The Origin of the Notion of Manifold: From Riemann's Habilitationsvortrag Onward

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1 Introduction

It goes without saying that the notion of manifold is a very important foundation of modern geometry, or even of modern mathematics in general. The invention of this notion is usually attributed to Riemann. In fact, the term "Mannigfaltigkeit", of which the word "manifold" is an English translation, appeared for the first time in the world of mathematics in Riemann's famous Habilitationsvortrag. There are other English translations such as "multiplicity" or "variety" in the literature. Prior to Riemann, the word "Mannigfaltigkeit" was already used in a non-mathematical context. There is even a poem by Schiller entitled "Mannigfaltigkeit." Still, if we read the text of Riemann today, we find that his description of manifolds is rather literary, and there is no clear, rigorous definition. On the other hand, his text is very suggestive and has fertile content. This is one of the reasons why his text inspired many mathematicians, and also philosophers. Even today, we can learn something new reading his text although it is not so easy to reach its depth.

New notions in mathematics quite often take much time for their maturation and consolidation. The notion of manifold is one such example, and it took nearly half a century from the first invention by Riemann to get to the modern definition we know today. The purpose of this article is to give an overview of this slow process where the notion of manifold was born, clarified and developed, taking a look at both mathematical and philosophical aspects. We shall start with the philosophical background of the time when Riemann's Habilitationsvortrag was written, focusing on Kant's metaphysics. Then we shall turn to Riemann's paper, showing how he described manifolds there and quoting some passages from his text directly. We

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shall also see how Riemann's worldview diverges from the Kantian. It was Poincaré who put for the first time Riemann's idea into clear mathematical terms. We shall look at Poincaré's celebrated paper "Analysis Situs", where he gave two definitions of manifold, and also at his philosophical position, conventionalism. Finally, we shall see the process in which the notion of manifold became mathematically rigorous, and reached the notion we understand today. There are many mathematicians involved in this process, but we just pick up prominent ones among them, looking at the work of Hilbert, Weyl, Kneser, Veblen-Whitehead and Whitney.

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2 Kantian Worldview

At the time when Riemann gave birth to the notion of manifold in the world, or before, what should be called a Kantian worldview was prevailing. This view was widely held by scientists and mathematicians. To understand the philosophical aspects of the papers of Riemann and Poincaré, it is very helpful first to take a glance at Immanuel Kant's work.

It is well known that Kant tried to give philosophical foundations to mathematics and the natural science. His position was quite different from naïve empiricism or even a "sceptical empiricism" à la David Hume, and also from continental rationalism. If we allow ourselves to use modern jargon, his work should be regarded as a trial to "aufheben" both.

He supposed that there are a priori things in human thinking which make natural science possible. In the Kritik der reinen Vernunft [7] (we use the abbreviation KrV from now on), he included space, which should be understood to be a three-dimensional Euclidean space or as subset of it, or the human recognition of space, into the category of 'synthetic a priori.' (Here the qualifier "synthetic" is opposed to "analytic", the latter referring to a tautological kind of propositions.) In spite of his non-empiricist view, in contrast to conventionalism which was to become preponderant afterwards, Kant regards space as neither a human invention nor a framework which we created, but something given to us from the beginning and preceding all other form of recognition and knowledge.

Let us take a look at how Kant describes "space" in the first chapter of KrV. For English translations of all quotations from KrV, we use the Cambridge version [8].

> Der Raum ist kein empirischer Begriff, der von äußeren Erfahrungen abgezogen worden.

> Space is not an empirical concept that has been drawn from outer experiences.

This is an outright negation of the empiricists' view that our notion of space and its perception derive from our experiences and observations. Moreover, he regarded space as a foundation of all our perceptions.

> Der Raum ist eine notwendige Vorstellung a priori, die allen äußeren Anschauungen zum Grunde liegt.
>
> Space is a necessary representation, a priori, which is the ground of all outer intuitions.
>
> Der Raum ist kein diskursiver oder, wie man sagt, allgemeiner Begriff von Verhältnissen der Dinge überhaupt sondern eine reine Anschauung. Denn erstlich kann man sich nur einen einigen Raum vorstellen, und wenn man von vielen Räumen redet, so versteht man darunter nur Teile eines und desselben alleinigen Raumes.
>
> Space is not a discursive or, as is said, general concept of relations of things in general, but a pure intuition. For, first, one can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same unique space.

The preceding claim is remarkable for us, mathematicians, although the present author is not sure whether Kant allowed this "unique space" to be disconnected.

> Der Raum wird als eine unendliche gegebene Größe vorgestellt.
>
> Space is represented as a given infinite magnitude.

The meaning of infinity for Kant is rather ambiguous if we consider it from a mathematical viewpoint, but it may be most natural to interpret this notion as the unboundedness of space with respect to some metric. Kant gave these claims with some justification, which looks like a kind of reductio ad absurdum.

In addition to these basic theses on space by Kant, it is worthwhile to look at what he claimed to be antinomies regarding the nature of space. He posed the following two theses which contradict each other.

