Chapter 7
Combining the Results of Measurements

7.1 Introductory Remarks

Measurements of the same quantity are often performed in different laboratories and, therefore, under different conditions and by different methods. Sometimes there arises the problem of combining these measurement data to find the most accurate estimate of the measured quantity.

In many cases, in the investigation of new phenomena, measurements of the quantities involved take a great deal of time. By grouping measurements performed over a limited time, intermediate estimates of the measurand can be obtained in the course of the measurements. It is natural to find the final result of a measurement by combining the intermediate results.

These examples show that the problem of combining the results of measurements is of great significance for metrology. At the same time, it is important to distinguish situations in which one is justified in combining results from those in which one is not justified in doing so. It is pointless to combine results of measurements of quantities that in their essence have different magnitude.

We should note that when comparing results of measurements, the data analysis is often performed based on the intuition of the experimenters without using formalized procedures. It is interesting that in the process, as a rule, the correct conclusions are drawn. On the one hand, this indicates that modern measuring instruments are of high quality and on the other hand that the experimenters, who by estimating the errors determine all sources of error, are usually highly qualified.

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7.2 Theoretical Principles

The following problem has a mathematically rigorous solution. Consider L groups of measurements of the same quantity A. Estimates of the measurand $\bar{x}_1, \ldots, \bar{x}_L$ were made from the measurements of each group, and

$$E[\bar{x}_1] = \cdots = E[\bar{x}_L] = A.$$

The variances of the measurements in each group $\sigma_1^2, \ldots, \sigma_L^2$ and the number of measurements in each group $n_1, \ldots, n_L$ are known. The problem is to find an estimate of the measured quantity based on data from all groups of measurements. This estimate is denoted as $\bar{\bar{x}}$ and is called the *combined average*. Because the combined average is commonly obtained as a linear combination of group averages, it is often referred to as the *weighted mean*.

We shall seek $\bar{\bar{x}}$ as a linear combination of $\{\bar{x}_j\}$, that is, as their weighted mean:

$$\bar{\bar{x}} = \sum_{j=1}^{L} g_j x_j. \tag{7.1}$$

Therefore, the problem reduces to finding the weights g_j. As $E[\bar{x}_j] = A$ for all j, and we obviously want $E[\bar{\bar{x}}] = A$, we obtain from (7.1)

$$E[\bar{\bar{x}}] = E\left[\sum_{j=1}^{L} g_j \bar{x}_j\right] = \sum_{j=1}^{L} g_j E[\bar{x}_j], \quad A = A\sum_{j=1}^{L} g_j.$$

Therefore,

$$\sum_{j=1}^{L} g_j = 1 \tag{7.2}$$

Next, we require that $\bar{\bar{x}}$ be an efficient estimate of A; that is, $V[\bar{\bar{x}}]$ must be minimum. $V[\bar{\bar{x}}]$ can be found using the formula

$$\begin{aligned} V[\bar{\bar{x}}] &= V\left[\sum_{j=1}^{L} g_j \bar{x}_j\right] = \sum_{j=1}^{L} g_j^2 V[\bar{x}_j] \\ &= g_1^2\sigma^2(\bar{x}_1) + g_2^2\sigma^2(\bar{x}_2) + \cdots + g_L^2\sigma^2(\bar{x}_L). \end{aligned} \tag{7.3}$$

We shall now find the weights g_j under which $V[\bar{\bar{x}}]$ reaches a minimum. Using the condition (7.2), we substitute $g_L = 1 - g_1 - g_2 - \cdots - g_{L-1}$ into (7.3), and then differentiate the resulting expression with respect to each g_j and equate each derivative to 0:

$$2g_1\sigma^2(\bar{x}_1) - 2(1 - g_1 - g_2 - \cdots - g_{L-1})\sigma^2(\bar{x}_L) = 0,$$
$$2g_2\sigma^2(\bar{x}_2) - 2(1 - g_1 - g_2 - \cdots - g_{L-1})\sigma^2(\bar{x}_L) = 0,$$
$$\cdots$$
$$2g_{L-1}\sigma^2(\bar{x}_{L-1}) - 2(1 - g_1 - g_2 - \cdots - g_{L-1})\sigma^2(\bar{x}_L) = 0,$$

