Chapter 8
Examples of Measurements and Measurement Data Processing

8.1 Voltage Measurement with a Pointer-Type Voltmeter

Our first example concerns a measurement of voltage with a pointer-type voltmeter. Such a measurement clearly represents an example of a single direct measurement. We shall study two examples of such measurements with a Class 1.0 pointer-type DC voltmeter that operates using the energy of the source of the voltage being measured. Note that the energy consumption by the voltmeter causes interaction between the voltmeter and the object under study.

Let the voltmeter have the following characteristics:

1. The upper limits of measurement ranges are 3 V, 7.5 V, 15 V, 30 V, and so on, up to 300 V.
2. The scale of the instrument has 75 graduations and starts at the 0 marker.
3. The limits of permissible intrinsic error are $\pm 1.0\%$ of a span (it is a fiducial error).
4. Full deflection of the pointer corresponds to the current of $15 \times 10^{-6}\text{A} \pm 1\%$.
5. Reference conditions include temperature of $+20 \pm 5\ ^{\circ}\text{C}$ and the requirement that the measurement be performed with the instrument positioned horizontally.
6. Additional errors are as follows. A deviation of the temperature from the reference range causes the indications of the instrument to change by no more than $\pm 1.0\%$ for each 10 °C change in temperature. Inclination of the instrument by 5° from the horizontal position changes the indications by not more than $\pm 1\%$.

The original version of this chapter was revised. An erratum to this chapter can be found at https://doi.org/10.1007/978-3-319-60125-0_11

S.G. Rabinovich, *Evaluating Measurement Accuracy*, Springer Series in Measurement Science and Technology, DOI 10.1007/978-3-319-60125-0_8

8.1.1 Single Measurement Under Reference Condition of the Voltmeter

The value of the measured quantity supposedly will be less than 3 V, so the 3 V range is used on the voltmeter. Let the indication of the voltmeter be 62.3 graduations at the range 3 V. Hence, the voltage indicated by the voltmeter is

$$U = 62.3\frac{3}{75} = 2.492\text{V}.$$

The accuracy of a measurement under reference condition is determined by the limits of intrinsic error of the instrument involved, and it does not have additional errors. But before the limits of intrinsic error of the instrument involved are re-calculated for the indication point, it is necessary to estimate the effect of interaction between the voltmeter and the object under study.

The goal of the measurement is to find the voltage between the two points on the electrical circuit ("the voltage source") to which the voltmeter is connected. But the voltmeter shows the voltage on its terminals, which is always lower than the voltage being measured due to the voltage drop across the resistance of the voltage source. In other words, the measuring instrument (the voltmeter) interacts with the object of study (the voltage source), and this interaction affects the observed value of the measurand (causes voltage drop). The extent of this voltage drop depends on the relation between the internal resistance of the voltmeter, R_V, and the resistance of the source of the voltage being measured, R. From the parameters of the voltmeter, the internal resistance of the voltmeter at the range 3 V can be found as

$$R_V = \frac{3}{15 \times 10^{-6}} = 2 \times 10^5 \Omega.$$

Let the source resistance R be 1 kΩ and constant. Then we can estimate the absolutely constant error caused by the above interaction between the voltmeter and the voltage source. The voltmeter indication U shows the voltage on its terminals. Without the above interaction it would have been $E>U$. Let I denote the power of current flowing through the voltmeter and the external circuit under study. Then the relative error δ_U of the indication of voltmeter due to the voltage drop is

$$\delta_U = \frac{IR_V - I(R_V + R)}{I(R_V + R)} = \frac{-R}{R_V + R}.$$

Then

$$\delta_U = -(1\text{k}\Omega/201\text{k}\Omega)^* 100\% = -0.5\%,$$

or in absolute form,

$$\Delta_U = -2.492 * 0.5 * 10^{-2} = -0.0124\text{V}.$$

This error can be removed by correction

$$C_U = +0.0124\text{V}.$$

The corrected indication is therefore

$$U_C = 2.492 + 0.012 = 2.504\text{V}.$$

In our case, the limit of permissible intrinsic error in relative form is equal to the fiducial error $\gamma = 1\%$, and the fiducial value $x_f = 3\ V$. The limit of absolute error Δ_{in} is computed in accordance to Sect. 4.6 as follows:

$$\Delta_{in} = \gamma \frac{x_f}{100} = \frac{1*3}{100} = 0.03\text{V}.$$

The number of significant figures in this error shows that the (corrected) indication U_C has one extra significant figure. After rounding off, it becomes

$$U_C = 2.50\text{V}.$$

The intrinsic error in absolute form is the same across the entire scale of the chosen range of the instrument. This allows us to find the limit of relative error of the indication:

$$\delta_{in} = \frac{0.03}{2.50} * 100 = 1.18\%.$$

Another source of error, characteristic of an analog measurement device with analog scale, is reading error δ_r, which is the error with which the experimenter reads the indication of the device in the course of the measurement. While this error generally depends on the indication (it disappears when the indication happens to fall on a graduation mark, and is larger for indications between marks), it does not exceed 0.25 of a graduation, and we will assume for simplicity this upper bound in our calculations. The value of one graduation is $\frac{3}{75} = 0.04\text{V}$ and at the reading point the 0.25 of a graduation produces

$$\delta_\text{r} = \frac{0.25*0.04*100}{2.50} = 0.39\%.$$

The measurement uncertainty in relative form is computed according to formula (4.3) for confidence probability 0.95:

$$\theta_{0.95,rel} = k_{0.95}\sqrt{{\delta_{in}}^2 + {\delta_r}^2} = 1.1\sqrt{1.18^2 + 0.39^2} = 1.36\%.$$

In absolute form, this becomes

$$\theta_{0.95} = 1.36*10^{-2}*2.50\text{V} = 0.034\text{V}.$$

Since as noted earlier, the accuracy of the measuring instrument stipulates the measurement result with no more than three significant figures, the value of measurement uncertainty must also be rounded to three significant figures, leading to $\theta_{0.95,rel} = 1.4\%$ and $\theta_{0.95} = 0.03\text{V}$. Finally, the result of the measurement is presented in the form

$$U = (2.50 \pm 0.03)\text{V} \text{ or } \tilde{U} = 2.50\text{V} \pm 1.4\%.$$

8.1.2 Single Measurement Under Rated Condition of the Voltmeter

The inaccuracy of measurement under rated condition is determined not just by the intrinsic error but also by additional errors. Let us consider the example of 8.1.1, but now under rated rather than reference condition. Assume that the instrument indicated has been observed to be 63.1 graduations, which translates into

$$U' = 63.1*\frac{3}{75} = 2.524\text{V}.$$

The electrical resistance of the voltmeter R_V and R of the chain are the same. Therefore the absolutely constant error caused by interaction between voltmeter and the chain is the same as it was in the measurement under reference condition and it may be removed by the correction $C_U = +0.0124\text{V}$ as obtained in Sect. 1.1. The measurement result after correction is

$$U'_C = 2.524 + 0.012 = 2.54\text{V}.$$

Now we need to calculate the accuracy of this result. The sources of error are as follows:

1. The intrinsic error of the voltmeter
2. The reading error
3. The temperature error
4. The error introduced by the inclination of the instrument
5. The error caused by the limited accuracy of internal resistance of the voltmeter.

All errors listed above are conditionally constant. We shall now estimate the errors of the measurement.

