From Riemannian to Relativistic Diffusions

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Abstract We first introduce Euclidean and Riemannian Brownian motions. Then considering Minkowski space, we present the Dudley relativistic diffusion. Finally we construct a family of covariant relativistic diffusions on a generic Lorentz manifold, the quadratic variation of which can be locally determined by the curvature (which allows the interpretation of the diffusion effect on a particle by its interaction with the ambient space-time). Examples are considered, in some classical space-time models: Schwarzschild, Gödel and Robertson-Walker manifolds.

1 Introduction

One of the four celebrated brilliant articles Einstein published in 1905 was devoted to Brownian Motion. He was seeing it as a consequence of the kinetic theory of gases: infinitely many small shocks on a given tiny particle move it in a Brownian way. Together with Langevin, Einstein then made the relation with the heat transport.

The mathematical construction of Brownian Motion, especially in terms of its law on $\mathbb{R}^d$-valued continuous paths, was performed by N. Wiener (1925–30). Namely, since Wiener this is a rigorously defined continuous stochastic process which has independent and homogeneous Gaussian increments, also called the Wiener process. Its trajectories are nowhere differentiable, which corresponds to the infinitely many small shocks specified by Einstein.

This gave rise to a huge literature, by among so many others P. Lévy, K. Itô, G. A. Hunt, J. L. Doob, S. R. S. Varadhan, M. Yor, to quote only very few probabilists, about $\mathbb{R}^d$-valued Brownian Motion and its relations to martingales, potential theory, heat equation and kernel, etc. Defining stochastic (Itô) integrals and then solving stochastic differential equations led in particular to the larger notion of so-called "diffusion" (after the physical corresponding phenomenon), namely continuous Markov process:

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the evolution of such a process, from a given state, depends only on this precise state and on the underlying law, but not on the past.

Thus Brownian Motion appeared soon as a very important physical and mathematical object, related to several theories as well as having its own interest. It originated in Biology (and was named after the biologist Brown) and soon enough, after Bachelier, progressively took a big importance in mathematical finance (and then insurance) too.

A following step was the extension of Brownian Motion to Riemannian manifolds. This was performed around 1970 by D. Elworthy (with J. Eells) and P. Malliavin, using Itô's Calculus and the Cartan moving frame method. It is worth underlining that this construction yields the stochastic parallel transport as well, and also stresses the intimate relation between Brownian Motion and the Laplace-Beltrami operator Δ. An important feature is that the latter makes the Riemannian Brownian Motion into a geometrical object, which is covariant with respect to the isometries of the underlying manifold, as well as a physical one, in the sense of Einstein (and Langevin).

A considerable amount of work has been achieved since, and still goes on, to exploit this relationship between probability theory and Riemannian geometry. To give only some examples and references: functional inequalities—such as isoperimetric, concentration or log-Sobolev—curvature-dimension inequalities, the study of harmonic maps [10], estimates about the heat kernel and its gradient [3, 16, 17], geometry of paths [9], gradient flows, optimal mass transportation [25], new proofs of the Gauss-Bonnet and Atiyah-Singer index Theorems (by Patodi, Bismut). Time-evolving Riemannian metrics are also considered now [2], in connection with Ricci flows.

Thus the extension of Brownian motion from a Euclidean to a Riemannian object allows us to understand it as a geometrical object, and explains its repeated use as a geometrical tool. To a certain extent, at this stage it remains a physical object too, as Einstein had in mind already in 1905, since its description in terms of kinetics of gases remains valid in a Riemannian context as well.

In this spirit, it is fairly natural to ask what counterpart Brownian motion might have in the relativistic framework, thereby bringing together two brilliant contributions by Einstein in 1905. The answer is not obvious, since a priori the property of Brownian paths to have an unbounded mean velocity (without any instantaneous velocity in the strict sense, since Brownian paths are nowhere differentiable) looks contradictory with the relativistic constraint of never exceeding the velocity of light.

During a long time several attempts were made, without any success, in order to define a reasonable "relativistic Brownian motion". The first real progress in this direction arrived in 1965, when R. M. Dudley showed that a relativistic diffusion, i.e., a Lorentz-covariant Markov diffusion process, cannot exist on the base space, even in the Minkowski framework of special relativity. This is indeed the precise mathematical counterpart of the fact that in a relativistic setting a Brownian motion cannot be physical any longer, since it can run at arbitrary large mean velocities. On the contrary, Dudley [8] showed that a relativistic diffusion makes sense at the level of the tangent bundle of Minkowski space, and he then specified the asymptotic behaviour of his (Dudley) diffusion. Moreover he showed that his relativistic diffusion is unique (in law), hence canonical as well as in the Euclidean setting, under the natural (at least geometrically) constraint to be covariant under the action of the Lorentz group.

A similar construction in the generic framework of General Relativity, that is, on the unit tangent bundle of a generic Lorentzian manifold, thereby attempting to relate further two major contributions by Einstein, was then made by Y. Le Jan and the author ([13], 2007). The related relativistic diffusion can be seen as a random perturbation of the geodesic flow, as well as the stochastic geometric development of the Dudley diffusion over a fixed tangent space of the Lorentzian manifold. It still enjoys the covariance with isometries, but therefore cannot be seen as resulting from a kinetic theory of gases, contrary to the maybe more physical process of [7], which is not covariant.

As in the Riemannian non-flat case, other intrinsic diffusions exist in the Lorentzian non-flat case. They enjoy the same geometrical invariance in law as the basic one, and could maybe be seen as more physical, as their quadratic variation is locally determined by their velocity and the curvature of the space, and vanishes in flat or in Ricci-flat (empty) regions ([14], 2011).

An important difference between the Riemannian and the Lorentzian (i.e., relativistic) settings is that the former gives naturally rise to elliptic and often self-adjoint infinitesimal generators, whereas the latter produces only hypoelliptic and non-self-adjoint generators, the analysis of which is much harder. Moreover, in the Riemannian framework the fibre of the frame bundle is compact, whereas it is not any longer in the Lorentzian one.

Lorentzian geometry is also at the heart of [19, 23, 24], in this same volume. Namely, in this relativistic framework, Hermann and Humbert and Nicolas address equations of hyperbolic type ([24] deals mainly with the wave equation) and elliptic type ([19] deals with the Yamabe equation); whereas the present chapter is concerned by equations of parabolic type, since diffusion processes are strongly associated with heat equations. Of course, tensor fields on space-time and the same Einstein equation are central for these four chapters.

In order to understand what relativistic diffusions look like, the best is to study them in some basic examples of General Relativity models, which exactly solve the Einstein equations, beyond the Minkowski space. The maybe most known such models are the following ones: the Schwarzschild space-time, which is intended to describe the physical space surrounding an isolated black hole or very massive star; the Robertson-Walker manifolds, which are intended to model an expanding (or shrinking) universe resulting from a "Big-Bang", as ours; the Gödel universe, which is a striking model where global causality does not hold (rending theoretically possible to return into the past after a long trajectory). Note that [19, 23, 24] (in this same volume) also particularize at some extend to the same basic examples of Minkowski and Schwarzschild.

The use of relativistic diffusions to address geometrical questions about Lorentzian geometry or analysis is still at its very beginning [4, 6, 12], and seems to be much harder than in the elliptic (Riemannian) case.

This chapter is intended to be a survey, relying mainly on [11, 13–15], written on the kind request of the editors Lizhen Ji, Athanase Papadopoulos and Sumio Yamada, for the volume of Springer "From Riemann to differential geometry and relativity". It is addressed not only to probabilists, and hopefully could also interest geometers and mathematical physicists. The proofs are omitted here, but can be found in the above quoted references.

2 Euclidean Brownian Motion

Basically, this is a continuous $\mathbb{R}^d$-valued stochastic process which has independent and homogeneous Gaussian increments. A precise definition (for $d = 1$ first) is as follows.

Definition 2.1 A real Brownian motion (or Wiener process) is a real valued continuous process $(B_t)_{t\geq 0}$ such that for any $n \in \mathbb{N}^*$ and $0 = t_0 < \cdots < t_n$, the random variables $(B_{t_j} - B_{t_{j-1}})$ are independent, and the law of $(B_{t_j} - B_{t_{j-1}})$ is $\mathcal{N}(0, t_j - t_{j-1})$, i.e., centred Gaussian with variance $(t_j - t_{j-1})$.

A slightly different formulation of the second part of the definition is:
The increments of (B_t) are independent, and stationary: $(B_t - B_s) \overset{\text{law}}{\equiv} B_{t-s}$ for any $s \leq t \in \mathbb{R}_+^*$, and moreover the law of B_t is $\mathcal{N}(0, t)$.

The construction of (B_t) can be done either as a limit of symmetrical conveniently normalized random walks, or by means of a multi-scale series, for example the Fourier expansion $\big($in terms of independent standard $\mathcal{N}(0, 1)$ Gaussian variables $(\xi_k\,,\ k \in \mathbb{N})\big)$:

$$B_t = \xi_0\, t + \frac{\sqrt{2}}{\pi} \sum_{k\in\mathbb{N}^*} \xi_k\, \frac{\sin(\pi k t)}{k}\,, \quad \text{for any } 0 \leq t \leq 1\,.$$

The (probability) law of such a process is clearly unique, and is known as the *Wiener measure* on the space of real continuous functions indexed by $\mathbb{R}_+$ and vanishing at 0.

The following property is straightforward from the definition, since the law of a Gaussian process is prescribed by its mean and its covariance.

Proposition 2.2 *The standard real Brownian motion (B_t) is the unique real process which is Gaussian centred with covariance function* $\mathbb{R}_+^2 \ni (s, t) \longmapsto \mathbb{E}(B_s B_t) = \min\{s, t\}$.

The processes $t \mapsto B_{a+t} - B_a$, $t \mapsto c^{-1} B_{c^2 t}$, $t \mapsto t\, B_{1/t}$, and $t \mapsto (B_T - B_{T-t})$ (for $0 \leq t \leq T$) satisfy the same conditions. We therefore deduce the following properties:

Corollary 2.3 *The standard real Brownian motion (B_t) satisfies*

(1) the Markov property: for all $a \in \mathbb{R}_+$, $(B_{a+t} - B_a)$ *is also a standard Brownian motion, and is independent from the "past"* σ*-field* $\mathcal{F}_a := \sigma\{B_s \,|\, 0 \le s \le a\}$ *;*
(2) the self-similarity: for any $c > 0$, $(c^{-1} B_{c^2 t})$ *is also a standard real Brownian motion ;*
(3) $(-B_t)$ *and* $(t\, B_{1/t})$ *are also standard real Brownian motions;*
(4) for any fixed $T > 0$, $(B_T - B_{T-t})_{0 \le t \le T}$ *is also a standard real Brownian motion.*

An $\mathbb{R}^d$-valued process $B_t := \left(B_t^1, \ldots, B_t^d\right)$ made of d independent standard Brownian motions (B_t^j) is called a *d-dimensional Brownian motion*. For $v \in \mathbb{R}^d$, $(v + B_t)$ is also called a d-dimensional Brownian motion, starting at v. The law of a d-dimensional Brownian motion is covariant with respect to Euclidean isometries of $\mathbb{R}^d$: if f is such an isometry then $f(v + B_t)$ is another d-dimensional Brownian motion, starting at $f(v)$.

The fundamental formula of Stochastic Calculus is due to K. Itô (around 1945).

Theorem 2.4 (Itô's Formula) *Let* $B \equiv (B^1, \ldots, B^d)$ *be a Brownian motion in* $\mathbb{R}^d$*, and* F *a* C^2 *function on* $\mathbb{R}^d$*. Then* $F \circ B$ *is a so-called semi-martingale, and precisely, we almost surely have: for all* $t \in \mathbb{R}_+$,

$$F(B_t) = F(B_0) + \sum_{j=1}^{d} \int_0^t \partial_j F(B_s)\, dB_s^j + \tfrac{1}{2} \int_0^t \Delta F(B_s)\, ds \ .$$

The half Laplacian $\frac{1}{2}\Delta$ appears naturally here, as a particular case of *infinitesimal generator*, namely that of the Brownian motion B. The stochastic so-called Itô integrals $\int_0^t \partial_j F(B_s)\, dB_s^j$ constitute the (local) *martingale part* of the above right hand side. They are pairwise orthogonal in L^2, and obey the following fundamental isometric Itô identity (physicists often understand "$(dB_s^j)^2 = ds$"):

$$\mathbb{E}\left[\left(\int_0^t \varphi_j(B_s)\, dB_s^j\right)^2\right] = \mathbb{E}\left[\int_0^t \varphi_j(B_s)^2\, ds\right].$$

Accordingly, the finite-variation process $\int_0^t \varphi_j(B_s)^2\, ds$ is called the *quadratic variation* of the *martingale* $\int_0^t \varphi_j(B_s)\, dB_s^j$ (which can be approached by stochastic Riemann-like sums of type $\sum_k Z_{s_{k-1}}(B_{s_k}^j - B_{s_{k-1}}^j)$, where each $Z_{s_{k-1}}$ is a functional of $\{B_s^j \,|\, 0 \le s \le s_{k-1}\}$).

Furthermore, the above class of *semi-martingales*, i.e., the sums of a (local) continuous (Brownian) martingale and of a *drift* term having finite variation, is closed under C^2 mappings, and a similar Itô formula holds, with a second order elliptic differential operator (generalizing $\frac{1}{2}\Delta$) as the infinitesimal generator. A given infinitesimal generator, together with a given starting point, specify the whole law of an associated diffusion. See for example [20] (for an exhaustive exposition) or [15] (for a shorter one).

