Riemann Surfaces: Reception by the French School

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La vérité réside dans l'imaginaire

Eugene Ionesco
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Abstract Riemann introduced in his doctoral dissertation (1851) the concept of Riemann surface as a new ground space for meromorphic functions and in particular as a domain for a multi-valued function defined by an algebraic equation such that this function becomes single-valued when its is defined on its associated Riemann surface. It took several years to the mathematical community to understand the concept of Riemann surface and the related major results that Riemann proved, like the so-called *Riemann existence theorem* stating that on any Riemann surface—considered as a complex one-dimensional manifold—there exists a non-constant meromorphic function. In this chapter, we discuss how the concept of Riemann surface was apprehended by the French school of analysis and the way it was presented in the major French treatises on the theory of functions of a complex variable, in the few decades that followed Riemann's work. Several generations of outstanding French mathematicians were trained using these treatises. At the same time, this will allow us to talk about the remarkable French school that started with Cauchy and expanded in the second half of the nineteenth century. We also comment on the relations between the French and the German mathematicians during that period.

Keywords Riemann surface · Nineteenth century mathematics · Elliptic integral · Algebraic equation

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1 Introduction

The notion of Riemann surface, discovered by Riemann and introduced in his doctoral dissertation (1851), is the culmination of a series of investigations done before him, by Cauchy and others, on the theory of functions of a complex variable. With this discovery, Riemann made a complete transformation of the field of complex analysis, merging it with topology and algebraic geometry. He also paved the way to the methods of hyperbolic geometry combined with group theory that gave rise to automorphic forms, developed by Poincaré, Klein and others, and to many other developments.

In Chap. 7 of the present volume (cf. [77]), we discussed the results of Cauchy and Puiseux on line integrals and their dependence on homotopy classes of paths, and we also mentioned other related results that were available to Riemann when he wrote his doctoral dissertation. Although the problems he addressed were in the continuity of the works of his predecessors, the complete novelty of his ideas, with proofs that rely largely on geometric intuition, sometimes with arguments from physics, led to the fact that these ideas were sometimes poorly understood by Riemann's contemporaries and immediate successors. In particular, this led Klein to spend a substantial part of his life explaining Riemann's work and trying to make it more accessible. He did this in numerous lectures and books, including the well-known treatise *Über Riemanns Theory der algebraischen Funktionen und ihrer Integrale* (On Riemann's theory of algebraic functions and their integrals) (1882) [58].

France, in the few years preceding the publication of Riemann's first memoir, saw the rise of a remarkable school of analysis whose major representative was Cauchy. Among the immediate followers of Cauchy, one has to mention Liouville, Puiseux, Hermite, Briot, and Bouquet, and then came another generation of analysts, including Jordan, Halphen, Goursat, Appell, Tannery, Lacour, Molk, Picard, Darboux, Simart, Fatou, and there are others. All these mathematicians had a great admiration for Riemann and had no doubt about the importance of his ideas, even if they did not fully make use of them in their works. Riemann's collected papers, translated into French, appeared in 1889, with a preface by Hermite [98], who starts with the following[1]:

> The work of Bernhard Riemann is the most beautiful and greatest one in analysis in our epoch. It has been consecrated by a unanimous admiration and will leave an imperishable mark in Science. [...] Never before that, in any mathematical publication, the gift of invention appeared with more power, never had anybody asked for such beautiful conquests in the most difficult questions in analysis.[2]

One notion which was particularly painful to accept by the French analysts is that of Riemann surface. Most of the treatises on the theory of functions of a complex

[1] In the present chapter, the translations from the French are mine.

[2] L'œuvre de Bernhard Riemann est la plus belle et la plus grande de l'Analyse à notre époque: elle a été consacrée par une admiration unanime, elle laissera dans la Science une trace impérissable. [...] Jamais, dans aucune publication mathématique, le don de l'invention n'était apparu avec plus de puissance, jamais on n'avait demandé autant de belles conquêtes dans les plus difficiles questions de l'analyse.

variable that were used in teaching in the French universities or at the École Polytechnique, in the few decades that followed Riemann's death, were based exclusively on the methods of Cauchy, missing the essential relevance of Riemann surfaces. As a general rule, Riemann's ideas were absorbed very slowly, and it was only around the turn of the twentieth century that the French treatises included the theory of Riemann surfaces in their full strength.

In the present chapter, we review this fascinating page of the history of complex analysis. This will also give us the occasion of surveying briefly the lives and works of several prominent mathematicians from this exceptional period, and of discussing the relations between the French and the German mathematical schools.

The plan of the rest of this chapter is the following.

In Sect. 2, we comment on the notion of Riemann surface and on Riemann's existence theorem and how these concepts were received when Riemann introduced them.

In Sect. 3, we review the way Riemann's ideas on this subject are presented in the famous French treatises on analysis, including those of Briot-Bouquet, Briot, Hermite, Jordan, Appell-Goursat, Goursat, Picard, Picard-Simart, Appel-Goursat-Fatou, Halphen, Tannery-Molk and Appell-Lacour. Elliptic functions constitute the central theme of several of these treatises. At the same time, we give some biographical information on the authors of these treatises, highlighting relations among them. The overall picture is that of a coherent group, forming a "school," which was probably the first French school of mathematics. Several doctoral dissertations were written under the same advisor, and the dissertation committees often consisted of the same persons: Darboux, Hermite, Bouquet, with some small variations.

In Sect. 4, we review the content of the doctoral dissertation of Georges Simart, which is entirely dedicated to a presentation of Riemann's work on Riemann surfaces and Abelian functions. To complete the picture, we have included a section, Sect. 5, in which we review a few French doctoral dissertations and other works of the period considered which contributed to the diffusion of other major ideas of Riemann: the zeta function, minimal surfaces and integration.

In Sect. 6, we take the opportunity of the topic discussed in this chapter to say a few words on the relationship between the French and the German schools of mathematics, in particular in the few years that followed the 1870 devastating French-German war.

The concluding section, Sect. 7, contains some additional notes on the relationship between the French and the German schools in the period considered.

2 Riemann Surfaces

In his doctoral dissertation [92], Riemann introduced Riemann surfaces as ramified coverings of the complex plane or of the Riemann sphere. He further developed his ideas on this topic in his paper on Abelian functions [94]. This work was motivated in particular by problems posed by multi-valued functions $w(z)$ of a complex variable z defined by algebraic equations of the form

$$f(w, z) = 0, \tag{1}$$

where f is a two-variable polynomial in w and z.

Cauchy, long before Riemann, dealt with such functions by performing what he called "cuts" in the complex plane, in order to obtain surfaces (the complement of the cuts) on which the various determinations of the multi-valued functions are defined. Instead, Riemann assigned to a multi-valued function a surface which is a ramified covering of the plane and which becomes a domain of definition of the function such that this function, defined on this new domain, becomes single-valued (or "uniform"). Riemann's theory also applies to transcendental functions. He also considered ramified coverings of surfaces that are not the plane.

Together with introducing Riemann surfaces associated with algebraic functions, Riemann considered the inverse problem: Given a Riemann surface obtained geometrically by gluing a certain finite number of pieces of the complex plane along some curves (which are equivalent to the "cuts" in the sense of Cauchy), can we find an algebraic relation such as (1) with which this Riemann surface is associated? This can also be formulated as the problem of finding on an arbitrary Riemann surface a meromorphic function with prescribed position and nature of its singularities (poles and branch points). The idea, contained in Riemann's 1851 dissertation [92], is natural, since a polynomial is described by its roots, and a rational function by its zeros and poles. Riemann showed that the general question has a positive answer, and in his solution to the problem, he proved that a meromorphic function is determined by its singularities. This result is one form of what is usually called the *Riemann existence theorem*, a theorem that had a tremendous impact on complex geometry. For instance, it was the main motivation for what became known as the Riemann–Roch theorem. In his paper on Abelian functions [94], Riemann proved one part of that theorem, namely, that given m points on a closed Riemann surface of genus p, the dimension of the complex vector space of meromorphic functions on this surface having at most poles of first order at the m points is $\geq m - p + 1$. In his paper [101] (1865), Gustav Roch, a student of Riemann, transformed this inequality into an equality, which became known as the Riemann–Roch theorem. Riemann's result relies on his existence theorem, the description of a meromorphic function by its singularities allowing a dimension count. The proof that Riemann gave of his inequality relies on the Dirichlet principle and it was considered non-rigorous. This initiated works by several mathematicians, some of them with the aim of finding alternative proofs Riemann's results that are based on this principle, and others with the goal of giving a solid foundation to the Dirichlet principle. Thus, an important activity was generated as an indirect consequence of Riemann's existence theorem.

The discussion around Riemann's existence theorem is spread in several sections of Riemann's doctoral dissertation [92] and his paper on Abelian function, [94], in particular in Section III of the preliminary part of the latter, entitled *Determination of a function of a complex variable magnitude by the conditions it fulfills relatively to the boundary and to the discontinuities*. Later in the same paper, an *existence result* is given in the case of functions defined by integrals of algebraic functions.

Riemann's use of the Dirichlet principle was harshly criticized by Weierstrass [109], and these critiques spread a doubt not only on the validity of Riemann's proof of his existence theorem but also of other theorems. It is important to emphasize this fact, because it explains in part why Riemann's results on Riemann surfaces were not used by his immediate followers. Klein writes in his *Development of mathematics in the 19th century* ([59] p. 247 of the English translation):

> With this attack by Weierstrass on Dirichlet's principle, the evidence to which Dirichlet, and after him, Riemann, had appealed, became fragile: Riemann's existence theorems[3] were left in the air.
>
> It is interesting to observe the positions mathematicians took with respect to Riemann's existence theorem and Weierstrass's critique.
>
> The majority of mathematicians turned away from Riemann; they had no confidence in the existence theorems, which Weierstrass's critique had robbed of their mathematical supports. They sought to salvage their investigations of algebraic functions and their integrals by again proceeding from a given equation $F(\zeta, z) = 0$ [...] Riemann's central existence theorem for algebraic functions on a given Riemann surface fell from its place, leaving only a vacuum.

It is also interesting to note Riemann's attitude toward Weierstrass's critique as recorded by Klein in the same book ([59] pp. 247–48 of the English translation):

> Riemann had a quite different opinion. He fully recognized the justice and correctness of Weierstrass's critique; but he said, as Weierstrass once told me "that he appealed to the Dirichlet principle only as a convenient tool that was right at hand, and that his existence theorems are still correct."

Concerning the notion of Riemann surface, Klein writes, in the same work ([59] p. 245 of the English translation):

> The most important point is that, according to Riemann's considerations, to any given Riemann surface there corresponds one (and only one) class (a "field") of algebraic functions (with their Abelian integrals). For Riemann a "class" of algebraic functions means the totality of functions $R(\zeta, z)$ that can be rationally expressed in terms of ζ and z; the term "field" was introduced later by Dedekind. This is a theorem that could not have been obtained in another way. At this point Riemann's theory remained, for the time being, ahead of all the others which started from the equation $F(\zeta, z) = 0$.

Riemann not only considered Riemann surfaces as associated with individual multi-valued functions or with meromorphic function in general, but he also considered them as objects in themselves, on which function theory can be developed

[3] The plural will be explained later, when we shall talk about Picard's work.

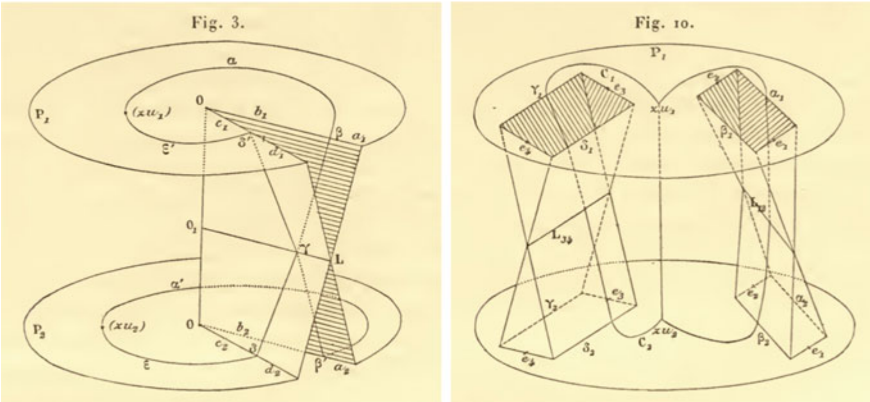

Fig. 1 A drawing of a Riemann surface, from the treatise *Théorie des fonctions algébriques* (1895) by Goursat and Appell

in the same way as the classical theory of functions is developed on the complex plane. Riemann's existence theorem for meromorphic functions with specified singularities on a Riemann surfaces is also an important factor in this setting of abstract Riemann surfaces. Riemann conceived the idea of an abstract Riemann surface, but his immediate followers did not. During several decades after Riemann, mathematicians (analysts and geometers) perceived Riemann surfaces as objects embedded in three-space, with self-intersections, instead of thinking of them abstractly. They tried to build branched covers by gluing together pieces of the complex plane cut along some families of curves, to obtain surfaces with self-intersections embedded in three-space. In his 1913 book *Idee der Riemannschen Fläche* (The concept of a Riemann surface), [110] (p. 16 of the English translation), Weyl writes about these spatial representations:

> In essence, three-dimensional space has nothing to do with analytic forms, and one appeals to it not on logical-mathematical grounds, but because it is closely associated with our sense-perception. To satisfy our desire for pictures and analogies in this fashion by forcing inessential representations on objects instead of taking them as they are could be called an anthropomorphism contrary to scientific principles.

Hilbert, in his 1903 paper [50], considers surfaces that are not embedded in a Euclidean space.[4]

The example of a Riemann surface in Fig. 1 is extracted from the treatise *Théorie des fonctions algébriques* (Theory of algebraic functions) by Paul Appell and Edouard Goursat (1895) in which the authors explain Riemann's ideas and on which we shall comment later in the present chapter. The authors explain that in the picture, the "sheets traverse each other," but that the reader should imagine that these "sheets are infinitely close to each other." We shall survey the treatise by Appell and Goursat in Sect. 3 below.

[4] I thank K. Ohshika for this reference.

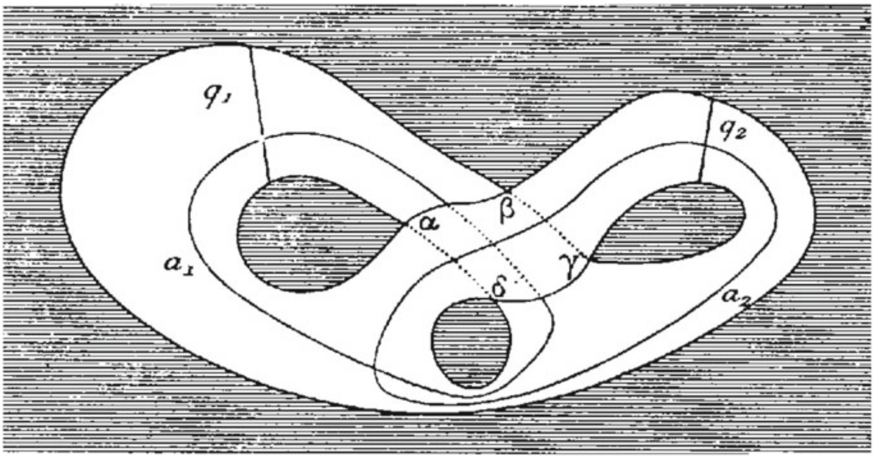

Fig. 2 A drawing from Riemann's paper on Abelian functions

In 1909, Hadamard, in his survey on topology entitled *Notions élémentaires sur la géométrie de situation* (Elementary notions of geometry of situation),[5] talking about Riemann surfaces, still considers lines along which the leaves cross each other (cf. [39] p. 204).

It was difficult to conceive these surfaces without the intersections of the sheets in 3-dimensional space. One had to wait several years before these surfaces were freed from their three-dimensional prison. Weyl, writes in his 1913 book ([110] p. 16 of the English translation): "The concept of 'two-dimensional manifold' or 'surface' will not be associated with points in three-dimensional space; rather it will be a much more general abstract idea." Figure 2 represents a more abstract drawing in the tradition of Riemann. It is extracted from the French version of Riemann's works [98].

Klein considers that around the year 1881, at least some of Riemann's important ideas were already understood in France. He writes in his *Development of mathematics in the 19th century* [59] p. 258:

> Working on the subject of automorphic functions, from 1881 on, I came into close touch with Poincaré; this was also the time when Riemann's modes of thoughts were transplanted to France and there found firm ground.

In the next section, we review the way Riemann surfaces are treated in some of the major French treatises on complex analysis that were published in the few years that followed Riemann's work.

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[5] "Geometry of situation" was one of the various names given to topology, before the word "topology" became universally accepted.

3 The Nineteenth-Century French Treatises on Analysis

In this section, we review some of the nineteenth-century French treatises on analysis, in relation with the notion of Riemann surface and some associated notions like elliptic and Abelian integrals and their periods. As we shall see, there was a great variety of important treatises of various levels of difficulty, covering a large spectrum of topics. Let us note that independently of the work of Riemann, it is interesting to review these treatises, because these were the textbooks in which the French mathematicians of that epoch were trained. These mathematicians constituted a consistent and very strong school of analysis whose imprint is still felt today. The next table is a list of the treatises that we shall mention, in an approximate chronological order. It is difficult to make a precise chronological order, because several of these treatises consist of several volumes, with a time lapse of several years between the first and the last volume. In the commentary that follows this table, the order takes into account the connections between the ideas rather than the chronology.

