A New Approach for Solving CCR Data Envelopment Analysis Model Under Uncertainty

Bindu Bhardwaj, Jagdeep Kaur and Amit Kumar

Abstract Wang and Chin (Expert Syst Appl, 38:11678–11685, 2011 [25]) proposed an optimistic as well as pessimistic fuzzy CCR data envelopment analysis (DEA) model and an approach for solving it to evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of decision making units (DMUs). In this chapter, it is shown that the fuzzy CCR models, proposed by Wang and Chin, are not valid and hence cannot be used to evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs. To resolve the flaws of the fuzzy CCR DEA models, proposed by Wang and Chin, new fuzzy CCR DEA models are proposed. Also, a new approach is proposed to solve the proposed fuzzy CCR DEA models for evaluating the relative geometric crisp efficiency of DMUs.

Keywords Data envelopment analysis · Fuzzy input and fuzzy output data · Fuzzy efficiency

1 Introduction

DEA is a non-parametric approach for measuring the relative efficiency of DMUs when the production process presents a structure of multiple inputs and outputs. DEA has found surprising development due to its wide range of applications to real world problems. The conventional CCR and BCC DEA models [1, 2] require accurate measurement of both the inputs and outputs.

In conventional DEA models, all the data is assumed to be exactly known. However, inputs and outputs of DMUs in real world problems may be imprecise. Imprecise evaluations may be the result of unquantifiable, incomplete and non-obtainable information. In recent years, fuzzy set theory has been proven to be useful as a way to quantify imprecise and vague data in DEA models. The DEA

B. Bhardwaj · J. Kaur (✉) · A. Kumar
Thapar University, Patiala 147004, Punjab, India
e-mail: sidhu.deepi87@gmail.com

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model with fuzzy data, called "fuzzy DEA" models, can more realistically represent real world problems than the conventional DEA models.

Several authors [1, 3–24, 27, 28] have proposed methods to solve fuzzy DEA models and to solve fuzzy DEA models two approaches are used: (1) by transforming the fuzzy fractional programming model into a fuzzy linear programming model (2) by transforming fuzzy DEA models into two respective pessimistic and optimistic crisp DEA models using α-cut technique. Wang et al. [26] pointed out that the former ignores the fact that a fuzzy fractional programming cannot be transformed into a linear programming model as we do for a crisp fractional programming; while the latter requires the solution of a series of linear programming models based on different α-level sets and therefore it requires much computational efforts to get the fuzzy efficiencies of DMUs. Thus Wang et al. [26] proposed methods to solve fuzzy DEA models without using the aforementioned transformations (1) and (2) and later extended their existing method by considering uncertain weights [25]. In this chapter, the shortcomings of the existing method [25] are pointed out and a new method is proposed for the same.

The rest of the chapter is organized as follows. In Sect. 2, some basic definitions and arithmetic operations on fuzzy numbers are presented. In Sect. 3, the existing method [25] for solving fuzzy DEA problems is reviewed. In Sect. 4, the flaws of the existing method [25] are pointed out. To overcome these flaws, a new method is proposed in Sect. 5 for solving the proposed fuzzy DEA models. The proposed method is illustrated with the help of a real world problem in Sect. 6 and the obtained results are discussed in Sect. 7. Finally, the conclusions are discussed in Sect. 8.

2 Basic Definitions and Arithmetic Operations

In this section, some basic definitions and arithmetic operations on fuzzy numbers are reviewed [29].

2.1 Basic Definitions

In this section, some basic definitions are reviewed [29].

Definition 1 A fuzzy number $\tilde{A} = (a^L, a^M, a^R)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x - a^L)}{(a^M - a^{L)}}, & a^L \le x < a^M \\ 1 & x = a^M \\ \dfrac{(x - a^R)}{(a^M - a^R)}, & a^M < x \le a^R \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2 A triangular fuzzy number $\tilde{A}=(a^L,a^M,a^R)$ is said to be non-negative triangular fuzzy number if and only if $a^L\geq 0$.

Definition 3 A triangular fuzzy number $\tilde{A}=(a^L,a^M,a^R)$ is said to be positive triangular fuzzy number if and only if $a^L>0$.

Definition 4 A ranking function is a function $\mathfrak{R}\colon F(R)\rightarrow R$, where $F(R)$ is a set of fuzzy numbers defined on the set of real numbers, which maps each fuzzy number into real line where a natural order exists.

Let $\tilde{A}=(a^L,a^M,a^R)$ and $\tilde{B}=(b^L,b^M,b^R)$ be two triangular fuzzy numbers. Then,

(i) $\tilde{A}\succcurlyeq\tilde{B}$ iff $\mathfrak{R}(\tilde{A})\geq\mathfrak{R}(\tilde{B})$
(ii) $\tilde{A}\approx\tilde{B}$ iff $\mathfrak{R}(\tilde{A})=\mathfrak{R}(\tilde{B})$

where, $\mathfrak{R}(\tilde{A})=\dfrac{a^L+2a^M+a^R}{4}$ and $\mathfrak{R}(\tilde{B})=\dfrac{b^L+2b^M+b^R}{4}$.

2.2 Arithmetic Operations on Triangular Fuzzy Numbers

Let $\tilde{A}=(a^L,a^M,a^R)$ and $\tilde{B}=(b^L,b^M,b^R)$ be two arbitrary triangular fuzzy numbers. Then,

(i) $\tilde{A}+\tilde{B}=(a^L+b^L,a^M+b^M,a^R+b^R)$
(ii) $\tilde{A}-\tilde{B}=(a^L-b^R,a^M-b^M,a^R-b^L)$
(iii) $\tilde{A}\tilde{B}=(a^Lb^L,a^Mb^M,a^Rb^R)$, where $\tilde{A}$ and $\tilde{B}$ are non-negative triangular fuzzy numbers.
(iv) $\dfrac{\tilde{A}}{\tilde{B}}=\left(\dfrac{a^L}{b^R},\dfrac{a^M}{b^M},\dfrac{a^R}{b^L}\right)$, where $\tilde{A}$ is a non-negative triangular fuzzy number and $\tilde{B}$ is a positive triangular fuzzy number.

3 An Overview of the Existing Fuzzy DEA Approach

Wang and Chin [25] proposed the optimistic fuzzy CCR DEA model (1) and pessimistic fuzzy CCR DEA model (2) to evaluate the best relative fuzzy efficiency $\left(\tilde{E}_p^B\right)$ and worst relative fuzzy efficiency $\left(\tilde{E}_p^W\right)$ respectively of pth DMU by considering input data, output data and weights as trapezoidal fuzzy numbers.

$$\begin{aligned}&\text{Maximize}\left[\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)\left(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U\right)]}\right]\\&\text{Subject to}\\&\quad\frac{\sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)\left(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U\right)]} \preccurlyeq (1,1,1,1), \forall j\end{aligned} \tag{1}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i=1, \ldots, m, r=1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

$$\begin{aligned}&\text{Minimize}\left[\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)\left(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U\right)]}\right]\\&\text{Subject to}\\&\quad\frac{\sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)\left(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U\right)]} \succcurlyeq (1,1,1,1), \forall j\end{aligned} \tag{2}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i=1, \ldots, m, r=1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Wang and Chin [25] proposed the following method to evaluate best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, geometric crisp efficiency of DMUs.

Step 1: Using the product of trapezoidal fuzzy numbers, defined in Sect. 2.2, the optimistic fuzzy CCR DEA model (1) and the pessimistic fuzzy CCR DEA model (2) can be transformed into optimistic fuzzy CCR DEA model (3) and the pessimistic fuzzy CCR DEA model (4) respectively.

$$\begin{aligned}&\text{Maximize}\left[\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right]\\&\text{Subject to}\\&\quad\frac{\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right)} \preccurlyeq (1,1,1,1), \forall j\end{aligned} \tag{3}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i=1, \ldots, m, r=1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

$$
\begin{array}{l}
\text{Minimize}\left[\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right) \approx \dfrac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right] \\
\text{Subject to} \\
\quad \dfrac{\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right)} \succcurlyeq (1,1,1,1), \forall j
\end{array}
\tag{4}
$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m, r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 2: The optimistic fuzzy CCR DEA model (3) and the pessimistic fuzzy CCR DEA model (4) can be transformed into optimistic fuzzy CCR DEA model (5) and the pessimistic fuzzy CCR DEA model (6) respectively.

$$
\begin{array}{l}
\text{Maximize}\left[\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) \approx \dfrac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L,\ \sum_{r=1}^{s} u_r^M y_{rp}^M,\ \sum_{r=1}^{s} u_r^N y_{rp}^N,\ \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L,\ \sum_{i=1}^{m} v_i^M x_{ip}^M,\ \sum_{i=1}^{m} v_i^N x_{ip}^N,\ \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right] \\
\text{Subject to} \\
\quad \dfrac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L,\ \sum_{r=1}^{s} u_r^M y_{rj}^M,\ \sum_{r=1}^{s} u_r^N y_{rj}^N,\ \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L,\ \sum_{i=1}^{m} v_i^M x_{ij}^M,\ \sum_{i=1}^{m} v_i^N x_{ij}^N,\ \sum_{i=1}^{m} v_i^U x_{ij}^U\right)} \preccurlyeq (1,1,1,1), \forall j
\end{array}
\tag{5}
$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m, r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

$$
\begin{array}{l}
\text{Minimize}\left[\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right) \approx \dfrac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L,\ \sum_{r=1}^{s} u_r^M y_{rp}^M,\ \sum_{r=1}^{s} u_r^N y_{rp}^N,\ \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L,\ \sum_{i=1}^{m} v_i^M x_{ip}^M,\ \sum_{i=1}^{m} v_i^N x_{ip}^N,\ \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right] \\
\text{Subject to} \\
\quad \dfrac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L,\ \sum_{r=1}^{s} u_r^M y_{rj}^M,\ \sum_{r=1}^{s} u_r^N y_{rj}^N,\ \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L,\ \sum_{i=1}^{m} v_i^M x_{ij}^M,\ \sum_{i=1}^{m} v_i^N x_{ij}^N,\ \sum_{i=1}^{m} v_i^U x_{ij}^U\right)} \succcurlyeq (1,1,1,1), \forall j
\end{array}
\tag{6}
$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m, r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 3: The optimistic fuzzy CCR DEA model (5) and the pessimistic fuzzy CCR DEA model (6) can be transformed into optimistic crisp CCR DEA model (7) and the pessimistic crisp CCR DEA model (8) respectively.