1. Die Welt hat einen Anfang in der Zeit, und ist dem Raum nach auch in Grenzen eingeschlossen.
   The world has a beginning in time, and in space it is also enclosed in boundaries.
2. Die Welt hat keinen Anfang, und keine Grenzen im Raume, sondern ist, sowohl in Ansehung der Zeit, als des Raumes, unendlich.
   The world has no beginning and no bounds in space, but is infinite with regard to both time and space.

For today's mathematicians, what Kant means by words like "infinite" or "bounds" may seem rather cryptic. For instance, we may wonder in the same way as for the previous quotation if he meant by 'infinite' space non-compactness of a topological space or unboundedness of a metric space. Since the notion of topological space did not exist in the time of Kant and no metrics were known except for the Euclidean one, it is more natural to interpret this notion of infinity as unboundedness with respect to the Euclidean metric. Still, we are allowed to imagine that if Kant had known the existence of closed 3-manifolds, then he would have posed his antinomies in a quite different way.

It should be mentioned here that Kant also used the term "Mannigfaltigkeit" in KrV. For him, however, this word is an abstract noun which means a condition of things being "manifold" rather than an object, and hence may be translated as "manifoldness." In contrast, the word "Mannigfaltige," which is a noun derived from the adjective “mannigfaltig,” appears in KrV in a sense a bit closer to the later use of “Mannigfaltigkeit” by Riemann etc., although its meaning is not mathematical. We cite here a couple of passages from KrV. The first is taken from a section where he talks about “time,” and there he poses an infinite line as an analogy of time. The use of the word “Mannigfaltige” in this passage is comparable to Riemann’s description of one-dimensional “Mannigfaltigkeit.”

> Und, eben weil diese innere Anschauung keine Gestalt gibt, suchen wir auch diesen Mangel durch Analogien zu ersetzen, und stellen die Zeitfolge durch eine ins Unendliche fortgehende Linie vor, in welcher das Mannigfaltige eine Reihe ausmacht, die nur von einer Dimension ist, und schließen aus den Eigenschaften dieser Linie auf alle Eigenschaften der Zeit, außer dem einigen, daß die Teile der ersteren zugleich, die der letzteren aber jederzeit nacheinander sind.

> And just because this inner intuition yields no shape we also attempt to remedy this lack through analogies, and represent the temporal sequence through a line progressing to infinity, in which the manifold constitutes a series that is of only one dimension, and infer from the properties of this line to all the properties of time, with the sole difference that the parts of the former are simultaneous but those of the latter always exist successively.

The second passage is taken from a section where Kant explains what the transcendental logic means. Here the meaning of the word “Mannigfaltige” is more abstract.

> Dagegen hat die transzendentale Logik ein Mannigfaltiges der Sinnlichkeit a priori vor sich liegen, ... Raum und Zeit enthalten nun ein Mannigfaltiges der reinen Anschauung a priori, ...

> Transcendental logic, on the contrary, has a manifold of sensibility that lies before it a priori, ... Now space and time contain a manifold of pure a priori institution, ...

In the paragraph next to the one containing this passage, Kant also uses the word “Mannigfaltigkeit.” In the Cambridge English translation, this word is translated as “manifoldness” as below.

> Ich verstehe aber unter Synthesis in der allgemeinsten Bedeutung die Handlung, verschiedene Vorstellungen zueinander hinzuzutun, und ihre Mannigfaltigkeit in einer Erkenntnis zu begreifen.

> By synthesis in the most general sense, I understand the action of putting different representations together with each other and comprehending their manifoldness in one cognition.

To sum up, as can be seen from these examples, Kant’s use of the word “Mannigfaltige” or “Mannigfaltigkeit” is close to our daily use of words like “variety” and “diversity”, and it does not indicate some concrete mathematical object as in Riemann’s paper. Still, we can imagine that this notion motivated Riemann’s choice of the word “Mannigfaltigkeit”, presumably via the work of Herbart, who was an indirect successor of Kant in Königsberg.

3 Riemann’s Habilitationsvortrag

Riemann’s Habilitationsvortrag, which was published only posthumously in 1867, is the content of his habilitation lecture given in 1854. It was required for Riemann to submit three different topics for his habilitation, from which the faculty would choose one. Riemann proposed one work on trigonometric series, another one on a system of quadratic equations, and a third one entitled 'Über Hypothesen, welche der Geometrie zu Grunde liegen', and Gauss chose the last one.

3.1 Philosophical Aspects

Reading Riemann's Habilitationsvortrag [18] today, it is clear that this work has a strong philosophical connotation. Although Riemann did not mention the name of Kant there, he talked about the philosophy of Herbart. In the introduction of this paper, Riemann writes the following. (There are several English translations of Riemann's text. We shall mainly use the one by Spivak [20], but also refer to the one by Clifford [19] occasionally.)