Author	**Title**	**Year (1st ed.)**
Ch.-A. Briot and J.-C. Bouquet	Théorie des fonctions doublement périodiques et, en particulier, des fonctions elliptiques	1859
Ch. Hermite	Cours d'analyse de l'École Polytechnique	1873
Ch.-A. Briot	Théorie des fonctions Abéliennes	1879
C. Jordan	Cours d'analyse de l'École Polytechnique	1882–1897
Ch. Hermite	Cours à la faculté des sciences de Paris	1882
G.-H. Halphen	Traité des fonctions elliptiques et de leurs applications	1886–1891
É. Picard	Traité d'analyse	1891–1896
J. Tannery and J. Molk	Éléments de la théorie des fonctions elliptiques	1893–1902
P. Appell and É. Goursat	Théorie des fonctions algébriques et de leurs intégrales	1895
É. Picard and G. Simart	Théorie des fonctions algébriques de deux variables indépendantes	1897–1906
P. Appell and É. Lacour	Principes de la théorie des fonctions elliptiques et applications	1897
É. Goursat	Cours d'analyse mathématique	1902–1905
P. Appell, É. Goursat and P. Fatou	Étude des fonctions analytiques sur une surface de Riemann	1929

Briot and Bouquet

We start with the treatise *Théorie des fonctions doublement périodiques et, en particulier, des fonctions elliptiques* (Theory of doubly periodic functions, and in particular, elliptic functions) [17] by Briot and Bouquet. This treatise, whose first edition appeared in 1859, became one of the major references on the theory of functions of a complex variable in France during the second half of the nineteenth century. As the name of the treatise indicates, the stress is on elliptic functions and their generalizations to doubly periodic functions. We recall that elliptic functions have (at most) two independent periods; they are essentially functions defined on the torus. We start by recalling a few facts about these functions. For a glimpse into the history of elliptic integrals, which are at the origin of the general theory of elliptic functions, the reader is referred to Chap. 1 of the present volume [75].

Before Riemann, elliptic functions had occupied the greatest mathematicians: Euler, Gauss, Dirichlet, Legendre and others. In France, the first mathematician who made a thorough study of these functions is Legendre, who wrote treatises comprising several volumes on the subject, cf. [63, 64]. The subject became fashionable in France only after his death. It is interesting in this respect to quote a letter from Legendre to Jacobi, dated February 9, 1828, in which Legendre complains that in France, mathematicians, at his time, were not enough interested in elliptic functions. Responding to a letter in which Jacobi makes for him a summary of Abel's article *Recherches sur les fonctions elliptiques* (Researches on elliptic functions) [1] published in 1827, Legendre writes ([54], t. 1, p. 407):

> I was already aware of the beautiful work of Mr. Abel inserted in Crelle's Journal. But I was very pleased by the analysis you have given me in your own language, which is closer to mine. For me, it is a big satisfaction to see two young geometers, like you and him, cultivating with success a branch of analysis which has been for such a long time my favorite subject of study, and which is not as much welcome in my own country as it deserves to be.[6]

By the time of Briot and Bouquet published their treatise, that is, thirty years after this letter was written, the study of elliptic functions was already a very active field in France. Cauchy has already introduced line integrals in the field of functions of a complex variable, and elliptic integrals constituted a new class of functions with interesting properties. The known functions of a complex variable, before this class, were limited to polynomials, exponentials, logarithms, trigonometric functions, and some other special functions introduced by Euler. Several questions concerning these functions, motivated by the work of Legendre, Abel and Jacobi, constituted the basis of several research topics. Furthermore, elliptic functions were known to have numerous applications in geometry, number theory, mechanics and physics.

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[6] J'avais déjà connaissance du beau travail de M. Abel inséré dans le *Journal de Crelle*. Mais vous m'avez fait beaucoup de plaisir de m'en donner une analyse dans votre langage qui est plus rapproché du mien. C'est une grande satisfaction pour moi de voir deux jeunes géomètres, comme vous et lui, cultiver avec succès une branche d'analyse qui a fait si longtemps l'objet de mes études favorites et qui n'a point été accueillie dans mon propre pays comme elle le méritait.

A few words about Briot and Bouquet may be useful, before talking about their treatise. Although they were great analysts and remarkable teachers, their names are rather unknown today.

In 1842, Charles-Auguste Briot (1817–1882) submitted at the Faculté des Sciences de Paris, a dissertation on mechanics whose title was *Sur le mouvement d'un corps solide autour d'un point fixe* (On the motion of a solid body around a fixed point) [15]. The aim of this dissertation was to provide complete proofs of results on mechanics that were stated by Poinsot in his memoir *Théorie nouvelle de la rotation des corps* (A new theory for the rotation of bodies) [91]. Briot then taught at the Sorbonne and at the École Normale Supérieure, but also, for several years, in two *lycées*[7] in Paris: Bourbon and Saint-Louis. These were among the famous lycées preparing for the highly competitive entrance examination of the École Polytechnique and the École Normale Supérieure. Having good teachers in such lycées was a tradition in France, and some of these teachers were excellent mathematicians.[8] Briot, like Riemann, Cauchy and many mathematicians of his generation, was highly interested in physics, in particular, heat, light and electricity, three topics which were particularly dear to Riemann. Briot's research in these fields was based on his theories of aether, and in his research on these topics he was strongly influenced by Louis Pasteur. He wrote a large number of textbooks for students, encompassing analysis, algebra,

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[7]The lycées where Briot (and several other mathematicians we encounter in the present chapter) taught are high-schools whose curricula included an additional year of study after the high-school diploma (*baccalauréat*). During that year, called *Mathématiques spéciales*, the *élèves* (pupils) are prepared for the entrance examinations (*concours d'entrée*) to some highly competitive schools which, in the period we are interested in, were essentially the École Polytechnique and the École Normale Supérieure. In principle, only gifted and hard-working élèves were admitted in such classes.

Only a small percentage of the élèves were accepted into these schools (2–5%) at the first trial. The others usually returned to the lycée and spent another year in the class of Mathématiques spéciales where they deepened their knowledge and their training. The chances of entering one of the two schools after this second year were about 25%. Some of the élèves, after a second failure, repeated a third time the class of Mathématiques spéciales, and the chances of success, for those who tried the *concours d'entrée* after a third year, were about 50%. (These figures are extracted from the article [85] by Pierpont in which the author compares the French and the American mathematical education systems by the end of the nineteenth century.)

These classes still exist today in France, they are called *Classes préparatoires aux Grandes Écoles*, and include two years, *Mathématiques supérieures* and *mathématiques spéciales*. They prepare to the entrance examinations of a large number of schools.

[8]The list includes Briot, Bouquet, Darboux, Bertrand, Hoüel, Valiron, Châtelet, Tannery, Boutroux, Lacour, Lucas, Lichnerowicz, and there are others. The following story is related by Picard, in his *Eulogy* of Jules Tannery [84]: "Bouquet used to relate that after he graduated from the École [Normale Supérieure], and while he was in charge of the class of "mathématiques spéciales" at Marseille's lycée, he received the visit of the father of one of his élèves, who wanted that his son be prevented from working in mathematics, because they lead to noting good. He asked for a professor who would give a course which is enough bad so that his son does not enter the École Polytechnique, after which one gains less money than in business. [Bouquet aimait à raconter que, chargé à sa sortie de l'École, du cours de mathématiques spéciales au Lycée de Marseille, il avait eu la visite du père d'un de ses élèves, qui voulait qu'on empêchât son fils de travailler les mathématiques qui ne mènent à rien de bon. Il demandait que le professeur fit un assez mauvais cours pour que son fils n'entrât pas à l'École Polytechnique au sortir de laquelle on gagne moins d'argent que dans le commerce.]

analytic geometry, mechanics and physics. Having textbooks written by outstanding and devoted teachers was traditional in France in that period.

Jean-Claude Bouquet (1819–1885) defended his doctoral dissertation in 1842, the same year as Briot. The subject was the calculus of variations, and the title was *Sur la variation des intégrales doubles* (On the variation of double integrals) [14]. Bouquet first taught at a lycée in Marseille and then became, at the age of 26, professor at the University of Lyon. Seven years later he moved to Paris where he became professor at Lycée Bonaparte, and then Lycée Louis-le-Grand. In 1868, he became the successor of Puiseux at the École Normale Supérieure, and in 1885 the successor of Serret at the Chair of differential and integral calculus of the Faculté des Sciences de Paris. Bouquet's successor at that chair was Émile Picard.

Briot and Bouquet published, separately and as co-authors, several important articles and treatises on the theory of functions of a complex variable and on elliptic and Abelian functions. It might be useful to recall that in the period considered, joint mathematical works were rare, and for this reason the long-term collaboration of Briot and Bouquet stands as a singular spot in the history of mathematics. In 1856, Briot and Bouquet published a joint paper entitled *Étude des fonctions d'une variable imaginaire* (Study of functions of an imaginary variable) [16] in which they present in a comprehensive way Cauchy's theory of functions of a complex variable. In the introduction to that memoir, they write:

> This first memoir contains the principles of Cauchy's theory of an imaginary variable. We shall adopt the definition given by Mr. Cauchy, and we shall explain it by examples. We then study the properties of the functions defined by series ordered according to the increasing integer powers of the variable. This will allow us to establish, in a clear and precise manner, the necessary and sufficient conditions for a function to be expanded as a convergent series according to the increasing integer powers of the variable. In this way, we shall get rid of the clouds that still obfuscate the beautiful theorem of Mr. Cauchy.[9]

This paper, together with two other papers by Briot and Bouquet, became the bulk of their famous treatise *Théorie des fonctions doublement périodiques et, en particulier, des fonctions elliptiques* which we consider now. In that treatise, Cauchy's work is at the forefront. This treatise became famous especially by its second edition (1875), which carried the simpler name *Théorie des fonctions elliptiques*, cf. [19]. In the preface, the authors start by pointing out the importance of transcendental functions, recalling that Legendre spent almost all his life in trying to understand them. They then mention the works of Abel and Jacobi, declaring that Abel, around the year 1826, was the first to consider elliptic functions from the right point of view and to realize that these functions are doubly periodic. According to their account, Jacobi's *Fundamenta nova theoriæ functionum ellipticarum* [52], published three years later,

[9]Ce premier mémoire contient les principes de la théorie des fonctions d'une variable imaginaire. Nous adoptons la définition donnée par M. Cauchy, et nous l'expliquons par des exemples. Nous étudions ensuite les propriétés des fonctions définies par des séries ordonnées suivant les puissances entières et croissantes de la variable. Ceci nous permet d'établir, d'une manière nette et précise, les conditions nécessaires et suffisantes pour qu'une fonction se développe en série convergente suivant les puissances entières et croissantes de la variable. Nous faisons disparaître ainsi les nuages qui obscurcissent encore le beau théorème de M. Cauchy.

contains nothing essential which Abel had not discovered before. They declare that the difference between the two mathematicians is that Abel tried to prove the main results on the theory of elliptic functions from their double periodicity prorperty, whereas Jacobi did the same using algebraic reasonings which have the disadvantage of hiding the reason behind the results and which do not lead to interesting developments. Briot and Bouquet then write ([19] p. xviii of the Preface):

> Despite the remarkable works of these two great geometers, the theory of elliptic functions was still in the dark, and very complicated. Neither the double periodicity was recognized clearly, not the function itself was defined rigorously. To shed light on this theory, one had to introduce a new mathematical idea, and it is to the famous Cauchy that we owe this important progress.[10]

In this treatise, single-valued functions are called *monotropic* (monotropes) and multi-valued ones are called *polytropic* (polytropes). This terminology is introduced in the first pages of the second edition of the treatise (p. 9 and 11 of the 1875 edition). It indicates clearly that the authors think of these functions in terms of paths. (The word "tropos" in Greek means path.) Riemann's work (or, at least, its existence) is known to the authors, but they prefer to rely on Cauchy, completed by Puiseux. They write in the preface of the 1875 edition:

> In Cauchy's theory, the excursion of the imaginary variable is represented by the motion of a point in the plane. To represent the functions which acquire several values for the same value of the variable, Riemann used to look at the plane as composed of several sheets which are superposed and joined by weldings, in such a way that the variable can pass from a sheet to another by passing a junction line ("ligne de raccordement"). The conception of many-sheeted surfaces presents some difficulties; in spite of the beautiful results that Riemann reached by this method, it appeared to us that it has no advantage regarding the object we have in mind. Cauchy's idea is very well fit to the presentation of multiple functions; it suffices to attach to the value of the variable the corresponding value of the function, and, when the variable describes a closed curve and the value of the function changes, to indicate this change by an index.[11]

The authors acknowledge in the preface that they were influenced by Liouville's course at the Collège de France on elliptic functions, based on the double periodicity

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[10] Malgré les remarquables travaux de ces deux grands géomètres, la théorie des fonctions elliptiques restait fort obscure et très-compliquée; ni la double périodicité n'avait été reconnue d'une manière nette, ni la fonction elle-même définie d'une manière rigoureuse. Il fallait, pour éclairer cette théorie, l'introduction d'une idée nouvelle en mathématiques, et c'est à l'illustre Cauchy que l'on doit cet important progrès.

[11] Dans la théorie de Cauchy, la marche de la variable imaginaire est figurée par le mouvement d'un point sur un plan. Pour représenter les fonctions qui acquièrent plusieurs valeurs pour une même valeur de la variable, Riemann regardait le plan comme formé de plusieurs feuillets superposés et réunis par des soudures, de manière que la variable puisse passer d'un feuillet à un autre en traversant une ligne de raccordement. La conception des surfaces à feuillets multiples présente quelques difficultés; malgré les beaux résultats auxquels Riemann est arrivé par cette méthode, elle ne nous a paru présenter aucun avantage pour l'objet que nous avions en vue. L'idée de Cauchy se prête très bien à la représentation des fonctions multiples; il suffit de joindre à la valeur de la variable la valeur correspondante de la fonction, et, quand la variable a décrit une courbe fermée et que la valeur de la fonction a changé, d'indiquer ce changement par un indice.

of these functions. A set of notes by Liouville on lectures he gave in 1847 on doubly periodic functions were published 33 years later,[12] cf. [66]. It seems that Liouville considered that Briot and Bouquet stole his ideas, and he treated them as "unworthy robbers," see [78], p. 232.

Bottazzini reports in [55] (p. 244) that in 1861, Riemann lectured on complex function theory following Cauchy's point of view as contained in Briot and Bouquet's treatise. A German translation of this treatise was published in 1862 [18].

Briot

In 1879, Briot published a treatise entitled *Théorie des fonctions abéliennes* (Theory of Abelian functions) [20]. His goal in this new book is to explain Riemann's theory of Abelian functions. These are integrals of algebraic differentials on Riemann surfaces that generalize elliptic functions (which are defined on surfaces of genus one, that is, tori), and they played a major role in the development of complex analysis and of algebraic geometry. In the introduction to his treatise, Briot recalls that Riemann was the first to study these functions, and that he found beautiful theorems concerning them. He nevertheless declares that the methods of Riemann present enormous difficulties and he describes them as lacking of clearness and rigor. He announces that, in his treatise, he relies on the works of Clebsch and Gordan,[13] but leaving aside some of their geometric considerations. Sofia Kovalevskaya did not like Briot's treatise. In a letter to Mittag-Leffler, sent on January 8, 1881 quoted by the latter in his 1900 Paris ICM talk [67], she writes:

> Isn't it surprising how, at the time being, the theory of Abelian functions with all the particularities of its own method and which make it rightly one of the most beautiful branches

[12]The notes were taken by C. W. Borchardt, the editor of the Journal für die reine und angewandte Mathematik. In a footnote to the article, Borchardt writes about these notes: "When, in the first half of the year 1847 I stayed in Paris at the same time of my late friend Ferdinand Joachimstahl, Mr. Liouville accepted to give, at his home, for the two of us, a few lessons on his method for treating the theory of doubly periodic functions. I collected Mr. Liouville's lessons, and when, back in Berlin, I have completed writing them up, I sent him a copy of my manuscript which he authorized me to communicated to Jacobi and Lejeune-Dirichlet. [...] In communicating to the geometers a work done more than thirty years ago and without the intention of publishing it, I think nevertheless that I can assure that in general my redaction reproduces faithfully the lessons of Mr. Liouville. [Lorsque dans la première moitié de l'année 1847 j'ai fait un séjour à Paris en même temps que mon ami bien regretté Ferdinand Joachimstahl, M. Liouville a bien voulu nous faire chez lui à nous deux quelques leçons sur sa méthode de traiter la théorie des fonctions doublement périodiques. [...] En communiquant aux géomètres un travail fait il y a plus de trente ans et sans l'intention de le faire imprimer, je crois néanmoins pouvoir assurer qu'en général ma rédaction reproduit fidèlement les leçons de M. Liouville.

[13]The work of Clebsch and Gordan which was a major reference at that time is their treatise *Theorie der Abelschen Funktionen* (Theory of Abelian functions), 1866 [24]. One of the major results of Clebsch is a classification of algebraic curves using Riemann's theory of Abelian functions and based on his notion of birational transformation. Clebsch's ideas were further developed by Brill and Noether.

> of analysis, is still poorly studied and poorly understood everywhere else than in Germany? I was really outraged in reading, for instance, the *Traité des fonctions abéliennes* by Briot, which I had not seen before. How can one present such beautiful material in such a dry and with so little benefits for the students? I am almost not surprised any more that our Russian mathematicians, who know this theory only through Neumann's[14] book and that of Briot, profess such a profound indifference to the study of these functions.[15]

This book by Briot is the only treatise that he authored alone. The book won the Poncelet prize.

The works of Briot and Bouquet were influential on Poincaré who, in his *Analysis* of his own works (*Analyse des travaux scientifiques de Henri Poincaré faite par lui-même*), [88], declares that the starting point of his research on differential equations—which was his first topic of investigation—were the works of Cauchy, Fuchs, Briot, Bouquet and Kovalevskaya.

Appell and Goursat

We now consider the treatise *Théorie des fonctions algébriques et de leurs intégrales* (Theory of Abelian functions and their integrals) by Appell and Goursat, [4]. This treatise was published in 1895, that is, thirty-six years after the first edition of Briot and Bouquet's *Théorie des fonctions doublement périodiques et, en particulier, des fonctions elliptiques*. The treatise carries the subtitle *Étude des fonctions analytiques sur une surface de Riemann* (A study of analytic functions on a Riemann surface). A few biographical notes on the authors are in order; both of them are important representatives of the nineteenth century French school of analysis.