3 Riemannian Brownian Motion

Let $\mathcal{M}$ be a d-dimensional oriented smooth Riemannian manifold, equipped with its Levi-Civita connection ∇. Denote by $O\mathcal{M}$ its direct orthonormal frame bundle, whose fibers are modelled on the special orthogonal group $\mathrm{SO}(d)$. Let $H_1, \ldots, H_d$ be the canonical horizontal vector fields on $O\mathcal{M}$, and π denote the canonical projection from $O\mathcal{M}$ onto $\mathcal{M}$. The Bochner horizontal Laplacian is $\mathcal{G} := \sum_{j=1}^{d} H_j^2$. The proofs relating to this section can be found for example in [9, 17, 22]. The following simple fact is crucial.

Lemma 3.1 *The Bochner horizontal Laplacian $\mathcal{G}$ acts on C^2 functions on $\mathcal{M}$, and induces the Beltrami Laplacian Δ: for any $F \in C^2(\mathcal{M})$, we have $\mathcal{G}(F \circ \pi) = (\Delta F) \circ \pi$ on $O\mathcal{M}$. Besides, in local coordinates (x^i, e_j^k), with $e_j = e_j^k \frac{\partial}{\partial x^k}$, denoting by Γ_{jk}^{ℓ} the Christoffel coefficients of the Levi-Civita connexion ∇, for $0 \le i, j \le d$ we have:*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k(x) \frac{\partial}{\partial x^k} \quad \textit{and} \quad H_j = e_j^k \frac{\partial}{\partial x^k} - e_j^k e_i^m \Gamma_{km}^{\ell}(x) \frac{\partial}{\partial e_i^{\ell}}.$$

The construction of the Riemannian Brownian motion uses the Cartan moving frame method, by means of a stochastic development, to produce a stochastic flow on the frame bundle $O\mathcal{M}$, putting white noises dB_s^j on the horizontal (velocity) vectors. The resulting diffusion will project to a diffusion on the base manifold $\mathcal{M}$, due to Lemma 3.1.

To proceed, let us fix $\Phi_0 \in O\mathcal{M}$ and an $\mathbb{R}^d$-valued Brownian motion $B = (B_s^j)$. The following theorem defines the Riemannian Brownian motion (X_s), with possibly some positive explosion time.

Theorem 3.2 (see [9, 17, 22]) (i) *The $O\mathcal{M}$-valued Stratonovitch stochastic differential equation*

$$(*) \qquad \Phi_s = \Phi_0 + \int_0^s \sum_{j=1}^{d} H_j(\Phi_t) \circ dB_t^j$$

defines a Riemannian Brownian motion $(X_s) := \pi(\Phi_s)$ on $\mathcal{M}$ (starting from $\pi(\Phi_0)$): this is a continuous Markovian (i.e., diffusion) process whose infinitesimal generator is $\frac{1}{2}\Delta$.

(ii) *The stochastic parallel transport of a vector $V_0 \in T_{X_0}\mathcal{M}$ along the Brownian path (X_s) is given by $V_s = \Phi_s V_0 \in T_{X_s}\mathcal{M}$.*

Remark 3.3 (o) The Stratonovitch integral (using it, Itô's Formula takes on the usual chain rule form of classical calculus) is deduced from the Itô one by the following defining rule:

$$\int_0^t \varphi(B_s^1,\dots,B_s^d)\circ dB_s^j := \int_0^t \varphi(B_s^1,\dots,B_s^d)\,dB_s^j + \tfrac{1}{2}\int_0^t \partial_j\varphi(B_s^1,\dots,B_s^d)\,ds\,.$$

(i) In local coordinates (x^i, e_j^k), $\Phi_s = \big(X_s\,;\, e_1(s), .., e_d(s)\big)$, Equation $(*)$ reads:

$$dX_s^i = e_j^i(s)\circ dB_s^j\ ;\quad de_j^k(s) = -\Gamma_{il}^k(X_s)e_j^i(s)e_m^l(s)\circ dB_s^m\ .$$

This means that the Riemannian Brownian motion (X_s) is the stochastic development of the $T_{X_0}\mathcal{M}$-valued Brownian motion $(\Phi_0 B_s)$.
(ii) The $O\mathcal{M}$-valued diffusion (Φ_s) admits the half Bochner horizontal Laplacian $\frac{1}{2}\,\mathcal{G}$ as its infinitesimal generator: for any $F\in C_b^2(O\mathcal{M})$, $F(\Phi_s)-\frac{1}{2}\int_0^s \mathcal{G}F(\Phi_t)\,dt$ is a martingale.
(iii) In the Itô form and in local coordinates, we have the following differential equation:

$$dX_s^i = (g^{-1/2})_j^i(X_s)\,dB_s^j - \tfrac{1}{2}\,g^{kl}(X_s)\,\Gamma_{kl}^i(X_s)\,ds\,.$$

(iv) The Riemannian Brownian motion (X_s) is covariant with respect to the isometries of $\mathcal{M}$: for any such isometry f, the process $(f\circ X_s)$ is another Riemannian Brownian motion, starting at $f(X_0)$.
(v) This construction offers the strong advantage of providing the *stochastic parallel transport* (Φ_s) along the Brownian curve (X_s) together with the Brownian motion itself.

4 The Relativistic Dudley Diffusion in Minkowski Space

Let us consider an integer $d\geq 2$, the Minkowski space $\mathbb{R}^{1,d} := \{\xi = (\xi^o,\vec{\xi}\,)\in \mathbb{R}\times\mathbb{R}^d\}$, endowed with its canonical basis $(e_0,\dots,e_d)$ and the Minkowski pseudo-metric $\langle\xi,\xi\rangle := |\xi^o|^2 - \|\vec{\xi}\,\|^2$.

Let $G = \mathrm{PSO}(1,d)$ denote the Lorentz-Möbius group, i.e., the connected component of the identity in the pseudo-orthogonal group $\mathrm{O}(1,d)$ (of linear mappings preserving $\langle\cdot,\cdot\rangle$), and denote by $\mathbb{H}^d := \{p\in\mathbb{R}^{1,d}\mid p^o>0 \text{ and } \langle p,p\rangle = 1\}$ the positive half of the unit pseudo-sphere.

The opposite of the Minkowski pseudo-metric induces a Riemannian metric on $\mathbb{H}^d$, namely the hyperbolic one, so that $\mathbb{H}^d$ is a model for the d-dimensional hyperbolic space. A convenient parametrization of $\mathbb{H}^d$ is $(\varrho,\theta)\in\mathbb{R}_+\times\mathbb{S}^{d-1}$, given by $\varrho := \mathrm{argch}(p^o)$ and $\theta := \vec{p}\Big/\sqrt{|p^o|^2-1}$. In these polar coordinates the hyperbolic metric reads $d\varrho^2 + \mathrm{sh}^2\varrho\,|d\theta|^2$, and the hyperbolic Laplacian is $\Delta^{\mathbb{H}} := \dfrac{\partial^2}{\partial\varrho^2} + (d-1)\coth\varrho\,\dfrac{\partial}{\partial\varrho} + \mathrm{sh}^{-2}\varrho \times \Delta_\theta$, Δ_θ denoting the Laplacian of $\mathbb{S}^{d-1}$. The associated volume measure is $|\mathrm{sh}\,\varrho|^{d-1} d\varrho\, d\theta$.

The group G acts isometrically on $\mathbb{R}^{1,d}$ and on $\mathbb{H}^d$, and the Casimir operator $\mathcal{C}$ on G induces the hyperbolic Laplacian on $\mathbb{H}^d$.

Fix $\sigma > 0$, and denote by $\mathcal{L}_\sigma$ the σ-relativistic Laplacian, defined on $\mathbb{R}^{1,d} \times \mathbb{H}^d$ by

$$\mathcal{L}_\sigma f(\xi, p) := p^o \frac{\partial f}{\partial \xi^o}(\xi, p) + \sum_{j=1}^d p^j \frac{\partial f}{\partial \xi^j}(\xi, p) + \tfrac{\sigma^2}{2}\, \Delta^{\mathbb{H}}_{(p)} f\,(\xi, p)\,,$$

that is to say,

$$\mathcal{L}_\sigma f := \langle p, \mathrm{grad}_{(\xi)} f\rangle + \tfrac{\sigma^2}{2}\, \Delta^{\mathbb{H}}_{(p)} f\,.$$

This is a hypoelliptic operator.

Given any $(\xi_0, p_0) \in \mathbb{R}^{1,d} \times \mathbb{H}^d$, there exists a unique (in law) diffusion process (ξ_s, p_s), $s \in \mathbb{R}_+$, such that for any compactly supported $f \in C^2(\mathbb{R}^{1,d} \times \mathbb{H}^d)$,

$$f(\xi_s, p_s) - f(\xi_0, p_0) - \int_0^s \mathcal{L}_\sigma f(\xi_t, p_t)\, dt \text{ is a martingale.}$$

Note that p_s is a hyperbolic Brownian motion, and that $\xi_s = \xi_0 + \int_0^s p_t\, dt$.

Remark 4.1 (1) The relativistic trajectories $(\xi_s \mid s \in \mathbb{R}_+)$ we get in Minkowski space are fully causal: since their spacetime velocities $\frac{d\xi_s}{ds} = p_s$ belong to $\mathbb{H}^d$, they are timelike, hence locally causal; moreover they satisfy $\frac{d\xi^o_s}{ds} = p^o_s > 0$, which ensures that $t(s) = \xi^o_s$ increases always strictly. Hence they are globally causal: in the terminology of [18], they satisfy the "causality condition": they cannot be closed.
(2) Note that ξ_s is parametrized by its arc length. Mechanically, ξ_s describes the trajectory of a relativistic particle of small mass indexed by its proper time, submitted to a white noise acceleration (in proper time). Its law is invariant under any Lorentz transformation.

If we denote by (e^*_j) the dual base of the canonical base $(e_0, e_1, .., e_d)$ (with respect to $\langle\ ,\ \rangle$), the matrices $E_j := e_0 \otimes e^*_j + e_j \otimes e^*_0$ belong to the Lie algebra $\mathrm{so}(1,d)$ of G, and generate the so-called *boost* transformations. Given d independent real Wiener processes w^j_s, $p_s = (p^o_s, \vec{p}_s)$ can be defined by $p_s := \Lambda_s e_0$, where the matrix $\Lambda_s \in G$ is defined by the following stochastic differential equation:

$$\Lambda_s = \Lambda_0 + \sigma \sum_{j=1}^d \int_0^s \Lambda_t\, E_j \circ dw^j_t\,.$$

This means that the relativistic diffusion process (ξ_s, p_s) is in fact the projection of some diffusion process having independent increments, namely a Brownian motion with drift, living in the Poincaré group. This group is the analogue in the present Lorentz-Minkowski setup of the classical group of rigid motions, and can be seen as the group of $(d+2, d+2)$ real matrices having the form $\begin{pmatrix} \Lambda & \xi \\ 0 & 1 \end{pmatrix}$, with $\Lambda \in G$, $\xi \in \mathbb{R}^{1,d}$ (written as a column), and $0 \in \mathbb{R}^{1+d}$ (written as a row). Its Lie algebra is the set of matrices $\begin{pmatrix} \beta & x \\ 0 & 0 \end{pmatrix}$, with $\beta \in \mathrm{so}(1,d)$ and $x \in \mathbb{R}^{1,d}$. The Brownian motion with drift we consider on the Poincaré group solves the stochastic differential equation $d\begin{pmatrix} \Lambda_s & \xi_s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda_s & \xi_s \\ 0 & 1 \end{pmatrix} \circ d\begin{pmatrix} \beta_s & e_0\, s \\ 0 & 0 \end{pmatrix}$, where $\big(\beta_s = \sigma \sum_{j=1}^d E_j\, w_s^j\big)$ is a Brownian motion on $\mathrm{so}(1,d)$. This equation is equivalent to $d\Lambda_s = \Lambda_s \circ d\beta_s$ and $d\xi_s = \Lambda_s e_0\, ds$, so that (Λ_s) is a Brownian motion on G. On functions of $p = \Lambda e_0$, its infinitesimal generator $\sum_{j=1}^d (\mathcal{L}_{E_j})^2$ coincides with a Casimir operator, and induces the hyperbolic Laplacian, so that $(p_s = \Lambda_s e_0)$ is a Brownian motion on $\mathbb{H}^d$, as required.

Then it is well known that $\theta_s := \vec{p}_s \Big/ \sqrt{|p_s^o|^2 - 1}$ converges almost surely in $\mathbb{S}^{d-1}$ to some random limit θ_∞, and that p_s^o increases to infinity. We also set $\varrho_s := \mathrm{argch}(p_s^o)$.

The Euclidean trajectory $Z(t)$ is defined by $\vec{\xi}_{s(t)}$, where $s(t)$ is determined by $\xi^o_{s(t)} = t$.
Let us note that the Euclidean velocity $dZ(t)/dt = \theta_{s(t)}\, \mathrm{th}\, \varrho_{s(t)}$ has norm <1, 1 being here the velocity of light, beyond which the relativistic diffusion cannot actually go. Moreover we have the following.

Remark 4.2 The mean Euclidean velocity $Z(t)/t$ converges almost surely to $\theta_\infty \in \mathbb{S}^{d-1}$.