1. Intrinsic error δ_{in}. Its limits, as derived in the previous section, are:

$$\delta_{\text{in}} = \frac{0.03*100}{2.54} = 1.18\%.$$

2. Reading error δ_r. This error was also derived in the previous section and is

$$\delta_r = \frac{0.25*0.04*100}{2.54} = 0.39\%.$$

3. Additional temperature error δ_T. The maximum deviation of the temperature from the normal value is 5 °C. Therefore,

$$\delta_T = 0.5\%.$$

4. Additional inclination error δ_l. Because of the 5° inclination of the instrument, the additional error will be 1% of the instrument indication:

$$\delta_l = 1\%.$$

Since all above errors are conditionally constant, we may sum them using Eq. (4.3). The resulting sum for confidence probability $\alpha = 0.95$ will be the uncertainty limits in relative form:

$$\theta_{0.95,rel} = 1.1\sqrt{1.18^2 + 0.39^2 + 0.5^2 + 1^2} = 1.1\sqrt{2.79} = 1.84\%.$$

In the absolute form, this uncertainty limit will be $\theta_{0.95} = 2.54 * 1.84 * 10^{-2} = 0.047\text{V}$. After rounding, the obtained uncertainties will become $\theta'_{0.95,rel} = 2\%$ and $\theta'_{0.95} = 0.05\text{V}$. Thus, the inaccuracy of the result of measurement under rated condition of the voltmeter is expressed as uncertainty with confidence probability 0.95. The result of measurement should be presented in the form

$$U = (2.54 \pm 0.05)\text{V} \;\text{ or }\; \tilde{U} = 2.54\text{V} \pm 2\%.$$

8.2 Voltage Measurement with a Potentiometer and a Voltage Divider

Potentiometers with manual control are highly accurate and universal. For these reasons, they are frequently used in scientific laboratories, although they have started to be displaced by digital multirange voltmeters in recent years. The latter are in essence automated potentiometers.

A voltage measurement with a potentiometer requires a two-phase measurement procedure. First, a standard cell is connected to the potentiometer, and the current through the potentiometer is adjusted using the potentiometer's set of accurate measuring resistors so that the voltage drop on the section of the circuit with these resistors would balance the EMF of the standard cell. Next, a special potentiometer switch is used to disconnect the standard cell, and we connect the voltage to be measured to the potentiometer.

When the voltage to be measured exceeds the range of the potentiometer, a voltage divider can be used, which allows only a known fraction of the voltage to be applied to the potentiometer. We should point out that a voltage divider contains electrical resistors and thus consumes a certain amount of power from the voltage source to which it connects. For this reason, a voltage divider can only be used if the power it consumes is so low that the resulting affect on the measured voltage is negligible. We assume that this is the case in our example.

The measurement of voltage with a potentiometer is a direct measurement. However, when the errors of the potentiometer and the errors of the standard cell are rated separately, and when a voltage divider is involved, the error produced by such a chain of measuring instruments is estimated with methods that are specifically designed for indirect measurements. We discussed these methods in Sect. 5.10. Here, we shall consider an example of a single measurement with individual inaccuracy estimation.

To be specific, we will consider the measurement of voltage using a class 0.005 potentiometer, a class 0.005 voltage divider, and a standard cell with voltage accuracy of ±10 μV. In particular, we will consider a P309 potentiometer and P35 voltage divider, which were manufactured in the former USSR. The measuring resisters in P309 potentiometer are organized in six blocks called decades. Each decade produces certain decimal digits in the measurement result. For example, if the measured voltage is 1.256316 V, the digits "1.2 V" are produced by indication "12" of decade "×100 mV," the digit "0.05 V" by indication "5" of decade "×10 mV," and so on.

Let the current through the potentiometer be I_p and the resistance of the section of the circuit with the accurate resistors after the adjustment in the first phase be R_{sc}. Since the voltage drop on the section of the circuit with the resistance R_{sc} balances the EMF of the standard cell, U_{sc}, we have in this case:

$$I_p = U_{\mathrm{SC}}/R_{\mathrm{SC}}.$$

When the standard cell is disconnected and a certain voltage, U_p, is connected to the potentiometer circuit, a fraction of the resistors of the potentiometer is introduced into the comparison circuit such that the voltage drop on their resistance R_p would compensate U_p; i.e., $U_p = I_p R_p$. Then

$$U_p = \frac{R_p}{R_{\mathrm{SC}}} U_{\mathrm{SC}},$$

and knowing the EMF of the standard cell and the ratio R_p/R_{sc}, we can find U_p. Finally, assuming that the division coefficient of the voltage divider is equal to K_d, the voltage to be measured, U, is determined from the formula $U = K_d U_p$. Therefore, we can write the measurement equation in this measurement in the form:

$$U = K_d \frac{R_p}{R_{SC}} U_{SC}. \quad (8.1)$$

The indications of the potentiometer are proportional to R_p, but its error is determined not by the errors of the resistances R_p and R_{sc}, but by the error of the ratio R_p/R_{sc}. The uncertainty associated with the operations of comparing the voltages can be neglected, because the smoothness of the resistance regulation in the potentiometer and the sensitivity of its zero indicator were designed specifically to keep this uncertainty extremely small compared to other errors.

The potentiometer has six decades and a built-in self-balancing amplifier. The limit of permissible error as a function of the measured voltage U_p is calculated using the formula (given in the manufacturer's documentation):

$$\Delta U_p = \pm(50U_p + 0.04) \times 10^{-6}\text{V}.$$

The error of the potentiometer does not exceed the above limits if the ambient air temperature ranges from +15 to +30 °C and differs by not more than 2.5 °C from the temperature at which the measuring resistors of the potentiometer were adjusted (the P309 potentiometer has built-in calibration and adjusting systems).

The EMF of the standard cells can be determined with an error of ± 10 μV that in relative form is $\pm 1 \times 10^{-3}$%. The effect of the temperature is taken into account using a well-known formula, which describes accurately the temperature dependence of the EMF in a standard cell. Thus, temperature does not introduce additional errors to the EMF of the standard cell.

Assume that in three repeated measurements of certain voltage, performed using a voltage divider whose voltage division ratio was set to 1:10, the following potentiometer indications were obtained:

$$x_1 = 1.256316\text{V}, \;\; x_2 = 1.256321\text{V}, \;\; x_3 = 1.256318\text{V}.$$

The limit of permissible error of the potentiometer in this case is

$$\Delta U_p = \pm\ (50*1.26 + 0.04)*10^{-6} = \pm\ 63\ \mu\text{V}.$$

For this reason, the difference of 5 μV between the results of the three observations above can be regarded as resulting from the random error of the measurement, and the magnitude of this error is negligible. In the calculation, therefore, any one of these results or their average value can be used.

Assume that in the process of adjusting the measuring resistors before the measurement, the corrections of the higher order decades were estimated. Let the correction for the indication "12" of the decade "×100 mV" be $+15 \times 10^{-6}$ V, and the correction for the indication "5" of the decade "×10 mV" be -3×10^{-6} V. Each correction is determined with an error of $\pm 5 \times 10^{-8}$ V.

The corrections for the other decades are so small that they are of no interest. Indeed, the indication of all the remaining decades is 0.0063V; the limit of permissible error corresponding to this indication in accordance with the formula given above is

$$\Delta U_p = \pm\,(50 \times 0.0063 + 0.04) \times 10^{-6} = \pm\,0.32 \times 10^{-6}\ \text{V}.$$

This error is already two orders of magnitude smaller than the permissible error of the higher decades, and it can be neglected without further corrections.

Further, it is necessary to take into account the possible change in the air temperature in the room. If this change falls within permissible limits, then according to the specifications of the potentiometer, the error can change approximately by one-forth of the permissible limit, i.e., by 16 μV.

We shall take for the result the average value of the observations performed, correcting it by the amount C = (15–3) $\times 10^{-6} = 12 \times 10^{-6}$ μV:

$$U_p = \bar{x} = 1.256318 + 0.000012 = 1.256330\text{V}.$$

The errors of the potentiometer, which enter into this result, include the error due to temperature ($\pm 16 \times 10^{-6}$ V), the error of correction of the higher decades ($\pm\,5 \times 10^{-8}$ V), and the error due to the lower decades ($\pm 0.32 \times 10^{-6}$ V). Clearly, these errors are dominated by the error due to temperature, and the remaining errors can be neglected. Thus, the limits of error of the potentiometer are

$$\theta_p = \pm 16 \times 10^{-6}\text{V}.$$

Next, we must estimate the errors from the standard cell and the voltage divider. The error of the class 0.005 voltage divider can reach 5×10^{-3}%. But the actual division coefficient of the divider can be found and taken into account, which is precisely what we must do in the case at hand. In the given measurement, assume that this coefficient has been found to be $K_d = 10.0003$ and the error in determining K_d falls within the range $\pm 2 \times 10^{-3}$%. Finally, the discrepancy between the real and the nominal value of the EMF of the standard cell falls within the limits of error of the standard cell (±10 μV).

