Comprehending the Connection of Things: Bernhard Riemann and the Architecture of Mathematical Concepts

Arkady Plotnitsky

Abstract This chapter is an essay on the conceptual nature of Riemann's thinking and its impact, as conceptual thinking, on mathematics, physics, and philosophy. In order to fully appreciate the revolutionary nature of this thinking and of Riemann's practice of mathematics, one must, this chapter argues, rethink the nature of mathematical or scientific concepts in Riemann and beyond. The chapter will attempt to do so with the help of Deleuze and Guattari's concept of philosophical concept. The chapter will argue that a fundamentally analogous concept of concept is also applicable in mathematics and science, specifically and most pertinently to Riemann, in physics, and that this concept is exceptionally helpful and even necessary for understanding Riemann's thinking and practice, and creative mathematical and scientific thinking and practice in general.

1 Introduction

This chapter is an essay on the conceptual nature of Bernhard Riemann's thinking and its impact, as conceptual thinking, on mathematics, physics, and philosophy. In order to fully appreciate the revolutionary nature of this thinking and of Riemann's practice of mathematics, one must, I argue, rethink the nature and structure, architecture, of mathematical or scientific concepts in Riemann and beyond. I shall attempt to do so here with the help of Gilles Deleuze and Félix Guattari's concept of philosophical concept, as defined in *What Is Philosophy*? [8], the culminating work of Deleuze's philosophy, on which I shall comment presently. I argue that a fundamentally analogous concept of concept is also applicable in mathematics and science, specifically and most pertinently to Riemann, in physics, and that this concept is exceptionally helpful and even necessary for understanding Riemann's thinking and practice, and creative mathematical and scientific thinking and practice in general. While I shall address Riemann's work in physics, I shall, given my scope, be less concerned with

A. Plotnitsky (✉)
Theory and Cultural Studies Program, Purdue University, W. Lafayette, IN 47907, USA
e-mail: plotnits@purdue.edu

L. Ji et al. (eds.), *From Riemann to Differential Geometry and Relativity*,
DOI 10.1007/978-3-319-60039-0_11

physical concepts as such.[1] The concept of concept in question here is discussed in Sect. 2, which follows this Introduction. Section 3 considers Riemann's concept of manifold(ness) [*Mannigfaltigkeit*] and his understanding of space and geometry, grounded in this concept. Finally, Sect. 4 addresses the relationships between mathematics and physics in Riemann. Thus, Sect. 2 is more philosophical, Sect. 3 is more mathematical, and Sect. 4 more physical. When it comes to Riemann, however, philosophy, mathematics, and physics are never far from each other; and the workings of all three in Riemann's thought will be addressed throughout this article.

I would like to begin, by way of a prologue, with the 1907 article, "*La logique et lintuition en mathématique,*" by Émile Borel, who questioned the logicists' philosophy of mathematics, which theorized mathematics as an extension of logic and which, championed by, among others, Bertrand Russell, was in vogue at the time. Borel opens his argument by considering polynomial identities, such as $(x+1)(x-1) = x^2 + 1$ familiar to all of us since our school days. One can imagine such identities to be produced mechanically and entirely correctly, by a logical mechanism or machine. One can then imagine one of these identities, say, number 35,427, being $4P^3 = Q^2 + 27R^2$, satisfied when $P = x^2 - x + 1$, $Q = (2x-1)(x+1)(x-2)$, and $R = x(x-1)$. While the previous or the following one in this sequence may be entirely uninteresting mathematically, this one is interesting and even special, first, because it expresses a cube as a (weighted) sum of two squares, and, although Borel did not mention this fact, this identity is also important in the theory of elliptic functions [2, pp. 273–274] [12, pp. 402–403]. Logic, Borel argues, cannot capture the importance of this identity. Borel gave other examples of this failure of logic to capture the essence of mathematical thought, such as the fact that the formula expressing the invariance of the cross-ratio of four points on a line under a perspectivity is easy to find, but it took a Chasles to see in it the key to projective geometry. Borel also considered, with the same aim, the case of another polynomial identity (the icosahedral equation) that could only have been found to be valuable if discovered by a nonmechanical route, namely through Felix Klein's unification of Evariste Galois's theory with the theory of the symmetries of the icosahendron. This is, I think, one of those findings in which the essence of modern mathematics is manifested, not the least because it brings together concepts and calculation, very much in the spirit of Riemann, which often guided Klein's work. This finding and its generalizations have profound connections to Riemann's ideas concerning Riemann surfaces, in the so-called Belyi theorem and Grothendieck's *Dessin d'Enfants* theory. Borel's view, then, was that a truly fertile invention in mathematics consists of the discovery of a new point of view from which to classify and interpret the facts, followed by a search for the necessary proofs by plausible reasoning (later considered by George Pólya [30]), and only in the third and final stage does logic take over. As Jeremy Gray notes, "Borel's criticisms are not quite the staples they might seem, and not just because they had a specific resonance in the France of the time. They point quite clearly toward a problem that has not gone away in philosophers' treatment of mathematics: a tendency to reduce it to some essence that not only deprives it of

[1] I have considered physical concepts from this perspective in [27, pp. 2–11].

purpose but is false to mathematical practice. The logical enterprise, even if it had succeeded, would only have been an account of part of mathematics—its deductive skeleton" [11, pp. 202–203].

Nor, I would contend, would one do much better on that score by using the axiomatic approach to foundations of mathematics, sometimes juxtaposed to logicism, or by using formalism, which assumes that mathematics is not a body of propositions representing an abstract sector of reality but is much more akin to a game, and as such, allows one to capture more of creative mathematical thinking [34]. Henri Poincaré, it is worth noting, was as skeptical as Borel, and on similar grounds, concerning the capacity of these approaches to capture creative mathematical thinking [11, 29, pp. 261–265, 390–391]. However, as I argue here, things are different when it comes to the conceptual aspects of mathematics, which may well be primarily, even if not solely, responsible for truly fertile inventions in mathematics, if we understand properly what mathematical concepts are and how they work. Riemann's thinking and practice provide a particularly helpful guidance for this understanding, especially, I would like to propose, if one sees Riemann's understanding and practice of concepts through the optics of Deleuze and Guattari's concept of a philosophical concept in *What Is Philosophy*? [8]. Gilles Deleuze (1925–1995), the main driving force behind this concept of concept, was one of the most creative, as well as the most controversial, French philosophers of the second half of the twentieth century, both in his own work and in his collaboration with Félix Guattari (1930–1992). Deleuze's thinking, including that concerning the nature of concepts, was influenced by Riemann, especially by Riemann's concept of manifold and the resulting rethinking of spatiality, which Deleuze saw as heralding a revolutionary change in philosophy, mathematics, and physics alike, and, I would add, in the relationships among them [7, p. 483].[2]

Deleuze and Guattari's concept of philosophical concept can and here will be transferred, partially and against their own grain, into our understanding of mathematical or scientific, such as physical, concepts. Riemann's contribution to physics in this regard and in general was, if more limited, nearly as revolutionary as his contribution to mathematics. In this chapter, I shall primarily focus on Riemann's mathematics, to which I shall mainly refer at the moment as well. Most of my argument concerning concepts, will equally apply to Riemann's physics, which I shall consider in Sect. 4, or, by definition, to Riemann's philosophical thinking. Riemann's mathematical thinking was, I argue, fundamentally defined by the invention of new concepts in Deleuze and Guattari's sense, now applied to mathematical rather than philosophical concepts. This transfer of their concept of concept to mathematics is partial because mathematics cannot be defined only by the invention of mathematical concepts, given the roles of other aspects of mathematical thinking, such as logical and axiomatic reasoning, which Deleuze and Guattari see as most defining in mathematics. Accordingly, it becomes a matter of the relative precedence of these components at different junctures, with, however, the invention of concepts having

[2] On this influence, see F. Jedrzejewski's contribution to this volume [14] and an earlier article by the present author [26].

the main role in creative mathematical or scientific thought. But then, one could argue that philosophy cannot be limited to the invention of new concepts either, and that there, too, it is a matter of the relative precedence of different aspects of philosophical thinking at different junctures, with the invention of concepts having the primary role in creative philosophical thought. This transfer is against Deleuze and Guattari's own grain, because, following Georg W. F. Hegel and other post-Kantians, Deleuze and Guattari associate their concept of concept primarily and even uniquely with philosophical thinking. The concept [in this sense], they say, belongs to philosophy and only to philosophy [8, pp. 11–12, 33–34].