$$
\begin{array}{l}
\text{Minimize}\left[\Re\left(\tilde{E}_p^B\right) = \Re\left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) = \Re\left[\dfrac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L,\ \sum_{r=1}^{s} u_r^M y_{rp}^M,\ \sum_{r=1}^{s} u_r^N y_{rp}^N,\ \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L,\ \sum_{i=1}^{m} v_i^M x_{ip}^M,\ \sum_{i=1}^{m} v_i^N x_{ip}^N,\ \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]\right] \\
\text{Subject to} \\
\quad \Re\left[\dfrac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L,\ \sum_{r=1}^{s} u_r^M y_{rj}^M,\ \sum_{r=1}^{s} u_r^N y_{rj}^N,\ \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L,\ \sum_{i=1}^{m} v_i^M x_{ij}^M,\ \sum_{i=1}^{m} v_i^N x_{ij}^N,\ \sum_{i=1}^{m} v_i^U x_{ij}^U\right)}\right] \le \Re(1,1,1,1), \forall j \\
\quad 0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, i = 1, \ldots, m, r = 1, \ldots, s.
\end{array}
\tag{7}
$$

$$\begin{aligned}&\text{Minimize}\left[\Re\left(\tilde{E}_p^W\right)=\Re\left(E_{p1}^W,E_{p2}^W,E_{p3}^W,E_{p4}^W\right)=\Re\left[\frac{\left(\sum_{r=1}^s u_r^L y_{rp}^L,\ \sum_{r=1}^s u_r^M y_{rp}^M,\ \sum_{r=1}^s u_r^N y_{rp}^N,\ \sum_{r=1}^s u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^m v_i^L x_{ip}^L,\ \sum_{i=1}^m v_i^M x_{ip}^M,\ \sum_{i=1}^m v_i^N x_{ip}^N,\ \sum_{i=1}^m v_i^U x_{ip}^U\right)}\right]\right]\\&\text{Subject to}\\&\quad\Re\left[\frac{\left(\sum_{r=1}^s u_r^L y_{rj}^L,\ \sum_{r=1}^s u_r^M y_{rj}^M,\ \sum_{r=1}^s u_r^N y_{rj}^N,\ \sum_{r=1}^s u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^m v_i^L x_{ij}^L,\ \sum_{i=1}^m v_i^M x_{ij}^M,\ \sum_{i=1}^m v_i^N x_{ij}^N,\ \sum_{i=1}^m v_i^U x_{ij}^U\right)}\right]\geq\Re(1,1,1,1),\forall j\\&\quad 0\leq u_r^L\leq u_r^M\leq u_r^N\leq u_r^U;0\leq v_i^L\leq v_i^M\leq v_i^N\leq v_i^U,i=1,\ldots,m,r=1,\ldots,s.\end{aligned}\tag{8}$$

Step 4: The optimistic crisp CCR DEA model (7) and the pessimistic crisp CCR DEA model (8) can be transformed into optimistic crisp CCR DEA model (9) and the pessimistic crisp CCR DEA model (10) respectively.

$$\begin{aligned}&\text{Maximize}\left[\Re\left(\tilde{E}_p^B\right)=\Re\left(\left(E_{p1}^B,E_{p2}^B,E_{p3}^B,E_{p4}^B\right)\right)=\left[\frac{\Re\left(\sum_{r=1}^s u_r^L y_{rp}^L,\ \sum_{r=1}^s u_r^M y_{rp}^M,\ \sum_{r=1}^s u_r^N y_{rp}^N,\ \sum_{r=1}^s u_r^U y_{rp}^U\right)}{\Re\left(\sum_{i=1}^m v_i^L x_{ip}^L,\ \sum_{i=1}^m v_i^M x_{ip}^M,\ \sum_{i=1}^m v_i^N x_{ip}^N,\ \sum_{i=1}^m v_i^U x_{ip}^U\right)}\right]\right]\\&\text{Subject to}\\&\quad\frac{\Re\left(\sum_{r=1}^s u_r^L y_{rj}^L,\ \sum_{r=1}^s u_r^M y_{rj}^M,\ \sum_{r=1}^s u_r^N y_{rj}^N,\ \sum_{r=1}^s u_r^U y_{rj}^U\right)}{\Re\left(\sum_{i=1}^m v_i^L x_{ij}^L,\ \sum_{i=1}^m v_i^M x_{ij}^M,\ \sum_{i=1}^m v_i^N x_{ij}^N,\ \sum_{i=1}^m v_i^U x_{ij}^U\right)}\leq\Re(1,1,1,1),\forall j\\&\quad 0\leq u_r^L\leq u_r^M\leq u_r^N\leq u_r^U;0\leq v_i^L\leq v_i^M\leq v_i^N\leq v_i^U,i=1,\ldots,m,r=1,\ldots,s.\end{aligned}\tag{9}$$

$$\begin{aligned}&\text{Minimize}\left[\Re\left(\tilde{E}_p^W\right)=\Re\left(E_{p1}^W,E_{p2}^W,E_{p3}^W,E_{p4}^W\right)=\frac{\Re\left(\sum_{r=1}^s u_r^L y_{rp}^L,\ \sum_{r=1}^s u_r^M y_{rp}^M,\ \sum_{r=1}^s u_r^N y_{rp}^N,\ \sum_{r=1}^s u_r^U y_{rp}^U\right)}{\Re\left(\sum_{i=1}^m v_i^L x_{ip}^L,\ \sum_{i=1}^m v_i^M x_{ip}^M,\ \sum_{i=1}^m v_i^N x_{ip}^N,\ \sum_{i=1}^m v_i^U x_{ip}^U\right)}\right]\\&\text{Subject to}\\&\quad\frac{\Re\left(\sum_{r=1}^s u_r^L y_{rj}^L,\ \sum_{r=1}^s u_r^M y_{rj}^M,\ \sum_{r=1}^s u_r^N y_{rj}^N,\ \sum_{r=1}^s u_r^U y_{rj}^U\right)}{\Re\left(\sum_{i=1}^m v_i^L x_{ij}^L,\ \sum_{i=1}^m v_i^M x_{ij}^M,\ \sum_{i=1}^m v_i^N x_{ij}^N,\ \sum_{i=1}^m v_i^U x_{ij}^U\right)}\geq\Re(1,1,1,1),\forall j\\&\quad 0\leq u_r^L\leq u_r^M\leq u_r^N\leq u_r^U;0\leq v_i^L\leq v_i^M\leq v_i^N\leq v_i^U,i=1,\ldots,m,r=1,\ldots,s.\end{aligned}\tag{10}$$

Step 5: The optimistic crisp CCR DEA model (9) and pessimistic crisp CCR DEA model (10) can be transformed into optimistic crisp CCR DEA model (11) and pessimistic crisp CCR DEA model (12) respectively.

$$\begin{aligned}&\text{Maximize}\left[\Re\left(\tilde{E}_p^B\right)=\frac{\left(\sum_{r=1}^s u_r^L y_{rp}^L+\sum_{r=1}^s u_r^M y_{rp}^M+\sum_{r=1}^s u_r^N y_{rp}^N+\sum_{r=1}^s u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^m v_i^L x_{ip}^L+\sum_{i=1}^m v_i^M x_{ip}^M+\sum_{i=1}^m v_i^N x_{ip}^N+\sum_{i=1}^m v_i^U x_{ip}^U\right)}\right]\\&\text{Subject to}\\&\quad\frac{\left(\sum_{r=1}^s u_r^L y_{rj}^L+\sum_{r=1}^s u_r^M y_{rj}^M+\sum_{r=1}^s u_r^N y_{rj}^N+\sum_{r=1}^s u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^m v_i^L x_{ij}^L+\sum_{i=1}^m v_i^M x_{ij}^M+\sum_{i=1}^m v_i^N x_{ij}^N+\sum_{i=1}^m v_i^U x_{ij}^U\right)}\leq 1,\forall j\\&\quad 0\leq u_r^L\leq u_r^M\leq u_r^N\leq u_r^U;0\leq v_i^L\leq v_i^M\leq v_i^N\leq v_i^U,i=1,\ldots,m,r=1,\ldots,s.\end{aligned}\tag{11}$$

$$\begin{aligned}&\text{Minimize}\left[\Re\left(\tilde{E}_p^W\right)=\frac{\left(\sum_{r=1}^s u_r^L y_{rp}^L+\sum_{r=1}^s u_r^M y_{rp}^M+\sum_{r=1}^s u_r^N y_{rp}^N+\sum_{r=1}^s u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^m v_i^L x_{ip}^L+\sum_{i=1}^m v_i^M x_{ip}^M+\sum_{i=1}^m v_i^N x_{ip}^N+\sum_{i=1}^m v_i^U x_{ip}^U\right)}\right]\\&\text{Subject to}\\&\quad\frac{\left(\sum_{r=1}^s u_r^L y_{rj}^L+\sum_{r=1}^s u_r^M y_{rj}^M+\sum_{r=1}^s u_r^N y_{rj}^N+\sum_{r=1}^s u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^m v_i^L x_{ij}^L+\sum_{i=1}^m v_i^M x_{ij}^M+\sum_{i=1}^m v_i^N x_{ij}^N+\sum_{i=1}^m v_i^U x_{ij}^U\right)}\geq 1,\forall j\\&\quad 0\leq u_r^L\leq u_r^M\leq u_r^N\leq u_r^U;0\leq v_i^L\leq v_i^M\leq v_i^N\leq v_i^U,i=1,\ldots,m,r=1,\ldots,s.\end{aligned}\tag{12}$$

Step 6: The optimistic crisp CCR DEA model (11) and pessimistic crisp CCR DEA model (12) can be transformed into optimistic crisp CCR DEA model (13) and pessimistic crisp CCR DEA model (14) respectively.

