The Riemann Mapping Theorem and Its Discrete Counterparts

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Abstract We introduce some of the recent work on discrete versions of the Riemann mapping theorem and the uniformization theorem.

Keywords Conformal maps · Riemann mapping · Uniformization theorem · Circle packing · Discrete conformality · Polyhedral surfaces

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1 Introduction

The Riemann mapping theorem was formulated by B. Riemann in 1851. It states that given any two simply connected open sets U_1, U_2 in the complex plane $\mathbb{C}$ with $U_i \neq \mathbb{C}$, there exists an analytic bijection (i.e., *conformal*) map $f : U_1 \to U_2$. In particular, if one takes U_2 or U_1 to be the open unit disk, then the map f is called a *Riemann mapping*. The Riemann mapping theorem is one of the most important results in complex analysis. It relates geometry (e.g. open sets) to analysis (e.g. complex analytic functions).

The uniformization theorem of Poincaré and Koebe generalizes the Riemann mapping theorem to Riemann surfaces. By definition, a Riemann surface is a connected orientable surface Σ with a special collection of charts (analytic charts) covering Σ so that the transitions functions are complex analytic maps. The essential feature of Riemann surfaces is that one can measure angles between curves on them. Riemann surfaces are ubiquitous in mathematics. For instance connected open sets in $\mathbb{C}$, smooth orientable surfaces with Riemannian metrics, smooth algebraic curves and polyhedral surfaces are naturally Riemann surfaces. In 1907, Poincaré and Koebe independently proved the uniformization theorem which states that any simply con-

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nected Riemann surface is conformally diffeomorphic to the complex plane $\mathbb{C}$, the open unit disk $\mathbb{D}$, or the Riemann sphere $\mathbb{S}^2$. The conformal diffeomorphism is called a *uniformization map*. This result is a pillar in mathematics and has a wide range of applications within and outside mathematics.

Computing the Riemann mapping or the uniformization mapping is not easy. For instance, the boundary of a tetrahedron is naturally a Riemann surface. Here the analytic charts consist of unions of two open triangle faces together with their common open edges and the orientation preserving isometric embedding, and charts at vertices are of the form $(U, z^{2\pi/\alpha})$ where U is a small neighborhood of a vertex of cone angle α. Using the uniformization theorem, one concludes that it is conformal to the Riemann sphere $\mathbb{S}^2$ with four marked points $\{0, 1, \infty, z\}$ corresponding to the four vertices. However, there is no algorithm to compute the conformal invariant z directly from the 6 edge lengths of the tetrahedron. There are powerful algorithms computing the Riemann mapping for simply connected domains. For instance the Schwarz–Christoffel algorithm developed by Trefethen and Driscoll [33] and the circle packing algorithm developed by Thurston and Stephenson [31] are powerful tools. However, computing the uniformization map for a simply connected surface with a non-flat Riemannian metric has been difficult. Our recent work [11, 18, 19] produces an algorithm to compute the uniformization maps, and shows that the uniformization maps are computable.

Over the years, there have been many research activities on establishing various discrete versions of the uniformization theorem and the Riemann mapping theorem. The purpose of this chapter is to introduce some of these works and their proofs. We will also discuss several open problems in the discrete setting.

The following two topics will be discussed in this chapter. These are: (1) the Koebe–Andreev–Thurston's circle packing version of the Riemann mapping theorem and (2) our recent work with Gu, Sun, Wu and Guo ([11, 12, 18, 19]) on a discrete uniformization theorem for polyhedral surfaces.

We remark that this is not a survey of works on discrete Riemann mapping theorems and we have left many important works untouched.

The chapter is organized as follows. Section 2 discusses circle packings and Sect. 3 covers a discrete uniformization theorem for polyhedral surfaces.

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2 Koebe–Andreev–Thurston's Circle Packing Theorem

We will discuss a simple form of the circle packing theorem in this section. For more details on circle packing, one may consult the nice book by Stephenson [31].

A *circle packing* on the Riemann sphere or the plane is a collection of closed round disks $D_1, \ldots, D_k$ with disjoint interiors. Its *nerve* is a finite graph on the 2-sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ or the plane $\mathbb{C}$ with one vertex for each disk D_i and an edge between two vertices if the corresponding disks are tangent (Fig. 1).

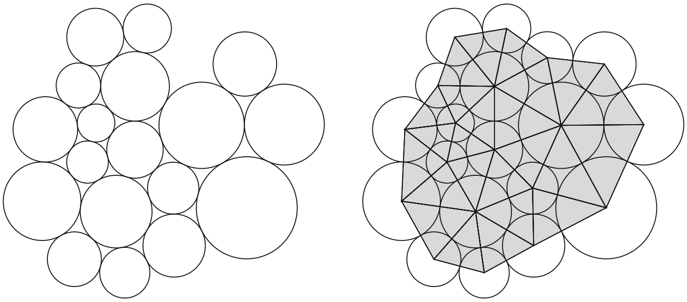

Fig. 1 A circle packing and its nerve. The picture is produced by K. Stephenson

Theorem 2.1 (Koebe–Andreev–Thurston) *Suppose $\mathcal{T}$ is a simplicial triangulation of the 2-sphere $\mathbb{S}^2$. There exists a circle packing $D_1, ..., D_n$, unique up to Möbius transformations, on the Riemann sphere $\mathbb{S}^2$ such that its nerve is isomorphic to the 1-skeleton $\mathcal{T}^{(1)}$ of $\mathcal{T}$.*

The theorems proved by Andreev and Thurston are more general allowing circles to intersect at angles at most $\pi/2$. For more details, see [3, 25, 35] or others.

There are many proofs of Theorem 2.1. See [3, 21, 31, 35] and others. Below we give a proof using ideas from [7, 21].

Following Marden-Rodin [21], we first reduce the circle packing on $\mathbb{S}^2$ to a circle packing on the plane $\mathbb{C}$. Removing a triangle face τ_0 from the triangulation $\mathcal{T}$, one produces a simplicial triangulation $\mathcal{T}_1$ of the (topological) triangle $T = \mathbb{S}^2 - int(\tau_0)$. To prove Theorem 2.1, it suffices to produce a circle packing on the plane whose nerve is the 1-skeleton $\mathcal{T}_1^{(1)}$. Indeed, if $D_1, ..., D_n$ is a circle packing on the plane whose nerve is the 1-skeleton $\mathcal{T}_1^{(1)}$, then $D_1, ..., D_n$ is a circle packing on $\mathbb{S}^2$ whose nerve is $\mathcal{T}^{(1)}$. Conversely suppose $D_1, ..., D_n$ is a circle packing on $\mathbb{S}^2$ whose nerve is $\mathcal{T}^{(1)}$ such that D_1, D_2, D_3 correspond to the three vertices of τ_0. Applying a Möbius transformation to $\{D_1, ..., D_n\}$ so that infinity is in the triangle region in $\mathbb{S}^2 - \cup_i D_i$ bounded by the circles $\partial D_1, \partial D_2, \partial D_3$, then the circle packing $\{D_1, ..., D_n\}$ on the plane $\mathbb{C}$ has nerve $\mathcal{T}_1^{(1)}$.

It is known that given three pairwise tangent closed disks D_1, D_2 and D_3 in the plane, there exists a Möbius transformation sending D_1, D_2 and D_3 to three disks of radii 1. Therefore, Theorem 2.1 is equivalent to producing a circle packing on $\mathbb{C}$ whose nerve is isomorphic to the 1-skeleton of a triangulation $\mathcal{T}_1$ of a triangle $T = \Delta v_1 v_2 v_3$ so that the three circles corresponding to three vertices v_i are of radii 1.

Thurston's approach to Theorem 2.1 uses polyhedral metrics on surfaces. Let V and E be the sets of all vertices and edges in $\mathcal{T}_1$ so that $v_1, v_2, v_3 \in V$ are the boundary vertices (i.e., vertices of τ_0). To produce a circle packing, Thurston assigns each vertex v a positive number $r(v)$, called the radius. The radius assignment is a function $r : V \to \mathbb{R}_{>0}$. For each radius assignment r, construct a polyhedral metric d on the triangulated triangle $(T, \mathcal{T}_1)$ by making each triangle in $\mathcal{T}_1$ a Euclidean triangle of edge lengths $l(vv') = r(v) + r(v')$ where $v, v' \in V$ and $vv' \in E$. The *discrete curvature* of d is the function $K_d : V \to (-\infty, 2\pi)$ sending each vertex $v \in V - \{v_1, v_2, v_3\}$ to 2π minus the sum of all angles at v and sending v_i $(i = 1, 2, 3)$ to π minus the sum of all angles at v_i. It is well known that the Gauss-Bonnet theorem holds, i.e., $\sum_{v\in V} K_d(v) = 2\pi$. The goal is to find a radius assignment $r \in \mathbb{R}^V_{>0}$ so that its discrete curvatures at all $v \in V - \{v_1, v_2, v_3\}$ are zero, i.e., (T, d) is a flat surface. Since the triangle T is simply connected, the developing map for the flat structure produces an isometric immersion $\Phi : (T, d) \to \mathbb{C}$ where the plane has the standard Euclidean metric. The map Φ sends the boundary ∂T to a triangle in $\mathbb{C}$. In particular, $\Phi|_{\partial T}$ is injective. This implies that $\Phi : (T, d) \to \mathbb{C}$ is an isometric embedding. Let the images of V under Φ be $\{v'_1, v'_2, ..., v'_m\}$ on the plane $\mathbb{C}$. Then by the construction, the circle packing $\{B(v'_1, r(v_1)), ..., B(v'_m, r(v_m))\}$ has nerve isomorphic to $\mathcal{T}_1^{(1)}$ where $B(c, r)$ is the ball of radius r centered at c.

