \geqslant 0,\ X \in [-1,1]\setminus\{h_0\}). \tag{1.70}$$

The *Lagrangian mass change of coordinates* consists in replacing the space variable ξ in (1.67) by

$$x = \Psi(t,\xi), \qquad \Psi(t,\xi) = \int_{h(t)}^{\xi} \varrho(t,y)\,\mathrm{d}y \qquad (\xi \in [-1,1]). \tag{1.71}$$

From (1.70) and (1.71) it follows that

$$\Psi(t,\chi(t,\eta)) = \int_{h_0}^{\eta} \rho_0(Y)\,\mathrm{d}Y \qquad (\eta \in [-1,1]\setminus\{h_0\}). \tag{1.72}$$

Using the facts that $\chi(-t,1) = -1$, $\chi(t,1) = 1$ and $\chi(t,h(t)) = 0$, it follows that

$$\Psi(t,-1) = -r_1, \qquad \Psi(t,1) = r_2, \qquad (t \geqslant 0), \tag{1.73}$$

where

$$r_1 = \int_{-1}^{h_0} \rho_0(\eta)\,\mathrm{d}\eta, \qquad r_2 = \int_{h_0}^{1} \rho_0(\eta)\,\mathrm{d}\eta.$$

On the other hand, using the fact that the right hand side of (1.72) is time independent, together with (1.69), we obtain that

$$\partial_t \Psi(t,\xi) + \partial_\xi \Psi(t,\xi) w(t,\xi) = 0 \qquad (t \geqslant 0, \xi \in [-1,1]\setminus\{h(t)\}),$$

so that

$$\partial_t \Psi(t,\xi) = -\rho(t,\xi) w(t,\xi) \qquad (t \geqslant 0, \xi \in [-1,1]\setminus\{h(t)\}). \tag{1.74}$$

Using the above properties, it follows that, for every $t \geqslant 0$, $\Psi(t,\cdot)$ is a diffeomorphism from $[-1,1]$ onto $[-r_1, r_2]$, with $\partial_\xi \Psi(t,\xi) = \varrho(t,\xi) > 0$ for every $t \geqslant 0$ and for every $\xi \in [-1,1]\setminus\{h(t)\}$.

For each $t \geqslant 0$ we denote by $\Phi(t, \cdot) = \Psi^{-1}(t, \cdot)$. The *specific volume in mass lagrangian coordinates* is defined by

$$v(t,x) = \frac{1}{\varrho(t, \Phi(t,x))}, \quad \varrho(t,\xi) = \frac{1}{v(t, \Psi((t,\xi))} \quad (t \geqslant 0 \quad x \in [-r_1, r_2] \setminus \{0\},\ \xi \in [-1,1] \setminus \{h(t)\}). \tag{1.75}$$

Similarly, the *velocity and temperature field in lagrangian mass coordinates* writes

$$u(t,x) = w(t, \Phi(t,x)),\ w(t,\xi) = u(t, \Psi(t,\xi)) \quad (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\},\ \xi \in [-1,1] \setminus \{h(t)\}), \tag{1.76}$$

$$\theta(t,x) = \vartheta(t, \Phi(t,x)),\ \vartheta(t,\xi) = \theta(t, \Psi(t,\xi)) \quad (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\},\ \xi \in [-1,1] \setminus \{h(t)\}) \tag{1.77}$$

From (1.71) and (1.74) it follows that for every $t \geqslant 0$ and every $\xi \in [-1,1] \setminus \{h(t)\}$ we have

$$\partial_\xi \varrho(t,\xi) = -\frac{1}{v^3(t, \Psi((t,\xi))}\partial_x v(t, \Psi(t,\xi)), \tag{1.78}$$

$$\partial_t \varrho(t,\xi) = -\frac{1}{v^2(t, \Psi((t,\xi))}\partial_t v(t, \Psi(t,\xi)) + \frac{1}{v^3(t, \Psi((t,\xi))}\partial_x v(t, \Psi(t,\xi)) u(t, \Psi(t,\xi)). \tag{1.79}$$

From (1.71) we have for every $t \geqslant 0$ and every $\xi \in [-1,1] \setminus \{h(t)\}$ we have

$$\partial_\xi w(t,\xi) = \partial_x u(t, \Psi(t,\xi))\varrho(t,\xi) = \frac{\partial_x u(t, \Psi(t,\xi))}{v(t, \Psi(t,\xi))} \qquad (\xi \in [-1,1] \setminus \{h(t)\}). \tag{1.80}$$

By combining (1.78), (1.79) and (1.80) it follows that for every $t \geqslant 0$ and every $\xi \in [-1,1] \setminus \{h(t)\}$ we have

$$\partial_t \varrho(t,\xi) + \partial_\xi (\varrho(t,\xi)\, w(t,\xi)) = -\frac{1}{v^2(t, \Psi((t,\xi)))}\left(\partial_t v(t, \Psi(t,\xi)) - \partial_x u(t, \Psi(t,\xi))\right).$$

Consequently, using Lagrangian mass coordinates, Eq. $(1.67)_1$ writes

$$\partial_t v(t,x) - \partial_x u(t,x) = 0, \qquad (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}). \tag{1.81}$$

Using again (1.71) and (1.74), together with (1.80) it follows that

$$\partial_{\xi\xi} w(t,\xi) = \frac{\partial_{xx} u(t,\Psi(t,\xi))}{v^2(t,\Psi(t,\xi))} - \frac{\partial_x u(t,\Psi(t,\xi))\partial_x v(t,\Psi(t,\xi))}{v^3(t,\Psi(t,\xi))} \quad (\xi \in [-1,1]\backslash\{h(t)\}), \tag{1.82}$$

$$\partial_t w(t,\xi) = \partial_t u(t,\Psi(t,\xi)) - \frac{1}{v(t,\Psi(t,\xi)}\partial_x u(t,\Psi(t,\xi))u(t,\Psi(t,\xi)) \qquad (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}. \tag{1.83}$$

The derivatives of ϑ and θ satisfy formulas similar to those satisfied by those of w and of u, that is

$$\partial_\xi \vartheta(t,\xi) = \partial_x\theta(t,\Psi(t,\xi))\varrho(t,\xi) = \frac{\partial_x\theta(t,\Psi(t,\xi))}{v(t,\Psi(t,\xi))} \qquad (\xi \in [-1,1] \setminus \{h(t)\}), \tag{1.84}$$

$$\partial_{\xi\xi}\vartheta(t,\xi) = \frac{\partial_{xx}\theta(t,\Psi(t,\xi))}{v^2(t,\Psi(t,\xi))} - \frac{\partial_x\theta(t,\Psi(t,\xi))\partial_x v(t,\Psi(t,\xi))}{v^3(t,\Psi(t,\xi))} \qquad (\xi \in [-1,1] \setminus \{h(t)\}), \tag{1.85}$$

$$\partial_t\vartheta(t,\xi) = \partial_t\theta(t,\Psi(t,\xi)) - \frac{1}{v(t,\Psi(t,\xi)}\partial_x\theta(t,\Psi(t,\xi))u(t,\Psi(t,\xi)) \qquad (\xi \in [-1,1] \setminus \{h(t)\}). \tag{1.86}$$

By combining (1.78), (1.80), (1.82)–(1.84) we obtain that

$$\varrho(t,\xi)\left(\partial_t w(t,\xi) + w(t,\xi)\ \partial_\xi w(t,\xi)\right) - \partial_{\xi\xi} w(t,\xi) + \partial_\xi[\varrho(t,\xi)\vartheta(t,\xi)] = \frac{1}{v(t,\Psi(t,\xi))}\left[\partial_t u(t,\Psi(t,\xi)) - \partial_x\left(\frac{1}{v}\partial_x u - \frac{\theta}{v}\right)(t,\Psi(t,\xi))\right].$$

Consequently, $(1.67)_2$ can be written as

$$\partial_t u(t,x) - \partial_x\left(\frac{\partial_x u}{v} - \frac{\theta}{v}\right)(t,x) = 0 \qquad (t \geqslant 0,\ \ x \in [-r_1, r_2] \setminus \{0\}). \tag{1.87}$$

To write $(1.67)_3$ in mass Lagrangian coordinates we combine (1.80), (1.84)–(1.86) to get

$$\rho(t,\xi)\left(\partial_t\vartheta(t,\xi) + \partial_\xi\vartheta(t,\xi)w(t,\xi)\right) - \partial_{\xi\xi}\vartheta - \left(\partial_\xi w\right)^2 + \varrho(t,\xi)\vartheta(t,\xi)\partial_\xi w$$
$$= \frac{1}{v(t,\Psi(\xi,t)}\left[\partial_t\theta(t,\Psi(t,\xi)) - \partial_x\left(\frac{1}{v}\partial_x\theta\right)(t,\Psi(t,\xi)) - \left(\frac{1}{v}\,|\partial_x u|^2\right)(t,\Psi(t,\xi)) + \left(\frac{\theta}{v}\,\partial_x u\right)(t,\Psi(t,\xi))\right].$$

From the above formula, it follows that $(1.67)_3$ is satisfied iff

$$\partial_t\theta(t,x)-\partial_x\left(\frac{1}{v}\partial_x\theta\right)(t,x)-\frac{1}{v}(\partial_x u)^2-\frac{\theta}{v}\partial_x u(t,x)=0, \quad (t\geqslant 0,\ x\in[-r_1,r_2]\backslash\{0\}). \tag{1.88}$$

The fourth equation in (1.67) can obviously be rewritten as

$$u(t,\pm 0)=\dot{h}(t), \qquad \frac{\partial\theta}{\partial x}(t,\pm 0)=Q(t) \qquad (t\geqslant 0). \tag{1.89}$$

As for $(1.67)_5$, using (1.80), we have

$$m\ddot{h}(t)=\left[\frac{1}{v}\partial_x u-\frac{\theta}{v}\right](t,0), \quad \dot{Q}(t)=\left[\frac{1}{v}\partial_x\theta\right](t,0) \qquad (t\geqslant 0). \tag{1.90}$$

Using (1.84), it is easily seen that $(1.67)_6$ write in mass Lagrangian coordinates as

$$u(t,-r_1)=u(t,r_2)=0, \qquad \frac{\partial\theta}{\partial x}(-r_1,t)=\frac{\partial\theta}{\partial x}(r_1,t)=0 \qquad (t\geqslant 0). \tag{1.91}$$

Putting together (1.81) and (1.87)–(1.90), it follows that the system (1.67) writes in Lagrangian mass coordinates as

$$\begin{aligned}
&\partial_t v-\partial_x u=0, && (t\geqslant 0,\ x\in[-r_1,r_2]\setminus\{0\}),\\
&\partial_t u-\partial_x\left(\frac{1}{v}\partial_x u\right)+\partial_x\left(\frac{\theta}{v}\right)=0, && (t\geqslant 0,\ x\in[-r_1,r_2]\setminus\{0\}),\\
&\partial_t\theta-\partial_x\left(\frac{1}{v}\partial_x\theta\right)-\frac{1}{v}(\partial_x u)^2+\frac{\theta}{v}\partial_x u=0, && (t\geqslant 0,\ x\in[-r_1,r_2]\setminus\{0\}),\\
&u(t,\pm 0)=\dot{h}(t), \qquad \theta(t,0\pm)=Q(t), && (t\geqslant 0),\\
&m\ddot{h}(t)=\left[\frac{1}{v}\partial_x u-\frac{\theta}{v}\right](t,0), && (t\geqslant 0),\\
&\dot{Q}(t)=\left[\frac{1}{v}\partial_x\theta\right](t,0), && (t\geqslant 0),\\
&u(t,-r_1)=u(t,r_2)=0, \quad \partial_x\theta(t,-r_1)=\partial_x\theta(t,r_2)=0, && (t\geqslant 0),\\
&v(0,x)=v_0(x), \quad u(0,x)=u_0(x), \quad \theta(0,x)=\theta_0(x), && x\in[-r_1,r_2]\setminus\{0\},\\
&h(0)=h_0, \quad \dot{h}(0)=g_0,
\end{aligned} \tag{1.92}$$

where

$$v_0(x):=\frac{1}{\varrho_0(\Phi(0,x))}, \quad u_0(x)=w_0(\Phi(0,x)), \quad \theta_0(x)=\vartheta_0(\Phi(0,x)). \tag{1.93}$$

1.2.2 Motion of a Rigid Body in a Viscous Incompressible Fluid

In this section we describe the system modelling the motion of a rigid body immersed in a viscous incompressible fluid. Let us assume that the fluid and the rigid body are contained in a bounded domain with smooth boundary. At time $t \geqslant 0$, the rigid body occupies a smooth domain $\Omega_S(t) \subset \Omega$. We assume that

$$d(\Omega_S(0), \partial\Omega) > 0. \tag{1.94}$$

We denote by $\Omega_F(t) = \Omega \setminus \Omega_S(t)$ the domain occupied by the fluid. The motion of the fluid is given by

$$\partial_t u + (u \cdot \nabla)u - \operatorname{div} \sigma(u,p) = 0, \quad \operatorname{div} u = 0, \quad x \in \Omega_F(t), t \in [0,T], \tag{1.95}$$

where the Cauchy stress tensor $\sigma(u,p)$ is defined by

$$\sigma(u,p) = 2\nu Du - pI_3, \quad Du = \frac{1}{2}(\nabla u + \nabla u^T),$$

and I_3 is the identity matrix.

At time $t \geqslant 0$, let $h(t) \in \mathbb{R}^3$, $Q(t) \in \mathbb{M}_{3\times3}(\mathbb{R})$ and $\omega(t) \in \mathbb{R}^3$ denote the position of the center of mass, the orthogonal matrix giving the orientation of the solid and the angular velocity of the rigid body. Therefore we have,

$$\dot{Q}(t)Q(t)^{-1}y = A(\omega(t))y = \omega(t) \times y, \quad \forall y \in \mathbb{R}^3, \tag{1.96}$$

where the skew-symmetric matrix $A(\omega)$ is given by

$$A(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \omega \in \mathbb{R}^3.$$

Without loss of generality we can assume that

$$h(0) = 0 \quad \text{and} \quad Q(0) = I. \tag{1.97}$$

At time t, the domain occupied by the structure $\Omega_S(t)$ is defined by

$$\Omega_S(t) = \chi_S(t, \Omega_S(0)) \tag{1.98}$$

where χ_S denotes the flow associated to the motion of the structure:

$$\chi_S(t,y) = h(t) + Q(t)y, \quad \forall y \in \Omega_S(0), \ \forall t > 0, \tag{1.99}$$

For each $t > 0$, $\chi_S(t,\cdot) : \Omega_S(0) \mapsto \Omega_S(t)$ is invertible and

$$\chi_S(t,\cdot)^{-1}(x) = Q(t)^{-1}(x - h(t)), \quad \forall x \in \Omega_S(t). \tag{1.100}$$

Thus the Eulerian velocity u_S of the structure is given by

$$u_S(t,x) = \partial_t \chi_S(t,\cdot) \circ \chi_S(t,\cdot)^{-1}(x) = \dot{h}(t) + \dot{Q}(t)Q(t)^{-1}(x - h(t)), \quad \forall x \in \Omega_S(t). \tag{1.101}$$

Therefore

$$u_S(t,x) = \dot{h}(t) + \omega(t) \times (x - h(t)), \quad \forall x \in \Omega_S(t). \tag{1.102}$$

We also assume the continuity of velocities at the fluid-solid interface, i.e.,

$$u(t,x) = \dot{h}(t) + \omega(t) \times (x - a(t)), \quad x \in \partial\Omega_S(t). \tag{1.103}$$

On the boundary of Ω we prescribe no-slip boundary condition for fluid, i.e.,

$$u(t,x) = 0, \quad x \in \partial\Omega. \tag{1.104}$$

We denote by $m > 0$ the mass of rigid structure and $J(t) \in \mathcal{M}_{3\times 3}(\mathbb{R})$ its tensor of inertia at time t. This tensor is given by

$$J(t)a \cdot b = \int_{\Omega_S(0)} \rho_S(y)(a \times Q(t)y) \cdot (b \times Q(t)y)\, dy, \qquad \forall a, b \in \mathbb{R}^3, \tag{1.105}$$

where $\rho_S > 0$ is the density of the structure. One can check that

$$J(t)a \cdot a \geqslant C_J |a|^2 > 0, \tag{1.106}$$

where C_J is independent of $t > 0$. The equation of the structure is given by

$$\begin{aligned} m\ddot{h} &= -\int_{\partial\Omega_S(t)} \sigma(u,p)n\, d\gamma, \\ J\dot{\omega} &= (J\omega) \times \omega - \int_{\partial\Omega_S(t)} (x - h(t)) \times \sigma(u,p)n\, d\gamma \end{aligned} \tag{1.107}$$

where $n(t,x)$ the unit normal to $\partial\Omega_S(t)$ at the point x directed toward the interior of the rigid body. The above system is completed by the following initial conditions

$$u(0) = u_0, \quad \text{in } \Omega_F(0), \quad h(0) = 0, \quad \dot{h}(0) = g_0, \quad \omega(0) = \omega_0. \tag{1.108}$$

1.2.3 Motion of a Solid in a Compressible Fluid

In this section, we consider a rigid structure immersed in a viscous compressible fluid. In this case, we assume that, the fluid and the rigid body are contained in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. At time $t \geqslant 0$, the rigid body occupies a smooth bounded domain $\Omega_S(t)$. We assume that

$$d(\Omega_S(0), \partial\Omega) > 0. \tag{1.109}$$

For any time $t \geqslant 0$, $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ denotes the region occupied by the fluid. Let $h(t) \in \mathbb{R}^3$, $Q(t) \in \mathbb{M}_{3\times 3}(\mathbb{R})$ and $\omega(t) \in \mathbb{R}^3$ denote the position of the center of mass, the orthogonal matrix giving the orientation of the solid and the angular velocity of the rigid body satisfying (1.96) and (1.97). Let m denote the mass of the rigid body and $J(t) \in \mathcal{M}_{3\times 3}(\mathbb{R})$ its tensor of inertia at time t given by (1.105). The system modelling the motion of rigid body in a viscous compressible fluid can be written as

$$\begin{aligned}
&\partial_t\rho + \operatorname{div}(\rho u) = 0, \quad t \in (0,T), x \in \Omega_F, (t)\\
&\rho(\partial_t u + (u\cdot\nabla)u) - \operatorname{div}\sigma(u,p) = 0, \quad t \in (0,T), x \in \Omega_F, (t)\\
&u(t,x) = \dot{h}(t) + \omega(t) \times (x - a(t)), \quad t \in (0,T), x \in \partial\Omega_S(t),\\
&m\ddot{h} = -\int_{\partial\Omega_S(t)} \sigma(u,p)n\, d\gamma, \quad t \in (0,T)\\
&J\dot{\omega} = (J\omega)\times\omega - \int_{\partial\Omega_S(t)} (x - h(t)) \times \sigma(u,p)n\, d\gamma\\
&u(t,x) = 0, \quad t \in (0,T), x \in \partial\Omega,\\
&\rho(0) = \rho_0, \quad u(0) = u_0, \quad \text{in } \Omega_F(0),\\
&h(0) = 0, \quad \dot{h}(0) = g_0, \quad \omega(0) = \omega_0,
\end{aligned} \tag{1.110}$$

where

$$\sigma(u,p) = 2\mu Du + (\alpha \operatorname{div} u - p)I_3, \quad Du = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\mu \geqslant 0 \text{ and } \alpha + \frac{2}{3}\mu \geqslant 0, \quad p = \rho^\gamma, \quad \gamma \geqslant 1.$$