As the second term is identical in each equation, we obtain

$$g_1\sigma^2(\bar{x}_1) = g_2\sigma^2(\bar{x}_2) = \cdots = g_{L-1}\sigma^2(\bar{x}_{L-1}).$$

Furthermore, if instead of g_L we eliminated another weighting coefficient from (7.3), we would have included the similar term with g_L into the above relation. Thus, we arrive at the following condition:

$$g_1\sigma^2(\bar{x}_1) = g_2\sigma^2(\bar{x}_2) = \cdots = g_L\sigma^2(\bar{x}_L),$$

or equivalently,

$$g_1 : g_2 : \cdots : g_L = \frac{1}{\sigma^2(\bar{x}_1)} : \frac{1}{\sigma^2(\bar{x}_2)} : \cdots : \frac{1}{\sigma^2(\bar{x}_L)}. \tag{7.4}$$

The relations (7.2) and (7.4) represent two conditions for the weights to compute the combined average. To find weight g_j, it is necessary to know either the variances of the arithmetic means or the ratio of the variances. If we have the variances $\sigma^2(\bar{x}_1)$ then we can set $g'_j = 1/\sigma^2(\bar{x}_1)$ We then obtain

$$g_j = \frac{g'_j}{\sum_{j=1}^{L} g'_j} \tag{7.5}$$

As the weights are nonrandom quantities, it is not difficult to determine the variance for $\bar{\bar{x}}$. According to relation (7.3), we have

$$V[\bar{\bar{x}}] = \sum_{j=1}^{L} g_j^2 V[\bar{x}_j] = \frac{\sum_{j=1}^{L} \left(g'_j\right)^2 V[\bar{x}_j]}{\left(\sum_{j=1}^{L} g'_j\right)^2} = \frac{\sum_{j=1}^{L} \left(\frac{1}{\sigma^2(\bar{x}_j)}\right)^2 \sigma^2(\bar{x}_j)}{\left(\sum_{j=1}^{L} \frac{1}{\sigma^2(\bar{x}_j)}\right)^2} = \frac{1}{\sum_{j=1}^{L} \frac{1}{\sigma^2(\bar{x}_j)}} \tag{7.6}$$

Let us now consider an important particular case when the variances (7.6) of the measurements are the same for all groups, although their estimates might still be different because the number of observations in the groups may be different. In this case, one can combine the measurements of all groups into one large group of measurements. The number of measurements in the combined group is $N = \sum_{j=1}^{L} n_j$ and the combined average will be

$$\bar{\bar{x}} = \frac{\sum\limits_{j=1}^{L} \sum\limits_{i=1}^{n_j} x_{ji}}{N}. \tag{7.7}$$

Expanding the numerator gives

$$\begin{aligned}\bar{\bar{x}} &= \frac{(x_{11} + x_{12} + \cdots + x_{1n_1}) + (x_{21} + x_{22} + \cdots + x_{2n_2}) + \cdots}{N} \\ &= \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \cdots + n_L\bar{x}_L}{N} = \sum_{j=1}^{L} g_j\bar{x}_j,\end{aligned}$$

where g_j is the weight of the j th arithmetic mean:

$$g_j = n_j/N \tag{7.8}$$

The variance of the weighted mean in this case (i.e., when measurement results in each group have equal variances) can be estimated by considering the weighted mean as the average of the combined group of all the measurements:

$$S^2(\bar{\bar{x}}) = \frac{\sum\limits_{k=1}^{N} (x_k - \bar{\bar{x}})^2}{N(N-1)}.$$

We gather the terms in the numerator by groups

$$S^2(\bar{\bar{x}}) = \frac{\sum\limits_{j=1}^{L} \sum\limits_{i=1}^{n_j} \left(x_{ij} - \bar{\bar{x}}\right)^2}{N(N-1)}.$$

and perform simple transformations of the numerator to simplify the calculations:

$$\begin{aligned}\sum_{j=1}^{L}\sum_{i=1}^{n_j}\left(x_{ji} - \bar{\bar{x}}\right)^2 &= \sum_{j=1}^{L}\sum_{i=1}^{n_j}\left(x_{ji} - \bar{x}_j + \bar{x}_j - \bar{\bar{x}}\right)^2 \\ &= \sum_{j=1}^{L}\sum_{i=1}^{n_j}\left(x_{ji} - \bar{\bar{x}}\right)^2 + 2\sum_{j=1}^{L}\sum_{i=1}^{n_j}\left(x_{ji} - \bar{x}_j\right)\left(\bar{x}_j - \bar{\bar{x}}\right) + \sum_{j=1}^{L}\sum_{i=1}^{n_j}\left(\bar{x}_j - \bar{\bar{x}}\right)^2.\end{aligned}$$