Proof We have $\lim\limits_{t \nearrow \infty} s(t) = +\infty$, so that $\mathrm{th}\, \varrho_{s(t)} = \sqrt{1 - (p^o_{s(t)})^{-2}}$ approaches 1. Thus we get almost surely $\lim\limits_{t\to\infty} \dfrac{dZ(t)}{dt} = \theta_\infty$, and the result follows easily. ⋄

The Poisson boundary of Minkowski's space has been determined, as follows.

Theorem 4.3 ([4]) (i) *As proper time s goes to infinity, the quantity* $\langle \xi_s\,, e_0 + \theta_\infty \rangle$ *converges almost surely to a random variable* ζ_∞.
(ii) *The limiting random variable* $(\theta_\infty\,, \zeta_\infty)$ *contains all the asymptotic information regarding the Dudley diffusion* $(\xi_s, \dot{\xi}_s \equiv p_s)$, *that is, generates its invariant* σ*-field and its tail* σ*-field as well. Equivalently, the bounded* $\mathcal{L}_\sigma$*-harmonic functions are precisely the functions which admit a Choquet representation* $(\xi, p) \mapsto \mathbb{E}_{(\xi,p)}\big[h(\theta_\infty\,, \zeta_\infty)\big]$, *for some bounded measurable function h.*

5 The Lorentzian Frame Bundle $G(\mathcal{M})$ over $(\mathcal{M}, g)$

We aim at presenting the extension of the relativistic diffusion, from the Minkowski space to a generic Lorentzian manifold, framework for any General Relativity model. This will be as well the Lorentzian counterpart of the Riemannian Brownian motion, as the Dudley diffusion was the relativistic counterpart of the Euclidean Brownian motion.

The leading idea is to proceed similarly as what was done to get from the Euclidian setting to the Riemannian one, that is to say, to rely again on the Cartan moving frame method (recall Sect. 3). To begin, in this section we introduce the necessary geometrical material and background, the frames and the canonical vector fields being somewhat more complicated in the Lorentzian (or pseudo-Riemannian) setting than in the Riemannian one.

Let $\mathcal{M}$ be a C^∞ time-oriented $(1+d)$-dimensional Lorentz manifold, with pseudo-metric g having signature $(+, -, \dots, -)$, Levi-Civita connection ∇, and let $T^1\mathcal{M}$ denote the positive half of its pseudo-unit tangent bundle. Let $G(\mathcal{M})$ be the bundle of direct pseudo-orthonormal frames, with first element in $T^1\mathcal{M}$ and with fibers modelled on the Lorentz-Möbius group G. Let $\pi_1 : u \mapsto \big(\pi(u), e_0(u)\big)$ denote the canonical projection from $G(\mathcal{M})$ onto the unit tangent bundle $T^1\mathcal{M}$, which to each frame $\big(e_0(u), \dots, e_d(u)\big)$ associates its first vector $e_0(u)$.

The action of $\mathrm{SO}(d)$ on $(e_1, \dots, e_d)$ induces the identification $T^1\mathcal{M} \equiv G(\mathcal{M})/\mathrm{SO}(d)$.

Let $H_0, H_1, \dots, H_d$ be the canonical horizontal vector fields on $G(\mathcal{M})$, and $V_{e_i \wedge e_j}$ (for $0 \le i < j \le d$) the canonical vertical vector fields on $G(\mathcal{M})$. In particular, we have $T\pi(H_k) = e_k$. To abbreviate the notation, we shall write V_j for $V_{e_0 \wedge e_j}$, i.e., the vector field associated with the previous matrix $E_j \in \mathrm{so}(1, d)$.

The canonical vectors H_k , $V_{e_i \wedge e_j}$ span $TG(\mathcal{M})$, the horizontal (resp. vertical) sub-bundle of $TG(\mathcal{M})$ being spanned by the H_ks (resp. the $V_{e_i \wedge e_j}$s). Note that H_0 generates the geodesic flow, that $V_1, \dots, V_d$ generate the *boosts*, and that the $V_{e_i \wedge e_j}$ $(1 \le i, < j \le d)$ generate rotations. We have:

$$[V_{e_i \wedge e_j}, H_k] = \langle e_i, e_k \rangle\, H_j - \langle e_j, e_k \rangle\, H_i\,, \qquad \text{for } 0 \le i, j, k \le d\,,$$

and

$$[H_i, H_j] = \sum_{0 \le k < \ell \le d} \mathcal{R}_{ij}{}^{k\ell}\, V_{e_k \wedge e_\ell}\,,$$

where the $((\mathcal{R}_{ij}{}^{k\ell}))$ are the entries of the Riemann *curvature tensor*. The associated *curvature operator* satisfies: for any C^1 vector fields X, Y, Z, A ,

$$\langle \mathcal{R}\,(X \wedge Y)\,, A \wedge Z \rangle = \Big\langle \big([\nabla_X\,, \nabla_Y] - \nabla_{[X,Y]}\big)\, Z\,,\ A \Big\rangle_g\,.$$

The *Ricci tensor* and *Ricci operator* are defined, for $0 \le i, k \le d$, by:

$$R_i^k := \sum_{j=0}^d \mathcal{R}_{ij}{}^{kj}\,, \quad \text{and} \quad \mathrm{Ricci}_\xi\big(e_i(u)\big) := \sum_{k=0}^d R_i^k\, e_k(u)\,, \quad \text{for any } \; u \in \pi^{-1}(\xi)\,.$$

The *scalar curvature* is: $R := \sum_{k=0}^d R_k^k\,.$

The indices of the curvature tensor $((\mathcal{R}_{ij}{}^{k\ell}))$ and of the Ricci tensor $((R_i^k))$ are lowered or raised by means of the Minkowski tensor $((\eta_{ab} := \langle e_a, e_b\rangle))$ and its inverse $((\eta^{ab}))$. For example, we have: $R_{ij} = R_i^k\, \eta_{kj}\,.$

The *energy-momentum* tensor $((T_j^k))$ and operator T_ξ are defined as:

$$T_j^k := R_j^k - \tfrac{1}{2}\, R\, \delta_j^k \quad \text{and} \quad T_\xi := \mathrm{Ricci}_\xi - \tfrac{1}{2}\, R\,. \tag{1}$$

Note that $\sum_{j=0}^d T_j^j = -\frac{d-1}{2}\, R\,$. The *energy* at any line-element $(\xi, \dot\xi) \in T^1\mathcal{M}$ is

$$\mathcal{E}(\xi, \dot\xi) := \langle T_\xi(\dot\xi), \dot\xi\rangle_{g(\xi)} = T_{00}(\xi, \dot\xi)\,. \tag{2}$$

The *weak energy condition* (see [18]) stipulates that $\mathcal{E}(\xi, \dot\xi) \geq 0$ on the whole $T^1\mathcal{M}$. This is also the content of ([21], (94, 10)).

5.1 Expressions in Local Coordinates

Consider local coordinates (ξ^i, e_j^k) for $u = (\xi, e_0, \ldots, e_d) \in G(\mathcal{M})$, with $e_j = e_j^k\, \dfrac{\partial}{\partial \xi^k}\,.$

As in the Riemannian case, for $0 \leq i, j \leq d$ we have:

$$\nabla_{\frac{\partial}{\partial \xi^i}} \frac{\partial}{\partial \xi^j} = \Gamma_{ij}^k(\xi)\, \frac{\partial}{\partial \xi^k} \quad \text{and} \quad H_j = e_j^k\, \frac{\partial}{\partial \xi^k} - e_j^k\, e_i^m\, \Gamma_{km}^\ell(\xi)\, \frac{\partial}{\partial e_i^\ell}\,.$$

The Christoffel coefficients of ∇ are computed by: $\Gamma_{ij}^k = \frac{1}{2}\, g^{k\ell}\Big(\frac{\partial g_{\ell j}}{\partial \xi^i} + \frac{\partial g_{i\ell}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^\ell}\Big)$, or equivalently, by the fact that geodesics solve $\ddot\xi^k + \Gamma_{ij}^k\, \dot\xi^i\, \dot\xi^j = 0\,.$

Then $V_{e_i \wedge e_j} = e_i^k\, \dfrac{\partial}{\partial e_j^k} - e_j^k\, \dfrac{\partial}{\partial e_i^k}$ and $V_j = e_0^k\, \dfrac{\partial}{\partial e_j^k} + e_j^k\, \dfrac{\partial}{\partial e_0^k}$, for $1 \leq i, j \leq d\,.$

The curvature operator is expressed in a local chart as: for $0 \leq m, n, p, q \leq d$,

$$\widetilde{\mathcal{R}}_{mnpq} := \Big\langle \mathcal{R}\Big(\tfrac{\partial}{\partial \xi^m} \wedge \tfrac{\partial}{\partial \xi^n}\Big), \tfrac{\partial}{\partial \xi^p} \wedge \tfrac{\partial}{\partial \xi^q} \Big\rangle_g = g_{mr}\bigg(\Gamma_{ps}^r\, \Gamma_{nq}^s - \Gamma_{qs}^r\, \Gamma_{np}^s + \frac{\partial \Gamma_{nq}^r}{\partial \xi^p} - \frac{\partial \Gamma_{np}^r}{\partial \xi^q}\bigg). \tag{3}$$

Then, the Ricci operator can be computed similarly, as: for $0 \le m, p \le d$,

$$\tilde{R}_{mp} := \Big\langle \mathrm{Ricci}(\tfrac{\partial}{\partial \xi^m}), \tfrac{\partial}{\partial \xi^p} \Big\rangle_g = \widetilde{\mathcal{R}}_{mnpq}\, g^{nq} = \Gamma^n_{nq} \Gamma^q_{mp} - \Gamma^n_{pq}\, \Gamma^q_{mn} + \frac{\partial \Gamma^n_{mp}}{\partial \xi^n} - \frac{\partial \Gamma^n_{mn}}{\partial \xi^p}\,. \tag{4}$$

The scalar curvature and the energy-momentum operator can be computed by:

$$R = \tilde{R}_{ij}\, g^{ij} \quad \text{and} \quad \tilde{T}_{\ell m} = \tilde{R}_{\ell m} - \tfrac{1}{2} R\, g_{\ell m} \text{ (Einstein equations).} \tag{5}$$

To summarize, the Riemann curvature tensor $((\mathcal{R}_{ij}{}^{k\ell}))$ is made of the coordinates of the curvature operator $\mathcal{R}$ in an orthonormal moving frame, and its indices are lowered or raised by means of the Minkowski tensor $((\eta_{ab}))$, while the curvature tensor $((\tilde{\mathcal{R}}_{mnpq}))$ is made of the coordinates of the curvature operator in a local chart, and its indexes are lowered or raised by means of the metric tensor $((g_{ab}))$.
To go from one tensor to the other, note that by (3) we have
$\mathcal{R}\big(\frac{\partial}{\partial \xi^m} \wedge \frac{\partial}{\partial \xi^n}\big) = \frac{1}{2}\, \widetilde{\mathcal{R}}_{mn}{}^{ab}\, \frac{\partial}{\partial \xi^a} \wedge \frac{\partial}{\partial \xi^b}$, whence: $e^k_i\, e^\ell_j\, \widetilde{\mathcal{R}}_{k\ell}{}^{pq} = \mathcal{R}_{ij}{}^{mn}\, e^p_m\, e^q_n$, or equivalently:

$$\mathcal{R}_{ijab} = \widetilde{\mathcal{R}}_{k\ell rs}\, e^k_i\, e^\ell_j\, e^r_a\, e^s_b\,, \qquad \text{or also :} \quad \widetilde{\mathcal{R}}^{rspq} = \mathcal{R}^{abmn}\, e^r_a\, e^s_b\, e^p_m\, e^q_n\,.$$

5.2 *Example of a Perfect Fluid*

The energy-momentum tensor T (of (1), or equivalently $\tilde{T}$, recall (5)) is associated to a *perfect fluid* (see [18]) if it has the form:

$$\tilde{T}_{k\ell} = q\, U_k\, U_\ell - p\, g_{k\ell}\,, \tag{6}$$

for some C^1 field U in $T^1\mathcal{M}$ (which represents the velocity of the fluid), and some C^1 functions p, q on $\mathcal{M}$. By Einstein's equations (5), (6) is equivalent to:

$$\tilde{R}_{k\ell} = q\, U_k\, U_\ell + \tilde{p}\, g_{k\ell}\,, \quad \text{with} \quad \tilde{p} = (2p - q)/(d-1), \tag{7}$$

or also, by (4), to:

$$\langle \mathrm{Ricci}(V), V\rangle_\eta = q \times g(U, V)^2 + \tilde{p} \times g(V, V)\,, \quad \text{for any} \quad V \in T\mathcal{M}\,. \tag{8}$$

The quantity $\langle U(\xi_s), \dot{\xi}_s\rangle$ is the hyperbolic cosine of the distance, on the unit hyperboloid at ξ_s identified with the hyperbolic space, between the space-time velocities of the fluid and of the path; it will be denoted by $\mathcal{A}_s$ or $\mathcal{A}(\xi_s, \dot{\xi}_s)$. Note that necessarily $\mathcal{A}_s \ge 1$. By Formulas (2) and (6), the energy equals:

$$\mathcal{E}(\xi, \dot{\xi}) = q(\xi)\, \mathcal{A}(\xi, \dot{\xi})^2 - p(\xi). \tag{9}$$

The energy of the fluid is simply: $\tilde{T}_{k\ell}\,U^k\,U^\ell = q - p$, and the scalar curvature equals $R = 2\,[(d+1)\,p - q]/(d-1)$. By (9), the weak energy condition reads here: $q \geq p^+$.