The above discussion shows that Theorem 2.1 is a consequence of the following:

Proposition 2.2 *Suppose $\mathcal{T}_1$ is a triangulation of a triangle $T = \Delta v_1 v_2 v_3$ such that there are only three vertices v_1, v_2, v_3 of $\mathcal{T}_1$ in the boundary ∂T. Then there exists a unique radius assignment $r : V \to \mathbb{R}_{>0}$ with $r(v_i) = 1$ for $i = 1, 2, 3$ such that the associated circle packing metric on T has zero discrete curvatures at all $v \in V - \{v_1, v_2, v_3\}$.*

2.1 A Variational Principle Associated to Circle Packing

The following variational principle was first established by Colin de Verdière in [7].

Proposition 2.3 *(Colin de Verdière) Let $\Delta A_1 A_2 A_3$ be a Euclidean triangle such that the length of edge $A_i A_j$ is $e^{x_i} + e^{x_j}$ and the angle at A_i is $\theta_i = \theta_i(x_1, x_2, x_3)$. Let $x = (x_1, x_2, x_3)$. Then*

(a) $\sum_{i=1}^3 \theta_i(x) dx_i$ is a closed 1-form such that $\frac{\partial \theta_i}{\partial x_j} > 0$ for $i \neq j$;

(b) the function $f(x) = \int_0^x \sum_{i=1}^3 \theta_i(x) dx_i$ is a well defined concave function in $x \in \mathbb{R}^3$ such that $\frac{\partial f}{\partial x_i} = \theta_i$ and f is strictly concave when restricted to the plane $P_c = \{x \in \mathbb{R}^3 | x_1 + x_2 + x_3 = c\}$ for any $c \in \mathbb{R}$;

(c) if $a_1, a_2, a_3 > 0$ such that $a_1 + a_2 + a_3 = \pi$, then $g(x) = \int_0^x \sum_{i=1}^3 (\theta_i(x) - a_i) dx_i$ satisfies that $\lim_{\max_{i,j} |x_i - x_j| \to \infty} g(x) = -\infty$ and $g(x + (t, t, t)) = g(x)$ for all $t \in \mathbb{R}$.

In [6, 17], this variational principle is generalized to the case of three circles intersecting at angles and more general polyhedral surfaces.

Proof Recall that the cosine law for triangles states that $\cos(\theta_i) = \frac{y_j^2 + y_k^2 - y_i^2}{2 y_j y_k}$ where y_k is the length of $A_i A_j$ and $\{i, j, k, \} = \{1, 2, 3\}$. Let the area of the triangle $\Delta A_1 A_2 A_3$ be A. Taking derivatives of the cosine law, we obtain (see Lemma A-1 in the appendix of [6])

$$\frac{\partial \theta_i}{\partial y_i} = \frac{y_i}{2A} > 0, \tag{2.1}$$

$$\frac{\partial \theta_i}{\partial y_k} = -\frac{\partial \theta_i}{\partial y_i}\cos(\theta_j). \tag{2.2}$$

Now to see part (a), it suffices to show that $\frac{\partial \theta_i}{\partial x_j} = \frac{\partial \theta_j}{\partial x_i} > 0$. By definition $y_i = e^{x_i} + e^{x_k}$. Therefore,

$$\frac{\partial \theta_i}{\partial x_j} = \frac{\partial \theta_i}{\partial y_i}\frac{\partial y_i}{\partial x_j} + \frac{\partial \theta_i}{\partial y_k}\frac{\partial y_k}{\partial x_j} = \frac{\partial \theta_i}{\partial y_i}e^{x_j} - \frac{\partial \theta_i}{\partial y_i}\cos(\theta_j)e^{x_j} = \frac{(1-\cos(\theta_j))y_i e^{x_j}}{2A}.$$

Let R be the radius of the inscribed circle in the triangle. Then $e^{x_j} = R\cot(\theta_j/2)$. Using the relation $1 - \cos(\theta_j) = 2\sin^2(\theta_j/2)$, we see that

$$\frac{(1-\cos(\theta_j))y_i e^{x_j}}{2A} = \frac{R\sin(\theta_j)y_i}{2A} = \frac{R}{y_k} > 0$$

and that $\frac{\partial \theta_i}{\partial x_j}$ is symmetric in i, j.

To see part (b), since $\sum_{i=1}^{3}\theta_i dx_i$ is closed in $\mathbb{R}^3$, the integral $\int_0^x \sum_{i=1}^{3}\theta_i dx_i$ is independent of the choice of paths and therefore $f(x)$ is well defined. Furthermore, $\frac{\partial f}{\partial x_i} = \theta_i$ follows from the definition of f. The Hessian of f is the 3×3 matrix $[h_{rs}]$ ($h_{rs} = \partial\theta_r/\partial x_s$) which satisfies the condition that $h_{ij} = h_{ji} > 0$ and $h_{1i} + h_{2i} + h_{3i} = \frac{\partial(\theta_1+\theta_2+\theta_3)}{\partial x_i} = \frac{\partial \pi}{\partial x_i} = 0$. It follows that the matrix $-[h_{rs}]$ is a diagonally dominated matrix whose kernel consists of vectors $\lambda[1, 1, 1]^t$. Hence $[h_{rs}]$ is negative semi-definite. This implies that the function $f(x)$ is concave in $\mathbb{R}^3$ and is strictly concave when restricted to the affine plane P_c.

To see part (c), given a_1, a_2, a_3, there exists a Euclidean triangle $\Delta B_1 B_2 B_3$ whose inner angles are a_1, a_2, a_3. Let C be the inscribed circle to $\Delta B_1 B_2 B_3$ and e^{u_i} be the distance from B_i to $C \cap B_i B_j$. Then by construction, the length of $B_i B_j$ is $e^{u_i} + e^{u_j}$. This shows that the point (u_1, u_2, u_3) is a critical point of the function $g(x)$ on $\mathbb{R}^3$ since $\frac{\partial g}{\partial x_i}(u) = \theta_i - a_i = 0$. Since $g(x)$ is strictly concave with a critical point in the plane P_c where $c = u_1 + u_2 + u_3$, it follows that $\lim_{x \in P_c, x\to\infty} g(x) = -\infty$. On the other hand, for any $b \in \mathbb{R}$, by definition and $\theta_i(x + (b, b, b)) = \theta_i(x)$, we have $g(x + (b, b, b)) = g(x)$. To see this,

$$g(x + (b, b, b)) - g(x) = \int_x^{x+(b,b,b)} \sum_{i=1}^{3}(\theta_i - a_i)dx_i$$

$$= \int_0^1 \sum_{i=1}^{3}(\theta_i(x + t(b, b, b)) - a_i)b dt = b\sum_{i=1}^{3}(\theta_i(x) - a_i) = 0.$$

For each vector $v \in \mathbb{R}^3$, let $\Pi(v) = v + (t, t, t) \in P_c$ be the orthogonal projection to P_c. Then a sequence of vectors $x(n) = (x_1(n), x_2(n), x_3(n)) \in \mathbb{R}^3$ satisfies $\max_{i,j} |x_i(n) - x_j(n)| \to \infty$ if and only if $\pi(x(n)) \to \infty$. Thus $g(x(n)) = g(\pi(x(n))) \to -\infty$ when $\max_{i,j} |x_i(n) - x_j(n)| \to \infty$. □

2.2 A Proof of Koebe–Andreev–Thurston's Theorem

We now prove Proposition 2.2 using Colin de Verdière's variational principle (see [7]).

To set up an appropriate variational framework, one needs the concept of an *angle structure* on a triangulated surface introduced in [7]. Suppose $(S, \mathcal{T})$ is a triangulated surface. An angle structure on $(S, \mathcal{T})$ assigns each vertex v in each triangle $\tau \in \mathcal{T}$ a positive number $a(v, \tau) \in \mathbb{R}_{>0}$, called the angle, such that (a) the sum of the three angles in each triangle is π and (b) the sum of all angles at each interior vertex v is 2π. Using linear programming, Colin de Verdière ([7]) proved that each simplicial triangulation of the triangle admits an angle structure. Another way to see it is to note that each geometric triangulation of a flat surface has a natural angle structure, i.e., $a(v, \tau)$ is the angle of the Euclidean triangle τ at v.

Lemma 2.4 *If $\mathcal{T}_1$ is an abstract simplicial triangulation of a triangle T with three vertices in ∂T, then there exists a geometric triangulation $\mathcal{T}'$ of an equilateral Euclidean Δ such that $\mathcal{T}'$ is isomorphic to $\mathcal{T}$.*

This lemma follows easily from Steinitz's theorem ([37]) that any 3-connected graph on $\mathbb{S}^2$ can be realized as the 1-skeleton of a compact convex polytope in $\mathbb{R}^3$. Indeed, by Steinitz's theorem, there exists a compact convex polytope P whose boundary with an open 2-cell Q removed is isomorphic $\mathcal{T}_1$. Project $\partial P - Q$ onto a plane from a point outside P and close to Q. The result is a geometric triangulation $\mathcal{T}''$ of a triangle such that $\mathcal{T}''$ is isomorphic to $\mathcal{T}_1$. Finally sending the triangle to the equilateral triangle Δ by an affine map produces the required $\mathcal{T}'$.