At the same time, as noted above, Riemann's ideas and specifically his concept of manifoldness had a major impact on their philosophy, in part in juxtaposition to Hegel, in a different context, that of dialectical thinking, overturned, they argued, by thinking in terms of manifold-like concepts in philosophy [7, p. 483]. This is not inconsistent. Hegel is close to Riemann when it comes to their respective concepts of concept, although Hegel, too, associates his concept of concept [*Begriff*] primarily and even uniquely with philosophical thought. My argument here, however, is only against Deleuze and Guattari's, or Hegel's, own grain, not against their thinking. Neither Deleuze and Guattari nor Hegel (who made an even stronger claim to this effect) are ultimately able to deny mathematical and scientific concepts most of the essential features of the architecture that they associate with philosophical concepts, and are, more expressly, Deleuze and Guattari, compelled to reinstate this architecture to mathematical and scientific concepts [8, pp. 217–218]. Riemann, by contrast, fuses philosophy, mathematics, and physics in his thinking and in the architecture of his concepts, and thus also offers us a better way of understanding the nature and workings of concepts in all three domains, and in the relationships among them.

The conceptual character of Riemann's mathematics has been noted and even emphasized in commentaries on Riemann, especially in contrast to mathematics based in calculations (e.g. [9, 16]). These commentaries have not, however, paid sufficient attention to the architecture of Riemann's concepts, or of fundamental mathematical concepts in general. They have not asked the question "What is a concept?" They either take the concept of concept for granted or adopt a conventional view of concepts as generalizations from particulars. As a result, they miss the architectural complexity of Riemann's concept of concepts and, along with it, the deeper nature of Riemann's conceptual thinking, which was not defined most essentially by its juxtaposition to calculation, but rather by his concept of concept.[3] It is not, it is worth noting, that Riemann did not do calculations. But he gave priority to concepts (in his sense), even in doing calculations, grounded in and made more effective by his use of concepts. This is manifested, for example, in his work on functions of a complex variable, where the invention of new concepts, such as that of Riemann surface, created new possibilities for calculations, the potential of which has been explored throughout the subsequent history and is still far from exhausted. It may indeed be inexhaustible.

[3] K. Ohshika's chapter in this volume [20] is a notable exception as a reflection on the architecture of Riemann's concepts.

Although arising from Riemann's work in general, my argument here is based most essentially on Riemann's Habilitation lecture, "On the Hypotheses That Lie at the Foundations of Geometry" [*Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*] [31], given in 1854 but first published by Richard Dedekind only in 1868, two years after Riemann's death.[4] The lecture offered a radical rethinking of space and geometry, as against the preceding history of both, from before Euclid to the discovery of non-Euclidean geometry around 1830. This rethinking was based on the concept of manifold or manifoldness [*Mannigfaltigkeit*], a major mathematical innovation. Establishing the possibility of non-Euclidean geometry was a great mathematical discovery, with profound implications for mathematics, physics, and philosophy, and indeed culture. However, as Riemann argued in his lecture, the pre-Riemannian conceptions of non-Euclidean geometry did not sufficiently depart from previous thinking concerning space and geometry, with a possible exception of certain ideas of Karl Friedrich Gauss, Riemann's teacher and precursor. The role of the discovery of the non-Euclidean geometry of Nikolai Lobachevsky and János Bolyai in Riemann's thinking is not clear, and the consensus appears to be that it was not important. Indeed, part of my argument here concerns some of the reasons why it might not have been. Lobachevsky's and Bolyai's work did not figure in Riemann's lecture and, unlike Gauss, neither was mentioned there. The question of the fifth (parallel) postulate of Euclid, central to the history that led to the discovery of non-Euclidean geometry, played no role in Riemann's argument either. Riemann pursued a different way of thinking, in part following Gauss's ideas concerning the curvature of two-dimensional surfaces. Riemann's thinking was also, and correlatively, problematic rather than axiomatic-theorematic (as was that of Lobachevsky and Bolyai), the juxtaposition that I explain below, merely noting for the moment that the axioms of geometry were neither Riemann's starting point nor figured significantly, if at all, in his investigation. Riemann's approach led him beyond a single alternative to Euclidean geometry to an uncontainable multiplicity of geometries and, in principle, to an even greater multiplicity of possible spaces, because some among them would not admit geometry. The latter circumstance became important for the development of topology, which, unlike geometry, deals with the architecture of a given space and associating numerical and algebraic entities with this architecture, rather than, as geometry, with the measurement of distances. Riemann's ideas, beginning with his concept of Riemann surfaces, which are two-dimensional manifolds with a particular type of topological structure, played a major role in the emergence of topology as a mathematical discipline.

The implications for physics, extending those of the discovery of a single non-Euclidean geometry, were dramatic as well, and as indicated above and as will be discussed in detail in Sect. 4, Riemann's contribution to physics in the lecture was nearly as revolutionary as his contribution to mathematics. For the moment, before the discovery of non-Euclidean geometry, one and only one geometry would be available for a geometrical description of physical space (which is, again, how the term space

[4] On Riemann's subsequent developments of his geometrical ideas and their application to physics and beyond, see Athanase Papadopoulos's contribution to this volume [22].

was understood at the time) if one assumes that space or, in Riemann's more rigorous phrasing, "the reality underlying space" could be described geometrically [31, p. 33]. This had been a grounding assumption of modern physics, from Galileo and Newton on, or of modern philosophy, from Descartes on. Kant's epistemology of phenomena (referring to appearances or representations constructed by our minds) vs. noumena or things-in-themselves (referring to how things, material or mental, exist independently of how we perceive or represent them) qualified this assumption [15]. While important, this qualification did not change the essential import of this assumption for physics. Kant's epistemology does not affect measurements that allow us to ascertain the observable properties of space, specifically whether the observable space is Euclidean (flat) or non-Euclidean (curved), or is three-dimensional (the only space we can phenomenally imagine) or not. Kant's epistemology even helps. Possible deviations from the flatness of physical space (or in any event, from what we refer to as physical space) could be established indirectly by using measuring instruments, while our phenomenal experience of space could still be Euclidean. Riemann reflected on this situation in his lecture, leaving the question of the ultimate nature of physical space or of the reality underlying space to the future physics. Kant, by contrast, ultimately assumed that physical space (or, again, whatever can be possibly referred to as physical space) is Euclidean and three-dimensional, or again, that it was unlikely to be anything else. He allowed that such alternatives are logically possible, but saw them as improbable, in part on theological grounds [23, pp. 3–4, 7]. These grounds have not deterred Riemann, who was originally trained in theology, which might, however, have affected his mathematical and philosophical thinking otherwise, possibly even his concept of manifold.[5] In Riemann's view, that space was a three-dimensional Euclidean manifold was a hypothesis, reasonably well confirmed by the measurements performed at the time, but a hypothesis nevertheless, the truth of which cannot be ascertained by reason alone, as Kant ultimately thought possible, although his position has further complexities, which cannot be addressed here.