> Diese Dunkelheit wurde auch von Euklid bis auf Legendre, um den berühmtesten neueren Bearbeiter der Geometrie zu nennen, weder von den Mathematikern, noch von den Philosophen, welche sich damit beschäftigten, gehoben. Es hatte dies seinen Grund wohl darin, dass der allgemeine Begriff mehrfach ausgedehnter Grössen, unter welchem die Raumgrössen enthalten sind, ganz unbearbeitet blieb.
>
> From Euclid to Lagrange, the most famous of the modern reformers of geometry, this darkness has been dispelled neither by the mathematicians nor by the philosophers who have concerned themselves with it. This is undoubtedly because the general concept of a multiply extended quantities, which includes spatial quantities, remains unexplored.

It is quite natural to imagine that Kant was included among the philosophers whom Riemann was talking about. With these words, Riemann began to elucidate the concept of space underlying geometry. The "multiply extended quantities" mentioned here would turn out to be "manifolds (Mannigfaltigkeiten)", whose study was supposed to be the subject of this Habilitationsvortrag.

There are phrases in Riemann's text which should be regarded as "anti-Kantian." The following is one example.

> …, dass die Sätze der Geometrie sich nicht aus allgemeinen Grössenbegriffen ableiten lassen, sondern dass diejenigen Eigenschaften, durch welche sich der Raum von anderen denkbaren dreifach ausgedehnten Grössen unterscheidet, nur aus der Erfahrung entnommen werden können.
> …, it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of quantity, but that those properties that distinguish space from other conceivable triply extended quantities are only to be deduced from experience.

Riemann claimed that there are more possibilities for multiply extended quantities than we can usually imagine and that the space as we intuitively grasp it is just one of them. Consequently, this choice is not a priori, but based on our experience. This position should be contrasted with the Kantian 'a priorism', and has rather a flavour of empiricism. We shall see later that this position is also significantly different from that of Poincaré, whose doctrine is explicitly anti-empiricist.

3.2 Mannigfaltigkeit

Now we proceed to look at the main part of the Habilitationsvortrag. Riemann introduced the notion of "Mannigfaltigkeit" (manifold or multiplicity) as a kind of set which "Bestimmungsweisen" (instances or specialisations) form and he said that this Mannigfaltigkeit can be either discrete or continuous. Here is the quotation of the part where the word "Mannigfaltigkeit" appears for the first time in the text.

> Grössenbegriffe sind nur da möglich, wo sich ein allgemeiner Begriff vorfindet, der verschiedene Bestimmungsweisen zulässt. Je nachdem unter diesen Bestimmungsweisen von einer zu einer andern ein stetiger Uebergang stattfindet oder nicht, bilden sie eine stetige oder discrete Mannigfaltigkeit; ...
>
> Notions of quantity are possible only when there already exists a general concept that admits particular instances. These instances form either a continuous or a discrete manifold, depending on whether or not a continuous transition of instances can be found between any two of them; ...

Therefore, in modern terminology, this notion of manifold should be interpreted as a set parametrised by n-tuples of real numbers. There is no formal definition in this part, such as the one using charts contained in modern textbooks. We can interpret what Riemann had in mind in several ways. For instance, it is possible to imagine that he allowed singularities to exist in a Mannigfaltigkeit. Riemann talked about the set of colours as an example of a Mannigfaltigkeit, which was said to have three dimensions. Of course he also mentioned a "Riemann" surface as an example in mathematics. It would not be so anachronic to expect that he also regarded the moduli space of Riemann surfaces as an example. In fact, Grothendieck referred to the "mulitiplicités modulaires" in [3], apparently in homage to Riemann. Unfortunately we cannot find any hint of Riemann's thinking along this line in his papers.

To justify the possibility of thinking of n-fold extended Mannigfaltigkeit, Riemann gave a detailed explanation on how to construct the entity of dimension n as follows. He started from a one-dimensional manifold:

> Geht man bei einem Begriffe, dessen Bestimmungsweisen eine stetige Mannigfaltigkeit bilden, von einer Bestimmungsweise auf eine bestimmte Art zu einer andern über, so bilden die durchlaufenen Bestimmungsweisen eine einfach ausgedehnte Mannigfaltigkeit, deren wesentliches Kennzeichen ist, dass in ihr von einem Punkte nur nach zwei Seiten, vorwärts order rückwärts, ein stetiger Fortgang möglich ist.
>
> In a concept whose instances form a continuous manifold, if one passes from one instance to another in a well-determined way, the instances through which one has passed form a simply extended manifold, whose essential characteristic is that from any point in it a continuous movement is possible in only tow directions, forwards and backwards.

Then he observed that it is possible to increase the dimension one by one, and to construct an n-dimensional Mannigfaltigkeit inductively. Thus he wrote:

Wenn man, anstatt den Begriff als bestimmbar, seinen Gegenstand als veränderlich betrachtet, so kann diese Construction bezeichnet werden als eine Zusammensetzung einer Veränderlichkeit von $n+1$ Dimensionen aus einer Veränderlichkeit von n Dimensionen und aus einer Veränderlichkeit von Einer Dimension.