Label triangles in $\mathcal{T}'$ by $\Delta_1, \Delta_2, ..., \Delta_m$, let the vertices of Δ_i be v_{i1}, v_{i2}, v_{i3} and the inner angle of Δ_i at v_{ij} be a_{ij}, i.e., $\{a_{ij}\}$ is an angle structure on $\mathcal{T}'$. For each $x \in \mathbb{R}^V$, define

$$W(x) = \sum_{i=1}^{m} g_{\Delta_i}(x)$$

where $g_{\Delta_i}(x) = \int_0^{(x(v_{i1}), x(v_{i2}), x(v_{i3}))} \sum_{j=1}^{3} (\theta_{ij} - a_{ij}) dx(v_{ij})$ is the Colin de Verdière's function in Proposition 2.3 associated to the triangle Δ_i with radius assignment $e^{x(v_{i1})}$, $e^{x(v_{i2})}$, and $e^{x(v_{i3})}$ such that the angle in Δ_i at v_{ij} is θ_{ij}.

By definition and Proposition 2.3, the function $W(x)$ is concave in $\mathbb{R}^V$ since it is a sum of concave functions. Also, $W(x + t(1, 1, 1, ..., 1)) = W(x)$ due to Proposition 2.3(c). Furthermore, since each g_{Δ_i} is bounded from above, $W(x)$ is bounded from above. We claim that W is a proper function when restricted to

$P = \{x \in \mathbb{R}^V \mid \sum_{v \in V} x(v) = 0\}$, i.e., $\lim_{x \in P, x \to \infty} W(x) = -\infty$. Indeed, if $x \in P$ such that $x \to \infty$, then $\max_{i,j,j'} |x(v_{ij}) - x(v_{ij'})|$ converges to ∞. Therefore, by Proposition 2.3(c), we see that $W(x) \to -\infty$. This shows that $W|_P$ has a critical point $u \in P$. Since $W(x + (t, t, t, ..., t)) = W(x)$, this shows the point u is a critical point of W.

For this critical point u, suppose v_i, $i > 3$, is an interior vertex and $x_i = x(v_i)$. Then by Proposition 2.3(a), $\frac{\partial W}{\partial x_i}(u) = \sum_j (\theta_{n_i,j} - a_{n_i,j}) = -K(v_i)$ where $\theta_{n_i,j}$ and $a_{n_i,j}$ are the angles at the vertex v_i and $\sum_j a_{n_i,j} = 2\pi$. This shows that the circle packing metric associated to u is flat. At vertices v_i with $i = 1, 2, 3$, the same calculation shows $K(v_i) = 2\pi/3$ due to the choices of a_{ij} (i.e., Δ is an equilateral triangle). This implies $u_1 = u_2 = u_3$.

To prove uniqueness, if $\tilde{u} \in \mathbb{R}^V_{>0}$ comes from the radii of a circle packing whose nerve is isomorphic to $\mathcal{T}_1$ such that the associated polyhedral surface is an equilateral triangle, then the above calculation shows that $\tilde{u}$ is a critical point of W. Since W is concave, all critical points of W are maximum points. Therefore, it suffices to prove that the restriction of the function W to P is strictly concave. Indeed, otherwise there exist two distinct points $x, y \in P$ such that the function $h(t) = W(tx + (1-t)y)$ is linear in $t \in [0, 1]$. This implies that for each triangle Δ_i, $g_{\Delta_i}(tx + (1-t)y)$ is linear in t. By Proposition 2.3, this implies there is a vector $u_i(1, 1, 1) \in \mathbb{R}^3$, one for each triangle Δ_i, such that

$$(x(v_{i1}), x(v_{i2}), x(v_{i3})) = (y(v_{i1}), y(v_{i2}), y(v_{i3})) + u_i(1, 1, 1). \tag{2.3}$$

We claim that $u_i = u_j$ for all i, j. Indeed, consider two triangles Δ_i and Δ_j sharing a vertex v. Then (2.3) at v shows $u_i = u_j$. Since any two triangles Δ_i and Δ_j can be linked by a sequence of triangles $\Delta_{n_1} = \Delta_i, \Delta_{n_2}, ..., \Delta_{n_k} = \Delta_j$ such that Δ_{n_r} and $\Delta_{n_{r+1}}$ share a common vertex, we see that $u_i = u_j$. It follows that the two vectors x, y differ by a vector of the form $t(1, 1, ..., 1) \in \mathbb{R}^V$. On the other hand, both $x, y \in P$, therefore $t = 0$, i.e., $x = y$ which contradicts the choice of x, y.

2.3 Thurston's Conjecture on Circle Packing And Rodin-Sullivan's Work

The relationship between the Koebe–Andreev–Thurston's theorem and the Riemann mapping theorem was explored by W. Thurston in early 1980s. The basic idea is that since conformal maps send infinitesimal circles (circles in the tangent space) to circles, a circle packing should be a good approximation to conformal maps.

Here is Thurston's conjecture which was proved by Rodin-Sullivan in [29].

Given a bounded simply connected domain Ω in the complex plane $\mathbb{C}$ and a point $p \in \Omega$, for each large integer n, let P_n be a maximum (hexagonal) circle packing by disks of radii $1/n$ inside Ω and p_n be a circle in P_n within distance $1/n$ to p. Here maximum means that one cannot add another $1/n$ radius disk in Ω to P_n such that

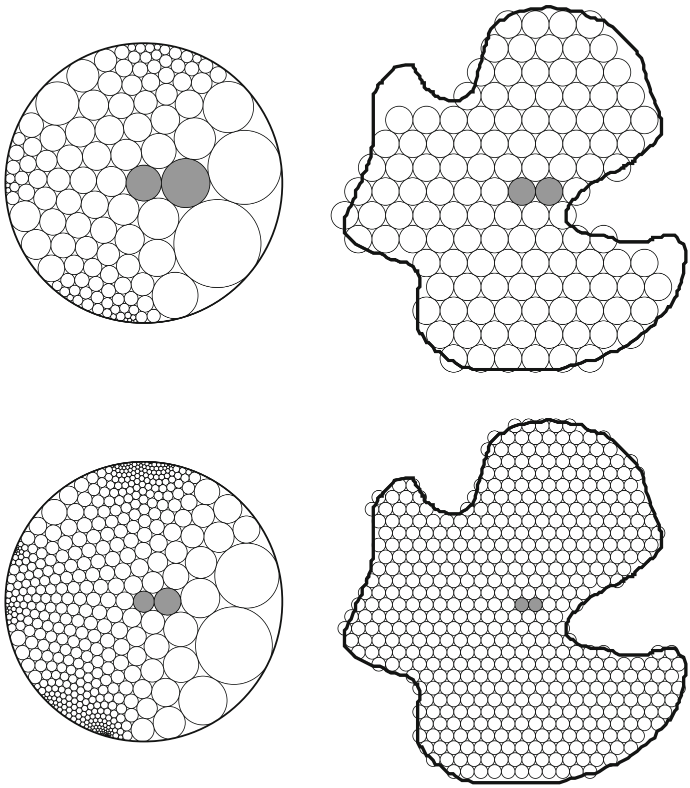

Fig. 2 Thurston's conjecture, Rodin-Sullivan's theorem, on convergence of circle packing to the Riemann mapping. The picture is produced by K. Stephenson

its nerve is the 1-skeleton of a topological triangulation $\mathcal{T}_n$ of a disk. Let p_n' be the circle in P_n adjacent to p_n from the right. Modify $\mathcal{T}_n$ to be a triangulation $\mathcal{T}_n^*$ of the 2-sphere $\mathbb{S}^2$ by adding one vertex v_∞ and edges from v_∞ to all boundary vertices of $\mathcal{T}_n$. Now by Koebe–Andreev–Thurston's theorem, there exists a circle packing Q_n of the Riemann sphere such that (a) its nerve is isomorphic to the 1-skeleton of $\mathcal{T}_n^*$; (b) the disk corresponding to v_∞ is the complement of the unit disk $\mathbb{D}$; (c) the disk corresponding to p_n is centered at 0, (d) the disk in Q_n corresponding to p_n' is centered in the positive x-axis (Fig. 2).

Let f_n be the piecewise linear map constructed as follows. It sends the center of a circle in P_n to the center of the corresponding circle in Q_n and f_n is linear on each triangles. Thurston's conjecture, proved by Rodin-Sullivan, is that as $n \to \infty$, f_n converges to the Riemann mapping $f : \Omega \to D$ uniformly on compact subsets of Ω.