Now we rewrite the above system in fixed domain. Here we use Lagrangian change of variable as it is well suited for the compressible fluids. We consider the characteristics X associated to the velocity fluid u, that is the solution of the Cauchy problem

$$\begin{cases} \partial_t X(t,y) = u(t, X(t,y)) \quad (t > 0),\\ X(0,y) = y \in \Omega. \end{cases} \tag{1.111}$$

Assume that $X(t,\cdot)$ is a C^1-diffeomorphism from $\Omega_F(0)$ onto $\Omega_F(t)$ for all $t \in (0,T)$ (see Lemma 1.60). For each $t \in (0,T)$, we denote by $Y(t,\cdot) = [X(t,\cdot)]^{-1}$ the inverse of $X(t,\cdot)$. We consider the following change of variables

$$\tilde{\rho}(t,y) = \rho(t, X(t,y)), \quad \tilde{u}(t,y) = Q^{-1}(t)u(t, X(t,y)), \tag{1.112}$$

$$\tilde{g}(t) = Q^{-1}(t)\dot{h}(t), \quad \tilde{\omega}(t) = Q^{-1}(t)\omega(t), \tag{1.113}$$

for $(t,y) \in (0,T) \times \Omega_F(0)$. In particular,

$$\rho(t,x) = \tilde{\rho}(t, Y(t,x)), \quad u(t,x) = Q(t)\tilde{u}(t, Y(t,x)), \tag{1.114}$$

for $(t,x) \in (0,T) \times \Omega_F(t)$. The system satisfied by $(\tilde{\rho}, \tilde{u}, \tilde{\ell}, \tilde{\omega})$ reads as follows

$$\begin{aligned}
&\partial_t \tilde{\rho} + \rho_0 \text{div}\, \tilde{u} = \mathcal{F}_1, \quad \text{in } (0,T) \times \Omega_F(0),\\
&\partial_t \tilde{u} - \frac{\mu}{\rho_0}\Delta \tilde{u} - \frac{\alpha+\mu}{\rho_0}\nabla(\text{div}\, \tilde{u}) = \mathcal{F}_{2,1} + \mathcal{F}_{2,2} \quad \text{in } (0,T) \times \Omega_F(0),\\
&\tilde{u} = 0 \quad \text{on } (0,T) \times \partial\Omega, \quad \tilde{u} = g + \omega \times y \quad \text{on } (0,T) \times \partial\Omega_S(0),\\
&m\frac{d}{dt}\tilde{g} = -\int_{\Omega_S(0)} \left(\mu \nabla \tilde{u} + \mu \nabla \tilde{u}^\top + \alpha \text{div}\, \tilde{u} I\right) n \, d\gamma + \mathcal{G}_1,\\
&J(0)\frac{d}{dt}\tilde{\omega} = -\int_{\Omega_S(0)} y \times \left(\mu \nabla \tilde{u} + \mu \nabla \tilde{u}^\top + \alpha \text{div}\, \tilde{u} I\right) n + \mathcal{G}_2,\\
&\tilde{\rho}(0) = \rho_0, \quad \tilde{u}(0) = u_0, \quad \text{in } \Omega_F(0),\\
&\tilde{g}(0) = g_0, \tilde{\omega}_0 = \omega_0,
\end{aligned} \tag{1.115}$$

where

$$Q(t) = I + \int_0^t Q(s)(\tilde{\omega}(s) \times I), \quad Q^T = Q^{-1} \tag{1.116}$$

$$X(t,y) = y + \int_0^t Q(s)\tilde{u}(s)\, ds, \quad \text{and} \quad J_Y = J_X^{-1}, \tag{1.117}$$

$$\mathcal{F}_1(\tilde{\rho}, \tilde{u}, \tilde{g}, \tilde{\omega}) = -(\tilde{\rho} - \tilde{\rho}_0)\text{div}\tilde{u} - \tilde{\rho}(Q-I)\nabla \tilde{u} : J_Y^\top - \tilde{\rho} \nabla \tilde{u} : (J_Y^\top - I), \tag{1.118}$$

$$\begin{aligned}
\mathcal{F}_{2,1}(\tilde{\rho}, \tilde{u}, \tilde{g}, \tilde{\omega}) = -\frac{\tilde{\rho}}{\rho_0}Q'(t)\tilde{u} - \frac{\tilde{\rho} - \rho_0}{\rho_0}Q(t)\partial_t \tilde{u} - (Q(t) - I)\partial_t \tilde{u} - \gamma \frac{\tilde{\rho}^{\gamma-1}}{\rho_0} J_Y^\top \nabla \tilde{\rho}\\
- \frac{\mu}{\rho_0}(Q-I)\Delta \tilde{u},
\end{aligned}$$

$$(\mathcal{F}_{2,2})_i(\tilde{\rho},\tilde{u},\tilde{g},\tilde{\omega}) = \frac{\mu}{\rho_0}\sum_{j,k,l=1}^{3} \partial_{y_l}(\partial_{y_k}(Q\tilde{u})_i[(J_Y)_{kj} - \delta_{kj}])(J_Y)_{lj}$$
$$+ \frac{\mu}{\rho_0}\sum_{k,l=1}^{3}(\partial^2_{y_l y_k}(Q\tilde{u})_i)\,[(J_Y)_{lk} - \delta_{lk}]$$
$$+ \frac{\alpha+\mu}{\rho_0}\sum_{j,k,l=1}^{3} \partial_{y_l}(\partial_{y_k}(Q\tilde{u})_j\left[(J_Y)_{kj}) - \delta_{kj}\right])(J_Y)_{li}$$
$$+ \frac{\alpha+\mu}{\rho_0}\sum_{l,j=1}^{3}(\partial^2_{y_l y_j}(Q\tilde{u})_j)\,[(J_Y)_{li} - \delta_{li}] + (Q^\top - I) : \partial_{y_i}\nabla\tilde{u}, \tag{1.119}$$

$$\mathcal{F}_3(\tilde{\rho},\tilde{u},\tilde{g},\tilde{\omega}) = -m(\tilde{\omega}\times\tilde{\ell}) - \int_{\Omega_S(0)} \mathcal{G}n,$$
$$\mathcal{F}_4(\tilde{\rho},\tilde{u},\tilde{g},\tilde{\omega}) = J(0)\tilde{\omega}\times\tilde{\omega} - \int_{\Omega_S(0)} y\times\mathcal{G}n,$$
$$\mathcal{G}(\tilde{\rho},\tilde{u},\tilde{g},\tilde{\omega}) = (Q^\top - I)\left(\mu\left(Q\nabla\tilde{u}J_Y + J_Y^\top\nabla\tilde{u}^\top Q^\top\right) + \left(\alpha J_Y^\top : Q\nabla\tilde{u}^\top - \tilde{p}\right)I\right)\mathrm{cof}J_X$$
$$+ \left(\mu\left(Q\nabla\tilde{u}J_Y + J_Y^\top\nabla\tilde{u}^\top Q^\top\right) + \left(\alpha J_Y^\top : Q\nabla\tilde{u}^\top - \tilde{p}\right)I\right)(\mathrm{cof}J_X - I)$$
$$+ \mu(Q-I)\nabla\tilde{u}J_Y + \mu\nabla\tilde{u}(J_Y - I) + \mu(J_Y^\top - I)\nabla\tilde{u}^\top Q^\top + \mu\nabla\tilde{u}^\top(Q^\top - I)$$
$$+ (\alpha(Q-I)\nabla\tilde{u} : J_Y^\top)I + (\alpha\nabla\tilde{u} : (J_Y^\top - I))I - R\widetilde{\rho}^\gamma I \tag{1.120}$$

1.3 Short Introduction to $\mathcal{R}$-Sectorial Operators

Let X be a Banach space and A be a closed, densely defined linear unbounded operator in X with domain $D(A)$. We shall consider the abstract Cauchy problem

$$\dot{u}(t) = Au(t) + f(t), \quad t > 0, \quad u(0) = 0, \tag{1.121}$$

where $f : \mathbb{R}^+ \mapsto X$ is a given function.

Definition 1.11 (Maximal L^p-Regularity) Let $1 < p < \infty$. The problem (1.121) has *maximal L^p-regularity* on $[0, T)$, $0 < T \leqslant \infty$, if for every $f \in L^p([0, T); X)$, there exists a unique u satisfying the above equation almost everywhere and such that $\dot{u} \in L^p([0, T); X)$. In this case $Au \in L^p([0, T); X)$ as well.

Remark 1.12 In the above definition we do not assume that $u \in L^p(0, T; X)$. In fact, if $T < \infty$ or $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, $\dot{u} \in L^p(0, T; X)$ can be replaced by $u \in W^{1,p}(0, T; X)$ [13].

Our aim is to give sufficient condition on the operator A so that the problem (1.121) has a maximal L^p-regularity. It is well known that, if (1.121) has a maximal L^p-regularity, then A generates an analytic semigroup in X (see [9] and [13]). On the other hand, if X is a Hilbert space, the above condition is enough to obtain maximal L^p-regularity, i.e., if A generates an analytic semigroup in X, then (1.121) has a maximal L^p-regularity (see de Simon [11] for more details). In fact, De Simon used Plancherel's theorem which is valid only in the Hilbert space setting and cannot be generalized. Since then, there has been considerable work in the general Banach space framework [10, 14, 20, 23, 24, 29]. We are interested in the recent result obtained by Weis [37]. He obtained a necessary and sufficient condition for maximal L^p-regularity when X is a UMD Banach space in terms of $\mathcal{R}$-boundedness of the operator A (for the precise definition of UMD spaces and $\mathcal{R}$-boundedness we refer to the next section). This result is very useful in order to obtain maximal $L^p - L^q$ regularity of linearized fluid structure interaction problem.

In this chapter we recall some basic definitions and results on $\mathcal{R}$-sectorial operators and we prove a lemma, which seems to be new, on the $\mathcal{R}$ sectoriality of a class of matrices of linear operators.

1.3.1 Basic Definitions

In this section we recall some basic definitions and results concerning maximal regularity and $\mathcal{R}$-boundedness in Banach spaces. For detailed information on these subjects we refer to [8, 12, 37] and references therein.

Definition 1.13 Let X be a Banach space. The Hilbert transform of a function $f \in \mathcal{S}(\mathbb{R}; X)$, the Schwartz space of X-valued rapidly decreasing functions, is defined by

$$Hf(t) = \frac{1}{\pi} \lim_{\epsilon \mapsto 0} \int_{|s|>\epsilon} \frac{f(t-s)}{s}\, ds, \quad t \in \mathbb{R}.$$

A Banach space X is said to be of class $\mathcal{HT}$, if the Hilbert transform is bounded on $L^p(\mathbb{R}; X)$ for some (thus all) $1 < p < \infty$.

These spaces are also called *UMD* Banach spaces, where *UMD* stands for *unconditional martingale differences*. Hilbert spaces, all closed subspaces and quotient spaces of $L^q(\Omega)$ with $1 < q < \infty$ are examples of *UMD* spaces. In fact, $X \in \mathcal{HT}$ implies that X is reflexive (see [1]). We next introduce the notion of $\mathcal{R}$-boundedness of family of operators and $\mathcal{R}$-sectoriality of a densely defined linear operator.

Definition 1.14 ($\mathcal{R}$-Bounded Family of Operators) Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for every $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$,

$\{x_j\}_{j=1}^n \subset X$ and for all sequences $\{r_j(\cdot)\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$ valued random variables on $[0,1]$, we have

$$\left\|\sum_{j=1}^n r_j(\cdot)T_jx_j\right\|_{L^p([0,1];Y)} \leqslant C\left\|\sum_{j=1}^n r_j(\cdot)x_j\right\|_{L^p([0,1];X)}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$ and denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Remark 1.15

(1) If $\mathcal{T} \subset \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded then it is uniformly bounded with

$$\sup\{\|T\| \mid T \in \mathcal{T}\} \leqslant \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}).$$

(2) If $X = Y = L^q(\Omega)$ for some open set $\Omega \subset \mathbb{R}^N$, then $\mathcal{T} \subset \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded if and only if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{x_j\}_{j=1}^n \subset L^q(\Omega)$ the following estimate holds:

$$\left\|\left(\sum_{j=1}^n |T_jx_j|^2\right)^{1/2}\right\|_{L^q(\Omega)} \leqslant C\left\|\left(\sum_{j=1}^n |x_j|^2\right)^{1/2}\right\|_{L^q(\Omega)}.$$

(3) If X and Y are Hilbert spaces every set $\mathcal{T}$ bounded in $\mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded.

For $0 < \varepsilon \leqslant \pi/2$, and $\gamma \geqslant 0$ we define the sector $\Sigma_{\varepsilon,\gamma}$ in the complex plane by

$$\Sigma_{\varepsilon,\gamma} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leqslant \pi - \varepsilon, \quad |\lambda| > \gamma\}. \tag{1.122}$$

When $\gamma = 0$, $\Sigma_{\varepsilon,\gamma}$ will be denoted by Σ_ε.

Definition 1.16 ($\mathcal{R}$-Sectorial Operator) Let A be a densely defined closed linear operator on a Banach space X with domain $\mathcal{D}(A)$. Then A is $\mathcal{R}$-sectorial operator in X if $\Sigma_{\varepsilon,\gamma}$ contained in the resolvent set $\rho(A)$ for some $\varepsilon \in (0,\pi/2), \gamma \geqslant 0$ and $\left\{\lambda(\lambda I - A)^{-1} \mid \lambda \in \Sigma_{\varepsilon,\gamma}\right\}$ is $\mathcal{R}$ bounded on $\mathcal{L}(X)$ with $\mathcal{R}$-bound M. In this case, the set $\left\{A(\lambda I - A)^{-1} \mid \lambda \in \Sigma_{\varepsilon,\gamma}\right\}$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound at most $1 + M$.

We now state several useful propositions concerning $\mathcal{R}$-boundedness.

Proposition 1.17

(1) Let X and Y be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X,Y)$. Then $\mathcal{T} + \mathcal{S}$ is also $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, and

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leqslant \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) Let X, Y and Z be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$ respectively. Then $\mathcal{S}\mathcal{T}$ is $\mathcal{R}$-bounded on $\mathcal{L}(X, Z)$, and

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \leqslant \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

(3) Let $q \in (1, \infty)$, let Ω be a bounded domain in $\mathbb{R}^n$ and let Λ be a domain in $\mathbb{C}$. Let $m(\lambda)$ be a bounded function defined on Λ and let $M_m(\lambda) \in \mathcal{L}(L^q(\Omega))$, defined by $M_m(\lambda)f = m(\lambda)f$, for any $f \in L^q(\Omega)$. Then $\{M_m(\lambda) \mid \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded and

$$\mathcal{R}_{\mathcal{L}(L^q(\Omega))}\{M_m(\lambda) \mid \lambda \in \Lambda\} \leqslant C_{n,q,\Omega}\|m\|_{L^\infty(\Lambda)}. \tag{1.123}$$

Proof The proof of first two statement follows easily from the definition of $\mathcal{R}$-boundedness. The proof of Proposition 1.17 (3) follows from Remark 1.15 (2). □

1.3.2 Weis' Theorem

In this section we will discuss Weis' theorem concerning maximal L^p-regularity of the Cauchy problem (1.121). First, we will prove a proposition due to Kunstmann and Weis [22], which states that $\mathcal{R}$-sectoriality is preserved by A small perturbations.

Proposition 1.18 *Let A be a $\mathcal{R}$-sectorial in a Banach space X with domain $\mathcal{D}(A)$. Assume that $\Sigma_{\varepsilon_0,\gamma_0} \subset \rho(A)$, for some $\varepsilon_0 \in (0, \pi/2)$, $\gamma_0 \geqslant 0$ and*

$$\mathcal{R}_{\mathcal{L}(X)}\left(\left\{A(\lambda I - A)^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_0}\right\}\right) \leqslant a < \infty. \tag{1.124}$$

Let B be a linear operator such that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and

$$\|Bx\| \leqslant \delta_1\|Ax\| + \delta_2\|x\|, \tag{1.125}$$

with $\delta_1 < 1/a$. Then there exists $\gamma_1 \geqslant \gamma_0$ such that

$$\mathcal{R}_{\mathcal{L}(X)}\left(\left\{\lambda(\lambda I - (A + B))^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_1}\right\}\right) < \infty. \tag{1.126}$$

Proof From the definition of $\mathcal{R}$-boundedness, we have

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(X)}\left\{B(\lambda I - A)^{-1}\right\} &\leqslant \delta_1\mathcal{R}_{\mathcal{L}(X)}\left\{A(\lambda I - A)^{-1}\right\} + \delta_2\mathcal{R}_{\mathcal{L}(X)}\left\{(\lambda I - A)^{-1}\right\} \\ &\leqslant \left(\delta_1 a + \frac{\delta_2 a}{|\lambda|}\right).\end{aligned}$$

Thus there exists $\gamma_1 \geqslant \gamma_0$ such that $\left(\delta_1 a + \dfrac{\delta_2 a}{|\lambda|}\right) \leqslant \delta < 1$ for $\lambda \in \Sigma_{\varepsilon_0,\gamma_1}$ and hence $I - B(\lambda I - A)^{-1}$ is invertible for $\lambda \in \Sigma_{\varepsilon_0,\gamma_1}$. Now

$$\begin{aligned}(\lambda I - (A+B)^{-1} &= (\lambda I - A)^{-1}\left(I - B(\lambda I - A)^{-1}\right)^{-1}\\ &= (\lambda I - A)^{-1}\sum_{j=0}^{\infty}\left(B(\lambda I - A)^{-1}\right)^j.\end{aligned}$$

By induction

$$\mathcal{R}\left(\lambda(\lambda I - A)^{-1}\left(B(\lambda I - A)^{-1}\right)^j\right) \leqslant \mathcal{R}\left(\lambda(\lambda I - A)^{-1}\right)\delta^j$$

Therefore

$$\mathcal{R}_{\mathcal{L}(X)}\left(\left\{\lambda(\lambda I - (A+B))^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_1}\right\}\right) \leqslant \frac{a}{1-\delta}.$$

□

The Theorem of Weis is the following:

Theorem 1.19 *Let X be a Banach space of class $\mathcal{HT}$, $1 < p < \infty$ and let A be a closed, densely defined unbounded operator with domain $D(A)$. Let A generates a bounded analytic semigroup on X, i.e.,*

$$\|\lambda(\lambda I - A)^{-1}\| \leqslant C, \text{ for } \lambda > 0.$$

Then the following statements are equivalent.

(i) The Cauchy problem (1.121) has maximal L^p-regularity.
(ii) The set $\left\{\lambda(\lambda - A)^{-1} \mid \lambda \in \Sigma_\varepsilon\right\}$ is $\mathcal{R}$ bounded for some $\varepsilon \in (0, \pi/2)$.

1.3.3 Abstract Framework Corresponding to Linear Fluid-Solid Interaction Problems

In this section, we introduce an abstract framework which will correspond to the linear fluid-solid interactions problems. The main idea in elaborating this approach is that linearized fluid-solid interaction problems can be viewed as boundary controlled fluid systems with dynamic boundary feedback. To this aim we first recall, following [34, Chap. 10], some background on boundary control systems.

Systems described by linear partial differential equations with nonhomogeneous boundary conditions can be written in the form:

$$\dot{z}(t) = Lz(t), \qquad Gz(t) = u(t). \tag{1.127}$$

Often L is a differential operator and G is a boundary trace operator. In the sequel, we assume that U, Z and X are reflexive Banach spaces such that

$$Z \subset X,$$

with continuous embedding. We shall call U the *input space*, Z the *solution space* and X the *state space*.