The second term in the last expression is equal to zero because, by virtue of the properties of the arithmetic mean, $\sum\limits_{i=1}^{n_j} \left(x_{ji} - \bar{x}_j\right) = 0$. For this reason,

$$S^2(\bar{\bar{x}}) = \frac{1}{N(N-1)}\left(\sum_{j=1}^{L}\sum_{i=1}^{n_j}\left(x_{ij}-\bar{x}\right)^2 + \sum_{j=1}^{L}\sum_{i=1}^{n_j}\left(\bar{x}_j-\bar{\bar{x}}\right)^2\right)$$

Note that

$$\sum_{i=1}^{n_j}\left(x_{ji}-\bar{x}_j\right)^2 = n_j\left(n_j-1\right)S^2\left(\bar{x}_j\right),$$

where $S^2\left(\bar{x}_j\right)$ is the estimate of the variance of arithmetic mean of the j th group, or, equivalently, $S^2\left(\bar{x}_j\right) = \frac{1}{n_j\left(n_j-1\right)}\sum_{i=1}^{n_j}\left(x_{ij}-\bar{x}_j\right)^2$.

Further,

$$\sum_{j=1}^{n_j}\left(\bar{x}_j-\bar{\bar{x}}\right)^2 = n_j\left(\bar{x}_j-\bar{\bar{x}}\right)^2$$

Thus, we obtain

$$S^2(\bar{\bar{x}}) = \frac{1}{N(N-1)}\left[\sum_{j=1}^{L}\left(n_j-1\right)n_jS^2\left(\bar{x}_j\right) + \sum_{j=1}^{L}n_j\left(\bar{x}_j-\bar{\bar{x}}\right)^2\right]. \tag{7.9}$$

Equation (7.9) can be expressed differently. Moving N in the denominator inside the square brackets, we have

$$S^2(\bar{\bar{x}}) = \frac{1}{N-1}\left[\sum_{j=1}^{L}\left(n_j-1\right)\frac{n_j}{N}S^2\left(\bar{x}_j\right) + \sum_{j=1}^{L}\frac{n_j}{N}\left(\bar{x}_j-\bar{\bar{x}}\right)^2\right].$$

Finally, using (7.8), we obtain:

$$S^2(\bar{\bar{x}}) = \frac{1}{N-1}\left[\sum_{j=1}^{L}g_j\left(n_j-1\right)S^2\left(\bar{x}_j\right) + \sum_{j=1}^{L}g_j\left(\bar{x}_j-\bar{\bar{x}}\right)^2\right]. \tag{7.10}$$

The first term in the above formula characterizes the spread in the measurements within groups, and the second term characterizes the spread of the arithmetic means of the groups.

7.3 Effect of the Error of the Weights on the Error of the Weighted Mean

Looking at (7.1) determining the weighted mean, one would think that, because the weights g_j and the weighted values of $\bar{x}_j$ appear in it symmetrically, they must be found with the same accuracy. In practice, however, the weights are usually expressed by numbers with one or two significant figures. How is the uncertainty of the weights reflected in the error of the weighted mean?

We shall consider weights g_j in (7.1) to be fixed, constant values. In addition, as usual, we shall assume that the weights add up to one [that is, condition (7.2) holds]. This condition is also satisfied for the inaccurately determined weight estimates, that is, for $\tilde{g}_j$. Therefore,

$$\sum_{j=1}^{L} \Delta g_j = 0,$$

where Δg_j is the error in determining the weight g_j.