Rodin-Sullivan's proof of convergence is beautiful and elegant. The readers are strongly recommended to read the original paper [29]. There are two steps involved in the proof. In the first step, they showed that there exists a constant $K > 0$ so that all approximation functions f_n are K-quasi-conformal. This is a consequence of Rodin-Sullivan's ring lemma which states that in a hexagonal circle packing, the ratio of the radii of any two adjacent circles is at most 1000. One can deduce the ring lemma by inspection. Now uniform K-quasiconformality follows since inner angles in a Euclidean triangle of edge lengths $r_1 + r_2, r_2 + r_3, r_3 + r_1$ with $\frac{r_i}{r_j} \leq 1000$ cannot be too small. Since f_n are uniformly K-quasi-conformal, it has a convergent subsequence. Let f be the limit of the subsequence. The claim is that f is the Riemann mapping. To establish conformality of f, Rodin-Sullivan proved that the hexagonal circle packing in the plane is unique. To be more precise, if $\{D_i\}$ is a locally finite collection of disks in $\mathbb{C}$ with disjoint interiors such that each D_i is tangent to exactly six other disks D_j's and $\mathbb{C} - \cup D_i$ is a disjoint union of open triangles whose boundary are in $\cup \partial D_i$, then all D_i have the same size.

Rigidity of hexagonal circle packing is the first rigidity theorem proved for infinite circle packing. This work has inspired and initiated many research activities. For instance Schramm [30] proved that any locally finite infinite circle packing of $\mathbb{C}$ is rigid. See also the works of He [13], He-Schramm [14] and many others.

3 A Discrete Uniformization Theorem

One form of the uniformization theorem states that each Riemann surface admits a complete Riemannian metric of constant curvature -1, 0, or 1 within its conformal class. Furthermore, the metric is unique unless the Riemann surface is conformal to the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C} - \{0\}$, the sphere $\mathbb{S}^2$, or tori $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some $\tau \notin \mathbb{R}$. In this section, we introduce our recent work on discrete conformal geometry for compact polyhedral surfaces and discuss a discrete version of uniformization theorem for compact polyhedral surfaces.

Polyhedral surfaces are ubiquitous due to digitization (e.g. 3D scan). Classifying them according to some discrete conformality should be useful in organizing polyhedral surfaces. Circle packing can be considered as a discrete conformality if one allows the changing of radii. However not all polyhedral surfaces can be canonically packed by circles. A discrete conformality for all polyhedral surfaces was introduced in [11, 12]. The main features of the discrete conformality are the following. First, the discrete conformality is algorithmic; second the corresponding discrete uniformization theorem holds for compact surfaces; third there exists a finite dimensional (convex) variational principle to find the discrete uniformization metric; and fourth discrete conformality is closely related to the convex ideal hyperbolic polyhedra in the 3-dimensional hyperbolic space $\mathbb{H}^3$. Similar to Thurston's conjecture on the convergence of circle packing metrics, we have recently proved a convergence result [18] which shows that the discrete conformality converges to smooth conformality when the triangulations are suitably chosen. Several conjectures about a discrete uniformization for non-compact polyhedral surfaces will be discussed at the end of this section.

3.1 Discrete Conformality of Polyhedral Surfaces

A closed surface S together with a non-empty finite subset of points $V \subset S$ will be called a *marked surface*. A triangulation $\mathcal{T}$ of a marked surface (S, V) is a topological triangulation of S such that the vertex set of $\mathcal{T}$ is V. We use $E = E(\mathcal{T})$, $V = V(\mathcal{T})$ to denote the sets of all edges and vertices in $\mathcal{T}$ respectively. A (Euclidean) *polyhedral metric* on (S, V), to be called a *PL metric* on (S, V) for simplicity, is a flat cone metric on (S, V) with cone points contained in V. We call the triple (S, V, d) a polyhedral surface. All PL metrics are obtained by isometric gluing of Euclidean triangles along pairs of edges. For instance boundaries of convex polytopes are PL metrics on $(\mathbb{S}^2, V)$. The *discrete curvature* of a PL metric d is the function $K_d : V \to (-\infty, 2\pi)$ sending a vertex v to 2π minus the cone angle at v. For a closed surface S, it is well known that the Gauss-Bonnet theorem $\sum_{v \in V} K_d(v) = 2\pi\chi(S)$ holds. If $\mathcal{T}$ is a triangulation of (S, V) with a PL metric d for which all edges in $\mathcal{T}$ are geodesic, we say $\mathcal{T}$ is *geometric* in d and d is a PL metric on $(S, V, \mathcal{T})$. In this case, we can represent the PL metric d by the length function $l_d : E(\mathcal{T}) \to \mathbb{R}_{>0}$ sending an edge to its length. Thus the polyhedral surface (S, V, d) can be represented by $(S, \mathcal{T}, l)$. This is a way of coding a polyhedral surface by a finite-dimensional vector $l_d \in \mathbb{R}^E$.

In general, a polyhedral surface (S, V, d) admits infinitely many different geometric triangulations. However, each polyhedral surface (S, V, d) has a natural *Delaunay triangulation* $\mathcal{T}_d$ which is a geometric triangulation with vertices V such that for each edge, the sum of two angles facing e is at most π. Delaunay triangulations are the most commonly used triangulations in scientific computing. It can be constructed from the Voronoi decomposition $\{R(v)|v \in V\}$ of (S, V, d) as follows. Here a Voronoi 2-cell $R(v)$ for $v \in V$ is defined to be $\{x \in S|d(x, v) \leq d(x, v'), \forall v' \in V\}$. The Delaunay tessellation of (S, V, d) is the dual cell decomposition of $\{R(v)|v \in V\}$ whose vertices are V and each 1-dimensional connected component of $R(v) \cap R(v')$ corresponds to a (geodesic) edge from v to v'. A Delaunay triangulation is a subdivision of the Delaunay tessellation into triangles without introducing extra vertices. Any two Delaunay triangulations of (S, V, d) are related by a sequence of Delaunay triangulations such that adjacent ones differ by a diagonal switch along an edge. See for instance [4].

Suppose d is a PL metric on a triangulated surface $(S, \mathcal{T})$ whose edge length function is $l_d : E(\mathcal{T}) \to \mathbb{R}_{>0}$. For a positive function $u : V(\mathcal{T}) \to \mathbb{R}_{>0}$, the vertex scaling of l_d by u is the new function $u * l_d : E(\mathcal{T}) \to \mathbb{R}_{>0}$ such that $u * l_d(vv') = u(v)u(v')l_d(vv')$ where vv' is an edge with end points v, v'. If d, d' are two PL metrics on $(S, \mathcal{T})$, then they differ by a *vertex scaling* if $l_d = u * l_{d'}$ for some $u : V \to \mathbb{R}_{>0}$. The notation of vertex scaling change of PL metrics was introduced in [28] and in [16].

The definition of discrete conformality involves Delaunay triangulations and vertex scaling.

Definition 3.1 ([11]) Two PL metrics d and d' on a marked closed surface (S, V) are *discrete conformal* if there is a sequence of PL metrics $d_1 = d, d_2, ..., d_n = d'$ and a sequence of triangulations $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n$ of (S, V) such that

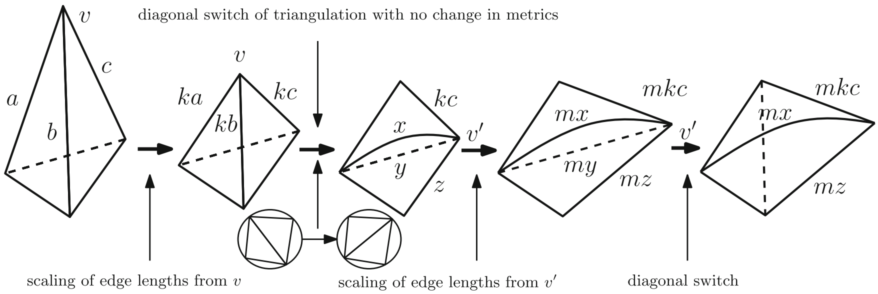

Fig. 3 Discrete conformal change of PL metrics from an arbitrary tetrahedron to one with constant curvature π. All triangulations involved are Delaunay

(a) each $\mathcal{T}_i$ is Delaunay in d_i,

(b) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then there is an isometry h_i from (S, V, d_i) to (S, V, d_{i+1}) such that h_i is homotopic to the identity map on (S, V), and

(c) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there is a function $u_i : V \to \mathbb{R}_{>0}$ such that for each edge $e = vv'$ in $\mathcal{T}_i$, the lengths $l_{d_i}(vv')$ and $l_{d_{i+1}}(vv')$ of e in d_i and d_{i+1} are related by

$$l_{d_{i+1}}(vv') = u_i(v)u_i(v')l_{d_i}(vv'), \tag{3.1}$$

i.e., $l_{d_{i+1}} = u_i * l_{d_i}$.

The original motivation in [16] for introducing vertex scaling $u * l_d$ as an approximation to conformal change is the following. Since a PL polyhedral metric l_d on $(S, \mathcal{T})$ is a discretization of a Riemannian metric g and a function $u : V(\mathcal{T}) \to \mathbb{R}_{>0}$ is a discretization of a positive function λ on S, the conformal Riemannian metric λg should be approximated by the PL metric defined by $u * l_d$. The deeper reason for $u * l_d$ to be a discrete conformal change is due to the following observation in Riemannian geometry ([18]). Given a Riemannian metric g on a compact connected manifold M and $\lambda : M \to \mathbb{R}_{>0}$, there exists a constant $C = C(M, g, \lambda)$ such that for any $p, q \in M$, we have

$$|d_{\lambda^4 g}(p, q) - \lambda(p)\lambda(q)d_g(p, q)| \leq C d_g(p, q)^3$$

where $d_g(p, q)$ is the distance between p, q in the metric g.