$$\begin{aligned}
&\text{Maximize}\left[\Re\left(\tilde{E}_p^B\right) = \textstyle\sum_{r=1}^{s} u_r^L y_{rp}^L + \sum_{r=1}^{s} u_r^M y_{rp}^M + \sum_{r=1}^{s} u_r^N y_{rp}^N + \sum_{r=1}^{s} u_r^U y_{rp}^U\right]\\
&\text{Subject to}\\
&\quad \textstyle\sum_{i=1}^{m} v_i^L x_{ip}^L + \sum_{i=1}^{m} v_i^M x_{ip}^M + \sum_{i=1}^{m} v_i^N x_{ip}^N + \sum_{i=1}^{m} v_i^U x_{ip}^U = 1\\
&\quad \left(\textstyle\sum_{r=1}^{s} u_r^L y_{rj}^L + \sum_{r=1}^{s} u_r^M y_{rj}^M + \sum_{r=1}^{s} u_r^N y_{rj}^N + \sum_{r=1}^{s} u_r^U y_{rj}^U\right)\\
&\quad -\left(\textstyle\sum_{i=1}^{m} v_i^L x_{ij}^L + \sum_{i=1}^{m} v_i^M x_{ij}^M + \sum_{i=1}^{m} v_i^N x_{ij}^N + \sum_{i=1}^{m} v_i^U x_{ij}^U\right) \le 0, \forall j\\
&\quad 0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, i = 1, \ldots, m, r = 1, \ldots, s.
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\text{Maximize}\left[\Re\left(\tilde{E}_p^B\right) = \textstyle\sum_{r=1}^{s} u_r^L y_{rp}^L + \sum_{r=1}^{s} u_r^M y_{rp}^M + \sum_{r=1}^{s} u_r^N y_{rp}^N + \sum_{r=1}^{s} u_r^U y_{rp}^U\right]\\
&\text{Subject to}\\
&\quad \textstyle\sum_{i=1}^{m} v_i^L x_{ip}^L + \sum_{i=1}^{m} v_i^M x_{ip}^M + \sum_{i=1}^{m} v_i^N x_{ip}^N + \sum_{i=1}^{m} v_i^U x_{ip}^U = 1\\
&\quad \left(\textstyle\sum_{r=1}^{s} u_r^L y_{rj}^L + \sum_{r=1}^{s} u_r^M y_{rj}^M + \sum_{r=1}^{s} u_r^N y_{rj}^N + \sum_{r=1}^{s} u_r^U y_{rj}^U\right)\\
&\quad -\left(\textstyle\sum_{i=1}^{m} v_i^L x_{ij}^L + \sum_{i=1}^{m} v_i^M x_{ij}^M + \sum_{i=1}^{m} v_i^N x_{ij}^N + \sum_{i=1}^{m} v_i^U x_{ij}^U\right) \ge 0, \forall j\\
&\quad 0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U i = 1, \ldots, m, r = 1, \ldots, s.
\end{aligned} \tag{14}$$

Step 7: Find the optimal value $\Re\left(\tilde{E}_p^B\right) = E_p^B$, representing the best relative crisp efficiency of pth DMU, by solving optimistic crisp CCR DEA model (13).

Step 8: Find the optimal value $\Re\left(\tilde{E}_p^W\right) = E_p^W$, representing the worst relative crisp efficiency of pth DMU, by solving pessimistic crisp CCR DEA model (14).

Step 9: Find the relative geometric average crisp efficiency $E_P^{GEOMETRIC}$ of pth DMU by putting the values E_p^B and E_p^W, obtained in Step 4 and Step 5, in $E_P^{GEOMETRIC} = \sqrt{E_p^B \times E_p^W}$.

4 Flaws in the Existing Method

If (a^L, a^M, a^N, a^U) & (b^L, b^M, b^N, b^U). are two trapezoidal fuzzy numbers then

$$\Re\left(\frac{(a^L, a^M, a^N, a^U)}{(b^L, b^M, b^N, b^U)}\right) = \Re\left(\frac{a^L}{b^U}, \frac{a^M}{b^N}, \frac{a^N}{b^M}, \frac{a^U}{b^L}\right) = \frac{1}{4}\left(\frac{a^L}{b^U} + \frac{a^M}{b^N} + \frac{a^N}{b^M} + \frac{a^U}{b^L}\right) \text{and}$$

$$\frac{\Re(a^L, a^M, a^N, a^U)}{\Re(b^L, b^M, b^N, b^U)} = \frac{a^L + a^M + a^N + a^U}{b^L + b^M + b^N + b^U}$$

It is obvious that

$$\Re\left(\frac{(a^L, a^M, a^N, a^U)}{(b^L, b^M, b^N, b^U)}\right) \neq \frac{\Re(a^L, a^M, a^N, a^U)}{\Re(b^L, b^M, b^N, b^U)}$$

However, Wang and Chin [25] have used the property

$$\Re\left(\frac{(a^L, a^M, a^N, a^U)}{(b^L, b^M, b^N, b^U)}\right) = \frac{\Re(a^L, a^M, a^N, a^U)}{\Re(b^L, b^M, b^N, b^U)}$$

in Step 4 of their proposed method. Therefore, the method, proposed by Wang and Chin [25], is not valid.

5 Proposed Fuzzy CCR DEA Approach

In this section, to resolve the flaws of the existing optimistic fuzzy CCR DEA model (1) and pessimistic fuzzy CCR DEA model (2), proposed by Wang and Chin [25], new optimistic fuzzy CCR DEA model (15) and pessimistic fuzzy CCR DEA model (16) are proposed.

$$\begin{aligned} &\text{Maximize}\left[\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)\left(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U\right)]}\right] \\ &\text{Subject to} \\ &\quad \sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)\left(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U\right) \preccurlyeq \sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U\right), \forall j \end{aligned} \tag{15}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m, r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

$$\begin{aligned} &\text{Minimize}\left[\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U)]}\right] \\ &\text{Subject to} \\ &\quad \sum_{r=1}^{s}\left(u_r^L, u_r^M, u_r^N, u_r^U\right)\left(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U\right) \succcurlyeq \sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U\right), \forall j \end{aligned} \tag{16}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m, r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

To evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and relative geometric crisp efficiency of DMUs considering the optimistic fuzzy CCR DEA model (15) and pessimistic fuzzy CCR DEA model (16) can be obtained by using the following steps:

Step 1: Using the product of trapezoidal fuzzy numbers, defined in Sect. 2.2, the optimistic fuzzy CCR DEA model (15) and pessimistic fuzzy CCR DEA model

(16) can be transformed into optimistic fuzzy CCR DEA model (17) and pessimistic fuzzy CCR DEA model (18) respectively.

$$\begin{aligned}&\text{Maximize}\left[\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right]\\&\text{Subject to}\\&\quad\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right) \preccurlyeq \sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right), \forall j\end{aligned} \tag{17}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i=1, \ldots, m, r=1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

$$\begin{aligned}&\text{Minimize}\left[\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right]\\&\text{Subject to}\\&\quad\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right) \succcurlyeq \sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right), \forall j\end{aligned} \tag{18}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i=1, \ldots, m, r=1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 2: Using the division of trapezoidal fuzzy numbers, defined in Sect. 22, the optimistic fuzzy CCR DEA model (17) and pessimistic fuzzy CCR DEA model (18) can be transformed into optimistic fuzzy CCR DEA model (19) and pessimistic fuzzy CCR DEA model (20) respectively.

$$\begin{aligned}&\text{Maximize}\left[\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) \approx \left(\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N}, \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L}\right)\right]\\&\text{Subject to}\\&\quad\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)\\&\quad\preccurlyeq \left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right), \forall j\end{aligned} \tag{19}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i=1, \ldots, m, r=1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

$$\begin{aligned}&\text{Minimize}\left[\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right) \approx \left(\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N}, \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L}\right)\right]\\&\text{Subject to}\\&\quad\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)\\&\quad\succcurlyeq \left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right), \forall j\end{aligned} \tag{20}$$

$\left(u_r^L, u_r^M, u_r^N, u_r^U\right)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i=1, \ldots, m, r=1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 3: Using the relation $(a^L, a^M, a^N, a^U) \le (b^L, b^M, b^N, b^U)$, $a^L \le b^L, a^M \le b^M$, $a^N \le b^N, a^U \le b^U$, the optimistic fuzzy CCR DEA model (19) and pessimistic fuzzy CCR DEA model (20) can be transformed into optimistic fuzzy CCR DEA model (21) and pessimistic fuzzy CCR DEA model (22) respectively.

$$\begin{aligned}
&\text{Maximize}\left[\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right) \approx \left(\frac{\sum_{r=1}^s u_r^L y_{rp}^L}{\sum_{i=1}^m v_i^U x_{ip}^U}, \frac{\sum_{r=1}^s u_r^M y_{rp}^M}{\sum_{i=1}^m v_i^N x_{ip}^N}, \frac{\sum_{r=1}^s u_r^N y_{rp}^N}{\sum_{i=1}^m v_i^M x_{ip}^M}, \frac{\sum_{r=1}^s u_r^U y_{rp}^U}{\sum_{i=1}^m v_i^L x_{ip}^L}\right)\right]\\
&\text{Subject to}\\
&\quad \sum_{r=1}^s u_r^L y_{rj}^L \le \sum_{i=1}^m v_i^L x_{ij}^L, \ \forall j\\
&\quad \sum_{r=1}^s u_r^M y_{rj}^M \le \sum_{i=1}^m v_i^M x_{ij}^M, \ \forall j\\
&\quad \sum_{r=1}^s u_r^N y_{rp}^N \le \sum_{i=1}^m v_i^N x_{ip}^N, \ \forall j\\
&\quad \sum_{r=1}^s u_r^U y_{rp}^U \le \sum_{i=1}^m v_i^U x_{ip}^U, \ \forall j\\
&\quad 0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, \ i = 1, \ldots, m, \ r = 1, \ldots, s.
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\text{Minimize}\left[\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right) \approx \left(\frac{\sum_{r=1}^s u_r^L y_{rp}^L}{\sum_{i=1}^m v_i^U x_{ip}^U}, \frac{\sum_{r=1}^s u_r^M y_{rp}^M}{\sum_{i=1}^m v_i^N x_{ip}^N}, \frac{\sum_{r=1}^s u_r^N y_{rp}^N}{\sum_{i=1}^m v_i^M x_{ip}^M}, \frac{\sum_{r=1}^s u_r^U y_{rp}^U}{\sum_{i=1}^m v_i^L x_{ip}^L}\right)\right]\\
&\text{Subject to}\\
&\quad \sum_{r=1}^s u_r^L y_{rj}^L \ge \sum_{i=1}^m v_i^L x_{ij}^L, \ \forall j\\
&\quad \sum_{r=1}^s u_r^M y_{rj}^M \ge \sum_{i=1}^m v_i^M x_{ij}^M, \ \forall j\\
&\quad \sum_{r=1}^s u_r^N y_{rp}^N \ge \sum_{i=1}^m v_i^N x_{ip}^N, \ \forall j\\
&\quad \sum_{r=1}^s u_r^U y_{rp}^U \ge \sum_{i=1}^m v_i^U x_{ip}^U, \ \forall j\\
&\quad 0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, i = 1, \ldots, m, r = 1, \ldots, s.
\end{aligned} \tag{22}$$