Definition 1.20 A *boundary control system* on U, Z and X is a pair of operators (L, G), where

$$L \in \mathcal{L}(Z, X), \qquad G \in \mathcal{L}(Z, U),$$

if there exists a $\beta \in \mathbb{C}$ such that the following properties hold:

(i) G is onto,
(ii) $\mathrm{Ker} G$ is dense in X,
(iii) $\beta I - L$ restricted to $\mathrm{Ker} G$ is onto,
(iv) $\mathrm{Ker}(\beta I - L) \cap \mathrm{Ker} G = \{0\}$.

With the assumptions of the last definition, we introduce the closed subspace X_1 of Z and the operator A by

$$X_1 = \mathrm{Ker}\, G, \qquad A = L|_{X_1}. \tag{1.128}$$

Obviously, X_1 is a closed subspace of Z and $A \in \mathcal{L}(X_1, X)$. Condition (iii) means that $\beta I - A$ is onto. Condition (iv) means that $\mathrm{Ker}(\beta I - A) = \{0\}$. Thus, (iii) and (iv) together are equivalent to the fact that $\beta \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, so that

$$(\beta I - A)^{-1} \in \mathcal{L}(X).$$

In fact, $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$, so that the norm on X_1 is equivalent to the norm

$$\|z\|_1 = \|(\beta I - A)z\|_X.$$

It is easy to see that $\|\cdot\|_{X_1}$ on X_1 is equivalent to the graph norm of A. Therefore, by closed graph theorem, $(X_1, \|\cdot\|_1)$ is complete. Also, for any $\beta' \in \rho(A)$, we have an equivalent norm on X_1. We define the space X_{-1} as the completion of X with respect to the norm

$$\|z\|_{-1} = \|(\beta I - A)^{-1} z\|.$$

The space X_{-1} does not depend on the specific value of β. We have $X_1 \subset X \subset X_1$, with continuous and dense embedding. Then A has an extension, also denoted by A, such that $A \in \mathcal{L}(X, X_{-1})$.

Let X' denotes the dual of X. Let $A^* : D(A)^* \mapsto X'$ denotes the adjoint of the operator $(A, D(A))$. We endow $D(A^*)$ with the graph norm

$$\|z^*\|_{1,*} = \|(\beta I - A^*)z^*\|_{X'},$$

where $\beta \in \rho(A) = \rho(A^*)$. The following theorem holds (see [16, Chap. 2, Sect. 5]).

Theorem 1.21 *Let X be a reflexive Banach space. Then X_{-1} is isomorphic to $D(A^*)'$.*

Also, if X is reflexive and if $(S(t))_{t\geqslant 0}$ is a C^0-semigroup on X with generator A, then the adjoint semigroup $(S(t)^*)_{t\geqslant 0}$ of $(S(t))_{t\geqslant 0}$ is a C^0-semigroup on X' with generator A^* ([28, Corollary 10.6]).

Proposition 1.22 *Let (L, G) be a boundary control system on U, Z and X. Let A and X_{-1} be as introduced earlier. Then there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that*

$$L = A + BG, \tag{1.129}$$

where A is regarded as an operator from X to X_{-1}. For every $\beta \in \rho(A)$ we have that $(\beta I - A)^{-1}B \in \mathcal{L}(U, Z)$ and

$$G(\beta I - A)^{-1}B = I, \tag{1.130}$$

so that in particular, B is bounded from below.

Proof Since G is onto, it has at least one bounded right inverse $H \in \mathcal{L}(U, Z)$. We put

$$B = (L - A)H. \tag{1.131}$$

From $G(I - HG) = 0$ we see that the range of $I - HG$ is in Ker$G = X_1$, so that $(L - A)(I - HG) = 0$. Thus we get that $BG = (L - A)HG = L - A$, as required in (1.129). It is easy to see that B is unique. To prove (1.130), first we rewrite (1.131) in the form

$$(\beta I - A)H - (\beta I - L)H = B.$$

If we apply $(\beta I - A)^{-1}$ to both sides, we get

$$H - (\beta I - A)^{-1}(\beta I - L)H = (\beta I - A)^{-1}B,$$

which shows that indeed $(\beta I - A)^{-1}B \in \mathcal{L}(U, Z)$. Therefore, we can apply G to both sides above and then the second term on the left-hand side disappears, due to $X_1 = \mathrm{Ker} G$. Since $GH = I$, we obtain (1.130). □

When L, G, A and B are as in the above proposition, we say that A is the *generator* of the boundary control system (L, G) and B is the *control operator* of (L, G).

Remark 1.23 The following fact is an easy consequence of Proposition 1.22 (we use the notation of the proposition): For every $v \in U$ and every $\beta \in \rho(A)$, the vector $z = (\beta I - A)^{-1}Bv$ is the unique solution of the "abstract elliptic problem"

$$Lz = \beta z\,, \qquad Gz = v\,.$$

For many L and G, this problem has a well known solution, and it is easier to describe $z \in X$ than to describe $Bv \in X_{-1}$, since X is usually a more "natural" space than X_{-1} (see the other sections of this chapter).

We are now in a position to write a class of linearized fluid-structure interaction problems as boundary control systems with dynamic feedback.

Let Z, X, U be reflexive Banach spaces of class $\mathcal{HT}$. Let (L, G) be a boundary control system on U, Z and X. Let X_1 and X_{-1} are defined as before. Let $A = L|_{X_1}$ generates a C^0 semigroup in X. Let K be a densely defined, closed unbounded operator in U with domain $D(K)$ and K generates a C^0 semigroup in U. Finally, let $C \in \mathcal{L}(Z, U)$. We consider the following abstract system

$$\begin{aligned} &\dot{z} = Lz, \quad Gz = u,\\ &\dot{u} = Ku + Cz, \\ &z(0) = z_0, \quad u(0) = u_0. \end{aligned} \tag{1.132}$$

Let us introduce the operator $(\mathcal{A}, D(\mathcal{A}))$ in $X \times U$ by

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ u \end{bmatrix} \in Z \times D(K) \mid \ Gz = u \right\} \tag{1.133}$$

and

$$\mathcal{A} \begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} Lz \\ Ku + Cz \end{bmatrix}. \tag{1.134}$$

Lemma 1.24 *The map*

$$\begin{bmatrix} z \\ u \end{bmatrix} \mapsto \|z\|_Z + \|u\|_{D(K)}.$$

is a norm on $D(\mathcal{A})$ equivalent to the graph norm.

Proof For $(z,u) \in D(\mathcal{A})$, and $\beta \in \rho(A)$, we note that

$$\begin{aligned}
&\|z\|_Z + \|u\|_{D(K)} \\
&\leqslant \|z-(\beta I-A)^{-1}Bu\|_{X_1} + \|(\beta I-A)^{-1}Bu\|_Z + \|u\|_U + \|Ku\|_U \\
&\leqslant c(\|(\beta I-A)z-Bu\|_X + \|u\|_U + \|Ku\|_U) \\
&\leqslant c(\|z\|_X + \|Az+Bu\|_X + \|u\|_U + \|Ku\|_U)) \leqslant c\left\|\begin{bmatrix} z \\ u \end{bmatrix}\right\|_{D(\mathcal{A})},
\end{aligned}$$

where c is a strictly positive constant, possibly depending on β. Since the reverse inequality is an obvious one, we obtain the claimed norm equivalence. □

The theorem below shows that if the operator A from (1.128) is $\mathcal{R}$-sectorial and if the operator C from the second equation in (1.132) is "small" with respect to A then the semigroup generator describing the system (1.132) is also $\mathcal{R}$-sectorial. In the applications we are interested in the first equation in (1.132) describes the fluid, with some boundary input. The second equation describes the motion of the structure. Our result below can be interpreted as asserting that, in some sense, the fluid structure system can be seen as a perturbation of the equations describing the fluid alone.

Theorem 1.25 *Let Z, X, U be reflexive Banach spaces of class $\mathcal{HT}$. Let (L,G) be a boundary control system on U, Z and X. Assume that $A = L|_{X_1}$ and K are $\mathcal{R}$-sectorial operators in X and U, respectively. More precisely, assume that there exists $\varepsilon_1, \varepsilon_2 \in (0,\pi/2)$ and $\gamma_1, \gamma_2 \geqslant 0$ such that*

$$\mathcal{R}_{\mathcal{L}(X)}\left\{\lambda(\lambda I-A)^{-1} \mid \lambda \in \Sigma_{\varepsilon_1,\gamma_1}\right\} < \infty, \quad \mathcal{R}_{\mathcal{L}(U)}\left\{\lambda(\lambda I-K)^{-1} \mid \lambda \in \Sigma_{\varepsilon_2,\gamma_2}\right\} < \infty. \tag{1.135}$$

We also suppose that $C \in \mathcal{L}(Z,U)$ satisfies the following condition: for every $\delta > 0$, there exists $C(\delta) > 0$ such that

$$\|Cz\|_U \leqslant \delta\|z\|_Z + C(\delta)\|z\|_X \qquad (z \in Z). \tag{1.136}$$

Then the operator $(\mathcal{A}, D(\mathcal{A}))$ defined as in (1.134) is $\mathcal{R}$-sectorial in $X \times U$, i.e., here exists $\varepsilon_0 \in (0,\pi/2)$ and $\gamma_0 \geqslant 0$ such that

$$\mathcal{R}_{\mathcal{L}(X\times U)}\left\{\lambda(\lambda I-\mathcal{A})^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_0}\right\} < \infty. \tag{1.137}$$

Proof To prove this theorem we write $\mathcal{A}$ in the form $\mathcal{A} = \mathcal{A}_1 + \mathcal{B}$, where

$$\mathcal{A}_1\begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} Lz \\ Ku \end{pmatrix}, \quad \mathcal{B}\begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ Cz \end{pmatrix}.$$

We first show that $(\mathcal{A}_1, D(\mathcal{A}))$ is a $\mathcal{R}$-sectorial operator in $X \times U$. Observe that

$$(\lambda - \mathcal{A}_1)\begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} x \\ v \end{pmatrix}$$

if and only if, $\lambda z - Az - Bu = x$ and $\lambda u - Ku = v$. Thus, for $\lambda \in \Sigma_{\varepsilon_1,\gamma_1} \cap \Sigma_{\varepsilon_2,\gamma_2}$,

$$u = (\lambda - K)^{-1}v \quad \text{and} \quad z = (\lambda - A)^{-1}(x + B(\lambda - K)^{-1}v).$$

Fix $\beta \in \rho(A)$ and set $D = (\beta I - A)^{-1}B$. Thus $D \in \mathcal{L}(U, Z)$. Therefore, for every $v \in U$,

$$\begin{aligned}(\lambda I - A)^{-1}Bv &= (\lambda I - A)^{-1}(\beta I - A)Dv \\ &= -\lambda(\lambda I - A)^{-1}Dv + \beta(\lambda I - A)^{-1}Dv + Dv.\end{aligned}$$

This yields

$$\begin{aligned}&\lambda(\lambda I - \mathcal{A}_1)^{-1} \\ &= \begin{pmatrix} \lambda(\lambda I - A)^{-1} & (\beta - \lambda)(\lambda I - A)^{-1}D\lambda(\lambda I - K)^{-1} + D\lambda(\lambda I - K)^{-1} \\ 0 & (\lambda I - K)^{-1} \end{pmatrix}\end{aligned}$$

Using Proposition 1.17 and (1.135), we can easily verify that, there exists $\varepsilon_3 \in (0, \pi/2)$ and $\gamma_3 > 0$, such that

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(X\times U)}\left\{\lambda(\lambda I - \mathcal{A}_1)^{-1} \mid \lambda \in \Sigma_{\varepsilon_3,\gamma_3}\right\} < \infty, \\ \mathcal{R}_{\mathcal{L}(X\times U)}\left\{\mathcal{A}_1(\lambda I - \mathcal{A}_1)^{-1} \mid \lambda \in \Sigma_{\varepsilon_3,\gamma_3}\right\} < \infty.\end{aligned}$$

Now, Lemma 1.24 and (1.136) gives, for any $\delta > 0$

$$\left\|\mathcal{B}\begin{pmatrix} z \\ u \end{pmatrix}\right\|_{X\times U} \leqslant M\delta\left\|\mathcal{A}_1\begin{pmatrix} z \\ u \end{pmatrix}\right\|_{X\times U} + C(\delta)\left\|\begin{pmatrix} z \\ u \end{pmatrix}\right\|_{X\times U}. \tag{1.138}$$

Therefore, by Proposition 1.18, $\mathcal{A}$ is a $\mathcal{R}$-sectorial operator in $X \times U$ and (1.137) holds. □

1.3.4 Bibliographical Notes

The importance of the maximal regularity property of linearized Navier-Stokes type systems in order to obtain existence and uniqueness for the original nonlinear problems is known for a long time (see Clément and Prüss [7] for an early reference). As previously mentioned, in a Hilbert space setting, this property holds if A generates an analytic semigroup, see [11]. In a Banach space context, the analyticity of the semigroup is no longer sufficient to guarantee the maximal regularity property, see Kalton and Lancien [21]. In our notes we choose to remind the important necessary and sufficient condition for maximal regularity on $\mathcal{HT}$ spaces due to Weis [37]. In our approach, an important role in passing from the maximal, regularity of linearized fluid problems to maximal regularity of associated fluid-structure systems is played by perturbations methods. The main results we have presented in this direction are Proposition 1.18, which is given in Kunstmann and Weis [22] and Theorem 1.25, which seems to be new.

1.4 Existence and Uniqueness Results

1.4.1 Some Background

In this section we will prove local in time existence and uniqueness results for the systems introduced in Sects. 1.1.2 and 1.2. The proofs of the local in time existence and uniqueness results are based on Banach fixed point theorem which is applied to the systems written in fixed spatial domain. In order to apply the Banach fixed point theorem, we need to study the regularity of linear systems with nonhomogeneous source term and non zero initial data on a compact time interval. In fact, to obtain local in time existence and uniqueness of solution, it is important to obtain estimate of solutions in terms of source term and initial data with precise dependence of the continuity constant with respect to time. To this aim, we first recall some basic facts about real interpolation spaces. The proofs can be found in [4, 18, 33].

Let X_0 and X_1 are two complex Banach spaces. The pair (X_0, X_1) is called interpolation couple if there is a linear, complex Hausdorff space Y such that $X_0, X_1 \hookrightarrow Y$ with continuous embeddings.

Lemma 1.26 *Let (X_0, X_1) be an interpolation couple. Then $X_0 \cap X_1$ with the norm*

$$\|x\|_{X_0 \cap X_1} = \max\left(\|x\|_{X_0}, \|x\|_{X_1}\right),$$

and $X_0 + X_1$ with the norm

$$\|x\|_{X_0+X_1} = \inf_{\substack{x=x_0+x_1 \\ x_j \in X_j}} \left(\|x\|_{X_0} + \|x\|_{X_1}\right),$$

are Banach spaces.

We now introduce the real interpolation space $(X_0, X_1)_{\theta,q}$ via K method. Let (X_0, X_1) be an interpolation couple. Fot $0 < t < \infty$, $x \in X_0 + X_1$,

$$K(t, x) = K(t, x, X_0, X_1) = \inf_{\substack{x=x_0+x_1 \\ x_j \in X_j}} \left(\|x\|_{X_0} + t\|x\|_{X_1}\right),$$

is an equivalent norm in $X_0 + X_1$.

Definition 1.27 Let (X_0, X_1) be an interpolation couple. Let $0 < \theta < 1$ and $1 < q < \infty$. Then

$$(X_0, X_1)_{\theta,q} = \left\{ x \in X_0 + X_1 \mid \|x\|_{(X_0,X_1)_{\theta,q}} := \left(\int_0^\infty (t^{-\theta} K(t,x))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

It is easy to verify that $\| \cdot \|_{(X_0,X_1)_{\theta,q}}$ is a norm and that $(X_0, X_1)_{\theta,q}$ is a linear subspace of $X_0 + X_1$. We recall some important properties of the space $(X_0, X_1)_{\theta,q}$

Proposition 1.28

(1) It holds that

$$(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}.$$

(2) There exists a constant $C_{\theta,q}$, $0 < \theta < \infty$, $1 < q < \infty$ such that for all $x \in X_0 \cap X_1$

$$\|x\|_{(X_0,X_1)_{\theta,q}} \leqslant C_{\theta,q} \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta}.$$

Now we introduce another definition of interpolation spaces.

Definition 1.29 Let (X_0, X_1) be an interpolation couple, $\alpha \in \mathbb{R}$ and $1 < q < \infty$. Then

$$\begin{aligned} W(q, \alpha, X_0, X_1) = \Big\{ & u(t) \mid u(t) \text{ locally integrable functions defined on } (0, \infty) \text{ with} \\ & \text{values in } X_0 + X_1 \text{ such that } t^\alpha u(t) \in L^q(0, \infty, X_0), \\ & t^\alpha \dot{u} \in L^q(0, \infty, X_1) \Big\}, \end{aligned}$$

where the derivative $\dot{u} = \dfrac{du}{dt}$ is the distributional derivative of u.
The space $W(q, \alpha, X_0, X_1)$ endowed with the norm

$$\|u\|_{W(q,\alpha,X_0,X_1)} := \|t^\alpha u\|_{L^q(0,\infty;X_0)} + \|t^\alpha \dot{u}\|_{L^q(0,\infty;X_1)}$$

is a Banach space. We define the space of traces as follows

Definition 1.30 Let (X_0, X_1) be an interpolation couple. Let $\alpha \in \mathbb{R}$ and $1 < q < \infty$ are such that $0 < \alpha + q^{-1} < 1$. Then we define

$$T(q, \alpha, X_0, X_1) := \{x \in X_0 + X_1 \mid \text{ there exists } u \in W(q, \alpha, X_0, X_1) \text{ with } u(0) = x\}.$$

The space $T(q, \alpha, X_0, X_1)$ endowed with the norm

$$\|x\|_{T(q,\alpha,X_0,X_1)} := \inf \left\{ \|u\|_{W(q,\alpha,X_0,X_1)} \mid u(0) = x \right\},$$

is a Banach space. The following theorem shows the connection between the interpolation spaces $(X_0, X_1)_{\theta,q}$ and $T(q, \alpha, X_0, X_1)$. This theorem will help us to determine the required space of initial conditions in order to obtain maximal $L^p - L^q$ regularity for linear systems.