We estimate the voltage being measured U as

$$\tilde{U} = K_d U_p = 10.0003 \times 1.256330 = 12.56368\,\text{V}.$$

To estimate the measurement error, we shall employ the usual calculation. First, we shall take the logarithm of the measurement Eq. (8.1). Then we find the differential of both sides of the equation and replace them by increments – measurement errors. This process gives

$$\frac{\Delta U}{U} = \frac{\Delta K_d}{K_d} + \frac{\Delta\left(R_p/R_{\mathrm{SC}}\right)}{R_p/R_{\mathrm{SC}}} + \frac{\Delta U_{\mathrm{SC}}}{U_{\mathrm{SC}}}.$$

For the terms on the right side of the above formula, we only have estimates of the limits, and not the values of the error. Thus, we shall estimate the limits of the measurement error on the left side. We can use formula (4.3) for this purpose. First, all components must be represented in the form of relative errorss. The limits of the relative error of the potentiometer, in percent, will be

$$\theta_{p,rel} = \pm\frac{16 \times 10^{-6} \times 100}{1.26} = \pm 1.3 \times 10^{-3}\%.$$

The limits of the relative error of the voltage divider were estimated directly as $\theta_K = \pm 2 \times 10^{-3}\%$. The limits of error in determining the EMF of the standard cell in the form of a relative error are known:

$$\theta_{SC,rel} = \pm 1 \times 10^{-3}\%.$$

We now find the limit of the measurement error according to (4.3):

$$\theta_{\alpha,rel} = k_\alpha\sqrt{1.3^2 + 2^2 + 1^2} * 10^{-3} = k_\alpha * 2.6 * 10^{-3}\%$$

Let $\alpha = 0.95$. Then $k_\alpha = 1.1$ and

$$\theta_{0.95,rel} = 1.1 * 2.6 * 10^{-3} = 2.9 * 10^{-3}\%.$$

Finally, we must check the number of significant figures in the result of measurement. To this end, we shall express the above limit $\theta_{0.95}$ in the absolute form:

$$\theta_{0.95} = \pm 2.9 \times 10^{-3} \times 10^{-2} \times 12.6 = \pm 37 \times 10^{-5}\mathrm{V}.$$

As this is an accurate measurement, the error of the result is expressed by two significant figures (see Sect. 1.8), and there are no extra figures in the obtained result to be rounded off. The final result is (omitting alternative representations from now on) as follows:

$$U_{0.95} = (12.56368 \pm 0.00037)\mathrm{V}.$$

If the measurement was performed with universal estimation of the errors, then the errors of all components would have to be set equal to $5 \times 10^{-3}\%$ and the limit of the measurement error would be

$$\theta'_{0.95,rel} = 1.1 \times 10^{-3}\sqrt{3 \times 5^2} = 5.8 \times 10^{-3}\%.$$

Then, in absolute form, $\theta'_{0.95} = \pm 12.6 \times 5.8 \times 10^{-5} = 0.0007\mathrm{V}$ and the result of measurement would have to be written with fewer significant figures:

$$U_{0.95} = (12.5637 \pm 0.0007)\mathrm{V}.$$

8.3 Comparison of Mass Measures

Let us consider the calibration of a 1-kg mass measure by comparing it with the reference standard measure of mass with the same nominal value using a balance. Assume that the comparison was repeated ten times. Column 1 of Table 8.1 lists the measurement results obtained from the comparison of the measures. Our goal is to produce the final measurement result and estimate its inaccuracy.

Assume that the measurement was performed by the methods of precise weighing, which eliminated the error caused by the arms of the balance not having precisely equal length. Thus, it can be assumed that there are no systematic errors.

Table 8.1 presents the input and intermediate data involved in producing the final measurement result and estimating its inaccuracy. Since the systematic errors were eliminated, the measurement results in column 1 can be viewed to be random independent quantities $\{x_i\}$, $i, = 1, \ldots, n$ and $n = 10$, and therefore, the probability of all x_i, is the same and equal to $1/n$. To simplify the computations, column 2 presents only the varying last three digits of x_i, denoted as x_{i0}.

Their mean value is

$$\bar{x}_{i0} = \frac{1}{n}\sum_{i=1}^{n} x_{i0} = \frac{1}{10} \cdot 7210 \times 10^{-6} = 721 \times 10^{-6}\mathrm{g}.$$

Table 8.1 Input measurement data and intermediate processing steps in the measurement of the mass of a weight

x_i g	$x_{i0} \times 10^{-6}$g	$(x_{i0} - \bar{x}_{i0}) \times 10^{-6}$g	$(x_{i0} - \bar{x}_{i0})^2 \times 10^{-12}\mathrm{g}^2$
999.998738	738	+17	289
999.998699	699	−22	484
999.998700	700	−21	441
999.998743	743	+22	484
999.998724	724	+3	9
999.998737	737	+16	256
999.998715	715	−6	36
999.998738	738	+17	289
999.998703	703	−18	324
999.998713	713	−8	64
Sum	7.210	0	2.676

Thus, the estimate of the value of the mass is

$$\bar{x} = 999.998000 + \bar{x}_{i0} = 999.998721\text{g}.$$

We can now obtain the estimate of the variance

$$S^2(x_i) = \frac{1}{n-1}\sum_{i=1}^{n}(x_{i0} - \bar{x}_0)^2.$$

Hence, the standard deviation is

$$S(x_i) = \sqrt{\frac{2676}{9} \times 10^{-12}} = 17 \times 10^{-6}\text{g}.$$

An estimate of the standard deviation of the obtained value of the mass measure is

$$S_{\bar{x}} = \frac{17 \times 10^{-6}}{\sqrt{10}} = 5 \times 10^{-6}\text{g}.$$

We shall find the uncertainty of the result using Student's distribution for confidence probability $\alpha = 0.95$; then, from Table A.2, we find the coefficient t_q for the degree of freedom $v = 10-1 = 9$ and $q = 1 - \alpha = 0.05$: $t_{0.05} = 2.26$. In accordance with formula (3.20), we obtain the uncertainty of measurement result:

$$u_{0.95} = 2.26 \times 5 \times 10^{-6} = 11 \times 10^{-6}\text{g}.$$

Thus, with the confidence probability $\alpha = 0.95$, the mass m of the measure studied lies in the interval

$$999.998710\ \text{g} \leq m \leq 999.998732\text{g}.$$

This result can be written more compactly as

$$m_{0.95} = \left(999.998\ \ 721 \pm 11 \times 10^{-6}\right)\text{g}..$$

Note that if the data above were processed by the nonparametric methods, the estimate of the measurand would be practically the same but its uncertainty would be much wider (see Sect. 3.10).

8.4 Measurement of Electric Power at High Frequency

Consider a measurement of electric power generated in a resistor by high-frequency current. The measurement is conducted by a single measurement of the current and resistance of the resistor, after which the value of the electric power is computed using equation $P = I^2R$, where P is the power to be measured, I is the effective current and R is the active resistance of the resistor. This is an example of a single indirect measurement.