The relationship between discrete conformal geometry and hyperbolic geometry is the following [5, 11]. Given a Delaunay triangulated polyhedral surface $(S, \mathcal{T}, d)$ with $V = V(\mathcal{T})$, one can naturally associate to d a cusped hyperbolic metric d^* on $S - V$. Here is the construction. Take a Euclidean triangle τ in $(\mathcal{T}, d)$ considered as the Euclidean convex hull of vertices $v_1, v_2, v_3 \in \mathbb{C}$. Let τ^* be the hyperbolic convex hull $C_{\mathbb{H}}(v_1, v_2, v_3)$ of v_1, v_2, v_3 in the hyperbolic 3-space $\mathbb{H}^3$. Here we consider $\mathbb{C}$ to be in the sphere at infinity of the upper half-space model $\mathbb{C} \times \mathbb{R}_{>0}$ of $\mathbb{H}^3$. Now if σ and τ are two Euclidean triangles in $\mathcal{T}$ glued by a Euclidean isometry f along an edge, since each isometry f of $\mathbb{C}$ extends naturally to an isometry f^* of $\mathbb{H}^3$, we glue

τ^* and σ^* along the corresponding edge by the isometry f^*. In this way, we obtain a complete finite area hyperbolic metric d^* on $S - V$. It follows from the construction that d^* is independent of the choices of the Delaunay triangulations $\mathcal{T}$ used in the construction. It is proved in [11, Theorem 43] that two PL metrics d_1 and d_2 on a closed marked surface (S, V) are discrete conformal in the sense of Definition 3.1 if and only if their associated hyperbolic metrics d_1^* and d_2^* are isometric by an isometry homotopic to the identity (respecting V). Conversely, if S is a closed surface and $\hat{d}$ is a complete finite area hyperbolic metric on $S - V$, then there exists a polyhedral metric d on (S, V) such that $d^* = \hat{d}$. Thus for closed surfaces, there exists a bijection between the space of all discrete conformal classes of polyhedral metrics on (S, V) and the Teichmüller space of cusped metrics on $S - V$.

By this construction, if $\mathcal{T}$ is a Delaunay triangulation of the plane $(\mathbb{C}, d_{st})$ with $V = V(\mathcal{T})$ and d_{st} being the standard flat metric on $\mathbb{C}$, then the associated hyperbolic metric d_{st}^* is the boundary of the convex hull $C_{\mathbb{H}}(V)$ of V in $\mathbb{H}^3$. To see this, we note that codimension-1 faces of $C_{\mathbb{H}}(V)$ correspond to the circum-disks of triangles $\tau \in \mathcal{T}$ due to the Delaunay condition. This shows the relationship between discrete conformal geometry and convex hull construction in the hyperbolic 3-space $\mathbb{H}^3$ and the essential role of Delaunay condition in discrete conformality.

The main theorems proved in [11, 12] are:

Theorem 3.2 ([11]) *Given any PL metric d on a closed marked surface* (S, V) *and any* $K^* : V \to (-\infty, 2\pi)$ *such that* $\sum_{v \in V} K^*(v) = 2\pi\chi(S)$*, there exists a PL metric* d^* *on* (S, V)*, unique up to scaling and isometries homotopic to the identity map on* (S, V)*, such that*

(a) d^* *is discrete conformal to d, and*

(b) the discrete curvature K_{d^*} *is equal to* K^**.*

Furthermore, the PL metric d^* *can be found by a finite-dimensional variational principle.*

For the constant function $K^* = 2\pi\chi(S)/|V|$ in Theorem 3.2, we obtain a constant curvature PL metric d^*, unique up to scaling and isometries homotopic to the identity, discrete conformal to d. We call d^* *the discrete uniformization metric* associated to d. The existence and uniqueness of d^* is a discrete version of the uniformization theorem for closed surfaces.

Theorem 3.2 for the torus $S = \mathbb{S}^1 \times \mathbb{S}^1$ with $K^* = 0$ is equivalent to a theorem of Fillastre [9]. Theorem 3.2 shows that every polyhedral torus $(\mathbb{S}^1 \times \mathbb{S}^1, V, d)$ is discrete conformal to a flat torus $(\mathbb{S}^1 \times \mathbb{S}^1, V, d_{flat})$. Translating it into the language of hyperbolic metrics, we can replace d by any cusped hyperbolic metric $\hat{d}$ on the punctured torus $\mathbb{S}^1 \times \mathbb{S}^1 - V$. The hyperbolic metric associated to $(\mathbb{S}^1 \times \mathbb{S}^1, V, d_{flat})$ is constructed as follows. Take a lattice $L = \mathbb{Z} + \tau\mathbb{Z}$ in $\mathbb{C}$ and consider the boundary $\partial C_{\mathbb{H}}(V^*)$ of the convex hull of V^* in $\mathbb{H}^3$ where V^* is a discrete set in $\mathbb{C}$ invariant under the action of L. Then by the discussion above, d_{flat}^* is isometric to the cusped hyperbolic metric $\partial C_{\mathbb{H}}(V^*)/L$. Furthermore, the lattice L is unique up to complex linear transformations. This is the result proved in [9]. To be more precise, Fillastre proved the following version of convex embedding theorem. For any cusped hyperbolic metric $\hat{d}$ on $\mathbb{S}^1 \times \mathbb{S}^1 - V$, there exist a lattice $L \subset \mathbb{C}$ and a finite set V' in the conformal infinite of the hyperbolic manifold $\mathbb{H}^3/L$ such that $\hat{d}$ is isometric to the boundary of the convex hull of V' in $\mathbb{H}^3/L$.

This shows a close connection between discrete conformal geometry and the convex isometric embedding program of Weyl, Alexandrov, Nirenberg, Pogorelov and others.

Theorem 3.3 ([12]) *Given two PL metrics on a closed marked surface (S, V) such that the lengths of edges are algebraic numbers, there exists an algorithm to decide if they are discrete conformal.*

Theorem 3.3 is proved in our joint work with Ren Guo in [12]. The counterpart of Theorem 3.2 for hyperbolic polyhedral surfaces is also proved in [12].

An important question is whether discrete conformality defined above approximates smooth conformality. To this end, let us recall discrete conformal maps between polyhedral surfaces [5, 18]. Given a closed polyhedral surface (S, V, d). Let d^* be the hyperbolic metric on $S - V$ associated d constructed using ideal hyperbolic triangles associated to Euclidean triangles. The vertical projection of the ideal hyperbolic triangle $\tau^* = C_{\mathbb{H}}(v_1, v_2, v_3)$ to the Euclidean triangle $\tau = C_{\mathbb{E}}(v_1, v_2, v_3)$ produces a piecewise projective homeomorphism Φ_d from $(S - V, d^*)$ to $(S - V, d|_{S-V})$. If d_1, d_2 are two discrete conformal PL metrics on (S, V), then the *discrete conformal map* from (S, V, d_1) to (S, V, d_2) is defined to be (the extension to S) of the composition $\Phi_{d_2} \circ \Psi \circ \Phi_{d_1}^{-1}$ where $\Psi : (S - V, d_1^*) \to (S - V, d_2^*)$ is the isometry homotopic to the identity. Discrete conformal maps are piecewise projective.

Our recent work with Sun and Wu [18] shows that discrete conformality does converge to the smooth conformality. Given a simply connected marked polygonal domain with a PL metric (D, V, d) and three boundary vertices $p, q, r \in V$, let the metric double of (D, V, d) along the boundary be the marked polyhedral 2-sphere $(\mathbb{S}^2, V', d')$. Using Theorem 3.2, one produces a new polyhedral surface $(\mathbb{S}^2, V', d^*)$ such that (1) $(\mathbb{S}^2, V', d^*)$ is discrete conformal to $(\mathbb{S}^2, V', d')$, (2) the discrete curvatures of d^* at p, q, r are $4\pi/3$, (3) the discrete curvatures of d^* at all other vertices are zero and (4) its area is $\sqrt{3}/2$. Thus $(\mathbb{S}^2, V', d^*)$ is the metric double of an equilateral triangle ΔABC of edge length 1. Here A, B, C correspond to p, q, r. Let $F : (\mathbb{S}^2, V', d') \to (\mathbb{S}^2, V', d^*)$ be the associated discrete conformal map sending $\{p, q, r\}$ to $\{A, B, C\}$ respectively. Due to the uniqueness part of Theorem 3.2, we see that $f = F|_D : D \to \Delta ABC$ sending p, q, r to the vertices A, B, C respectively. We call f the *discrete uniformization map* associated to $(D, \mathcal{T}, l, \{p, q, r\})$

Theorem 3.4 ([18]) *Suppose Ω is a Jordan domain in the complex plane with three distinct points p, q, r in the boundary of Ω. There exists a sequence of simply connected polygonal domains $(\Omega_n, \mathcal{T}_n, \{p_n, q_n, r_n\})$ with triangulations $\mathcal{T}_n$ by equilateral triangles of edge lengths converging to 0 where p_n, q_n, r_n are three boundary vertices such that the following hold*

(i) $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \cup_{n=1}^{\infty} \Omega_n$,

(ii) the discrete uniformization maps f_n associated to $(\Omega_n, \mathcal{T}_n, \{p_n, q_n, r_n\})$ converge uniformly on compact sets to the Riemann mapping $f : (\Omega, \{p, q, r\}) \to (\Delta ABC, \{A, B, C\})$.