In any event, the situation changed with the discovery of non-Euclidean geometry. That actual physical space may not be Euclidean (the three-dimensional nature of space was not contested) made this discovery a major event, even though no measurements made at the time showed any deviation from Euclidean geometry. The situation changed even more radically with Riemann, because his argument implied that an infinite number of possible geometries or, in principle, even topologies (some of which or rather the corresponding mathematical entities, which we now call spaces, would not possess geometry) could be associated with physical space or, again, the reality underlying space, that is, the reality underlying what appears to us as physical

[5] Although the term "*Mannigfaltigkeit*" was not uncommon in German philosophical literature, including in Leibniz and Kant, it is worth noting that the German word for the Trinity is "*Dreifaltigkeit*," thus, etymologically, suggesting a kind of "three-folded-ness," which could not have been missed by Riemann, or, for that matter, Leibniz and Kant. See [20] on the use of the term "*Mannigfaltigkeit*" in Kant vs. Riemann.

space.[6] Any such association is a hypothesis (this is what the hypotheses of Riemann's title refers to) and as such is subject to testing, verification, qualification, refinement, and so forth, which can rule out some among possible geometries or require different geometries at different scales, as indeed happens in modern physics. Thus, with the help of Einstein's general relativity (his non-Newtonian theory of gravity), we know reasonably well certain local physical geometries, say, the one, curved, in the vicinity of the solar system, and even more global geometries, say, that (on average flat) in the Milky Way. Or, to speak more rigorously and, arguably, closer to Riemann's way of thinking, the corresponding argumentation works well in physics and astronomy as things stand now. It is, however, much more difficult to be sure concerning the ultimate geometry of the Universe, although the current data seems to suggest that it is, on the average, flat, as far as we can observe it. Locally, space could be curved by gravity, in accordance with general relativity. This disregards possible quantum aspects of the reality underlying space, for example, as permeated by quantum fields, which may make this reality discrete, the possibility Riemann entertained in his lecture. This discreteness remains conjectural and it is not inherent in quantum field theory as currently constituted (in most interpretations), but it is envisioned in some versions of it, or its extrapolations beyond its current scope, for example, to the Planck scale. Still other versions or interpretations of quantum field theory or even quantum mechanics (the currently standard nonrelativistic, low energy, form of quantum theory) suggests that the reality of the ultimate constitution of nature, including possibly the reality underlying space are beyond representation or even conception, discrete or continuous, the possibility on which I shall further comment below. It is also possible (there are physical considerations that suggest this as well) that there are other Universes with different geometries and topologies. Riemann did not envision this possibility, which in part arises from quantum considerations concerning the Big-Bang origin of the Universe that we observe. In fact, he rejected, perhaps too hastily, any consideration of the Universe on such a scale, "in the infinitely large," as an idle speculation [31, p. 23]. However, his discovery of an infinite number of possible geometries is in accord with this idea, the genealogy of which goes back to Leibniz's concept of (com)possible worlds, although for Leibniz there is only one, the best one, in which we live and which is monadologically reflected in our thought. On the other hand, Riemann argued that "the reality underlying space" "in the infinitely small" is an important question: this reality may have a dimension higher than three (possibly even be infinite-dimensional), be discrete rather than continuous, and so forth. This question, he argued, could only be answered by physics, because this reality is defined by matter and forces acting upon it, rather than on the basis of purely philosophical considerations or "traditional prejudices" [31, p. 33]. This question, which was given new dimensions by quantum theory, is still with us,

[6]It is true that Riemann never considered or even mentioned this possibility, arguably, first expressly investigated by Poincaré, and it is not my aim to make a historical claim to the contrary. My point instead is that this possibility and, as will be explained below, the concept of topological space may be seen as conceptual implications of his argument. It is conceivable, especially given his concept of a Riemann surface, that Riemann entertained this type of idea, just as he (admittedly, expressly) entertained the idea that the reality underlying space may be discrete.

testifying to the remarkable reach and lasting significance of Riemann's thought for mathematics, physics, and philosophy.

2 Philosophy: Planes of Thought and the Architecture of Concepts

In *What Is Philosophy*? Deleuze and Guattari view thought [*la pensée*] as a confrontation between the brain and chaos. On the surface, this view is hardly surprising: much of our thinking (in the sense of mental states and processes) may be understood as this type of confrontation. Deleuze and Guattari, however, have in mind a special form of this confrontation, defined by their conception of thought as different from merely thinking and manifested especially in philosophy, art, and mathematics and science. While unremittingly at war with chaos, thought is also working together with chaos, rather than only protecting us against chaos, as do certain other forms of thinking, in particular, opinion. Deleuze and Guattari see chaos (which they also understand in a particular way, explained below) not only as an enemy but also as a friend of thought, its greatest friend and its best ally in a yet greater struggle, that against opinion, an enemy only, "like a sort of umbrella that protects us from chaos." As they say:

> [The] struggle against chaos does not take place without an affinity with the enemy, because another struggle develops and takes on more importance—the struggle against opinion, which claims to protect us from chaos... [T]he struggle with chaos is only the instrument in a more profound struggle against opinion, for the misfortune of people comes from opinion... But art, science, and philosophy require more: they cast planes over chaos. These three disciplines are not like religions that invoke dynasties of gods, or the epiphany of a single god, in order to paint the firmament on the umbrella, like the figures of an Urdoxa from which opinions stem. Philosophy, science, and art want us to tear open the firmament and plunge into chaos. And what would thinking be if it did not confront chaos? [8, pp. 203, 206, 202].

Chaos itself is "defined [by them] not so much by its disorder as by the infinite speed with which every form taking shape in it vanishes. It is a void that is not a nothingness but a virtual, containing all possible particles and drawing out all possible forms, which spring up only to disappear immediately, without consistency or reference, without consequence. Chaos is an infinite speed of birth and disappearance" [8, p. 118]. This is an unusual conception of chaos. Indeed, it does not appear to have been previously used in philosophy. It originates in quantum field theory and the concept of virtual particle formation there, as is suggested by the terms "particle" and "virtual," although "virtual" is also Deleuze's own philosophical concept [24]. This conception gives a particular form to thought's interaction with chaos. Thought extracts more stable forms of order from speedily disappearing forms of order inhabiting chaos, analogously to the way our measuring technology in high-energy physics extracts "real particles," as they are called, from the "foam" of continuously transforming "virtual particles": electrons into positrons or electron-positron pairs, either to photons, and for forth, in the case of quantum electrodynamics. The picture becomes still more complex (involving neutrinos, electroweak bosons, quarks, Higgs bosons, and so forth) in higher-energy quantum regimes, governed by other forms of quantum field theory [27, pp. 226–238].

Given the essentially creative nature of thought, thus defined, it is not surprising that philosophy, art, and science are among the primary means, and for Deleuze and Guattari are even the primary means, for thinking to become thought [8, p. 208]. Philosophy engages with chaos by creating concepts and planes of immanence; art by creating affects and planes of composition; and mathematics and science by creating functions and propositions, and planes of reference or coordination in science. These conceptions are intricate, and their fuller meaning will become apparent in the course of the discussion to follow. It suffices to say for the moment that the corresponding planes of immanence, composition, and reference, are defined by the movement of thought in each field, while concepts in philosophy, compositions in art, and functions (or other mathematical entities) and propositions in mathematics and science emerge from and are created by this movement.