6 The Basic Relativistic Diffusion

The following (where $\mathcal{C}$ denotes the Casimir operator) is analogous to Lemma 3.1.

Lemma 6.1 *The operators H_0, $\sum_{j=1}^d V_j^2$, $\mathcal{C}$, $H_0 + \frac{\sigma^2}{2}\sum_{j=1}^d V_j^2$ do act on C^2 functions on the pseudo-unit tangent bundle $T^1\mathcal{M}$, inducing respectively: the vector field $\mathcal{L}_0$ generating the geodesic flow on $T^1\mathcal{M}$, the so-called vertical Laplacian Δ_v $\big($i.e., the Laplacian on $T^1_\xi\mathcal{M}$ equipped with the hyperbolic metric induced by $g(\xi)\big)$, Δ_v again, and the generator $\mathcal{H}^1 := \mathcal{L}_0 + \frac{\sigma^2}{2}\Delta_v$. More precisely, for any $F \in C^2(T^1\mathcal{M})$, on $G(\mathcal{M})$ we have:*

$$(\mathcal{L}_0 F)\circ\pi_1 \;=\; H_0(F\circ\pi_1)\;, \quad (\Delta_v F)\circ\pi_1 \;=\; \mathcal{C}(F\circ\pi_1)\;.$$

Besides, in local coordinates (x^i, e_j^k) such that $e_j = e_j^k\,\frac{\partial}{\partial x^k}$ we have $V_j = e_j^k\,\frac{\partial}{\partial e_0^k} + e_0^k\,\frac{\partial}{\partial e_j^k}$, and denoting the inverse matrix of the pseudo-Riemannian metric of $\mathcal{M}$ by (g^{kl}) in these coordinates, we have:

$$(\Delta_v F)\circ\pi_1 = \sum_{j=1}^d V_j^2(F\circ\pi_1) = \Big((e_0^k e_0^l - g^{kl})\,\frac{\partial^2}{\partial e_0^k \partial e_0^l} + d\,e_0^k\,\frac{\partial}{\partial e_0^k}\Big) F\circ\pi_1\;.$$

Now, according to Sect. 4, the relativistic motion we will consider lives on $T^1\mathcal{M}$ and admits as infinitesimal generator the operator $\mathcal{H}^1 = \mathcal{L}_0 + \frac{\sigma^2}{2}\Delta_v$ of Lemma 6.1 above. If $\mathcal{M}$ is the Minkowski flat space of special relativity, it coincides with the (Dudley) diffusion defined in Sect. 4 above.

To construct this general relativistic diffusion, we use a kind of stochastic development to produce a stochastic flow on the bundle $G(\mathcal{M})$, as for the Riemannian Brownian motion in Sect. 3. But we have now to project on $T^1\mathcal{M}$ and no longer on the base manifold $\mathcal{M}$ (this cannot work here), and to put the white noises on the acceleration, i.e., on the vertical vectors, and no longer on the velocity, i.e., on the horizontal vectors.

To proceed, let us fix $\Psi_0 \in G(\mathcal{M})$ and an $\mathbb{R}^d$-valued Brownian motion $w = (w_s^j)$. By Lemma 6.1, the stochastic flow (Ψ_s) defined by $(**)$ in the theorem below, possibly till some explosion time, commutes with the action of $\mathrm{SO}(d)$ on $G(\mathcal{M})$, thereby allowing to project it on $T^1\mathcal{M}$. This projection is precisely the relativistic diffusion we intended to define and construct. The vector field $\mathcal{L}_0$ denotes the generator of the geodesic flow, which operates on the position ξ-component, and

Δ_v denotes the vertical Laplacian (restriction to $T^1\mathcal{M}$ of the Casimir operator on $G(\mathcal{M})$), which operates on the velocity $\dot{\xi}$-component.

Theorem 6.2 ([13]) (i) *The $G(\mathcal{M})$-valued Stratonovitch stochastic differential equation*

$$(**)\qquad \Psi_s = \Psi_0 + \int_0^s H_0(\Psi_t)\,dt + \sigma \int_0^s \sum_{j=1}^d V_j(\Psi_t)\circ dw_t^j$$

defines a diffusion $(\xi_s, \dot{\xi}_s) := \pi_1(\Psi_s)$ on $T^1\mathcal{M}$, with generator $\mathcal{H}^1 = \mathcal{L}_0 + \frac{\sigma^2}{2}\,\Delta_v$.
(ii) *If $\overleftarrow{\xi}(s) : T_{\xi_s}\mathcal{M} \to T_{\xi_0}\mathcal{M}$ denotes the inverse parallel transport along the C^1 curve $(\xi_{s'}\,|\,0 \le s' \le s)$, then $\zeta_s := \overleftarrow{\xi}(s)\,\dot{\xi}_s$ is a hyperbolic Brownian motion on $T_{\xi_0}\mathcal{M}$.*
Therefore the path (ξ_s) is almost surely the development of a relativistic (Dudley) diffusion path in the Minkowski space $T_{\xi_0}\mathcal{M}$.

The infinitesimal generator of the $G(\mathcal{M})$-valued relativistic diffusion (Ψ_s) is $H_0 + \frac{\varrho^2}{2}\sum_{j=1}^d V_j^2$, which by Lemma 6.1 projects under π_1 to the infinitesimal generator of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$; namely the relativistic operator expressed by:

$$\mathcal{H}^1 = \mathcal{L}^0 + \frac{\varrho^2}{2}\,\Delta^v = \dot{\xi}^k \frac{\partial}{\partial \xi^k} + \Big(\frac{d\,\varrho^2}{2}\,\dot{\xi}^k - \dot{\xi}^i \dot{\xi}^j\,\Gamma_{ij}^k(\xi)\Big)\frac{\partial}{\partial \dot{\xi}^k} + \frac{\varrho^2}{2}\,(\dot{\xi}^k\dot{\xi}^\ell - g^{k\ell}(\xi))\,\frac{\partial^2}{\partial \dot{\xi}^k \partial \dot{\xi}^\ell}\,. \quad (10)$$

The relativistic diffusion $(\xi_s, \dot{\xi}_s)$ is parametrized by proper time $s \ge 0$ (since $g_{\xi_s}(\dot{\xi}_s, \dot{\xi}_s) = 1$), possibly till some positive explosion time.

As in the Riemannian case (recall Remark 3.3), on the one hand this construction uses the Cartan moving frame method and provides the stochastic parallel transport (Ψ_s) along the relativistic Brownian curve $(\xi_s, \dot{\xi}_s)$ together with the curve itself, and on the other hand, the relativistic Brownian motion $(\xi_s, \dot{\xi}_s)$ is covariant with respect to the isometries of $\mathcal{M}$: for any such (Lorentzian) isometry f, the process $\big(f\circ(\xi_s, \dot{\xi}_s)\big)$ is another relativistic Brownian motion, starting at $f(\xi_0, \dot{\xi}_0)$.

In local coordinates (ξ^i, e_j^k), setting $\Psi_s = (\xi_s^i, e_j^k(s))$, Equation $(**)$ becomes locally equivalent to the following system of Itô equations:

$$d\xi_s^k = \dot{\xi}_s^k\,ds = e_0^k(s)\,ds\ ;\quad d\dot{\xi}_s^k = -\Gamma_{i\ell}^k(\xi_s)\,\dot{\xi}_s^i\,\dot{\xi}_s^\ell\,ds + \varrho \sum_{i=1}^d e_i^k(s)\,dw_s^i + \frac{d\,\varrho^2}{2}\,\dot{\xi}_s^k\,ds\,,\quad \text{and}$$

$$de_j^k(s) = -\Gamma_{i\ell}^k(\xi_s)\,e_j^\ell(s)\,\dot{\xi}_s^i\,ds + \varrho\,\dot{\xi}_s^k\,dw_s^j + \frac{\varrho^2}{2}\,e_j^k(s)\,ds\,,\quad \text{for } 1 \le j \le d\,,\ 0 \le k \le d\,.$$

Furthermore, on $T^1\mathcal{M}$ we have:

$$\sum_{j=1}^{d} V_j^2 \mathcal{E} = 2(d+1)\,\mathcal{E} - 2\,\mathrm{Tr}(T) = 2(d+1)\,\mathcal{E} + (d-1)\,R\,.$$

As an application, a direct computation yields the following evolution of the energy.

Remark 6.3 The random energy process $\mathcal{E}_s = \mathcal{E}(\xi_s, \dot{\xi}_s)$ associated to the basic relativistic diffusion $\pi_1(\Psi_s) = (\xi_s, \dot{\xi}_s)$ satisfies the following equation (where $\nabla_V := V^j \nabla_j$):

$$d\mathcal{E}_s = \nabla_{\dot{\xi}_s} \mathcal{E}_s\, ds + \varrho^2 \left[(d+1)\mathcal{E}_s + \tfrac{d-1}{2}\, R(\xi_s)\right] ds + dM_s^{\mathcal{E}}\,,$$

with the quadratic variation of its martingale part $dM_s^{\mathcal{E}}$ given by:

$$[d\mathcal{E}_s, d\mathcal{E}_s] = [dM_s^{\mathcal{E}}, dM_s^{\mathcal{E}}] = 4\varrho^2\, [\mathcal{E}_s^2 - \langle \tilde{T}\dot{\xi}_s, \tilde{T}\dot{\xi}_s\rangle]\, ds\,.$$

Note that generally the energy $\mathcal{E}_s$ is not a Markov process.

6.1 Example: The Schwarzschild Solution (After [13])

This space-time is commonly used in physics to model the complement of a spherical body, star or black hole; see for example ([21], Sect. 97). It is a basic example of space-time, i.e., of exact solution to the Einstein equations. This is actually the unique such solution which is both radial and empty (the latter amounts to having a vanishing Ricci tensor); see ([23], Theorems 3.1, 3.4, 3.7) for more specific statements in this direction.

Take $\mathcal{M} = \mathcal{S}_0 := \left\{\xi = (t, r, \theta) \in \mathbb{R} \times [R, +\infty[\times \mathbb{S}^2\right\}$, where $R \in \mathbb{R}_+$ is a parameter of the central body, endowed with the radial pseudo-metric:

$$(1 - \tfrac{R}{r})\, dt^2 - (1 - \tfrac{R}{r})^{-1} dr^2 - r^2 |d\theta|^2\,.$$

The coordinate t represents the absolute time, and r the distance from the origin. The Ricci tensor vanishes, the space $\mathcal{S}_0$ being empty. A theorem by Birkhoff asserts that there is no other radial pseudo-metric in $\mathcal{S}_0$ which satisfies this constraint.
Take as local coordinates the global spherical coordinates: $\xi \equiv (\xi^0, \xi^1, \xi^2, \xi^3) := (t, r, \varphi, \psi)$. According to the above, the system of Itô stochastic differential equations governing the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ becomes:

$$dt_s = e_0^0(s)\, ds\,,\ \ dr_s = e_0^1(s)\, ds\,,\ \ d\varphi_s = e_0^2(s)\, ds\,,\ \ d\psi_s = e_0^3(s)\, ds\,,$$

$$de_0^0(s) = \tfrac{3\sigma^2}{2}\, e_0^0(s)\, ds - \tfrac{R}{r_s(r_s - R)}\, e_0^0(s)\, e_0^1(s)\, ds + dM_s^0\,,$$

$$de_0^2(s) = \tfrac{3\sigma^2}{2}\, e_0^2(s)\, ds - \tfrac{2}{r_s}\, e_0^1(s)\, e_0^2(s)\, ds + \sin\varphi_s \cos\varphi_s\, e_0^3(s)^2\, ds + dM_s^2\ ,$$

$$de_0^3(s) = \tfrac{3\sigma^2}{2}\, e_0^3(s)\, ds - \tfrac{2}{r_s}\, e_0^1(s)\, e_0^3(s)\, ds - 2\,\mathrm{cotg}\,\varphi_s\, e_0^2(s)\, e_0^3(s)\, ds + dM_s^3\ ,$$

where the martingale $M_s := (M_s^0, M_s^1, M_s^2, M_s^3)$ has the following rank 3 quadratic covariation matrix: $K_s = \sigma^2\,(e_0(s)\,{}^t e_0(s) - g^{-1}(\xi_s))$.