If one considers the process as one in which the objects vary, instead of regarding the concept as fixed, then this construction can be characterised as a synthesis of a variability of $n+1$ dimensions from a variability of n dimensions and a variability of one dimension.

Here again the description is rather intuitive, and there is no formal definition of dimension. Still, we can see that Riemann understood dimension as a number of parameters which can vary independently. It should be noted that n-dimensional Euclidean space itself had already been introduced by Grassmann [2] in 1844, as a vector space, using a basis. What is new in Riemann's argument lies in the fact that he considered dimension as something which can be applied to much more general spaces.

Riemann's definition of metrics on manifolds is more explicit than that of manifolds themselves. He considered how the lengths of curves can be measured, and he used the symbol ds to denote the length element in manifolds, as we still do today. He also defined (rather intuitively) the sectional curvature by considering a surface spanned by geodesics in the manifold, as follows.

Um die Krümmungsmass einer n-fach ausgedehnten Mannigfaltigkeit in einem gegebenen Punkte und einer gegebenen durch ihn gelegten Flächenrichtung eine greifbare Bedeutung zu geben, muss man davon ausgehen, dass eine von einem Punkte ausgehende kürzeste Linie völlig bestimmt ist, wenn ihre Ansfangsrichtung gegeben ist. Hienach wird man eine bestimmte Fläche erhalten, wenn man sämmtliche von dem gegebenen Punkte ausgehenden und in dem gegebenene Flächenelement liegenden Anfangsrichtungen zu kürzesten Linien verlängert, und diese Fläche hat in dem gegebenen Punkte ein bestimmtes Krümmungsmass, welches zugleich das Krümmungsmass der n-fach ausgedehnten Mannigfaltigkeit in dem gegebenen Punkte und der gegebenen Flächenrichtung ist.

To give a tangible meaning to the curvature of an n-fold extended manifold, at a given point, and in a given surface direction, we first mention that a shortest line emanating from a point is completely determined if its initial direction is given. Consequently, we obtain a certain surface if we prolong all the initial directions from the given point which lie in the given surface element, into shortest lines; and this surface has a definite curvature at the given point, which is equal to the curvature of the n-fold extended manifold at the given point, in the given surface direction.

4 Poincaré's Analysis Situs

In contrast to Riemann's paper, the definition of a manifold by Poincaré is clear-cut. In his celebrated paper "Analysis situs" which was published in the Journal d'Ecole Polytechnique in 1895 [13], Poincaré gave explicitly two different definitions of manifolds. This paper begins with an audacious declaration which reads (the translation is by the present author):

> La Géométrie à n dimensions a un objet réel; personne n'en doute aujourd'hui. Les êtres de l'hyperespace sont susceptibles de définitions précises comme ceux de l'espace ordinaire, et si nous ne pouvons nous les représenter, nous pouvons les concevoir et les étudier. Si donc, par exemple, la Mécanique à plus de trois dimensions doit être condamnée comme dépourvue de tout objet, il n'en est pas de même de l'Hypergéométrie.

> The geometry of n-dimensions has a real object, nobody doubts this today. The things in hyperspace are susceptible of precise definitions in the same way as those in the ordinary space, and even if we cannot represent them, we can conceive and study them. So if, for instance, Mechanics in more than three dimensions should be condemned as lacking any object, it is not the case for hypergeometry.

4.1 *Poincaré's Definitions of Manifold*

In the first two sections after the introduction, Poincaré gives his definitions of manifolds. The first definition is as follows. We consider the n-dimensional Euclidean space with coordinates $x_1, \dots, x_n$. Suppose that we are given $p + q$ functions $F_1, \dots, F_p; \varphi_1, \dots, \varphi_q$ with $p \leq n$, which are assumed to be continuous and have continuous derivatives. Assume moreover that the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial F_p}{\partial x_1} & \frac{\partial F_p}{\partial x_2} & \cdots & \frac{\partial F_p}{\partial x_n} \end{pmatrix}$$

is p at every point in the domain where the F_i are defined. We then consider a system of equations and inequalities as follows,

$$\begin{cases} F_1(x_1, \dots, x_n) = 0 \\ \quad\vdots \\ F_p(x_1, \dots, x_n) = 0 \\ \varphi_1(x_1, \dots, x_n) > 0 \\ \quad\vdots \\ \varphi_q(x_1, \dots, x_n) > 0. \end{cases}$$

Poincaré defines a manifold of dimension $n - p$ as the set of points satisfying the above system of equations and inequalities. By the implicit function theorem, such a set constitutes a differentiable manifold in the modern sense. In fact, Poincaré himself showed that a manifold according to this definition is automatically a manifold according to the second definition which we shall describe below, using an argument involving a kind of implicit function theorem. Notice that the first definition does not use charts and local coordinates.