Assume single measurements of the electric current and resistance of the resistor have produced the estimates $\tilde{I} = 500$ mA and $\tilde{R} = 10.0\ \Omega$. We know also that these measurements were conducted under reference conditions. The limits of error of $\tilde{I}$ and $\tilde{R}$ are estimated using the procedure for direct single measurements under reference conditions (see Sect. 4.6). Assume these limits in relative form are:

$$\delta_I = 0.5\% \text{ and } \delta_R = 1\%.$$

Substituting values $\tilde{I}$ and $\tilde{R}$ in the measurement equation, we obtain the estimate of the measurand $\tilde{P}$:

$$\tilde{P} = (0.5)^2 * 10.0 = 2.50 \text{ W}.$$

We now estimate the accuracy of the measurement result. The measurement equation follows the structure of Eq. (5.10) and we have the limits of measurement errors of arguments represented in relative form. Thus, according to the discussion following Eq. (5.10), we know the influence coefficients of the arguments: $l_I = 2$ and $l_R = 1$. Therefore we can transform the limits of measurement errors of the arguments into the limits of elementary errors of the indirect measurement, $\theta_{I,rel}$ and $\theta_{R,rel}$ in relative form:

$$\theta_{I,rel} = 2\delta_I \text{ and } \theta_{R,rel} = \delta_R.$$

We can now combine these elementary errors using formula (5.28). For confidence probability 0.95, coefficient $k_{0,95} = 1.1$. Then, we obtain:

$$\theta_{0.95,rel} = 1.1\sqrt{4\delta_I^2 + \delta_R^2} = 1.1\sqrt{4*0.25 + 1} = 1.5\%.$$

Since the number of elementary errors is small, we need to compare the above probabilistic uncertainty with the arithmetic sum of the two arithmetic errors. The arithmetic sum is $\theta_{I,rel} + \theta_{R,rel} = 2\%$, which is greater than $\theta_{0.95,rel} = 1.5\%$. Thus, we take the latter as the estimate of the measurement inaccuracy, which in the absolute form is

$$\theta_{0.95} = 2.50 * 1.5 * 10^{-2} = 0.0375\,W \approx 0.04\ \mathrm{W}.$$

Finally, the measurement result and its uncertainty are recorded as:

$$\tilde{P}_{0.95} = (2.50 \pm 0.04)\mathrm{W} \ \text{ or } \ \tilde{P} = 2.50\mathrm{W} \pm 1.5\%(0.95).$$

8.5 An Indirect Measurement of the Electrical Resistance of a Resistor

Consider the measurement of electrical resistance using an ammeter and a voltmeter. This is an indirect measurement with measurement equation $R = U/I$, where R is the electrical resistance of the resistor, U is the voltage drop on the resistor, and I is the strength of the current. Furthermore, it is a dependent indirect measurement because the value of I depends on the value of U.

The connections of the instruments and the resistor are shown in Fig. 8.1. Assume that the measurement was performed under reference conditions for the instruments, and that the input resistance of the voltmeter is so high that its influence on the accuracy of the measurement can be neglected.

Both voltage and current have been measured with the limits of error ±0.1% of the indications. The results of measurements of the strength of current and voltage are given in Table 8.2. In accordance with the discussion from Sect. 5.2 and 5.6, all results presented in the table were obtained in pairs: the results with the same subscript belong to the same measurement vector.

We can use in this example both the traditional method and the method of reduction. Let us use each in turn and compare the calculations and results.

8.5.1 Application of the Traditional Method

The traditional method of experimental data processing for dependent indirect measurements was described in Sect. 5.4.

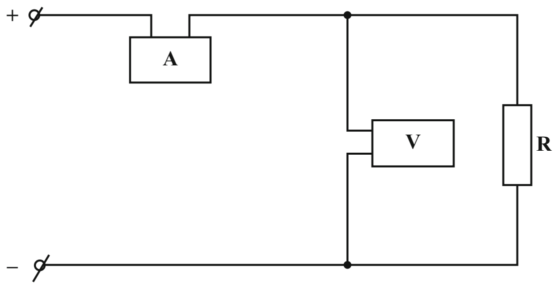

Fig. 8.1 The schema for indirect measurement of an electrical resistance

Table 8.2 Input measurement data in indirect measurement of a resistor

Num.	I_i (A)	U_i (V)
1	0.05996	6.003
2	0.06001	6.001
3	0.05998	5998
4	0.06003	6.001
5	0.06001	5.997
6	0.05998	5.999
7	0.06003	6.004
8	0.005995	5.997
9	0.06002	6.001
10	0.06001	6.003
11	0.05999	5.998

Table 8.3 Data processing for indirect measurement of electrical resistance using the traditional method

Num.	I_i A	U_i V	$(I_i - \bar{I})$ $\times 10^{-5}$ A	$(I_i - \bar{I})^2$ $\times 10^{-10}$ A^2	$(U_i - \bar{U})$ $\times 10^{-3}$ V^3	$(U_i - \bar{U})^2$ $\times 10^{-6}$ V^2	$(I_i - \bar{I})(U_i - \bar{U})$ $\times 10^{-8}$ AV
1	2	3	4	5	6	7	8
1	0.05996	6.003	−3.7	13.69	+2.82	7.95	−10.4
2	0.06001	6.001	+1.3	1.69	+0.82	0.67	+1.1
3	0.05998	5.998	−1.7	2.89	−2.18	4.75	+3.7
4	0.06003	6.001	+3.3	10.89	+0.82	0.67	+2.7
5	0.06001	5.997	+1.3	1.69	−3.18	10.11	−4.1
6	0.05998	5.999	−1.7	2.89	−1.18	1.39	+2.0
7	0.06003	6.004	+3.3	10.89	+3.82	14.59	+12.6
8	0.05995	5.997	−4.7	22.09	−3.18	10.11	+14.9
9	0.06002	6.001	+2.3	5.29	+0.82	0.67	+1.9
10	0.06001	6.003	+1.3	1.69	+2.82	7.95	+3.7
11	0.05999	5.998	−0.7	0.49	−2.18	4.75	+1.5
Sum	0.65997	66.002		74.19		63.61	29.6

The calculations are illustrated by Table 8.3, which also repeats the input measurement data for convenience. Using the values of U_i, and I_i, we obtain the estimates of the arguments:

$$\bar{U} = 66.002/11 = 6.00018\text{V}, \ \ \bar{I} = 0.65997/11 = 0.059997\text{A}.$$

We can now compute the estimate of the measurand R. But because the number of measurements of the arguments is the same, one can avoid the inaccuracy of calculation of the argument estimates by obtaining R from the sums of the individual measurement results of the arguments (given in columns 2 and 3, the last row of Table 8.3) rather than from their estimates:

$$\tilde{R} = \bar{U}/\bar{I} = \frac{\sum_{i=1}^{n} U_i}{\sum_{i=1}^{n} I_i} = 66.002/0.65997 = 100.0075\Omega.$$

Now we must calculate the variance and the standard deviation of this result.

First, we will estimate the variances of $\bar{I}, \bar{U}$, their standard deviations, and the correlation coefficient. According to the discussion in Sect. 5.2, we obtain

$$S^2(\bar{I}) = \frac{\sum_{i=1}^{n}(I_i - \bar{I})^2}{n(n-1)} = \frac{74.19 \times 10^{-10}}{11 \times 10} = 0.674 \times 10^{-10}\text{A}^2,$$

$$S^2(\bar{U}) = \frac{\sum_{i=1}^{n}(U_i - \bar{U})^2}{n(n-1)} = \frac{63.61 \times 10^{-6}}{11 \times 10} = 0.578 \times 10^{-6}\text{V}^2.$$

The estimates of standard deviations are

$$S(\bar{I}) = 0.82 \times 10^{-5}\text{A}, S(\bar{U}) = 0.76 \times 10^{-3}\text{V}.$$

The estimate of the correlation coefficient is

$$r_{I,U} = \frac{\sum_{i=1}^{n}(I_i - \bar{I})}{n(n-1)S(I)S(U)} = \frac{29.6 \times 10^{-8}}{110 \times 0.82 \times 10^{-5} \times 0.76 \times 10^{-3}} = 0.43.$$