Theorem 1.31 *Let* (X_0, X_1) *be an interpolation couple. Let* $\alpha \in \mathbb{R}$ *and* $1 < q < \infty$ *are such that* $0 < \alpha + q^{-1} < 1$. *Then we have*

$$T(q, \alpha, X_0, X_1) \cong (X_0, X_1)_{\theta,q}. \tag{1.139}$$

As discussed earlier in this chapter, we are now going to study the regularity of linear systems with nonhomogeneous source term and non zero initial data. Let $0 < T < \infty$. We consider the initial value problem

$$\dot{u}(t) = Au(t) + f(t), \quad t \in [0, T], \quad u(0) = u_0. \tag{1.140}$$

As a consequence of Theorem 1.19, we have the following result:

Theorem 1.32 *Let* X *be a Banach space of class* $\mathcal{HT}$, $1 < p < \infty$, *and let* A *be a closed, densely defined unbounded operator in* X *with domain* $D(A)$. *Let* A *be an* $\mathcal{R}$*-sectorial operator in* X, *i.e., there exists* $\varepsilon_0 \in (0, \pi/2)$ *and* $\gamma_0 \geqslant 0$ *such that*

$$\mathcal{R}_{\mathcal{L}(X)} \left\{ \lambda(\lambda - A)^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_0} \right\} < \infty. \tag{1.141}$$

Then for every $u_0 \in (X, D(A))_{1-1/p,p}$ *and for every* $f \in L^p(0, T; X)$, (1.140) *admits a unique solution in* $L^p(0, T; D(A)) \cap W^{1,p}(0, T; X)$. *Moreover, there exists a constant* C *independent of* T *such that the following estimate holds*

$$\begin{aligned} &\|u\|_{L^p(0,T;D(A))} + \|u\|_{W^{1,p}(0,T;X)} \\ &\qquad \leqslant C(1 + 2\gamma_0)e^{2\gamma_0 T} \left(\|u_0\|_{(X,D(A))_{1-1/p,p}} + \|f\|_{L^p(0,T;X)} \right). \end{aligned} \tag{1.142}$$

Proof Let us set

$$f_T = \begin{cases} f & \text{if } 0 \leqslant t \leqslant T, \\ 0 & \text{if } t > T, \end{cases}$$

and

$$A_{\gamma_0} = A - 2\gamma_0 I, \quad f_{\gamma_0}(t) = e^{-2\gamma_0 t} f_T. \tag{1.143}$$

Therefore, obviously we have

$$D(A_{\gamma_0}) = D(A), \quad \mathcal{R}_{\mathcal{L}(X)} \left\{ \lambda(\lambda I - A_{\gamma_0})^{-1} \mid \lambda \in \Sigma_{\varepsilon_0} \right\} < \infty$$

and f_{γ_0} belongs to $L^p(0,\infty;X)$. Let us consider the problem

$$\dot{u}_{\gamma_0}(t) = A_{\gamma_0}u_{\gamma_0}(t) + f_{\gamma_0}(t), \quad t \geqslant 0, \quad u_{\gamma_0}(0) = u_0. \tag{1.144}$$

According to Theorem 1.31,

$$(X, D(A))_{1-1/p,p} \cong \left\{u(0) \mid u \in L^p(0,\infty;D(A)) \cap W^{1,p}(0,\infty;X))\right\}.$$

Therefore for every $u_0 \in (X, D(A))_{1-1/p,p}$, there exists $u_1 \in L^p(0,\infty;D(A)) \cap W^{1,p}(0,\infty;X)$ such that $u_1(0) = u_0$ and $\dot{u}_1 - A_{\gamma_0}u_1$ belongs to $L^p(0,\infty;X)$. By Theorem 1.19 and using the fact that $0 \in \rho(A_{\gamma_0})$, we obtain existence and uniqueness strong solution $u_2 \in L^p(0,\infty;D(A)) \cap W^{1,p}(0,\infty;X)$ to

$$\dot{u}_2 = A_{\gamma_0}u_2 + (f - \dot{u}_1 + A_{\gamma_0}u_1), \quad u_2(0) = 0.$$

Hence, $u_{\gamma_0} = u_1 + u_2$ belongs to $L^p(0,\infty;D(A)) \cap W^{1,p}(0,\infty;X)$ and u_{γ_0} solves (1.144). By closed graph theorem, there exists a constant $C > 0$ such that

$$\|u_{\gamma_0}\|_{L^p(0,\infty;D(A))} + \|u_{\gamma_0}\|_{W^{1,p}(0,\infty;X)} \leqslant C(\|f_{\gamma_0}\|_{L^p(0,\infty;X)} + \|u_0\|_{(X,D(A))_{1-1/p,p}}). \tag{1.145}$$

Define

$$u(t) = e^{2\gamma_0 t}u_{\gamma_0}(t), \quad 0 \leqslant t \leqslant T.$$

Then u belongs to $L^p(0,T;D(A)) \cap W^{1,p}(0,T;X)$ and u solves (1.140). Moreover,

$$\begin{aligned}
&\|u\|_{L^p(0,T;D(A))} + \|u\|_{W^{1,p}(0,T;X)} \\
&\leqslant (1 + 2\gamma_0)e^{2\gamma_0 T}(\|u_{\gamma_0}\|_{L^p(0,T;D(A))} + \|u_{\gamma_0}\|_{W^{1,p}(0,T;X)}) \\
&\leqslant (1 + 2\gamma_0)e^{2\gamma_0 T}(\|u_{\gamma_0}\|_{L^p(0,\infty;D(A))} + \|u_{\gamma_0}\|_{W^{1,p}(0,\infty;X)}).
\end{aligned}$$

Finally, by using the above estimate and (1.145), we obtain (1.142). □

For a smooth bounded domain $\Omega \subset \mathbb{R}^n$, the Sobolev spaces of order $s > 0$ are denoted by $W^{s,q}(\Omega)$. Let $m \in \mathbb{N}$. For every $0 < s < m$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, we define Besov spaces by real interpolation of Sobolev spaces

$$B^s_{q,p}(\Omega) = (L^q(\Omega), W^{m,q}(\Omega))_{s/m,p}. \tag{1.146}$$

In particular if $p = q = 2$, then $B^s_{2,2}(\Omega) = W^{s,2}(\Omega)$. We introduce the space

$$W^{2,1}_{q,p}((0,T) \times \Omega) := L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)), \tag{1.147}$$

and

$$\|u\|_{W^{2,1}_{q,p}((0,T)\times\Omega)} := \|u\|_{L^p(0,T;W^{2,q}(\Omega))} + \|u\|_{W^{1,p}(0,T;L^q(\Omega))}. \tag{1.148}$$

We now state an important embedding theorem

Theorem 1.33 ([1, Theorem 4.10.2]) *Let X_0 and X_1 are two Banach spaces such that X_1 is densely embedded in X_0. Let $0 < T \leqslant \infty$ and fix $p \in (1, \infty)$. Then*

$$L^p([0,T); X_1) \cap W^{1,p}([0,T); X_0) \hookrightarrow C([0,T); (X_0, X_1)_{1-1/p,p}).$$

As a consequence of the above theorem, we obtain the following proposition

Proposition 1.34 ([31, Proposition 4.2]) *Let $1 < p, q < \infty$ and T be any positive number. Let Ω be a smooth domain in $\mathbb{R}^n$. Then for any $u \in W^{2,1}_{q,p}((0,T)\times\Omega)$,*

$$\sup_{t\in(0,T)} \|u(t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leqslant C\left(\|u(0)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|u\|_{W^{2,1}_{q,p}((0,T)\times\Omega)}\right), \tag{1.149}$$

where the constant C is independent of time T. In particular, if $p = q = 2$, then for any $u \in L^2(0,T; W^{2,2}(\Omega)) \cap W^{1,2}(0,T; L^{2(\Omega)})$

$$\sup_{t\in(0,T)} \|u(t)\|_{W^{1,2}(\Omega)} \leqslant C\left(\|u(0)\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(0,T;W^{2,2}(\Omega))} + \|u\|_{W^{1,2}(0,T;L^2(\Omega))}\right), \tag{1.150}$$

where the constant C is independent of T.

For any $1 < p < \infty$, p' denotes the conjugate of p, i.e., $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We recall some basic embedding estimates

$$\begin{aligned} \|f\|_{L^p(0,T)} &\leqslant T^{1/p-1/r}\|f\|_{L^r(0,T)}, \quad \text{for all } f \in L^r(0,T), r > p \\ \|f\|_{L^\infty(0,T)} &\leqslant T^{1/p'}\|f\|_{W^{1,p}(0,T)}, \quad \text{for all } f \in W^{1,p}(0,T), f(0) = 0. \end{aligned} \tag{1.151}$$

Let $\Omega(t)$ be a time dependent domain. We define Sobolev spaces in the time dependent domain $\Omega(t)$ as follows.

Definition 1.35 We say that $u \in W^{s_1,p}(0,T; W^{s_2,q}(\Omega(\cdot)))$ if for almost every $t \in (0,T)$, $u(t)$ belongs to $W^{s_2,q}(\Omega(t))$ and $t \mapsto \|u(t,\cdot)\|_{W^{s_2,q}(\Omega(t))}$ is in $W^{s_1,p}(0,T)$.
Other type of Sobolev spaces in the time dependent domain $\Omega(t)$ can be defined similarly. Finally we recall the following useful lemma

Lemma 1.36 ([32, Chap. 3, Lemma 2.1]) *Let X_i, $i = 1, 2, 3$ be Banach spaces with continuous inclusions*

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3.$$

Assume that X_1 is compactly embedded in X_2. Then for any given $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that for all $x \in X_1$

$$\|x\|_{X_2} \leqslant \epsilon \|x\|_{X_1} + C(\epsilon)\|x\|_{X_3}.$$

1.4.2 Back to the Toy Problem

In this section we consider again the toy problem (1.9), this time in a $L^p - L^q$ framework. The main result asserts the local in time existence and uniqueness of solutions for system (1.9) in this context. Let us set $\Omega_h(t) = (-1, 1) \setminus \{h(t)\}$ and $\Omega_{h_0} = (-1, 1) \setminus \{h_0\}$. For every $1 < p < \infty$ and $1 < q < \infty$ the set $\mathcal{I}_{p,q}$ is defined by

$$\mathcal{I}_{p,q} = \left\{(z_0, h_0, g_0) \mid z_0 \in B^{2(1-1/p)}_{q,p}(\Omega_{h_0}), h_0 \in (-1, 1), g_0 \in \mathbb{R}\right\} \tag{1.152}$$

and

$$\|(z_0, h_0, g_0)\|_{\mathcal{I}_{p,q}} := \|z_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_{h_0})} + |h_0| + |g_0|.$$

For every $p, q \in (1,\infty)$ satisfying the condition $\dfrac{1}{p} + \dfrac{1}{2q} \neq 1$, we introduce the space of initial data

$$\mathcal{I}^{cc}_{p,q} = \begin{cases} \mathcal{I}_{p,q} & \text{if } \frac{1}{p} + \frac{1}{2q} > 1, \\ \left\{(z_0, h_0, g_0) \in \mathcal{I}_{p,q} \mid z_0(h_0) = g_0,\ z_0(-1) = z_0(1) = 0\right\} & \text{if } \frac{1}{p} + \frac{1}{2q} < 1. \end{cases} \tag{1.153}$$

The main result of this section states as follows.

Theorem 1.37 *Let $1 < p, q < \infty$ satisfying the condition $\dfrac{1}{p} + \dfrac{1}{2q} \neq 1$. Assume that (v_0, h_0, g_0) belongs to $\mathcal{I}^{cc}_{p,q}$. Then there exists a $T > 0$ such that the system (1.9) admits a unique strong solution*

$$v \in L^p(0, T; W^{2,q}(\Omega_h(\cdot))) \cap W^{1,p}(0, T; L^q(\Omega_h(\cdot))) \cap C([0, T]; B^{2(1-1/p)}_{q,p}(\Omega_h(\cdot))),$$

$$h \in W^{2,p}(0, T).$$

Moreover, $h(t) \in (-1, 1)$ for all $t \in [0, T]$.

In view of Proposition 1.2, it is enough to show local in time existence and uniqueness of solutions for system (1.12) which holds in a fixed spatial domain. Therefore, in this section, we prove the following theorem

Theorem 1.38 *Let* $1 < p, q < \infty$ *satisfying the condition* $\frac{1}{p} + \frac{1}{2q} \neq 1$. *Assume that* (z_0, h_0, g_0) *belongs to* $\mathcal{I}^{cc}_{p,q}$. *Then there exists a* $T > 0$ *such that the system* (1.12) *admits a unique strong solution*

$$z \in L^p(0, T; W^{2,q}(\Omega_{h_0})) \cap W^{1,p}(0, T; L^q(\Omega_{h_0})) \cap C([0, T]; B^{2(1-1/p)}_{q,p}(\Omega_{h_0})),$$
$$h \in W^{2,p}(0, T).$$

Moreover, $h(t) \in (-1, 1)$ *for all* $t \in [0, T]$.

In order to prove the above theorem, we first rewrite (1.12) as follows

$$\begin{aligned}
&\dot{z} - z_{xx} = \mathcal{F}_1(z, g, h), \quad t \in (0, T),\ x \in (-1, 1) \setminus h_0,\\
&z(t, -1) = z(t, 1) = 0, \quad z(t, h_0) = g(t), \quad t \in (0, T)\\
&\dot{g} = [z_x]\,(t, h_0) + \mathcal{F}_2(z, g, h), t \in (0, T)\\
&z(0, x) = z_0(x) \quad x \in (-1, 1), \qquad h(0) = h_0, \qquad g(0) = g_0.
\end{aligned} \tag{1.154}$$

where

$$\begin{aligned}
&h(t) = h_0 + \int_0^t g(s)\, ds,\\
&\mathcal{F}_1(z, g, h) = \frac{k(h - h_0)}{1 - kh}\left[2 + \frac{k(h - h_0)}{1 - kh}\right] z_{xx} + \frac{1 - kx}{1 - kh} g z_x - \frac{1 - kh_0}{1 - kh} z z_x,\\
&\mathcal{F}_2(z, g, h) = (h - h_0)\left[\frac{kz_x}{1 - kh}\right](t, h_0).
\end{aligned} \tag{1.155}$$

We consider the following linear system

$$\begin{aligned}
&\dot{z} - z_{xx} = f_1, \quad t \in (0, T),\ x \in (-1, 1) \setminus h_0,\\
&z(t, -1) = z(t, 1) = 0, \quad z(t, h_0) = g(t), \quad t \in (0, T)\\
&\dot{g} = [z_x]\,(t, h_0) + f_2, t \in (0, T)\\
&z(0, x) = z_0(x) \quad x \in (-1, 1) \setminus \{h_0\}, \qquad g(0) = g_0.
\end{aligned} \tag{1.156}$$

We want to rewrite the above system as an evolution equation in an appropriate Banach space. Let $\Omega_{h_0} = (-1, 1) \setminus \{h_0\}$ and $q > 1$. We introduce the following spaces

$$Z = W^{2,q}(\Omega_{h_0}) \cap W^{1,q}_0(-1, 1), \quad X = L^q(\Omega_{h_0}), \quad U = \mathbb{R}.$$

Let $L \in \mathcal{L}(Z, X)$, $G \in \mathcal{L}(Z, U)$ and $C \in \mathcal{L}(Z, U)$ are defined as follows

$$Lz = z_{xx}, \quad Gz = z(h_0), \quad Cz = [z_x](h_0) \tag{1.157}$$

Let us introduce the unbounded operator $(\mathscr{A}, D(\mathcal{A})$ in $X \times U$ by

$$D(\mathcal{A}) = \{(z, g) \in X \times U \mid Lz \in X,\ Gz = g\} \tag{1.158}$$

and

$$\mathcal{A}\begin{pmatrix} z \\ g \end{pmatrix} = \begin{pmatrix} Lz \\ Cz \end{pmatrix} \tag{1.159}$$

Thus (1.156) can be written as

$$\frac{d}{dt}\begin{pmatrix} z \\ g \end{pmatrix} = \mathcal{A}\begin{pmatrix} z \\ g \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \begin{pmatrix} z(0) \\ g(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ g_0 \end{pmatrix}. \tag{1.160}$$

Proposition 1.39 *There exists* $\varepsilon \in (0, \pi/2)$ *and* $\gamma_0 > 0$ *such that*

$$\mathcal{R}_{\mathcal{L}(X\times U)}\left\{\lambda(\lambda - \mathcal{A})^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_0}\right\} < \infty. \tag{1.161}$$

Proof Let us set $X_1 = \mathrm{Ker}G = \{z \in Z \mid z(h_0) = 0\}$, and $A = L|_{X_1}$. Then, by Denk et al. [12, Theorem 8.2], A is $\mathcal{R}$-sectorial in X and there exists ε_0 and $\tilde{\gamma} \geqslant 0$ such that

$$\mathcal{R}_{\mathcal{L}(X)}\left\{\lambda(\lambda - A)^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\tilde{\gamma}}\right\} < \infty. \tag{1.162}$$

We also have, for $s \in (1/q, 1)$

$$\|Cz\|_U \leqslant C\|z_x\|_{W^{s,q}(\Omega_{h_0})} \leqslant C\|z\|_{W^{1+s}(\Omega_{h_0})}. \tag{1.163}$$

Since $W^{2,q}(\Omega_{h_0}) \hookrightarrow_{compact} W^{1+s,q}(\Omega_{h_0})$, we obtain for any $\delta > 0$, there exists

$$\|z\|_{W^{1+s}(\Omega_{h_0})} \leqslant \delta\|z\|_{W^{2,q}(\Omega_{h_0})} + C(\delta)\|z\|_{L^q(\Omega_{h_0})},$$

holds for arbitrary small δ. This completes the proof of the proposition. □

Combining the above proposition and Theorem 1.32, we obtain the following result

Theorem 1.40 *Let* $1 < p, q < \infty$ *and* $h_0 \in (-1, 1)$. *Then for every* $(z_0, g_0) \in (Z, D(\mathcal{A}))_{1-1/p,p}$ *and for every* $(f_1, f_2) \in L^p(0, T; L^q(\Omega_{h_0})) \times L^p(0, T)$, *the system* (1.156) *admits a unique strong solution satisfying*

$$\begin{aligned}&\|z\|_{L^p(0,T;W^{2,q}(\Omega_{h_0}))} + \|z\|_{W^{1,p}(0,T;L^q(\Omega_{h_0}))} + \|g\|_{W^{1,p}(0,T)} \\ &\leqslant C(1 + e^{2\gamma_0 T})\Big(\|(z_0, h_0, g_0)\|_{(Z,D(\mathcal{A}))_{1-1/p,p}} + \|f_1\|_{L^p(0,T;L^q(\Omega_{h_0}))} + \|f_2\|_{L^p(0,T)}\Big).\end{aligned}$$

In order to prove our local in time existence and uniqueness result we combine the above theorem with a fixed point procedure. In the above theorem, one requires initial conditions from a real interpolation space between $\mathcal{D}(\mathcal{A})$ and Z. In order to identify this interpolation space, we prove the following lemma:

Lemma 1.41 *Let $p, q \in (1, \infty)$ satisfying the condition $\frac{1}{p} + \frac{1}{2q} \neq 1$. Let us assume that $h_0 \in (-1, 1)$ and (z_0, g_0) belongs to $(Z, D(\mathcal{A}))_{1-1/p,p}$. Then (z_0, h_0, g_0) belongs to $\mathcal{I}^{cc}_{p,q}$, where $\mathcal{I}^{cc}_{p,q}$, defined as in (1.153).*

Proof For proof we refer to [33, Sect. 4.3.3] and [2, Theorem 2.2]. □

Next for $T > 0$, we define

$$\mathcal{B}_T = \Big\{(f_1, f_2) \in L^p(0, T; L^q(\Omega_{h_0})) \times L^p(0, T) \mid \\ \|f_1\|_{L^p(0,T;L^q(\Omega_{h_0}))} + \|f_2\|_{L^p(0,T)} \leqslant 1\Big\}. \tag{1.164}$$

Proposition 1.42 *Let $p, q \in (1, \infty)$ satisfying the condition $\frac{1}{p} + \frac{1}{2q} \neq 1$. Assume that (z_0, h_0, g_0) belongs to $\mathcal{I}^{cc}_{p,q}$. Let $M > 0$ be such that*

$$\|(z_0, h_0, g_0)\|_{\mathcal{I}_{p,q}} \leqslant M. \tag{1.165}$$

Then for every $(f_1, f_2) \in \mathcal{B}_T$, the system (1.156) admits a unique strong solution on $[0, T]$. *Moreover, there exists a constant C depending only on M such that*

$$\|z\|_{L^p(0,T_*;W^{2,q}(\Omega_{h_0}))} + \|z\|_{W^{1,p}(0,T_*;L^q(\Omega_{h_0}))} + \|g\|_{W^{1,p}(0,T_*)} \leqslant C, \tag{1.166}$$

$$\|z\|_{L^\infty(0,T_*;L^q(\Omega_{h_0}))} \leqslant C, \quad \|g\|_{L^\infty(0,T_*)} \leqslant C, \tag{1.167}$$

$$\|z_x\|_{L^p(0,T_*;L^\infty(\Omega_{h_0}))} \leqslant CT_*^{(1-s)/2p}, \quad s \in (1/q, 1), \tag{1.168}$$

holds for all $T_* \in (0, 1]$.