Step 4: The fuzzy optimal value $\tilde{E}_p^B \approx \left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right)$, representing the best relative fuzzy efficiency of pth DMU, as well as the fuzzy optimal value $\tilde{E}_p^W \approx \left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right)$, representing the worst relative fuzzy efficiency of pth DMU, can be obtained by solving the optimistic fuzzy CCR DEA model (21) and pessimistic fuzzy CCR DEA model (22) as follows:

Step 4(a): Find the optimal value $\left(E_{p1}^B\right)$ and $\left(E_{p1}^W\right)$ of the optimistic crisp CCR DEA model (23a) and pessimistic CCR DEA model (24a) by solving optimistic crisp CCR DEA model (23b) and pessimistic CCR DEA model (24b) equivalent to optimistic crisp CCR DEA model (23a) and pessimistic CCR DEA model (24a) respectively.

$$\begin{aligned}
&\text{Maximize}\left[E_{p1}^B = \frac{\sum_{r=1}^s u_r^L y_{rp}^L}{\sum_{i=1}^m v_i^U x_{ip}^U}\right]\\
&\text{Subject to}\\
&\quad \frac{\sum_{r=1}^s u_r^L y_{rp}^L}{\sum_{i=1}^m v_i^U x_{ip}^U} \le 1,\\
&\quad \text{\& all the constraints of model 21.}
\end{aligned} \tag{23a}$$

$$\begin{aligned}&\text{Maximize}\left[E_{p1}^{B}=\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}\right]\\&\text{Subject to}\\&\qquad\sum_{i=1}^{m}v_i^{U}x_{ip}^{U}=1,\\&\qquad\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}\le\sum_{i=1}^{m}v_i^{U}x_{ip}^{U},\\&\qquad\&\text{ all the constraints of model 21.}\end{aligned}\tag{23b}$$

$$\begin{aligned}&\text{Minimize}\left[E_{p1}^{W}=\frac{\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}}{\sum_{i=1}^{m}v_i^{U}x_{ip}^{U}}\right]\\&\text{Subject to}\\&\qquad\frac{\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}}{\sum_{i=1}^{m}v_i^{U}x_{ip}^{U}}\ge1,\\&\qquad\&\text{ all the constraints of model 22.}\end{aligned}\tag{24a}$$

$$\begin{aligned}&\text{Minimize}\left[E_{p1}^{W}=\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}\right]\\&\text{Subject to}\\&\qquad\sum_{i=1}^{m}v_i^{U}x_{ip}^{U}=1,\\&\qquad\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}\ge\sum_{i=1}^{m}v_i^{U}x_{ip}^{U},\\&\qquad\&\text{ all the constraints of model 22.}\end{aligned}\tag{24b}$$

Step 4(b): Find the optimal value $\left(E_{p2}^{B}\right)$ and $\left(E_{p2}^{W}\right)$ of the optimistic crisp CCR DEA model (25a) and pessimistic CCR DEA model (26a) by solving optimistic crisp CCR DEA model (25b) and pessimistic CCR DEA model (26b) equivalent to optimistic crisp CCR DEA model (25a) and pessimistic CCR DEA model (26a) respectively.

$$\begin{aligned}&\text{Maximize}\left[E_{p2}^{B}=\frac{\sum_{r=1}^{s}u_r^{M}y_{rp}^{M}}{\sum_{i=1}^{m}v_i^{N}x_{ip}^{N}}\right]\\&\text{Subject to}\\&\qquad\frac{\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}}{\sum_{i=1}^{m}v_i^{U}x_{ip}^{U}}=E_{p1}^{B},\\&\qquad E_{p1}^{B}\le\frac{\sum_{r=1}^{s}u_r^{M}y_{rp}^{M}}{\sum_{i=1}^{m}v_i^{N}x_{ip}^{N}}\le1,\\&\qquad\&\text{ all the constraints of model (21).}\end{aligned}\tag{25a}$$

$$\begin{aligned}&\text{Maximize}\left[E_{p2}^{B}=\sum_{r=1}^{s}u_r^{M}y_{rp}^{M}\right]\\&\text{Subject to}\\&\qquad\sum_{i=1}^{m}v_i^{N}x_{ip}^{N}=1,\\&\qquad\sum_{r=1}^{s}u_r^{L}y_{rp}^{L}=E_{p1}^{B}\left(\sum_{i=1}^{m}v_i^{U}x_{ip}^{U}\right),\\&\qquad E_{p1}^{B}\left(\sum_{i=1}^{m}v_i^{N}x_{ip}^{N}\right)\le\sum_{r=1}^{s}u_r^{M}y_{rp}^{M}\le\sum_{i=1}^{m}v_i^{N}x_{ip}^{N},\\&\qquad\&\text{ all the constraints of model (21).}\end{aligned}\tag{25b}$$

$$\begin{aligned}
&\text{Minimize}\left[E_{p2}^{W}=\frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\sum_{i=1}^{m}v_i^N x_{ip}^N}\right]\\
&\text{Subject to}\\
&\qquad \frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\sum_{i=1}^{m}v_i^U x_{ip}^U}=E_{p1}^{W},\\
&\qquad E_{p1}^{W}\le\frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\sum_{i=1}^{m}v_i^N x_{ip}^N},\\
&\qquad \&\ \text{all the constraints of model (22).}
\end{aligned}\tag{26a}$$

$$\begin{aligned}
&\text{Maximize}\left[E_{p2}^{W}=\sum_{r=1}^{s}u_r^M y_{rp}^M\right]\\
&\text{Subject to}\\
&\qquad \sum_{i=1}^{m}v_i^N x_{ip}^N=1,\\
&\qquad \sum_{r=1}^{s}u_r^L y_{rp}^L=E_{p1}^{W}\left(\sum_{i=1}^{m}v_i^U x_{ip}^U\right),\\
&\qquad E_{p1}^{W}\left(\sum_{i=1}^{m}v_i^N x_{ip}^N\right)\le\sum_{r=1}^{s}u_r^M y_{rp}^M,\\
&\qquad \&\ \text{all the constraints of model (22).}
\end{aligned}\tag{26b}$$

Step 4(c): Find the optimal value $\left(E_{p3}^{B}\right)$ and $\left(E_{p3}^{W}\right)$ of the optimistic crisp CCR DEA model (27a) and pessimistic CCR DEA model (28a) by solving optimistic crisp CCR DEA model (27b) and pessimistic CCR DEA model (28b) equivalent to optimistic crisp CCR DEA model (27a) and pessimistic CCR DEA model (28a) respectively.

$$\begin{aligned}
&\text{Maximize}\left[E_{p3}^{B}=\frac{\sum_{r=1}^{s}u_r^N y_{rp}^N}{\sum_{i=1}^{m}v_i^M x_{ip}^M}\right]\\
&\text{Subject to}\\
&\qquad \frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\sum_{i=1}^{m}v_i^U x_{ip}^U}=E_{p1}^{B},\\
&\qquad \frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\sum_{i=1}^{m}v_i^N x_{ip}^N}=E_{p2}^{B},\\
&\qquad E_{p2}^{B}\le\frac{\sum_{r=1}^{s}u_r^N y_{rp}^N}{\sum_{i=1}^{m}v_i^M x_{ip}^M}\le 1,\\
&\qquad \&\ \text{all the constraints of model (21).}
\end{aligned}\tag{27a}$$

$$\begin{aligned}
&\text{Maximize}\left[E_{p3}^{B}=\sum_{r=1}^{s}u_r^N y_{rp}^N\right]\\
&\text{Subject to}\\
&\qquad \sum_{i=1}^{m}v_i^M x_{ip}^M=1,\\
&\qquad \sum_{r=1}^{s}u_r^L y_{rp}^L=E_{p1}^{B}\left(\sum_{i=1}^{m}v_i^U x_{ip}^U\right),\\
&\qquad \sum_{r=1}^{s}u_r^M y_{rp}^M=E_{p2}^{B}\left(\sum_{i=1}^{m}v_i^N x_{ip}^N\right),\\
&\qquad E_{p2}^{B}\left(\sum_{i=1}^{m}v_i^M x_{ip}^M\right)\le\sum_{r=1}^{s}u_r^N y_{rp}^N\le\sum_{i=1}^{m}v_i^M x_{ip}^M,\\
&\qquad \&\ \text{all the constraints of model (21).}
\end{aligned}\tag{27b}$$

$$\begin{aligned}&\text{Minimize}\left[E_{p3}^{W}=\frac{\sum_{r=1}^{s}u_r^N y_{rp}^N}{\sum_{i=1}^{m}v_i^M x_{ip}^M}\right]\\&\text{Subject to}\\&\qquad\frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\sum_{i=1}^{m}v_i^U x_{ip}^U}=E_{p1}^{W},\\&\qquad\frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\sum_{i=1}^{m}v_i^N x_{ip}^N}=E_{p2}^{W},\\&\qquad E_{p2}^{W}\le\frac{\sum_{r=1}^{s}u_r^N y_{rp}^N}{\sum_{i=1}^{m}v_i^M x_{ip}^M},\\&\qquad\&\text{ all the constraints of model (22).}\end{aligned}\tag{28a}$$

$$\begin{aligned}&\text{Minimize}\left[E_{p3}^{W}=\sum_{r=1}^{s}u_r^N y_{rp}^N\right]\\&\text{Subject to}\\&\qquad\sum_{i=1}^{m}v_i^M x_{ip}^M=1,\\&\qquad\sum_{r=1}^{s}u_r^L y_{rp}^L=E_{p1}^{W}\left(\sum_{i=1}^{m}v_i^U x_{ip}^U\right),\\&\qquad\sum_{r=1}^{s}u_r^M y_{rp}^M=E_{p2}^{W}\left(\sum_{i=1}^{m}v_i^N x_{ip}^N\right),\\&\qquad E_{p2}^{W}\left(\sum_{i=1}^{m}v_i^M x_{ip}^M\right)\le\sum_{r=1}^{s}u_r^N y_{rp}^N,\\&\qquad\&\text{ all the constraints of model (22).}\end{aligned}\tag{28b}$$