In our example, inserting the obtained values into (5.20) we can calculate the desired estimation of standard deviation $S(\tilde{R})$. But first we have to calculate the influence coefficients. Thus, the calculations are

$$\begin{aligned}
w_1 &= \frac{\partial R}{\partial U} = \frac{1}{I}, \quad w_2 = \frac{\partial R}{\partial I} = \frac{U}{I^2}, \\
S^2(\tilde{R}) &= \left(\frac{\bar{U}}{\bar{I}^2}\right) \times S^2(\bar{I}) + \frac{1}{\bar{I}^2} \times S^2(\bar{U}) - r_{I,U}\frac{\bar{U}}{\bar{I}^2} \times \frac{1}{I} \times S(\bar{I})S(\bar{U}) \\
&= \left(\tfrac{6}{36\times 10^{-4}}\right)^2 \times 0.674 \times 10^{-10} + \frac{1}{36 \times 10^{-4}} \times 0.578 \times 10^{-6} \\
&\quad -2 \times 0.43 \times \frac{6 \qquad 1}{36 \times 10^{-4} 6 \times 10^{-2}} \times 0.82 \times 10^{-5} \times 0.76 \times 10^{-3} \\
&= 1.87 \times 10^{-4} + 1.61 \times 10^{-4} - 1.49 \times 10^{-4} \\
&= 1.99 \times 10^{-4}\Omega^2,
\end{aligned}$$

and

$$S(\bar{R}) = \sqrt{S^2(R)} = 1.41 \times 10^{-2}\Omega.$$

The next step is to find the uncertainty of the obtained result. Unfortunately, we have the standard deviation, but no information about the distribution function of the measurement error, and it is unclear how to find the degree of freedom of the measurement result to account for the dependency between the arguments. Thus, with dependent indirect measurements, we have to use standard deviation of the measurement result as the indication of measurement accuracy rather than its uncertainty. Furthermore, because in the traditional method, we are unable to calculate the random uncertainty of that measurement and hence cannot combine it with the systematic uncertainty, we did not calculate the latter.

8.5.2 Application of the Method of Reduction

We now turn to the method of reduction described in Sect. 5.6. Table 8.4 lists the intermediate data involved in the calculations. The initial data are again provided in columns 2 and 3.

According to the method of reduction, we first compute values of the measurand using the measurement equation for each measurement vector. The calculated values of R_i, ($i = 1, \ldots, 11$) are given in column 4. Treating these values as if they were obtained by direct measurements, we obtain immediately the estimate of R as

Table 8.4 Data processing for indirect measurement of electrical resistance using the method of reduction

Num.	I_i A	U_i V	R_i Ω	$(R_i - \bar{R})$ Ω	$(R_i - \bar{R})$ $\times 10^{-2}$ Ω^2
1	2	3	4	5	6
1	0.05996	6.003	100.117	+0.109	1.188
2	0.06001	6.001	100.000	0.002	0.000
3	0.05998	5.998	100.000	0.002	0.000
4	0.06003	6.001	99.967	0.041	0.168
5	0.06001	5.997	99.933	0.075	0.562
6	0.05998	5.999	100.017	+0.009	0.008
7	0.06003	6.004	100.017	+0.009	0.008
8	0.05995	5.997	100.033	+0.025	0.0625
9	0.06002	6.001	99.983	–0.025	0.0625
10	0.06001	6.003	100.033	+0.025	0.0625
11	0.05999	5.998	99.983	–0.025	0.0625
Sum			1100.083		2.184

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i = 100.0075\Omega$$

and the estimates of its variance and standard deviation as

$$S^2(\bar{R}) = \frac{1}{n(n-1)}\sum_{i=1}^{n}(R_i - \bar{R})^2 = \frac{2.184 \times 10^{-2}}{11 \times 10} = 1.99 \times 10^{-4}\Omega^2,$$
$$S(\bar{R}) = 1.41 \times 10^{-2}\Omega.$$

As one can see from this example, the calculations using the method of reduction are much simpler than using the traditional method, even in this case with a measurement equation having only two arguments. More importantly, we now have a set of output data $\{R_i\}$ that does not differ in any way from data obtained in direct measurements. Thus, we know the degree of freedom $v = 11–1 = 10$ and can compute the uncertainty of the measurement result. Using confidence probability $\alpha = 0.95$ we find the corresponding value of Student's coefficient $t_q = 2.23$ and uncertainty

$$u_{0.95} = 2.23{*}1.41{*}10^{-2} = 3.14{*}10^{-2}\Omega.$$

Turning to the systematic error, the measurement equation conforms to the structure studied in Sect. (5.10) and therefore we immediately know the influence coefficients: $l_U = 1$ and $l_I = -1$. Then, keeping in mind that for our chosen confidence probability $\alpha = 0.95$, $k_{0.95} = 1.1$, we obtain:

$$\theta_{0.95,rel} = 1.1\sqrt{1^2 + 1^2}{*}10^{-3} = 1.1{*}1.41{*}10^{-3} = 1.55{*}10^{-3},$$

and

$$S_{\vartheta,rel} = \frac{1}{\sqrt{3}}{*}\sqrt{1^2 + 1^2}{*}10^{-3} = 0.82{*}10^{-3}.$$

In the absolute form,

$$\theta_{0.95} = 1.55{*}10^{-3}{*}100\,\Omega = 0.155\,\Omega \text{ and } S_{\vartheta} = 0.82{*}10^{-3}{*}100\,\Omega = 0.082\,\Omega.$$

We now combine the random and systematic components of the measurement uncertainty according to Sect. 4.10. The combined standard deviation is computed by formula (4.19):

$$S_c = \sqrt{\left(1.41{*}10^{-2}\right)^2 + \left(8.2{*}10^{-2}\right)^2} = 8.31{*}10^{-2}\Omega.$$

Coefficient t_c – by formula (4.22):

$$t_c = \frac{3.14*10^{-2} + 15.5*10^{-2}}{1.41*10^{-2} + 8.31*10^{-2}} = 1.92.$$

Now, by formula (4.20) we obtain the overall uncertainty of the measurement result:

$$U_c = 1.92*8.31*10^{-2} = 0.16\,\Omega.$$

The final measurement result is recorded as

$$R_{0.95} = (100.01 \pm 0.16)\Omega.$$

8.6 Measurement of the Density of a Solid Body

The accurate measurement of the density of a solid body can serve as an example of a multiple nonlinear independent indirect measurement. The density of a solid body is given by the formula

$$\rho = m/V,$$

where m is the mass of the body and V is the volume of the body. In the experiment considered, the mass of the body was measured by methods of precise weighing using a balance and a collection of standard weights whose errors did not exceed 0.01mg. The volume of the body was determined by the method of hydrostatic weighing using the same set of weights. The results of measurements are presented in Table 8.5 in columns 2 and 5.

The difference between the observational results of the body mass is explained by the random error of the balance and the inevitable fluctuations of the environmental conditions. As follows from the data presented, this error is so much larger than the systematic errors in the masses of the weights that the latter errors can be neglected.

8.6.1 Application of the Traditional Method

As the mass of the solid body and its volume are constants, to estimate the density of the body, the mass and volume of the body must be estimated with the required accuracy and their ratio must be formed. For this reason, we find the average values of the measurement results of the arguments and estimates of the standard deviations of these averages (Table 8.5 lists intermediate results for these calculations –