Assuming that the exact value of the weighted mean is y, we estimate the error of its estimate:

$$\Delta y = \sum_{j=1}^{L} \tilde{g}_j \bar{x}_j - \sum_{j=1}^{L} g_j \bar{x}_j = \sum_{j=1}^{L} \Delta g_j \bar{x}_j.$$

We shall express Δg_1 with the other errors:

$$\Delta g_1 = -(\Delta g_2 + \cdots + \Delta g_L)$$

and substitute it into the expression for Δy:

$$\Delta y = (\bar{x}_2 - \bar{x}_1)\Delta g_2 + (\bar{x}_3 - \bar{x}_1)\Delta g_3 + \cdots + (\bar{x}_L - \bar{x}_1)\Delta g_L$$

or in the form of relative error

$$\frac{\Delta y}{y} = \frac{g_2(\bar{x}_2 - \bar{x}_1)\frac{\Delta g_2}{g_2} + \cdots + g_L(\bar{x}_L - \bar{x}_1)\frac{\Delta g_L}{g_L}}{\sum_{j=1}^{L} g_j \bar{x}_j}.$$

The errors of the weights $\Delta g_j/g_j$ are unknown. But let us assume that we can estimate their limits, and let $\Delta g/g$ be the largest absolute value of these limits. Replacing all relative errors $\Delta g_j/g_j$ with $\Delta g/g$, we obtain the upper limit of the relative error of the weighted mean:

$$\frac{\Delta y}{y} \leq \frac{\Delta_g}{g} \left(\frac{[|g_2(\bar{x}_2 - \bar{x}_1) + g_3(\bar{x}_3 - \bar{x}_1) + \cdots + g_L(\bar{x}_L - \bar{x}_1)|]}{\sum\limits_{j=1}^{L} g_j \bar{x}_j} \right).$$

The numerator on the right-hand side of the inequality can be put into the following form:

$$\begin{aligned} &g_2(\bar{x}_2 - \bar{x}_1) + g_3(\bar{x}_3 - \bar{x}_1) + \cdots + g_L(\bar{x}_L - \bar{x}_1) \\ &\quad = g_2\bar{x}_2 + g_3\bar{x}_3 + \cdots + g_L\bar{x}_L - (g_2 + g_3 + \cdots + g_L)\bar{x}_1. \end{aligned}$$

But $g_2 + g_3 + \cdots + g_L = 1 - g_1$, so that

$$g_2(\bar{x}_2 - \bar{x}_1) + g_3(\bar{x}_3 - \bar{x}_1) + \cdots + g_L(\bar{x}_L - \bar{x}_1) = \sum_{j=1}^{L} g_j\bar{x}_j - \bar{x}_1 = y - \bar{x}_1.$$

Thus,

$$\frac{\Delta y}{y} \leq \frac{\Delta g}{g} \frac{|y - \bar{x}_1|}{y}.$$

It is obvious that if the entire derivation is repeated, but in so doing the error not in the coefficient g_1 but in some other weight is eliminated, then a weighted value other than $\bar{x}_1$ will appear on the right-hand side of the inequality. Therefore, the above inequality holds for every $\bar{x}_j$; the obtained result can be represented in the form

$$\frac{\Delta\bar{\bar{x}}}{\bar{\bar{x}}} \leq \frac{\Delta g}{g} \frac{|\bar{\bar{x}} - \bar{x}_j|}{\bar{\bar{x}}}.$$

This inequality shows that the error introduced into the weighted mean as a result of the error of the weights is many times smaller than the error of the weights itself. The cofactor $|\bar{\bar{x}} - \bar{x}_j|/\bar{\bar{x}}$ can be assumed to be of the same order of magnitude as the relative error of the measurement results $\bar{x}_j$ produced by each group. Thus, if this error is of the order of 0.01, then the error introduced into the weighted mean as a result of the error of the weights will be at least 100 times smaller than the latter.

7.4 Combining the Results of Measurements with Predominately Random Errors

We shall now study a scenario of combining measurement results where measurements in each group have negligibly small systematic errors. Each result being combined in this case is usually the arithmetic mean of the measurements in the corresponding group, and the differences between them are explained by the random spread of the averages of the groups.

Before attempting to combine these results, one must verify that the same quantity is measured in each case and there are no systematic shifts between the measurement results produced by each group. This verification is equivalent to checking that the true value of the measured quantity is the same for all groups and is accomplished by the methods presented in Chap. 3.