Proof The first estimate follows directly from Theorem 1.40. Notice that,

$$\|z - z_0\|_{L^\infty(0,T;L^q(\Omega_{h_0}))} \leqslant T^{1/p'} \|z\|_{W^{1,p}(0,T;L^q(\Omega_{h_0}))},$$

which yields,

$$\|z\|_{L^\infty(0,T_*;L^q(\Omega_{h_0}))} \leqslant C, \quad T_* \in (0, 1]. \tag{1.169}$$

Similarly, we can show $\|g\|_{L^\infty(0,T_*)} \leqslant C$. Since $1 < q < \infty$, we have $z_x \in L^p(0, T_*, W^{1,q}(\Omega_{h_0})) \hookrightarrow L^p(0, T_*, L^\infty(\Omega_{h_0}))$. Let us fix, $s \in (1/q, 1)$. Therefore, we have

$$\|z_x(t, \cdot)\|_{L^\infty(\Omega_{h_0})} \leqslant C\|z_x(t, \cdot)\|_{W^{s,q}(\Omega_{h_0})} \leqslant C\|z(t, \cdot)\|^{(1+s)/2}_{W^{2,q}(\Omega_{h_0})} \|z(t, \cdot)\|^{(1-s)/2}_{L^q(\Omega_{h_0})}.$$

Thus, using (1.169) and Hölder's inequality we get

$$\begin{aligned}\|z_x\|_{L^p(0,T_*;L^\infty(\Omega_{h_0}))} &\leqslant C\|z\|^{(1-s)/2}_{L^\infty(0,T_*;L^q(\Omega_{h_0}))}\left(\int_0^{T_*}\|z(t,\cdot)\|^{(1+s)p/2}_{W^{2,q}(\Omega_{h_0})}\,dt\right)^{1/p}\\ &\leqslant CT_*^{(1-s)/2p}.\end{aligned}$$

□

Lemma 1.43 *Let* $p, q \in (1,\infty)$ *satisfying the condition* $\dfrac{1}{p} + \dfrac{1}{2q} \neq 1$*. For* $T_* \in (0,1]$*, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.164)*. Let* (z_0, h_0, g_0) *and* M *as in Proposition 1.42. Given* $(f_1, f_2) \in \mathcal{B}_{T_*}$*, let* (z, g) *be the solution of* (1.156) *on* $[0, T_*]$ *constructed in Proposition 1.42.*

Then there exists a constant $C > 0$*, depending only on* M*, such that*

$$\begin{aligned}&|h(t) - h(0)| \leqslant CT_*^{1/p'}, \quad t \in [0, T_*]\\ &\|\mathcal{F}_1(z, g, h)\|_{L^p(0,T_*;L^q(\Omega_{h_0}))} \leqslant C(T_*^{1/p'} + T_*^{(1-s)/2p}), \quad s \in (1/q, 1) \qquad (1.170)\\ &\|\mathcal{F}_2(z, g, h)\|_{L^p(0,T_*)} \leqslant CT_*^{1/p'},\end{aligned}$$

where $h, \mathcal{F}_1$ *and* $\mathcal{F}_2$ *have been defined in* (1.155)*.*

Proof Using (1.166), we get for all $t \in [0, T_*]$,

$$|h(t) - h_0| \leqslant \int_0^{T_*}|g(s)|\,ds \leqslant T_*^{1/p'}\|g\|_{L^p(0,T_*)} \leqslant CT_*^{1/p'}. \tag{1.171}$$

Using the above estimate it is easy to see that, for all $t \in [0, T_*]$

$$\left|\frac{1}{1-\kappa h}\right| \leqslant C, \tag{1.172}$$

where the constant C is independent of T_*. Using (1.166), (1.171) and (1.172), the first term of $\mathcal{F}_1(z, g, h)$ can be estimated as follows

$$\begin{aligned}&\left\|\frac{k(h-h_0)}{1-kh}\left[2 + \frac{k(h-h_0)}{1-kh}\right]z_{xx}\right\|_{L^p(0,T_*;L^q(\Omega_{h_0}))}\\ &\leqslant C|h-h_0|\|z_{xx}\|_{L^p(0,T_*;L^q(\Omega_{h_0}))} \leqslant CT_*^{1/p'}. \qquad (1.173)\end{aligned}$$

Using (1.167), (1.168) and (1.172), it is easy to see that the second term of $\mathcal{F}_1$ satisfy the following estimate

$$\begin{aligned}&\left\|\frac{1-kx}{1-kh}gz_x\right\|_{L^p(0,T_*;L^q(\Omega_{h_0}))} \leqslant C\|g\|_{L^\infty(0,T_*)}\|z_x\|_{L^p(0,T_*;L^q(\Omega_{h_0}))}\\ &\leqslant C\|z_x\|_{L^p(0,T_*;L^\infty(\Omega_{h_0}))} \leqslant CT_*^{(1-s)/2p}. \qquad (1.174)\end{aligned}$$

Similarly, using (1.167), (1.168) and (1.172), we obtain

$$\begin{aligned} &\left\| \frac{1-kh_0}{1-kh} z z_x \right\|_{L^p(0,T_*;L^q(\Omega_{h_0}))} \leqslant C\|z\|_{L^\infty(0,T_*;L^q(\Omega_{h_0}))}\|z_x\|_{L^p(0,T_*;L^\infty(\Omega_{h_0}))} \\ &\leqslant CT_*^{(1-s)/2p}. \end{aligned} \tag{1.175}$$

Combining (1.173), (1.174) and (1.175), we get

$$\|\mathcal{F}_1(z,g,h)\|_{L^p(0,T_*;L^q(\Omega_{h_0}))} \leqslant C(T_*^{1/p'} + +T_*^{(1-s)/2p}).$$

Finally, using (1.166), (1.171) and (1.172), one has

$$\begin{aligned} \|\mathcal{F}_2(z,g,h)\|_{L^p(0,T_*)} &= \left\| (h-h_0)\left[\frac{kz_x}{1-kh}\right](\cdot,h_0)\right\|_{L^p(0,T_*)} \\ &\leqslant C\|h-h_0\|_{L^\infty(0,T_*)}\|z_x\|_{L^p(0,T_*;W^{1,q}(\Omega_{h_0}))} \leqslant CT_*^{1/p'}. \end{aligned}$$

□

Lemma 1.44 *Let $p,q \in (1,\infty)$ satisfying the condition $\dfrac{1}{p}+\dfrac{1}{2q} \neq 1$. For $T_* \in (0,1]$, let $\mathcal{B}_{T_*}$ be the ball defined in (1.164). Let (z_0,h_0,g_0) and M as in Proposition 1.42. Given $(f_1^j,f_2^j) \in \mathcal{B}_{T_*}$, for $j=1,2$, let (z^j,g^j) be the solution of (1.156) on $[0,T_*]$ constructed in Proposition 1.42.*

Then there exist a constant $C>0$ depending only on M and $\delta>0$ depending only on p and q such that

$$\begin{aligned} &|h^1(t)-h^2(t)| + \|\mathcal{F}_1(z^1,g^1,h^1)-\mathcal{F}_1(z^2,g^2,h^2)\|_{L^p(0,T_*;L^q(\Omega_{h_0}))} \\ &\qquad + \|\mathcal{F}_2(z^1,g^1,h^1)-\mathcal{F}_2(z^2,g^2,h^2)\|_{L^p(0,T_*)} \\ &\qquad \leqslant CT_*^{\delta}\left(\|f_1^1-f_1^2\|_{L^p(0,T_*;L^q(\Omega_{h_0}))} + \|f_2^1-f_2^2\|_{L^p(0,T_*)}\right) \end{aligned} \tag{1.176}$$

where h, $\mathcal{F}_1$ and $\mathcal{F}_2$ have been defined in (1.155).

We are now in a position to prove our main theorem.

Proof of Theorem 1.38 We consider the map

$$\begin{cases} \mathcal{N}:\mathcal{B}_{T_*}\to\mathcal{B}_{T_*}, \\ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{bmatrix}, \end{cases}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ have been defined in (1.155). We want to show $\mathcal{N}$ is a strict contraction of $\mathcal{B}_{T_*}$, with a Lipschitz constant $\dfrac{1}{2}$ for small T_*. We first note that from

Lemma 1.43 , we obtain, for all $(f_1, f_2) \in \mathcal{B}_{T_*}$

$$\|\mathcal{N}(f_1, f_2)\|_{L^p(0,T_*;L^q(\Omega_{h_0}))\times L^p(0,T_*)} \leqslant C(T_*^{1/p'} + T_*^{(1-s)/2p}),$$

where C is a constant depending only on M. Therefore by choosing $T_* \leqslant 1$ small enough

$$\|\mathcal{N}(f_1, f_2)\|_{L^p(0,T_*;L^q(\Omega_{h_0}))\times L^p(0,T_*)} < 1. \tag{1.177}$$

Therefore $\mathcal{N}$ maps $\mathcal{B}_{T_*}$ into $\mathcal{B}_{T_*}$ for small enough T_*. Next from Lemma 1.44, there exists $C > 0$, depending only on M such that

$$\begin{aligned}&\|\mathcal{N}(f_1^1, f_2^1) - \mathcal{N}(f_1^2, f_2^2)\|_{L^p(0,T_*;L^q(\Omega_{h_0}))\times L^p(0,T_*)}\\ &\leqslant C(T_*^{1/p'} + T_*^{(1-s)/2p}) \left\|(f_1^1, f_2^1) - (f_1^2, f_2^2)\right\|_{L^p(0,T_*;L^q(\Omega_{h_0}))\times L^p(0,T_*)}.\end{aligned}$$

Thus by choosing T_* small enough we obtain $\mathcal{N}$ is a strict contraction, which implies the existence and uniqueness result.

1.4.3 A More Realistic 1D Model

In this section, we shall prove local in time existence and uniqueness of solutions for the system (1.67). Let us set $\Omega_h(t) = (-1, 1) \setminus \{h(t)\}$ and $\Omega_{h_0} = (-1, 1) \setminus \{h_0\}$. For every $1 < p < \infty$ and $1 < q < \infty$ the set $\mathcal{I}_{p,q,\Omega_{h_0}}$ is defined by

$$\begin{aligned}\mathcal{I}_{p,q,\Omega_{h_0}} = \Big\{(\varrho_0, w_0, \vartheta_0, h_0, g_0, Q_0) \mid \varrho_0 \in W^{1,q}(\Omega_{h_0}), \quad w_0, \vartheta_0 \in B^{2(1-1/p)}_{q,p}(\Omega_{h_0}),\\ h_0 \in (-1, 1), \quad g_0 \in \mathbb{R}, \quad Q_0 \in \mathbb{R}, \quad \min_{\xi\in\overline{\Omega_{h_0}}} \varrho_0(\xi) > 0\Big\},\end{aligned}$$

and

$$\begin{aligned}\|(\varrho_0, w_0, \vartheta_0, h_0, g_0, Q_0)\|_{\mathcal{I}_{p,q}} := \|\varrho_0\|_{W^{1,q}(\Omega_{h_0})} + \|w_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_{h_0})} + \|\vartheta_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_{h_0})}\\ +|h_0| + |g_0| + |Q_0|.\end{aligned}$$

Let p and q satisfy one of the following conditions:

$$\begin{aligned}&\text{either } 1 < q < \infty \text{ and } 2 < p < \infty \text{ satisfying } \frac{1}{p} + \frac{1}{2q} \neq 1 \text{ and } \frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2},\\ &\text{or } p = q = 2.\end{aligned} \tag{1.178}$$

Let p, q satisfy the condition (1.178). We introduce the space of initial data

$$\mathcal{I}^{cc}_{p,q,\Omega_{h_0}} = \begin{cases} \mathcal{I}_{p,q,\Omega_{h_0}} & \text{if } \frac{1}{p} + \frac{1}{2q} > 1, \\ \Big\{(\varrho_0, w_0, \vartheta_0, h_0, g_0, Q_0) \in \mathcal{I}_{p,q,\Omega_{h_0}} \mid w_0(h_0) = g_0, \\ \qquad w_0(-1) = w_0(1) = 0,\ \vartheta_0(h_0) = Q_0\Big\} & \text{if } \frac{1}{2} < \frac{1}{p} + \frac{1}{2q} < 1, \\ \Big\{(\varrho_0, w_0, \vartheta_0, h_0, g_0, Q_0) \in \mathcal{I}_{p,q,\Omega_{h_0}} \mid w_0(h_0) = g_0, \\ \quad w_0(-1) = w_0(1) = 0,\ \vartheta_0(h_0) = Q_0, \\ \quad \partial_\xi \vartheta_0(-1) = \partial_\xi \vartheta_0(1) = 0\Big\} & \text{if } \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}. \end{cases} \tag{1.179}$$

We prove the following theorem

Theorem 1.45 *Let p, q satisfy the condition (1.178). Assume that $(\varrho_0, w_0, \vartheta_0, h_0, g_0, Q_0)$ belongs to $\mathcal{I}^{cc}_{p,q,\Omega_{h_0}}$. Then there exists a $T > 0$ such that the system (1.67) admits a unique strong solution*

$$\varrho \in W^{1,p}(0,T; W^{1,q}(\Omega_h(\cdot))) \cap C([0,T]; W^{1,q}(\Omega_h(\cdot)))$$
$$w, \vartheta \in L^p(0,T; W^{2,q}(\Omega_h(\cdot))) \cap W^{1,p}(0,T; L^q(\Omega_h(\cdot))) \cap C([0,T]; B^{2(1-1/p)}_{q,p}(\Omega_h(\cdot))),$$
$$h \in W^{2,p}(0,T), \quad Q \in W^{1,p}(0,T).$$

Moreover, $h(t) \in (-1,1)$ for all $t \in [0,T]$ and $\min\limits_{\xi \in \overline{\Omega_{h_0}}} \varrho(t,\xi) > 0$ for all $t \in [0,T]$, $\xi \in \Omega_h(t)$.

Due to the change of variable introduced in Sect. 1.2.1, it is enough to prove local in time existence and uniqueness for the system (1.92). To this aim, let us set

$$\Omega = (-r_1, r_2) \setminus \{0\}.$$

Let p, q satisfy the condition (1.178). We introduce following space of initial conditions for system (1.92),

$$\mathcal{I}^{cc}_{p,q,\Omega} = \begin{cases} \mathcal{I}_{p,q,\Omega} & \text{if } \frac{1}{p} + \frac{1}{2q} > 1, \\ \Big\{(v_0, u_0, \theta_0, h_0, g_0, Q_0) \in \mathcal{I}_{p,q,\Omega} \mid u_0(0) = g_0, \\ \qquad u_0(-r_1) = u_0(r_2) = 0,\ \theta_0(0) = Q_0\Big\} & \text{if } \frac{1}{2} < \frac{1}{p} + \frac{1}{2q} < 1, \\ \Big\{(v_0, u_0, \theta_0, h_0, g_0, Q_0) \in \mathcal{I}_{p,q,\Omega} \mid u_0(0) = g_0, \\ \quad u_0(-r_1) = u_0(r_2) = 0,\ \theta_0(0) = Q_0, \\ \quad \partial_x \theta_0(-r_1) = \partial_x \theta_0(r_2) = 0\Big\} & \text{if } \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}. \end{cases} \tag{1.180}$$

In this section, we prove the following theorem

Theorem 1.46 *Let p, q satisfy the condition (1.178). Assume that $(v_0, u_0, \theta_0, h_0, g_0, Q_0)$ belongs to $\mathcal{I}^{cc}_{p,q,\Omega}$. Then there exists a $T > 0$ such that the system (1.92) admits a unique strong solution*

$$v \in W^{1,p}(0,T; W^{1,q}(\Omega)) \cap C([0,T]; W^{1,q}(\Omega))$$

$$u, \theta \in L^p(0,T; W^{2,q}(\Omega)) \cap W^{1,p}(0,T; L^q(\Omega)) \cap C([0,T]; B^{2(1-1/p)}_{q,p}(\Omega)),$$

$$h \in W^{2,p}(0,T), \quad Q \in W^{1,p}(0,T).$$

Moreover, $h(t) \in (-r_1, r_2)$ for all $t \in [0,T]$ and $\min_{x\in\overline{\Omega}} v(t,x) > 0$ for all $t \in [0,T]$, $x \in \Omega$.

To prove the above theorem, we rewrite (1.92) as follows

$$\begin{aligned}
&\partial_t v - \partial_x u = 0, && (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}),\\
&\partial_t u - \partial_x\left(\frac{1}{v_0}\partial_x u\right) = \mathcal{F}_1(v,u,\theta), && (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}),\\
&\partial_t \theta - \partial_x\left(\frac{1}{v_0}\partial_x \theta\right) = \mathcal{F}_2(v,u,\theta), && (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}),\\
&u(t,\pm 0) = \dot{h}(t), \qquad \theta(t, 0\pm) = Q(t), && (t \geqslant 0),\\
&m\ddot{h}(t) = \left[\frac{1}{v_0}\partial_x u\right](t,0) + \mathcal{F}_3(v,u,\theta), && (t \geqslant 0),\\
&\dot{Q}(t) = \left[\frac{1}{v_0}\partial_x \theta\right](t,0) + \mathcal{F}_4(v,u,\theta), && (t \geqslant 0),\\
&u(t,-r_1) = u(t,r_2) = 0, \quad \partial_x\theta(t,-r_1) = \partial_x\theta(t,r_2) = 0, && (t \geqslant 0),\\
&v(0,x) = v_0(x), \quad u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x), && x \in [-r_1, r_2] \setminus \{0\},\\
&h(0) = h_0, \quad \dot{h}(0) = g_0,
\end{aligned} \tag{1.181}$$

where

$$\begin{aligned}
&\mathcal{F}_1(v,u,\theta) = \partial_x\left(\left(\frac{1}{v} - \frac{1}{v_0}\right)\partial_x u\right) - \partial_x\left(\frac{\theta}{v}\right)\\
&\mathcal{F}_2(v,u,\theta) = \partial_x\left(\left(\frac{1}{v} - \frac{1}{v_0}\right)\partial_x \theta\right) + \frac{1}{v}(\partial_x u)^2 - \frac{\theta}{v}\partial_x u\\
&\mathcal{F}_3(v,u,\theta) = \left[\left(\frac{1}{v} - \frac{1}{v_0}\right)\partial_x u - \frac{\theta}{v}\right](t,0), \quad \mathcal{F}_4(v,u,\theta) = \left[\left(\frac{1}{v} - \frac{1}{v_0}\right)\partial_x \theta\right](t,0).
\end{aligned} \tag{1.182}$$

We consider the following linear system

$$\begin{aligned}
&\partial_t v - \partial_x u = 0, && (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}),\\
&\partial_t u - \partial_x \left(\frac{1}{v_0}\partial_x u\right) = f_1, && (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}),\\
&\partial_t \theta - \partial_x \left(\frac{1}{v_0}\partial_x \theta\right) = f_2, && (t \geqslant 0,\ x \in [-r_1, r_2] \setminus \{0\}),\\
&u(t, \pm 0) = \dot{h}(t), \qquad \theta(t, 0\pm) = Q(t), && (t \geqslant 0),\\
&m\ddot{h}(t) = \left[\frac{1}{v_0}\partial_x u\right](t, 0) + f_3, && (t \geqslant 0),\\
&\dot{Q}(t) = \left[\frac{1}{v_0}\partial_x \theta\right](t, 0) + f_4, && (t \geqslant 0),\\
&u(t, -r_1) = u(t, r_2) = 0, \quad \partial_x\theta(t, -r_1) = \partial_x\theta(t, r_2) = 0, && (t \geqslant 0),\\
&v(0, x) = v_0(x), \quad u(0, x) = u_0(x), \quad \theta(0, x) = \theta_0(x), && x \in [-r_1, r_2] \setminus \{0\},\\
&h(0) = h_0, \quad \dot{h}(0) = g_0.
\end{aligned} \tag{1.183}$$