Paul Appell (1855–1930) was born in Strasbourg. He started studying mathematics at the University of this city, but had to flee from there, in order to remain French, after the annexion of Alsace by Germany, in 1870.[16] His brother, who stayed in occupied Alsace, was later convicted for "anti-German activities." Appell wrote his

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[14]The book by Neumann which is referred to in this quote is certainly his treatise *Vorlesungen über Riemann's Theorie der Abel'schen Integrale* (Lectures on Riemann's theory of Abelian integrals), published in 1865, [69]. Unlike the French treatises, Neumann's book was written in the spirit of Riemann.

[15]N'est-il pas étonnant vraiment comme, à l'heure qu'il est, la théorie des fonctions abéliennes avec toutes les particularités de la méthode qui lui sont propres et qui en font justement une des plus belles branches de l'Analyse, est encore peu étudiée et peu comprise partout ailleurs qu'en Allemagne ? J'ai été vraiment indignée en lisant, par exemple, le *Traité des fonctions abéliennes* par Briot, qui jusqu'à présent ne m'était pas tombé sous les yeux. Peut-on exposer une aussi belle matière d'une manière aussi aride et aussi peu profitable pour l'étudiant ? Je ne m'étonne presque plus que nos mathématiciens russes, qui ne connaissent toute cette théorie que par le livre de Neumann et celui de Briot, professent une indifférence aussi profonde pour l'étude de ces fonctions.

[16]In a chronicle on Appell which appeared in *Le petit parisien* (18/02/1929) it is reported that when he came back to Strasbourg, after the Second World War, he whispered: "I thought I was becoming crazy when I saw the French flag fleeting on our old cathedral. On that day, my life was filled. I could well have died." [Je croyais devenir fou en voyant le drapeau tricolore flotter sur notre chère cathédrale, murmure-t-il. Ce jour-là, ma vie était comblée. J'aurais pu mourir.]

doctoral dissertation under Chasles, on projective geometry. The title of this dissertation is *Sur la propriété des cubiques gauches et le mouvement hélicoïdal d'un corps solide* (On the properties of skew cubics and on the helocoidal motion of a solid body) [2]. The thesis was published in the Annales de l'École Normale Supérieure, [3]. Besides being a mathematician, Appell was the rector of the *Académie de Paris* from 1920 to 1925, and he became secretary general of France at the League of Nations. He is also the founder of the Paris *Cité Universitaire Internationale*. He married a niece of Bertrand and Hermite, and his daughter became the wife of Emile Borel. Appell, like many other French mathematicians of his generation (see Chap. 7 of the present volume, [77]), was profoundly religious.[17] There is an interesting correspondence between Appell and Poincaré, see [86].

Édouard Goursat (1858–1936) had as teachers Briot, Bouquet and Darboux. Goursat started as a teaching assistant ("agrégé préparateur") at the École Normale Supérieure in 1879, and one year later he was appointed at the Faculté des Sciences de Paris, taking over the position of Picard who was appointed at Toulouse. In 1881 he submitted a doctoral dissertation bearing the title *Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique* (On the linear differential equation that admits as integral the hypergeometric series), [32]. The thesis committee consisted of Bouquet, Darboux and Tannery. It was published in the Annales de l'École Normale Supérieure [33]. This dissertation, written under Darboux, is based on results of Jacobi and Riemann, and it uses Cauchy's theory. Among other things, Goursat simplifies a proof of a theorem given by Riemann in his memoir of the hypergeometric function [97] (Second part of Goursat's dissertation). After his dissertation, he took a position at the Faculté des Sciences de Toulouse, as the successor of Picard who returned to Paris. In 1885, he came back to the École Normale Supérieure, replacing Bouquet. In 1897, he took over again Picard's position at the Chair of Differential and Integral Calculus at the Faculté des Sciences de Paris. The name of Goursat is attached to a theorem in complex function theory, which is usually referred to as the *Cauchy-Goursat* theorem. It says that given a holomorphic function on a simply connected domain in the plane, the integral of this function over a loop contained in the interior of the domain is zero. The first step of the proof is a lemma, called the Goursat lemma, which is a particular case of the theorem in which the loop bounds a rectangle. The result is contained in the 1814 paper of Cauchy [23] but under some unnecessary strong hypotheses on the function. Goursat's proof is contained in a paper that appeared in Acta Mathematica entitled "Proof of Cauchy's theorem" [36].

Unlike the case of the treatise of Briot and Bouquet, Riemann's theory is well present in the treatise *Théorie des fonctions algébriques et de leurs intégrales* by Goursat and Appell. Hermite wrote the preface of that treatise. In this preface, he starts by giving an overall summary of the work of Puiseux on algebraic functions,

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[17] In a biography of Hermite, written by his grand-daughter (the manuscript, kept in the Archives of the Académie des Sciences de Paris) quoted in [48] p. 79, we read that Hermite told Appell once, "Can you imagine, my dear Appell, that after our death, we shall at last contemplate, face to face, the number π and the number e?" [Songez-vous, mon cher Appell, qu'après la mort nous contemplerons enfin face à face le nombre π et le nombre e ?].

which, he says "opened the field of research which led to the great discoveries of our epoch." He declares that this work transformed the field of analysis by giving it new bases.[18] Hermite, in his introduction, also mentions the influence of Cauchy. After that, he passes to the work of Riemann, praising this work and announcing that the treatise is based on the latter's ideas. Hermite writes in this introduction:

> The works of Puiseux were followed, in 1857, by those of Riemann, received with a unanimous admiration, as the most considerable event in analysis of our times. The present treatise is dedicated to the exposition of the work of this great geometer, and to the researches and the discoveries to which it led.
>
> A remarkably original concept is at their foundation. These are the surfaces to which is attached the name of their discoverer. They are constituted of superposed planes, whose number is equal to the degree of an algebraic equation, connected among themselves by crossing lines, which we obtain by joining in a certain manner the critical points. The establishment of these lines is a first question of great importance, which later on was made much simpler and easier by a beautiful theorem of Mr. Lüroth. After that, we are offered the notion of connected surfaces, their order of connection, the theorems on the lowering, using cuts, which lead the surface to a simply connected one. From these profound and delicate considerations follows a geometric representation, which is an element of the greatest power for the study of the algebraic functions. It would be too long to recall all the discoveries that carry the seal of the greatest mathematical genius, to which it led Riemann. [...][19]

In their treatise, Goursat and Appell present Riemann's topological theory of surfaces and their dissection, his theory of the complex-analytic Riemann surfaces, and his theory of Abelian integrals. Cauchy's calculus of residues is used, as well as Puiseux' method of dealing with multiple branch points of algebraic functions. The treatise also contains an exposition of Riemann–Roch's theorem, of the Brill-Noether law of reciprocity, of Abel's theorem and of the theory of moduli of algebraic curves. Jacobi's inversion problem of Abelian integrals, and a problem of Briot and Bouquet on the uniformization of solutions algebraic differential equations are addressed. W. F. Osgood published an extensive review of Appell and Goursat's treatise in the Bulletin of the AMS, see [72].

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[18] The reader may find details on the work of Puiseux, and its relations to the works of Cauchy, Hermite and others, in Chap. 7 of the present volume [77].

[19] Aux travaux de Puiseux succèdent, en 1857, ceux de Riemann accueillis par une admiration unanime, comme l'événement le plus considérable dans l'analyse de notre temps. C'est à l'exposition de l'œuvre du grand géomètre, des recherches et des découvertes auxquelles elle a donné lieu qu'est consacré cet ouvrage.

Une conception singulièrement originale leur sert de fondement, celle des surfaces auxquelles est attaché le nom de l'inventeur, formées de plans superposés, en nombre égal au degré d'une équation algébrique, et reliés par des lignes de passage, qu'on obtient en joignant d'une certaine manière les points critiques. L'établissement de ces lignes est une première question de grande importance, rendue depuis beaucoup plus simple et plus facile par un beau théorème de M. Lüroth. S'offre ensuite la notion des surfaces connexes, de leurs ordres de connexion, les théorèmes sur l'abaissement par des coupures qui ramènent la surface à être simplement connexe. De ces considérations profondes et délicates résulte une représentation géométrique, qui est un instrument de la plus grande puissance pour l'étude des fonctions algébriques. Il serait trop long de rappeler toutes les découvertes portant l'empreinte du plus grand génie mathématique, auxquelles elle conduit Riemann [...].

Goursat

Goursat is mostly known today for his *Cours d'analyse mathématique* (A course in mathematical analysis) [34], a treatise which became a reference for all French students in mathematics. The first edition of that book, in two volumes, was published in 1902 and 1905. A second edition, in three volumes, appeared between 1910 and 1915, a third edition in 1917–1923, a fourth edition in 1923–1927, a fifth edition in 1933–1942, and there were several later editions after Goursat's death in 1936. The treatise was translated into English, cf. [35]. The whole treatise is a systematic treatment of analysis, including integration and differential equations. The subtitles of the various volumes of Goursat's *Cours* give an idea of the content. They are (in the final three-volume version): Volume I: *Dérivées et différentielles. Intégrales définies. Développements en séries. Applications géométriques.* (Derivatives and differentials. Definite integrals. Series expansions. Geometrical applications). Volume II: *Théorie des fonctions analytiques. Equations différentielles. Equations aux dérivées partielles du premier ordre.* (Theory of analytic functions. Differential equations. First order partial differential equations). Volume III: *Intégrales infiniment voisines. Équations aux dérivées partielles du second ordre. Équations intégrales. Calcul des variations* (Infinitely close integrals. Second order partial differential equations. Integral equations. Calculus of variations).

In his treatise, Goursat, in presenting the theory of functions of a complex variable, relies on Cauchy's methods on the theory of complex integration and on the existence of solutions for ordinary and partial differential equations. Weierstrass's methods are also presented, in particular for what concerns singular points and series of analytic functions, and the calculus of variations. Riemann's theories are briefly addressed in Volume III, Chap. XXVII, in relation with the Laplace equation. The author discusses, besides the methods of Riemann, those of Neumann, Schwarz and others, in relation with conformal mappings.

Osgood wrote two reviews for the Bulletin of the AMS, [73, 74], on Goursat's first edition (two volumes) of his treatise. As a conclusion to his review of Volume I, Osgood writes the following:

> When the future historian inquires how the calculus appeared to the mathematicians of the close of the nineteenth century, he may safely take Professor Goursat's book as an exponent of that which is central in the calculus conceptions and methods of this age.

Goursat's treatise lost its prestige with the advent of Bourbaki, and it was replaced in the French university curricula by the more rigorous (in the modern standards) treatises of Dieudonné, Cartan, Schwartz, etc.

Picard

Emile Picard (1856–1941) was one of those mathematicians whose work, encompassing a period straddling the nineteenth and the twentieth centuries, exerted an important influence on mathematics by giving it a new direction. In 1877, he submitted a doctoral dissertation on the geometry of Steiner surfaces, written under the guidance of Darboux. The title of the dissertation is *Application de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches* (Application of the theory of linear complexes to surfaces and skew curves) [80]. Picard's thesis was also published in the Annales de l'École Normale, [81]. Picard had a long career during which he worked on ordinary and partial differential equations, algebraic geometry, algebra, mechanics, elasticity, heat, electricity, relativity, astronomy and on other subjects of mathematics and theoretical physics. But he was above all an analyst. His name is attached in particular to two theorems he obtained in 1879 which exerted a tremendous influence on analysis. One of these theorems says that a non-constant entire function takes every complex value an infinite number of times, possibly with one exception. Picard's proof of this result uses Hermite's theory of elliptic modular functions. It is short, elegant but indirect. Giving simpler proofs and generalizations of that theorem gave rise to a large number of works done by several generations of mathematicians, including Borel, Hadamard, Montel, Julia, Bloch, Carathéodory, Landau, Lindelöf, Milloux, Schottky, Valiron, Nevanlinna, Ahlfors and several others. These works resulted in a thorough investigation of the nature of holomorphic functions and they led to a whole field of mathematics called *value distribution theory*. When the young Picard (he was 23) published his two theorems, he attracted the attention of Hermite, and they soon became friends. Two years later (in 1881), Picard married Hermite's daughter. Between 1895 and 1937, Picard taught mechanics at an engineering school in Paris, the École Centrale des Arts et Manufactures. Picard was also a philosopher and a historian of science. In 1917, Picard lost his son (who was therefore Hermite's grand-son) at the war.

In 1891, Picard published the first volume of his *Traité d'analyse* (Treatise on analysis) [79], a treatise in three volumes (the second volume was published in 1893 and the third one in 1896). This treatise was acclaimed as one of the important writings of its epoch. In a 27-page review of the first two volumes published by T. Craig in the Bulletin of the AMS, the author writes:

> One of the ablest of American mathematicians said to the writer not long ago, 'we have waited fifty years for the book!'

Cauchy's theory and all the introductory material on functions of a complex variable are presented in Volume I of Picard's *Traité* (1891). Riemann's ideas play a central role in Volume II (1893). Picard writes in the introduction to that volume:

> This volume contains the lessons I gave at the Sorbonne during the last two years. It is primarily dedicated to harmonic and to analytic functions. Without leaving aside Cauchy's point of view on the theory of analytic functions, I mainly dwell on a thorough study of harmonic functions, i.e., of the Laplace equation; a large section of this volume is dedicated to that famous equation, on which depends all the theory of analytic functions. I also dwell at length on the principle of Dirichlet, which plays such a big role in the works of Riemann, and which is as much important for mathematical physics as for analysis.
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> Among the particular functions I study, I note the algebraic functions and the Abelian integrals. A chapter deals with Riemann surfaces, whose study has been too much left over in France. It is possible, by a convenient geometric representation, to make intuitive the main

results of this theory. Once this clear view of the Riemann surface is obtained, all the applications are conducted with the same facility as the classical Cauchy theory relative to the ordinary plane. But it is important to judge according to its real value the beautiful conception of Riemann. It would be an incomplete view to regard it only as a simplified method of presenting the theory of algebraic functions. No matter how important is the simplification brought in this study by the consideration of surfaces with many leaves, it is not there that the interest of Riemann's ideas lies. The essential point of his theory is the *a priori* conception of the connected surface formed by a finite number of plane leaves, and in the fact that to such a surface conceived in full generality corresponds a class of algebraic curves. Thus, we did not want to mutilate the profound thought of Riemann, and we have dedicated a chapter to the capital and difficult question of the existence of analytic functions on an arbitrarily given Riemann surface. The problem itself is susceptible of generalization, if we take an arbitrary closed surface in space and if we consider the corresponding Beltrami equation.[20]

Riemann surfaces are introduced in Chap. XIII of Volume II. They are associated with algebraic equations of the form $f(u,z)=0$ where f is a polynomial in the two variables u and z. Their construction uses the method of paths and the analysis of permutations of roots developed by Puiseux which we describe in Chap. 7 of the present volume [77]. On the resulting Riemann surface, we have a single-valued function u of z. Picard writes that "the algebraic function u is uniform: to each point on that surface is associated a single value of u, which is the value corresponding to the leaf on which we find the point that we consider." He proves that the surface obtained by this construction is connected, and he spends some time explaining how one obtains a simply-connected surface from an arbitrary Riemann surface by performing a certain number of cuts. Picard refers to Riemann's article on Abelian functions [94], to Simart's dissertation [103] which we consider below, and to papers

[20] Ce second volume contient les leçons que j'ai faites à la Sorbonne ces deux dernières années. Il est principalement consacré aux fonctions harmoniques et aux fonctions analytiques. Sans négliger le point de vue de Cauchy dans la théorie de ces dernières fonctions, je me suis surtout attaché à une étude approfondie des fonctions harmoniques, c'est-à-dire de l'équation de Laplace; une grande partie de ce volume est consacrée à cette équation célèbre, dont dépend toute la théorie des fonctions analytiques. Je me suis arrêté longuement sur le principe de Dirichlet, qui joue un si grand rôle dans les travaux de Riemann, et qui est aussi important pour la physique mathématique que pour l'analyse.

Parmi les fonctions particulières que j'étudie, je signalerai les fonctions algébriques et les intégrales abéliennes. Un chapitre traite des surfaces de Riemann, dont l'étude a été laissée un peu trop de côté en France; on peut, par une représentation géométrique convenable, rendre intuitifs les principaux résultats de cette théorie. Cette vue claire de la surface de Riemann une fois obtenue, toutes les applications se déroulent avec la même facilité que dans la théorie classique de Cauchy relative au plan simple. Mais il importe de juger à sa véritable valeur la belle conception de Riemann. Ce serait une vue incomplète que de la regarder seulement comme une méthode simplificative pour présenter la théorie des fonctions algébriques. Si importante que soit la simplification apportée dans cette étude par la considération de la surface à plusieurs feuillets, ce n'est pas là ce qui fait le grand intérêt des idées de Riemann. Le point essentiel de sa théorie est dans la conception a priori de la surface connexe formée d'un nombre limité de feuillets plans, et dans le fait qu'à une telle surface conçue dans toute sa généralité correspond une classe de courbes algébriques. Nous n'avons donc pas voulu mutiler la pensée profonde de Riemann, et nous avons consacré un chapitre à la question difficile et capitale de l'existence des fonctions analytiques sur une surface de Riemann arbitrairement donnée; le problème même est susceptible de se généraliser, si l'on prend une surface fermée arbitraire dans l'espace et que l'on considère l'équation de Beltrami qui lui correspond.

by Clebsch and Lüroth. Chapter XIV of Volume II of Picard's treatise concerns periods of Abelian integrals, another topic which was dear to Riemann. Chapter XVI contains several results on meromorphic functions on Riemann surfaces, including the Riemann–Roch theorem. These are the famous *Riemann existence theorems*.[21] The title of this chapter is: "General theorems relative to the existence of functions on Riemann surfaces." Picard summarizes first the work he did in the previous chapters ([79], Vol. II, beginning of Chap. XVI). To an algebraic equation $f(x, y) = 0$ as above, a Riemann surface is associated, and on that surface, functions and integrals are studied. The problem addressed now is the converse: one starts with a connected Riemann surface which, Picard says, is defined a priori and "in a purely geometrical manner," taking a certain number of leaves and joining them by a certain number of "intersection curves" (*lignes de croisement*). One wishes to associate with such an abstract surface a class of algebraic curves, and to show a priori the existence of the functions of the type considered before. After formulating this problem, Picard writes: "We thus enter in the profound thought of Riemann." He declares that the previous chapters diverged from Riemann's ideas, in that one started there from a curve, or from an algebraic relation, whereas now, "the starting point is the m-sheeted Riemann surface." He adds (p. 459):

> Unfortunately, Riemann's method, which was so simple for establishing general existence theorems, does not have the rigor which we require today in the theory of functions. It relies on the consideration of the minimum of certain integrals which are very similar to those we already studied in the Dirichlet problem, and the same objections were addressed to him. Another way had to be found, and Mr. Neumann and Mr. Schwarz reached it independently.[22]

Picard mentions the references [69] (pp. 388–471) and [102] (p. 303), and from there he reconstructs completely the proof. In Sects. 6–13 of this chapter, the author studies the existence of harmonic functions on Riemann surfaces. These functions are used in the proof of the existence theorem. We note incidentally that for several decades, all the proofs of Riemann's existence theorem were based, like the one of Riemann, on potential theory. Picard states the main result of that chapter as a "fundamental theorem" ([79] Tome II, Chap. XVI, §18):

> *To an arbitrary Riemann surface there corresponds a class of algebraic curves.*

Another "fundamental theorem" is stated in §28 of the same chapter:

> *To a surface in space having p holes, corresponds uniformly an algebraic curve of genus p.*

Without entering into the technical definition of the genus of an algebraic curve, let us simply say that this is a birational invariant and that the equality between a

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[21] Picard indeed uses the plural for Riemann's existence theorems.