It is important to note that this verification can fail for two reasons: different quantities could have been measured in different groups or there are systematic shifts between the means of the groups. In the former case, it is pointless to combine the measurements. In the latter case the measurements can still be combined but with the help of another method, which we will discuss in the next section. The distinction between these two causes of verification failure must be clear from the physical essence of the measurement and its purpose; one cannot draw this distinction from statistical methods.

Only if the data pass the above verification can we combine the measurements by applying the approach from Sect. 7.2. Indeed, the absence or negligible size of the systematic errors is a necessary condition for the validity of this approach. One may notice that our verification only checks for the absence of the systematic shift between the groups, not the absence of the systematic errors themselves. This is inevitable; if measurements in all the groups have the same systematic error, this error is impossible to detect with statistical methods and it will also be present in the combined measurement result. Fortunately, this situation rarely occurs in practice. Recall that different groups of measurements are typically collected in different laboratories. Any systematic error that is so pervasive that it is the same across all the laboratories is likely to have been eliminated during calibration of the instruments involved.

The theory of calculating the weighted mean of several groups of measurements that we considered in Sect. 7.2 assumes that the variance of the measurement results in each group is known. However, the experimental data only allow one to obtain the estimates of these variances. Thus, one has to use the estimates in places of true variances throughout the calculations. In particular, the variance estimate of the weighted mean is computed by the following formula, modified from (7.6):

$$S^2(\bar{\bar{x}}) = \frac{1}{\sum_{j=1}^{L} \frac{1}{S^2(\bar{x}_j)}}. \tag{7.11}$$

In the case of equal variances in all the groups, (7.9) and (7.10) already contain estimates of the group variance, and so these formulas can be used directly. Note that one can check if the estimates of the variances of measurement groups are the estimates of the same variance using the methods from Chap. 3.

Given this variance estimate, the uncertainty of the weighted mean can be calculated by considering the combination of the group averages as a linear indirect measurement and thus by applying (5.23) to calculate the effective degrees of freedom.

Example 7.1 The mass of some body is being measured. In one experiment, the value $\tilde{m}_1 = 409.52\,\mathrm{g}$ is obtained as the arithmetic mean of $n_1 = 15$ measurements. The variance of the group of measurements is estimated to be $S_1^2 = 0.1\,\mathrm{g}^2$. In a different experiment, the value $\tilde{m}_2 = 409.44\,\mathrm{g}$ was obtained with $n_2 = 10$ and $S_2^2 = 0.03\,\mathrm{g}^2$. It is known that the systematic errors of the measurements are negligibly small, and the measurement results in each experiment can be assumed normally distributed. It is necessary to estimate the mass of the body and the variance of the result using data from both experiments.

We shall first determine whether the unification is justified, that is, whether an inadmissible difference exists between the estimates of the measured quantity in each group. Following the method described in Sect. 3.9,

$$\begin{aligned}
S^2(\bar{x}_1) = \frac{S_1^2}{n_1} = \frac{0.1}{15} = 0.0067,\quad S^2(\bar{x}_2) = \frac{0.03}{10} = 0.003,\\
S^2(\bar{x}_1 - \bar{x}_2) = S^2(\bar{x}_1) + S^2(\bar{x}_2) = 0.0097,\\
S(\bar{x}_1 - \bar{x}_2) = 0.098,\\
\bar{x}_1 - \bar{x}_2 = \tilde{m}_1 - \tilde{m}_2 = 0.08.
\end{aligned}$$

Assuming that the confidence probability $\alpha = 0.95$, Table A.1 gives $z_{\frac{1+\alpha}{2}} = 1.96$. Then, $z_{\frac{1+\alpha}{2}} S(\bar{x}_1 - \bar{x}_2) = 1.96 \times 0.098 = 0.19$. As $0.08 < 0.19$, the unification is possible.

To decide if we can use the simpler method based on (7.8, 7.9, and 7.10), we shall check whether both groups of observations have the same variance. We do so using Fisher's test from Sect. 3.9. We compute:

$$F = S_1^2 / S_2^2 = 0.1 : 0.03 = 3.3.$$

The degrees of freedom are $v_1 = 14$ and $v_2 = 9$. We shall assume the significance level of 2%. Then, $q = 0.01$ and $F_q = 5$ (see Table A.5). As $F < F_q$, it can be assumed that the variances of the groups are equal.