Let us introduce the angular momentum $\vec{b} := r^2\,\theta \wedge \dot{\theta}$, the energy $a := (1 - \frac{R}{r})\,\dot{t}$, and the norm of $\vec{b}$: $b := |\vec{b}| = r^2\, U$, with $U := |\dot{\theta}|$. Let us also set $T := \dot{r}$, and accordingly

$$T_s := \dot{r}_s = e_0^1(s)\ ,\quad U_s := |\dot{\theta}_s| = \sqrt{e_0^2(s)^2 + \sin^2\varphi_s\, e_0^3(s)^2}\ ,\quad \text{and}\quad D := \min\{s > 0 \,|\, r_s = R\}\ .$$

Standard stochastic calculus computations yield the following:

Proposition 6.1.1 (i) *The unit pseudo-norm relation (which expresses that the parameter* s *is precisely the arc length, i.e., the so-called proper time) is given by*

$$T_s^2 = a_s^2 - (1 - R/r_s)(1 + b_s^2/r_s^2)\ .$$

(ii) *The process* (r_s, a_s, b_s, T_s) *is a degenerate diffusion, with lifetime* D*, which solves the following system of stochastic differential equations:*

$$dr_s = T_s\, ds\ ,\quad dT_s = dM_s^T + \tfrac{3\,\sigma^2}{2}\, T_s\, ds + (r_s - \tfrac{3}{2}R)\, \frac{b_s^2}{r_s^4}\, ds - \frac{R}{2r_s^2}\, ds\ ,$$

$$da_s = dM_s^a + \tfrac{3\,\sigma^2}{2}\, a_s\, ds\ ,\quad db_s = dM_s^b + \tfrac{3\,\sigma^2}{2}\, b_s\, ds + \frac{\sigma^2\, r_s^2}{2\, b_s}\, ds\ ,$$

with quadratic covariation matrix of the local martingale (M^a, M^b, M^T) *given by*

$$K_s' := \sigma^2 \begin{pmatrix} a_s^2 - 1 + \frac{R}{r_s} & a_s\, b_s & a_s\, T_s \\ a_s\, b_s & b_s^2 + r_s^2 & b_s\, T_s \\ a_s\, T_s & b_s\, T_s & T_s^2 + 1 - \frac{R}{r_s} \end{pmatrix}.$$

We get in particular the following statement, in which the dimension is reduced.

Corollary 6.1.2 *The process* (r_s, b_s, T_s) *is a diffusion, with lifetime* D *and infinitesimal generator*

$$\mathcal{G}' := T\frac{\partial}{\partial r} + \frac{\sigma^2}{2}\,(b^2 + r^2)\,\frac{\partial^2}{\partial b^2} + \frac{\sigma^2}{2b}\,(3b^2 + r^2)\,\frac{\partial}{\partial b} + \sigma^2 bT\,\frac{\partial^2}{\partial b\partial T}$$

$$+ \frac{\sigma^2}{2}\left(T^2 + 1 - \tfrac{R}{r}\right)\frac{\partial^2}{\partial T^2} + \left(\frac{3\,\sigma^2}{2}\,T + (r - \tfrac{3}{2}R)\,\frac{b^2}{r^4} - \frac{R}{2r^2}\right)\frac{\partial}{\partial T}\ .$$

In the geodesic case $\sigma = 0$ five types of behaviour can occur, owing to the trajectory of (r_s) ; it can be:

- running from R to $+\infty$, or in the opposite direction;
- running from R to R in finite proper time;
- running from $+\infty$ to $+\infty$;
- running from R to some R_1 or from R_1 to $+\infty$, or idem in the opposite direction;
- running endlessly in a bounded region away from R.

Though the stochastic case $\sigma \neq 0$ can be seen as a perturbation of the geodesic case $\sigma = 0$, the asymptotic behaviour classification regarding it is quite different.

Theorem 6.1.3 *([13])* (i) *For any initial condition, the radial process* (r_s) *almost surely hits* R *within a finite proper time* D *or goes to* $+\infty$ *as* $s \to +\infty$.
(ii) *Both events in (1) occur with positive probability, from any initial condition.*
(iii) *Conditionally on the event* $\{D = \infty\}$ *of non-hitting the central body, the Schwarzschild relativistic diffusion* $(\xi_s, \dot{\xi}_s)$ *goes almost surely to infinity in some random asymptotic direction of* $\mathbb{R}^3$, *asymptotically with the velocity of light.*

In particular, the relativistic diffusion almost surely cannot explode before a finite proper time D.

The Schwarzschild relativistic diffusion has been analyzed further in [13], using the (Kruskal-Szekeres) maximal extension of the Schwarzschild spacetime (also considered by [23, 24]): not only its behaviour till the hitting of the singularity can be thoroughly specified, but also a continuation of the diffusion thereafter makes sense, at least mathematically, and can be analyzed for proper time running the whole $\mathbb{R}_+$.

6.2 Example: The Gödel Universe (After [11])

The Gödel universe was intended by K. Gödel to object to the Einstein general theory of 1915: while being an exact solution to the Einstein equations, it presents the striking particularity of excluding global causality, since it has closed future-directed timelike continuous paths, which makes theoretically possible to access to one's own past after a long travel. That particular feature soon made this universe famous, and it is still the object of numerous developments.

The Gödel universe G is the manifold $\mathbb{R}^4$, endowed with coordinates $\xi := (t, x, y, z)$, and with the pseudo-metric g defined for some positive constant ω, by:

$$ds^2 := dt^2 - dx^2 + \tfrac{1}{2}\, e^{2\sqrt{2}\,\omega\, x}\, dy^2 + 2\, e^{\sqrt{2}\,\omega\, x}\, dt\, dy - dz^2 \,.$$

Along any timelike curve (t_s, x_s, y_s, z_s), the unit pseudo-norm relation, defining proper time s , is:

$$1 + \dot{t}_s^2 + \dot{x}_s^2 + \dot{z}_s^2 = \tfrac{1}{2}\left[e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s + 2\, \dot{t}_s\right]^2.$$

Gödel's universe can be viewed as a matrix group, on which Gödel's metric g happens to be the left-invariant metric generated by the Lorentz metric g^0 on the Lie algebra $\mathcal{G}$: $\langle \mathcal{L}_A, \mathcal{L}_A \rangle_g = \langle A, A \rangle_{g^0}$ for any $A \in \mathcal{G}$. This group structure is given by the following: for any $\xi_0 = (t_0, x_0, y_0, z_0)$, $\xi = (t, x, y, z) \in G$, $\xi_0 \times \xi = (t + t_0, x + x_0, y\, e^{-\sqrt{2}\,\omega\, x_0} + y_0, z + z_0)$.

Proposition 6.2.1 *The Gödel universe G is piece-wise geodesically transitive: any two points of it can be linked by a piece-wise geodesic future-directed timelike continuous path.*

The relativistic diffusion $(\xi_s, \dot{\xi}_s)$, in coordinates $\xi = (t, x, y, z)$, solves the following system of stochastic differential equations:

$$dt_s = \dot{t}_s\, ds\ ; \quad dx_s = \dot{x}_s\, ds\ ; \quad dy_s = \dot{y}_s\, ds\ ; \quad dz_s = \dot{z}_s\, ds\ ;$$

$$d\dot{t}_s \;=\; -2\sqrt{2}\,\omega\, \dot{t}_s\, \dot{x}_s\, ds - \sqrt{2}\,\omega\, e^{\sqrt{2}\,\omega\, x_s}\, \dot{x}_s\, \dot{y}_s\, ds + \tfrac{3\sigma^2}{2}\, \dot{t}_s\, ds + \sigma\, dM_s^t\ ;$$

$$d\dot{x}_s \;=\; -\sqrt{2}\,\omega\, e^{\sqrt{2}\,\omega\, x_s}\, \dot{t}_s\, \dot{y}_s\, ds - (\omega/\sqrt{2})\, e^{2\sqrt{2}\,\omega\, x_s}\, \dot{y}_s^2\, ds + \tfrac{3\sigma^2}{2}\, \dot{x}_s\, ds + \sigma\, dM_s^x\ ;$$

$$d\dot{y}_s \;=\; 2\sqrt{2}\,\omega\, e^{-\sqrt{2}\,\omega\, x_s}\, \dot{t}_s\, \dot{x}_s\, ds + \tfrac{3\sigma^2}{2}\, \dot{y}_s\, ds + \sigma\, dM_s^y\ ;$$

$$d\dot{z}_s \;=\; \tfrac{3\sigma^2}{2}\, \dot{z}_s\, ds + \sigma\, dM_s^z\ ;$$

where the $\mathbb{R}^4$-valued martingale $M_s := (M_s^t, M_s^x, M_s^y, M_s^z)$ has (rank 3) quadratic covariation matrix:

$$((K_s^{ij})) := \frac{\langle dM_s^i, dM_s^j \rangle}{ds} = \begin{pmatrix} \dot{t}_s^2 + 1 & \dot{t}_s\, \dot{x}_s & \dot{t}_s\, \dot{y}_s - 2\, e^{-\sqrt{2}\,\omega\, x_s} & \dot{t}_s\, \dot{z}_s \\ \dot{t}_s\, \dot{x}_s & \dot{x}_s^2 + 1 & \dot{x}_s\, \dot{y}_s & \dot{x}_s\, \dot{z}_s \\ \dot{t}_s\, \dot{y}_s - 2\, e^{-\sqrt{2}\,\omega\, x_s} & \dot{x}_s\, \dot{y}_s & \dot{y}_s^2 + 2\, e^{-2\sqrt{2}\,\omega\, x_s} & \dot{y}_s\, \dot{z}_s \\ \dot{t}_s\, \dot{z}_s & \dot{x}_s\, \dot{z}_s & \dot{y}_s\, \dot{z}_s & \dot{z}_s^2 + 1 \end{pmatrix}.$$

The following quantities, as $\dot{z}_s$, are constant along each geodesic:

$$a_s \;:=\; \dot{t}_s + e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s \qquad \text{and} \qquad b_s \;:=\; e^{\sqrt{2}\,\omega\, x_s}\, (2\, \dot{t}_s + e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s)\, .$$

The quantity a_s^2 represents an energy. Then we have:

$$da_s \;=\; \tfrac{3\sigma^2}{2}\, a_s\, ds + \sigma\, dM_s^a \;=\; \tfrac{3\sigma^2}{2}\, a_s\, ds + \sigma\, (dM_s^t + e^{\sqrt{2}\,\omega\, x_s}\, dM_s^y)\ ;$$

and

$$db_s \;=\; \tfrac{3\sigma^2}{2}\, b_s\, ds + \sigma\, dM_s^b \;=\; \tfrac{3\sigma^2}{2}\, b_s\, ds + \sigma\, e^{\sqrt{2}\,\omega\, x_s}\, (2\, dM_s^t + e^{\sqrt{2}\,\omega\, x_s}\, dM_s^y)\, .$$

Moreover we have:

$$d\dot{x}_s = (\omega/\sqrt{2})\, e^{-2\sqrt{2}\,\omega\, x_s}\, b_s^2\, ds - \sqrt{2}\,\omega\, e^{-\sqrt{2}\,\omega\, x_s}\, a_s\, b_s\, ds + \tfrac{3\sigma^2}{2}\, \dot{x}_s\, ds + \sigma\, dM_s^x\,,$$

and the $\mathbb{R}^4$-valued martingale $\tilde{M}_s := (M_s^a, M_s^b, M_s^x, M_s^z)$ has (rank 3) quadratic covariation matrix:

$$((\tilde{K}_s^{ij})) = \begin{pmatrix} a_s^2 - 1 & a_s\, b_s - 2\, e^{\sqrt{2}\,\omega\, x_s} & a_s\, \dot{x}_s & a_s\, \dot{z}_s \\ a_s\, b_s - 2\, e^{\sqrt{2}\,\omega\, x_s} & b_s^2 - 2\, e^{2\sqrt{2}\,\omega\, x_s} & b_s\, \dot{x}_s & b_s\, \dot{z}_s \\ a_s\, \dot{x}_s & b_s\, \dot{x}_s & \dot{x}_s^2 + 1 & \dot{x}_s\, \dot{z}_s \\ a_s\, \dot{z}_s & b_s\, \dot{z}_s & \dot{x}_s\, \dot{z}_s & \dot{z}_s^2 + 1 \end{pmatrix}.$$

From this, we deduce the following, which allows the asymptotic study, as proper time s goes to infinity, of relativistic paths.

Corollary 6.2.2 *The (7-dimensional) relativistic diffusion* $(\xi_s, \dot{\xi}_s)$ *admits the following sub-diffusions:* (a_s) ; $(\dot{z}_s)$; $(a_s, \dot{z}_s)$; $(x_s, \dot{x}_s, a_s, b_s)$.

The unit pseudo-norm relation can be written as: $1 + \dot{x}_s^2 + \dot{z}_s^2 + \frac{1}{2}\,(2\, a_s - e^{-\sqrt{2}\,\omega\, x_s}\, b_s)^2 = a_s^2$.
Hence the phase space $\mathcal{E}$ of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ can be written equivalently:

$$\mathcal{E} = \Big\{(t, x, y, z, a, b, \dot{x}, \dot{z}) \in \mathbb{R}^8 \,\Big|\, 1 + \dot{x}^2 + \dot{z}^2 + \tfrac{1}{2}\,(2\, a - e^{-\sqrt{2}\,\omega\, x}\, b)^2 = a^2\Big\},$$

in which the particular phase subspace $\mathcal{E}_0$ has to be distinguished:

$$\mathcal{E}_0 = \mathcal{E} \cap \Big\{a^2 = 1 + \dot{z}^2 \;;\; 2\, a = e^{-\sqrt{2}\,\omega\, x}\, b \;;\; \dot{x} = 0\Big\} = \mathcal{E} \cap \Big\{a^2 = 1 + \dot{z}^2\Big\}.$$

Remark 6.2.3 The phase space $\mathcal{E}$ splits into two connected components: $\mathcal{E} = \mathcal{E}^+ \sqcup \mathcal{E}^-$, with $\mathcal{E}^+ := \mathcal{E} \cap \{a \geq 1,\ b > 0\}$ and $\mathcal{E}^- := \mathcal{E} \cap \{a \leq -1,\ b < 0\}$. Similarly, $\mathcal{E}_0 = \mathcal{E}_0^+ \sqcup \mathcal{E}_0^-$, with $\mathcal{E}_0^+ := \mathcal{E}_0 \cap \mathcal{E}^+$ and $\mathcal{E}_0^- := \mathcal{E}_0 \cap \mathcal{E}^-$. Note that since $2\, \dot{t}_s + e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s = e^{-\sqrt{2}\,\omega\, x_s}\, b_s$, the paths in $\mathcal{E}^+$ are always future-directed. Since the symmetry $(a, b) \mapsto (-a, -b)$ exchanges $(\mathcal{E}^+, \mathcal{E}_0^+)$ and $(\mathcal{E}^-, \mathcal{E}_0^-)$, from now on, we can restrict the phase space of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ to $\mathcal{E}^+$ (its behaviour on $\mathcal{E}^-$ being trivially related).