[22] Malheureusement, la méthode si simple de Riemann pour établir les théorèmes généraux d'existence ne présente pas la rigueur qu'on exige aujourd'hui dans la théorie des fonctions. Elle repose sur la considération du minimum de certaines intégrales tout à fait analogues à celles que nous avons déjà étudiées dans le problème de Dirichlet et on lui a adressé les mêmes objections. Il a donc fallu chercher dans une autre voie. M. Neumann et M. Schwarz y sont parvenus, chacun de son côté.

notion from birational geometry and a topological notion is one of the major ideas of Riemann. It is interesting to read Picard's footnote to the theorem:

> This theorem was stated by Mr. Klein in his work which we quoted several times on the *Theory of Riemann surfaces*. The method of proof of Mr. Klein is extremely interesting, even though it does not pretend to be rigorous from the analytical viewpoint. The author borrows the elements of his proof to a fictive electrical experience performed on the surface. Thus, the existence of potential functions together with their various singularities is, in some way, proved experimentally.[23]

Section 5 of Chap. XVI concerns moduli of algebraic curves. Picard starts by addressing a preliminary question raised by Riemann: Suppose we are given in the complex plane of the variable z, the $2(m + p - 1)$ ramification points of a Riemann surface of genus p with m sheets. (The count was carried on in §19 of Chap. XIII of Picard's treatise.) The question is to find the number of such surfaces. Picard notes that this number is finite, and that Hurwitz found it for small values of m. The question then is to find the number of arbitrary parameters on which a Riemann surface of some fixed genus p "essentially" depends. This is the famous moduli problem raised by Riemann and solved in a satisfactory manner by Teichmüller in his seminal paper [108]. Picard describes two methods, which are both due to Riemann, for computing these moduli. One of them relies on the Riemann–Roch theorem, and the other one uses a conformal representation of a Riemann surface onto a polygon, using an integral of the first kind, and a count of the number of periods of such integrals. The result of each of these methods is Riemann's count of the number of moduli, that is, $3p - 3$, for a closed surface of genus p.

Picard concludes this important chapter by explaining how these ideas are used in the conformal representation of multiply-connected surfaces.

Picard-Simart

We now consider Picard and Simart's *Théorie des fonctions algébriques de deux variables indépendantes* (Theory of algebraic functions of two independent variables) [83], a treatise in two volumes, published in 1897 and 1906 respectively. The level of difficulty is higher than most of the other French treatises of the same period on the same subject, and the topics treated are more specialized. The introduction in each volume is written by Picard. In the introduction to the first volume, Picard declares that since a long time he had the intention to resume his ancient research on algebraic functions of two variables and to present them in a didactical form. He writes that he realized that, for more clarity, it was necessary to take into account the classical work

[23] Ce théorème a été énoncé par M. Klein dans son ouvrage déjà bien des fois cité sur la *Théorie des surfaces de Riemann*. Le mode de démonstration de M. Klein est extrêmement intéressant, quoi qu'il ne prétende pas à être rigoureux au point de vue analytique. C'est à une expérience électrique fictive faite sur la surface que l'illustre auteur emprunte les éléments de ses démonstrations. L'existence des fonctions potentielles avec leurs singularités diverses se trouve ainsi démontrée en quelque sorte expérimentalement.

of Mr. Noether as well as several works done in Italy on the same subject. The book contains indeed sections on invariants of algebraic surfaces and integrals of total differentials, including a study of the invariants introduced by Clebsch and Noether, and an exposition of the works of Castelnuovo and Enriques. Picard declares that his co-author and himself by all means "do not have the pretentiousness of going deeply into all the questions that are addressed in this "very difficult theory," but that their unique goal is "to give the state of the art on a question that deserves the effort of several researches."[24]

In the first volume, the authors develop Riemann's ideas on integrals of Abelian differentials and on Riemann surfaces, from the topological viewpoint. The title of the first chapter is *On multiple integrals of functions of several variables*. The theories of multiple integrals and integrals of total differentials constitute a link between several questions addressed in this treatise. They are generalizations of the Abelian integrals that were studied by Riemann, and they lead Picard and Simart to study hypersurfaces in a five-dimensional space. This is why the authors are led, in Chap. 2, to questions of topology in an n-dimensional space. Indeed, the second chapter is dedicated to *geometry of situation* (topology). By the time Picard and Simart's treatise was written, Poincaré had already published his famous paper with this title, two years before, in the Journal de l'École Polytechnique [87]. Picard and Simart show in particular that the genus of a Riemann surface is determined by the number of linear independent integrals of the first kind on such a surface. At the beginning of this chapter, they write (p. 19):

> This theory was founded by Riemann, who gave the name. In his study of Abelian functions, the great geometer considers only two-dimensional spaces, but later on he generalized his researches to an arbitrary number of dimensions, as is shown by his notes published after his death in the volume containing his Complete Works. Independently of Riemann, Betti studied various orders of connectivity in n-dimensional spaces, and he published a fundamental memoir on this subject.[25] In his memoir on algebraic functions of two variables, Mr. Picard showed the usefulness of such considerations in the study of algebraic surfaces. Very recently, Mr. Poincaré[26] took up in a general manner this question of *Analysis situs*, and after completing it and making more precise the results obtained by Betti, he drew attention to the considerable differences that the theories present, the two-dimensional and the higher-dimensional ones.[27]

[24] Nous n'avons certes pas la prétention d'approfondir toutes les questions qui se posent dans cette théorie difficile; notre seul but est de donner une idée de l'état actuel de la science sur un sujet dont l'étude mérite de tenter l'effort de nombreux chercheurs.

[25] Annali di Mathematica, t. IV (1870–71).

[26] Journal de Mathématiques (1899).

[27] Cette théorie a été fondée par Riemann, qui lui a donné ce nom; dans ses études sur les fonctions abéliennes, le grand géomètre ne considère que les espaces à deux dimensions, mais il a ensuite généralisé ses recherches pour un nombre quelconque de dimensions, comme le montrent les notes publiées après sa mort dans le volume renfermant ses œuvres complètes. Indépendamment de Riemann, Betti avait de son côté étudié les divers ordres de connexion dans les espaces à n dimensions, et publié un mémoire fondamental sur ce sujet. Dans son mémoire sur les fonctions algébriques, M. Picard avait montré l'intérêt que présentent des considérations de ce genre dans l'étude des surfaces algébriques. Tout récemment, M. Poincaré a repris d'une manière générale cette question

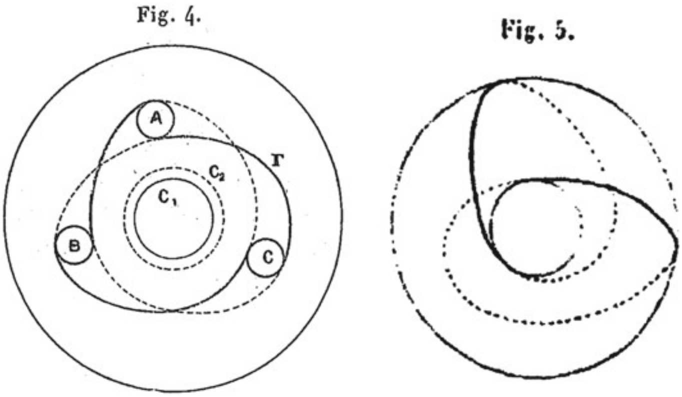


Fig. 3 Simple closed curves on surfaces, from the treatise by Picard and Simart [83]

On p. 22 of the same volume, the authors consider a Riemann surface as "a 2-dimensional manifold in 3-dimensional space," allowing the surfaces to traverse each other. The authors show that a surface defined by algebraic equations and inequalities is necessarily orientable. They introduce boundaries, Betti numbers, and their relations with multiple integrals. Homotopy classes of simple closed curves on orientable topological surfaces are drawn (cf. Fig. 3). The authors prove, at the end of Chap. 2, that for a general closed "multiplicity" (a word used by Riemann), the first and the last Betti numbers are equal, which is a special case of the result of Poincaré saying that two Betti numbers which are equidistant from the extreme ones are equal.

The 3rd chapter is dedicated to the extension of Cauchy's theorem to double integrals of functions of two variables, an extension due to Poincaré, and to residues of double integrals of rational functions. The 4th chapter concerns the reduction of singularities of an algebraic surface, and the study of its topological invariants. The authors prove in particular that any algebraic surface is birationally equivalent to a nonsingular surface embedded in the 5-dimensional space. Chapters 5 and 6 concern integrals of total differentials, and Chap. 7, double integrals.

In Volume II of the treatise, published nine years after the first one, the authors present the recent results, obtained by Picard, Castelnuovo, Enriques and others, on questions that were already addressed in the first volume and their extensions. In particular, the reduction theory for singularities of an algebraic surface is revisited, as well as the theory of double integrals of the second kind, in particular, their invariants and their periods.

(Footnote 27 continued)
dans l'*Analysis situs*, et, après avoir complété et précisé les résultats obtenus par Betti, a appelé l'attention sur les différences considérables que présentent ces théories, suivant qu'il s'agit d'un espace à deux dimensions ou d'un espace à un plus grand nombre de dimensions.

Appel-Goursat-Fatou

Riemann surfaces are also thoroughly studied in the first volume of the treatise *Théorie des fonctions algébriques et de leurs intégrales et des transcendantes qui s'y rattachent* (Theory of algebraic functions and their integrals, and their related transcendentals) [6] by Appell, Goursat and Fatou, which appeared in 1929. In reality, the treatise is a revised edition, by Fatou, of the treatise [4] by Appell and Goursat. Fatou was at the same time a mathematician and an astronomer. In 1906, he defended a thesis entitled *Séries trigonométriques et séries de Taylor* (Trigonometric series and Taylor series), [29, 30], whose subject is Lebesgue's integration theory, which in some sense is a refinement of Riemann's integration theory (see Sect. 5 below). It is in this thesis that we find the famous Fatou Lemma (also called the Fatou-Lebesgue Lemma) on the comparison between the integral of a lower limit of positive measurable functions and the lower limit of their integrals. The lemma is a key element in the proof of the Dominated Convergence Theorem. In the same year, Fatou started his work on the iteration of rational maps of the plane, a work that was revived in the last two decades of the twentieth century by Sullivan, Thurston and others. Fatou also worked on the dynamics of transcendental functions.

The title of the first volume of the treatise by Appell, Goursat and Fatou is *Étude des fonctions analytiques sur une surface de Riemann* (Study of analytic functions on a Riemann surface) [6]. In that treatise, Riemann surfaces are still represented, like in the 19th-century treatises, in an anthropomorphic fashion, (using Weyl's expression; see Sect. 2 of the present article). Figure 4 is extracted from that volume, and is already contained in the first edition by Appell and Goursat (Fig. 1 in Sect. 2 above). The authors declare, concerning the surface considered: "This surface is analogous to that represented in Fig. 10, with the difference that, in reality, the two leaves are infinitely close and the apertures are infinitely narrow.

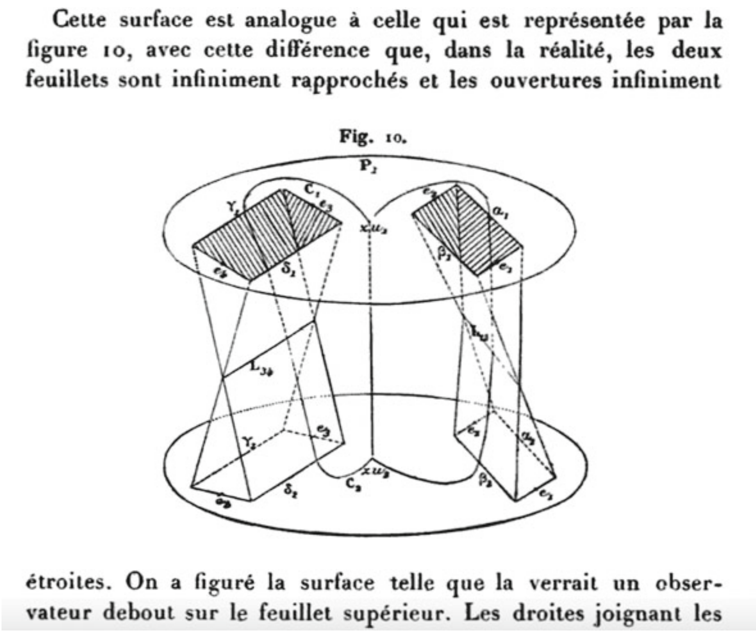
Chapter III of this volume is entitled *Connexion des surfaces à deux feuillets. Périodicité des intégrales hyperelliptiques* (Connectivity of two-sheeted surfaces and periodicity of hyperelliptic integrals). The authors start by saying (p. 99):

> In what follows, we consider surfaces as leaves without thickness, in such a way that a point or a line drawn on that surface will be visible for a observer situated on one side or the other. These surfaces will be considered as *perfectly elastic and rip-stop*.[28]

Halphen

Among the other treatises that are related to Riemann surfaces, we mention Halphen's *Traité des fonctions elliptiques et de leurs applications* (A treatise on elliptic fonctions and their applications) in 3 volumes, published in 1886, 1888 and 1891 [41].

[28] Dans ce qui suit, nous considérons des surfaces comme des feuillets sans épaisseur, de sorte qu'un point ou une ligne tracée sur la surface seront visibles pour un observateur placé d'un côté ou de l'autre. Les surfaces seront en outre regardées comme *parfaitement élastiques et indéchirables*.



Cette surface est analogue à celle qui est représentée par la figure 10, avec cette différence que, dans la réalité, les deux feuillets sont infiniment rapprochés et les ouvertures infiniment

Fig. 10.

étroites. On a figuré la surface telle que la verrait un observateur debout sur le feuillet supérieur. Les droites joignant les

Fig. 4 A picture from the treatise by Appell, Goursat and Fatou [6]. One can read in the text above the figure: "This surface is analogous to the one presented in Fig. 10, with the difference that, in reality, the two sheets are infinitely close, etc.," and, below the figure: "We have represented the surface in the way an observer standing on the upper sheet would see it."

This treatise had a certain impact on students in algebra and analysis. Georges-Henri Halphen, was a graduate of the École Polytechnique,[29] and he started with a career in the army. He submitted a doctoral dissertation on 1878, titled *Sur les invariants différentiels* (On differential invariants) [40], in which he determined the invariants of planar or skew curves under projective transformations. His thesis committee consisted of Hermite, Bouquet and Darboux. Haplhen participated to the 1870 French-German war. In 1872, he was appointed *répétiteur*[30] at the École Polytechnique. He was a specialist, among other things, of differential invariants, elimination theory, and singularities of algebraic curves. Picard, in biography of Halphen [82], writes the following (p. x of the Introduction):

> Riemann, in his theory of Abelian functions, had introduced the major notion of genus of elliptic curves, and he classified them into different classes, two curves being in the same class whenever there is a uniform correspondence between them. The famous geometer, who liked the great horizons, passed quickly on more than one difficult point, in particular, for what concerns higher singularities. Halphen gave a general formula, which applies to all cases, for the determination of the genus of an algebraic curve. Then, passing to the study of curves belonging to the same class, he went deeper into a remarkable proposition of

[29]We remind the reader that the École Polytechnique is a military school.

[30]A kind of a teaching assistant.