Table 8.5 Data processing for measurement of the density of a solid body

Num.	Body mass, $m_i \times 10^{-3}$ kg	$(m_i - \bar{m}) \times 10^{-7}$kg	$(m_i - \bar{m})^2 \times 10^{-14}$kg^2	Body volume, $V_i \times 10^{-6}$ m^3	$(V_i - \bar{V}) \times 10^{-10}$m^3	$(V_i - \bar{V})^2 \times 10^{-20}$m^6
1	2	3	4	5	6	7
1	252.9119	−1	1	195.3799	+1	1
2	252.9133	+13	169	195.3830	+32	1024
3	252.9151	+31	961	195.3790	−8	64
4	252.9130	+10	100	195.3819	+21	441
5	252.9109	−11	121	195.3795	−3	9
6	252.9094	−26	676	195.3788	−10	100
7	252.9113	−7	49	195.3792	−6	36
8	252.9115	−5	25	195.3794	−4	16
9	252.9119	−1	1	195.3791	−7	49
10	252.9115	−5	25	195.3791	−7	49
11	252.9118	−2	4	195.3794	−4	16
Sum			2132			1805

the deviations of individual measurements from their mean as well as the squares of these deviations):

$$\bar{m} = 252.9120 \times 10^{-3}\text{kg}, \quad \bar{V} = 195.3798 \times 10^{-6}\text{m}^3,$$

$$S^2(\bar{m}) = \frac{1}{n_1(n_1-1)}\sum_{i=1}^{n_1}(m_i - \bar{m})^2 = \frac{2132 \times 10^{-14}}{11 \cdot 10} = 19.38 \times 10^{-14}\text{kg}^2,$$

$$S^2(\bar{V}) = \frac{1}{n_2(n_2-1)}\sum_{i=1}^{n_2}(V_i - \bar{V})^2 = \frac{1805 \times 10^{-20}}{11 \cdot 10} = 16.41 \times 10^{-20}\text{m}^6.$$

The standard deviations of the measurement results of the arguments in the relative form are as follows:

$$S_{rel}(\bar{m}) = \frac{\sqrt{19.38 * 10^{-14}}}{252.9 * 10^{-3}} = 1.74 * 10^{-6}.$$

$$S_{rel}(\bar{V}) = \frac{\sqrt{16.41 * 10^{-20}}}{195.4 * 10^{-6}} = 2.08 * 10^{-6}.$$

We can now find the uncertainty of the obtained estimates of the arguments. Both were measured 11 times. Therefore, their degree of freedom is $v = 10$. Exploiting the robustness of Student's distribution, we will make use of this distribution. We thus obtain, for confidence probability $\alpha = 0.95$ and the corresponding value of Student's coefficient $t_q = 2.23$, the following confidence limits in relative form:

$$u_{0.95,rel}(\bar{m}) = 2.23 * 1.74 * 10^{-6} = 3.88 * 10^{-6},$$

$$u_{0.95,rel}(\bar{V}) = 2.23 * 2.08 * 10^{-6} = 4.64 * 10^{-6}.$$

The estimate of the measurand is

$$\widetilde{\rho} = \frac{\bar{m}}{\bar{V}} = \frac{252.9120 \times 10^{-3}}{195.3798 \times 10^{-6}} = 1.2944634 \times 10^3\text{kg/m}^3.$$

To calculate the uncertainty of the overall measurement result we use here the traditional method of linearization. It is not difficult to see that, in our example, using just the first term from the Taylor series is sufficient. (To this end, one must estimate the remainder R_2 of the Tailor series according to (5.15); we omit these details here.)

We shall now find the uncertainty of the result. This can be done in two ways: using the root sum of the squares formula (5.24) or by taking advantage of the fact that due to the expansion into the Taylor series, the measurement error of the result took the form of a linear combination of the measurement errors of the arguments, making it possible to compute the effective degree of freedom. In the first method, according to (5.24), the combined uncertainty in relative form is as follows:

$$u_{0.95,rel}(\overline{\rho}) = \sqrt{u_{rel}^2(\bar{m}) + u_{rel}^2(\bar{V})} = \sqrt{3.88^2 + 4.64^2} * 10^{-6} = 6.0 * 10^{-6}.$$

In absolute form the uncertainty is:

$$u(\overline{\rho}) = 1.29 \times 10^3 \times 6 \times 10^{-6} = 7.7 \times 10^{-3}\ \mathrm{kg/m^3}.$$

The measurement result, including its uncertainty in absolute form, can be expressed as:

$$\rho_{0.95} = \left(1.294463 * 10^3 \pm 7.7 * 10^{-3}\right)\mathrm{kg/m^3}.$$

Uncertainty in that result represents the random error of the measurement. The systematic error of it is negligible because the errors of the used weights were sufficiently small.

In principle, one could combine random errors of argument measurements after the linearization of the measurement equation, using Welch- Satterthwaite formula. But this formula is only applicable if the errors can be considered as normally distributed, which in our case would be unfounded. Another possibility is to apply corrections as discussed in Sect. 5.5. But these corrections are also only possible when the errors being combined can be considered normally distributed. Thus, neither method could be used to reduce the uncertainty of the measurement result.

8.6.2 Application of Method of Enumeration

Let us consider again a precise measurement of density of a solid body, with measurement data from Sect. 6.1. The sought density ρ is determined using measurement equation

$$\rho = m/V$$

and is computed from the measurements of the mass of the body m and its volume V. The estimate of the measurand (density) was obtained in Sect. 6.1:

$$\overline{\rho} = 1.294463 \times 10^3 \mathrm{kg/m^3}.$$

Now we will apply the method of enumeration to data processing in this measurement example using the procedure described in Sect. 5.7.

Note that the measurement results of the arguments are analogous to realizations $a_{i,k}$ of the discrete random quantities η_i from Sect. 5.3. The measurements of the mass are analogous to $a_{1,k}$ and of the volume to $a_{2,k}$. Substituting all possible combinations of $a_{i,k}$ and $a_{2,k}$ (for $k = 1,\ldots,11$) we obtain 121 values of density ρ_t similarly to how we obtained values of a_t in Sect. 5.3.

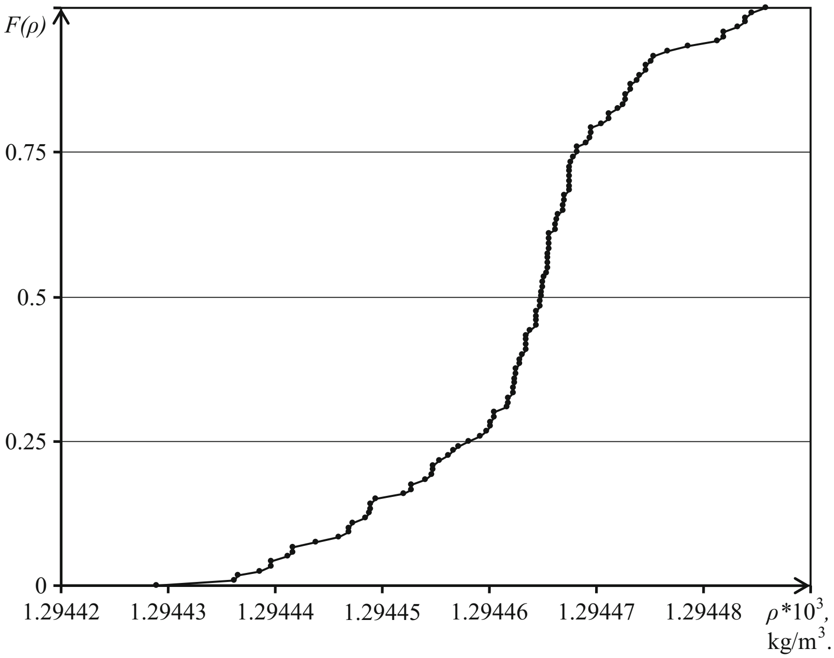

Fig. 8.2 The cumulative distribution function of realizations ρ_t

All 11 realizations of each argument have equal probability hence the probability of each is 1/11, and the probability of each of the obtained values ρ_t is 1/121. Sorting these values in the increasing order and knowing the probability of each value, we construct the stepped and then linear approximation of the cumulative distribution function of the realizations of measurand ρ. The resulting CDF is shown in Fig. 8.2.

To estimate the parameters of this experimental distribution, following the procedure from Sect. 5.7, we generate an independent sample from this distribution of size $K = 1{,}000$ by sequentially going through the probability interval [0,1] with step 0.001 and taking realizations of the measurand corresponding to each probability. From this sample, we obtain the estimates of the parameters of the distribution

$$\overline{\rho}_t = 1.29446299 \times 10^3 \mathrm{kg/m^3} \quad \text{and} \quad S^2(\rho_t) = 1.163 \times 10^{-4} \left(\mathrm{kg/m^3}\right)^2.$$

After rounding-off, the estimate $\overline{\rho}_t$ precisely matches the rounded-off estimate of the measurand obtained in Sect. 6.1.