The specificity of the workings of thought in each field makes them different from each other; and part of the project of *What Is Philosophy*? is to explore this specificity and this difference, in order to answer or (it might not be possible to ever answer it) to pose the title question of the book more sharply. However, the affinities and relationships among art, science, and philosophy are just as significant, and reflect a more complex landscape of thought, in which these fields and the interactions among them are positioned. Here, I shall address conceptual thought in philosophy and mathematics and science. I argue, again, against the grain of Deleuze and Guattari's argument, that creative thought in mathematics and science, and Riemann's thought in particular, are defined as much by the invention of new mathematical and scientific (in Riemann's case, specifically physical) concepts as is creative thought in philosophy by the invention of new philosophical concepts.[7] This argumentation does not negate that of Deleuze and Guattari. First of all, planes of reference, and mathematical entities, such as functions, or logical propositions, are unavoidable in and crucial to mathematics and science. Secondly, as noted from the outset, Deleuze and Guattari are ultimately unable to unconditionally maintain this distinction either. In particular, they are compelled to address the interferences among philosophy, art, and science, interferences essential even for the workings of any single field itself [8, pp. 216–218]. The present argument, which moves beyond only such interferences (found in Riemann's thinking as well), makes this distinction even less definitive and by doing so becomes even more open to the interactions between these fields (again, leaving art aside for the moment).

I shall now explain Deleuze and Guattari's *concept* of a philosophical concept. A concept is not only a generalization from particulars (which is commonly assumed to define concepts) or merely "a [single] general or abstract idea," although a concept may contain such generalizations and abstract ideas [8, pp.11–12, 24]. (Abstract

[7]Mathematics, science, and philosophy also involve the creation of compositions, found in artistic thought, and the latter may, conversely, involve planes of immanence and the creation of concepts, or planes of reference. For one thing, concepts thus defined are composed. More pertinently here, Riemann's concept of manifold is compositional because it defines a manifold as composed of local spaces [26].

ideas invoked here are not the same as abstract mathematical formations, which are, in the present view, concepts in Deleuze and Guattari's sense.) A concept is a multi-component entity, defined by the specific *organization* of its components, which may be general or particular, and some of these components are concepts in turn: "there are no simple concepts. Every concept has components and is defined by them. It is a multiplicity. There is no concept with only one component" [8, p. 16]. Each concept is a multi-component conglomerate of concepts (in their conventional senses), figures, metaphors, and so forth, which are conjoint in a heterogeneous, but interactive, architecture, and this multiplicity that does not amount to a unity, even if it is the unity of the multiple [8, pp.12–13]. It is the *relational organization* of a concept's components that defines it. The role of the multiple in the architecture of concepts is thus crucial. Some unification could take place within the architecture of a concept, but, again, without necessarily fully encompassing the multiplicity, at least a potential multiplicity, of this architecture. It is rare for a concept to have only one component, and ultimately impossible to do so. A single-component concept is only a product of a provisional cut-off of its multi-component organization. In practice, there are always cut-offs in delineating a concept, which results from assuming some of the components of this concept to be primitive entities whose structure is not specified. These components could, however, be specified by alternative delineations, leading to a new overall concept, containing a new set of primitive (unspecified) components. The history of a concept, and every concept has a history, is a history of such successive specifications and changes in previous specifications [8, p.17].

Consider the concept of "bird," beginning with its use in daily life. On the one hand, it may be seen as a single generalization. On the other hand, what makes this concept that of "bird" is the implied presence of components or sub-concepts, such as "wings," "feathers," and "beak," and the relationships among them. The concept acquires further features and components, and thus becomes a different concept, in zoology or biology (as reflected, for example, in the evolutionary relationships between birds and theropod dinosaurs). A philosophical concept of a bird is yet something else. According to Deleuze and Guattari: "a [philosophical] concept of a bird is found not in its genus or species but in the composition of its postures, colors, and songs, something indiscernible that is not so much synesthetic as syneidetic" (a product of the synthesis of the eidos, form, of each concept it absorbs) [8, p. 20].

Each concept is also defined as a problem (as multifaceted as the concept is), a definition that has a mathematical genealogy. A problem is not something that, like a theorem (in the direct sense of the term), is derived from assumed axioms by means of strict logical rules, but is something that is posed, created, along with a concept. A mathematical theorem could also be a problem, when it arises, as in Riemann, from mathematical concepts, rather than from axioms. A problem in this sense, while it must be solved, does not disappear in its solutions: it is "determined at the same time as it is solved" and is "at once both transcendent and immanent in relation to its solution," insofar as it leads to ever-new problems and concepts [6, p. 163]. This persistence helps to make a problem and the concept associated to it "always new," to live on [8, p. 5]. The invention and exploration of new, "always new," concepts, has, Deleuze and Guattari argue, defined the practice of philosophical thought from the pre-Socratics on.

I contend that the same type of argument could be made for the concepts invented in creative mathematics and science. Each mathematical concept (1) emerges from the cooperative confrontation between mathematical thought and chaos; (2) is multi-component; (3) is related to or is a problem; and (4) has a history. Thus, consider the concept of space, historically suspended between mathematics and physics (provisionally putting its philosophical aspects aside), with its constitutive concept-components, point, line, plane, distance, and so forth, each of which, just as the overall concept of space, has a long history of modifications, transformations, redefinitions, and so forth. To mark some of its junctures, by symbolically placing Riemann at the center of this history, this history extends from Euclid (who does not define space, but defines the components just listed) to René Descartes (a coordinate space) to Riemann (a space defined as a manifold) to Felix Hausdorff (topological space) to Alexandre Grothendieck (topos).

It is sometimes difficult to perceive this multi-component architecture of concepts in mathematics and science, because this complexity could be circumvented in their technical practice, in this respect in contrast to philosophy. A more conventional understanding of concepts (such as a generalization from particulars), joined with mathematical and scientific formulas and propositions, tends to suffice. This may be one of the reasons why Deleuze and Guattari (almost) deny that concepts in their sense are found in mathematics and science. They even declare (I think, quite misleadingly) that "it is pointless to say that there are concepts in science [including mathematics]" and adding "even when science is concerned with the same 'objects' [as philosophy] it is not from the view point of the concept; it is not by creating concepts" [8, p. 33]. This includes Riemann's thinking concerning manifolds and spatiality, even as they, at the same time, invoke "a Riemannian concept of space peculiar to philosophy," possibly also in Riemann's philosophical thought, but not, as I argue here, his mathematical or physical thought [8, p. 61]. For them, mathematical and scientific thought is limited to planes of reference, linked to the invention of functions (or other mathematical entities, for example, again, in Riemann) and propositions, and lacks planes of immanence, which make philosophical concepts possible [8, pp. 33–34, 132, 161]. Although their view is more ambivalent and complex than this brief summary and these unequivocal statements by them suggest, they do not extend their concept of concept or their conception (it is not quite a concept in their sense) of the plane of immanence to mathematics and science. By contrast, I argue that planes of immanence and the creation of concepts in this type of sense play central roles not only in creative philosophical thought but also in creative mathematical and scientific thought. In fact, very little of what they say about the *architecture* of philosophical concepts does not apply to mathematical and scientific concepts. Mathematics or science, certainly that of Riemann, is concerned with its objects (shared with philosophy or not) *from the viewpoint of concepts, by creating concepts.*

I am not disputing that mathematical and scientific thought also works with planes of reference, and, via planes of reference, with functions, propositions, and so forth.

