# Some Remarks on "A Contribution to Electrodynamics" by Bernhard Riemann

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**Abstract** Around 1850, the idea originated that electromagnetic forces between moving charges in circuits are propagated with the velocity of light. After such a speculation by C. F. Gauss in 1845, B. Riemann, in 1858, suggested the inhomogeneous wave equation in 3-dimensional space for the modeling of this propagation. He found a particular solution replacing Coulomb's potential, now called the retarded potential. His attempt failed to derive from this solution Weber's action-at-a-distance potential. Riemann withdrew his pertinent paper before it became printed. After a description of some aspects of research by Gauss, Weber and Riemann, a likely reason for Riemann's withdrawal is specified differing from recent suggestions by historians of mathematics.

## 1 Introduction

After James Clerk Maxwell's equations for electrodynamics, suggested already in 1864, had been generally accepted, the early contributions to this field by other mathematicians and physicists like Ludvig Lorenz (1829–1891), Franz Neumann (1798–1895), Rudolf Clausius (1822–1888), Hermann von Helmholtz (1821–1894), and Carl Neumann (1832–1925) have been largely forgotten by physicists during the 20th century. It is left to historians of science to maintain the memory of these men and of their achievements (cf. [7], [15]). The reason for this situation is twofold: since Maxwell, field theory with its "near"—interaction has supplanted the previous particle theories with their instantaneous interaction at-a-distance. Secondly, as an invariance group the Poincaré group has replaced the Galilei group ("relativistic" theories).

Surprisingly, within the then reigning view of electromagnetism as a particle theory, we can note a relativistic input, made by the famous mathematician Bernhard Riemann (1826–1866): His introduction of the retarded scalar potential into

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theoretical electrodynamics is still valid, but remains unknown to the overwhelming majority of today's theoretical physicists (Sect. 2). In this chapter, we will try to answer several questions: why Riemann has withdrawn the relevant paper from publication during his lifetime, what brought him to the discovery of the retarded potential, and why did he not further use this potential in his course on electricity and magnetism. Up to now, mathematicians have held accountable a trivial mistake in his paper for the withdrawal by Riemann, i.e., a forbidden interchange of integrations ([1], pp 54–56). Occasionally, it is also claimed that Riemann did make inadmissible approximations in his calculations ([17], p. 265). After recalling ideas of C. F. Gauss and W. Weber concerning a possible propagation of what now is called the electromagnetic field (Sect. 3), we will point to a more serious flaw in Riemann's paper, very likely discovered by the author himself soon after handing in his manuscript to the Royal Academy of Science in Göttingen (Sect. 4). Some helpful concepts of Maxwell's theory, a special relativistic theory of the electromagnetic field, leading to the retarded potential are introduced in Appendix 1.

## 2 Riemann's New Result of 1858: The Retarded Potential

Riemann's manuscript of 1858 "A contribution to electrodynamics" [25], became published only after his death in 1867 in the journal Annalen der Physik [24], with the same volume also containing the paper by L. Lorenz [18] who, in addition, displayed the retarded *vector* potential. A footnote in the English translation of Riemann's note [23] stated: "This paper was laid before the Royal Academy of Sciences at Göttingen on the 10th of February 1858, by the author [..], but appears, from a remark added to the title by the then Secretary to have been subsequently withdrawn."

The gist of his paper is stated right at its beginning:

> I have found that the electrodynamic actions of galvanic currents may be explained by assuming that the action of one electrical mass on the others is not instantaneous, but is propagated to them with a constant velocity which, within the limits of errors of observation, is equal to that of light.[1]

Moreover, he concluded that "[..] the differential equation for the propagation of the electrical force is the same as that for light and of radiant heat."

His idea was to derive Weber's law for the force between two pointlike electrical charges from a partial differential equation in the same way as Coulomb's potential $V$ had been a consequence of Poisson's equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 4\pi\rho \,, \tag{1}$$

[1]The translation is taken from [23], p. 368. If not indicated otherwise, translations are made by myself.

where $\rho$ is the electrical charge density. He knew that in order to allow for propagation, the PDE ought to be of the hyperbolic type. As to the type of propagation, in the same lecture course Riemann had also dealt with the parabolic diffusion equation:

$$\alpha\frac{\partial u}{\partial t}+\rho+\beta^2[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}]=0\;. \tag{2}$$

Already in the term 1854/55, in his first course on PDEs and their application to problems of physics, Riemann had studied the 1-dimensional wave equation [28]

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

(§43, S. 111), oscillations of a strained string in §74, §75 and solutions by D’Alembert (p. 188).