3.2 Vertex Scaling and Its Associated Variational Principle

A key property established in [16] for the vertex scaling is the following variational principle (See Lemma 3.5(a) below).

Lemma 3.5 *Suppose $\Delta v_1 v_2 v_3$ is a Euclidean triangle of edge lengths l_1, l_2, l_3 such that v_i is opposite to the edge of length l_i. Let $l_1 e^{x_2+x_3}, l_2 e^{x_1+x_3}, l_3 e^{x_1+x_2}$ be the edge lengths of a vertex scaled Euclidean triangle whose inner angle at v_i is $\theta_i = \theta_i(x_1, x_2, x_3)$.*

(a)([16]) There exists a locally concave function $F(x_1, x_2, x_3)$ such that

$$\frac{\partial F}{\partial x_i} = \theta_i \tag{3.2}$$

and the kernel of the positive semidefinite symmetric matrix $[\frac{\partial \theta_i}{\partial x_j}]_{3\times 3}$ consists of column vectors $(a, a, a)^t$ and

(b) If $e^{x_1} \to \infty$ and e^{x_2} is bounded, then e^{x_3} is bounded and $\theta_1(x) \to 0$.

Proof To see part (a), it suffices to show that the 3×3 matrix $[\frac{\partial \theta_r}{\partial x_s}]_{3\times 3}$ is symmetric and negative semi-definite. Let the area of the triangle $\Delta v_1 v_2 v_3$ be A and $y_i = l_i e^{x_j + x_k}$ be the length of the edge $v_j v_k$ where $\{i, j, k\} = \{1, 2, 3\}$. Note that $\frac{\partial y_i}{\partial x_j} = y_i$ and $y_j = y_k \cos(\theta_i) + y_i \cos(\theta_k)$. By (2.1) and (2.2), we have

$$\frac{\partial \theta_i}{\partial x_j} = \frac{\partial \theta_i}{\partial y_i}\frac{\partial y_i}{\partial x_j} + \frac{\partial \theta_i}{\partial y_k}\frac{\partial y_k}{\partial x_j} = \frac{y_i(y_i - y_k\cos(\theta_j))}{2A} = \frac{y_i y_j \cos(\theta_k)}{2A} = \cot(\theta_k)$$

and

$$\frac{\partial \theta_i}{\partial x_i} = \frac{\partial \theta_i}{\partial y_j}\frac{\partial y_j}{\partial x_i} + \frac{\partial \theta_i}{\partial y_k}\frac{\partial y_k}{\partial x_i} = -\frac{\partial \theta_i}{\partial y_i}(y_j\cos(\theta_k) + y_k\cos(\theta_j)) = -\frac{y_i^2}{2A} - \frac{\sin\theta_i}{\sin(\theta_j)\sin(\theta_k)}.$$

This shows that the matrix $[\frac{\partial \theta_r}{\partial x_s}]_{3\times 3}$ is symmetric and can be written as $-DGD^t$ where $G = [g_{rs}]_{3\times 3}$ is the Gram matrix of the triangle and D is a diagonal matrix. Here $g_{ii} = -1$ and $g_{ij} = -\cos(\theta_k)$. Let n_i be the unit outward normal vector to the triangle at edge $v_j v_k$ and (u, v) be the inner product in $\mathbb{R}^3$. Then the Gram matrix G is the same as $[(n_r, n_s)]_{3\times 3}$ which is well known to be positive semi-definite with kernel (t, t, t). Thus part (a) follows.

To see part (b), note that the triangle of edge lengths $e^{x_2+x_3}l_1, e^{x_1+x_3}l_2, e^{x_1+x_2}l_3$ is similar to the Euclidean triangle of edge lengths $l_1 e^{-x_1}, l_2 e^{-x_2}, l_3 e^{-x_3}$. In particular, we have the triangle inequality that $l_2 e^{-x_2} < l_1 e^{-x_1} + l_3 e^{-x_3}$. This implies that x_3 must be bounded. Since $l_1 e^{-x_1} \to 0$ and $l_2 e^{-x_2}, l_3 e^{-x_3}$ are bounded away from 0, it follows $\theta_1 \to 0$. □

The identity in Lemma 3.5(a) can be considered as a 2-dimensional analogue of the Schlaefli formula. This variational principle has been generalized in the work of [5]. In particular, an explicit formula for the function F was found in [5] using Lobachevsky function $-\int_0^x \ln(|2\sin(t)|)dt$.

3.3 Basic Idea of the Proof of Theorem 3.2

There are two steps involved in the proof. The first step is to understand discrete conformality using hyperbolic metrics. The goal is to show that given any PL metric d on (S, V), the space $DC([d])$ of all PL metrics on (S, V) discrete conformal to d is C^1-diffeomorphic to the Euclidean space $\mathbb{R}^V$. The second step is to show that the discrete curvature map $K : DC([d]) \to \{x \in (-\infty, 2\pi)^V \mid \sum_{v\in V} K(v) = 2\pi\chi(S)\}$ is a bijection up to scalings. This is achieved by showing that the discrete curvature map K is the gradient of a convex function using Lemma 3.5(a) and the work of [1].

The first step is achieved by using Penner's theorey of decorated Teichmüller space. Let us first recall that two PL metrics on (S, V) are *Teichmüller equivalent* if they are isometric by an isometry homotopic to the identity in (S, V). For instance the condition (b) in Definition 3.1 says that (S, V, d_i) is Teichmüller equivalent to (S, V, d_{i+1}). The PL Teichmüller space $T_{pl} = T_{pl}(S, V)$ is the space of all Teichmüller equivalence classes of PL metrics on (S,V). The space $T_{pl}(S, V)$ is known to be a real analytic manifold diffeomorphic to a Euclidean space by the work of Troyanov [34]. The discrete conformality is an equivalence relation on $T_{pl}(S, V)$. The discrete curvature $K : T_{pl}(S, V) \to (-\infty, 2\pi)^V$ is a real analytic map. There exists a natural action of the set of positive real numbers $\mathbb{R}_{>0}$ on $T_{pl}(S, V)$ by scaling. The discrete curvature is well defined on the quotient space $K : T_{pl}(S, V)/\mathbb{R}_{>0} \to \{x \in (-\infty, 2\pi)^V \mid \sum_{v\in V} x(v) = 2\pi\chi(S)\}$.

Given a metric $[d] \in T_{pl}(S, V)$, let $DC([d]) = \{[d'] \in T_{pl} \mid d'$ is discrete conformal to $d\}$ be the discrete conformal class associated to $[d]$. Theorem 3.2 is equivalent to the statement that the restriction of the discrete curvature map K to $DC([d])/\mathbb{R}_{>0}$ is a bijection from $DC([d])/\mathbb{R}_{>0}$ onto $\{x \in (-\infty, 2\pi)^V \mid \sum_v x(v) = 2\pi\chi(S)\}$. We prove that K is a C^1 diffeomorphism.

Let $T(S - V)$ be the Teichmüller space of complete hyperbolic metrics of finite area on $S - V$ and $T_D = T(S - V) \times \mathbb{R}^V_{>0}$ be Penner's decorated Teichmüller space [23]. Recall that a decorated hyperbolic metric on $S - V$ is a complete finite area hyperbolic metric together with a horoball at each cusp. By measuring the lengths of the boundaries of the horoballs, one can write a decorated hyperbolic metric as a pair (d, u) where $u \in \mathbb{R}^V_{>0}$. This shows that the space of all decorated hyperbolic metrics modulo the natural equivalence relation is $T(S - V) \times \mathbb{R}^V_{>0}$. Decorated hyperbolic metrics on an ideal triangulated surface $(S - V, \mathcal{T})$ can be constructed by isometrically gluing decorated ideal hyperbolic triangles along edges. Here a decorated ideal hyperbolic triangle is an ideal triangle with a horoball at each vertex. Since all ideal hyperbolic triangles are isometric, a decorated ideal triangle is determined up to isometries preserving decoration by the three lengths of horocycles inside it. Another way to parameterize a decorated ideal triangle is to use the edge lengths. If e is an edge of a decorated ideal triangle, then the length $l(e)$ of e is the distance between the two horoballs B_1, B_2 at its end points if $B_1 \cap B_2 = \emptyset$, and is the negative of the length of the interval $e \cap (B_1 \cap B_2)$ if $B_1 \cap B_2 \neq \emptyset$. Penner defines the λ-length of an edge e is defined to be $e^{l(e)/2}$. Given any three positive real numbers, there exists a unique decorated ideal triangle whose λ-lengths are the given numbers. In particular, given any Euclidean triangle σ of edge lengths l_1, l_2, l_3, one can associate a decorated ideal triangle σ^* of λ-length l_1, l_2, l_3 to σ. Given a PL metric d represented as $(S, \mathcal{T}, l)$ (i.e., $\mathcal{T}$ is geometric in d), one assigns a decorated hyperbolic metric $\Phi_{\mathcal{T}}(d)$ on $S - V$ as follows. Each Euclidean triangle $\sigma \in \mathcal{T}$ is replaced by its decorated ideal triangle counterpart σ^*. These decorated ideal triangles are glued along edges by isometries preserving decorations. The resulting decorated hyperbolic metric is $\Phi_{\mathcal{T}}(d)$. See [5]. We prove the following theorem.