Planes of reference give rise to these formations, which define the disciplinary nature of mathematics and science, and essentially shape mathematical and scientific thinking and practice—essentially, but, I would argue, not completely or even most centrally, at least in creative mathematics and science. One might say that creative or, to adopt Thomas Kuhn's language, revolutionary, thought in philosophy and mathematics or science alike is defined by planes of immanence and creation of concepts in Deleuze and Guattari's sense, which always carry individual signatures underneath them [8, p. 50]. Just as there are Plato's ideas, Descartes's cogito, and Leibniz's monads, there are Gauss's curvature, Riemann's manifolds, Dedekind's ideals, and Grothendieck's topoi in mathematics, or Einstein's spaces curved by gravity in general relativity and Heisenberg's matrix variables or Dirac's spinors in quantum physics. This is true even though the functioning of mathematical and scientific concepts does require planes of reference, functions (or other formal mathematical entities), propositions, and so forth.

The difference between philosophy and mathematics or science may instead be, to stay with Kuhn's idiom, in the nature of *normal*, rather than *revolutionary*, practice in each domain. In philosophy, the normal practice consists primarily in understanding, interpreting, and commenting on concepts, while in mathematics and science, the normal practice consists primarily in creating, by means of planes of references, frames of reference, functions and other mathematical or scientific formations, propositions, and so forth. It is true that for Deleuze and Guattari creative, revolutionary philosophical practice is the only true philosophy. However, leaving aside an arguably too restrictive character of this view of philosophy, this is not in conflict with the view of creative mathematics or science advocated here.

Deleuze and Guattari do allow that creative mathematical and scientific thought, such as that of Riemann (one of their primary examples), could have philosophical or artistic, compositional, aspects. But they appear to associate these aspects with *philosophical* or, in Deleuze's language, *inexact* (but philosophically rigorous) thought within mathematical or scientific thought. This philosophical thought is either operative alongside mathematical and scientific thought or enters by way of interference (in the positive sense of interfering wave fronts rather than in the more negative sense of inhibition) between mathematics and philosophy [8, pp. 217–218]. Both types of association are pertinent and important, certainly in Riemann's case. My argument is different, however. I argue that creative *technical*, *exact* mathematical and scientific thought is defined by planes of immanence and multi-component mathematical or scientific concepts, the architecture of which is analogous to that of philosophical concepts in Deleuze and Guattari's sense. That is, even apart from their philosophical strata, *mathematical* and *scientific* planes of immanence and the nature of mathematical or scientific concepts are *analogous* to philosophical thought. It is not only a matter of mathematical and scientific thought becoming philosophical at certain junctures, but a matter of the mathematical and philosophical thought creating parallel homomorphic (partially corresponding to each other), although not isomorphic (fully corresponding to each other), architectures of mathematical and philosophical concepts. They are not isomorphic because of technical, exact, aspects of mathematical and scientific concepts, demanded by the disciplinary nature of mathematics and science, aspects, generally, not found in philosophical concepts.

To bring this point home, I need to say more about Deleuze and Guattari's conceptions of planes of immanence and reference. The plane of immanence, as the plane of the movement of thought, is not "a concept that is or can be thought," but is "the image of thought, the image that thought gives itself of what it means to think" [8, p. 37]. As they say: "Concepts are like multiple waves, rising and falling, but the plane of immanence is the single wave that roles them up and unrolls them" [8, p. 36]. The present argument aims to extend, rather than to juxtapose, the plane of immanence (and the relationship between it and concepts) to mathematical and scientific thought, and to join this plane with the plane of reference. For Deleuze and Guattari, mathematics or science "relinquishes the infinite, infinite speed [of thought], in order to a gain *a reference able to actualize the virtual* [of chaos]. [It] gives reference to the virtual, a reference that actualizes the virtual through functions [or other mathematical objects]" [8, p. 118; translation modified]. Thought's enactment of this process constitutes a plane of reference. Planes of reference do play a major, indeed irreducible, role in mathematical and scientific thinking, especially in the disciplinary functioning of mathematics and science. This, however, is not inconsistent with the view that planes of immanence and the creation of concepts are found in mathematics and science. Mathematical and scientific thought combines both planes (sometimes, as does philosophy, also adding planes of composition) and creates its concepts from this fusion. The processes of thought defined the plane of immanence and specifically the creation of concepts (in Deleuze and Guattari's sense) are equally found in mathematics and science, and, again, define creative thinking there most essentially, analogously to the way it happens in philosophy. Such mathematical planes of immanence and the concepts they give rise to may coexist and interact with philosophical planes of immanence and concepts, but they are not reducible to philosophical planes and concepts. This is because of the equally irreducible interaction between these mathematical planes and concepts with technical, exact aspects of mathematical and scientific thinking.

Thus, in his rethinking of spatiality and geometry, Riemann, not only laid out a philosophical plane of immanence that gives rise to philosophical concepts and architectures, as Deleuze and Guattari rightly argue [7, pp. 483–486] but also introduced *a new mathematical plane of immanence*, which gives rise to multicomponent mathematical concepts, alongside *a mathematical plane of reference*. Riemann's thought shaped the plane of immanence of modern mathematical thought arguably more than that of any other mathematician (although Newton, Gauss, and Galois before Riemann, and Poincaré and Hilbert after him offer some competition). This plane extends well beyond geometry. Riemann made major transformative contributions, especially of a conceptual nature, to many areas of modern mathematics: geometry, topology, analysis, algebra, and number theory–not the least by bringing these fields to bear on each other, and his contributions to physics or philosophy were part of this interactive thinking and practice.

The interactive heterogeneity of Riemann's practice is a crucial aspect of Riemann's mathematics, and it extends beyond mathematics and shapes its mathematical operation from this exterior. Thus, the plane of immanence of Riemann's thought also has a more strictly philosophical dimension and thus creates more strictly philosophical concepts, emphasized by Deleuze and Guattari, rather than only mathematical concepts analogous to philosophical concepts by virtue of their multi-component architecture, stressed here. Riemann's philosophical thought was uncommonly and even nearly uniquely significant for his mathematical or physical thought, although one could think of a few competing cases, such as Weyl, who might have been inspired by Riemann in this respect as well, as he was by many other aspects of Riemann's thought. So were the physical dimensions of Riemann's thought, and this, too, is shared by Weyl, or, earlier Gauss, although this is more common. This role of physics in their work is also essentially connected to the role of philosophy there, which is, again, uncommon, if one speaks of such essential connections, making one's philosophical thinking a constitutive part of one's mathematical thinking, rather than of general philosophical reflections concerning mathematics and science, on the part of mathematicians and scientists.[8] The creation of new mathematics in Riemann was enabled by all three dimensions—mathematical, physical, and philosophical—of Riemann's thought and concepts. Riemann lays out a new plane of immanence of mathematical thought by changing both how to think about geometry or space and how to pursue thinking differently mathematically, via bringing together different mathematical fields and combining them with physical and philosophical concepts. A similar claim could also be made about his thinking concerning physics.