In §43 the general solution with initial conditions at $t=0$

$$u=f(x)\;,\;\frac{\partial u}{\partial t}=F(x)$$

is written down:

$$u=1/2[f(x+at)+f(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}F(\lambda)d\lambda\;\;. \tag{3}$$

(His formula (III) on p. 113.) It should not have been a problem for Riemann to generalize the 1-dimensional wave equation to three space-dimensions and to replace the argument $x-at$ by $r-at$ with $r^2=x^2+y^2+z^2$. However, the new physics comes from the combination with d’Alembert’s inhomogeneous PDE:

$$\frac{\partial^2 V}{\partial t^2}-\alpha^2(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2})+\alpha^2 4\pi\rho=0\;, \tag{4}$$

and this is exactly the equation he wrote down in his paper for the Royal Academy. Without giving a calculation, he presented as a particular solution of (4) what is now called the “retarded potential”:

$$V=\frac{f(t-\frac{r}{\alpha})}{r}\;, \tag{5}$$

with $r=(x^2+y^2+z^2)^{\frac{1}{2}}$ , and $\alpha$ a velocity.[2] He thus had chosen the correct physical solution by leaving aside the advanced potential $V=\frac{f(t+\frac{r}{\alpha})}{r}$.

[2]In today’s view, he used $\rho=f\delta(r)$, where $\delta$ is Dirac’s distribution; cf. Appendix 1.

Maxwell's reaction to the retarded potentials of Riemann and Lorenz when they were published in 1867 was entirely negative:

> We are unable to conceive of a propagation in time except either as the flight of a material substance through space or as the propagation of a condition of motion or stress in a medium already existing in space. (Quoted from [22], p. 185.)

For him, the scalar potential was not an observable of the "state" of the electrical field unlike Lorentz's vector potential. Leaving aside the question of observability, there in fact is an epistemological problem when the switch from Weber's theory for point particles to a field theory lying behind the new concept of propagation is to be made: the aether problem.[3]

## 3 Gauss, Weber, and Riemann on Electrodynamic Interaction

In the first half of the 19th century, from electrostatics and magnetism as generated by electrical currents, electrodynamics developed. For the sources of electricity, the picture of an electrical fluid became replaced by the concept of charged electrical particles. In a first approach, still within theories with action at-a-distance, potentials depending on the velocity of such particles were introduced by C. F. Gauss, W. Weber, F. Neumann, and R. Clausius.[4] At the time, from experiments no convincing conclusion could be drawn as to which of these potentials described the phenomena best. A comparison by help of thought experiments or exemplary calculations was rarely tried; the dissertation by a student of Clausius is an example [30].

### *3.1 Gauss*

Riemann attained the idea that a force between electrical currents need not be instantaneous but propagates from Gauss via Wilhelm Weber. In fact, in his letter of 19 March 1845 to Weber, Gauss wrote:

> Without doubt, I would have given notice of my investigations a long time ago, had I not missed at the time when I stopped them what I considered the real cap stone. To wit: the derivation of the additional forces (supervening the forces of the mutual interaction of electrical parts at rest when they are in motion) from the action which is not instantaneous but propagated in time (similarly as with light) ([9], p. 627–629.)

[3]We do not dwell here on Riemann's ideas about the nature of the medium through which the electrical forces are propagated. Cf. [20], p. 529, 532, 534 with the pagination after the 2nd edition of Riemann's collected papers of (1892).

[4]Even before R. Clausius, H. Grassmann had suggested the same potential as Clausius [11], ([19], III, 203–210).

But Gauss had more up his sleeve. In his unpublished notes, we find a remark entitled "Fundamental law for all interactions of galvanic currents (found in July 1835)" ([9], p. 616–617). Let, $e$, $e'$ be the electric charges, $x$, $y$, $z$ and $x'$, $y'$, $z'$ their coordinates, $r^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$. For the mutual action (repulsive force) of the charges in motion, Gauss then gave the expression:

$$\frac{ee'}{r^2}\{1 + k[(\frac{d(x'-x)}{dt})^2 + (\frac{d(y'-y)}{dt})^2 + (\frac{d(z'-z)}{dt})^2 - \frac{3}{2}(\frac{dr}{dt})^2]\}\ , \tag{6}$$

where "$\sqrt{\frac{1}{k}}$ represents a determined speed". The corresponding potential has been correctly reported in [8] to be:

$$\phi = \frac{Q}{4\pi\epsilon_0 r}[1 + (\frac{\vec{v}_{rel}}{c})^2 - \frac{3}{2c^2}(\vec{v}_{rel} \cdot \frac{\vec{x}}{|\vec{x}|})^2]\ . \tag{7}$$