We introduce the following spaces

$$Z_1 = W^{2,q}(\Omega) \cap W_0^{1,q}(-r_1, r_2), \quad Z_2 = \left\{\theta \in \times W^{2,q}(\Omega) \mid \partial_x\theta(-r_1) = \partial_x\theta(r_2) = 0\right\},$$
$$Z = W^{1,q}(\Omega) \times Z_1 \times Z_2, \quad X = W^{1,q}(\Omega) \times L^q(\Omega) \times L^q(\Omega), \quad U = \mathbb{R}^2.$$

Let $L \in \mathcal{L}(Z, X)$, $G \in \mathcal{L}(Z, U)$ and $C \in \mathcal{L}(Z, U)$ are defined as follows

$$L\begin{bmatrix} v \\ u \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & \partial_x & 0 \\ 0 & \partial_x\left(\frac{1}{v_0}\partial_x\right) & 0 \\ 0 & 0 & \partial_x\left(\frac{1}{v_0}\partial_x\right) \end{bmatrix}\begin{bmatrix} v \\ u \\ \theta \end{bmatrix}, \qquad G\begin{bmatrix} v \\ u \\ \theta \end{bmatrix} = \begin{bmatrix} u(0) \\ \theta(0) \end{bmatrix},$$
$$C\begin{bmatrix} v \\ u \\ \theta \end{bmatrix} = \begin{bmatrix} m^{-1}\left[\frac{1}{v_0}\partial_x u\right](0) \\ \left[\frac{1}{v_0}\partial_x \theta\right](0) \end{bmatrix} \tag{1.184}$$

Let us introduce the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in $X \times U$ by

$$D(\mathcal{A}) = \left\{(v, u, \theta, g, Q) \in Z \times U \mid G(v, u, \theta)^T = (g, Q)^T\right\} \tag{1.185}$$

and

$$\mathcal{A}\begin{pmatrix} v \\ u \\ \theta \\ g \\ Q \end{pmatrix} = \begin{pmatrix} L\begin{bmatrix} v \\ u \\ \theta \end{bmatrix} \\ C\begin{bmatrix} v \\ u \\ \theta \end{bmatrix} \end{pmatrix}. \tag{1.186}$$

Set $\dot{h}(t) = g(t)$. Then (1.183) can be written as

$$\frac{d}{dt}\begin{pmatrix} v \\ u \\ \theta \\ g \\ Q \end{pmatrix} = \mathcal{A}\begin{pmatrix} v \\ u \\ \theta \\ g \\ Q \end{pmatrix} + \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \quad \begin{pmatrix} v(0) \\ u(0) \\ \theta(0) \\ g(0) \\ Q(0) \end{pmatrix} = \begin{pmatrix} v_0 \\ u_0 \\ \theta_0 \\ g_0 \\ Q_0 \end{pmatrix}. \tag{1.187}$$

Proposition 1.47 *Let* $1 < q < \infty$ *and* v_0 *belongs to* $W^{1,q}(\Omega)$ *such that* $v_0(y) > 0$ *for all* $y \in [-r_1, r_2]$. *The operator* $(\mathcal{A}, D(\mathcal{A}))$ *is* $\mathcal{R}$*-sectorial in* $X \times U$*, i.e., there exists* $\varepsilon \in (0, \pi/2)$ *and* $\gamma_0 > 0$ *such that*

$$\mathcal{R}_{\mathcal{L}(X\times U)}\left\{\lambda(\lambda - \mathcal{A})^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_0}\right\} < \infty. \tag{1.188}$$

Proof Let us set $X_1 = \mathrm{Ker} G = \{(v, u, \theta) \in Z \mid u(0) = 0 = \theta(0)\}$, and $A = L|_{X_1}$. We rewrite A as $A = A_1 + B$, where

$$A_1 = \begin{pmatrix} 0 & \partial_x & 0 \\ 0 & \frac{1}{v_0}\partial_{xx} & 0 \\ 0 & 0 & \frac{1}{v_0}\partial_{xx} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\partial_x v_0}{v_0}\partial_x & 0 \\ 0 & 0 & -\frac{\partial_x v_0}{v_0}\partial_x \end{pmatrix} \tag{1.189}$$

By Denk et al. [12, Theorem 8.2], we first obtain that the operator A_1 with $D(A_1) = X_1$ is $\mathcal{R}$-sectorial in $X \times U$. Next, using Lemma 1.36, it is easy to see that the operator B with $D(B) = D(A_1)$ satisfies the condition (1.125). Thus, by Proposition 1.18, we obtain A is $\mathcal{R}$-sectorial in $X \times U$. Again, using Lemma 1.36 one can check that, the operator C satisfies the condition (1.136). Thus the $\mathcal{R}$-sectoriality of the operator $(\mathcal{A}, D(\mathcal{A}))$ follows from Theorem 1.25. □

Theorem 1.48 *Let* $1 < p, q < \infty$. *Then for every* $(v_0, u_0, \theta_0, g_0, Q_0) \in (Z, D(\mathcal{A}))_{1-1/p,p}$ *and for every* $(f_1, f_2, f_3, f_4) \in L^p(0, T; L^q(\Omega)) \times L^p(0, T; L^q(\Omega)) \times$

$L^p(0,T) \times L^p(0,T)$, *the system* (1.183) *admits a unique strong solution satisfying*

$$\|v\|_{W^{1,p}(0,T;W^{1,q}(\Omega))} + \|u\|_{W^{2,1}_{q,p}((0,T)\times\Omega)} + \|\theta\|_{W^{2,1}_{q,p}((0,T)\times\Omega)} + \|h\|_{W^{2,p}(0,T)}$$
$$+ \|Q\|_{W^{1,p}(0,T)} \leqslant C(1+e^{2\gamma_0 T})\Big(\|(v_0,u_0,\theta_0,g_0,Q_0)\|_{(Z,D(\mathcal{A}))_{1-1/p,p}}$$
$$+ \|f_1\|_{L^p(0,T;L^q(\Omega))} + \|f_2\|_{L^p(0,T;L^q(\Omega))} + \|f_3\|_{L^p(0,T)} + \|f_4\|_{L^p(0,T)}\Big),$$

where the constant C is independent of time T.

Now we are in a position to get estimates required for fixed point argument. As before, at first we want to identify the space of initial conditions. We have the following lemma.

Lemma 1.49 *Let p, q satisfy the condition* (1.178). *Let us assume that* $(v_0, u_0, \theta_0, g_0, Q_0)$ *belongs to* $(Z, D(\mathcal{A}))_{1-1/p,p}$. *Then* $(v_0, u_0, \theta_0, g_0, Q_0)$ *belongs to* $\mathcal{I}^{cc}_{p,q,\Omega}$, *where* $\mathcal{I}^{cc}_{p,q,\Omega}$, *defined as in* (1.180).

Proof For proof we refer to [33, Sect. 4.3.3] and [2, Theorem 2.2]. □

For $T > 0$, we define the space $\mathcal{B}_T$ as follows

$$\mathcal{B}_T = \Big\{(f_1,f_2,f_3,f_4) \in L^p(0,T;L^q(\Omega)) \times L^p(0,T;L^q(\Omega)) \times L^p(0,T) \times L^p(0,T) \mid \|f_1\|_{L^p(0,T;L^q(\Omega))} + \|f_2\|_{L^p(0,T;L^q(\Omega))} + \|f_3\|_{L^p(0,T)} + \|f_4\|_{L^p(0,T)} \leqslant 1\Big\}. \tag{1.190}$$

Proposition 1.50 *Let p, q satisfy the condition* (1.178). *Assume that* $(v_0, u_0, \theta_0, h_0, g_0, Q_0)$ *belongs to* $\mathcal{I}^{cc}_{p,q,\Omega}$. *Let $M > 0$ be such that*

$$\|(v_0, u_0, \theta_0, h_0, g_0, Q_0)\|_{\mathcal{I}_{p,q,\Omega}} \leqslant M, \quad \frac{1}{M} \leqslant v_0(x) \leqslant M. \tag{1.191}$$

Then for every $(f_1,f_2,f_3,f_4) \in \mathcal{B}_T$, *the system* (1.183) *admits a unique strong solution on* $[0,T]$. *Moreover, there exist* $\widetilde{T} \leqslant 1$ *a constant C, both depending only on M such that*

$$\|v\|_{W^{1,p}(0,T_*;W^{1,q}(\Omega))} + \|v\|_{L^\infty(0,T_*;W^{1,q}(\Omega))} \leqslant C, \tag{1.192}$$

$$\frac{1}{C} \leqslant v(t,x) \leqslant C, \qquad t \in (0,\widetilde{T}), x \in (-r_1, r_2) \tag{1.193}$$

$$\|u\|_{W^{2,1}_{q,p}((0,T_*)\times\Omega)} + \|\theta\|_{W^{2,1}_{q,p}((0,T_*)\times\Omega))} \leqslant C, \tag{1.194}$$

$$\|u\|_{L^\infty(0,T_*;W^{1,q}(\Omega))} + \|\theta\|_{L^\infty(0,T_*;W^{1,q}(\Omega))} \leqslant C, \tag{1.195}$$

$$\|u\|_{L^p(0,T_*;L^\infty(\Omega))} + \|\theta\|_{L^p(0,T_*;L^\infty(\Omega))} \leqslant CT_*^{(2-s)/2p}, \quad s \in (1/q, 1), \tag{1.196}$$

$$\|\partial_x u\|_{L^p(0,T_*;L^\infty(\Omega))} + \|\partial_x \theta\|_{L^p(0,T_*;L^\infty(\Omega))} \leqslant CT_*^{(1-s)/2p}, \quad s \in (1/q, 1), \tag{1.197}$$

holds for all $T_* \in (0, \widetilde{T})$.

Proof From Theorem 1.48, there exists a constant C depending only on M such that

$$\|v\|_{W^{1,p}(0,T;W^{1,q}(\Omega))} + \|u\|_{W^{2,1}_{q,p}((0,T)\times\Omega)} + \|\theta\|_{W^{2,1}_{q,p}((0,T)\times\Omega)} \leqslant C, \quad T_* \in (0, 1].$$

Since $1 < q < \infty$, we also have

$$\|v\|_{W^{1,q}(0,T_*;L^\infty(\Omega))} \leqslant C\|v\|_{W^{1,p}(0,T;W^{1,q}(\Omega))} \leqslant C, \quad T_* \in (0, 1].$$

Notice that, for every $T_* \in (0, 1]$

$$\sup_{t\in(0,T_*)} \|v(t, \cdot) - v_0\|_{L^\infty(\Omega)} \leqslant T_*^{1/p'} \|v\|_{W^{1,q}(0,T_*;L^\infty(\Omega))} \leqslant CT_*^{1/p'}. \tag{1.198}$$

Thus there exist $\widetilde{T} \leqslant 1$ a constant C, both depending only on M such that

$$\frac{1}{C} \leqslant v(t, x) \leqslant C, \qquad t \in (0, \widetilde{T}),\ x \in (-r_1, r_2).$$

To prove (1.195), note that $W^{1,q}(\Omega) \hookrightarrow B^{2(1-1/p)}_{q,p}(\Omega)$ provided $2 < p < \infty$. Thus, Proposition 1.34 yields (1.195). In view of (1.150), estimate (1.195) also holds when $p = q = 2$. Proof of other estimates are similar to proof of estimates in Proposition 1.42. □

Lemma 1.51 *Let* p, q *satisfy the condition* (1.178). *For* $T_* \in (0, \widetilde{T}]$, *where* $\widetilde{T}$ *is a constant in Proposition 1.50, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.190). *Let* $(v_0, u_0, \theta_0, h_0, g_0, Q_0)$ *and* M *as in Proposition 1.50. Given* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_{T_*}$, *let* (v, u, θ, h, Q) *be the solution of* (1.183) *on* $[0, T_*]$ *constructed in Proposition 1.50.*

Then there exist a constant $C > 0$ *depending only on* M *and a constant* δ *depending only on p and q, such that*

$$\|\mathcal{F}_1(v, u, \theta)\|_{L^p(0,T_*;L^q(\Omega))} + \|\mathcal{F}_2(v, u, \theta)\|_{L^p(0,T_*;L^q(\Omega))} + \|\mathcal{F}_3(v, u, \theta)\|_{L^p(0,T_*)} + \|\mathcal{F}_4(v, u, \theta)\|_{L^p(0,T_*)} \leqslant CT_*^{\delta}. \tag{1.199}$$

where $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ *and* $\mathcal{F}_4$ *have been defined in* (1.182).

Proof Using the estimates (1.192)–(1.197) and (1.198), we obtain the following estimate of $\mathcal{F}_1$

$$
\begin{aligned}
&\|\mathcal{F}_1(v,u,\theta)\|_{L^p(0,T_*;L^q(\Omega))} \\
&= \left\|\partial_x\left(\left(\frac{1}{v}-\frac{1}{v_0}\right)\partial_x u\right)\right\|_{L^p(0,T_*;L^q(\Omega))} + \left\|\partial_x\left(\frac{\theta}{v}\right)\right\|_{L^p(0,T_*;L^q(\Omega))} \\
&\leqslant \left\|\frac{v-v_0}{vv_0}\right\|_{L^\infty((0,T)\times\Omega)} \|\partial_{xx}u\|_{L^p(0,T_*;L^q(\Omega))} \\
&\quad + \left(\left\|\frac{\partial_x v}{v^2}-\frac{\partial_x v_0}{v_0^2}\right\|_{L^\infty(0,T_*;L^p(\Omega))}\right)\|\partial_x u\|_{L^p(0,T_*;L^\infty(\Omega))} \\
&\quad + \left\|\frac{1}{v}\right\|_{L^\infty(0,T_*;L^q(\Omega))}\|\partial_x\theta\|_{L^p(0,T_*;L^\infty(\Omega))} + \left\|\frac{\partial_x v}{v^2}\right\|_{L^\infty(0,T_*;L^q(\Omega))}\|\theta\|_{L^p(0,T_*;L^\infty(\Omega))} \\
&\leqslant C(T_*^{1/p'} + T_*^{(2-s)/2p} + T_*^{(1-s)/2p}), \quad s\in(1/q,1).
\end{aligned}
$$

Estimates of first and third term of $\mathcal{F}_2$ are similar to the above estimate. Using (1.193), (1.194) and (1.197), it is easy to see that the second term of $\mathcal{F}_2$ satisfy the following estimate

$$
\begin{aligned}
&\left\|\frac{1}{v}(\partial_x u)^2\right\|_{L^p(0,T_*;L^q(\Omega))} \\
&\leqslant \left\|\frac{1}{v}\right\|_{L^\infty((0,T)\times\Omega)} \|\partial_x u\|_{L^p(0,T_*;L^\infty(\Omega))}\|\partial_x u\|_{L^\infty(0,T_*;L^q(\Omega))} \\
&\leqslant CT_*^{(1-s)/2p}, \quad s\in(1/q,1).
\end{aligned}
$$

Thus there exist a constant $C>0$ depending only on M and a constant δ depending only on p and q, such that

$$\|\mathcal{F}_2(v,u,\theta)\|_{L^p(0,T_*;L^q(\Omega))} \leqslant T_*^\delta. \tag{1.200}$$

Notice that

$$\left\|\left[\left(\frac{1}{v}-\frac{1}{v_0}\right)\partial_x u - \frac{\theta}{v}\right](\cdot,0)\right\|_{L^p(0,T_*)} \leqslant C\left\|\left(\frac{1}{v}-\frac{1}{v_0}\right)\partial_x u - \frac{\theta}{v}\right\|_{L^p(0,T_*;W^{1,q}(\Omega))}.$$

Therefore, from the estimate of $\mathcal{F}_1$ we obtain

$$\|\mathcal{F}_3(v,u,\theta)\|_{L^p(0,T_*)} \leqslant T_*^\delta, \tag{1.201}$$

where $C > 0$ depends only on M and δ depends only on p and q. The estimate of $\mathcal{F}_4$ is similar. □

Lemma 1.52 *Let p, q satisfy the condition (1.178). For $T_* \in (0, \widetilde{T}]$, where $\widetilde{T}$ is a constant in Proposition 1.50, let $\mathcal{B}_{T_*}$ be the ball defined in (1.190). Let $(v_0, u_0, \theta_0, h_0, g_0, Q_0)$ and M as in Proposition 1.50. Given $(f_1^i, f_2^i, f_3^i, f_4^i) \in \mathcal{B}_{T_*}$, $i = 1, 2$, let $(v^i, u^i, \theta^i, h^i, Q^i)$ be the solution of (1.183) on $[0, T_*]$ constructed in Proposition 1.50.*

Then there exist a constant $C > 0$ depending only on M and a constant δ depending only on p and q, such that

$$\begin{aligned}
&\|\mathcal{F}_1(v^1, u^1, \theta^2) - \mathcal{F}_1(v^2, u^2, \theta^2)\|_{L^p(0,T_*;L^q(\Omega))} \\
&\qquad + \|\mathcal{F}_2(v, u, \theta) - \mathcal{F}_2(v^2, u^2, \theta^2)\|_{L^p(0,T_*;L^q(\Omega))} \\
&+ \|\mathcal{F}_3(v, u, \theta) - \mathcal{F}_3(v^2, u^2, \theta^2)\|_{L^p(0,T_*)} + \|\mathcal{F}_4(v, u, \theta) - \mathcal{F}_4(v^2, u^2, \theta^2)\|_{L^p(0,T_*)} \\
&\qquad \leqslant CT_*^{\delta}\Big(\|f_1^1 - f_1^2\|_{L^p(0,T_*;L^q(\Omega))} + \|f_2^1 - f_2^2\|_{L^p(0,T_*;L^q(\Omega))} \\
&\qquad\qquad + \|f_3^1 - f_3^2\|_{L^p(0,T_*} + \|f_4^1 - f_3^2\|_{L^p(0,T_*}\Big)
\end{aligned} \tag{1.202}$$

where $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ have been defined in (1.182).

Proof of Theorem 1.46 We consider the map

$$\begin{cases} \mathcal{N} : \mathcal{B}_{T_*} \to \mathcal{B}_{T_*}, \\ \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \mathcal{F}_4 \end{bmatrix}, \end{cases}$$

where $\mathcal{F}_1$, $\mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ have been defined in (1.182). From Lemmas 1.51 and 1.52 and by choosing $T_* \leqslant \widetilde{T}$, small enough it is easy to see that $\mathcal{N}$ is a strict contraction of $\mathcal{B}_{T_*}$, with a Lipschitz constant $\frac{1}{2}$. This implies the existence and uniqueness result.

1.4.4 Motion of a Solid in a Compressible Fluid

In this section, we prove local in time existence and uniqueness of solution for the system (1.110). For every $1 < p < \infty$ and $1 < q < \infty$, the space $\mathcal{I}_{p,q,\Omega_F(0)}$ is defined by

$$\mathcal{I}_{p,q,\Omega_F(0)} = \Big\{(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \mid \rho_0 \in W^{1,q}(\Omega_F(0)) \cap L^\infty(\Omega_F(0)),$$
$$u_0 \in B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3, g_0 \in \mathbb{R}^3, \omega_0 \in \mathbb{R}^3, \min_{y \in \overline{\Omega}_F(0)} \rho_0(y) > 0\Big\}, \tag{1.203}$$

and

$$\|(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)\|_{\mathcal{I}_{p,q,\Omega_F(0)}} = \|\rho_0\|_{W^{1,q}(\Omega_F(0))} + \|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))} + \|g_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3}.$$

We now state our main result.