Riemann's Habilitation lecture is a magnificent unfolding of this plane (a geometrical metaphor of "plane" is fitting here). It is a major contribution not only to mathematics but also to physics and philosophy, especially to the philosophy of mathematics and physics, but far from exclusively so, as, for example, Deleuze and Guattari's use of Riemann's concepts shows. It is difficult to overestimate the significance and impact of Riemann's thinking concerning spatiality and geometry in mathematics and physics, in shaping the planes of immanence of thought of both. In mathematics the list of even major areas of impact is nearly inexhaustible, and I shall only mention a few of, arguably, the most important ones. First of all, Riemann's rethinking of geometry in terms of manifoldness crucially expanded the idea of the multiplicity of spaces and geometries themselves. Riemann's view of geometry as, in Deleuze and Guattari's language, topology and typology of manifolds led from the late-nineteenth century on to the extraordinary (a still ongoing) progress of geometry, beginning with the work of Sophus Lie and Felix Klein, and a bit later Élie Cartan [7, p. 483]. This work also connected differential geometry of manifolds and the theory of groups, specifically Lie groups. These connections eventually proved to have a major significance for quantum theory, especially in the theory of elementary

[8]Poincaré's extensive (much more extensive than Riemann's) philosophical works (e.g. [28, 29]), while influenced, as were his mathematical works in geometry, by Riemann's Habilitation lecture, may be seen along these lines. I realize that this claim may be challenged, and make it with caution. I would, nevertheless, argue that Riemann's philosophical thinking plays a greater constitutive role in his mathematical thinking than Poincaré's philosophical thinking in his mathematical thinking. The situation is of course different when it comes to physics, which is a major part of Riemann's and Poincaré's mathematical thinking alike, and both made major contributions to physics.

particles, which are classified by using Lie groups as symmetry groups, although in connection with the infinite-dimensional spaces. The concept of manifold was also crucial for the development of topology. Initially, it was Riemann's earlier work on Riemann surfaces that had a greater impact. Eventually, topology came to be defined by understanding, which, as I shall explain, follows Riemann, of topological spaces as composed of local neighborhoods of points and (open) subspaces of a given space. It is true that Riemann did not have a concept of topological space. My main concern, however, is the development, transformation of Riemann's concepts (such as manifold) leading to new concepts, such as topological space, which makes Riemann's concepts alive, makes them "live on" in new concepts.

3 Mathematics: Space, Geometry, and the Concept of Manifold

The significance of Riemann's lecture for mathematics, physics, and philosophy, and its impact in all three fields were immense. This impact was delayed until its publication, in 1868, two years after Riemann's death, and fourteen years after it was presented in 1854, although some of Riemann's key ideas contained there and in Riemann's related works became known and had their impact earlier. One can only surmise (a tantalizing surmise!) the consequences for the history of mathematics and physics if the lecture was published more immediately. Riemann opens as follows:

> As is well known, geometry presupposes the concept of space, as well as assuming the basic principles for construction in space. It gives only nominal definitions of these things, while their essential specification appears in the form of axioms. The relationship between these presuppositions is left in the dark; we do not see whether, or to what extent, any connection between them is necessary, or a priori whether any connection between them is even possible.
>
> From Euclid to Lagrange this darkness has been dispelled neither by the mathematicians nor the philosophers who have concerned themselves with it. The reason [ground] [*Grund*] for this is undoubtedly because the general concept of multiply extended magnitudes [*Grösse*], which includes spatial magnitudes, remains completely unexplored. I have therefore first set myself the task of constructing the concept of a multiply extended magnitude from general notions of magnitude. It will be shown that a multiply extended magnitude is susceptible of various metric relations, so that space constitutes only a special case of a triply extended magnitude. From this, however, it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of magnitude, but that those properties that distinguish space from other conceivable triply extended magnitudes can only be deduced from experience. Thus arises the problem of seeking out the simplest data from which the metric relations of space can be determined, a problem that by its very nature is not completely determined, for there may be several systems of simple data that suffice to determine the metric relations of space; for the present purposes, the most important system is that laid down as a foundation of geometry by Euclid. These data are—like all data—not logically necessary, but only of empirical certainty, they are hypotheses [*Hypothesen*]; one can therefore investigate their likelihood, which is certainly very great within the bounds of observation, and afterwards decide on the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large [*Unmessbargrosse*] and in the direction of the immeasurably small [*Unmessbarkleinen*]. [31, p. 23; translation modified]

These introductory reflections are already profound and far-reaching, and Riemann develops the ideas suggested here quite a bit further in the lecture. First of all, Riemann does not appear to be interested in the axiomatic approach to geometry and is even suspicious of axioms of geometry. In contrast to most previous works on non-Euclidean geometry, the parallel postulate is not the starting point of his investigation. The reason is clearly that the concepts of space and of measuring distances in space are not adequately defined. This is what leaves "the relationships between [axioms] in the dark. We do not see whether, or to what extent, any connection between them is necessary, or a priori whether any connection between them is even possible." This questioning of the axioms of geometry was unusual at Riemann's time, and it was much deeper than customary doubts concerning the parallel postulate. The axiomatic approach to non-Euclidean geometry is, thus, abandoned and even implicitly questioned by Riemann, even though it was this approach that led to the discovery of non-Euclidean geometry by Lobachevsky and Bolyai. This may have been one of the reasons why only Gauss, rather than either Lobachevsky (highly regarded by Gauss) or Bolyai, was expressly invoked in the lecture.[9] It is true that Gauss was ultimately unable to establish a possible existence (in the sense of logical consistency) of non-Euclidean geometry, because, unlike Lobachevsky and Bolyai, he did not do this for the three-dimensional case. In spirit, however, Gauss's work on the geometry of two-dimensional surfaces and his concept of curvature as intrinsic to a given surface was much closer to Riemann. For Riemann, Gauss was pursuing a trajectory of thought better suited for the foundations of geometry and more fruitful for its conceptual development. In his famous extraordinary theorem (*theorema egrerium*, as he called it), Gauss proved that the curvature of a surface, which he defined as well, was intrinsic to the surface. More precisely, the theorem states that the Gaussian curvature of a surface does not change if one only bends the surface but does not stretch it. This means that curvature can be fully determined by measuring angles and distances on the surface itself, without considering the way it is embedded in the ambient (three-dimensional) Euclidean space or, in Riemann's terms, manifold, which makes the Gaussian curvature an intrinsic invariant of a given surface.

One nearly has here a two-dimensional conceptual architecture that suggests and even approaches that of Riemann, although major additional thinking is necessary to give this architecture the type of generality Riemann is able to do. Gauss's theorem suggests that one could see the surface as an independent curved manifold and then to generalize this concept to higher dimensions via "the concept of a multiply extended magnitude," mentioned in this passage and the concept of manifold, which is what Riemann did. These concepts also helped him to generalize to higher dimensions Gauss's concept of curvature. To do so required yet another new concept, another

[9] There were other, more extrinsic, reasons, beginning with the fact that Gauss was Riemann's mentor and the chair of his Habilitation committee. Indeed, Gauss selected this topic among three proposed by Riemann (following the rules). The philosopher R. H. Lotze, a fervent opponent of non-Euclidean geometry, was a member of the philosophy faculty, to which Riemann's Habilitation was presented. Later on, Lotze criticized Riemann's approach anyway, as part of his general critique of non-Euclidean geometry (see [16, pp. 222–226] and [33, pp. 97–112]).

great invention of Riemann, the tensor of curvature, and a new form of differential calculus, tensor calculus on manifolds, a generalization of differential calculus. This calculus was fully developed later on, and it played a major role in Einstein's general relativity. The independence of the curvature or of geometry of a given manifold of any dimension from its embedding also allows one to determine, at least in principle, intrinsically whether this space is flat or curved, which is crucial for determining the nature of the physical space we inhabit. For example, as indicated earlier, space is curved in the immediate vicinity of the solar system, because of the gravity of the Sun, planets, and other material entities in the solar system (or locally around other stars), but appears to be on the overage flat on the scale of the observable Universe.[10] In any event, by virtue of inventing his concepts and by using them, Riemann provided a general rigorous grounding to all geometry, Euclidean or non-Euclidean, rather than merely establishing, as Lobachevsky and Bolyai did, the logical possibility of the non-Euclidean geometry of (constant) negative curvature in three dimensions. The absence of a concept, such as that of manifold, leaves us in "the dark," because "we do not see whether, or to what extent, any connection between [axioms] is necessary, or a priori whether any connection between them is even possible."