Here, $Q$ is the electric charge, $c$ the velocity of light in vacuum, and $\vec{v}_{rel}$ the relative velocity of the two charges. Hence it is to be noted, that a velocity-dependent potential already occurred in the work of Gauss, but remained unpublished during his lifetime.

## 3.2 Weber

As Gauss had done more than a decade earlier, in 1846 Wilhelm Weber derived his law for the absolute value of the force between two charges in relative motion from Ampère's law[5] [37], [35]:

$$\frac{e_1 e_2}{r^2}\{1 + \frac{r\ddot{r}}{c^2} - \frac{\dot{r}^2}{2c^2}\}\ , \tag{8}$$

where $r = |\vec{r}| = |\vec{r_1} - \vec{r_2}|$. In order to do so, assumptions about the distribution and velocity of the charges in the currents had to be made such as: (1) positive and negative charges move with the same speed; (2) In each volume element, always the same amount of positive and negative charges must be present. (8) can be obtained from a Lagrangian:

$$V = \frac{1}{c^2}\frac{e_1 e_2}{r}(1 - (\frac{dr}{dt})^2)\ . \tag{9}$$

Weber's approach was criticized immediately by H. Helmholtz on the false premise that it would violate conservation of energy [13] and, with the same argument, by

[5] Instead of by expression (8), Weber's force also is given in the form resulting from the substitution $c \rightarrow \sqrt{2}c$. In Weber's original paper [37] the coefficient of $\dot{r}^2$ had been $\frac{a^2}{16}$. This was changed later into $c^2$ by Weber, but his $c$ corresponds to $(\sqrt{2})^{-1}\times$velocity of light.

Thomson and Tait in their influential textbook [34][6] until the mistake became obvious. Maxwell rejected Weber's law because it followed from electromagnetism as described by a theory of particles with interaction at-a-distance; he preferred a field theoretic description [6].[7]

Despite his work within a theory of instantaneous action at-a-distance, Weber, besides Kirchhoff, was first in correctly describing the propagation with the velocity of light of oscillations of the electric current in wires of negligible resistance [36], [16]. He also determined the velocity of light in vacuum by electrodynamic measurements with highest precision [2], [4], [3].

## *3.3 Riemann*

B. Riemann joined Weber in his description of an electric current by moving point-like electrical charges and the interaction with other currents as an interaction at-a-distance between two charges (2-body forces). He introduced a further potential ("Riemann's potential") ([28], p. 326) containing only the relative velocity of the particles:

$$V^* = \frac{e_1 e_2}{r}\{(\frac{dx_1}{dt} - \frac{dx_2}{dt})^2 + (\frac{dy_1}{dt} - \frac{dy_2}{dt})^2 + (\frac{dz_1}{dt} - \frac{dz_2}{dt})^2\} . \qquad (10)$$

As seen above in (6), Gauss already had thought about this term. Yet, in all likelihood, Riemann was unaware of the expression given by Gauss. As is clear from letters to his sister Ida and his brother Wilhelm, he had to guess what the results of Gauss were. Already on 28 December 1853 he wrote to Wilhelm:

> Right after my Habilitationsschrift, I had taken up again my further investigations about the connection of electricity, galvanism, light and gravity. I reached the point that I can publish them in this form without risk. But in the course of this work, I became ever more sure that Gauss works in this field since a couple of years and has told some friends, e.g., A. Weber, of it under the promise to keep it secret [..]. ([27], p. 547.)

Five years later, when he had submitted his results, he still had not learned more on the work by Gauss and let Ida know:

> I have handed over to the Royal Soc. my discovery about the connection between electricity and light. From some utterances which I heard, I must conclude that Gauss, in this context, has set up a theory different from mine. But I am fully convinced that mine is the correct one, [..]. (Letter to Ida early in 1858 [26], p. 585.)

The potential (10) can be found already in Riemann's course on "The mathematical theory of electricity and magnetism" of summer 1858 [33].[8]

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[6]Translated into German by H. Helmholtz and G. Wertheim [14].