> Mr. Nœther according to which one may find in every class curves that have only ordinary singularities [...][31]

The first part of Halphen's treatise concerns the general theory of elliptic functions. The second part makes this treatise special compared to the other treatises on the same subject: it concerns the applications of elliptic functions to various branches of mathematics and physics. The subtitle of that volume is *Applications à la mécanique, à la physique, à la géodésie, à la géométrie et au calcul intégral* (Applications to mechanics, physics, geodesy, geometry and integral calculus). It was known since the eighteenth century, that is, since the birth of the theory of elliptic functions, that these functions have many applications in physics. It suffices to recall in this respect that these functions are in some sense generalizations of the familiar trigonometric functions, and that they can be used to represent a large class of periodic phenomena. For instance, whereas the small oscillations of a pendulum are represented by the sine functions (which is the inverse function of the elliptic integral $\int_0^x \frac{dt}{\sqrt{1-t^2}}$), for large oscillations, one needs (inverses of) more general elliptic integrals. By the time of Riemann, elliptic integrals were used in problems of gravitation and electromagnetism. We recall in this respect that the famous treatise of Legendre, *Exercices de calcul intégral* (Exercises of integral calculus) [63] contains a substantial part on elliptic integrals and their applications to problems in geometry and mechanics. We also note that the subtitle of the first volume of Legendre's *Traité des fonctions elliptiques et des intégrales eulériennes* (Treatise on elliptic functions and Eulerian integrals) [64] is: *Contenant la théorie des fonctions elliptiques et son application à différents problèmes de géométrie et de mécanique* (Containing the theory of elliptic functions and its application to various problems of geometry and mechanics). One may also mention in this respect that expressions of the lengths of arcs of an ellipse (which are precisely given by elliptic integrals) are obviously useful in celestial mechanics, since Kepler's first law says that orbits of planets in the solar system are ellipses with the Sun at one of their two foci. His second law says that a segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. We also recall that Gauss was also an astronomer, and his interest in elliptic functions was motivated by his work on the trajectories of planets. Finally, Abel's 1827 famous paper on elliptic functions that we already mentioned, starts by mentioning the "beautiful properties" of Abelian functions "and their applications." He writes ([1] p. 101):

> Since a long time, the logarithmic functions, and the exponential and circular functions were the only transcendental functions that attracted the attention of the geometers. It is only in

[31] Riemann, dans sa théorie des fonctions abéliennes, avait introduit la notion capitale du genre des courbes algébriques, et partagé celles-ci en différentes classes, deux courbes étant de la même classe quand elles se correspondent uniformément. L'illustre géomètre, qui aimait les grands horizons, avait peu insisté sur plus d'un point difficile, en particulier sur ce qui concerne les singularités élevées. Halphen donne une formule générale, applicable à tous les cas, pour la détermination du genre d'une courbe algébrique; puis, passant à l'étude des courbes d'une même classe, il approfondit une proposition remarquable donnée par M. Noether, d'après laquelle on peut trouver dans toute classe des courbes n'ayant que des singularités ordinaires [...].

> recent times that some other functions started to be considered. Among them one has to distinguish the so-called elliptic functions, at the same time because of their beauty and of their use in the various branches of mathematics."[32]

The applications to geodesy mentioned by Halphen concern the geodesics on an ellipsoid of revolution whose ratio of major to minor axis is close to 1. Such a body is a representation of the shape of the Earth. It is also well known that Gauss was highly interested in geodesy. The applications of elliptic functions to geodesy were also considered by Jacobi in his paper [53]. In that paper, Jacobi solves a problem in geodesy which was addressed by Gauss. More details on elliptic functions are given in Chap. 1 of the present volume [76].

The third volume of Halphen's treatise contains fragments on elliptic functions which were collected after Halphen's death and published by Stieltjes.[33] The volume also contains Picard's biography of Halphen [82] which we already quoted. Picard declares there that Halphen was "one of the most eminent geometers in Europe."

Tannery and Molk

We now review the 4-volume treatise *Éléments de la théorie des fonctions elliptiques* (Elements of the theory of elliptic functions) [107] by Tannery and Molk. A few words on the authors are in order.

Jules Tannery (1848–1910) was a geometer, philosopher and writer. He edited the correspondence between Lagrange and d'Alembert.

In 1874, Tannery defended a doctoral dissertation whose title is *Propriétés des intégrales des équations différentielles linéaires à coefficients variables* (Properties of the integrals of linear differential equations with variable coefficients) [105] and [106]. The thesis committee consisted of Hermite, Briot and Bouquet. The dissertation starts with the following:

> The study of functions of an imaginary variable defined by an equation, a study which was substituted to the research, often unworkable, of the explicit form of these functions, profoundly renewed analysis in this century. It is well known that the glory of having shown this new way goes to Cauchy. The works of Mr. Puiseux on the solutions of algebraic equations, those of Messrs. Briot and Bouquet on doubly periodic functions and on differential equations, have largely proved the fertility of the idea of Cauchy in France. In Germany, the

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[32]Depuis longtemps les fonctions logarithmiques, et les fonctions exponentielles et circulaires ont été les seules fonctions transcendantes qui ont attiré l'attention des géomètres. Ce n'est que dans les derniers temps qu'on a commencé à en considérer quelques autres. Parmi celles-ci il faut distinguer les fonctions, nommées elliptiques, tant pour leurs belles propriétés analytiques que pour leur application dans les diverses branches des mathématiques.

[33]Thomas Johannes Stieltjes (1856–1894) was Dutch but he decided to live in France. He acquired the French citizenship and in 1886 he became professor at the Faculté des Sciences de Toulouse. Stieltjes is known for several works on analysis and number theory, in particular on the so-called Stieltjes integral, elliptic functions, Dirichlet series, and is considered as the founder of the analytic theory of continued fractions. Stieltjes is also remembered for a failed attempt to prove the Riemann hypothesis, which he announced in his paper [104].

beautiful discoveries of Riemann have accelerated the scientific movement which, since that time, did not slow down.

Those who love science and who have too many reasons for distrusting their invention capacities, still have a useful role to play, that of clarifying the others' researches and disseminating them. This is what I tried to do in the present work.[34]

There is a beautiful biography of Tannery by Picard [84]. The latter, as the *secrétaire perpétuel* of the *Académie des Sciences* had to write several such biographies and reports, and many of them give us a lively image of the French mathematical life in France at his epoch. In his report on Tannery, describing his teachers—Puiseux, Bouquet and Hermite—at the École Normale, Picard writes, concerning the latter:

What stroke Tannery above all in the teaching of Hermite is that he was able to give to mathematical abstractions color and life. He used to show how functions transform into one another, like a naturalist would do, in recounting the evolution of human beings.[35]

Jules Tannery was the thesis advisor of Hadamard. His brother, Paul Tannery, (1843-1904) was also a mathematician and (probably the most important French) historian of mathematics.

Jules Molk was Alsacian. He was born in 1857 in Strasbourg, where he studied at the Protestant Gymnasium founded by Jean Sturm in 1538. From 1874 to 1877 he studied at Zürich's Eidgenössische Technische Hochschule. His teachers there included Méquet, Geiser and Frobenius. After obtaining his diploma he went to Paris, where he followed courses by Hermite, Bouquet, Bonnet, Tisserand and Tannery. In 1882, he moved to Berlin, where he followed the courses of Weierstrass, Helmholtz, Kirchhoff and Kronecker. He obtained his doctorate in 1884 in Berlin under Kronecker. The title of his doctoral dissertation is: *Sur une notion qui comprend celle de la divisibilité et sur la théorie générale de l'élimination* (On a notion which included that of divisibility and on the general theory of elimination). The dissertation was published in Acta Mathematica, [68]. In the introduction, Molk writes that his goal is to unravel some points of Kronecker's memoir *Gründzüge einer arithmetischen Theorie der algebraischen Grössen* (Principles of an arithmetic theory of algebraic magnitudes) [51] published in 1882. He declares that this memoir seems to have been designed to give a new direction to algebra, and that his aim in his thesis is

[34]L'étude des fonctions d'une variable imaginaire définies par une équation, étude qui s'est substituée à la recherche, souvent impraticable, de la forme explicite de ces fonctions, a, dans notre siècle, profondément renouvelé l'analyse. C'est, comme on le sait, à Cauchy que revient la gloire d'avoir frayé cette voie nouvelle. Les travaux de M. Puiseux sur la recherche des racines des équations algébriques, ceux de MM. Briot et Bouquet sur les fonctions doublement périodiques et sur les équations différentielles ont, en France, amplement prouvé la fécondité de l'idée de Cauchy. En Allemagne, les belles découvertes de Riemann ont accéléré un mouvement scientifique qui, depuis lors, ne s'est pas ralenti.

Ceux qui aiment la science et qui ont trop de raisons pour se défier de leurs facultés d'invention, ont encore un rôle utile à jouer, celui d'élucider les recherches des autres et de les répandre: c'est ce que j'ai essayé de faire dans ce travail.

[35]Ce qui frappa surtout Tannery dans l'enseignement d'Hermite, c'est qu'il donnait aux abstractions mathématiques la couleur et la vie; il montrait les fonctions se transformant les unes dans les autres, comme l'eût fait un naturaliste retraçant l'évolution des êtres vivants.

to call the geometers to go thoroughly into Kronecker's difficult memoir. Molk died in Nancy in 1914. He was a specialist of elliptic functions, but he is mostly known for his collaboration with Klein to the edition of an encyclopedia of mathematics, which appeared in two versions, a German and a French one. The first volume of the German edition appeared in 1898 (Teubner, Leipzig) and the first volume of the French one in 1904 (Gauthier-Villars, Paris). The German name of the encyclopedia is *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihre Anwendungen* (Encyclopedia of mathematical sciences including their applications). The French title is *Encyclopédie des sciences mathématiques pures et appliquées* (Encyclopedia of the pure and applied mathematical sciences). The French version comprises 22 volumes. More than a hundred mathematicians and physicists from Germany, France, Italy and England collaborated to the project. Their names include Abraham, Appell, Bauer, Borel, Boutroux, É. Cartan, Darwin, Ehrenfest, Enriques, Esclangon, Fano, Fréchet, Furtwängler, Goursat, Hadamard, Hilbert, Klein, Langevin, Montel, Painlevé, Pareto, Perrin, Runge, Schoenflies, Schwarzschild, Sommerfeld, Steinitz, Study, Vessiot, Zermelo, and there are others. The publication of the encyclopedia is a remarkable example, at the turn of the twentieth century, of a trans-border collaboration between mathematicians, especially French and German. The publication date also corresponds to the period where the International Congresses of Mathematicians started. The French edition is modeled on the German one, but it is not an exact translation of it. It contains several original articles, and several of the German articles, in the French version, are expanded. It is interesting to quote some excerpts from a letter from Molk to Poincaré, sent on December 12, 1901; cf. [90] pp. 188–189, in which he describes the project. This is also a testimony of the collaboration between mathematicians of the two countries.

> Our *Encyclopedia* will not be a translation of the German edition; it will be a *new edition* of that encyclopedia. We shall be free to insert new articles, to present the German articles according to our French habits, to add to them notes and complements. Each article will be published with the mark: exposed by (the French author) following (the German author), and the notes [or complemets] *added* by the French author will be, furthermore, mentioned in a special way, with the goal of *reserving our rights*, in the case where the French edition will be followed – which is most probable – by an English-American one, or a German one, or even other editions. [...] The Germans have very remarkable qualities in careful scholarship; we shall take advantage of those that they highlight in their German edition. Their exposition qualities may be less remarkable; we shall try to do our best in this regard. We shall may be succeed in helping them: this would be something! In any case, it would be dangerous to not to have in our country a research tool which is analogous to the one which is spreading more and more rapidly in their country [...] But there are also articles which *manifestly are missing* in the German edition. For instance, researches on the law of great numbers are hardly mentioned. Here, an additional article seems to be appropriate; the researches of Mr. Darboux, your own researches, those of Hadamard, should find their place in our edition. You will tell me if it is convenient for you to talk yourself about this subject, or if you find it appropriate to entrust this article to others.[36]

[36]Notre *Encyclopédie* ne sera pas une traduction de l'édition allemande; ce sera une *nouvelle édition* de cette encyclopédie. Nous serons libres d'intercaler de nouveaux articles, d'exposer, d'après nos habitudes françaises, les articles allemands, d'y ajouter des notes, des compléments. Chaque article sera publié avec la mention: exposé par (l'auteur français) d'après (l'auteur allemand), et les notes

Unfortunately, the French edition was interrupted during the First World War and the project was never resumed. We refer the reader who wishes to know more about this project to the article [31] by H. Gispert.

We now review the four volumes of the treatise *Éléments de la théorie des fonctions elliptiques* by Tannery and Molk [107]. They appeared in 1893, 1896, 1898 and 1902.

In the introduction, the authors explain why they "dared writing a book on elliptic functions, such a short time after the publication of Halphen's treatise." They say that they do not have any pretension of replacing or equating the work of the Master. But Halphen's work remained incomplete after his early death, and the missing part was long-awaited from the public. Tannery and Molk declare that the fragments edited by Stieltjes are difficult to be read by students and that their treatise is meant to compensate this fact. They write that their aim is that the student, after reading this treatise, becomes able to work on the applications—in particular those contained in the second volume of Halphen's treatise, and of reading without difficulty Schwarz's *Formeln und Lehrsäte zum Gebrauche der elliptischen Functionen* (Formulae and propositions for the use of elliptic functions)[37] which is based on the lessons and notes of Weierstrass, the fundamental memoirs of Abel and Jacobi, and the rest of the "rich and admirable literature on elliptic functions," in particular the researches of Kronecker and Hermite.

The first volume of the treatise by Tannery and Molk contains an exposition of infinite series and sums, with details on results of Weierstrass. The authors declare right at the beginning that they assume that Cauchy's theory of line integrals is known. The second volume is an exposition of ϑ functions and the general results on doubly periodic functions, deduced from the work of Hermite. The third volume is concerned with the problem of inversion of elliptic functions. One may recall here that the inverse functions of elliptic integrals are considered in some sense as a generalization of the familiar trigonometric functions. (The reader might recall that the the integral $\int_0^x \frac{dt}{\sqrt{1-t^2}}$ represents the inverse sine function.) The fourth

(Footnote 36 continued)
[ou compléments] *ajoutées* par l'auteur français seront, en outre, mentionnées d'une façon spéciale, afin de *réserver nos droits*, dans le cas où à l'édition française succéderait, ce qui est fort probable, une édition anglo-américaine, une nouvelle édition allemande, ou d'autres éditions encore. [...] Les Allemands ont des qualités d'érudition minutieuses très remarquables; nous profiterons de celles qu'ils ont mises en évidence dans leur édition allemande. Leurs qualités d'exposition sont peut-être moins remarquables; nous essayerons de faire mieux à cet égard. Nous parviendrons peut-être ainsi à leur rendre service; c'est quelque chose. Il serait en tous cas dangereux de ne pas avoir chez nous un instrument de recherche analogue à celui qui se répand de plus en plus rapidement chez eux. [...] Mais il y a aussi des articles qui *manquent manifestement* dans l'édition allemande. C'est à peine si l'on mentionne, par exemple, les recherches sur les lois des grands nombres. Là un article additionnel semblerait peut-être indiqué; les recherches de M. Darboux, les vôtres, celles d'Hadamard devraient trouver place dans notre édition. Vous me direz s'il vous convient d'en parler vous-même, ou si vous croyez bon de confier à d'autres cet article.

[37] Schwarz's treatise was also published in French, under the title *Formules et propositions pour l'emploi des fonctions elliptiques, d'après des leçons et des notes manuscrites de M. K. Weierstrass*, translated by Henri Padé, Gauthier-Villars, Paris, 1894. The translation was offered to Charles Hermite at the occasion of his seventieth birthday.

chapter of that volume is concerned with the applications. The authors declare in the introduction to Volume I (which serves as an introduction to the whole series) that the notation they use is that of Weierstrass. The fourth volume ends with a reprint of a long letter (9 pages), dated September 24, 1900, from Hermite to Tannery, preceded by a commentary (12 pages) by the authors on that letter. Hermite, in his letter, explains to the authors (at their demand) a result which he had published without proof in 1858, in two articles both entitled *Sur la résolution de l'équation du cinquième degré* [42, 43]. The authors refer to Hermite's result in their treatise, but they rely there on proofs by Weber and Dedekind, instead of the one of Hermite which was difficult to follow. They declare in their commentary that the reason for which they reproduce Hermite's proof is its beauty, and this explains the inclusion of that letter.

Jordan

We shall review Jordan's *Cours d'analyse de l'École Polytechnique* (Course in analysis of the École Polytechnique) [57] in three volumes, entitled respectively *Calcul différentiel* (Differential calculus), *Calcul intégral* (Integral calculus) and *Equations différentielles* (Differential equations). The first edition was published in 1882, 1883 and 1887 respectively. The courses given at the École Polytechnique had a large impact, because several French mathematicians were trained at that school. On the other hand, the *Cours* were intended to the students and had to comply with a specific official program, therefore they cannot be considered as a testimony of the research in mathematics that was conducted at that time. Still, the *Cours* by Jordan, like that by Hermite which we also consider below, contains enough interesting material related to the ideas of Riemann.

Jordan has been himself a student of the École Polytechnique (graduating in 1855). In 1860, he defended a doctoral dissertation entitled *Sur le nombre des valeurs des fonctions* (On the number of values of functions) [56]. The jury consisted of Duhamel, Serret and Puiseux. His second thesis[38] is entitled *Sur les périodes des fonctions inverses des intégrales des différentielles algébriques.* (On the periods of inverse functions of integrals of algebraic differentials). The subject was proposed to him by Puiseux. Jordan is mostly known for his results on topology and group theory, but he also worked on the theory of functions of a complex variable, and he was well aware of Riemann's work. Furthermore, he was among the first mathematicians to understand the impact of Galois' ideas, and he was also among the first who introduced group theory in the study of differential equations. Jordan was appointed examiner at the École Polytechnique in 1873, and then professor, at the chair of

[38]The French doctorate (until a reform which took place at the end of the 1980s) always involved a *second thesis*, on a subject which was proposed by the jury, about 3 months before the date of the thesis defense. The work done for that second thesis was not necessarily original, but it was an occasion for the student to familiarize himself with a subject which was not his main research subject.

analysis, in 1876. His last years were saddened by the loss of three of his sons in World War I.

Part of Jordan's *Cours d'analyse de l'École Polytechnique* is related to Riemann's theory. In fact, Jordan's treatise is concerned essentially with the (new) foundations of real analysis, but half of Volume II is on complex analysis. This volume is entitled *Calcul intégral* (Integral calculus). Chapter V (pp. 305–376) is on complex integration, Chap. VI (pp. 378–621) on elliptic functions, and Chap. VIII (pp. 619–693) on Abelian integrals.