Having the variance of the distribution, we now find the variance of the estimate of the mean for the sample of size 11:

$$S^2(\overline{\rho}_{11}) = S^2(\rho_t)/11 = 1.057 \times 10^{-5} \left(\text{kg/m}^3\right)^2.$$

Thus, the standard deviation of the mean is

$$S(\overline{\rho}_{11}) = 3.25 \times 10^{-3}\text{kg/m}^3,$$

and in relative form,

$$S_{rel}(\overline{\rho}_{11}) = \frac{S(\overline{\rho}_{11})}{\overline{\rho}_t} = 2.51 \times 10^{-6}.$$

The 0.95 quintile of normal distribution is $z_{0.95} = 1.96$ and thus the uncertainty of the measurement result is

$$u_{0.95,rel}(\overline{\rho}_{11}) = 1.96 \times 2.51 \times 10^{-6} = 4.92 \times 10^{-6}.$$

Uncertainties, as measurement errors, are conventionally written with no more than two significant figures and expressed as percentage. Thus, in the final form we have:

$$u_{0.95,rel}(\overline{\rho}_{11}) = 4.9{*}10^{-6} = 4.9 \times 10^{-4}\%.$$

Uncertainties of input experimental data were computed in Sect. 6.1:

$$u_{0.95,rel}(\bar{m}) = 3.88 \times 10^{-4}\%,$$
$$u_{0.95,rel}(\bar{V}) = 4.64 \times 10^{-4}\%.$$

The obtained uncertainty of measurement result slightly higher than the uncertainty of the experimental data, and this is quite natural. This measurement uncertainty is more than 10% less than the one obtained using the traditional method ($6.0 \times 10^{-4}\%$). We should also stress that the obtained uncertainty reflects all the information contained in the experimental data, unlike the traditional method where residual terms in the Taylor series are neglected.

Systematic errors in this example were negligibly small. In general, they exist and need to be accounted for. The methodology of accounting for these errors in computing the uncertainty of measurement result is given in the example in Sect. 8.5.2.

8.7 Measurement of Ionization Current

Accurate measurements of weak currents, for example, currents generated by γ rays from measurement standards of unit radium mass, are performed by the compensation method using an electrometer. Such currents are measured and compared, for example, in the process of calibration of these standards.

In the compensation method, a high-impedance resistor is inserted into the circuit with the current to be measured. This resistor is also connected in parallel to a capacitor, which is charged prior to being connected. The two connections are arranged so that the measured current and the discharge current from the capacitor flow in the opposite directions. The difference between the two currents creates voltage on the resistor, which is detected by the electrometer. When the electrometer indicator shows zero, the two currents are equal. The time from the start of the capacitor's discharge to when the two currents equalize is measured; this time depends on the dynamics of the capacitor discharge, which is determined by the time constant of the circuit containing the capacitor and resistor. This constant can be determined accurately because both the capacitance of the capacitor and the impedance of the resistor are found a priori with high accuracy. Thus, given the known charge on the capacitor before it is connected to the resistor, one can determine the ionization current by the discharge time until the moment of compensation.

The measured strength of current I is defined by the expression

$$I = CU/\tau,$$

where C is the capacitance of the capacitor used to compensate the ionization current; U is the initial voltage on the capacitor; and τ is the compensation time. As U and τ are dependent, it is a dependent measurement. This measurement equation has the form that is presented in Sect. 5.10. Therefore we know the influence coefficients $l_c = 1$, $l_u = 1$, and $l_c = -1$.

We shall examine the measurement of ionization current on the specific apparatus described in [34]. It employs a capacitor with capacitance C = 4006.3 pF, which is known to be within 0.005% of the above value. The voltage on the capacitor is established with the help of a class 0.1 voltmeter with a measurement range of 0–15 V. The time is measured with a timer whose scale is divided into tenths of a second. The results of a calibration of one standard of radium mass against another using this apparatus are presented in [34]; we will use these results to estimate the accuracy of the measurement of the ionization current involved in the calibration procedure.

The measurement described in [34] included 27 repeated observations. Each time the same indication of the voltmeter U = 7V was established and the compensation time was measured. The results of the 27 observations of time are given in the first column of Table 8.6. Using the measurement equation, we can compute the strength of the ionization current from the compensation time. The 27 values of the current corresponding to the measured compensation times are

Table 8.6 Measurement results and intermediate processing steps in the measurement of ionization current

τ (s)	$I_i \times 10^{-10}$A	$(I_i - \bar{I}) \times 10^{-14}$A	$(I_i - \bar{I})^2 \times 10^{-28}$A^2
74.4	3.7694	7	49
74.6	3.7593	−94	8,836
74.3	3.7745	58	3,364
74.6	3.7593	−94	8,836
74.4	3.7694	7	49
74.4	3.7694	7	49
74.4	3.7694	7	49
74.4	3.7694	7	49
74.4	3.7694	7	49
74.3	3.7745	58	3,364
74.5	3.7643	−44	1,936
74.4	3.7694	7	49
74.5	3.7643	−44	1,936
74.4	3.7694	7	49
74.6	3.7593	−94	8,836
74.2	3.7705	18	324
74.5	3.7643	−44	1,936
74.3	3.7745	58	3,364
74.4	3.7694	7	49
74.4	3.7694	7	49
74.5	3.7643	−44	1,936
74.5	3.7643	−44	1,936
74.3	3.7745	58	3,364
74.3	3.7745	58	3,364
74.3	3.7745	58	3,364
74.4	3.7694	7	49
74.5	3.7643	−44	1,936

listed in column 2 of the table. We now need to obtain the estimate of the result of this measurement and its inaccuracy.

Let us first obtain the estimate of the current. Because ionization currents are weak, one has to account for the so-called background current caused by the background radiation. The average background current is usually equal to $(0.5\text{–}1) \times 10^{-12}$A and can be measured to within 5%. In the measurement in question, the background current was found to be $\bar{I}_b = 0.75 \times 10^{-12}$A. The average value of current observations from Table 8.6 is $\bar{I} = 3.7687 \times 10^{-10}$A. Thus, the estimate the ionization current is

$$\tilde{I} = \bar{I} - \bar{I}_b = 3.7612 \times 10^{-10}\text{A}.$$

Now let us turn to the inaccuracy. First consider the conditionally constant systematic errors. For a class 0.1 voltmeter, its limit of error in indicating the voltage of 7V is $\theta_U = 0.1\% \times (15/7) = 0.21\%$. The limit of error of measuring compensation time with the timer that has the graduations of 0.1s is equal to half the graduation or 0.05 s. In relative form, for the time intervals of 74–75 s, this gives $\theta_\tau = (0.05/74) \times 100 = 0.067\%$. Although the capacitance of the capacitor is supposed to be known within 0.005%, the measurement was performed under rated rather than reference temperature conditions, leading to an additional error. Thus, the capacitance is known only with the limit of error of 0.05%. The limit of measurement error of the background current, which is within 0.5% of the value of the background current, is only 0.013% with respect to the ionization current estimate, and it can obviously be neglected compared to the error in voltage indication θ_U. Turning to formula (5.48) and taking confidence probability $\alpha = 0.95$,

$$\theta_{I,0.95,rel} = k_\alpha \sqrt{\theta_C^2 + \theta_U^2 + \theta_\tau^2} = 1.1\sqrt{0.05^2 + 0.21^2 + 0.067^2} = 0.24\%.$$

Now let us consider random errors. First we shall find an estimate of the standard deviation of the measurement result, which is

$$S(\bar{I}) = S(\bar{I}) = \sqrt{\frac{\sum_{i=1}^{27} (I_i - \bar{I})^2}{27 \times 26}} = 9 \times 10^{-14}\text{A}.$$

It is obvious that the random error can be neglected compared to the limit of the conditionally constant systematic error computed above, which in the absolute form is equal to

$$\theta_{I,0.95} = 3.7612*10^{-10} *0.24*10^{-2} = 0.90 \times 10^{-12}\text{A}.$$

The latter therefore determines the overall inaccuracy of the result. Therefore, our obtained estimate of the ionization current has one extra digit. Rounding it off, we arrive at the result of the measurement:

$$I_{0.95} = (3.761 \pm 0.009) \times 10^{-10}\text{A}.$$

Finally, as a side note, Table 8.6 shows that the random error of an individual observation in this measurement, which could be explained by the inaccuracy of the detection of the moment of the equality of the measured and compensating currents and of the setting of the initial voltage on the capacitor, can reach 0.25% (this can be seen as the deviation of individual observations in Table 8.6, column 2, from the average). However, repeating the measurement 27 times allowed us to reduce the error to the level where it could be neglected compared to the systematic errors.