[7]A comparison between Maxwell's and Weber's electrodynamics is presented in [5].

[8]In fact, in the notes by Eduard Sellin, Riemann's second course of summer 1858 on Selected physical problems is also mixed in.

## 4 Riemann's Paper

In his paper "A contribution to electrodynamics," Riemann set out from "the potential of the forces exerted by [the circuit] S upon S':

$$-\frac{2}{c^2}\int\int\frac{uu'+vv'+ww'}{r}dSdS' \;, \tag{11}$$

this integral being extended over the whole of the elements $dS$ and $dS'$ of the conductors S and S' ." Here, $u$, $v$, $w$; $u'$, $v'$, $w'$ are the components of the specific intensity of the currents. In the particle picture, with charges $e$, $e'$ and their velocities $\frac{dr}{dt}$, $\frac{d'r}{dt}$ in the conductors S and S', Riemann wrote (11) in the form:

$$V = \frac{1}{c^2}\Sigma\Sigma\frac{ee'}{r}\frac{dd'(r^2)}{dt^2}\;. \tag{12}$$

The summations are taken over the charges $e$ in conductor $S$ and the charges $e'$ in conductor $S'$. (12) is equal to Neumann's potential [21]. After some manipulations depending on an assumption concerning the motion of electric charges of different sign[9] Riemann arrived at:

$$V = \frac{1}{c^2}\Sigma\Sigma ee'r^2\frac{dd'(\frac{1}{r})}{dt^2}\;. \tag{13}$$

He intended to derive in a different way the expression integrated over time:

$$P = \frac{1}{c^2}\int_0^t d\tau\Sigma\Sigma ee'r^2\frac{dd'(\frac{1}{r})}{d\tau^2}\;. \tag{14}$$

At this point, Riemann's new retarded potential came in. By introducing the function

$$F(t,t') = r(t,t')^{-1} \tag{15}$$

with $r(t,t') = [(x_t - x'_{t'})^2 + (y_t - y'_{t'})^2 + (z_t - z'_{t'})^2]^{\frac{1}{2}}$, coordinates $x_t, y_t, z_t$ of charge $e$ at time $t$ and $x'_{t'}, y'_{t'}, z'_{t'}$ of charge $e'$ at time $t'$, he went over from (14) to

$$P = \frac{1}{c^2}\int_0^t d\tau\Sigma\Sigma ee'F(\tau - \frac{r}{\alpha}, \tau)\;. \tag{16}$$

[9]For the motion of the electrical particles I assume that for each part of the conductor the sum of the fundamental actions exerted by the particles with positive and negative electricity is still almost the same during a span of time in which a very large flow passes through. It is known that this assumption is justified as well by experience as by inspection of the electro-motoric forces ([29], Blatt 10).

Expression (16) is interpreted by him as: "the potential of the forces exerted by all masses $\epsilon$ [$= e$] of conductor $S$ on the masses $\epsilon'$ [$= e'$] of conductor $S'$ during the time 0 to $t$." ([29], Blatt 14, verso.) On the same page of these handwritten notes, another assumption is formulated: "It is now assumed that the electrical masses cover only a very small distance during the time of the force's propagation; the effect is considered during a time span with regard to which the time of propagation is vanishing."

On the two following pages of his paper of 1858, Riemann replaced $F(\tau - \frac{r}{\alpha}, \tau)$ by $-\int_0^{\frac{r}{\alpha}} d\sigma F(\tau - \sigma, \tau)$, inverted integrations, omited small terms ("it is easily seen...") and then claimed "The value of $P$ from our theory agrees with the experimental one (14), if we assume $\alpha^2 = \frac{1}{2}c^2$."

The flaw in this argument lies in (16): A comparison with (5) shows that Riemann has introduced retarded time also in the distance in the denominator. Thus he has lost his exact solution of the (inhomogeneos) wave equation. It seems that Emil Wiechert (1861-1828) who, independently from Alfred-Marie Liénard, also introduced the retarded potential, has seen this. In his paper of 1900 he wrote: "At first, a conjecture could have been that [..] for a single electron with charge $l$ and velocity $v$, one could simply set:

$$\phi_{t=t_0} = \frac{1}{r_{t=t_0-\frac{r}{v}}}\ , \Gamma_\nu = l(\frac{1}{r}\frac{v_\nu}{v})_{t=t_0-\frac{r}{v}}\ , \tag{17}$$

and in fact this was assumed at the time by Riemann. Yet this approach leads to contradictions with the fundamental assumptions of our theory." ([38], p. 563.) In (17) $\phi$ and $\Gamma_\nu$ denote the scalar and vector potentials.