We shall now find the weights of the arithmetic means. According to (7.8), we have $g_1 = 15/25 = 0.6$ and $g_2 = 10/25 = 0.4$. The weighted mean is $\bar{\bar{m}} = 0.6 \times 409.52 + 0.4 \times 409.44 = 409.49\mathrm{g}$. Next we find $S(\bar{\bar{m}})$. In accordance with (7.9), we have

$$S^2(\bar{\bar{m}}) = \frac{1}{25 \times 24}\left(14 \times 0.1 + 9 \times 0.03^2 + 15 \times 0.03^2 + 10 \times 0.05^2\right)$$
$$= 28 \times 10^{-4} g^2,$$
$$S^2(\bar{\bar{m}}) = 5.3 \times 10^{-2} g.$$

Having found the variance of the combined result, we can now calculate its uncertainty using Student's distribution with the effective degrees of freedom obtained from (5.23).

7.5 Combining the Results of Measurements Containing Both Systematic and Random Errors

In a general case, measurements within groups have not just random but also systematic error. The latter is typically a conditionally constant error or a sum of several conditionally constant errors. However, occasionally one may encounter absolutely constant systematic errors, such as methodological errors, as well. Let us start with considering measurements that do not have absolutely constant systematic errors.

Let us assume again that a quantity A is measured in L laboratories. Each laboratory produces the result $\bar{x}_j$ with error ς_j $(j = 1, \ldots, L)$:

$$\bar{x}_j = A + \varsigma_j.$$

The error ς_j is the sum of the conditionally constant error ϑ_j and random error ψ_j errors: $\varsigma_j = \vartheta_j + \psi_j$. As discussed in Chap. 4 (Sect. 4.3), the conditionally constant error is modeled as a uniformly distributed random quantity with limits θ_j, which are estimated analytically from the specifications of the instruments and measurement conditions: $|\vartheta_j| \leq \theta_j$ We will assume that the mathematical expectation of this error is zero: $E[\vartheta_j] = 0$ We will also assume that θ_j is symmetrical about $\bar{x}_j$. Occasionally, one can encounter cases of asymmetrical limits; the methodology of handling this asymmetry is given in Chap. 4.

The random error ψ_j is assumed to be a centered quantity; that is, $E[\psi_j] = 0$. Thus, when there are no absolutely constant errors, we have $E\left[\bar{x}_j\right] = A$.

To allow the unification of measurement results, each laboratory must report the result itself, $\bar{x}_j$, along with the estimates of the variance of this result that is due to the random error, $S^2(\psi_j)$ and the limit of the conditionally constant systematic error θ_j The former is calculated in the normal way:

$$S^2\left(\psi_j\right) = \frac{\sum\limits_{i=1}^{n_j}\left(x_{ij} - \bar{x}_j\right)^2}{n_j\left(n_j - 1\right)}.$$

The latter is equivalent to providing an estimate of the variance of this error, $S^2(\vartheta_j)$ since $S^2(\vartheta_j) = \theta_j^2/3$.

Similar to the case without systematic errors considered in Sect. 7.4, we will follow the theory of combining the results of measurements using the weighted mean while replacing variances with their estimates. As shown in Sect. 4.9, the estimate of the combined variance of the measurement result $\bar{x}_j$ is

$$S^2(\bar{x}_j) = S^2(\vartheta_j) + S^2(\psi_j). \tag{7.12}$$

Now, the weights of the results being combined can be derived from (7.2) and (7.4) by substituting the variances appearing in these relations with the estimates of these variances:

$$g_j = \frac{\frac{1}{S^2(\vartheta_j)+S^2(\psi_j)}}{\sum\limits_{j=1}^{L}\frac{1}{S^2(\vartheta_j)+S^2(\psi_j)}} \tag{7.13}$$

Knowing the weights, we can calculate the estimate of the combined result as the weighted mean of the results from each lab.