Step 4(d): Find the optimal value $\left(E_{p4}^{B}\right)$ and $\left(E_{p4}^{W}\right)$ of the optimistic crisp CCR DEA model (29a) and pessimistic CCR DEA model (30a) by solving optimistic crisp CCR DEA model (29b) and pessimistic CCR DEA model (30b) equivalent to optimistic crisp CCR DEA model (29a) and pessimistic CCR DEA model (30a) respectively.

$$\begin{aligned}&\text{Maximize}\left[E_{p4}^{B}=\frac{\sum_{r=1}^{s}u_r^U y_{rp}^U}{\sum_{i=1}^{m}v_i^L x_{ip}^L}\right]\\&\text{Subject to}\\&\qquad\frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\sum_{i=1}^{m}v_i^U x_{ip}^U}=E_{p1}^{B},\\&\qquad\frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\sum_{i=1}^{m}v_i^N x_{ip}^N}=E_{p2}^{B},\\&\qquad\frac{\sum_{r=1}^{s}u_r^N y_{rp}^N}{\sum_{i=1}^{m}v_i^M x_{ip}^M}=E_{p3}^{B},\\&\qquad E_{p3}^{B}\le\frac{\sum_{r=1}^{s}u_r^U y_{rp}^U}{\sum_{i=1}^{m}v_i^L x_{ip}^L}\le 1,\\&\qquad\&\text{ all the constraints of model (21).}\end{aligned}\tag{29a}$$

$$\begin{aligned}
&\text{Maximize}\left[E_{p4}^{B}=\sum_{r=1}^{s}u_r^U y_{rp}^U\right]\\
&\text{Subject to}\\
&\quad \sum_{i=1}^{m}v_i^L x_{ip}^L=1,\\
&\quad \sum_{r=1}^{s}u_r^L y_{rp}^L=E_{p1}^{B}\left(\sum_{i=1}^{m}v_i^U x_{ip}^U\right),\\
&\quad \sum_{r=1}^{s}u_r^M y_{rp}^M=E_{p2}^{B}\left(\sum_{i=1}^{m}v_i^N x_{ip}^N\right),\\
&\quad \sum_{r=1}^{s}u_r^N y_{rp}^N=E_{p3}^{B}\left(\sum_{i=1}^{m}v_i^M x_{ip}^M\right),\\
&\quad E_{p3}^{B}\left(\sum_{i=1}^{m}v_i^L x_{ip}^L\right)\le\sum_{r=1}^{s}u_r^U y_{rp}^U\le\sum_{i=1}^{m}v_i^L x_{ip}^L,\\
&\quad \&\text{ all the constraints of model (21).}
\end{aligned}\tag{29b}$$

$$\begin{aligned}
&\text{Minimize}\left[E_{p4}^{W}=\frac{\sum_{r=1}^{s}u_r^U y_{rp}^U}{\sum_{i=1}^{m}v_i^L x_{ip}^L}\right]\\
&\text{Subject to}\\
&\quad \frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\left(\sum_{i=1}^{m}v_i^U x_{ip}^U\right)}=E_{p1}^{W},\\
&\quad \frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\left(\sum_{i=1}^{m}v_i^N x_{ip}^N\right)}=E_{p2}^{W},\\
&\quad \frac{\sum_{r=1}^{s}u_r^N y_{rp}^N}{\sum_{i=1}^{m}v_i^M x_{ip}^M}=E_{p3}^{W},\\
&\quad E_{p3}^{W}\le\frac{\sum_{r=1}^{s}u_r^U y_{rp}^U}{\sum_{i=1}^{m}v_i^L x_{ip}^L},\\
&\quad \&\text{ all the constraints of model (22).}
\end{aligned}\tag{30a}$$

$$\begin{aligned}
&\text{Minimize}\left[E_{p4}^{W}=\sum_{r=1}^{s}u_r^U y_{rp}^U\right]\\
&\text{Subject to}\\
&\quad \sum_{i=1}^{m}v_i^L x_{ip}^L=1,\\
&\quad \sum_{r=1}^{s}u_r^U y_{rp}^U=E_{p1}^{W}\left(\sum_{i=1}^{m}v_i^U x_{ip}^U\right),\\
&\quad \sum_{r=1}^{s}u_r^M y_{rp}^M=E_{p2}^{W}\left(\sum_{i=1}^{m}v_i^N x_{ip}^N\right),\\
&\quad \sum_{r=1}^{s}u_r^N y_{rp}^N=E_{p3}^{W}\left(\sum_{i=1}^{m}v_i^M x_{ip}^M\right),\\
&\quad E_{p3}^{W}\left(\sum_{i=1}^{m}v_i^L x_{ip}^L\right)\le\sum_{r=1}^{s}u_r^U y_{rp}^U,\\
&\quad \&\text{ all the constraints of model (22).}
\end{aligned}\tag{30b}$$

Step 5: Using the values of $E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B$ and E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W, obtained in Step (4a) to Step (4d), find the fuzzy optimal value $\tilde{E}_p^B=\left(E_{p1}^B, E_{p2}^B, E_{p3}^B, E_{p4}^B\right)$ of optimistic fuzzy DEA model (21), representing the best relative fuzzy efficiency of pth DMU, as well as pessimistic fuzzy optimal value $\tilde{E}_p^W=\left(E_{p1}^W, E_{p2}^W, E_{p3}^W, E_{p4}^W\right)$ of pessimistic fuzzy DEA model (22), representing the worst relative fuzzy efficiency of pth DMU.

Step 5: Find the crisp optimal value $E_p^B = \mathfrak{R}\left(\tilde{E}_p^B\right)$, representing the best relative crisp efficiency of p^{th} DMU.

Step 6: Find the crisp optimal value $E_p^W = \mathfrak{R}\left(\tilde{E}_p^W\right)$, representing the worst relative crisp efficiency of p^{th} DMU.

Step 7: Find the relative geometric crisp efficiency $E_P^{GEOMETRIC}$ of pth DMU by putting the values E_p^B and E_p^W, obtained in Step 5 and Step 6, in $E_P^{GEOMETRIC} = \sqrt{E_p^B \times E_p^W}$.

6 Application to Real Life Problem

Wang and Chin [25] evaluated the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of by considering eight manufacturing enterprises (DMUs) of China with two inputs and two outputs shown in Table 1 and using the optimistic fuzzy CCR DEA model (1) as well as pessimistic fuzzy CCR DEA model (2). The eight manufacturing enterprises, all manufacture the same type of products but with different qualities. Both the gross output value (GOV) and product quality (PQ) are considered as outputs. Manufacturing cost (MC) and the number of employees (NOE) are considered as inputs. The data about the GOV and MC are uncertain due to the unavailability at the time of assessment and are therefore estimated as fuzzy numbers. The product quality is assessed by customers using fuzzy linguistic terms such as Excellent, Very Good, Average, Poor and Very Poor. The assessment results by customers are weighted and averaged.

However, as discussed in Sect. 4 that the optimistic fuzzy CCR DEA model (1) as well as pessimistic fuzzy CCR DEA model (2) are not valid. Therefore, the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of 8 manufacturing enterprises, evaluated by Wang and Chin [25], is not exact. In this section, to illustrate the proposed method the exact best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of Enterprise A is evaluated by using the proposed method.

The best relative fuzzy efficiency and worst relative fuzzy efficiency of DMU_A can be obtained by solving the optimistic fuzzy CCR DEA models (31) and pessimistic CCR DEA models (32).

Table 1 Input and output data for eight manufacturing enterprises [25]

(DMUs)	Inputs (Two)		Outputs (two)	
	MC	NOE	GOV	PQ
A	(2120, 2170, 2210)	1870	(14500, 14790, 14860)	(3.1, 4.1, 4.9)
B	(1420, 1460, 1500)	1340	(12470, 12720, 12790)	(1.2, 2.1, 3.0)
C	(2510, 2570, 2610)	2360	(17900, 18260, 18400)	(3.3, 4.3, 5.0)
D	(2300, 2350, 2400)	2020	(14970, 15270, 15400)	(2.7, 3.7, 4.6)
E	(1480, 1520, 1560)	1550	(13980, 14260, 14330)	(1.0, 1.8, 2.7)
F	(1990, 2030, 2100)	1760	(14030, 14310, 14400)	(1.6, 2.6, 3.6)
G	(2200, 2260, 2300)	1980	(16540, 16870, 17000)	(2.4, 3.4, 4.4)
H	(2400, 2460, 2520)	2250	(17600, 17960, 18100)	(2.6, 3.6, 4.6)

$$
\begin{aligned}
&\text{Maximize}\Big[\tilde{E}_A^B = \left(E_{A1}^B, E_{A2}^B, E_{A3}^B\right) \\
&\quad\approx \left[\frac{(14500, 14790, 14860)\left(u_1^L, u_1^M, u_1^U\right) + (3.1, 4.1, 4.9)\left(u_2^L, u_2^M, u_2^U\right)}{(2120, 2170, 2210)\left(v_1^L, v_1^M, v_1^U\right) + (1870, 1870, 1870)\left(v_2^L, v_2^M, v_2^U\right)}\right]\Big] \\
&\text{Subject to} \\
&(14500, 14790, 14860)\left(u_1^L, u_1^M, u_1^U\right) + (3.1, 4.1, 4.9)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (2120, 2170, 2210)\left(v_1^L, v_1^M, v_1^U\right) + (1870, 1870, 1870)\left(v_2^L, v_2^M, v_2^U\right) \\
&(12470, 12720, 12790)\left(u_1^L, u_1^M, u_1^U\right) + (1.2, 2.1, 3.0)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (1420, 1460, 1500)\left(v_1^L, v_1^M, v_1^U\right) + (1340, 1340, 1340)\left(v_2^L, v_2^M, v_2^U\right) \\
&(17900, 18260, 18400)\left(u_1^L, u_1^M, u_1^U\right) + (3.3, 4.3, 4.0)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (2510, 2570, 2610)\left(v_1^L, v_1^M, v_1^U\right) + (2360, 2360, 2360)\left(v_2^L, v_2^M, v_2^U\right) \\
&(14970, 15270, 15400)\left(u_1^L, u_1^M, u_1^U\right) + (2.7, 3.7, 4.6)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (2300, 2350, 2400)\left(v_1^L, v_1^M, v_1^U\right) + (2020, 2020, 2020)\left(v_2^L, v_2^M, v_2^U\right) \\
&(13980, 14260, 14330)\left(u_1^L, u_1^M, u_1^U\right) + (1.0, 1.8, 2.7)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (1480, 1520, 1560)\left(v_1^L, v_1^M, v_1^U\right) + (1550, 1550, 1550)\left(v_2^L, v_2^M, v_2^U\right) \\
&(14030, 14310, 14400)\left(u_1^L, u_1^M, u_1^U\right) + (1.6, 2.6, 3.6)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (1990, 2030, 2100)\left(v_1^L, v_1^M, v_1^U\right) + (1760, 1760, 1760)\left(v_2^L, v_2^M, v_2^U\right) \\
&(16540, 16870, 17000)\left(u_1^L, u_1^M, u_1^U\right) + (2.4, 3.4, 4.4)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (2200, 2260, 2300)\left(v_1^L, v_1^M, v_1^U\right) + (1980, 1980, 1980)\left(v_2^L, v_2^M, v_2^U\right) \\
&(17600, 17960, 18100)\left(u_1^L, u_1^M, u_1^U\right) + (2.6, 3.6, 4.6)\left(u_2^L, u_2^M, u_2^U\right) \\
&\quad\preccurlyeq (2400, 2460, 2520)\left(v_1^L, v_1^M, v_1^U\right) + (2250, 2250, 2250)\left(v_2^L, v_2^M, v_2^U\right),
\end{aligned}
\tag{31}
$$