Theorem 3.6 ([11]) *For any closed marked surface (S, V) such that $(S, V) \neq (\mathbb{S}^2, \{p\})$ or $(\mathbb{S}^2, \{p, q\})$, there exists a C^1 smooth diffeomorphism Φ from the PL Teichmüller space $T_{pl}(S, V)$ to the decorated Teichmüller space $T(S - V) \times \mathbb{R}^V_{>0}$ such that two PL metrics d and d' are discrete conformal if and only if the projections of $\Phi(d)$ and $\Phi(d')$ to the Teichmüller space $T(S - V)$ are the same.*

The map Φ is constructed in a piecewise smooth manner on the natural cell decompositions of T_{pl} and T_D. For each triangulation $\mathcal{T}$ of (S, V), define $D_{pl}(\mathcal{T})$ (and $D(\mathcal{T})$) to be the set of all PL metrics (and decorated hyperbolic metrics) $[d]$ in T_{pl} (and T_D) such that $\mathcal{T}$ is isotopic to a Delaunay triangulation in d. The important works of Rivin [26] and Penner [23] show that $D_{pl}(\mathcal{T})$ and $D(\mathcal{T})$ are cells and $T_{pl} = \cup_{\mathcal{T}} D_{pl}(\mathcal{T})$ and $T_D = \cup_{\mathcal{T}} D(\mathcal{T})$ are cell decompositions of the Teichmüller spaces invariant under the action of the mapping class group. The definition of Φ goes as follows. For each triangulation $\mathcal{T}$, define $\Phi_{\mathcal{T}} : D_{pl}(\mathcal{T}) \to T_D(S, V)$ by sending a PL metric $(S, \mathcal{T}, l)$ to $\Phi_{\mathcal{T}}(S, \mathcal{T}, l)$. By definition the two decorated metrics $\Phi_{\mathcal{T}}(S, \mathcal{T}, l)$ and $\Phi_{\mathcal{T}}(S, \mathcal{T}, w * l)$ have the same underlying hyperbolic metrics and differ only in decorations.

It is a straightforward calculation to see that Euclidean Delaunay condition is mapped to hyperbolic Delaunay condition, i.e., $\Phi_{\mathcal{T}}(D_{pl}(\mathcal{T})) \subset D(\mathcal{T})$. Penner observed that hyperbolic Delaunay condition implies the triangle inequality for (Euclidean) edge lengths, i.e., $\Phi_{\mathcal{T}}(D_{pl}(\mathcal{T})) = D(\mathcal{T})$. Furthermore, Penner's result that the Ptolemy identity holds for λ-lengths of decorated ideal quadrilaterals implies that for different triangulations $\mathcal{T}$ and $\mathcal{T}'$ of (S, V),

$$\Phi_{\mathcal{T}}|_{D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')} = \Phi_{\mathcal{T}'}|_{D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')}.$$

Thus these maps $\Phi_{\mathcal{T}}$ can be glued together to produce a homeomorphism $\Phi = \cup_{\mathcal{T}} \Phi_{\mathcal{T}} : T_{pl} \to T_D$. Note that the complete finite area hyperbolic metric $d_{\mathbb{H}}$ on $S - V$ associated to a PL metric d on (S, V) is $P \circ \Phi([d])$ where $P : T(S - V) \times \mathbb{R}^V_{>0} \to T(S - V)$ is the projection.

We prove that Φ is a C^1 diffeomorphism by using the following lemma on quadrilaterals.

Lemma 3.7 *Let Q be a convex Euclidean quadrilateral whose four edge lengths are x, y, z, w labelled cyclically and the length of a diagonal be u. Let $A(x, y, z, w, u)$ be the length of second diagonal and $B(x, y, z, w, u) = \frac{xz+yw}{u}$. If a point (x, y, z, w, u) satisfies $A(x, y, z, w, u) = B(x, y, z, w, u)$, i.e., Q is inscribed in a circle, then $DA(x, y, z, w, u)= DB(x, y, z, w, u)$ where DA is the derivative of A.*

In the second step, we examine the restriction $K|$ of the discrete curvature map to the space of discrete conformal classes $DC([d])$. By Theorem 3.6, $DC([d])$ is naturally a Euclidean space. Using Lemma 3.5(a), we show that the discrete curvature map on $DC([d])/\mathbb{R}_{>0}$ is the gradient of a strictly convex function. Thus, $K| : DC([d])/\mathbb{R}_{>0} \to Y := \{x \in (-\infty, 2\pi)^V | \sum_v x(v) = 2\pi\chi(S) \}$ is injective. On the other hand, by using Lemma 3.5(b) and a result of Akiyoshi [1] we show that the image $K(DC([d]))$ is closed in Y. Since both $DC([d])/\mathbb{R}_{>0}$ and Y are connected manifolds of the same dimension, we conclude that $K|$ is a homeomorphism and thus prove Theorem 3.2.

3.4 Basic Ideas of the Proof of Theorem 3.3

Suppose d, d' are two PL metrics on a marked closed surface (S, V) such that d, d' are given by the edge length functions $l_d : E(\mathcal{T}) \to \mathbb{A}$ and $l_{d'} : E(\mathcal{T}') \to \mathbb{A}$ where $\mathbb{A}$ is the set of all real algebraic numbers. Our goal is to use these two vectors l_d and $l_{d'}$ to decide whether d, d' are discrete conformal or not.

Using a well-known algorithm from computational geometry, we may assume that both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay in d and d' respectively. Now consider the associated decorated hyperbolic metrics $y = \Phi_{\mathcal{T}}(d)$ and $y' = \Phi_{\mathcal{T}'}(d')$ in Penner's decorated Teichmüller space. By Theorem 3.6, it suffices to check if y, y' have the same underlying hyperbolic metric. To this end, we use a theorem of Thurston and Mosher [22] that there is an algorithm which produces a sequence of triangulations $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, ..., \mathcal{T}_n = \mathcal{T}'$ of (S, V) such that for each i, $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ differ by a diagonal switch. Combining with Penner's Ptolemy identity, we find algorithmically the λ-length coordinates $z\,(= y)$ and z' of the decorated metrics y, y' in the same triangulation $\mathcal{T}$. For a triangulation $\mathcal{T}$, it is known by Penner's work that z, z' represent the same underlying hyperbolic metric if and only if their associated shear coordinates in the triangulation $\mathcal{T}$ are the same. Here the shear coordinate of $z : E(\mathcal{T}) \to \mathbb{R}_{>0}$ is the function $\phi(z) : E(\mathcal{T}) \to \mathbb{R}$ given by $\phi(z)(e) = \frac{ab}{cd}$ where a, b, c, d are the values of z at the four edges, ordered cyclically, of the quadrilateral in $\mathcal{T}$ formed by the two triangles adjacent to e. Therefore, we can check algorithmically if z and z' have the same underlying hyperbolic structure.

3.5 Basic Idea of the Proof of Convergence Theorem 3.4

The proof of Theorem 3.4 follows the basic strategy appeared in Rodin-Sullivan's work [29]. Namely, we prove that the approximating discrete conformal maps f_n are K-quasi-conformal for some K independent of n and a rigidity result about the hexagonal triangulations of the plane. Finally since Delaunay triangulations may change due to flip operations, we choose the approximation triangulations nicely to ensure that no flips occur.

The K-quasi-conformality is relatively easy to establish and is based on a ratio lemma first appeared in [36] and a non-degeneration lemma. The conditions which ensure no flips are technical and will not be addressed here. We will discuss the rigidity result in more details.

The rigidity theorem that we proved is the following,

Theorem 3.8 ([18]) *Suppose $(\mathbb{C}, \mathcal{T}, l)$ is a geometric Delaunay triangulation of an open set in the complex plane $\mathbb{C}$ such that (i) each vertex is adjacent to exactly six triangles and (ii) there exists a function $w : V(\mathcal{T}) \to \mathbb{R}_{>0}$ satisfying $l(vv') = w(v)w(v')$ for all edges vv'. Then the triangulation is the regular hexagonal triangulation, i.e., w is a constant function.*

This should be compared with Rodin-Sullivan's rigidity theorem for circle packing metric which can be stated as,

Theorem 3.9 (Rodin-Sullivan [29]) *Suppose $(\mathbb{C}, \mathcal{T}, l)$ is a geometric triangulation of an open set in the complex plane $\mathbb{C}$ such that (i) each vertex is adjacent to exactly six triangles and (ii) there exists a function $w : V \to \mathbb{R}_{>0}$ satisfying $l(vv') = w(v) + w(v')$ for all edges vv'. Then the triangulation is the regular hexagonal triangulation, i.e., w is a constant function.*

Recall that a PL metric on a triangulated surface $(S, \mathcal{T})$ can be represented by the edge length function $l : E(\mathcal{T}) \to \mathbb{R}_{>0}$ so that the triangle inequality $l(e_i) + l(e_j) >$

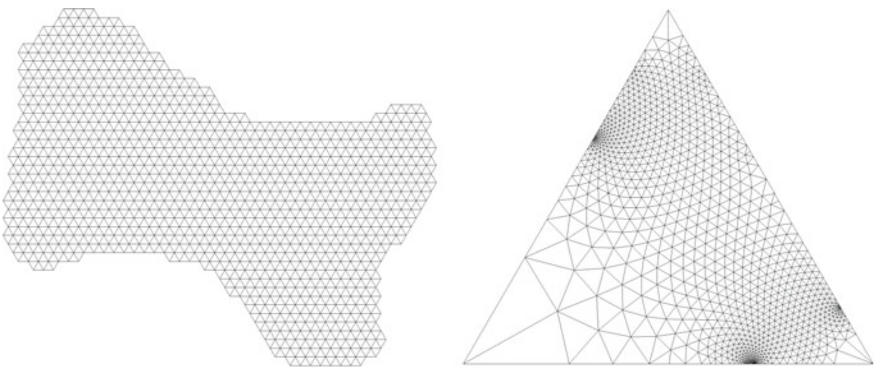

Fig. 4 Convergence of discrete conformality and approximation of the Riemann mapping

$l(e_k)$ holds for three edges e_i, e_j, e_k of a triangle. A *generalized PL metric* on $(S, \mathcal{T})$ is map $l : E(\mathcal{T}) \to \mathbb{R}_{>0}$ so that $l(e_i) + l(e_j) \geq l(e_k)$ holds for three edges e_i, e_j, e_k of a triangle. Since the edge lengths $l(e) > 0$ in a generalized PL metric, the inner angles, discrete curvatures and Delaunauy conditions are still defined for generalized PL metrics. A generalized PL metric is called *flat* if its curvature is zero at each vertex (Fig. 4).