Now we turn to the second definition. Again, we work in the n-dimensional Euclidean space with coordinates $x_1, \ldots, x_n$. We consider n functions θ_i of m variables $y_1, \ldots, y_m$ such as the following,

$$\begin{cases} x_1 = \theta_1(y_1, \ldots, y_m) \\ x_2 = \theta_2(y_1, \ldots, y_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = \theta_n(y_1, \ldots, y_m). \end{cases}$$

These functions θ_n are assumed first to be continuous, but Poincaré claims that they can be assumed to be analytic, by approximating the original functions arbitrarily closely by analytic ones. Furthermore, the Jacobian matrix of these functions is assumed to have rank m at every point. We cut out from this an open region denoted by V by adding inequalities $\psi_j(y_1, \ldots, y_m) > 0$. For convenience, let us call such a V a local manifold.

Poincaré next considers another system of functions $x_i = \theta'_i(y_1, \ldots, y_m)$ for $i = 1, \ldots, n$ which defines another local manifold V', supposing that $V \cap V'$ is non-empty. Then he claims that we can extend V to a larger manifold $V \cup V'$ by analytic continuation. He also observes that in the same way, we can continue the process by gluing local manifolds $V_1, \ldots, V_n$.

This definition is closer to the modern one which usually starts with charts, but it is not clear whether Poincaré assumed that two local manifolds which are not adjacent to each other are disjoint. Without such an assumption, this is a definition of an immersed manifold rather than an embedded manifold. Anyway, it should be noted that in both definitions, a manifold always lies in a Euclidean space. Manifolds without this environment were to appear later, in the 20th century.

4.2 *Poincaré's Conventionalism*

It is well known that Poincaré's position regarding the philosophical aspects of geometry is radical. His position is often labelled as "conventionalism," which opposes him to both empiricists and neo-Kantians. Generally, conventionalism refers to a philosophical doctrine which regards (both judicial and natural) laws and morals as mere conventions. We can think of Poincaré's views as conventionalism with regard to geometry. This can be seen typically in his text on the foundation of geometry [16]. (Its English translation by T. J. McCormack, which we use here together with the original, appeared in [14] a long time before the original French version.)

> La géométrie n'est pas une science expérimentale; l'expérience n'est pour nous que l'occasion de réfléchir sur les idées géométriques qui préexistent en nous.
>
> Geometry is not an experimental science; experience forms merely the occasion for our reflecting upon the geometric ideas which pre-exist in us.

This shows that Poincaré's position is far from the empiricists' view. On the other hand, the following passage shows that his thinking is also quite different from Kantians (or neo-Kantians). It is about how we choose one geometry, say Euclidean geometry, among other possible geometries.

> Notre choix ne nous est donc pas imposé par l'expérience. Il est simplement guidé par l'expérience. Mais il reste libre: nous choisissons cette géométrie-ci plutôt que celle-là, non parce qu'elle est plus vraie, mais parce qu'elle est plus commode.
>
> Our choice is therefore not imposed by experience. It is simply guided by experience. But it remains free: we choose this geometry rather than that geometry, not because it is more true, but because it is more convenient.

Therefore, for him, geometry is something like a tool, and our preference of one geometry over another does not depend on its validity, but on its usefulness. His view is quite close to our modern thinking, and we have little difficulty in understanding it. But we can imagine that for people at the turn of the 20th century, this view should have been outrageous, as we explain now.

To see the impact and the novelty of his view, it is worthwhile to take a look at the debate between him and B. Russell. (See Nabonnand's paper [11] for more details and for the significance of this debate in wider contexts.) Russell, who was in his mid 20s at that time, published a book entitled "An essay on foundation of geometry" in 1897 [21], which was based on his doctoral dissertation. In this book, although Russell criticised Kantian a-priority of geometry, he tried to separate what is given a priori and what should be tested empirically in geometry. For Russell, a-priority means what precedes every experience and what makes experiences possible. To make a clear distinction between what is a priori and what depends on experience, he gave axioms for projective geometry, which is a common ground for both Euclidean and non-Euclidean geometries, and those of "metric geometry." Russell considered that axioms of projective geometry (except for the one concerning the dimension of space) and some of metric geometry are a priori, but there are others in metric geometry, for instance those which distinguish Euclidean geometry from non-Euclidean, which can be verified or falsified by experiences.

Poincaré criticised this approach of Russell in two ways, in his review of the book [15]. The first criticism is purely mathematical: he pointed out the insufficiency of Russell's axioms on projective geometry. The second criticism is more philosophical: he denied the a-priority of these axioms. Poincaré examined the axioms proposed by Russell one by one, and gave a harsh judgement declaring that for most of the cases, Russell failed to show that they were indispensable for experience. He said that although Russell's statements with unclear terms made it difficult to see through, once it was cleared, then the illusion he gave would also disappear.

Poincaré also criticised Russell's claim that some of his axioms (for metric geometry) are empirical. In particular Poincaré argued, in opposing Russell's claim that Euclidean geometry can be empirically tested, that none of our experiences verifies Euclidean geometry and falsifies hyperbolic geometry.

As can be seen in this polemic, Poincaré's view on the foundations of geometry was quite clearly posed, and we can say that it was quite ahead of time, even compared with that of a much younger philosopher like Russell.