How did Riemann arrive at the expression (15)? This remains unclear even from Riemann's handwritten notes. At some point, he looked at

$$r^2 = a^2 + 2a(x'_{t'} - x_t) + (x'_{t'} - x_t)^2 + (y'_{t'} - y_t)^2 + (z'_{t'} - z_t)^2 \tag{18}$$

and expanded in terms of $\frac{r}{a}$ ([29], Blatt 11, verso). On another page he suggested that Poisson's equation be replaced by the (inhomogeneous) wave equation and in the next line wrote ([29], Blatt 16, recto, Blatt 17 recto):

$$rr = (r^2) = (x'_t - x_{t-\frac{r}{a}})^2 + (y'_t - y_{t-\frac{r}{a}})^2 + (z'_t - z_{t-\frac{r}{a}})^2\ , \tag{19}$$

and added "The assumption concerning the electrostatic effect by arbitrarily distributed electrical masses can be expressed as such." Riemann's fallacy thus can be localized in his notes: When he passed over from Poisson's PDE, with the particular solution $\frac{1}{r}$ written down by him, to the wave equation a particular solution of which he also had found, for reasons of similarity he was intrigued by the idea that the time-independent $r$ in Coulomb's potential must be replaced by (19). Apparently, he did not check whether this also was a solution of the wave equation, and he did not see a contradiction with the form of the retarded potential given in the same paper. Perhaps, he has been in a hurry: some of his calculations were made on sheets intended for letters dated January 28 and 29, 1858, i.e., just two weeks before he handed in his paper to the Academy.

## 5 Concluding Remarks

What then is the importance of Riemann’s paper of 1858? Three main points were made by him :

(1) The “electrical force” is propagated with the velocity of light and this propagation is the same as that for light and of radiant heat;
(2) For moving electrical charges the retarded potential replaces the Coulomb potential;
(3) Weber’s potential can be derived by help of the retarded potential.

The first two statements correspond precisely to what we accept today as consequences of Maxwell’s theory and are a remarkable anticipation of Maxwell. Only the third point is mistaken; this very likely is the reason why Riemann has withdrawn his paper from publication. In his attempted proof, Riemann started from an expression *different* from the retarded potential and consequently failed to establish a link between the retarded potential and Weber’s potential. The inadmissible inversion of two integrals was only a minor additional blemish. In his subsequent course of summer 1861 on “The mathematical theory of gravitation, electricity and magnetism,” about which notes by a student are available ([32], pp. 192–199), he changed his previous proof and derived Weber’s law with the help of energy conservation in the form of what he called the Lagrange principle—without mentioning the retarded potential at all (cf. also [15], p. 180–181).[10] We do not have the slightest documentary evidence about whether Riemann tried to re-do his calculation with the correct expression for the retarded potential just to conclude that he could not reach Weber’s potential in this way.

Another possible reason for the withdrawal might have been that, around the time of the submission of his paper, he had found his additional velocity-dependent (Riemann-)potential.[11] This would have weakened the importance of the suspected connection between the retarded potential and Weber’s potential. Some support may be seen in the report by Riemann’s colleague, the mathematician and astronomer Ernst Schering (1833–1897), that Riemann had expressed his satisfaction, that [his manuscript] back then had not been printed, because in the meantime he had found a specification of his law as a consequence of which it would satisfy certain general principles like the other fundamental laws for forces [31].

[10] By the same approach, Riemann’s potential could be derived as well. Thus Riemann had achieved what Gauss had had in mind, i.e., “the derivation of the additional forces [..] from the action”.

[11] As mentioned above, he first presented his potential in one of his two summer courses of 1858.

For Riemann, a possible relation between the retarded potential and Weber's potential apparently was more important than the study of the retarded potential for its own sake. Thus he missed the discovery of Lorentz invariance of the wave equation (4) (with $\rho = 0$). Unfortunately, his handwritten notes for the paper withdrawn do not reveal calculations showing how he arrived at (5). Perhaps, with his expertise in the field of PDEs, he had made the calculations already some years earlier; perhaps he had found the particular solution of the wave equation by pure intuition. That he failed to relate it to Weber's potential may have discredited the retarded potential in his eyes. In accord with his idea that the electromagnetic interaction between charges is propagated with the velocity of light, Riemann might have believed that Weber's potential already reflected this propagation. Despite his ingenuity, Riemann thus could not pave the way toward a relativistic electrodynamics for physics. This was left to H. Poincaré and H. A. Lorentz.