The idea of the proof of Theorem 3.8 is as follows. Suppose otherwise that w is not a constant, we will derived a contradiction by using a maximum principle and a spiral lemma.

Let $V = \mathbb{Z} + e^{2\pi i/3}\mathbb{Z}$ be the set of vertices of the standard hexagonal triangulation $\mathcal{T}_{st}$ with $l_{st} : V \to \{1\}$ being the edge length function. Consider those $u : V \to \mathbb{R}_{>0}$ so that $u * l_{st}$ are generalized PL metrics, i.e., $u * l(v_1v_2) + u * l(v_2v_3) \geq u * l(v_3v_1)$ for vertices $\{v_1, v_2, v_3\}$ of a triangle. The maximum principle says if $u : V \to \mathbb{R}_{>0}$ is a function so that $u * l_{st}$ is a flat generalized PL metric and has a maximum point, then u is a constant. This is essentially a consequence of Lemma 3.5(a). The ratio lemma says if $u * l_{st}$ is flat, then $\frac{u(v)}{u(v')} \leq 6$ for each edge $vv' \in \mathcal{T}$. The spiral lemma says for any non-constant linear function $\ln(u) : V \to \mathbb{R}$ so that $u * l_{st}$ is a generalized PL metric, then the metric $e^u * l_{st}$ is flat and furthermore, if $u * l_{st}$ contains a triangle of positive area, then the developing map for the $u * l_{st}$ metric sends two triangles to two triangles in $\mathbb{C}$ with overlapping interiors. Using these lemmas, one proves Theorem 3.8 as follows. We may assume without loss of generality that $\lambda = \sup\{\frac{w(v)}{w(v\pm 1)} | v \in V\} > 1$. By the ratio lemma, we know $\lambda < \infty$. Choose a sequence of vertices $v_n \in V$ so that, say, $\frac{w(v_n)}{w(v_n+1)} \to \lambda$. Now using the symmetry of the lattice $\mathbb{Z} + e^{2\pi i/3}\mathbb{Z}$, we produce a new sequence of function $w_n : V \to \mathbb{R}_{>0}$ obtained by shifting $v_n \in V$ to 0 and re-scaling so that $\{w_n\}$ contains a convergent subsequence converging to $w_\infty : V \to \mathbb{R}_{>0}$. The generalized PL metric $w_\infty * l_{st}$ is still flat since flatness is a closed condition. By the maximum principle applied to the generalized flat PL metric $w'_\infty * l_{st}$ where $\alpha'(v) = \alpha(v)/\alpha(v+1) : V \to \mathbb{R}_{>0}$, we see that $w_\infty(v) = \lambda w_\infty(v+1)$ for all $v \in V$. By the same argument applied to $\delta = sup\{\frac{w(v)}{w(v \pm e^{2\pi i/3})} | v \in V\}$ and taking subsequence of the subsequence, we can improve the result to a limit function $w_\infty : V \to \mathbb{R}_{>0}$ so that $w_\infty(v) = \lambda w_\infty(v+1)$ and $w_\infty(v) = \delta w_\infty(v + e^{2\pi i/3})$ for all v. Therefore, $\ln(w_\infty) : V \to \mathbb{R}$ is a non-constant linear function. We show that there exists a triangle in $w_\infty * l_{st}$ of positive area. By the spiral lemma, the developing map for the flat generalized PL metric $w_\infty * l_{st}$ sends two triangles to two triangles with overlapping interiors. On the other hand, by the construction, $w_\infty * l_{st}$ is the limit of $w_n * l_{st}$ which is a geodesic triangulation of $\mathbb{C}$ obtained from $(\mathbb{C}, \mathcal{T}, w * l_{st})$ by shifting base points and re-scaling. In particular, the developing map D_∞ of $w_\infty * l_{st}$ is the limit of injective maps where the convergence is uniform on compact sets. This shows that D_∞ cannot send two triangles to two triangles with overlapping interiors. The contradiction shows Theorem 3.8 holds.

Our proof of Theorem 3.8 also gives a new proof of Rodin-Sullivan's Theorem 3.9 since the similar maximum principle, the ratio lemma and the spiral lemma hold in the circle packing case. The spiral lemma in the circle packing case was first discovered by Peter Doyle and the phenomena is called the Doyle spiral.

The rigidity theorem proved in [18] also holds for any lattice in $\mathbb{C}$ instead of the regular hexagonal lattice.

3.6 Discrete Uniformization for Non-compact Simply Connected Polyhedral Surfaces

An essential step in Poincaré's and Koebe's proofs of the uniformization theorem is to establish that every simply connected non-compact Riemann surface is conformal to the plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. The corresponding statement for discrete uniformization is that every non-compact simply connected polyhedral surface (S, V, d) is discrete conformal to $(\mathbb{C}, V', d_{st})$ or $(\mathbb{D}, V', d_{st})$ for some discrete set $V' \subset \mathbb{C}$ or $V' \subset \mathbb{D}$ and the set V' is unique up to Möbius transformations. Here d_{st} is the standard flat Euclidean metric. Given a closed set $X \subset \mathbb{S}^2$, the convex hull of X in the hyperbolic 3-space $\mathbb{H}^3$ is denoted by $C_{\mathbb{H}}(X)$. Using geometry, discrete uniformization is equivalent to the statement that a hyperbolic metric d^* on $S - V$ (with cusp ends at each $v \in V$) is isometric to the boundary of the convex hull $\partial C_{\mathbb{H}}(V')$ or $\partial C_{\mathbb{H}}(V' \cup (\mathbb{S}^2 - \mathbb{D}))$. Furthermore, the set V' is unique up to Möbius transformations.

Recall that a closed set X in the Riemann sphere $\mathbb{S}^2$ is of *circle type* if each connected component of X is either a point or a closed round disk. For instance if V' is a discrete subset of $\mathbb{D}$, then $(\mathbb{S}^2 - \mathbb{D}) \cup V'$ is a circle type closed set. The generalization of the above discrete uniformization conjecture is the following. For any complete hyperbolic surface (Σ, g) of genus zero, there exists a circle type closed set Y, unique up to Möbius transformations, such that (Σ, g) is isometric to the boundary of the convex hull of $C_{\mathbb{H}}(Y)$ in $\mathbb{H}^3$. This can be rephrased using a theorem of Alexandrov [2] that any genus zero hyperbolic surface (Σ, g) is isometric to $\partial C_{\mathbb{H}}(X)$ for some closed set $X \subset \mathbb{S}^2$. Therefore, we have,

Conjecture 1 ([18]) *Given any closed set $X \subset \mathbb{S}^2$ with $\mathbb{S}^2 - X$ connected, there exists a circle type closed set Y such that the boundaries of $C_{\mathbb{H}}(X)$ and $C_{\mathbb{H}}(Y)$ are isometric.*

In particular, Conjecture 1 for X to be $V \cup \{\infty\}$ or $(\mathbb{S}^2 - \mathbb{D}) \cup V$ where V is discrete in $\mathbb{C}$ or $\mathbb{D}$ is the existence part of the discrete uniformization for non-compact simply connected polyhedral surfaces. In [19] we proved that

Theorem 3.10 *Conjecture 1 holds if the given closed set X has countably many connected components. In particular, the existence part of the discrete uniformization theorem holds.*

Conjecture 1 is a geometric form of the Koebe conjecture that any genus zero Riemann surface S is conformal to $\mathbb{S}^2 - Y$ for a circle type closed set Y.

Conjecture 2 ([18]) *Suppose X and Y are two circle type closed sets in $\mathbb{S}^2$ such that the boundary of $C_{\mathbb{H}}(X)$ is isometric to the boundary of $C_{\mathbb{H}}(Y)$. Then X and Y differ by a Möbius transformation.*

Here are some evidences supporting Conjectures 1 and 2. If the given set X is finite, Rivin [27] proved that both Conjectures 1 and 2 hold. If X is a disjoint union of a finite number of closed round disks, then Schlenker [30] proved that both Conjectures 1 and 2 hold. See also [20] for the case of a union of a closed round disk with a finite set of points. Theorem 3.8 is a very special case of Conjecture 2.

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