Recall that in a strongly causal space-time, it seems natural to use the causal boundary, in the sense of Penrose (see the conformal compactification considered in Sect. 2 of [24]), to classify lightlike geodesics by gathering in an equivalence class, called a *beam*, all geodesics which converge to a given causal boundary point (having asymptotically the same past, see ([18], Sect. 6.8)). On the contrary, in the present setting (recall Proposition 6.2.1) such a classification is totally inoperative. It seems that no alternative classification has been proposed before [11], which is relevant in a non-causal setting. Now, owing to the analysis of lightlike geodesics of G, the following alternative classification of lightlike geodesics into beams was adopted in [11], viewing then the 3-dimensional space of beams as an alternative notion of (non-causal, however conformal) boundary, as follows.

Definition 6.2.4 Let us call beam, or boundary point, of Gödel's universe, any equivalence class of lightlike geodesics, identifying those which have the same impact parameter $B = (\ell, \varrho, Y) \in \mathcal{B} = [-1, 1] \times \mathbb{R}_+^* \times \mathbb{R}$. Thus $\mathcal{B}$ will be the boundary of Gödel's universe.

Here the definition of the impact parameter B is exactly as in the following main result, though it is of course by far easier to obtain in the case of a lightlike geodesic.

Theorem 6.2.5 (i) *The relativistic diffusion is irreducible on its phase space* $\mathcal{E}^+ \backslash \mathcal{E}_0^+$.

(ii) *Almost surely, the relativistic diffusion path possesses a 3-dimensional asymptotic random variable* $B = (\ell, \varrho, Y) \in \mathcal{B}$. *Namely, it converges to this beam* B *in the sense that, as proper time* s *goes to infinity, we almost surely have:*

$$\dot{z}_s/a_s \longrightarrow \ell \in]-1, 0[\cup]0, 1[\,; \quad b_s/a_s \longrightarrow \varrho \in]0, \infty[\,; \quad Y_s := \frac{\sqrt{2}\,\dot{x}_s}{\omega\, b_s} + y_s \longrightarrow Y \in \mathbb{R}\,;$$

$$\left[\frac{\varrho}{2}\, e^{-\sqrt{2}\,\omega\, x_s} - 1\right]^2 + \left[\frac{\omega\,\varrho}{2}\,(y_s - Y)\right]^2 \longrightarrow \tfrac{1}{2}(1 - \ell^2)\,.$$

(iii) *The asymptotic random variable* (ℓ, ϱ, Y) *can be arbitrarily close to any given* $(\ell_0, \varrho_0, y) \in]-1, 1[\times]0, \infty[\times\mathbb{R}$, *with positive probability. Hence, the whole boundary (space of beams)* $\mathcal{B}$ *is the support of beams the relativistic diffusion can converge to.*

It remains an open question to establish whether the limiting random variable $B = (\ell, \varrho, Y)$ contains all asymptotic information, i.e., generates the invariant or the tail σ-field of the whole Gödel relativistic diffusion, and thereby is enough to represent all bounded harmonic functions (i.e., the *Poisson boundary*, recall Theorem 4.3) of the Gödel universe.

7 Covariant Ξ-relativistic Diffusions

We present here other intrinsic Lorentz-covariant diffusions, taking advantage of the curvature tensor (recall Sect. 5). They could be seen as maybe more physical than the basic relativistic diffusion presented till now, as their quadratic variation is locally determined by their velocity and the curvature of the space, and vanishes in flat or in Ricci-flat (empty) regions.

Let Ξ denote a non-negative smooth function on $G(\mathcal{M})$, invariant under the right action of $\mathrm{SO}(d)$ (so that it is identified with a function on $T^1\mathcal{M}$).

Our basic non-constant examples will be $\Xi = -\sigma^2 R$ and $\Xi = \sigma^2 \mathcal{E}$ (for a constant $\sigma > 0$).

We start with the following Stratonovitch stochastic differential equation on $G(\mathcal{M})$ (for a given $\mathbb{R}^d$-valued Brownian motion (w_s^j)):

$$d\Phi_s = H_0(\Phi_s)\, ds + \tfrac{1}{4} \sum_{j=1}^{d} V_j\, \Xi\, (\Phi_s) V_j(\Phi_s)\, ds + \sum_{j=1}^{d} \sqrt{\Xi(\Phi_s)}\, V_j(\Phi_s) \circ dw_s^j \,. \tag{11}$$

Note that all coefficients in this equation are clearly smooth, except $\sqrt{\Xi}$ on its vanishing set $\Xi^{-1}(0)$. However, $\sqrt{\Xi}$ is a locally Lipschitz function; see ([20], Proposition IV.6.2). Hence, Equation (11) does define a unique $G(\mathcal{M})$-valued diffusion (Φ_s). We have the following theorem, which defines the Ξ-*relativistic diffusion* (or Ξ-*diffusion*) (Φ_s) on $G(\mathcal{M})$ and $(\xi_s, \dot{\xi}_s)$ on $T^1\mathcal{M}$, possibly till some positive explosion time.

Theorem 7.1 (see [14]) (i) *The Stratonovitch stochastic differential equation (11) has a unique solution* $(\Phi_s) = (\xi_s;\, \dot{\xi}_s\,, e_1(s), \ldots, e_d(s))$*, possibly defined till some positive explosion time. This is a* $G(\mathcal{M})$*-valued covariant diffusion process, with generator*

$$\mathcal{H}_\Xi \;:=\; H_0 + \tfrac{1}{2} \sum_{j=1}^{d} V_j\, \Xi\, V_j \,. \tag{12}$$

(ii) *Its projection* $\pi_1(\Phi_s) = (\xi_s, \dot{\xi}_s)$ *defines a covariant diffusion on* $T^1\mathcal{M}$*, with* SO(d)*-invariant generator*

$$\mathcal{H}_\Xi^1 \;:=\; \mathcal{L}_0 + \tfrac{1}{2}\, \nabla^v\, \Xi\, \nabla^v \,, \tag{13}$$

∇^v *denoting the gradient on* $T_\xi^1\mathcal{M}$ *equipped with the hyperbolic metric induced by* $g(\xi)$.
(iii) *Moreover, the adjoint of* $\mathcal{H}_\Xi$ *with respect to the Liouville measure of* $G(\mathcal{M})$ *is*
$\mathcal{H}_\Xi^* := -H_0 + \frac{1}{2} \sum_{j=1}^{d} V_j\, \Xi\, V_j$. *In particular, if there is no explosion, then the Liouville measure is invariant. Furthermore, if* Ξ *does not depend on* $\dot{\xi}$*, i.e., is a function on* $\mathcal{M}$*, then the Liouville measure is preserved by the stochastic flow defined by Eq. (11).*

We specify right away how this looks in a local chart.

Corollary 7.2 *The* $T^1\mathcal{M}$*-valued* Ξ*-diffusion* $(\xi_s, \dot{\xi}_s)$ *satisfies* $d\xi_s = \dot{\xi}_s\, ds$*, and in any local chart, the following Itô stochastic differential equations: for* $0 \le k \le d$*, (denoting* $\Xi_s \equiv \Xi(\xi_s, \dot{\xi}_s)$*)*

$$d\dot{\xi}_s^k = dM_s^k - \Gamma_{ij}^k(\xi_s)\,\dot{\xi}_s^i\,\dot{\xi}_s^j\,ds + \tfrac{d}{2}\,\Xi_s\,\dot{\xi}_s^k\,ds + \tfrac{1}{2}\left[\dot{\xi}_s^k\,\dot{\xi}_s^\ell - g^{k\ell}(\xi_s)\right]\frac{\partial\Xi}{\partial\dot{\xi}^\ell}(\xi_s,\dot{\xi}_s)\,ds\,, \tag{14}$$

with the quadratic covariation matrix of the martingale term (dM_s) *given by:*

$$\left[d\dot{\xi}_s^k, d\dot{\xi}_s^\ell\right] = \left[\dot{\xi}_s^k\,\dot{\xi}_s^\ell - g^{k\ell}(\xi_s)\right]\Xi_s\,ds\,, \quad \textit{for}\ \ 0 \le k,\ell \le d\,.$$

Remark 7.3 (i) The vertical terms could be seen as an effect of the matter or the radiation present in the space-time $\mathcal{M}$. The Ξ-diffusion (Φ_s) reduces to the geodesic flow in the regions of the space where Ξ vanishes, which happens in particular for empty space-times $\mathcal{M}$ in the cases $\Xi = -\sigma^2 R(\xi)$, or $\Xi = \sigma^2\,\mathcal{E}(\xi,\dot{\xi})$, or also $\Xi = -\sigma^2 R(\xi)\,e^{\kappa\,\mathcal{E}(\xi,\dot{\xi})/R(\xi)}$ (for any positive constant κ) for example.

(ii) As for the basic relativistic diffusion, the law of the Ξ-relativistic diffusion is covariant with respect to any isometry of $(\mathcal{M}, g)$. The basic relativistic diffusion corresponds to $\Xi \equiv \sigma^2 > 0$, and the geodesic flow to $\Xi \equiv 0$.

(iii) In [5] is considered a general model for relativistic diffusions, which may be covariant or not. Up to enlarging it by allowing the "rest frame" (denoted by z in [5]) to have space vectors of non-unit norm, this model includes the generic Ξ-diffusion (compare the above Eq. (11) to (2.5), (3.3) in [5]).

7.1 Example 1: The R-diffusion

We assume here that the scalar curvature $R = R(\xi)$ is everywhere non-positive on $\mathcal{M}$, which is physically relevant: this is the *strong energy condition*, in the case where the energy-momentum tensor T has a timelike eigenvector (the so-called *type I*, e.g., a perfect fluid); see ([18] page 95); this is also the *dominant energy condition* in the terminology used by ([19], Sect. 2.2); up to the convention used for the sign. Consider the particular case corresponding to $\Xi = -\sigma^2 R(\xi)$, with a constant positive parameter σ.
In this case, as its central term clearly vanishes, Eq. (11) takes on the simple form:

$$d\Phi_s = H_0(\Phi_s)\,ds + \sigma\sum_{j=1}^{d}\sqrt{-R(\Phi_s)}\,V_j(\Phi_s)\circ dw_s^j\,.$$

7.2 Example 2: The $\mathcal{E}$-diffusion

We assume that the Weak Energy Condition (recall Sect. 5) holds (everywhere on $T^1\mathcal{M}$), which is physically relevant (see ([21], (94, 10)), [18]: this means that the energy has to be non-negative everywhere in the space-time), and consider the particular case corresponding to $\Xi = \sigma^2\,\mathcal{E} = \sigma^2\,\mathcal{E}(\xi,\dot{\xi}) = \sigma^2\,T_{00}$.

We call *energy relativistic diffusion* or *$\mathcal{E}$-diffusion* the $G(\mathcal{M})$-valued diffusion process (Φ_s) we get in this way, as well as its $T^1\mathcal{M}$-valued projection $\pi_1(\Phi_s)$.
The following computational lemma implies that the central drift term in Eq. (11) is a function of the Ricci tensor alone when Ξ is.

Lemma 7.2.1 *We have* $V_j R_i^k = \delta_{0i}\, R_j^k - \eta_{ij}\, R_0^k + \delta_0^k\, R_{ij} - \delta_j^k\, R_{0i}$ *, for* $0 \le i, k \le d$ *and* $1 \le j \le d$ *. In particular,* $V_j R = 0$*, and* $V_j \mathcal{E} = V_j T_{00} = V_j R_{00} = 2R_{0j}$ *.*

Lemma 7.2.1 and some more computation lead to the following, to be compared with Corollary 6.3. The notation $(\tilde{T}\dot{\xi})^k \equiv \tilde{T}_m^k \dot{\xi}^m$ below has the meaning of a matrix product.

Remark 7.2.2 The random energy $\mathcal{E}_s := \mathcal{E}(\xi_s, \dot{\xi}_s)$ associated to the $\mathcal{E}$-diffusion (Φ_s) satisfies the following equation (where $\nabla_V := V^j \nabla_j$):

$$d\mathcal{E}_s = \nabla_{\dot{\xi}_s} \mathcal{E}(\xi_s, \dot{\xi}_s)\, ds + (d+2)\, \sigma^2\, \mathcal{E}_s^2\, ds - 2\sigma^2\, g(\tilde{T}\dot{\xi}_s, \tilde{T}\dot{\xi}_s)\, ds + 2\sigma\, dM_s^{\mathcal{E}} \, ,$$

with the quadratic variation of its martingale part $dM_s^{\mathcal{E}}$ given by:

$$[d\mathcal{E}_s, d\mathcal{E}_s] = 4\sigma^2 \left[dM_s^{\mathcal{E}}, dM_s^{\mathcal{E}}\right] = 4\sigma^2 \left[\mathcal{E}_s^2 - g(\tilde{T}\dot{\xi}_s, \tilde{T}\dot{\xi}_s)\right] \mathcal{E}_s\, ds.$$

Remark 7.2.3 The case of Einstein Lorentz manifolds.