Theorem 1.53 *Let $2 < p < \infty$ and $3 < q < \infty$. Assume that $(\rho_0, u_0, g_0, \omega_0)$ belongs to $\mathcal{I}_{p,q,\Omega_F(0)}$ satisfying the compatibility condition*

$$u_0 = 0 \text{ on } \partial\Omega, \quad u_0 = g_0 + \omega_0 \times y \text{ on } \partial\Omega_S(0). \tag{1.204}$$

Let $M > 0$ be such that

$$\|(\rho_0, u_0, g_0, \omega_0)\|_{\mathcal{I}_{p,q,\Omega_F(0)}} \leqslant M, \quad \frac{1}{M} \leqslant \rho_0(x) \leqslant M \text{ for } x \in \Omega_F(0). \tag{1.205}$$

Then, there exists $T > 0$ such that the system (1.110) admits a unique strong solution

$$\rho \in W^{1,p}(0,T; W^{1,q}(\Omega_F(\cdot))) \cap C([0,T]; W^{1,q}(\Omega_F(\cdot))),$$
$$u \in L^p(0,T; W^{2,q}(\Omega_F(\cdot))^3) \cap W^{1,p}(0,T; L^q(\Omega_F(\cdot))^3) \cap C([0,T]; B^{2(1-1/p)}_{q,p}(\Omega_F(\cdot))^3),$$
$$h \in W^{2,p}(0,T; \mathbb{R}^3), \quad \omega \in W^{1,p}(0,T; \mathbb{R}^3).$$

Moreover, there exists a constant $M_T > 0$ such that $\frac{1}{M_T} \leqslant \rho(t,x) \leqslant M_T$ for all $t \in (0,T), x \in \Omega_F(t)$.

As before, we first prove our result for a equivalent system in a fixed spatial domain.

Theorem 1.54 *Let $2 < p < \infty$ and $3 < q < \infty$. Assume that $(\rho_0, u_0, g_0, \omega_0)$ belongs to $\mathcal{I}_{p,q,\Omega_F(0)}$ such that (1.204)–(1.205) holds. Then, there exists $T > 0$ such that the system (1.115)–(1.120) admits a unique strong solution*

$$\widetilde{\rho} \in W^{1,p}(0,T; W^{1,q}(\Omega_F(0))) \cap C([0,T]; W^{1,q}(\Omega_F(0))),$$
$$\widetilde{u} \in L^p(0,T; W^{2,q}(\Omega_F(0))^3) \cap W^{1,p}(0,T; L^q(\Omega_F(0))^3) \cap C([0,T]; B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3),$$
$$\widetilde{g} \in W^{1,p}(0,T; \mathbb{R}^3), \quad \widetilde{\omega} \in W^{1,p}(0,T; \mathbb{R}^3).$$

Moreover, there exists a constant $M_T > 0$*, such that* $\dfrac{1}{M_T} \leqslant \widetilde{\rho}(t,y) \leqslant M_T$*, for all* $t \in (0,T), y \in \Omega_F(0)$.

We start with the following linear system

$$\begin{aligned}
&\partial_t\tilde{\rho} + \rho_0 \mathrm{div}\, \tilde{u} = f_1, \quad \text{in } (0,T)\times\Omega_F(0),\\
&\partial_t\tilde{u} - \frac{\mu}{\rho_0}\Delta\tilde{u} - \frac{\alpha+\mu}{\rho_0}\nabla(\mathrm{div}\,\tilde{u}) = f_2 \quad \text{in } (0,T)\times\Omega_F(0),\\
&\tilde{u} = 0 \quad \text{on } (0,T)\times\partial\Omega, \quad \tilde{u} = g + \omega\times y \quad \text{on } (0,T)\times\partial\Omega_S(0),\\
&m\frac{d}{dt}\tilde{\ell} = -\int_{\Omega_S(0)} \big(\mu\nabla\tilde{u} + \mu\nabla\tilde{u}^\top + \alpha\mathrm{div}\,\tilde{u}I\big)n\, d\gamma + f_3, \quad t\in(0,T),\\
&J(0)\frac{d}{dt}\tilde{\omega} = -\int_{\Omega_S(0)} y\times\big(\mu\nabla\tilde{u} + \mu\nabla\tilde{u}^\top + \alpha\mathrm{div}\,\tilde{u}I\big)n + f_4, \quad t\in(0,T)\\
&\tilde{\rho}(0) = \rho_0, \quad \tilde{u}(0) = u_0, \quad \text{in } \Omega_F(0),\\
&\tilde{g}(0) = g_0, \quad \tilde{\omega}_0 = \omega_0.
\end{aligned} \tag{1.206}$$

We introduce the following spaces

$$\begin{aligned}
Z_1 = \Big\{ &z \in W^{2,q}(\Omega_F(0))^3 \mid z = 0 \text{ on } \partial\Omega,\ \exists \ell, k\in\mathbb{R}^3 \text{ such that}\\
&\qquad z = \ell + k\times y \text{ on } \partial\Omega_S(0)\Big\}\\
Z = W^{1,q}&(\Omega)\times Z_1, \quad X = W^{1,q}(\Omega)\times L^q(\Omega)^3, \quad U = \mathbb{R}^6.
\end{aligned}$$

Let $L \in \mathcal{L}(Z,X)$, $G\in\mathcal{L}(Z,U)$ and $C\in\mathcal{L}(Z,U)$ are defined as follows

$$L\begin{bmatrix}\widetilde{\rho}\\ \widetilde{u}\end{bmatrix} = \begin{bmatrix} 0 & \rho_0\mathrm{div} \\ 0 & \dfrac{\mu}{\rho_0}\Delta + \dfrac{\alpha+\mu}{\rho_0}\nabla(\mathrm{div})\end{bmatrix}\begin{bmatrix}\widetilde{\rho}\\ \widetilde{u}\end{bmatrix}, \qquad G\begin{bmatrix}\widetilde{\rho}\\ \widetilde{u}\end{bmatrix} = \begin{bmatrix}\ell\\ k\end{bmatrix},$$

$$C\begin{bmatrix}\widetilde{\rho}\\ \widetilde{u}\end{bmatrix} = \begin{bmatrix} -m^{-1}\displaystyle\int_{\Omega_S(0)} \big(\mu\nabla\tilde{u} + \mu\nabla\tilde{u}^\top + \alpha\mathrm{div}\,\tilde{u}I\big)n\,d\gamma \\ -J(0)^{-1}\displaystyle\int_{\Omega_S(0)} y\times\big(\mu\nabla\tilde{u} + \mu\nabla\tilde{u}^\top + \alpha\mathrm{div}\,\tilde{u}I\big)n \end{bmatrix} \tag{1.207}$$

Let us introduce the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in $X\times U$ by

$$D(\mathcal{A}) = \big\{(\widetilde{\rho},\widetilde{u},\widetilde{g},\widetilde{\omega})\in Z\times U \mid G(\widetilde{\rho},\widetilde{u})^T = (\widetilde{g},\widetilde{\omega})^T\big\} \tag{1.208}$$

and

$$\mathcal{A}\begin{pmatrix}\widetilde{\rho}\\ \widetilde{u}\\ \widetilde{g}\\ \widetilde{\omega}\end{pmatrix} = \begin{pmatrix} L\begin{bmatrix}\widetilde{\rho}\\ \widetilde{u}\end{bmatrix} \\ C\begin{bmatrix}\widetilde{\rho}\\ \widetilde{u}\end{bmatrix}\end{pmatrix}. \tag{1.209}$$

Then (1.206) can be written as

$$\frac{d}{dt}\begin{pmatrix}\widetilde{\rho}\\ \widetilde{u}\\ \widetilde{g}\\ \widetilde{\omega}\end{pmatrix} = \mathcal{A}\begin{pmatrix}\widetilde{\rho}\\ \widetilde{u}\\ \widetilde{g}\\ \widetilde{\omega}\end{pmatrix} + \begin{pmatrix}f_1\\ f_2\\ f_3\\ f_4\end{pmatrix}, \quad \begin{pmatrix}\widetilde{\rho}(0)\\ \widetilde{u}(0)\\ \widetilde{g}(0)\\ \widetilde{\omega}(0)\end{pmatrix} = \begin{pmatrix}\rho_0\\ u_0\\ g_0\\ \omega_0\end{pmatrix}. \tag{1.210}$$

Proposition 1.55 *Let* $3 < q < \infty$ *and* ρ_0 *belongs to* $W^{1,q}(\Omega_F(0))$ *such that* $\rho_0(y) > 0$ *for all* $y \in \overline{\Omega_F(0)}$. *The operator* $(\mathcal{A}, D(\mathcal{A}))$ *is* $\mathcal{R}$*-sectorial in* $X \times U$, *i.e., there exists* $\varepsilon \in (0, \pi/2)$ *and* $\gamma_0 > 0$ *such that*

$$\mathcal{R}_{\mathcal{L}(X\times U)}\left\{\lambda(\lambda - \mathcal{A})^{-1} \mid \lambda \in \Sigma_{\varepsilon_0,\gamma_0}\right\} < \infty. \tag{1.211}$$

Proof The proof is similar to the proof of Proposition 1.47. □

As a consequence of the above proposition and Theorem 1.32 we obtain the following theorem

Theorem 1.56 *Let* $3 < q < \infty$ *and* $1 < p < \infty$. *Then for every* $(\rho_0, u_0, g_0, \omega_0) \in (Z, D(\mathcal{A}))_{1-1/p,p}$ *and for every* $(f_1, f_2, f_3, f_4) \in L^p(0, T; L^q(\Omega_F(0))) \times L^p(0, T; L^q(\Omega_F(0))) \times L^p(0, T) \times L^p(0, T)$, *the system* (1.206) *admits a unique strong solution satisfying*

$$\begin{aligned}&\|\widetilde{\rho}\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|\widetilde{u}\|_{W^{2,1}_{q,p}((0,T)\times\Omega_F(0))} + \|\widetilde{g}\|_{W^{1,p}(0,T)} + \|\widetilde{\omega}\|_{W^{1,p}(0,T)}\\ &\quad\leqslant C(1 + e^{2\gamma_0 T})\Big(\|(\rho_0, u_0, g_0, \omega_0)\|_{(Z,D(\mathcal{A}))_{1-1/p,p}} + \|f_1\|_{L^p(0,T;L^q(\Omega_F(0)))}\\ &\qquad + \|f_2\|_{L^p(0,T;L^q(\Omega_F(0)))} + \|f_3\|_{L^p(0,T)} + \|f_4\|_{L^p(0,T)}\Big),\end{aligned}$$

where the constant C *is independent of time* T.

Now we characterize the space of initial conditions. As before, using [33, Sect. 4.3.3] and [2, Theorem 2.2] we obtain the following characterization of the initial conditions.

Lemma 1.57 *Let* $3 < q < \infty$ *and* $2 < p < \infty$. *Let us assume that* $(\rho_0, u_0, g_0, \omega_0)$ *belongs to* $(Z, D(\mathcal{A}))_{1-1/p,p}$. *Then* $(\rho_0, u_0, g_0, \omega_0)$ *belongs to* $\mathcal{I}_{p,q,\Omega_F(0)}$, *where* $\mathcal{I}_{p,q,\Omega_F(0)}$, *defined as in* (1.203) *satisfying the compatibility condition* (1.204).

For $T > 0$, we define the space $\mathcal{B}_T$ as follows

$$\mathcal{B}_T = \Big\{(f_1, f_2, f_3, f_4) \in L^p(0,T; L^q(\Omega_F(0))) \times L^p(0,T; L^q(\Omega_F(0))) \times L^p(0,T) \times L^p(0,T) \mid \|f_1\|_{L^p(0,T;L^q(\Omega_F(0)))} + \|f_2\|_{L^p(0,T;L^q(\Omega_F(0)))} + \|f_3\|_{L^p(0,T)} + \|f_4\|_{L^p(0,T)} \leqslant 1\Big\}. \tag{1.212}$$

Proposition 1.58 *Let* $3 < q < \infty$ *and* $2 < p < \infty$. *Assume that* $(\rho_0, u_0, g_0, \omega_0)$ *belongs to* $\mathcal{I}_{p,q,\Omega_F(0)}$ *such that* (1.204)–(1.205) *holds. Then for every* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_T$*, the system* (1.206) *admits a unique strong solution* $[0, T]$. *Moreover, there exists a constant C, depending only on M such that*

$$\|\widetilde{\rho}\|_{W^{1,p}(0,T_*;W^{1,q}(\Omega_F(0)))} + \|\widetilde{u}\|_{W^{2,1}_{q,p}((0,T_*)\times\Omega_F(0))} \leqslant C \tag{1.213}$$

$$\|\widetilde{g}\|_{W^{1,p}(0,T_*)} + \|\widetilde{\omega}\|_{W^{1,p}(0,T_*)} \leqslant C, \tag{1.214}$$

$$\|\widetilde{\rho} - \rho_0\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant CT_*^{1/p'}, \|\widetilde{\rho}\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant CT_*^{1/p}, \tag{1.215}$$

$$\|\widetilde{u}\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} + \|\widetilde{g}\|_{L^\infty(0,T_*)} + \|\widetilde{\omega}\|_{L^\infty(0,T_*)} \leqslant C, \tag{1.216}$$

$$\|u\|_{L^p(0,T_*;L^\infty(\Omega_F(0)))} \leqslant CT_*^{(2-s)/2p}, \quad s \in (3/q, 1), \tag{1.217}$$

$$\|\nabla\widetilde{u}\|_{L^p(0,T_*;L^\infty(\Omega_F(0)))}, \|\mathrm{div}\widetilde{u}\|_{L^p(0,T_*;L^\infty(\Omega_F(0)))} \leqslant CT_*^{(1-s)/2p}, \quad s \in (3/q, 1), \tag{1.218}$$

$$\|\widetilde{g}\|_{L^p(0,T_*)} + \|\widetilde{\omega}\|_{L^p(0,T_*)} \leqslant CT_*^{1/p} \tag{1.219}$$

holds for all $T_* \in (0, 1]$.

Proof The proof is similar to the proof of Proposition 1.50. The main difference here is $W^{1,q}(\Omega_F(0)) \hookrightarrow L^\infty(\Omega_F(0))$ if $3 < q < \infty$. □

Now we proof several lemmas required for fixed point argument.

Lemma 1.59 *Let* $3 < q < \infty$ *and* $2 < p < \infty$. *For* $T_* \in (0, 1]$*, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212). *Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and M as in Proposition 1.58. Given* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_{T_*}$*, let* $(\widetilde{\rho}, \widetilde{u}, \widetilde{g}, \widetilde{\omega})$ *be the solution of* (1.206) *constructed in Proposition 1.58. Let Q be defined as in* (1.116). *Then there exists a constant* $C > 0$ *depending only on M such that*

$$\|Q\|_{L^\infty(0,T_*)} \leqslant C, \quad \|Q^T\|_{L^\infty(0,T_*)} \leqslant C,$$

$$\|Q - I\|_{L^\infty(0,T_*)} \leqslant CT_*^{1/p'}, \quad \|Q^T - I\|_{L^\infty(0,T_*)} \leqslant CT_*^{1/p'}, \tag{1.220}$$

$$\|\partial_t Q\|_{L^\infty(0,T_*)} \leqslant C. \tag{1.221}$$

Proof From (1.116) and Proposition 1.58, we have

$$|Q(t)| \leqslant 1 + C\int_0^t |Q(s)|\, ds, \qquad \text{for all } t \in (0, T_*].$$

By Gronwall's lemma, we have

$$|Q(t)| \leqslant e^{Ct} \leqslant e^{C} \qquad \text{for all } t \in (0, T_*].$$

Similarly, from (1.116) and Proposition 1.58, we have

$$\|Q - I\|_{L^\infty(0,T_*)} \leqslant \|Q\|_{L^\infty(0,T_*)} \int_0^{T_*} |(\widetilde{\omega}(s) \times I| ds \leqslant CT_*^{1/p'}.$$

□

Lemma 1.60 *Let* $3 < q < \infty$ *and* $2 < p < \infty$*. For* $T_* \in (0, 1]$*, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212)*. Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and* M *as in Proposition 1.58. Given* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_{T_*}$*, let* $(\widetilde{\rho}, \widetilde{u}, \widetilde{g}, \widetilde{\omega})$ *be the solution of* (1.206) *constructed in Proposition 1.58. Let* X *be defined as in* (1.117)*. Then there exists a constant* $C > 0$*, depending only on* M *such that*

$$\|J_X - I\|_{L^\infty((0,T_*)\times\Omega_F(0))} \leqslant CT_*^{1/p'}. \tag{1.222}$$

Moreover, there exists $\widetilde{T} \leqslant 1$ *such that*

$$\|J_X - I\|_{L^\infty((0,T_*)\times\Omega_F(0)))} \leqslant \frac{1}{2}, \quad (T_* \in (0, \widetilde{T}]). \tag{1.223}$$

Proof From the definition of X and Proposition 1.58, we obtain

$$\sup_{t\in(0,T_*)} \|J_X(t, \cdot) - I\|_{W^{1,q}(\Omega_F(0))} \leqslant C\int_0^{T_*} \|\nabla\widetilde{u}\|_{W^{1,q}(\Omega_F(0))} \leqslant CT_*^{1/p'}.$$

Therefore

$$\|J_X - I\|_{L^\infty((0,T_*)\times\Omega_F(0)))} \leqslant C\|J_X - I\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant CT_*^{1/p'}.$$

□

Lemma 1.61 *Let* $3 < q < \infty$ *and* $2 < p < \infty$*. For* $T_* \in (0, \widetilde{T}]$*, where* $\widetilde{T}$ *is the constant in Lemma 1.60, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212)*. Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and* M *as in Proposition 1.58. Given* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_{T_*}$*, let* $(\widetilde{\rho}, \widetilde{u}, \widetilde{g}, \widetilde{\omega})$ *be the solution of* (1.206) *constructed in Proposition 1.58. Let* X *be defined as in* (1.117)*.*

Then there exists a constant $C > 0$ depending only on M such that

$$\begin{aligned}
&\|J_X\|_{W^{1,p}(0,T_*;W^{1,q}(\Omega_F(0)))} + \|J_X\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant C,\\
&\|\mathrm{cof} J_X\|_{W^{1,p}(0,T_*;W^{1,q}(\Omega_F(0)))} + \|\mathrm{cof} J_X\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant C\\
&\|\mathrm{det} J_X\|_{W^{1,p}(0,T_*;W^{1,q}(\Omega_F(0)))} + \|\mathrm{det} J_X\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant C\\
&\|J_Y\|_{W^{1,p}(0,T_*;W^{1,q}(\Omega_F(0)))} + \|J_Y\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant C,
\end{aligned} \tag{1.224}$$