We shall now estimate the uncertainty of the weighted mean. In solving this problem, because the errors of the weights are insignificant (see Sect.7.3), we shall assume that the weights of the combined measurement results are exact. A necessary prerequisite to find the uncertainty is to estimate the standard deviation. In principle, we accomplish this by replacing variances in (7.5) with their estimates from (7.12). However, for subsequent calculations we will need the components of the combined standard deviation contributed by the random and conditionally constant systematic errors, denoted respectively as $S_\psi(\bar{\bar{x}})$ and $S_\vartheta(\bar{\bar{x}})$. Thus, we will compute these components and then obtain the overall standard deviation by combining these components rather than from (7.5) and (7.12).

Following the calculation procedure of Sect. 4.8, and taking into account the weights, $S_\psi(\bar{\bar{x}})$ and $S_\vartheta(\bar{\bar{x}})$ are computed as follows:

$$\begin{aligned} S_\psi(\bar{\bar{x}}) &= \sqrt{\sum_{j=1}^{L} g_j^2 S^2(\psi_j)} \\ S_\vartheta(\bar{\bar{x}}) &= \sqrt{\sum_{j=1}^{L} g_j^2 S^2(\vartheta_j)} \end{aligned} \tag{7.14}$$

Now we can find the combined standard deviation of the weighted mean:

$$S(\bar{\bar{x}}) = \sqrt{S_\psi^2(\bar{\bar{x}}) + S_\vartheta^2(\bar{\bar{x}})}. \tag{7.15}$$

To move from the combined standard deviation to the uncertainty of the weighted mean, according to (4.20), we must obtain coefficient t_c. This coefficient can be found from (4.22), which requires the coefficient t_ϑ for the systematic component of error and the quantile t_q of Student's distribution for the random component. To find t_ϑ we must first calculate the uncertainty of the systematic component. The easiest way to do it is by using (4.3) with weights:

$$u_\vartheta(\bar{\bar{x}}) = k_\alpha \sqrt{\sum_{j=1}^{L} g_j^2 \theta_j^2}.$$

Coefficient k_α is determined by the desired confidence probability and is found from Table 4.1. Now we can find t_ϑ according to (4.21):

$$t_\vartheta = \frac{u_\vartheta(\bar{\bar{x}})}{S_\vartheta(\bar{\bar{x}})}.$$

Quantile t_q of Student's distribution can be found given the effective degrees of freedom using (5.23), which in this case obtains the form:

$$\nu_{\mathrm{eff}} = \frac{\left[\sum_{j=1}^{L} g_j^2 S^2(\psi_j)\right]^2}{\sum_{j=1}^{L} \left(g_j^4 S^4(\psi_j)/\nu_j\right)}$$

where $\nu_j = n_j - 1$. Note that both t_ϑ and t_q must be obtained for the same confidence probability.

Now we can apply (4.22) to compute coefficient t_c

$$t_c = \frac{t_q S_\psi(\bar{\bar{x}}) + t_\vartheta S_\vartheta(\bar{\bar{x}})}{S_\psi(\bar{\bar{x}}) + S_\vartheta(\bar{\bar{x}})}$$

and, finally, obtain the uncertainty of the weighted mean:

$$U_c = t_c S(\bar{\bar{x}}).$$

We should say a few words on the potential presence of absolutely constant systematic error. If among the groups being combined there is a group with such error, then the limit of this error must be re-calculated by taking into account the weight of this group. For instance if the only group with such error is group number 2 and its absolutely constant error is H_2 then the absolutely constant error of the weighted mean will be $H(\bar{\bar{x}}) = g_2 H_2$. If more than one group has such errors, their respective limits (again recalculated according to their groups' weights) are summed up arithmetically as in direct and indirect measurements. Then, the resulting limit is again summed up arithmetically with the confidence limit of the weighted mean computed using the methodology described here.

An example of a measurement where a weighted mean is used as the estimate of the measurand is a precise measurement of the activity of a source of alpha particles. A detailed treatment of this example is given in Chap. 8 (Sect. 8.8).

As a final note, when the results of measurements must be combined, it is always necessary to check the agreement between the starting data and the obtained result. If some contradiction is discovered, for example, the combined average falls outside the permissible limits of error of some group, then the reason for this must be determined and the contradiction must be eliminated. Sometimes this is difficult to do and may require special experiments. Great care must be exercised in combining the results of measurements because in this case information about the errors is employed to refine the result of the measurement and not to characterize its uncertainty, as is usually done.