Chapter V is an exposition of Cauchy's theory of integration, included in the new rigorous setting of analysis, with applications to algebraic functions. The theory is developed in the complex plane, and Riemann surfaces are not introduced. We refer the reader to Chap. 7 [77] of the present volume for a discussion of the relation between Cauchy's and Riemann's theories.

In Chap. VI, Jordan studies elliptic functions. He starts with the fact they have at most two (independent) periods. Group theory (in the language of "substitutions") is introduced in the study of linear transformations, and the language of determinants is used. Elliptic functions are considered, as in the modern point of view, as defined on the torus. Hermite's decomposition of elliptic functions into elementary functions is presented. This is an analogue of the decomposition theory of rational functions, and it is used in integration. Operations on elliptic functions (multiplication and division) are discussed in detail.

We now review Chap. VIII, on Abelian integrals. Jordan starts with a proposition which he attributes to Lüroth, concerning a canonical way of associating to an algebraic function a cut system of curves in the plane. He then introduces the connectivity of a Riemann surface in terms of such a canonical cut systems. The curves of such a system are called *retrosections*. The fact that a simple closed curves on a simply connected surface is homotopic to a point (Jordan says: "is equivalent to zero") is presented as a theorem. The definition of the genus of a surface is also given. The adjective *monodromic* ("one-path") for functions on a piece of a Riemann surface is introduced. A *synectic* function is monodromic with no critical point. A function is said to be *uniform* if it is synectic on the whole surface. Integrals of functions on Riemann surfaces are then introduced and studied. Using integrals, a function which is synectic on the whole Riemann surface is shown to be constant. A general expression is given for functions which are uniform on a Riemann surface and whose only critical points are poles. Abelian integrals are then studied, as integrals of the form $\int F dz$ where F is a rational function of two variables. Periods of these integrals are introduced, as integrals along certain paths. The number of times a rational function F takes a given value is independent of that value and is equal to the number poles of the function. From that, a proposition, called Abel's theorem, on the determination of Abelian integrals along some paths, is proved. Jordan gives then a theorem saying that an Abelian integral is determined up to a constant by some periods he calls the *first p cyclic periods*, and the location of its critical points together with some finite part of its expansion at each such point. Integrals of the first, second and third kind are introduced, and a strong form of Riemann's existence theorem, which Jordan calls the Riemann–Roch theorem, is obtained. ϑ functions and the inversion problem are introduced, and the solution of the inversion problem is presented. In particular, an expression of elementary integrals of the second and third type in terms of ϑ functions are given.

Appell and Lacour

In the treatise *Principes de la théorie des fonctions elliptiques et applications* (Principles of the theory of elliptic functions and applications) [5] (1897) by Appell and Lacour, the ideas of Riemann are hardly mentioned, but we include it in our series of commentaries because this treatise complements naturally those that we considered before.

Émile Lacour (1854–1913) was one of those good mathematicians who taught in the French *lycées*, namely, at the famous lycée Saint-Louis and at the fancy Parisian lycée Janson-de-Sailly. In 1895, he defended a thesis entitled *Sur des fonctions d'un point analytique à multiplicateurs exponentiels ou à périodes rationnelles* (On functions of an analytic point with exponential multipliers or with rational periods) [60]. The second thesis concerns the heat equation. The theory of Riemann surfaces of algebraic curves is used in this dissertation. The "analytic points" that are mentioned in the title are points on the Riemann surfaces of the functions considered. The "multipliers" are related to Riemann's theory of Abelian integrals, and they refer to the factors with which such an integral is multiplied when one traverses the cuts of a Riemann surface on which it is defined. In other words, they are periods. The functions considered (those that are referred to in the title) are generalizations of functions introduced by Appell which are analogues of the so-called doubly periodic functions of the third type. On of the simply connected surfaces obtained—in the tradition of Riemann—by cutting the Riemann surface along $2p$ arcs called "cuts", the multiplicative constants of the functions along the cuts are exponential, with an exponent being a linear function of p Abelian integrals of the first kind. The thesis contains results that make relations between, on the one hand, theorems of Abel on the zeros and singularities of algebraic functions and of Appell on the so-called "functions with multipliers," and on the other hand, results of Riemann on ϑ functions. We recall by the way that Riemann's solution of the inversion problem, given in his paper on Abelian functions, is based on the properties of the ϑ function in which the variables are replaced by the corresponding integrals of the first kind. The resulting functions become uniform when they are defined on their Riemann surfaces. In the last part of his dissertation, Lacour shows that the new functions he introduces are solutions of certain linear differential equations whose coefficients are rational functions.

In 1886, Lacour had Élie Cartan among his students, at the lycée Janson-de-Sailly. At the same time, he taught at the Faculté des Sciences de Paris. In 1901, he held the chair of differential and integral calculus at the University of Nancy, and he later moved to the University of Rennes. After Lacour left Nancy, he was replaced there by his former student Élie Cartan.

Appell and Lacour conceived their treatise as an elementary introduction to the subject, and as a preparation for the more advanced treatises (they refer to them as the "great treatises") of Briot-Bouquet, Halphen and Tannery-Molk. The treatise of Appell and Lacour also includes simple applications to geometry, mechanics and mathematical physics. The authors consider the theory of elliptic functions as a "higher-order trigonometry," in reference to the generalizations of the complex sine and cosine functions.

Hermite

To end this sequence of treatises, we say a few words on a treatise of Hermite, who was already mentioned several times in this chapter. This is his *Cours d'analyse de l'École Polytechnique*. We first mention a few biographical facts on Hermite, extracted from the Preface to Volume I of his collected works [49], written by Picard.

Charles Hermite (1822–1901) studied at the famous lycées Henri IV and Louis-le-Grand. His teacher at Louis-le-Grand was Richard, who, fifteen years before, had the young Galois as *élève*. Hermite, while he was still at Louis-le-Grand, used to go to the nearby library, the famous Bibliothèque Sainte-Geneviève, to read Lagrange's *Traité de la résolution des équations numériques*. He bought with his savings, in French translation, Gauss's *Recherches arithmétiques*. Later on, Hermite used to say that it was mainly in these two works that he learned algebra. In 1842, at the age of 20, Hermite entered the École Polytechnique, and the same year he published two papers in the new journal *Nouvelles annales de mathématiques*. One of these papers is on the impossibility of solving the fifth degree equation. A few months later, in January 1843, Hermite wrote to Jacobi, presenting his work on Abelian functions in which he extends results of Abel on the division of the argument of elliptic functions.The next year he sent another letter to Jacobi, on transformations on elliptic functions which included results on ϑ functions. Jacobi was so pleased by the letters of the young Hermite that he inserted them in his Collected Works. Later on, Hermite became mostly interested in number theory, and elliptic and Abelian functions continued to occupy his mind for the rest of his life. Jacobi's *Fundamenta nova* were always on his worktable. According to Picard, Hermite used to say that he will be until his last day a disciple of Gauss, Jacobi and Dirichlet.

Hermite taught at the École Polytechnique and he wrote, like many other professors at that school, a *Cours d'analyse de l'École Polytechnique* (1873) [44]. He also taught at the University of Paris, and lecture notes from his teaching, for the year 1882–1883, exist [45]. A large part of his course at the university is on elliptic integrals. The topics include the rectification of the parabola, ellipse and hyperbola, results of Fagnano, Graves and Chasles on arcs of ellipses whose difference is rectifiable (see Chap. 1 in the present volume for the work done on the rectifiability of these curves), and hyperelliptic integrals. Several results of Chebyshev are also presented together with Cauchy's theory on the dependence of a path integral on the homotopy class of the path. Riemann's method for the construction of holomorphic functions is also discussed, together with Green's theorem. Hermite also included in his course Riemann surfaces associated to multi-valued functions, periods of elliptic functions, doubly periodic functions, the transformation theory of elliptic functions, the ϑ function and other functions introduced by Jacobi.

4 Simart's Dissertation

Georges Simart (1846–1921) studied at the École Polytechnique. After that, he became a mathematician but he also worked as an officer in the Navy.[39] On the cover page of his doctoral dissertation, he is described as *Capitaine de vaisseau*.[40] On the one of his book with Picard, he is described as *Capitaine de frégate*[41] *et répétiteur*[42] *à l'École Polytechnique*. His dissertation is entitled *Commentaire sur deux mémoires de Riemann relatifs à la théorie générale des fonctions et au principe de Dirichlet* (A commentary on two memoirs of Riemann relative to the general theory of functions and to the principle of Dirichlet). It was defended on May 1, 1882, with a jury consisting of Hermite (acting as the president), Darboux and Bouquet. Simart had personal relations with Picard. In the introduction to Volume I of his *Traité d'analyse* [79], Picard writes that the volume was proof-read by Simart, "a dedicated friend and an invaluable collaborator" (un ami dévoué et un précieux collaborateur). We already mentioned the treatise that Picard and Simart wrote together, the *Théorie des fonctions algébriques de deux variables indépendantes* (Sect. 3). In the introduction to that work, Picard indicates that he wrote that book "with his friend, Georges Simart, who had helped him a lot in his *Traité d'analyse*."

Simart's thesis is a commentary on the two memoirs of Riemann on functions of a complex variable, namely, his doctoral dissertation [92] and his memoir on Abelian functions [94].

The first sentences of the thesis give us some hints on the status of Riemann's work among the French mathematicians at that epoch:

> We know the magnificent results obtained by Riemann in his two memoirs on the general theory of functions and on the theory of Abelian functions; but the methods he used, may be too briefly presented, are poorly known in France. On the other hand, reading these memoirs is particularly difficult and requires a heavy amount of work. Furthermore, the methods used by the famous geometer, and in particular his use of the Dirichlet principle, gave rise to several criticisms, whether in Germany or in France.[43]

[39] We remind the reader that the École Polytechnique is primarily a military school.

[40] A Captain in the Navy.

[41] A Frigate Captain. The progress is unusual because the rank of Capitaine de frégate is lower than that of Capitaine de vaisseau.

[42] See Footnote 30. From 1900 to 1906, Simart worked as a *répétiteur* at the École Polytechnique.

[43] On connaît les magnifiques résultats auxquels Riemann est parvenu dans ses deux mémoires relatifs à la théorie générale des fonctions et à la théorie des fonctions abéliennes; mais les méthodes qu'il a employées, peut-être trop succinctement exposées, sont peu connues en France. La lecture de ces mémoires est d'ailleurs singulièrement difficile et demande un travail approfondi. De plus,

The author then declares that his exposition is based on the works published in Germany by Königsberger, Neumann, Klein, Dedekind, Weber, Prym, Fuchs and a few others.[44] He declares that "reading these memoirs requires a knowledge of the so-called Riemann surfaces, whose use became classical in some German universities." He writes, at the end of the introduction, that at the moment he was achieving his work, he learnt about the existence of a booklet by Klein[45] in which the latter develops Riemann's ideas. Simart declares that Klein explains in that booklet that it is not necessary that Riemann surfaces be coverings of the plane ("des surfaces à plusieurs feuillets étendues sur le plan"), but that complex functions may be studied on arbitrary curved surfaces, in the same way as we do it on the plane. Simart also uses the work of Puiseux. We refer the reader to the description of the work of Puiseux given in Chap. 7 of the present volume, [77].

At the beginning of the dissertation, Simart shows how a Riemann surface is associated with an irreducible algebraic equation $F(s, z) = 0$ defining implicitly an algebraic function s of z. This surface is obtained using the distribution of the critical points and the poles, and it depends on the combinatorics of the (multi-)values of the function $s(z)$ at these points. This is considered as "the Riemann surface of the function s." This is the new domain on which the function s becomes uniform (that is, no more multi-valued). The construction of the surface is described on pp. 5–7 of the thesis. To the critical points (points z for which the given equation has multiple roots s) are associated products of cyclic transformations (permutations) obtained by winding around these values, in the tradition of Cauchy and Puiseux (see the review in [77]). The Riemann surface is obtained by gluing pieces of the complex plane using this combinatorial data. The pieces constitute the various "sheets" of the Riemann surface, which becomes a branched covering of the sphere. Each critical point gives rise to a certain number of ramification points of the covering, their number depending on the number of cyclic systems associated with the critical point. A ramification point of order μ corresponds to a cyclic permutation of $\mu + 1$ roots of the algebraic equation. Examples of gluing patterns for the various sheets are represented in Fig. 5. In this figure, the surface to the left (called Fig. 1 in the original drawing) represents a critical point of order 3, having a unique cycle. It corresponds to a unique ramification point of order 2. The surface in the middle (called Fig. 2) represents a critical point of order 4 having two cycles. It corresponds to two ramification points of order 1 each. The surface to the right (called Fig. 3) represents a critical point of order 4 having three cycles. It corresponds to three ramification points, one of order 1, and two others of order 0. The Riemann surface associated with the algebraic equation satisfies the following properties:

(Footnote 43 continued)
les procédés employés par l'illustre géomètre, en particulier l'application qu'il a faite du principe de Dirichlet, ont donné lieu à de nombreuses critiques tant en Allemagne qu'en France.

[44]Klein, in his *Development of mathematics in the 19th century* [59], gives a concise report on the contribution of these authors to the diffusion of Riemann's work.

[45]This should be Klein's *Über Riemanns Theorie der algebraischen Funktionen und ihrer Integrale* [58].

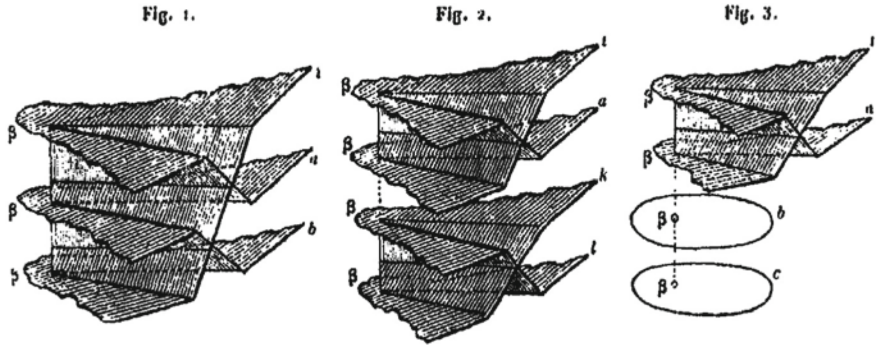

Fig. 5 Picture from Simart's thesis [103]

(1) Any rational function of s and z, when it is defined on the Riemann surface, is also a uniform function of z.
(2) The various integrals of the function s on this surface differ by a constant.

The rest of Part I of the dissertation is also purely topological. Simart recalls Riemann's definition and classification of surfaces according to their connectivity, and how an $n + 1$-connected surface may be transformed into an n-connected one by performing cuts. He declares that this theory was outlined by Riemann, but that the details were worked out by Königsberger. Simart then proves that a connected $(n + 1)$-connected surface is transformed by an arbitrary cut into an n-connected surface.

Part II of the dissertation concerns the study of the Laplace equation. We recall that Riemann, at the beginning of his doctoral dissertation, showed that if a function $w = u + iv$ of a complex variable $z = x + iy$ has the property that its derivative is independent of direction, then its real and imaginary parts satisfy the Laplace equation. This is one of the major tools that Riemann uses in the rest of his work. Using a system of coordinates that Riemann introduced in his dissertation and his memoir on Abelian functions, Simart proves an extension of Green's theorem to a region contained in an arbitrary Riemann surface bounded by an arbitrary finite number of curves. Riemann's use of the Dirichlet principle relies on that theorem. Simart gives the precise hypotheses on the functions which are concerned by Green's theorem, taking into account points of discontinuity and the points at infinity. The points of discontinuity of a function u are arranged, following Riemann's classification in §10 of his dissertation, into two species, according to whether the surface integral

$$\int\int \Big((\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2\Big) dT$$

is finite or not on a piece of surface containing this point.

Simart proves the following theorem, which he attributes to Riemann (§10 of Riemann's dissertation):

Let u be a function defined on a simply connected Riemann surface with boundary satisfying the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and suppose that the function satisfies furthermore the following conditions:

(1) *The set of points where this differential equation is not satisfied has dimension* ≤ 1.
(2) *The number of points where* u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ *are discontinuous is finite.*
(3) *At these discontinuity points, the magnitudes* $\rho \frac{\partial u}{\partial x}$, $\rho \frac{\partial u}{\partial y}$ *are infinitely small compared with* ρ, *where* ρ *is the distance to the singular point.*
(4) *There are no isolated discontinuities of* u *which correspond to an instantaneous change in value.*

Then u *as well as its partial derivatives are necessarily finite and continuous.*

Simart then proves (§11 of Riemann's dissertation) the uniqueness of a function u satisfying the Laplace equation on the interior of a domain, with a given value on the boundary.

Part III of Simart's dissertation concerns the Dirichlet principle (§16–18 of Riemann's disssertation), in connection with Riemann's determination of the functions discussed in Part II. We recall that Riemann uses this principle in his proof of the so-called Riemann mapping theorem, stated as follows (§21 of Riemann's dissertation and p. 78 of Simart's dissertation):

Given a simply connected Riemann surface T *with boundary, there exists a function* $\zeta(z)$ *defined on this surface such that the image by* ζ *of* T *is the unit disc.*

Part IV concerns Abelian integrals, as an approach to the Riemann existence problem: "To determine a function knowing its ramification points, its discontinuity points and the way in which it is discontinuous." The analytic forms of the so-called integrals of the first kind are given as well as the Riemann–Roch theorem.

More precisely, Simart addresses in this part the following two problems, for which he gives a complete solution:

Problem 1 (p. 80) Given an irreducible algebraic equation $F(s, z) = 0$ defining a multi-valued algebraic function s of z, find the associated Riemann surface, that is:

(1) determine the critical points of the function s, the number of ramification points that are above these critical points, and the order of each of these ramification points;
(2) transform this surface, using Riemann's "cuts," into a simply connected surface, evaluate the number of cuts, and then determine the connection of the surface.