$\left(u_r^L, u_r^M, u_r^N\right)$ and $\left(v_i^L, v_i^M, v_i^N\right), i = 1, 2, r = 1, 2,$ are non-negative triangular fuzzy numbers.

$$
\begin{aligned}
&\text{Minimize}\left[\tilde{E}_A^W = \left(E_{A1}^W, E_{A2}^W, E_{A3}^W\right)\right.\\
&\quad \left.\approx \left[\frac{(14500, 14790, 14860)\left(u_1^L, u_1^M, u_1^U\right) + (3.1, 4.1, 4.9)\left(u_2^L, u_2^M, u_2^U\right)}{(2120, 2170, 2210)\left(v_1^L, v_1^M, v_1^U\right) + (1870, 1870, 1870)\left(v_2^L, v_2^M, v_2^U\right)}\right]\right]\\
&\text{Subject to}\\
&(14500, 14790, 14860)\left(u_1^L, u_1^M, u_1^U\right) + (3.1, 4.1, 4.9)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (2120, 2170, 2210)\left(v_1^L, v_1^M, v_1^U\right) + (1870, 1870, 1870)\left(v_2^L, v_2^M, v_2^U\right)\\
&(12470, 12720, 12790)\left(u_1^L, u_1^M, u_1^U\right) + (1.2, 2.1, 3.0)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (1420, 1460, 1500)\left(v_1^L, v_1^M, v_1^U\right) + (1340, 1340, 1340)\left(v_2^L, v_2^M, v_2^U\right)\\
&(17900, 18260, 18400)\left(u_1^L, u_1^M, u_1^U\right) + (3.3, 4.3, 4.0)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (2510, 2570, 2610)\left(v_1^L, v_1^M, v_1^U\right) + (2360, 2360, 2360)\left(v_2^L, v_2^M, v_2^U\right)\\
&(14970, 15270, 15400)\left(u_1^L, u_1^M, u_1^U\right) + (2.7, 3.7, 4.6)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (2300, 2350, 2400)\left(v_1^L, v_1^M, v_1^U\right) + (2020, 2020, 2020)\left(v_2^L, v_2^M, v_2^U\right)\\
&(13980, 14260, 14330)\left(u_1^L, u_1^M, u_1^U\right) + (1.0, 1.8, 2.7)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (1480, 1520, 1560)\left(v_1^L, v_1^M, v_1^U\right) + (1550, 1550, 1550)\left(v_2^L, v_2^M, v_2^U\right)\\
&(14030, 14310, 14400)\left(u_1^L, u_1^M, u_1^U\right) + (1.6, 2.6, 3.6)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (1990, 2030, 2100)\left(v_1^L, v_1^M, v_1^U\right) + (1760, 1760, 1760)\left(v_2^L, v_2^M, v_2^U\right)\\
&(16540, 16870, 17000)\left(u_1^L, u_1^M, u_1^U\right) + (2.4, 3.4, 4.4)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (2200, 2260, 2300)\left(v_1^L, v_1^M, v_1^U\right) + (1980, 1980, 1980)\left(v_2^L, v_2^M, v_2^U\right)\\
&(17600, 17960, 18100)\left(u_1^L, u_1^M, u_1^U\right) + (2.6, 3.6, 4.6)\left(u_2^L, u_2^M, u_2^U\right)\\
&\quad \succcurlyeq (2400, 2460, 2520)\left(v_1^L, v_1^M, v_1^U\right) + (2250, 2250, 2250)\left(v_2^L, v_2^M, v_2^U\right),
\end{aligned} \tag{32}
$$

$\left(u_r^L, u_r^M, u_r^N\right)$ and $\left(v_i^L, v_i^M, v_i^N\right), i = 1, 2, r = 1, 2,$ are non-negative triangular fuzzy numbers.

Using the method, proposed in Sect. 5, the exact best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence relative geometric crisp efficiency of DMU_A can be obtained as follows:

Step 1: Using the product of triangular fuzzy numbers, defined in Sect. 2.2, the optimistic fuzzy CCR DEA model (31) and pessimistic fuzzy CCR DEA model (32) can be transformed into optimistic fuzzy CCR DEA model (33) and pessimistic fuzzy CCR DEA model (34) respectively.

$$
\begin{aligned}
&\text{Maximize}\left[\tilde{E}_A^B=\left(E_{A1}^B,E_{A2}^B,E_{A3}^B\right)\quad\approx\left[\frac{\left(14500u_1^L+3.1u_2^L,14790u_1^M+4.1u_2^M,14860u_1^U+4.9u_2^U\right)}{\left(2120v_1^L+4.9v_2^L,2170v_1^M+1870v_2^M,2210v_1^U+1870v_2^U\right)}\right]\right]\\
&\text{Subject to}\\
&\left(14500u_1^L+3.1u_2^L,14790u_1^M+4.1u_2^M,14860u_1^U+4.9u_2^U\right)\\
&\quad\preccurlyeq\left(2120v_1^L+1870v_2^L,2170v_1^M+1870v_2^M,2210v_1^U+1870v_2^U\right)\\
&\left(12470u_1^L+1.2u_2^L,12720u_1^M+2.1u_2^M,12790u_1^U+3.0u_2^U\right)\\
&\quad\preccurlyeq\left(1420v_1^L+1340v_2^L,1460v_1^M+1340v_2^M,1500v_1^U+1340v_2^U\right)\\
&\left(17900u_1^L+3.3u_2,18260u_1^M+4.3u_2^M,18400u_1^U+4.0u_2^U\right)\\
&\quad\preccurlyeq(2510v_1^L+2360v_2^L,2570v_1^M+2360v_2^M,2610v_1^U+2360v_2^U)\\
&\left(14970u_1^L+2.7u_2^L,15270u_1^M+3.7u_2^M,15400u_1^U+4.6u_2^U\right)\\
&\quad\preccurlyeq(2300v_1^L+2020v_2^L,2350v_1^M+2020v_2^M,2400v_1^U+2020v_2^U)\\
&\left(13980u_1^L+1.0u_2^L,14260u_1^M+1.8u_2^M,14330u_1^U+2.7u_2^U\right)\\
&\quad\preccurlyeq(1480v_1^L+1550v_2^L,1520v_1^M+1550v_2^M,1560v_1^U+1550v_2^U)\\
&\left(14030u_1^L+1.6u_2^L,14310u_1^M+2.6u_2^M,14400u_1^U+3.6u_2^U\right)\\
&\quad\preccurlyeq(1990v_1^L+1760v_2^L,2030v_1^M+1760v_2^M,2100v_1^U+1760v_2^U)\\
&\left(16540u_1^L+2.4u_2^L,16870u_1^M+3.4u_2^M,17000u_1^U+4.4u_2^U\right)\\
&\quad\preccurlyeq(2200v_1^L+1980v_2^L,2260v_1^M+1980v_2^M,2300v_1^U+1980v_2^U)\\
&\left(17600u_1^L+2.6u_2^L,17960u_1^M+3.6u_2^M,18100u_1^U+4.6u_2^U\right)\\
&\quad\preccurlyeq\left(2400v_1^L+2250v_2^L,2460v_1^M+2250v_2^M,2520v_1^U+2250v_2^U\right),\\
&0\le u_r^L\le u_r^M\le u_r^N\le u_r^U;0\le v_i^L\le v_i^M\le v_i^N\le v_i^U,i=1,2,r=1,2.
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&\text{Minimize}\left[\tilde{E}_A^W=\left(E_{A1}^W,E_{A2}^W,E_{A3}^W\right)\quad\approx\left[\frac{\left(14500u_1^L+3.1u_2^L,14790u_1^M+4.1u_2^M,14860u_1^U+4.9u_2^U\right)}{\left(2120v_1^L+4.9v_2^L,2170v_1^M+1870v_2^M,2210v_1^U+1870v_2^U\right)}\right]\right]\\
&\text{Subject to}\\
&\left(14500u_1^L+3.1u_2^L,14790u_1^M+4.1u_2^M,14860u_1^U+4.9u_2^U\right)\\
&\quad\succcurlyeq\left(2120v_1^L+1870v_2^L,2170v_1^M+1870v_2^M,2210v_1^U+1870v_2^U\right)\\
&\left(12470u_1^L+1.2u_2^L,12720u_1^M+2.1u_2^M,12790u_1^U+3.0u_2^U\right)\\
&\quad\succcurlyeq\left(1420v_1^L+1340v_2^L,1460v_1^M+1340v_2^M,1500v_1^U+1340v_2^U\right)\\
&\left(17900u_1^L+3.3u_2,18260u_1^M+4.3u_2^M,18400u_1^U+4.0u_2^U\right)\\
&\quad\succcurlyeq(2510v_1^L+2360v_2^L,2570v_1^M+2360v_2^M,2610v_1^U+2360v_2^U)\\
&\left(14970u_1^L+2.7u_2^L,15270u_1^M+3.7u_2^M,15400u_1^U+4.6u_2^U\right)\\
&\quad\succcurlyeq(2300v_1^L+2020v_2^L,2350v_1^M+2020v_2^M,2400v_1^U+2020v_2^U)\\
&\left(13980u_1^L+1.0u_2^L,14260u_1^M+1.8u_2^M,14330u_1^U+2.7u_2^U\right)\\
&\quad\succcurlyeq(1480v_1^L+1550v_2^L,1520v_1^M+1550v_2^M,1560v_1^U+1550v_2^U)\\
&\left(14030u_1^L+1.6u_2^L,14310u_1^M+2.6u_2^M,14400u_1^U+3.6u_2^U\right)\\
&\quad\succcurlyeq(1990v_1^L+1760v_2^L,2030v_1^M+1760v_2^M,2100v_1^U+1760v_2^U)\\
&\left(16540u_1^L+2.4u_2^L,16870u_1^M+3.4u_2^M,17000u_1^U+4.4u_2^U\right)\\
&\quad\succcurlyeq(2200v_1^L+1980v_2^L,2260v_1^M+1980v_2^M,2300v_1^U+1980v_2^U)\\
&\left(17600u_1^L+2.6u_2^L,17960u_1^M+3.6u_2^M,18100u_1^U+4.6u_2^U\right)\\
&\quad\succcurlyeq\left(2400v_1^L+2250v_2^L,2460v_1^M+2250v_2^M,2520v_1^U+2250v_2^U\right),\\
&0\le u_r^L\le u_r^M\le u_r^N\le u_r^U;0\le v_i^L\le v_i^M\le v_i^N\le v_i^U,i=1,2,r=1,2,
\end{aligned}
\tag{34}
$$