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## Appendix 1

Electric field $\vec{E}$ and magnetic field $\vec{B}$ are combined in the the field tensor of the electromagnetic field $F = F_{ik}dx^i \wedge dx^k$ $(i, k = 0, 1, 2, 3)$ , which can be expressed by the 4-potential $A = A_i dx^i$ as $F = dA$ with (in components) $A_i \simeq (\phi, -\vec{A})$ where $\phi$ is the scalar, $\vec{A}$ the vector potential. Thus, $F_{ik} = \partial_i A_k - \partial_k A_i$ , with $F_{0k} \simeq \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ , $F_{\mu\nu}$ $(\mu, \nu = 1, 2, 3) \rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ . From the first of Maxwell's equations :

$$\partial_l F^{il} = \frac{4\pi}{c} j^i \ , \quad \partial_l F^{*il} = 0 \tag{20}$$

with $F^{ik} = \eta^{ir}\eta^{ks}F_{rs}$ , $F^{*ik} = \frac{1}{2}\epsilon^{iklm}F_{lm}$ , the Minkowski metric $\eta_{ik}$, and the 4-current $J^i \simeq (c\rho, \vec{j})$, we obtain:

$$\partial^i \partial_l A^i - \partial_l \partial^l A^i = \frac{4\pi}{c} j^i \ . \tag{21}$$

As the vector potential is determined only up to gauge transformations $A \rightarrow A' = A + d\lambda$ with a scalar function $\lambda$, a so-called gauge condition may be added. Taking the *Lorenz gauge* $\partial_l A^l = 0$, from (2) the inhomogeneous wave equation follows:

$$\Box A^i = -\frac{4\pi}{c} j^i \tag{22}$$

with $\Box = \partial_s \partial^s = \eta^{rs} \partial_r \partial_s$. The Lorentz gauge condition then leads to $\partial_s j^s = 0$, i.e., to the equation for the conservation of electrical charge. For the scalar potential, then

$$\Box\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla}\phi = -4\pi\rho\ . \tag{23}$$

For a static electric field, Poisson's equation follows with the Coulomb potential

$$\phi(x) = \frac{1}{4\pi}\int d^3x' \frac{\rho(x')}{|\vec{x} - \vec{x}'|}\ . \tag{24}$$

The retarded potential is a particular solution of (23):

$$\phi(x) = \frac{1}{4\pi}\int d^3x' \frac{\rho(x', t - \frac{|\vec{x} - \vec{x}'|}{c})}{|\vec{x} - \vec{x}'|} \tag{25}$$

vanishing at spacelike infinity. It replaces Coulomb's potential for an arbitrarily time-dependent charge distribution.
Retarded and advanced solutions are combined in:

$$A^i = \frac{2}{c}\int d^4x' \theta(x' - x)\delta[(x^s - x'^s)(x_s - x'_s)]j^i \tag{26}$$

with Dirac's $\delta$-distribution and the characteristic function $\theta(x' - x) = 0, +1$ or $0, -1$ selecting directions into the future and past lightcone [10]. With the expression for the electrical current

$$j^i = ce\int_{-\infty}^{+\infty} ds\ u^i \delta^4(x - x')\ , \tag{27}$$

where $u^i \simeq \gamma(c, \vec{v})$, $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, and $e$ the electrical charge of a point particle, then the so-called Liénard-Wiechert potential results:

$$\phi = \frac{e}{|\vec{x} - \vec{x}'|}(1 - \frac{1}{c}\vec{n}\cdot\vec{v})^{-1}\ ,\ \vec{A} = \frac{e\vec{v}}{|\vec{x} - \vec{x}'|}(1 - \frac{1}{c}\vec{n}\cdot\vec{v})^{-1}\ , \tag{28}$$

with $\vec{v}, \vec{x}'$ taken at the retarded time; $\vec{n} = (\vec{x} - \vec{x}')(|\vec{x} - \vec{x}'|)^{-1}$.
(28) is different from Riemann's Ansatz (16) criticized by Wiechert; cf. (17) in Sect. 4.

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