Problem 2 (p. 97) Let T be the closed surface associated with the function $s(z)$ defined in Problem 1, and assume it is $2p + 1$-connected. Let T' be the simply connected surface obtained from s using $2p$ cuts. Find a function $w(z)$ which is uniform on T, continuous on T' except at certain points and along certain lines, and satisfying the following:

(1) Along each cut, the difference of the function from one side of the cut to the other is a constant; the real parts of these constants are given in advance.
(2) The function is discontinuous at a certain finite number of points, and at such points it has a finite expression of the form

$$A \log r + Br^{-1} + Cr^{-2} + \cdots$$

where the constants $A, B, \ldots$ are given and r is an arbitrary function of z which at the given point is infinitely small of the first order.
(3) With the set of points in (2), the surface is no more closed, and one has to draw new cuts joining these points to the boundary of the surface T'. The difference of the function w along both sides of each of these new cuts is constant for each such cut and equal to $2\pi A$.

In the solution of Problem 1, the Puiseux expansions and the techniques of the Puiseux-Newton polygon are thoroughly used.

The second problem is one of the main problems that were addressed by Riemann in his memoir on Abelian functions. The proof that Simart gives uses, in the tradition of Riemann, the Dirichlet principle.

To each critical point corresponds a certain number of ramification points which are determined by the system of circular points formed around that point. A ramification point of order μ is a point around which $\mu + 1$ roots are permuted. A ramification point of order 1 is a point around which 2 roots are permuted, and it is called a simple ramification point. There is a relation between the *order* and the *degree* of a critical point, and the orders of the corresponding ramification points above it. These considerations are in the tradition of the work of Puiseux; cf. the exposition in Chap. 7 of the present volume [77].

Simart's dissertation is one of the important French writings that contributed to the understanding of Riemann's ideas by the French mathematicians.

5 Other French Dissertations and Other Works of Riemann

In this section, we review briefly a few other works done in France in which the authors explain some major ideas of Riemann, including his work on the zeta function, on minimal surfaces, and on integration.

The Zeta Function

Eugène Cahen, defended in 1895, at the Faculté des Sciences de Paris, a doctoral dissertation titled *Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues* (On Riemann's $\zeta(s)$ function and on analogous functions) [21]. The dissertation is dedicated to a generalization of Riemann's zeta function to functions of the form $\sum \frac{\alpha_n}{n^s}$, in particular for sequences α_n which are periodic, and to the development of a theory of Dirichlet series. The dissertation was criticized as being faulty, but it contains the kind of mistakes which were a ferment for further research. For instance, Cahen gives, with an incomplete proof, an asymptotic value of the sum of the logarithms of prime numbers which are smaller than x. In his paper [38], Hadamard writes:

> In his memoir which was previously quoted, Mr. Cahen presents a proof of the theorem stated by Halphen: *The sum of the logarithms of the prime numbers which are at most x is asymptotic to x*. However, his reasoning depends on Stieltjes' proposition concerning the realness of the roots of $\zeta(\frac{1}{2}+ti)=0$. We shall see that by modifying slightly the author's analysis, we can establish the same result in all rigor.[46]

The mistakes in Cahen's dissertation are analyzed in E. Landau's review [61]. Landau corrected some of them. Cahen's dissertation was published in the Annales de l'École Normale, [22].

It is interesting to recall that in 1891, the Paris *Académie des Sciences* announced a prize for a competition whose subject was: "The determination of the number of prime numbers smaller than a given quantity." When the competition was announced, it was thought that the prize would be attributed to Stieltjes, who had claimed a proof of the Riemann hypothesis, but his proof turned out to be wrong. The prize went in 1892 to Hadamard, for completing Riemann's proof of the prime number theorem. Here is how Hadamard relates his discoveries, in his report on his own works [37]:

> The last ring in the chain of deductions which started in my thesis and continued in my crowned memoir led to the clarification of the most important properties of Riemann's $\zeta(s)$ function.
>
> By considering this function, Riemann determines the frequency asymptotic law of prime numbers. But his reasoning assumes: 1) that the function $\zeta(s)$ has finitely many zeros; 2) that the successive moduli of these zeros grow roughly like $n \log n$; 3) that, in the expression of the auxiliary function $\xi(t)$ in prime factors, no exponential factor is introduced.
>
> Since these propositions remained without proof, Riemann's results remained completely hypothetical, and it was not possible to find others in the same trend. As a matter of fact, no effort has been attempted in this respect since Riemann's memoir, with the exception of: (1) Halphen's note which I mentioned earlier, which was, after all, a research project for the case where Riemann's postulates would be established; (2) a note by Stieltjes in which this geometer announced a proof of the realness of the roots of $\zeta(t)$, a proof which was never produced since.
>
> Nevertheless the propositions whose statements I recalled before are only a trivial application of general theorems contained in my memoir.
>
> Once these propositions are established, the analytic theory of prime numbers was able, after a break which lasted thirty years, to take a new boom; since that time, it continued to grow rapidly.

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[46]Dans son mémoire précédemment cité, M. Cahen présente une démonstration du théorème énoncé par Halphen: *La somme des logarithmes des nombres premiers inférieurs à x est asymptotique à x*. Toutefois son raisonnement dépend de la proposition de Stieltjes sur la réalité des racines de $\zeta(\frac{1}{2}+ti)=0$. Nous allons voir qu'en modifiant légèrement l'analyse de l'auteur on peut établir le même résultat en toute rigueur.

This is how the knowledge of the genus[47] of $\zeta(s)$ allowed, first, Mr. von Mangoldt to establish in all rigor the final result of Riemann's memoir. Before that, Mr. Cahen had made a first step towards the solution of the problem addressed by Halphen; but he was not able to attain completely his goal: indeed, it was necessary, in order to achieve in an irrefutable way Halphen's reasoning, to prove once again that the ζ function has no zero on the line $\mathrm{R}(s) = 1$.

I was able to overcome this difficulty in 1896, while Mr. de la Vallée-Poussin reached independently the same result. But the proof which I gave is much quicker and Mr. de la Vallée-Poussin adopted it in his later publications. It uses only the simple properties of $\zeta(s)$.

At the same time, I extended the reasoning to Dirichlet series and, consequently, I determined the distribution law for prime numbers in an arbitrary arithmetic progression, then I showed that this reasoning may be used as such for quadratic forms with negative determinant. Since then, the same general theorems on entire functions allowed Mr. de la Vallée-Poussin to complete this cycle of proofs by treating the case of forms with positive $b^2 - ac$.[48]

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[47] Hadamard was studying, at the same period, a notion of genus for entire functions. In particular, he gave a formula for the growth of the moduli of the roots of such functions in terms of their power series expansion.

[48] Le dernier anneau de la chaîne de déductions commencée dans ma Thèse et continuée dans mon Mémoire couronné aboutit à l'éclaircissement des propriétés les plus importantes de la fonction $\zeta(s)$ de Riemann.

Par la considération de cette fonction, Riemann détermine la loi asymptotique de fréquence des nombres premiers. Mais son raisonnement suppose: (1) que la fonction $\zeta(s)$ a des zéros en nombre infini; (2) que les modules successifs de ces zéros croissent à peu près comme $n \log n$; (3) que, dans l'expression de la fonction auxiliaire $\xi(t)$ en facteurs primaires, aucun facteur exponentiel ne s'introduit.

Ces propositions étant restées sans démonstration, les résultats de Riemann restaient complètement hypothétiques, et il n'en pouvait être recherché d'autres dans cette voie. De fait, aucun essai n'avait été tenté dans cet ordre d'idées depuis le Mémoire de Riemann, à l'exception: (1) de la Note précédemment citée d'Halphen, qui était, en somme, un projet de recherches pour le cas où les postulats de Riemann seraient établis; (2) d'une Note de Stieltjes, où ce géomètre annonçait une démonstration de la réalité des racines de $\xi(t)$, démonstration qui n'a jamais été produite depuis.

Or les propositions dont j'ai rappelé tout à l'heure l'énoncé ne sont qu'une application évidente des théorèmes généraux contenus dans mon Mémoire.

Une fois ces propositions établies, la théorie analytique des nombres premiers put, après un arrêt de trente ans, prendre un nouvel essor; elle n'a cessé, depuis ce moment, de faire de rapides progrès.

C'est ainsi que la connaissance du genre de $\zeta(s)$ a permis, tout d'abord, à M. von Mangoldt d'établir en toute rigueur le résultat final du Mémoire de Riemann. Auparavant, M. Cahen avait fait un premier pas vers la solution du problème posé par Halphen; mais il n'avait pu arriver complètement au but: il fallait, en effet, pour achever de construire d'une façon inattaquable le raisonnement d'Halphen, prouver encore que la fonction ζ n'avait pas de zéro sur la droite $\mathrm{R}(s) = 1$.

J'ai pu vaincre cette dernière difficulté en 1896, pendant que M. de la Vallée-Poussin parvenait de son côté au même résultat. La démonstration que j'ai donnée est d'ailleurs de beaucoup la plus rapide et M. de la Vallée-Poussin l'a adoptée dans ses publications ultérieures. Elle n'utilise que les propriétés les plus simples de $\zeta(s)$.

En même temps j'étendais le raisonnement aux séries de Dirichlet et, par conséquent, déterminais la loi de distribution des nombres premiers dans une progression arithmétique quelconque, puis je montrais que ce raisonnement s'appliquait de lui-même aux formes quadratiques à déterminant négatif. Les mêmes théorèmes généraux sur les fonctions entières ont permis, depuis, à M. de la Vallée-Poussin d'achever ce cycle de démonstrations en traitant le cas des formes à $b^2 - ac$ positif.

Minimal Surfaces

Regarding Riemann's work on minimal surfaces (see [95, 96] cf. also Chap. 5 of the present volume [111]), we mention the thesis defended at the Faculté des Sciences de Paris on May 27, 1880, by Boleslas-Alexandre Niewenglowski [71]. The title is *Exposition de la méthode de Riemann pour la détermination des surfaces minima de contour donné* (Exposition of Riemann's method for the determination of minimal surfaces with a given contour). The thesis committee consisted of Hermite, Bonnet and Tannery. The author declares there that Riemann, in his work on minimal surfaces, was inspired by Bonnet. He writes, in his introduction:

> I would like to clarify, if I can, a remarkable memoir of Riemann, relative to minimal surfaces. The famous author had briefly indicated most of the results he obtained; I hope that I established them in a satisfactory way.
>
> Riemann makes use of imaginary variables which we immediately reduce to the variables that were used before him by Mr. O. Bonnet, in several important memoirs on the general theory of surfaces. Indeed, the logarithm of the variable μ, chosen by Riemann, is equal to $y + x\sqrt{-1}$ and, therefore, the logarithm of the conjugate variable μ' is equal to $y - x\sqrt{-1}$, where x et y are the independent variables adopted by Mr. O. Bonnet. I think that I am not exaggerating at all in claiming that the scholarly research of Mr. O. Bonnet inspired that of Riemann.[49]

In §6 of his dissertation, Niewenglowski recalls the partial differential equation that Riemann obtains to show that a surface is minimal (that is, has zero mean curvature), and he shows that this equation is contained in Bonnet's memoir [7]. We note by the way that Bonnet wrote several other articles on minimal surfaces; cf. e.g. [8–12]. In the first section of the second part of his dissertation, titled *Applications*, Niewenglowski considers the special case of minimal surfaces that contain two non-planar surfaces. He notes that the only such surface that Riemann indicates in his article is a surface that was known since a long time (a surface Niewenglowski calls "hélicoïde gauche à plan directeur.") Niewenglowski notes that Serret showed that there are other surfaces that satisfy this requirement and he describes them. Other examples of minimal surfaces given by Riemann are described from a new point of view. Niewenglowski's dissertation was published in the Annales de l'École Normale Supérieure, [70].

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[49] Je me propose d'élucider, s'il m'est possible, un mémoire remarquable de Riemann, relatif aux surfaces minima. L'illustre auteur a brièvement indiqué la plupart des résultats qu'il a obtenus; j'espère les avoir établis d'une manière satisfaisante.

Riemann se sert de variables imaginaires que l'on ramène immédiatement aux variables employées avant lui par M. O. Bonnet, dans plusieurs mémoires importants sur la théorie générale des surfaces. En effet, le logarithme népérien de la variable μ, choisie par Riemann, est égal à $y + x\sqrt{-1}$ et le logarithme de la variable conjuguée μ' est égal, par suite, à $y - x\sqrt{-1}$, x et y étant les variables indépendantes adoptées par M. O. Bonnet. Je pense ne rien exagérer en affirmant que les recherches savantes de M. O. Bonnet ont inspiré celles de Riemann.

The Riemann Integral

Finally, we talk about the fate of the Riemann integral in the French treatises on analysis of the period considered. It seems that it is only in the second edition of Jordan's *Cours d'analyse*, published in 1893, that this topic was considered for the first time. We note by the way that this second edition contains Jordan's theorem saying that a simple closed curve in the plane separates the plane into two regions.

Riemann introduced his theory of integration in his habilitation memoir on trigonometric series, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (On the representability of a function by a trigonometric series) [93]. The text was written in 1853 but was published only after Riemann's death. Darboux, in a letter to Hoüel, who had just translated Riemann's memoir into French, dated March 30, 1873 and quoted in [28], writes the following:

> It is very kind of you to have finished the Riemann. There is a pearl which everybody will discover there, I hope. This is the definition of the definite integral. It is from here that I extracted a large quantity of functions which do not have a derivative.[50]

Darboux and Hoüel were the two editors of the *Bulletin des sciences mathématiques et astronomiques*, and we mention incidentally that Hoüel translated into French, and published, other memoirs of Riemann, including his two Habilitation works, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (On the representability of a function by a trigonometric series) [93] and *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the hypotheses that lie at the bases of geometry) [99].

Two years after he wrote that letter, Darboux published a memoir on discontinuous functions [25] in which he uses Riemann's ideas. His memoir starts as follows:

> Until the appearance Riemann's memoir on trigonometric series, no doubts were raised on the existence of a derivative for continuous functions. Excellent and famous geometers, among whom one must count Ampère, had tried to provide rigorous proofs for the existence of a derivative. These attempts were without doubt far from being satisfying. But I repeat it: no doubt was even formulated on the existence of a derivative for continuous functions.
>
> The publication of Riemann's memoir concluded the question in the opposite way. At the occasion of trigonometric series, the famous geometer presents his ideas on the principle of infinitesimal calculus: he generalizes, with one of these views that belong only to first order minds, the notion of definite integral; he shows that it applies to discontinuous functions on any interval, and he states the necessary and sufficient conditions under which a function, continuous or discontinuous, can be integrated. As we shall see, the sole fact that there exist discontinuous functions that can be integrated suffices to prove that there are discontinuous functions that have no derivative, and this consequence of Riemann's works was soon admitted by the German geometers.

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[50] Vous êtes bien aimable d'avoir fini le Riemann. Il y a une perle que tout le monde y découvrira, je l'espère. C'est la définition de l'intégrale définie. C'est de là que j'ai tiré une foule de fonctions qui n'ont pas de dérivées.

> [...] In the work that will be read, I resume, providing all the necessary developments, the definitions of Riemann's definite integral after Riemann, and I show how this definition must lead to infinitely many continuous functions which have no derivative.[51]

Picard, in his *Notice* on Darboux, reports that the latter declared later on that his memoir "was poorly received by several among those who usually are interested by his works. They had dissuaded him to further cultivate this unproductive field of functions which do not have a derivative."[52]

Lebesgue, in a letter to Borel dated April 28, 1909, writes ([65] p. 189):

> I appreciate the last works of Riemann (I think he died young) as much as his dissertation on functions of a complex variable, whose importance, it seems to me, was exaggerated.[53]

One may mention here that the main idea that Lebesgue wanted to convey in that letter is that, from his point of view, the work of a mature mathematician is generally more important than the work he did when he was young. It is also true that Lebesgue found in Riemann's memoir on trigonometric series [93], which was written three years after his doctoral dissertation [92] (that is, he was more mature, in Lebesgue's wording), the bases of his integration theory, the work for which the name of Lebesgue is mostly remembered.

Lebesgue is the founder of measure theory, and he was inspired by Riemann's integration theory. In the introduction to his famous *Leçons sur l'intégration et la recherche des fonctions primitives* (Lessons on integration and on the search for primitive functions) [62], Lebesgue writes:

> [...] It is for the resolution of these problems, and not by love of complications, that I introduced in this book a definition of the integral which is more general than that of Riemann and which includes the latter as a special case.

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[51] Jusqu'à l'apparition du mémoire de Riemann sur les séries trigonométriques aucun doute ne s'était élevé sur l'existence de la dérivée des fonctions continues. D'excellents, d'illustres géomètres, au nombre desquels il faut compter Ampère, avaient essayé de donner des démonstrations rigoureuses de l'existence de la dérivée. Ces tentatives étaient loin sans doute d'être satisfaisantes; mais je le répète, aucun doute n'avait été formulé sur l'existence même d'une dérivée pour les fonctions continues.

La publication du mémoire de Riemann a décidé la question en sens contraire. À l'occasion des séries trigonométriques, l'illustre géomètre expose ses idées sur le principe du Calcul Infinitésimal: il généralise, par une de ces vues qui n'appartient qu'aux esprits de premier ordre, la notion d'intégrale définie; il montre qu'elle est applicable à des fonctions discontinues dans tout intervalle, et il énonce les conditions nécessaires et suffisantes pour qu'une fonction, continue ou discontinue, soit susceptible d'intégration. Ce seul fait, qu'il existe des fonctions discontinues susceptibles d'intégration, suffit à prouver, comme on le verra, qu'il y a des fonctions continues n'ayant pas de dérivée, et cette conséquence des travaux de Riemann n'a pas tardé à être admise par les géomètres allemands.

[...] Dans le travail qu'on va lire, je reprends, en donnant tous les développements nécessaires, les définitions de l'intégrale définie d'après Riemann, et je montre comment cette définition doit conduire à une infinité de fonctions continues n'ayant pas de dérivée.

[52] Ce Mémoire avait été froidement accueilli par plusieurs de ceux qui habituellement s'intéressaient à ses travaux. Ils l'avaient dissuadé de labourer plus longtemps le champ stérile des fonctions qui n'ont pas de dérivée.