5 Definitions Using Local Charts According to Hilbert, Weyl, Kneser and Veblen-Whitehead

As we saw in the previous section, although Poincaré gave two clear definitions of manifold, they are both different from the definition which we are familiar with. In this section, we shall see when and how the modern-day definition of topological manifold appeared for the first time.

Historically, the definition of a two-dimensional manifold precedes that for general dimensions. We first look at Hilbert's paper entitled "Ueber die Grundlagen der Geometrie" published in 1902 [5], where he tried to give axioms for a surface which he called "Ebene" i.e. a plane rather than a surface. Here are his axioms. A surface is defined to be a point set with bijections onto domains in the Cartesian plane with the following conditions. (The itemisation and the translation are by the present author.)

1. Zu jedem Punkte A unserer Ebene giebt es Jordan'sche Gebiete, in welchen der Bildpunkt von A liegt und deren sämmtliche Punkte ebenfalls Punkte unserer Ebene darstellen. Diese Jordan'schen Gebiete heissen Umgebungen des Punktes A.
2. Jedes in einer Umgebung von A enthaltene Jordan'sche Gebiet, welches den Punkt A einschliesst, ist wiederum eine Umgebung von A.
3. Ist B irgend ein Punkt in einer Umgebung von A, so ist diese Umgebung auch zugleich eine Umgebung von B.
4. Wenn A und B irgend zwei Punkte unserer Ebene sind, so giebt es stets eine Umgebung, die zugleich eine Umgebung von A und eine Umgebung von B ist.

1. For every point A on our surface, there is a Jordan domain in which the point corresponding to A lies and all of whose points represent points in our surface. These Jordan domains are called neighbourhoods of A.
2. Every Jordan domain contained in a neighbourhood of A which contains A is in turn a neighbourhood of A.
3. If B is any point contained in a neighbourhood U of A, then U is also a neighbourhood of B at the same time.
4. If A and B are any two points on our surface, then there is always a neighbourhood of A which is at the same time a neighbourhood of B.

We should keep in mind that this paper of Hilbert precedes the work of Hausdorff on abstract topological spaces, which appeared in his book "Grundzüge der Mengenlehre" [4]. Therefore, Hilbert could not start from a topological space, but from just a point set. Consequently, his axioms include those for neighbourhood systems in a topological space. Setting this part aside, these axioms are equivalent to the modern definition of topological manifold using charts. Thus we can say that two-dimensional topological manifolds as we know them today were formally defined for the first time by Hilbert in 1902.

There is one more remark which we should make: the last axiom implies that the surface is a Hausdorff space. This should be contrasted with the following definition by Weyl. (This subtle difference was also pointed out by Scholtz [22].)

Now, we turn to Weyl's book on Riemann surface [27]. A formal definition of a topological surface (a topological 2-dimensional manifold) which is close to the modern one is given in §4 of this book. It is somehow more common to attribute the first formal definition of a manifold using charts to this book rather than to Hilbert's paper. In Weyl's definition, the setting is quite similar to Hilbert's. We are given a point set $\mathfrak{F}$, and for each point $\mathfrak{p}$ of $\mathfrak{F}$, there is a system of subsets $\mathfrak{U}$ containing $\mathfrak{p}$ which are called neighbourhoods of $\mathfrak{p}$. For every neighbourhood $\mathfrak{U}_0$ of a point $\mathfrak{p}_0$ of $\mathfrak{F}$, there is a bijection from $\mathfrak{U}_0$ to an Euclidean open disc K_0 taking $\mathfrak{p}_0$ to the centre of K_0, and the following conditions hold.

1. ist $\mathfrak{p}$ irgend ein Punkt von $\mathfrak{U}_0$ und $\mathfrak{U}$ eine nur aus Punkten von $\mathfrak{U}_0$ bestehende Umgebung von $\mathfrak{p}$ auf $\mathfrak{F}$, so enthält das (durch jene Abbildung in K_0 entworfene) Bild von $\mathfrak{U}$ den Bildpunkt von $\mathfrak{p}$ im Innern; d. h. es läßt sich um den Bildpunkt p von $\mathfrak{p}$ eine Kreisfläche k beschreiben, sodaß jeder Punkt von k Bild eines Punktes von $\mathfrak{U}$ ist;
2. ist K das Innere irgend eines ganz in K_0 gelegenen Kreises mit dem Mittelpunkt p, so gibt es stets eine Umgebung $\mathfrak{U}$ von $\mathfrak{p}$ auf $\mathfrak{F}$, deren Bild ganz in K liegt.

1. If $\mathfrak{p}$ is any point of $\mathfrak{U}_0$ and $\mathfrak{U}$ is a neighbourhood of $\mathfrak{p}$ consisting only of points of $\mathfrak{U}_0$, then the image of $\mathfrak{U}$ (under the map from $\mathfrak{U}_0$ to K_0) contains that of $\mathfrak{p}$ as an interior point, i.e., there is an open disc k around the point p which is the image of $\mathfrak{p}$, such that every point of k is the image of a point of $\mathfrak{U}$.
2. If K is any circle with centre p contained entirely in K_0, then there is always a neighbourhood $\mathfrak{U}$ of $\mathfrak{p}$ on $\mathfrak{F}$ whose image entirely lies in K.