8.8 Measurement of the Activity of a Radioactive Source

We shall examine the measurement of the activity of a radioactive source by absolute counting of α particles emitted by the source. We will use the experiment described in [15], as well as the measurement data reported there, as the basis for our discussion. The measurement is performed using a detector that counts the particles arriving from the source through a diaphragm opening. The number of particles captured by the detector depends on the geometric configuration of the experimental setup – the diameter of the diaphragm, the distance between the detector and the source, and the diameter of the source (assuming the source is spherical). Following [15], these parameters can be encapsulated into a geometric factor G, which is calculated from the above quantities. Then the measured radioactivity is determined from the formula

$$A = GN_{0\eta},$$

where G is the geometric factor of the apparatus, N_0 is the α-particle counting rate, and η is the α-particle detection efficiency. In the course of the measurement, G does not change, so that errors of G create a systematic error of measurement of the activity A. Measurements of the numbers of α particles, however, have random errors.

To reduce the error arising from the error of the geometric factor, the measurements were performed for different values of this factor (by changing the distance between the source and detector and the diameter of the diaphragm). All measurements were performed using the same source ^{239}Pu.

All the arguments appear in the measurement equation with the same degree of 1. Thus, as discussed in Sect. 5.10 it is convenient to express their errors in relative form since all the influence coefficients will then be equal to 1. Table 8.7 gives measurement results for the five geometric configurations studied. In each case, 50 measurements were performed, and estimates of the measured quantity and their standard deviation, which are also presented in Table 8.7, were calculated. The standard deviations of the (conditionally constant) systematic errors of the results

Table 8.7 The results of measurements of the activity of nuclides using a setup with different geometric factors

Group number j	Source-detector distance (mm)	Diaphragm radius (mm)	Measurand estimate $x_j \times 10^5$	Estimates of standard deviation	
				Random errors (%)	Systematic errors (%)
1	97.500	20.017	1.65197	0.08	0.52
2	97.500	12.502	1.65316	0.10	0.52
3	397.464	30.008	1.66785	0.16	0.22
4	198.000	20.017	1.66562	0.30	0.42
5	198.000	30.008	1.66014	0.08	0.42

were calculated from the estimated limiting values of all error components under the assumption that they can be regarded as centered uniformly distributed random quantities.

The data in Table 8.7 show, first of all, that the systematic errors are much larger than the random errors, so that the number of measurements in the groups is sufficient. The observed difference between the obtained values of the activity of the nuclides in the groups can be mostly explained by their different systematic errors.

In the example studied, the same quantity was measured in all cases. Therefore, one can use the weighted mean as the overall estimate of the measurand. Based on the considerations from Sect. 7.5, we shall use (7.13) to calculate the weights. First, we shall calculate an estimate of the combined variance according to (7.12):

$$S^2(\bar{x}_j) = S^2_{\psi}(\bar{x}_j) + S^2_{\vartheta}(\bar{x}_j).$$

The results of the calculations are given in Table 8.8. As an example, we provide the calculation details of weight g_1:

$$g_1 = \frac{\frac{1}{0.28}}{\frac{1}{0.28} + \frac{1}{0.28} + \frac{1}{0.07} + \frac{1}{0.27} + \frac{1}{0.18}} = \frac{3.57}{30.7} = 0.12.$$

Now we find the weighted mean:

$$\tilde{A} = \bar{\bar{x}} = \sum_{j=1}^{5} g_j \bar{x}_j = 1.6625 \times 10^5.$$

Using estimates $S^2(\bar{x}_j)$ from Table 8.8 in accordance with (7.6), we obtain

$$S^2(\tilde{A}) = 0.33(\%)^2 \text{ and } S(\tilde{A}) = 0.182\%.$$

We can now estimate the uncertainty of the measurement result. To do this, we need to find, using (7.13), the standard deviations of the random and conditionally constant systematic components of the weighted mean and then, since $S(\tilde{A})$ has

Table 8.8 The estimate of combined variances and weights of measurement results in different groups

Group number j	Estimate of combined variance $S^2(\bar{x}_j)$ $(\%)^2$	Weight g_i
1	0.28	0.12
2	0.28	0.12
3	0.07	0.46
4	0.27	0.12
5	0.18	0.18

already been found, calculate t_c from (4.22). All data for these calculations are available in Tables 8.7 and 8.8.

The standard deviations of the random and systematic components of the weighted mean are as follows:

$$S_{\psi}^2(\bar{\bar{x}}) = \sum_{j=1}^{L} g_j^2 S_{\psi}^2(\overline{x_j}) = 71.58 \times 10^{-8} \text{ and } S_{\psi}(\bar{\bar{x}}) = 8.46 \times 10^{-4}$$

$$S_{\theta}^2(\bar{\bar{x}}) = \sum_{j=1}^{L} g_j^2 S_{\theta}^2(\overline{x_j}) = 261.7 \times 10^{-8} \text{ and } S_{\theta}(\bar{\bar{x}}) = 16.2 \times 10^{-4}$$

Next, we compute the uncertainty of the systematic component, θ_α. The easiest way to do it is by using (4.3). For this, however, we need to transfer from the standard deviations of the elementary systematic errors back to their limits, which as we know can be done using factor $\sqrt{3}$ (since $S^2 = \theta^2/3$). Thus,

$$\theta_\alpha = k_\alpha \sqrt{3 \sum_{j=1}^{L} g_j^2 S_{\vartheta}^2(\bar{x}_j)} = k_\alpha \sqrt{3 S_{\vartheta}^2(\bar{\bar{x}})}$$

Taking $\alpha = 0.95$, we have $k_\alpha = 1.1$ and $\theta_{0.95} = 1.1 \times 1.73 \times S_{\vartheta}(\bar{\bar{x}}) = 1.90 S_{\vartheta}(\bar{\bar{x}})$. From here, we obtain $t_{\vartheta} = \theta_{0.95}/S_{\vartheta}(\bar{\bar{x}}) = 1.90$. To find quantile t_q of Student's distribution for the selected confidence probability, we also need the degree of freedom. In general, when the measurement result represents a weighted mean of several measurements, the degree of freedom is obtained from (5.23) as an effective degree of freedom. In our case, however, we have five groups, each comprising a large number of observations ($n = 50$ in each group), so it is obvious even without calculations that the resulting distribution can be considered normal. Then, $t_q = z_{\frac{(1+\alpha)}{2}} = 1.96$.

We can now use formula (4.22) to find t_c:

$$t_c = \frac{t_{\vartheta} S_{\vartheta}(\bar{\bar{x}}) + t_q S_{\psi}(\bar{\bar{x}})}{S_{\vartheta}(\bar{\bar{x}}) + S_{\psi}(\bar{\bar{x}})} = 1.92..$$

Finally, we are ready to compute the uncertainty of the measurement result:

$$u_c = t_c S(\bar{\bar{x}}) = 1.92 \times 0.182 = 0.35\%.$$

In the form of absolute uncertainty, we obtain $u_{0.95} = 0.006 \times 10^5$. Thus, the result of the measurement can be given as follows:

$$\tilde{A}_{0.95} = (1.662 \pm 0.006) \times 10^5.$$