[53] J'apprécie autant les derniers travaux de Riemann (mort jeune je crois) que sa dissertation sur les fonctions de variable complexe dont l'importance m'a semblé parfois exagérée.

> I think that those who will read me carefully, even if they regret that things are not simpler, will grant me that this definition is necessary and natural. I dare say that in a certain sense it is simpler than that of Riemann, as much easy to grasp, and that only some previously acquired mental habits can make it appear more complicated. It is simpler because it highlights the most important properties of the integral, whereas Riemann's definition only highlights a computational mechanism. For this reason, it is almost always as much easy, and even easier, using the general definition of the integral, to prove a property for all the functions to which this definition applies, that is, the *summable* functions, than to prove it for all the integrable functions, relying on Riemann's definition. Even if one is only interested in the results relative to simple functions, it is therefore useful to be familiar with the notion of summable function because it suggests fast methods of proof.[54]

Chapter II of Lebesgue's treatise is entirely dedicated to Riemann's theory.

6 On the Relations Between the French and German Mathematicians

The impact of Riemann's work on the French mathematical school naturally leads to the question of the relation between the French and German schools of mathematics. We already addressed this issue, in particular in Sect. 3 above. The question has several sides, ranging from the attitude towards the so-called German tendency to abstraction, to the political aspect, taking into account the ravaging war that broke out 20 years after Riemann defended his dissertation. We recall that in 1870, a devastating war erupted between France and Germany, which resulted in the German annexion of the French provinces of Alsace and Moselle. This war clearly affected the relations between the two countries, but the French kept the great admiration they had for Riemann, Weierstrass and the German school of function theory. One must add that despite this admiration, some of Riemann's methods remained foreign to the French geometers. Darboux, in a letter to Hoüel, dated March 5, 1870, complains of the fact that the French mathematicians were still relying on the old methods. He writes ([26] p. 109):

[54][...] C'est pour la résolution de ces problèmes, et non par amour des complications, que j'ai introduit dans ce livre une définition de l'intégrale plus générale que celle de Riemann et comprenant celle-ci comme cas particulier.

Ceux qui me liront avec soin, tout en regrettant peut-être que les choses ne soient pas plus simples, m'accorderont, je le pense, que cette définition est nécessaire et naturelle. J'ose dire qu'elle est, en un certain sens, plus simple que celle de Riemann, aussi facile à saisir que celle-ci et que, seules, des habitudes d'esprit antérieurement acquises peuvent faire paraître plus compliquée. Elle est plus simple parce qu'elle met en évidence les propriétés les plus importantes de l'intégrale, tandis que la définition de Riemann ne met en évidence qu'un procédé de calcul. C'est pour cela qu'il est presque toujours aussi facile, parfois même plus facile, à l'aide de la définition générale de l'intégrale, de démontrer une propriété pour toutes les fonctions auxquelles s'applique cette définition, c'est-à-dire pour toutes les fonctions *sommables*, que de la démontrer pour toutes les fonctions intégrables, en s'appuyant sur la définition de Riemann. Même si l'on ne s'intéresse qu'aux résultats relatifs aux fonctions simples, il est donc utile de connaître la notion de fonction sommable parce qu'elle suggère des procédés rapides de démonstration.

> All our geometers, although very distinguished, seem to belong to another age. They are eminent scientists, belonging to a science which is twenty or thirty years old which they improve and develop with a lot of success, but all the modern branches remain inaccessible to them.[55]

One may naturally address the question of quoting the German mathematical literature by the French, and vice-versa, independently of the question of the difficulty of Riemann's ideas. Darboux, in another letter to Hoüel, complains about the fact that the Germans never quote Cauchy. In a letter written around the year 1870 (the letter does not carry a date), he writes (see [26] p. 89, Letter No. 3):

> People in France start studying extensively complex variables. It is odd that this theory, born in France with the work of Cauchy, received its most beautiful developments abroad, but, I don't know if you will be of the same opinion as me, I find that the Germans are not fair for what regards Cauchy. They take advantage of his work but never quote him.[56]

In another letter to Hoüel, talking again about the Germans ([26] p. 96, Letter No. 7, again with no date), Darboux writes:

> Their behavior concerning Cauchy is unworthy. All the copies of Cauchy['s writings] leave for Germany. Gauthier-Villars quite rightly said this to me. Nevertheless his work is never quoted.[57]

How was the situation in France? It is sometimes claimed that Poincaré was not keen on quoting the Germans. In a letter to Hermite (August 20, 1881), Mittag-Leffler ([46] p. 251, also quoted in Dugac [28], pp. 156–157), writes:

> Weiserstrass's work is prior to that of Merss. Briot and Bouquet, but Mr. Poincaré, who should have known this from the memoir of Mme Kowalewski—if ever he did not know about the work *Analytische Facultäten*—never said a word about it. Monsieur de Ramsey told me that he heard from Mr. Molk—the French student following Weierstrass's course in Berlin—that Mr. Poincaré hates the Germans, which I find very natural, and that he made it a principle to never quote any German author, which I find very bad if it were true.[58]

It is possible that Poincaré's passing over the German literature is simply due to his general ignorance about others' writings. Dieudonné, writes, in his article on Poincaré in the Dictionary of Scientific Biography ([27] Vol. 11, pp. 51–61):

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[55]Tous nos géomètres, quoique tous fort distingués, semblent appartenir à un autre âge. Ce sont des savants éminents restés à la science d'il y a vingt ou trente ans qu'ils perfectionnent, développent avec beaucoup de succès, mais toutes les branches modernes sont pour eux très accessoires.

[56][...] on commence à s'occuper beaucoup en France des variables complexes. Il est singulier que cette théorie née en France par le travail de Cauchy ait reçu les plus beaux développements à l'étranger, mais je ne sais si vous serez de mon avis, je trouve que les Allemands ne sont pas justes envers Cauchy. Ils profitent de ses travaux mais ne le citent presque jamais.

[57]Leur conduite vis à vis de Cauchy est indigne. Tous les exemplaires de Cauchy partent pour l'Allemagne. Gauthier-Villars me l'a bien dit et cependant il n'est jamais cité.

[58]Le travail de Weierstrass est antérieur à celui de Messieurs Briot et Bouquet, mais M. Poincaré qui devait savoir ça par le mémoire de Madame Kowalewski—s'il n'a pas connu le travail *Analytische Facultäten*—n'en dit pas un mot. Monsieur de Ramsey m'a raconté qu'il a entendu par M. Molk—l'étudiant français qui suit le cours de M. Weierstrass à Berlin—que M. Poincaré déteste les Allemands, ce que je trouve fort naturel, et qu'il a pour principe de ne jamais citer un auteur allemand ce qui serait fort mal si c'était vrai.

> Poincaré's ignorance of the mathematical literature, when he started his researches, is almost unbelievable. He hardly knew anything on the subject beyond Hermite's work on the modular functions; he certainly had never read Riemann, and by his own account had not even heard of the Dirichlet principle.

This may also be due to Poincaré's lack of time, although the contrary may also be supported, that is, Poincaré had so much energy that it is unlikely that he could not find time to read others' writings, especially on topics on which he was working. The explanation may come from the fact that Poincaré belongs to this small category of a mathematician who reconstructs his background by himself, without reading others' works.

As we already mentioned, despite the war, the French mathematicians had an immense admiration for German mathematics, even though they considered it too abstract. Let us quote a few passages on this subject from the correspondence between Hermite and Mittag-Leffler. Hermite writes in a letter dated October 6, 1884, [47]:

> Abstraction, which is a charm for the Germans, is bothering us; it draws a kind of veil on the consequences which stays hidden to us in part, until we have taken, to attain it, a path which is more adapted to us.[59]

In other letters, Hermite expresses his highest esteem for the German mathematicians. For example, on January 14, 1892, he writes [48]:

> History of science keeps for ever the memory of the relations between Legendre and Jacobi; something good and affectionate emerges from the correspondence between these great geometers, which exerted its influence on their heirs.[60] No division ever emerged among mathematicians of these two countries. It is in entertaining friendly relations that they followed the same path in their works, and Appell's *mémoire couronné*[61] is a shining example, by its exceptional merit, by the new light it sheds on Riemann, of the ultimate alliance of the genius of the two nations, for the advancement of science."[62]

In another letter to Mittag-Leffler, dated July 10, 1893, Hermite writes [48]:

> I wrote to the French ambassador a letter which Appell read, at my request, with great care, and to which he gave his complete assessment. I was expressing, in a natural way, the

[59] L'abstraction, qui est un charme pour les Allemands, nous gêne et jette sur les conséquences comme un voile qui nous dérobe une partie jusqu'à ce que nous ayons fait pour y parvenir un chemin plus à notre convenance.

[60] The correspondence is reproduced in Jacobi's *Collected Works*, [54] t. I, pp. 385–461, and in Crelle's Journal, 80 (1875), pp. 205–279.

[61] This is Paul Appell's memoir *Sur les intégrales de fonctions à multiplicateurs et leur application au développement des fonctions Abéliennes en séries trigonométriques* (mémoire couronné par S. M. le roi Oscar II, le 21 janvier 1889).

[62] L'histoire de la science garde à jamais le souvenir des relations de Legendre et de Jacobi; quelque chose de bon et d'affectueux se dégage de la correspondance entre ces grands géomètres, qui a exercé son influence sur leurs successeurs. Aucune division ne s'est jamais montrée entre les mathématiciens des deux pays; c'est en entretenant des relations d'amitié qu'ils ont suivi la même voie dans leurs travaux, et le mémoire couronné d'Appell est un témoignage éclatant, par son mérite hors ligne, par le lustre nouveau qu'il jette sur Riemann, de l'intime alliance des génies des deux nations, pour la marche en avant de la science.

> sympathy and the admiration that all of us vow to the geometers that are the pride and the glory of German science.[63]

We quote, as the last example (there are many others) a letter from Hermite to Poincaré, dated November 27, 1880. We already mentioned that Poincaré was not keen on reading other's mathematical papers. Hermite writes ([89] pp. 169-170):

> [...] Allow me to urge you most of all to familiarize yourself with the works of Mr. Kronecker who infinitely surpassed me in this kind of research and to whom we owe the most remarkable and the most productive discoveries. The notions of class and of genus in the theory of quadratic forms were entirely linked to analysis by the eminent geometer [...] Some of the beautiful results discovered by Mr. Kronecker, and published in the *Monatsbericht*, were translated into French, at my request, and they appeared, around 1859 or 1860 in the *Annales de l'École Normale Supérieure*. But you must read in the same issue of the *Monatsbericht* of the Academy of Sciences of Berlin, and without omitting anythings of them, everything written by the hand of the great geometer.[64]

It is well known that Klein, at several places of his published talks, classifies mathematicians into logicians, formalists, and intuitives, and he claims that this has to do with the fact they are of Latin, Hebraic or German descent. Jules Tannery, whom we mentioned several times in this chapter, says that "Klein modestly related the gift of *envisioning*, which was so generously allocated to him, to the Teutonic race, whose natural power for intuition is supposed to be a pre-eminent attribute."[65] (quoted by Picard in [84] p. xxviii). This is an indication of the admiration that the French had for Klein. There are many other examples. Thus, to the question of whether French and German mathematicians ignored each other because of that war, the answer is clearly no.

7 In a Way of Conclusion

In this chapter, we tried to convey the idea that it took a certain amount of time for the notion of Riemann surface to be understood and used by French mathematicians. We also wanted to give a broad picture of the French mathematical community,

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[63][...] J'ai écrit à l'ambassadeur de France une lettre qu'Appell a lue avec grande attention à ma demande, et à laquelle il a donné son plus complet assentiment. J'exprimais naturellement les sentiments que nous éprouvons tous de sympathie et d'admiration pour les géomètres qui sont à l'honneur et la gloire de la science allemande.

[64][...] Permettez-moi de vous engager à prendre surtout connaissance des travaux de Mr. Kronecker qui m'a infiniment dépassé dans ce genre de recherches et à qui l'on doit les découvertes les plus remarquables et les plus fécondes. Les notions de classes et de genres dans la théorie des formes quadratiques ont été entièrement rattachées à l'analyse par l'éminent géomètre [...] Quelques uns des beaux résultats découverts par Mr. Kronecker, et publiés dans les *Monatsbericht*, ont été à ma demande traduits en français et ont paru, vers 1859 ou 1860, dans les *Annales de l'École Normale Supérieure*. Mais il faut lire dans ce même recueil des *Monatsbericht* de l'Académie des Sciences de Berlin, et sans en rien omettre, tout ce qui est sorti de la plume du grand géomètre.

[65]Le don de *voir*, qui lui a été départi si généreusement, M. Klein le rapporte modestement à la race teutonique, dont la puissance naturelle d'intuition serait un attribut prééminent.

especially the branch on analysis, in the few decades following Riemann's work, and of the relations between the French mathematicians and their German colleagues. Let us quote again Hermite, from his preface to the French edition of Riemann's works [98], published in 1898. This is an interesting passage in which he summarizes the passage from Cauchy's ideas to Riemann's notion of Riemann surface.

> The notion of integration along a curve has been presented, in its simplest and easiest form, with numerous and important applications which showed their scope, since 1825, in a memoir by Cauchy entitled *Sur les intégrales définies prises entre des limites imaginaires* (On the definite integrals taken between imaginary limits). But it stays a property of the famous author. One had to wait for twenty-five years, until the works of Puiseux, Briot and Bouquet, so that it soars up and shines in Analysis. The profound notion of Riemann surface, whose access is very difficult, was soon introduced and it dominated Science, so as to remain there for ever.[66]

It is important to recall that in Germany, although Riemann's ideas were investigated since the beginning by several pre-eminent mathematicians, these ideas remained, to many, very cryptic. We may add that in Germany, Riemann's ideas were not always unanimously praised, and they were even subject to criticism. Bottazzini, in his ICM 2002 communication [13], reports on some notes written by Casorati during a visit he made to Berlin in 1864, at the time when Riemann was staying, for health reasons, in Italy (Pisa). Casorati writes ([13] p. 919) that "Riemann's things are creating difficulties in Berlin [...]" Bottazini quotes Casorati:

> Weierstrass claimed that "he understood Riemann, because he already possessed the results of his [Riemann's] research." As for Riemann surfaces, they were nothing other than "geometric fantasies." According to Weierstrass, "Riemann's disciples are making the mistake of attributing everything to their master, while many [discoveries] had already been made by and are due to Cauchy, etc.; Riemann did nothing more than to dress them in his manner for his convenience."

The mathematician and historian of science Leo Könisbsberger, who taught at the University of Heidelberg, recalls in his autobiography, *Mein Leben* (My life) published in 1919, that at the time he was a student in Berlin, the mathematics taught by Weierstrass was considered as the only mathematics that was rigorous. He writes: "All of us, the younger generation, had the impression that the ideas and methods of Riemann were not part of the rigorous mathematics of Euler, Lagrange, Gauss, Jacobi and Dirichlet" (p. 59). In his last course at the University of Berlin (1866), Weierstrass also declared that the theory of Riemann surfaces was a "pure fantasy." (From the manuscript course in the Humbolt-University in Berlin, quoted in [90], p. 131.) Regarding the same theory, Klein writes in his *Development of mathematics in the 19th century* (1926) ([59] p. 241):

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[66] La notion de l'intégration le long d'une courbe avait été exposée, sous la forme la plus simple et la plus facile, avec de nombreuses et importantes applications qui en montraient la portée, dès 1825, dans un Mémoire de Cauchy ayant pour titre *Sur les intégrales définies prises entre des limites imaginaires*; mais elle reste dans les mains de l'illustre Auteur; il faut attendre vingt-cinq ans, jusqu'aux travaux de Puiseux, de Briot, de Bouquet, pour qu'elle prenne son essor et rayonne dans l'Analyse. La notion profonde des surfaces de Riemann, qui est d'un accès difficile, s'introduit sans retard et domine bientôt la Science pour y rester à jamais.

> Even today, the beginning student of Riemann surfaces faces great difficulties: The "winding points," around which the various "sheets" hang together, are essential; the curves proceeding from these points along which the sheets intersect, are not—they can be arbitrarily shifted, as long as their ends remain fixed, and in any case, they occur only because we involuntarily make the construction in three-dimensional space.

Riemann visited Paris in April 1860, on the invitation of French mathematicians. In a letter to his sister Ida, he describes a social atmosphere that was not in accord with his restrained character. He writes[67]:

> In general I am satisfied with the results of my trip, even if my expectations which I had earlier attached to the journey must remain unfulfilled, necessitated by the shortness of time. In this regard it would have been of little value if I had remained one or two weeks longer in Paris. And so I preferred to return to Göttingen at the right time.
>
> I can not complain at all about a lack of friendliness on the part of the Parisian scholars. The first social occasion, in which I took part, was a tea at Herr Serret's, who had become a member of the institute a few weeks before. Such a tea or "Réunion" contrasts sharply with our socials. It begins at 9:00 pm, really gets going at 10:00 and goes till 1 o'clock in the morning. During this time guests continually come and go; many come right from the theatre, which in Paris seldom closes before 12:30. They consist of nothing but teal ice cream and a variety of sweet-meats (?), namely, glazed fruits and other sweets of that sort. It cannot be denied that this unrestrained manner has perverted many.
>
> The social gathering at Serret's consisted of 30 to 40 ladies and gentlemen, among whom were also several Germans or rather speakers of German. I conversed chiefly with them.

Bottazzini declares in [55] p. 244 that during that stay in Paris, Riemann met, among others, Hermite, Puiseux, Briot and Bouquet.

The German mathematicians had in general a great consideration for the French. We quote a passage from a letter from Weierstrass to Kovalevskaya, sent on June 14, 1882, after the latter informed him that she met Hermite (the letter is reproduced in Mittag-Leffeler's ICM lecture [67]): "You should now also enter into a relationship with other mathematicians: the young ones, Appell, Picard, Poincaré will be extremely interesting for you."

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[67] The translation is by Gallagher and M. Weissbach, cf. [100].

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