Step 2: The optimistic fuzzy CCR DEA model (33) and pessimistic fuzzy CCR DEA model (34) can be transformed into optimistic fuzzy CCR DEA model (35) and pessimistic fuzzy CCR DEA model (36) respectively.

$$\begin{aligned}
&\text{Maximize}\left[\tilde{E}_A^B = \left(E_{A1}^B, E_{A2}^B, E_{A3}^B\right) \approx \left[\frac{\left(14500u_1^L + 3.1u_2^L, 14790u_1^M + 4.1u_2^M, 14860u_1^U + 4.9u_2^U\right)}{\left(2120v_1^L + 4.9v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U\right)}\right]\right]\\
&\text{Subject to}\\
&14500u_1^L + 3.1u_2^L \le 2120v_1^L + 1870v_2^L, \quad 14790u_1^M + 4.1u_2^M \le 2170v_1^M + 1870v_2^M,\\
&14860u_1^U + 4.9u_2^U \le 2210v_1^U + 1870v_2^U, \quad 12470u_1^L + 1.2u_2^L \le 1420v_1^L + 1340v_2^L,\\
&12720u_1^M + 2.1u_2^M \le 1460v_1^M + 1340v_2^M, \quad 12790u_1^U + 3.0u_2^U \le 1500v_1^U + 1340v_2^U,\\
&17900u_1^L + 3.3u_2^L \le 2510v_1^L + 2360v_2^L, \quad 18260u_1^M + 4.3u_2^M \le 2570v_1^M + 2360v_2^M,\\
&18400u_1^U + 4.0u_2^U \le 2610v_1^U + 2360v_2^U, \quad 14970u_1^L + 2.7u_2^L \le 2300v_1^L + 2020v_2^L,\\
&15270u_1^M + 3.7u_2^M \le 2350v_1^M + 2020v_2^M \quad 15400u_1^U + 4.6u_2^U \le 2400v_1^U + 2020v_2^U,\\
&13980u_1^L + 1.0u_2^L \le 1480v_1^L + 1550v_2^L, \quad 14260u_1^M + 1.8u_2^M \le 1520v_1^M + 1550v_2^M,\\
&14330u_1^U + 2.7u_2^U \le 1560v_1^U + 1550v_2^U \quad 14030u_1^L + 1.6u_2^L \le 1990v_1^L + 1760v_2^L,\\
&14310u_1^M + 2.6u_2^M \le 2030v_1^M + 1760v_2^M, \quad 14400u_1^U + 3.6u_2^U \le 2100v_1^U + 1760v_2^U,\\
&16540u_1^L + 2.4u_2^L \le 2200v_1^L + 1980v_2^L, \quad 16870u_1^M + 3.4u_2^M \le 2260v_1^M + 1980v_2^M,\\
&17000u_1^U + 4.4u_2^U \le 2300v_1^U + 1980v_2^U, \quad 17600u_1^L + 2.6u_2^L \le 2400v_1^L + 2250v_2^L,\\
&17960u_1^M + 3.6u_2^M \le 2460v_1^M + 2250v_2^M, \quad 18100u_1^U + 4.6u_2^U \le 2520v_1^U + 2250v_2^U.\\
&\qquad 0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, i = 1, 2, r = 1, 2.
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\text{Minimize}\left[\tilde{E}_A^W = \left(E_{A1}^W, E_{A2}^W, E_{A3}^W\right) \approx \left[\frac{\left(14500u_1^L + 3.1u_2^L, 14790u_1^M + 4.1u_2^M, 14860u_1^U + 4.9u_2^U\right)}{\left(2120v_1^L + 4.9v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U\right)}\right]\right]\\
&\text{Subject to}\\
&14500u_1^L + 3.1u_2^L \ge 2120v_1^L + 1870v_2^L, \quad 14790u_1^M + 4.1u_2^M \ge 2170v_1^M + 1870v_2^M,\\
&14860u_1^U + 4.9u_2^U \ge 2210v_1^U + 1870v_2^U \quad 12470u_1^L + 1.2u_2^L \ge 1420v_1^L + 1340v_2^L,\\
&12720u_1^M + 2.1u_2^M \ge 1460v_1^M + 1340v_2^M, \quad 12790u_1^U + 3.0u_2^U \ge 1500v_1^U + 1340v_2^U,\\
&17900u_1^L + 3.3u_2^L \ge 2510v_1^L + 2360v_2^L, \quad 18260u_1^M + 4.3u_2^M \ge 2570v_1^M + 2360v_2^M,\\
&18400u_1^U + 4.0u_2^U \ge 2610v_1^U + 2360v_2^U, \quad 14970u_1^L + 2.7u_2^L \ge 2300v_1^L + 2020v_2^L,\\
&15270u_1^M + 3.7u_2^M \ge 2350v_1^M + 2020v_2^M, \quad 15400u_1^U + 4.6u_2^U \ge 2400v_1^U + 2020v_2^U,\\
&13980u_1^L + 1.0u_2^L \ge 1480v_1^L + 1550v_2^L, \quad 14260u_1^M + 1.8u_2^M \ge 1520v_1^M + 1550v_2^M,\\
&14330u_1^U + 2.7u_2^U \ge 1560v_1^U + 1550v_2^U, \quad 14030u_1^L + 1.6u_2^L \ge 1990v_1^L + 1760v_2^L,\\
&14310u_1^M + 2.6u_2^M \ge 2030v_1^M + 1760v_2^M \quad 14400u_1^U + 3.6u_2^U \ge 2100v_1^U + 1760v_2^U,\\
&16540u_1^L + 2.4u_2^L \ge 2200v_1^L + 1980v_2^L, \quad 16870u_1^M + 3.4u_2^M \ge 2260v_1^M + 1980v_2^M,\\
&17000u_1^U + 4.4u_2^U \ge 2300v_1^U + 1980v_2^U, \quad 17600u_1^L + 2.6u_2^L \ge 2400v_1^L + 2250v_2^L,\\
&17960u_1^M + 3.6u_2^M \ge 2460v_1^M + 2250v_2^M, \quad 18100u_1^U + 4.6u_2^U \ge 2520v_1^U + 2250v_2^U.\\
&\qquad 0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, i = 1, 2, r = 1, 2,
\end{aligned} \tag{36}$$

Step 3: The optimistic fuzzy CCR DEA model (35) and pessimistic fuzzy CCR DEA model (36) can be transformed into optimistic fuzzy CCR DEA model (37) and pessimistic fuzzy CCR DEA model (38) respectively.

$$\begin{aligned}&\text{Maximize}\left[\tilde{E}_A^B=\left(E_{A1}^B,E_{A2}^B,E_{A3}^B\right)\approx\left[\left(\frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U},\frac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M},\frac{14860u_1^U+4.9u_2^U}{2120v_1^L+4.9v_2^L}\right)\right]\right]\\&\text{Subject to}\\&\quad\text{All the constraints of model}\,(35).\end{aligned}\tag{37}$$

$$\begin{aligned}&\text{Minimize}\left[\tilde{E}_A^W=\left(E_{A1}^W,E_{A2}^W,E_{A3}^W\right)\approx\left[\left(\frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U},\frac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M},\frac{14860u_1^U+4.9u_2^U}{2120v_1^L+4.9v_2^L}\right)\right]\right]\\&\text{Subject to}\\&\quad\text{All the constraints of model}\,(36).\end{aligned}\tag{38}$$

Step 4: The fuzzy optimal value $\tilde{E}_A^B \approx \left(E_{A1}^B, E_{A2}^B, E_{A3}^B\right)$, representing the best relative fuzzy efficiency of DMU$_\text{A}$, as well as the fuzzy optimal value $\tilde{E}_A^W \approx \left(E_{A1}^W, E_{A2}^W, E_{A3}^W\right)$, representing the worst relative fuzzy efficiency of DMU$_\text{A}$, can be obtained by solving the optimistic fuzzy CCR DEA model (37) and pessimistic fuzzy CCR DEA model (38) as follows:

Step 4(a): The optimal value $\left(E_{A1}^B\right)$ and $\left(E_{A1}^W\right)$ of the optimistic crisp CCR DEA model (39a) and pessimistic CCR DEA model (40a) by solving optimistic crisp CCR DEA model (39b) and pessimistic CCR DEA model (39b) equivalent to optimistic crisp CCR DEA model (39a) and pessimistic CCR DEA model (40a) are 0.812 and 1 respectively.