Proof The estimate of J_X in $L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))$ norm follows from Lemma 1.60. Next we have,

$$\partial_t J_X = Q\nabla\widetilde{u}.$$

Therefore $\partial_t J_X \in L^p(0,T_*;W^{1,q}(\Omega_F(0)))$ and the estimate follows. The estimates of $\mathrm{cof} J_X$ and $\mathrm{det} J_X$ follows from the fact that $W^{1,p}(0,T_*;W^{1,q}(\Omega_F(0)))$ and $L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))$ are algebras for $p > 2$ and $q > 3$. In order to estimate the norms of J_Y we use the following relation

$$J_Y = \frac{1}{\mathrm{det} J_X}\mathrm{cof} J_X.$$

□

Lemma 1.62 *Let $3 < q < \infty$ and $2 < p < \infty$. For $T_* \in (0,\widetilde{T}]$, where $\widetilde{T}$ is the constant in Lemma 1.60, let $\mathcal{B}_{T_*}$ be the ball defined in (1.212). Let $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ and M as in Proposition 1.58. Given $(f_1,f_2,f_3,f_4) \in \mathcal{B}_{T_*}$, let $(\widetilde{\rho},\widetilde{u},\widetilde{g},\widetilde{\omega})$ be the solution of (1.206) constructed in Proposition 1.58. Let $\widehat{\mathcal{F}}_1$ be defined as in (1.118). Then there exists a constant $C > 0$ depending only on M such that*

$$\|\mathcal{F}_1\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant C T_*^{1/p'}. \tag{1.225}$$

Proof Let us recall

$$\mathcal{F}_1 = -(\widetilde{\rho} - \rho_0)\,\mathrm{div}\,\widetilde{u} - \widetilde{\rho}(Q - I)\nabla\widetilde{u} : J_Y^\top - \widetilde{\rho}\nabla\widetilde{u} : (J_Y^\top - I).$$

Notice that $W^{1,q}(\Omega_F(0))$ is an algebra for $q > 3$. Therefore, using Proposition 1.58, Lemmas 1.59 and 1.61 we estimate the first term of $\mathcal{F}_1$ as follows

$$\begin{aligned}
&\|(\widetilde{\rho} - \rho_0)\,\mathrm{div}\,\widetilde{u}\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))}\\
&\leqslant C\|\widetilde{\rho} - \rho_0\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))}\|\mathrm{div}\,\widetilde{u}\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))}\\
&\leqslant C T_*^{1/p'}.
\end{aligned}$$

Similarly, the second term of $\mathcal{F}_1$ can be estimated as follows

$$\begin{aligned}
&\|\widetilde{\rho}(Q-I)\nabla\widetilde{u} : J_Y^\top\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))} \\
&\leqslant C\|\widetilde{\rho}\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))}\|Q-I\|_{L^\infty(0,T_*)} \\
&\qquad\qquad \|\widetilde{u}\|_{L^p(0,T_*;W^{2,q}(\Omega_F(0)))}\|J_Y^\top\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \\
&\leqslant CT_*^{1/p'}.
\end{aligned}$$

The last term of $\mathcal{F}_1$ satisfies the following estimate

$$\begin{aligned}
&\|\widetilde{\rho}\nabla\widetilde{u} : (J_Y^\top - I)\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))} \\
&\leqslant C\|\widetilde{\rho}\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))}\|\widetilde{u}\|_{L^p(0,T_*;W^{2,q}(\Omega_F(0)))}\|J_Y^\top - I\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \\
&\leqslant CT_*^{1/p'}.
\end{aligned}$$

□

Lemma 1.63 *Let* $3 < q < \infty$ *and* $2 < p < \infty$*. For* $T_* \in (0,\widetilde{T}]$*, where* $\widetilde{T}$ *is the constant in Lemma 1.60, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212)*. Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and* M *as in Proposition 1.58. Given* $(f_1,f_2,f_3,f_4) \in \mathcal{B}_{T_*}$*, let* $(\widetilde{\rho},\widetilde{u},\widetilde{g},\widetilde{\omega})$ *be the solution of* (1.206) *constructed in Proposition 1.58. Let* $\mathcal{F}_{2,1}$ *be defined as in* (1.119)*. Then there exist a constant* $C > 0$ *depending only on* M *and a constant* δ *depending only on* p *and* q *such that*

$$\|\mathcal{F}_{2,1}\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \leqslant CT_*^\delta. \tag{1.226}$$

Proof

$$\mathcal{F}_{2,1} = -\frac{\tilde{\rho}}{\rho_0}\partial_t Q(t)\tilde{u} - \frac{\tilde{\rho}-\rho_0}{\rho_0}Q(t)\partial_t\tilde{u} - (Q(t)-I)\partial_t\tilde{u} - \gamma\frac{\tilde{\rho}^{\gamma-1}}{\rho_0}J_Y^\top\nabla\tilde{\rho}$$

Using Proposition 1.58 and Lemmas 1.59–1.61, we estimate the various terms of $\mathcal{F}_{2,1}$ as follows

$$\begin{aligned}
&\left\|\frac{\widetilde{\rho}}{\rho_0}\partial_t Q\widetilde{u}\right\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\leqslant C\|\widetilde{\rho}\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))}\|\partial_t Q\|_{L^\infty(0,T_*)}\|\widetilde{u}\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\leqslant CT_*^{1/p}\|\widetilde{u}\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant CT_*^{1/p'},
\end{aligned}$$

$$\begin{aligned}
&\left\|\frac{\widetilde{\rho}-\rho_0}{\rho_0}Q\partial_t\widetilde{u}\right\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\leqslant C\|\widetilde{\rho}-\rho_0\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))}\|Q\|_{L^\infty(0,T_*)}\|\partial_t\widetilde{u}\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\leqslant CT_*^{1/p'},
\end{aligned}$$

$$\|(Q-I)\partial_t\widetilde{u}\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \leqslant \|Q-I\|_{L^\infty(0,T_*)}\|\widetilde{u}\|_{W^{1,p}(0,T_*;L^q(\Omega_F(0)))} \leqslant CT_*^{1/p'},$$

$$\left\|\gamma\frac{\tilde{\rho}^{\gamma-1}}{\rho_0}J_Y^\top\nabla\tilde{\rho}\right\|_{L^p(0,T_*;L^q(\Omega_F(0)))}$$
$$\leqslant C\|\widetilde{\rho}\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))}\|J_Y^\top\|_{L^\infty((0,T_*)\times\Omega_F(0))}\|\nabla\tilde{\rho}\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \leqslant CT_*^{1/p}.$$

□

Lemma 1.64 *Let* $3 < q < \infty$ *and* $2 < p < \infty$. *For* $T_* \in (0,\widetilde{T}]$, *where* $\widetilde{T}$ *is the constant in Lemma 1.60, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212). *Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and* M *as in Proposition 1.58. Given* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_{T_*}$, *let* $(\widetilde{\rho}, \widetilde{u}, \widetilde{g}, \widetilde{\omega})$ *be the solution of* (1.206) *constructed in Proposition 1.58. Let* $\mathcal{F}_{2,2}$ *be defined as in* (1.119). *Then there exist a constant* $C > 0$ *depending only on* M *and a constant* δ *depending only on* p *and* q *such that*

$$\|(\mathcal{F}_{2,2})_i\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \leqslant CT_*^{1/p'}, \qquad i = 1,2,3. \tag{1.227}$$

Proof We have

$$(\mathcal{F}_{2,2})_i = \frac{\mu}{\rho_0}\sum_{j,k,l=1}^{3}\partial_{y_l}(\partial_{y_k}(Q\tilde{u})_i[(J_Y)_{kj}-\delta_{kj}])(J_Y)_{lj}$$
$$+\frac{\mu}{\rho_0}\sum_{k,l=1}^{3}(\partial^2_{y_ly_k}(Q\tilde{u})_i)\,[(J_Y)_{lk}-\delta_{lk}]$$
$$+\frac{\alpha+\mu}{\rho_0}\sum_{j,k,l=1}^{3}\partial_{y_l}(\partial_{y_k}(Q\tilde{u})_j\left[(J_Y)_{kj})-\delta_{kj}\right])(J_Y)_{li}$$
$$+\frac{\alpha+\mu}{\rho_0}\sum_{l,j=1}^{3}(\partial^2_{y_ly_j}(Q\tilde{u})_j)\,[(J_Y)_{li}-\delta_{li}] + (Q^\top - I):\partial_{y_i}\nabla\tilde{u},$$

Let us notice that

$$(\partial_{y_k}J_Y)(0,\cdot) = 0.$$

Therefore, using the estimates in Lemma 1.61, we get

$$\|\partial_{y_k}J_Y\|_{L^\infty(0,T_*;L^q(\Omega_F(0)))} \leqslant T_*^{1/p'}\|\partial_{y_k}J_Y\|_{W^{1,p}(0,T_*;L^q(\Omega_F(0)))} \leqslant CT_*^{1/p'}. \tag{1.228}$$

The first term can be estimated as follows,

$$\begin{aligned}
&\left\| \frac{\mu}{\rho_0} \sum_{j,k,l=1}^{3} \partial_{y_l}(\partial_{y_k}(\mathcal{Q}\widetilde{u})_i[(J_Y)_{kj} - \delta_{kj}])(J_Y)_{lj} \right\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\leqslant C \sum_{j,k,l=1}^{3} \Big(\|[\partial_{y_l y_k}(\mathcal{Q}\widetilde{u})_i][(J_Y)_{kj} - \delta_{kj}]\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\qquad\qquad + \|[\partial_{y_k}(\mathcal{Q}\widetilde{u})_i][\partial_{y_l}(J_Y)_{kj}]\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \Big) \\
&\leqslant C \Big(\|\mathcal{Q}\widetilde{u}\|_{L^p(0,T_*;W^{2,q}(\Omega_F(0)))} \|J_Y - I\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \\
&\qquad + \|\mathcal{Q}\widetilde{u}\|_{L^p(0,T_*;W^{2,q}(\Omega_F(0)))} \sum_{j,k,l=1}^{3} \|\partial_{y_l}(J_Y)_{kj}\|_{L^\infty(0,T_*;L^q(\Omega_F(0)))} \Big) \\
&\leqslant C T_*^{1/p'}.
\end{aligned}$$

Other terms in $\mathcal{F}_{2,2}$ can be estimated similarly. □

Lemma 1.65 *Let* $3 < q < \infty$ *and* $2 < p < \infty$*. For* $T_* \in (0, \widetilde{T}]$*, where* $\widetilde{T}$ *is the constant in Lemma 1.60, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212)*. Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and M as in Proposition 1.58. Given* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_{T_*}$*, let* $(\widetilde{\rho}, \widetilde{u}, \widetilde{g}, \widetilde{\omega})$ *be the solution of* (1.206) *constructed in Proposition 1.58.Let* $\mathcal{G}$ *be defined as in* (1.120)*. Then there exist a constant* $C > 0$ *depending only on M and a constant* δ *depending only on p and q such that*

$$\|\mathcal{G}\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant C T_*^{1/p'}. \tag{1.229}$$

Proof The proof is similar to the proof of Lemma 1.62. The only thing is left to check is the estimate $\mathrm{cof} J_X - I$ in $L^\infty(0, T_*; W^{1,q}(\Omega_F(0)))$ norm. Since, $(\mathrm{cof} J_X - I)(0, \cdot) = 0$, we have

$$\|\mathrm{cof} J_X - I\|_{L^\infty(0,T_*;W^{1,q}(\Omega_F(0)))} \leqslant T_*^{1/p'} \|\mathrm{cof} J_X - I\|_{W^{1,p}(0,T_*;W^{1,q})(\Omega_F(0))}.$$

With the help of above estimate we can proceed as the proof of Lemma 1.62 to complete the proof of this Lemma. □

Lemma 1.66 *Let* $3 < q < \infty$ *and* $2 < p < \infty$*. For* $T_* \in (0, \widetilde{T}]$*, where* $\widetilde{T}$ *is the constant in Lemma 1.60, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212)*. Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and M as in Proposition 1.58. Given* $(f_1, f_2, f_3, f_4) \in \mathcal{B}_{T_*}$*, let* $(\widetilde{\rho}, \widetilde{u}, \widetilde{g}, \widetilde{\omega})$ *be the solution of* (1.206) *constructed in Proposition 1.58. Let* $\mathcal{F}_3$ *and* $\mathcal{F}_4$ *be defined as* *in* (1.120)*. Then there exists a constant* $C > 0$ *depending only on* M *such that*

$$\|\mathcal{F}_3\|_{L^p(0,T_*)} \leqslant C(T_*^{1/p} + T_*^{1/p'})$$
$$\|\mathcal{F}_4\|_{L^p(0,T_*)} \leqslant C(T_*^{1/p} + T_*^{1/p'}). \tag{1.230}$$

Proof Let us recall

$$\mathcal{F}_3 = -m(\tilde{\omega} \times \tilde{\ell}) - \int_{\Omega_S(0)} \mathcal{G} n$$

Therefore

$$\begin{aligned}
&\|\mathcal{F}_3\|_{L^p(0,T_*)} \\
&\leqslant C\Big(\|\widetilde{\omega}\|_{L^p(0,T_*)}\|\widetilde{g}\|_{L^\infty(0,T_*)} + \|\mathcal{G}\|_{L^p(0,T_*;L^q(\partial\Omega_S(0)))}\Big) \\
&\leqslant C\Big(T_*^{1/p'} + \|\mathcal{G}\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))}\Big) \\
&\leqslant C T_*^{1/p'}.
\end{aligned}$$

The estimate of $\|\mathcal{F}_4\|_{L^p(0,T_*)}$ is similar. □

Proposition 1.67 *Let* $3 < q < \infty$ *and* $2 < p < \infty$*. For* $T_* \in (0, \widetilde{T}]$*, where* $\widetilde{T}$ *is the constant in Lemma 1.60, let* $\mathcal{B}_{T_*}$ *be the ball defined in* (1.212)*. Let* $(\rho_0, u_0, \vartheta_0, a_0, \omega_0)$ *and* M *as in Proposition 1.58. Given* $(f_1^j, f_2^j, f_3^j, f_4^j) \in \mathcal{B}_{T_*}$*,* $j = 1, 2$*, let* $(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{g}^j, \widetilde{\omega}^j)$ *be the solution of* (1.206) *constructed in Proposition 1.58.*

Let us set

$$\mathcal{F}_1^j = \mathcal{F}_1(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{g}^j, \widetilde{\omega}^j), \mathcal{F}_{2,1}^j = \mathcal{F}_{2,1}(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{g}^j, \widetilde{\omega}^j), \mathcal{F}_{2,2}^j = \mathcal{F}_{2,2}(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{g}^j, \widetilde{\omega}^j)$$
$$\mathcal{F}_3^j = \mathcal{F}_3(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{g}^j, \widetilde{\omega}^j), \mathcal{F}_4^j = \mathcal{F}_4(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{g}^j, \widetilde{\omega}^j) \text{ for } j = 1, 2. \tag{1.231}$$

Then there exists a constant $C > 0$ *depending only on* M *such that*

$$\begin{aligned}
&\|\mathcal{F}_1^1 - \mathcal{F}_1^2\|_{L^p(0,T_*;W^{1,q}(\Omega_F(0)))} + \|\mathcal{F}_{2,1}^1 - \mathcal{F}_{2,1}^2\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\quad + \|\mathcal{F}_{2,2}^1 - \mathcal{F}_{2,2}^2\|_{L^p(0,T_*;L^q(\Omega_F(0)))} + \|\mathcal{F}_3^1 - \mathcal{F}_3^2\|_{L^p(0,T_*)} + \|\mathcal{F}_4^1 - \mathcal{F}_4^2\|_{L^p(0,T_*)} \\
&\qquad \leqslant C T_*^\delta \Big(\|f_1^1 - f_1^2\|_{L^p(0,T_*;L^q(\Omega_F(0)))} + \|f_2^1 - f_2^2\|_{L^p(0,T_*;L^q(\Omega_F(0)))} \\
&\qquad\qquad + \|f_3^1 - f_3^2\|_{L^p(0,T_*} + \|f_4^1 - f_3^2\|_{L^p(0,T_*}\Big)
\end{aligned} \tag{1.232}$$

where $\delta > 0$ *is a positive constant depending only on* p *and* q*.*

Now we give the proofs of main theorems of this section.

Proof of Theorem 1.54 We consider the map

$$\begin{cases} \mathcal{N} : \mathcal{B}_{T_*} \to \mathcal{B}_{T_*}, \\ \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_{2,1} + \mathcal{F}_{2,2} \\ \mathcal{F}_3 \\ \mathcal{F}_4 \end{bmatrix}, \end{cases}$$

By virtue of Lemmas 1.62–1.66 and Proposition 1.67, it is easy to see that $\mathcal{N}$ is a strict contraction of $\mathcal{B}_{T_*}$, with a Lipschitz constant $\frac{1}{2}$ for small T_*. Thus the proof follows from the Banach fixed point theorem.

Proof of Theorem 1.53 Let us assume that $(\rho_0, u_0, g_0, \omega_0) \in \mathcal{I}_{p,q,\Omega_F(0)}$ satisfying (1.204)–(1.205). Let $\widehat{T} \leqslant \widetilde{T}$ be such that, $(\widetilde{\rho}, \widetilde{u}, \widetilde{g}, \widetilde{\omega})$ be the solution to the system (1.115)–(1.120) satisfying

$$\begin{gathered} \widetilde{\rho} \in W^{1,p}(0, \widehat{T}; W^{1,q}(\Omega_F(0))) \\ \widetilde{u} \in L^p(0, \widehat{T}; W^{2,q}(\Omega_F(0))^3) \cap W^{1,p}(0, \widehat{T}; L^q(\Omega_F(0))^3) \\ \widetilde{g} \in W^{1,p}(0, \widehat{T}; \mathbb{R}^3), \quad \widetilde{\omega} \in W^{1,p}(0, \widehat{T}; \mathbb{R}^3). \end{gathered}$$

Since $\widehat{T} \leqslant \widetilde{T}$, $X(t, \cdot)$ is C^1-diffeomorphism from $\Omega_F(0)$ into $\Omega_F(t)$. Therefore, there is a unique $Y(t, \cdot)$ from $\Omega_F(t)$ into $\Omega_F(0)$ such that $Y(t, \cdot) = X^{-1}(t, \cdot)$. We set

$$\begin{aligned} &\rho(t, x) = \widetilde{\rho}(t, Y(t, x)), \quad u(t, x) = Q(t)\widetilde{u}(t, Y(t, x)), \\ &\dot{h}(t) = Q(t)\widetilde{g}(t), \quad \omega(t) = Q(t)\widetilde{\omega}(t), \text{ for all } x \in \Omega_F(t), \ t \geqslant 0. \end{aligned} \tag{1.233}$$

We can easily check that $(\rho, u, \vartheta, h, \omega)$ satisfies the original system (1.110) and

$$\begin{gathered} \rho \in W^{1,p}(0, T; W^{1,q}(\Omega_F(\cdot))) \cap C([0, T]; W^{1,q}(\Omega_F(\cdot))), \\ u \in L^p(0, T; W^{2,q}(\Omega_F(\cdot))^3) \cap W^{1,p}(0, T; L^q(\Omega_F(\cdot))^3) \cap C([0, T]; B^{2(1-1/p)}_{q,p}(\Omega_F(\cdot))^3), \\ h \in W^{2,p}(0, T; \mathbb{R}^3), \quad \omega \in W^{1,p}(0, T; \mathbb{R}^3). \end{gathered}$$

The uniqueness for the solution of (1.110) follows from uniqueness of solution to the system (1.115)–(1.120). This completes the proof of Theorem 1.53.

1.4.5 Bibliographical Notes

As far as we know, the first mathematical analysis approach of piston problems similar to the ones we have introduced in Sect. 1.4.3 was performed in Shelukhin [30], where global in time existence and uniqueness of classical solutions have been given. Less regular solutions, in a Hilbert space setting have been given in Maity et al. [26], which was our main source in Sect. 1.4.3. Our approach of the three dimensional case in Sect. 1.4.4 should be seen as a simplification of the methodology proposed in Hieber and Murata [19], which is also considering the L^p-L^q setting. Earlier results in a Hilbert space framework, which require more derivability of the initial data, have been given in Boulakia and Guerrero [5].

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