We see that if we define a topology on $\mathfrak{F}$ using the given neighbourhood systems, then these two conditions guarantee that the map from $\mathfrak{U}_0$ to K_0 is a homeomorphism. Thus, this definition is equivalent to a modern definition of topological two-manifold. There is one subtle point: the definition does not contain the axiom of separability, which means that the surface defined as such may not be a Hausdorff space. In a later revised version of the same book and its English translation, this part was substantially revised. The definition has been divided into two parts: the first part is the definition of a Hausdorff space using neighbourhood systems whereas the second part is almost the same as the two conditions above. In particular the subtlety concerning the separability disappeared.

There is one more difference with Hilbert's definition. Weyl assumed the image of the map from a neighbourhood to be an open disc in the plane, not a general Jordan domain. In fact Schoenflies proved that any Jordan domain is homeomorphic to an open disc in 1906; this was after Hilbert's paper was published but before Weyl wrote his book.

As we have seen above, the definitions by Hilbert and Weyl only deal with two-dimensional manifolds. The same kind of definition for higher dimensional manifolds was given for the first time by Kneser [9] in 1926. After giving axioms of neighbourhood systems for a Hausdorff space, Kneser added the following two conditions as axioms for an n-dimensional manifold.

(a) Zu jedem Punkt gibt es eine Umgebung, die sich topologisch auf die offene Vollkugel des n-dimensionalen Zaahlenraumes: $x_1^2 + \cdots + x_n^2 < 1$ abbilden läßt.
(b) Unter den Umgebungen des den topologischen Raum definierenden oder eines äquivalenten Systems befinden sich nur abzählbar unendlich viele verschiedene Mengen.

(a) For every point there exists a neighbourhood which is mapped homeomorphically to an open ball $x_1^2 + \cdots + x_n^2 < 1$ in the n-dimensional Euclidean space.
(b) Among neighbourhoods defining the topological space or those of an equivalent system, there are only countably many different sets.

Thus, Kneser defined a topological manifold of general dimension with an axiom of countability, which is precisely what we do today. This definition appears at the beginning of Kneser's paper. The main topic of the paper is rather the triangulability of manifolds and the uniqueness of triangulations (up to subdivision), i.e., the Hauptvermutung. Of course he did not prove this result, but he posed the conjecture in clear terms for topological manifolds and gave a precise definition of combinatorial manifolds, which made the meaning of the conjecture clear. The Hauptvermutung itself for simplicial complexes had been given earlier by Steinitz [23] and Tietze [24], and a definition of a manifold using simplices was first given by Brouwer [1]. The originality of Kneser was to consider this problem for topological manifolds, and to try to unify the approach using neighbourhood systems with the simplicial approach pioneered by Brouwer.

Up to this point, we only talked about the definition of a *topological* manifold. Although Weyl's book contains the notion of differentiability or analyticity of functions on the surface, hence also differentiable structures and complex structures, we did not touch upon this part. In the rest of this section, we shall see how the definition of a differentiable manifold as we understand it today appeared.

Veblen and Whitehead published a paper entitled "A set of axioms for differential geometry" in 1931 [25], whose aim was to define a space axiomatically where differential geometry can be developed without the aid of global coordinates as in Euclidean space. (An expanded and detailed version of this theory can be found in their book [26].) There they gave a set of axioms of C^u (class u) manifolds. This was done by defining a set of coordinate systems, which they call allowable coordinate systems, such that the transition between any two of them with an overlapping domain is C^r. They did not assume the manifold to be a topological space at the beginning, but they gave a topology using the domains of coordinate systems, which are assumed to satisfy the Hausdorff separability.

From this definition of a differentiable manifold, it became possible for the first time to define differentiation, etc. on manifolds without assuming they lie in Euclidean spaces. From the work of Whitney [28], it follows that every differentiable manifold can be regarded as lying in a Euclidean space. In fact Whitney showed that every n-dimensional C^r-manifold can be embedded into a $2n + 1$-dimensional Euclidean space by a C^r-map. This showed the equivalence between Poincaré's definition of manifolds and the modern one using charts. He also defined notions like submanifolds, C^r-maps between manifolds and so on. Thus we can say that at this stage the foundations of the notion of manifold were really consolidated.

6 Conclusion: Philosophical Significance

Now, we return to the philosophical viewpoint, with which we started our exposition. It is obvious that thanks to the development of manifold theory, our worldview (both in a cosmological sense and in a more practical sense) has been widened. If we take into account the possibility that our universe is compact, then Kantian antinomies must be seriously reconsidered, even in the framework of classical physics. The possibility of a non-simply connected universe poses further problems of epistemology. In fact, we do not have any way to distinguish a non-simply connected universe and its (non-trivial) coverings, only by observations.