$$\begin{aligned}&\text{Maximize}\left[E_{A1}^B=\frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}\right]\\&\text{Subject to}\\&\quad\frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}\le 1,\\&\quad\&\ \text{all the constraints of model}\,(35).\end{aligned}\tag{39a}$$

$$\begin{aligned}&\text{Maximize}\left[E_{A1}^B=14500u_1^L+3.1u_2^L\right]\\&\text{Subject to}\\&\quad 2210v_1^U+1870v_2^U=1\\&\quad 14500u_1^L+3.1u_2^L\le 2210v_1^U+1870v_2^U,\\&\quad\&\ \text{all the constraints of model}\,(35).\end{aligned}\tag{39b}$$

$$\begin{aligned}&\text{Minimize}\left[E_{A1}^W=\frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}\right]\\&\text{Subject to}\\&\quad\frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}\ge 1,\\&\quad\&\ \text{all the constraints of model}\,(36).\end{aligned}\tag{40a}$$

$$\begin{aligned}&\text{Minimize}\left[E_{A1}^W=14500u_1^L+3.1u_2^L\right]\\&\text{Subject to}\\&\quad 2210v_1^U+1870v_2^U=1\\&\quad 14500u_1^L+3.1u_2^L\ge 2210v_1^U+1870v_2^U,\\&\quad\&\ \text{all the constraints of model}\,(36).\end{aligned}\tag{40b}$$

Step 4(b): The optimal value $\left(E_{A2}^{B}\right)$ and $\left(E_{A2}^{W}\right)$ of the optimistic crisp CCR DEA model (41a) and pessimistic CCR DEA model (42a) by solving optimistic crisp CCR DEA model (41b) and pessimistic CCR DEA model (42b) equivalent to optimistic crisp CCR DEA model (41a) and pessimistic CCR DEA model (42b) are 0.833 and 1.046 respectively.

$$\begin{aligned}
&\text{Maximize}\left[E_{A2}^{B}=\frac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M}\right]\\
&\text{Subject to}\\
&\qquad \frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}=0.812\\
&\qquad 0.812\leq\frac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M}\leq 1\\
&\qquad \&\text{ all the constraints of model }(38).
\end{aligned}\tag{41a}$$

$$\begin{aligned}
&\text{Maximize}\left[E_{A2}^{B}=14790u_1^M+4.1u_2^M\right]\\
&\text{Subject to}\\
&\qquad 2170v_1^M+1870v_2^M=1,\\
&\qquad 14500u_1^L+3.1u_2^L=(0.812)\left(2210v_1^U+1870v_2^U\right),\\
&\qquad (0.812)\left(2170v_1^M+1870v_2^M\right)\leq 14790u_1^M+4.1u_2^M\leq 2170v_1^M+1870v_2^M\\
&\qquad \&\text{ all the constraints of model }(35).
\end{aligned}\tag{41b}$$

$$\begin{aligned}
&\text{Minimize}\left[E_{A2}^{W}=\frac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M}\right]\\
&\text{Subject to}\\
&\qquad \frac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}=1,\\
&\qquad \frac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M}\geq 1\\
&\qquad \&\text{ all the constraints of model }(36).
\end{aligned}\tag{42a}$$

$$\begin{aligned}
&\text{Minimize}\left[E_{A2}^{B}=14790u_1^M+4.1u_2^M\right]\\
&\text{Subject to}\\
&\qquad 2170v_1^M+1870v_2^M=1,\\
&\qquad 14500u_1^L+3.1u_2^L=2210v_1^U+1870v_2^U,\\
&\qquad 2170v_1^M+1870v_2^M\leq 14790u_1^M+4.1u_2^M\\
&\qquad \&\text{ all the constraints of model }(36).
\end{aligned}\tag{42b}$$

Step 4(c): The optimal value $\left(E_{A2}^{B}\right)$ and $\left(E_{A2}^{W}\right)$ of the optimistic crisp CCR DEA model (43a) and pessimistic CCR DEA model (44a) by solving optimistic crisp CCR DEA model (43b) and pessimistic CCR DEA model (P29b) equivalent to optimistic crisp CCR DEA model (43a) and pessimistic CCR DEA model (44a) are 0.854 and 1.072 respectively.

$$\begin{array}{l}\text{Maximize}\left[E_{A3}^{B}=\dfrac{14860u_1^U+4.9u_2^U}{2120v_1^L+4.9v_2^L}\right]\\ \text{Subject to}\\ \quad\dfrac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}=0.812,\\ \quad\dfrac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M}=0.833,\\ \quad 0.833\le\dfrac{14860u_1^U+4.9u_2^U}{2120v_1^L+4.9v_2^L}\le 1,\\ \quad\&\text{ all the constraints of model (37).}\end{array}\tag{43a}$$

$$\begin{array}{l}\text{Maximize}\left[E_{A3}^{B}=14860u_1^U+4.9u_2^U\right]\\ \text{Subject to}\\ \quad 2120v_1^L+4.9v_2^L=1,\\ \quad 14500u_1^L+3.1u_2^L=(0.812)\left(2210v_1^U+1870v_2^U\right)\\ \quad 14790u_1^M+4.1u_2^M=(0.833)\left(2170v_1^M+1870v_2^M\right),\\ \quad (0.833)\left(2120v_1^L+4.9v_2^L\right)\le 14860u_1^U+4.9u_2^U\le 2120v_1^L+4.9v_2^L.\\ \quad\&\text{ all the constraints of model (37).}\end{array}\tag{43b}$$

$$\begin{array}{l}\text{Minimize}\left[E_{A3}^{W}=\dfrac{14860u_1^U+4.9u_2^U}{2120v_1^L+4.9v_2^L}\right]\\ \text{Subject to}\\ \quad\dfrac{14500u_1^L+3.1u_2^L}{2210v_1^U+1870v_2^U}=1,\\ \quad\dfrac{14790u_1^M+4.1u_2^M}{2170v_1^M+1870v_2^M}=1.046,\\ \quad\dfrac{14860u_1^U+4.9u_2^U}{2120v_1^L+4.9v_2^L}\ge 1.046,\\ \quad\&\text{ all the constraints of model (36).}\end{array}\tag{44a}$$

$$\begin{array}{l}\text{Minimize}\left[E_{A3}^{W}=14860u_1^U+4.9u_2^U\right]\\ \text{Subject to}\\ \quad 2120v_1^L+4.9v_2^L=1,\\ \quad 14500u_1^L+3.1u_2^L=2210v_1^U+1870v_2^U\\ \quad 14790u_1^M+4.1u_2^M=(1.046)\left(2170v_1^M+1870v_2^M\right),\\ \quad (1.046)\left(2120v_1^L+4.9v_2^L\right)\le 14860u_1^U+4.9u_2^U.\\ \quad\&\text{ all the constraints of model (36).}\end{array}\tag{44b}$$

Step 5: Using the values of $E_{A1}^B, E_{A2}^B, E_{A3}^B$ and $E_{A1}^W, E_{A2}^W, E_{A3}^W$, obtained in Step (4a) to Step (4c), the fuzzy optimal value $\tilde{E}_A^B = \left(E_{A1}^B, E_{A2}^B, E_{A3}^B\right)$ of optimistic fuzzy DEA model-4.16, representing the best relative fuzzy efficiency of DMU_A, is $\tilde{E}_A^B = \left(E_{A1}^B, E_{A2}^B, E_{A3}^B\right) = (0.812, 0.833, 0.854)$, as well as pessimistic fuzzy optimal value $\tilde{E}_A^W = \left(E_{A1}^W, E_{A2}^W, E_{A3}^W\right)$ of pessimistic fuzzy DEA model (31), representing the worst relative fuzzy efficiency of DMU_A, is $\tilde{E}_A^W = \left(E_{A1}^W, E_{A2}^W, E_{A3}^W\right) = (1, 1.046, 1.072)$.

Step 6: The crisp optimal value $E_A^B = \Re\left(\tilde{E}_A^B\right) = \Re(0.812, 0.833, 0.854)$, representing the best relative crisp efficiency of $\mathrm{DMU_A}$, is 0.833.

Step 7: The crisp optimal value $E_A^W = \Re\left(\tilde{E}_A^W\right) = \Re(1, 1.046, 1.072)$, representing the worst relative crisp efficiency of $\mathrm{DMU_A}$, is 1.041.

Step 8: The geometric average crisp efficiency $E_A^{GEOMETRIC}$ of Ath DMU by putting the values E_A^B and E_A^W, obtained in Step 5 and Step 6, in $E_A^{GEOMETRIC} = \sqrt{E_A^B \times E_A^W}$ is 0.931.

7 Results

The exact best relative fuzzy efficiency, exact worst relative fuzzy efficiency and relative geometric crisp efficiency of all the DMUs, obtained by using the proposed method are shown in Table 2.

It is obvious from Table 2 that $\Re\left(\tilde{E}_B\right) > R\left(\tilde{E}_E\right) > R\left(\tilde{E}_G\right) = \Re\left(\tilde{E}_F\right) > R\left(\tilde{E}_H\right) > R\left(\tilde{E}_A\right) > R\left(\tilde{E}_C\right) > R\left(\tilde{E}_D\right)$. Therefore, $\tilde{E}_B \succ \tilde{E}_E \succ \tilde{E}_G \approx \tilde{E}_F \succ \tilde{E}_A \succ \tilde{E}_H \succ \tilde{E}_C \succ \tilde{E}_D$.

Table 2 Results obtained by using the proposed method

DMU$_j$	Best relative fuzzy efficiency	Worst relative fuzzy efficiency	Relative geometric crisp efficiency
A	(0.81238, 0.833217, 0.85407)	(1, 1.04625, 1.07227)	0.931
B	(0.97498, 1, 1)	(1, 1.12689, 1.28793)	1.062
C	(0.79661, 0.815201, 0.83732)	(1, 1.01738, 1.055204)	0.913
D	(0.77643, 0.79636, 0.81927)	(1, 1, 1.028724)	0.898
E	(0.97303, 1, 1)	(1, 1, 1)	0.996
F	(0.83517, 0.85653, 0.879205)	(1, 1, 1.104026)	0.938
G	(0.87519, 0.89757, 0.92263)	(1, 1.07384, 1.1585)	0.938
H	(0.819529, 0.84089, 0.86447)	(1, 1.009602, 1.08548)	0.929

8 Conclusions

On the basis the present study, it can be concluded that there are flaws in the existing method [25] and hence, the existing method [25] cannot be used for evaluating the best relative geometric crisp efficiency of DMUs. Also, to resolve the flaws of the existing method [25], a new approach is proposed to solve the proposed fuzzy CCR DEA models for evaluating the best relative geometric crisp efficiency of DMUs.

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