Riemann's thinking is conceptual-problematic rather than axiomatic-theorematic: he grounds mathematics, as well as physics, in hypotheses and concepts, concepts-problems, such as the concept of manifold and its subconcepts (distance, curvature, tensor, and so forth). The problem of space and geometry is now posed in terms of finding concepts that are necessary to define space and to give it geometry. A general concept of manifold was assumed to be applicable to any possible space, while specific manifolds, flat or curved, define different spaces or subspaces. In this respect, as foundational thinking, that of Riemann is different from that conforming to Hilbert's concept of "foundations" [*Grundlagen*] in the sense of axiomatic foundations, an idea that Hilbert also tried to apply to physics. One of the problems of his famous 1900 list (Problem 6) was in fact that of the development of (mathematized) axiomatic foundations, system(s) of axioms, for physics, on the model of his own *Foundations of Geometry* [13].[11] That such a system is possible is inevitably a hypothesis. It became clear subsequently that the existence of such a system is a hypothesis even in mathematics, and in view of Gödel's incompleteness theorems in 1931, ultimately a wrong hypothesis, insofar as such a system cannot be proven to be free of contradiction, once it is large enough to include arithmetic, as geometry is.

---

[10] As I qualified earlier, at least this is a workable and widely accepted view, widely but not universally. It has never been established definitively or, in any event, agreed upon whether such a determination is ever rigorously possible, as opposed to having a practically effective and possibly, within its proper limits, the best available theory or, as Poincaré would have it, "convention," without making a real claim concerning "the reality underlying space" [31, p. 33]. Einstein had his doubts too, although he was ultimately inclined to accept the possibility of such a determination, at least in principle, as, it appears, was Riemann, but not Poincaré, with whose position Einstein, nevertheless, had to contend and which he tried to accommodate within his own (e.g. [11, pp. 324–328]).

[11] For the development of Hilbert's ideas, as reflected in different editions of the *Grundlagen*, see [4]. In the first version of the book, Hilbert was closer to Riemann, and he later returned to a more Riemannian view of geometry in the wake of general relativity to which he made important contributions.

While Hilbert's foundational thinking aimed to bring physics closer to mathematics, even to make it mathematics, by giving physics an axiomatic form, that of Riemann brings mathematics closer to physics by grounding it in hypotheses and concepts, rather than in axioms.[12]

Riemann's approach and his concepts arise from the plane of immanence, at once, mathematical, physical, and philosophical, a plane defined by thinking in terms of multi-component concepts, such as that of multiply extended magnitude and manifoldness, rather than in terms of axioms and propositions, an approach that has defined most thinking concerning geometry before and even after Riemann. Riemann's thought is defined by a plane of immanence by virtue of giving rise to multicomponent concepts, rather than only creating frames of reference, functions, or propositions, from a plane of reference, although this is necessary as well. In Riemann, both planes are joined, as they must be in the creation of mathematical or scientific concepts. Thus, functions define both local neighborhoods (as infinitesimally Euclidean) and how local spaces are connected or pass into each other, and metrical relations and curvature, although curvature is ultimately defined by tensors, which are more complex entities.

The parallel postulate is, again, never mentioned in the lecture, although Euclidean geometry is invoked there as the geometry of "flat space," merely a particular and very special case of geometry, where metrical relations take an especially simple form, defined by the Pythagorean theorem. In Gray's words, "[In Riemann] geometry no longer starts with Euclidean geometry" [11, p. 52]. It is the metrical relation characterizing a given space that defines a possible geometry. In other words, the character of this relation is a hypothesis on which a geometry could be based, a hypothesis to be tested physically in order to establish whether such a space corresponds to the actual physical space. Non-Euclidean geometry (Riemann, again, does not use the term) is introduced as such a possible case of geometry, that of a curved, rather than flat, space of either negative or positive curvature, defined by Riemann by the corresponding type of quadratic form determining the metric. Riemann, thus, not only introduces a more general concept of geometry, but also gives a more rigorous conceptual grounding to non-Euclidean geometry of either negative or positive curvature. While the non-Euclidean geometry of negative curvature (hyperbolic geometry) was discovered before Riemann, Riemann's lecture introduced the three-dimensional non-Euclidean geometry of positive curvature (elliptical geometry), keeping in mind that space for Riemann meant the three-dimensional physical space. (The two-dimensional spherical geometry was considered well before Riemann.) Although eclipsed by Riemann's overall achievement in the lecture, this

[12]One might challenge this argument on historical grounds because it would have been difficult, if not impossible, to present a concept such as that of manifold in axiomatic form at the time of Riemann's lecture. That may be true. My point, however, is that Riemann's alternative, conceptual-problematic rather than axiomatic-theorematic, thinking, could still be contrasted to that of Hilbert and lead to a different type of mathematical thinking. It is difficult to say how Riemann would have approached the foundations of geometry if he had means of axiomatizing his concepts. On the other hand, it is possible to argue, as I do here, *on historical grounds*, that Riemann, unlike his predecessors, Lobachevsky and Bolyai, was not pursuing an axiomatic approach to geometry.

was a major mathematical discovery with important cosmological implications, for example, in its anticipation of the idea, later considered by Einstein, that the universe may be unbounded and yet finite.[13]

Riemann defines the concept of space, again, understood as physical space (as against, the concept of manifold, which is mathematical), as a three-dimensional instance of the concept of continuous manifoldness, in accordance with the hypotheses that he assumed as likely given the experimental data then available. (There is still no definitive data to refute this view, unless perhaps in the very small, say, at the Planck scale.) While a manifold, as defined by Riemann, may be either discrete or continuous, the concept of continuous manifoldness has a richer and more complex architecture, and most of Riemann's lecture is devoted to it. Technically, continuous manifolds considered by Riemann were differentiable manifolds, which means that one can define differential calculus on them. Indeed, they are metrical manifolds, now called Riemannian, which allow for the concept of distance between any two points and thus for geometry.[14] I shall, however, speak of continuous manifolds, following Riemann and his juxtaposition between continuous and discrete manifoldness. In modern use, the term manifold more customarily refers to continuous (but not necessarily differentiable) manifolds, although one also refers to discrete manifolds, which have topological dimension zero, as zero-manifolds. As defined by Riemann, discrete and continuous manifolds do not appear to have that much in common, and in effect form two different concepts.[15]

Riemann's concept of continuous manifoldness was a new concept of geometrical multiplicity. It is a multiplicity of local subspaces, most specifically those, "neighborhoods," associated with each point, out of which a given space is com-

---

[13] It would be similar to the three-dimensional sphere. As I explained, the currently dominant view or hypothesis (which appears to be confirmed by cosmological measurements) is that the universe is on average flat and is expanding.

[14] As most of his contemporaries, Riemann did not distinguish continuous and differentiable manifolds. It became eventually clear, however, that not all continuous (also called topological) manifolds are differentiable. There are topological manifolds with no differentiable structure, and some with multiple non-diffeomorphic differentiable structures. Thus, there is a continuum of non-diffeomorphic differentiable structures of $\mathbf{R}^4$.