The Lorentz manifold $\mathcal{M}$ is said to be Einstein if its Ricci tensor is proportional to its metric tensor. Bianchi's contracted identities (see for example [18]), which entail the conservation equations $\nabla_k \tilde{T}^{jk} = 0$, force the proportionality coefficient $\tilde{p}$ to be constant on $\mathcal{M}$. Hence: $\tilde{R}_{\ell m}(\xi) = \tilde{p}\; g_{\ell m}(\xi)$, for any ξ in $\mathcal{M}$ and $0 \le \ell, m \le d$.

Then the scalar curvature is $R(\xi) = (d+1)\, \tilde{p}$, and by Einstein's Equations (5) we have:

$$\tilde{T}_{\ell m}(\xi) = (\Lambda - \tfrac{d-1}{2}\, \tilde{p})\, g_{\ell m}(\xi) \;=:\; -p\; g_{\ell m}(\xi)\, .$$

Hence Eq. (6) holds, with $q = 0$: we are in a limiting case of a perfect fluid. Moreover, R and $\mathcal{E}$ are constant, so that in an Einstein Lorentz manifold, the R-diffusion and the $\mathcal{E}$-diffusion coincide with the basic relativistic diffusion (of Sect. 6).

8 Example of Robertson-Walker (R-W) Manifolds

These important manifolds are intended to model an expanding (or shrinking) universe resulting from a "Big-Bang", as ours is believed to be. They admit an absolute time coordinate t ; in the classical terminology used in particular by ([19], Sect. 2), they are "globally hyperbolic".

They are particular cases of warped product: they can be written as $\mathcal{M} = I \times M$, where I is an open interval of $\mathbb{R}_+$ and $M \in \{\mathbb{S}^3, \mathbb{R}^3, \mathbb{H}^3\}$, with spherical coordinates $\xi \equiv (t, r, \varphi, \psi)$ (which are global in the case of $\mathbb{R}^3$, $\mathbb{H}^3$, and are defined separately on two hemispheres in the case of $\mathbb{S}^3$), and are endowed with the pseudo-norm:

$$g(\dot{\xi},\dot{\xi}) := \dot{t}^2 - \alpha(t)^2\left(\frac{\dot{r}^2}{1-kr^2} + r^2\dot{\varphi}^2 + r^2\sin^2\varphi\ \dot{\psi}^2\right), \tag{15}$$

where the constant scalar spatial curvature k belongs to $\{-1, 0, 1\}$ (note that $r \in [0, 1]$ for $k = 1$ and $r \in \mathbb{R}_+$ for $k = 0, -1$), and the expansion factor α is a positive C^2 function on I. The so-called *Hubble function* is: $H(t) := \alpha'(t)/\alpha(t)$.
Note that we necessarily have $\dot{t} \geq 1$ everywhere on $T^1\mathcal{M}$. The curvature operator is given (denoting by *X,Y,A,Z* vectors over *M* and by *h* the metric tensor of *M*) by:

$$\Big\langle \mathcal{R}\left((u\partial_t + X)\wedge(v\partial_t + Y)\right), (a\partial_t + A)\wedge(w\partial_t + Z)\Big\rangle_\eta$$

$$= \alpha\alpha''\, h(uY - vX, aZ - wA) - \alpha^2(\alpha'^{\,2} + k)\,[h(X, A)h(Y, Z) - h(X, Z)h(Y, A)]\,.$$

The Ricci tensor $(\!(\tilde{R}_{\ell m})\!)$ is diagonal, with diagonal entries:

$$\left(-3\,\frac{\alpha''(t)}{\alpha(t)}, \frac{A(t)}{1-kr^2}, A(t)\,r^2, A(t)\,r^2\sin^2\varphi\right), \quad \text{where } A(t) := \alpha(t)\,\alpha''(t) + 2\,\alpha'(t)^2 + 2k\,,$$

and the scalar curvature is $R = -6\,[\alpha(t)\,\alpha''(t) + \alpha'(t)^2 + k]\,\alpha(t)^{-2}$.
The Einstein energy-momentum tensor $\tilde{R}_{\ell m} - \frac{1}{2}\,R\,g_{\ell m} = \tilde{T}_{\ell m}$ is diagonal as well, with diagonal entries:

$$\left(3\,\frac{\alpha'(t)^2 + k}{\alpha(t)^2}, \frac{-\tilde{A}(t)}{1-kr^2}, -\tilde{A}(t)\,r^2, -\tilde{A}(t)\,r^2\sin^2\varphi\right), \text{ with } \tilde{A}(t) := 2\alpha(t)\alpha''(t) + \alpha'(t)^2 + k\,.$$

Hence, we have

$$\tilde{T}_{\ell m} - \alpha(t)^{-2}\tilde{A}(t)\,g_{\ell m} = 2\,[k\,\alpha(t)^{-2} - H'(t)]\,1_{\{\ell=m=0\}}\,.$$

Thus, we have here an example of perfect fluid: Eq. (6) holds, with

$$U_j \equiv \delta_j^0\,, \quad -p(\xi) = k\,\alpha(t)^{-2} + 2H'(t) + 3H^2(t)\,, \quad q(\xi) = 2\,[k\,\alpha(t)^{-2} - H'(t)]\,, \tag{16}$$

$$\tilde{p}(\xi) = -2\,[2k\,\alpha(t)^{-2} + H'(t) + 3H^2(t)]/(d-1)\,.$$

Note that

$$\mathcal{A}_s = U_i(\xi_s)\,\dot{\xi}_s^i = \dot{t}_s \quad \text{and} \quad \mathcal{E}_s = 2\,[k\,\alpha(t_s)^{-2} - H'(t_s)]\,\dot{t}_s^2 - p(\xi_s)\,. \tag{17}$$

The weak energy condition is equivalent to: $\alpha'^{\,2} + k \geq (\alpha\,\alpha'')^+$.

We shall consider only eternal Robertson-Walker space-times, which have their future-directed half-geodesics complete. This amounts to $I = \mathbb{R}_+^*$, together with $\displaystyle\int^\infty \frac{\alpha}{\sqrt{1+\alpha^2}} = \infty$. In the case of the basic relativistic diffusion (within such a Robertson-Walker model), we have in particular:

$$d\dot{t}_s = \sigma\sqrt{\dot{t}_s^2 - 1}\, dw_s + \tfrac{3\sigma^2}{2}\,\dot{t}_s\, ds - H(t_s)[\dot{t}_s^2 - 1]\, ds\,. \tag{18}$$

8.1 Ξ-relativistic Diffusions in an Einstein-De Sitter-Like Manifold

We consider henceforth the particular case $I =]0,\infty[$, $k = 0$, and $\alpha(t) = t^c$, with exponent $c > 0$. Note that such expansion functions α can be obtained by solving a proportionality relation between p and q (see [18] or [21]).
Thus $q = 2c\,t^{-2}$, $p = (2-3c)c\,t^{-2}$, $R = -6\,c\,(2c-1)\,t^{-2}$, $\mathcal{E} = c\,t^{-2}\,(2\,\dot{t}^2 + 3c - 2)$.
Note that the weak energy condition holds. The scalar curvature is non-positive if and only if $c \ge 1/2$, and the pressure p is non-negative if and only if $c \le 2/3$.

Note that the particular case $c = \frac{2}{3}$ corresponds to a vanishing pressure p, and is precisely known as that of *Einstein-de Sitter universe* (see for example [18]). And the analysis of [21] shows precisely both limiting cases $c = \frac{2}{3}$ and $c = \frac{1}{2}$.

8.1.1 Basic Relativistic Diffusion in an Einstein-De Sitter-Like Manifold

In order to compare with the other relativistic diffusions, we mention, first for the basic relativistic diffusion (of Sect. 6), the stochastic differential equations satisfied by the main coordinates $\dot{t}_s$ and $\dot{r}_s$, appearing in the 4-dimensional sub-diffusion $(t_s, \dot{t}_s, r_s, \dot{r}_s)$. By (18), we have, for independent standard real Brownian motions $w, \tilde{w}$:

$$d\dot{t}_s \;=\; \sigma\sqrt{\dot{t}_s^2 - 1}\, dw_s + \tfrac{3\sigma^2}{2}\,\dot{t}_s\, ds - \frac{c}{t_s}\,(\dot{t}_s^2 - 1)\, ds\,; \tag{19}$$

$$d\dot{r}_s = \frac{\sigma\,\dot{t}_s\,\dot{r}_s}{\sqrt{\dot{t}_s^2 - 1}}\, dw_s + \sigma\sqrt{\frac{1}{t_s^{2c}} - \frac{\dot{r}_s^2}{\dot{t}_s^2 - 1}}\, d\tilde{w}_s + \tfrac{3\sigma^2}{2}\,\dot{r}_s\, ds + \left[\frac{\dot{t}_s^2 - 1}{t_s^{2c}} - \dot{r}_s^2\right]\frac{ds}{r_s} - \frac{2c}{t_s}\,\dot{t}_s\,\dot{r}_s\, ds\,. \tag{20}$$

Almost surely (see [1]), $\lim\limits_{s\to\infty} \dot{t}_s = \infty$; moreover x_s/r_s and $\dot{x}_s/|\dot{x}_s|$ converge in $\mathbb{S}^2$, to the same random limit.

Further results are established in [1], where the whole diffusion is thoroughly considered. In particular, the Poisson boundary is determined in some sub-cases of interest, yielding in such a curved framework an analogue of Theorem 4.3.

8.1.2 R-diffusion in an Einstein-De Sitter-Like Manifold

With the above, Sect. 7.1 gives here, for the R-relativistic diffusion, when $c \geq 1/2$:

$$d\dot{\xi}_s = \sigma\, dM_s + 9c\,(2c-1)\,\sigma^2\, t_s^{-2}\, \dot{\xi}_s\, ds - \Gamma^{\cdot}_{ij}(\xi_s)\, \dot{\xi}_s^i\, \dot{\xi}_s^j\, ds\,, \tag{21}$$

with the quadratic covariation matrix of the martingale part dM_s given by:

$$\sigma^{-2}\left[d\dot{\xi}_s^k, d\dot{\xi}_s^\ell\right] = 6c\,(2c-1)\left[\dot{\xi}_s^k\, \dot{\xi}_s^\ell - g^{k\ell}(\xi_s)\right] t_s^{-2}\, ds\,, \quad \text{for } 0 \leq k, \ell \leq d\,.$$

In particular, we have for independent standard real Brownian motions $w, \tilde{w}$:

$$d\dot{t}_s = \frac{\sigma}{t_s}\sqrt{6\,c\,(2c-1)(\dot{t}_s^2-1)}\, dw_s + \frac{9\,\sigma^2 c\,(2c-1)}{t_s^2}\, \dot{t}_s\, ds - \frac{c}{t_s}(\dot{t}_s^2-1)\, ds\,; \tag{22}$$

$$d\dot{r}_s = \frac{\sigma\sqrt{6c\,(2c-1)}}{t_s}\left[\frac{\dot{t}_s\,\dot{r}_s}{\sqrt{\dot{t}_s^2-1}}\, dw_s + \sqrt{\frac{1}{t_s^{2c}} - \frac{\dot{r}_s^2}{\dot{t}_s^2-1}}\, d\tilde{w}_s\right] \tag{23}$$

$$+ \frac{9\,\sigma^2 c\,(2c-1)}{t_s^2}\, \dot{r}_s\, ds + \left[\frac{\dot{t}_s^2-1}{t_s^{2c}} - \dot{r}_s^2\right]\frac{ds}{r_s} - \frac{2c}{t_s}\, \dot{t}_s\, \dot{r}_s\, ds\,.$$

As the scalar curvature $R_s = 6c\,(1-2c)/t_s^2$ vanishes asymptotically, we expect that almost surely the R-diffusion behaves eventually as a timelike geodesic, and in particular that $\lim\limits_{s\to\infty} \dot{t}_s = 1$.