It is quite well known that Einstein used Riemannian geometry for his theory of general relativity. Modern physics makes use of much more sophisticated manifold theory, typically seen in string/super-string theory. Such a development might lead to the necessity to redefine the "universe" and distinguish the world where physical theory should work from the three-dimensional manifold in which we believe that we live in our day-to-day recognition. On the other hand, what is striking is that there still remains a long way to go to the complete solution of a naïve question asking what is the topological structure of our day-to-day universe. By virtue of the resolution of Thurston's geometrisation conjecture by Perelman, once we know the (average) sectional curvature of the universe, we can make our list of possible topologies of the universe shorter. Still, to the author's knowledge, it is not known even whether the universe is compact or not, and although there is some data bounding the sectional curvature of our universe (see e.g. [12]), it is a far cry from determining the geometric structure of the universe. On the other hand, the development of manifold theory started by Riemann gave a strong impact and an inspiration to contemporary philosophy, which is often dubbed "post-modern". Some aspects of this kind of influence on philosophy are illustrated in Jedrzejewski [6] and Plotnitsky [17] in this volume.

The epistemological revolution with regard to the universe started by Riemann's Habilitationsvortrag is not completely achieved yet. It is still going on.

References

1. L.E.J. Brouwer, Über Abbildung von Mannigfaltigkeiten. Math. Ann. **69**, 97–115 (1911)
2. H. Grassmann, *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik* (Otto Wigand, Leipzig, 1844)
3. A. Grothendieck, Esquisse d'un programme, London Math. Soc. Lecture Note Ser., 242, Geometric Galois actions, 1, 5–48, (Cambridge University Press, Cambridge, 1997)
4. F. Hausdorff, *Grundzüge der Mengenlehre* (Veit & Comp, Leipzig, 1914)

5. D. Hilbert, Ueber die Grundlagen der Geometrie. Math. Ann. **56**, 381–422 (1902)
6. F. Jedrzejewski, Deleuze et la géométrie riemainnienne: une topologie des multiplicités. In this volume, pp. 311–327
7. I. Kant, Critik der reinen Vernunft, Johann Friedrich Hartknoch, Riga 1781, 2 Aufl 1787
8. I. Kant, *Critique of Pure Reason*, translated by P. Guyer, A.W. Wood (Cambridge University Press, 1998)
9. H. Kneser, Die Topologie der Mannigfaltigkeiten. Jahresbericht d. Deutschen Mathem. Vereinigung **34**, 1–14 (1926)
10. D. Laugwitz, *Bernhard Riemann 1826–1866: Wendepunkte in der Auffassung der Mathematik* (Birkhäuser, Basel, 1996)
11. P. Nabonnand, La polémique entre Poincaré et Russell au sujet du statut des axiomes de la géométrie. Revue d'histoire des mathématiques **6**, 219–269 (2000)
12. Planck Collaboration, Planck 2015 results. XIII. Cosmological parameters. arXiv:1502.01589
13. H. Poincaré, Analysis Situs, Journal de l'École Polytechnique, 1–121 (1895)
14. H. Poincaré, On the foundations of geometry (transl. by T.J. McCormack). The Monist **9**, 1–43 (1899)
15. H. Poincaré, Des fondements de la géométrie, à propos d'un livre de M. Russell, Revue de méthaphysique et de morale **3**, 630–646 (1895)
16. H. Poincaré, Les fondements de la géométrie. Bulletin des sciences mqthé-matiques **26**, 249–272 (1902)
17. A. Plotnitsky, Comprehending the connection of things: Bernhard Riemann and the architecture of mathematical concepts. In this volume, pp. 329–362
18. B. Riemann, Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen **13**, 133–150 (1867)
19. B. Riemann, translated by W.K. Clifford, Nature, 8, 14–17, 36–37 (1873)
20. B. Riemann, translated by Spivak in *A Comprehensive Introduction to Differential Geometry*, vol. 2 (Publish or Perish, 1999)
21. B. Russell, *An Essay on the Foundations of Geometry* (Cambridge University Press, Cambridge, 1897)
22. E. Scholz, *The concept of manifold, 1850–1950, in History of Topology* ed. by I.M. James (Elsevier, 1999), pp. 25–64
23. E. Steinitz, Beisträge zur Analysis situt. Sitzungsberichte Berliner Mathematisches Gesellschaft **7**, 28–49 (1908)
24. H. Tietze, Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten. Monat. Math. **19**, 1–118 (1908)
25. O. Veblen, J.H.C. Whitehead, A set of axioms for differential geometry. Proc. Natl. Acad. Sci. **10**, 551–561 (1931)
26. O. Veblen, J.H.C. Whitehead, *The Foundations of Differential Geometry* (Cambridge University Press, Cambridge, 1932)
27. H. Weyl, *Die Idee der Riemannsches Fläsche, Teubner, Leipzig, 1913, and 3* (Aufl, Teubner Stuttgart, 1955)
28. H. Whitney, Differentiable manifolds. Ann. Math. **37**, 645–680 (1936)