8.1.3 $\mathcal{E}$-diffusion in an Einstein-De Sitter-Like Manifold

Similarly, using (16) and (17), we have here $\mathcal{E}\,\dot{\xi} - \tilde{T}\dot{\xi} = 2(0-H')(\dot{t}^2\dot{\xi} - \dot{t}\, U)$, so that Sect. 7.2 reads here, for the $\mathcal{E}$-diffusion:

$$d\dot{\xi}_s = \sigma\, dM_s + \tfrac{3\sigma^2 c}{2}\, t_s^{-2}\,(2\,\dot{t}_s^2 + 3c - 2)\, \dot{\xi}_s\, ds + 2\sigma^2 c\, t_s^{-2}(\dot{t}_s\, \dot{\xi}_s - U_s)\, \dot{t}_s\, ds - \Gamma^{\cdot}_{ij}(\xi_s)\, \dot{\xi}_s^i\, \dot{\xi}_s^j\, ds\,, \tag{24}$$

with the quadratic covariation matrix of the martingale part dM_s given by:

$$\sigma^{-2}\,[d\dot{\xi}_s^k, d\dot{\xi}_s^\ell] = c\,[\dot{\xi}_s^k\, \dot{\xi}_s^\ell - g^{k\ell}(\xi_s)]\,(2\,\dot{t}_s^2 + 3c - 2)\, t_s^{-2}\, ds\,, \quad \text{for } 0 \leq k, \ell \leq d\,.$$

In particular, we have for some standard real Brownian motion w:

$$d\dot{t}_s = \frac{\sigma\sqrt{c}}{t_s}\sqrt{(2\,\dot{t}_s^2 - 2 + 3c)(\dot{t}_s^2-1)}\, dw_s + c\left[5\sigma^2(\dot{t}_s^2 - 1 + \tfrac{9c}{10})\,\frac{\dot{t}_s}{t_s^2} - \frac{\dot{t}_s^2-1}{t_s}\right] ds\,; \tag{25}$$

$$d\dot{r}_s = \frac{\sigma\sqrt{c}}{t_s}\sqrt{2\,\dot{t}_s^2 - 2 + 3c}\left[\frac{\dot{t}_s\,\dot{r}_s}{\sqrt{\dot{t}_s^2 - 1}}\,dw_s + \sqrt{\frac{1}{t_s^{2c}} - \frac{\dot{r}_s^2}{\dot{t}_s^2 - 1}}\,d\tilde{w}_s\right] \tag{26}$$

$$+\,\sigma^2 c\,(5\,\dot{t}_s^2 - 3 + \tfrac{9c}{2})\,\frac{\dot{r}_s}{t_s^2}\,ds - \frac{2c}{t_s}\,\dot{t}_s\,\dot{r}_s\,ds + \left[\frac{\dot{t}_s^2 - 1}{t_s^{2c}} - \dot{r}_s^2\right]\frac{ds}{r_s}\,.$$

Remark 8.1.4 Comparison of Ξ-diffusions in an Einstein-de Sitter-like manifold. Along the preceding Sects. 8.1.1, 8.1.2, 8.1.3, we specified the various Ξ-diffusions we considered successively in Sects. 6, 7.1, 7.2 to an Einstein-de Sitter-like manifold. Restricting to the only equation related to the hyperbolic angle $\mathcal{A}_s = \dot{t}_s$, or in other words, to the simplest sub-diffusion $(t_s, \dot{t}_s)$, this yields Equations (19), (22), (25) respectively. We observe that even in this simple case, all these covariant relativistic diffusions differ notably, having pairwise distinct minimal sub-diffusions (with 3 non-proportional diffusion factors).

8.2 Asymptotic Behavior of the *R*-diffusion in an Einstein-De Sitter Manifold

We present here the asymptotic study of the R-diffusion of an Einstein-de Sitter-like manifold (recall Sects. 8.1, 8.1.2). We will focus our attention on the simplest sub-diffusion $(t_s, \dot{t}_s)$, and on the space component $x_s \in \mathbb{R}^3$. Recall from (17) that $\dot{t}_s = \mathcal{A}_s$ equals the hyperbolic angle, measuring the gap between the ambient fluid and the velocity of the diffusing particle. Recall also that, by the unit pseudo-norm relation, $\dot{t}_s$ controls the behavior of the whole velocity $\dot{\xi}_s$. We get as a consequence the asymptotic behavior of the energy $\mathcal{E}_s$. As quoted in Sect. 8.1.2, we must here have $c \geq \frac{1}{2}$.

Note that for $c = \frac{1}{2}$, the scalar curvature vanishes, and the R-diffusion reduces to the geodesic flow, whose equations are easily solved and whose time coordinate satisfies (for constants a and s_0):

$$s - s_0 = \sqrt{t_s\,(t_s + a^2)} - a^2 \log[\sqrt{t_s} + \sqrt{t_s + a^2}\,]\,, \quad \text{whence} \quad t_s \sim s\,.$$

The proofs in this section (and in the following one) repeatedly use the elementary fact that almost surely a continuous local martingale cannot go to infinity.

The following confirms a conjecture stated at the end of Sect. 8.1.2.

Proposition 8.2.1 *The process $\dot{t}_s$ goes almost surely to 1, and $\mathcal{E}_s \to 0$, as $s \to \infty$.*

The following reveals the asymptotic behavior of the space component (x_s) for $c > \frac{1}{2}$.

Proposition 8.2.2 *For $c > \frac{1}{2}$, the space component converges almost surely (as $s \to \infty$):*

$$x_s \to x_\infty \in \mathbb{R}^3.$$

This does not hold in the geodesic flow limiting case $c = \frac{1}{2}$, since then we have

$$r_s = \sqrt{b^2/a^2 + (a + o(1)) \log s} \sim \sqrt{a \log s} \quad \text{as } s \to \infty .$$

To compare the R-diffusion with geodesics, note that (as is easily seen; see for example [1]) along any timelike geodesic, we have $x_s = x_1 + \frac{\dot{x}_1}{|\dot{x}_1|} \int_1^s \frac{a\, d\tau}{t_\tau^{2c}}$ (and $\frac{\dot{x}_s}{|\dot{x}_s|} = \frac{\dot{x}_1}{|\dot{x}_1|}$), which converges precisely for $c > \frac{1}{2}$; and along any lightlike geodesic, we have $x_s = x_1 + \frac{\dot{x}_1}{|\dot{x}_1|} \int_{t_1}^{t_s} \frac{d\tau}{\tau^c} \sim V \times s^{\frac{1-c}{1+c}}$ (and $\frac{\dot{x}_s}{|\dot{x}_s|} = \frac{\dot{x}_1}{|\dot{x}_1|}$), which converges only for $c > 1$.

On the other hand, for $c \le 1$, the behavior of the basic relativistic diffusion is shown to satisfy (see [1]): $r_s \underset{s \to \infty}{\sim} \int_1^s \frac{a_\tau\, d\tau}{t_\tau^{2c}} \longrightarrow \infty$ (exponentially fast, at least for $c < 1$).

Hence, the R-diffusion behaves asymptotically more like a (timelike) geodesic than like the basic relativistic diffusion. However due to other facts, the asymptotic behavior of the R-diffusion seems to be somehow intermediate between those of the geodesic flow and of the basic relativistic diffusion.

8.3 Asymptotic Energy of the $\mathcal{E}$-diffusion in an Einstein-De Sitter Manifold

We consider here the case of Sect. 8.1.3, dealing with the energy diffusion in an Einstein-de Sitter-like manifold, and more precisely, with its absolute-time minimal sub-diffusion $(t_s, \dot{t}_s)$ satisfying Eq. (25), and with the resulting random energy:

$$\mathcal{E}_s = c\, t_s^{-2} (2\, \dot{t}_s^2 + 3c - 2) = 2c\, (\dot{t}_s / t_s)^2 + \mathcal{O}(s^{-2}).$$

Let us denote by ζ the explosion time: $\zeta := \sup\{s > 0 \,|\, \dot{t}_s < \infty\} \in]0, \infty]$.

Lemma 8.3.1 *We have almost surely: either* $\lim_{s \to \zeta} \dot{t}_s = 1$ *and* $\zeta = \infty$, *or* $\lim_{s \to \zeta} \dot{t}_s = \infty$.

The asymptotic behavior can, with positive probability, be partly opposite to that of the preceding R-diffusion:

Proposition 8.3.2 *From any starting point $(t_{s_0}, \dot{t}_{s_0})$, there is a positive probability that both $\mathcal{A}_s \equiv \dot{t}_s$ and the energy $\mathcal{E}_s$ explode. This happens with arbitrarily large probability, starting with $\dot{t}_{s_0}/t_{s_0}$ sufficiently large and t_0 bounded away from zero.*

On the other hand, there is also a positive probability that the hyperbolic angle $\mathcal{A}_s = \dot{t}_s$ does not explode and goes to 1, and then that the random energy $\mathcal{E}_s$ goes to 0. This happens actually with arbitrarily large probability, starting with sufficiently large $t_{s_0}/\dot{t}_{s_0}$.

9 Sectional Relativistic Diffusion

We turn now our attention towards a different class of intrinsic relativistic generators on $G(\mathcal{M})$, whose expressions derive directly from the commutation relations of Sect. 5, on canonical vector fields of $TG(\mathcal{M})$. They all project on the unit tangent bundle $T^1\mathcal{M}$ onto a unique relativistic generator $\mathcal{H}^1_{curv}$, whose expression involves the curvature tensor. Semi-ellipticity of $\mathcal{H}^1_{curv}$ requires the assumption of non-negativity of timelike sectional curvatures. Note that in general $\mathcal{H}^1_{curv}$ does not induce the geodesic flow in an empty space.

We shall actually consider, among these generators, those which are invariant under the action of $SO(d)$ on $G(\mathcal{M})$. To this aim, we introduce the following dual vertical vector fields, by lifting indexes: $V^{ij} := \eta^{im}\,\eta^{jn}\,V_{e_m\wedge e_n}$, so that $V^j \equiv V^{0j} = -V_j$ and $V^{ij} = V_{e_i\wedge e_j}$ for $1 \le i, j \le d$. We again fix a positive parameter σ.

Theorem 9.1 (i) *The following four* SO(d)*-invariant differential operators define the same operator $\mathcal{H}^1_{curv}$ on $T^1\mathcal{M}$:*

$$H_0 - \tfrac{\sigma^2}{2}\sum_{j=1}^{d}\Big([H_0, H_j]V^j + V^j[H_0, H_j]\Big)\,; \quad H_0 + \sigma^2\sum_{j=1}^{d}[H_j, H_0]V^j\,;$$

$$H_0 + \sigma^2\sum_{j=1}^{d} R_0^j\,V_j - \sigma^2\sum_{1\le j,k\le d}\mathcal{R}_0{}^{j0k}\,V_jV_k\,; \quad H_0 - \tfrac{\sigma^2}{4}\sum_{1\le i,j\le d}\Big([H_i, H_j]\,V^{ij} + V^{ij}\,[H_i, H_j]\Big);$$

(ii) *$(\mathcal{H}^1_{curv} - \mathcal{L}_0)$ is self-adjoint with respect to the Liouville measure of $T^1\mathcal{M}$.*
(iii) *In local coordinates, the so-defined second order operator $\mathcal{H}^1_{curv}$ on $T^1\mathcal{M}$ is given by:*

$$\mathcal{H}^1_{curv} = \dot\xi^j\,\frac{\partial}{\partial\xi^j} - \dot\xi^i\dot\xi^j\,\Gamma^k_{ij}\,\frac{\partial}{\partial\dot\xi^k} + \tfrac{\sigma^2}{2}\,\dot\xi^n\,\tilde R^k_n\,\frac{\partial}{\partial\dot\xi^k} - \tfrac{\sigma^2}{2}\,\dot\xi^p\dot\xi^q\,\widetilde{\mathcal{R}}_p{}^k{}_q{}^\ell\,\frac{\partial^2}{\partial\dot\xi^k\partial\dot\xi^\ell}$$

$$= \dot\xi^j\,\frac{\partial}{\partial\xi^j} - \dot\xi^i\dot\xi^j\,\Gamma^k_{ij}\,\frac{\partial}{\partial\dot\xi^k} + \tfrac{\sigma^2}{2}\,\dot\xi^m\,\widetilde{\mathcal{R}}_{mnpq}\left(g^{nq}\,g^{pk}\,\frac{\partial}{\partial\dot\xi^k} - \dot\xi^p\,g^{nk}g^{q\ell}\,\frac{\partial^2}{\partial\dot\xi^k\partial\dot\xi^\ell}\right).$$

The generator $\mathcal{H}^1_{curv}$ defined on $T^1\mathcal{M}$ by Theorem 9.1 is covariant with respect to any Lorentz isometry of $(\mathcal{M}, g)$. Hence, it is a candidate to generate a covariant "sectional" relativistic diffusion on $T^1\mathcal{M}$. Now, a necessary and sufficient condition, in order that such an operator be the generator of a well-defined diffusion, is that it be subelliptic.

We are thus led to consider the following negativity condition on the curvature:

$$\big\langle \mathcal{R}(u \wedge v), u \wedge v \big\rangle_\eta \leq 0\,, \qquad \text{for any timelike } u \text{ and any spacelike } v\,. \tag{27}$$

This condition is equivalent to the following lower bound on sectional curvatures of timelike planes $\mathbb{R}u + \mathbb{R}v$:

$$\frac{\langle \mathcal{R}(u \wedge v), u \wedge v \rangle_\eta}{g(u \wedge v, u \wedge v)} \geq 0\,,$$

since $g(u \wedge v, u \wedge v) := g(u,u)g(v,v) - g(u,v)^2 < 0$ for such planes.

When this negativity condition is fulfilled, we call the resulting covariant diffusion on $T^1\mathcal{M}$, which has generator $\mathcal{H}^1_{curv}$ given by Theorem 9.1, the *sectional relativistic diffusion*. Note that the sectional curvature classically plays a significant role in Lorentzian geometry, see for example ([23], Theorems 2.2 and 2.3).

Remark 9.2 Consider a Lorentz manifold $(\mathcal{M}, g)$ having the warped product form, for example a Robertson-Walker one. Then the sign condition (27) is equivalent to: $\alpha'' \leq 0$ on I, together with the following lower bound on sectional curvatures of the Riemannian factor (M, h) :

$$\inf_{X,Y \in TM} \frac{\langle \mathcal{K}(X \wedge Y)\,, X \wedge Y \rangle}{h(X,X)\,h(Y,Y) - h(X,Y)^2} \geq \sup_I \{\alpha\,\alpha'' - \alpha'^{\,2}\}.$$

In an Einstein-de Sitter-like manifold (recall Sect. 8.1), the sign condition (27) holds if and only if $\alpha'' \leq 0$, i.e., if and only if $c \leq 1$. The generator $\mathcal{H}^1_{curv}$ is fully computable, but has a complicated expression, even in such a simple example.

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