[15] These two concepts could, especially in modern understanding, be subsumed under the same concept. This is because all zero-dimensional manifolds, which are discrete manifolds in Riemann's terms, are continuous (topological) manifolds. In fact they are also differentiable manifolds, because transition functions for them are constant functions, which are continuous and even differentiable. In modern terminology, the distinction between continuous and discrete manifolds in Riemann's lecture would be interpreted as that of zero-dimensional manifolds and positive dimensional manifolds. I am grateful to Ken'ichi Ohshika for helping to clarify this point. It is not inconceivable that Riemann's thought along similar lines, which would explain his choice of the term manifold for both discrete and continuous manifolds, although the term had a more general use at the time. (Georg Cantor, possibly influenced by Riemann's lecture, initially referred to sets as *Mannigfaltigkeiten* but eventually switched to *Mengen*.) It is, however, difficult to be certain on the basis of his lecture or his other writings. I would argue that the difference between these two types of manifolds is still crucial, both in general and for Riemann, especially for his analysis of physical space and geometry. Riemann stressed the significance of the relationships between continuity and discontinuity for mathematics, physics, and philosophy (e.g. [32, pp. 515–524]; [9, pp. 77–80]).

posed. A continuous (differentiable) manifold is understood by Riemann on the model of two-dimensional surfaces, which, as explained earlier, were defined by Gauss in terms of their intrinsic geometry. Riemann defines first the concept of "*n*-dimensional magnitude," which allows one to determine a position in a manifold by *n* numerical determinations, in the same way a position is determined by coordinates in the Euclidean space of *n* dimensions. Riemann is rigorous to extend (scale down) the concept of manifold to one-dimensional manifolds, curves, which, however, also helps him to built up the concept *n*-dimensional manifold by analogy. He starts with the concept of "a simply extended [one-dimensional] manifold, whose essential characteristic is that from any point in it a continuous movement is possible in only two directions, forwards and backwards." Then, he defines a two-dimensional or "a doubly-extended manifold" by saying that "if one now imagines that this [one-dimensional manifold] passes to another, completely different one, and once again in a well-determined way, that is, so that every point passes to a well-determined point of the other, then the instances for, similarly, a double extended manifold" [31, p. 25]. In other words, one continuously "fills" a surface with curves. Then, one similarly defines a triply extended manifold by imagining a similar continuous passing of a doubly extended manifold to another, thus continuously filling a three-dimensional object with two dimensional-ones, and so forth. "This construction," Riemann says, "can be characterized as a synthesis of a variability of $n + 1$ dimensions from a variability of *n* dimensions and a variability of one dimension" [31, p. 25]. This construction may have been one of the reasons for his use of the term: a manifold is literally a continuous fold(er) of manifolds of lower dimensions. Conversely, one can unfold a variability of *n* dimensions, which allows one to determine a position in a manifold by *n* numerical determinations, generalizing the way a position is determined by coordinates in the Euclidean space of *n* dimensions.

The most defining feature of the concept of manifold (under the assumption than one can measure the length of line-segments, straight or curved) is that it is conceived as infinitesimally Euclidean. This makes a continuous manifold into a conglomerate of local, continuously connected, small open neighborhoods around each point. The concept of neighborhood, again, assumed to be infinitesimally flat and Euclidean, is a component-concept of the concept of manifoldness. This concept of manifold as composed out of local neighborhoods is extendable to a still more general concept of topological space, in which case local neighborhood need no longer be Euclidean and can be defined with a great degree of generality. It is true that Riemann did not have a concept of topological space, in contrast to his concept of a Riemann surface, which had a more direct and immediate impact on the development of topology as an independent mathematical discipline. I would argue, however, that the conceptual architecture defining topological spaces is a generalization of that of Riemann's conceptual architecture of manifoldness as a "space" composed of neighborhoods, or generalizing it even further to other "spaces," a conception that, as will be seen presently, extends to Grothendieck's topos theory.[16] Admittedly, this architecture is presented here in the spirit of modern axiomatic thinking rather than in the spirit of Riemann's conceptual-problematic thinking, but it does, I think, inherit Riemann's conceptual architecture defining his concept of manifold. The theory of Riemann surfaces, too, came to be recast in terms of manifolds. Weyl, who, as his title *The Concept of a Riemann Surface* stated, considered a Riemann surface to be a concept, was the first to expressly define Riemann surfaces as manifolds [20, 35]. Riemann, however, undoubtedly realized that they were manifolds, and they were part of the genealogy of the concept of manifold.

In the case of Riemannian manifolds, while each neighborhood is infinitesimally flat, Euclidean, the manifold as a whole is, in general, not, except in the special case of flat, Euclidean manifolds. A manifold may be negatively or positively curved, and, which is another major innovation of Riemann, this curvature can also be variable. Riemann defined the metric form as a quadratic differential form, by the only formula in his lecture (discounting the coordinate expression for the line element), and assumed that the transition from one local coordinate system to another was differentiable. Thus, he, again, de facto, considered differentiable manifolds with positive definite metrics, Riemannian manifolds. In modern terms, such a manifold is defined by using a differentiable section of positive-definite quadratic forms on the tangent-bundle. While, however, modern technical language can bring out deeper mathematical aspects of Riemann's concepts, it can also displace how Riemann thought, mathematically, physically, and, especially, philosophically, a displacement sometimes found in twentieth-century English translations of Riemann's works, including his Habilitation lecture. One is, accordingly, always in complex negotiations between Riemann's and contemporary technical language, even though and because Riemann is so often ahead of his time, so much our contemporary.

Another important and equally future-oriented conceptual aspect of Riemann's approach is that it allows one to define a geometrical or, more generally, topological space (in modern terminology) not as a multiple, say, a set of points, but as a space that could be covered by maps (Euclidean in the case of manifolds) and in its relation to other spaces. As just explained, in part following Riemann's way of thinking, topology describes a given space not only in terms of its points, continuously connected to each other, but also and most essentially in terms of its open neighborhoods around each point. These neighborhoods are subspaces of this space, the idea that, again, underlies Riemann's concept of manifold, in this case, however, giving each neighborhood a Euclidean geometry. The approach, again, enabled Riemann to define manifolds of any dimension, even infinite-dimensional ones, in terms of its inner properties rather than in relation to the ambient Euclidean space, where a manifold could be placed, against the flat Euclidean background. It is true that, if one appeals, as is usual even in considering Riemann, to open sets, this concept of

[16] It is also worth recalling in this connection that Grothendieck's initial primary areas of mathematical research concerned topological vector spaces, which suggests yet another genealogical line in the history of the (broadly) Riemannian problematics in question here.

space retains the concept of set (of points) as a primitive concept.[17] Riemann's way of thinking concerning manifolds, however, also suggests a possibility of thinking of and even defining a space in terms of its relations to other spaces, which allows one to use this structure as more primordial by replacing covering a space by *open sets (of points)* with covering it by *open spaces*. A topological space, defined above in set-theoretical terms, becomes a collection of open spaces as sub-spaces with certain (algebraic) rules for the relationships between them.

This way of defining space by its relation to other spaces (as opposed to their constitution as sets of points) leads all the way to Grothendieck's topos theory, inspired by Riemann's ideas of manifolds and of the so-called covering spaces, originating in Riemann's theory of Riemann surfaces. Although it extends far beyond the question of spatiality, including mathematical logic, topos theory is arguably the farthest and most abstract extension of the concept of spatiality available, if one can rigorously speak of spatiality in this case, given an essentially algebraic nature of the concept. It does, however, give important new dimensions to our understanding of spatiality, when we deal in mathematics and elsewhere with objects or concepts that are considered in spatial terms.

It would not be possible here to present topos theory in its proper abstractness and rigor, sometimes prohibitive even for those not trained in the field of algebraic geometry or mathematical logic, where the concept is used as well (e.g. [18]). The essential philosophical ideas involved may, however, be sketched, as an example of both a rich mathematical concept in its own terms and of Riemann's influence on modern mathematics.[18] First, very informally, consider the following way of endowing a space with a structure, generalizing the definition of topological space. One begins with an arbitrarily chosen space, X, potentially any given space, which may initially be left unspecified in terms of its properties and structure. What would be specified are the relationship between spaces applicable to X, such as mapping or covering one or a portion of one, by another. One calls this structure the arrow structure $Y \rightarrow X$ (X is the space under consideration), where the arrow designates the relationship(s) in question. One can also generalize the notion of neighborhood or of an open subspace of (the topology of) a topological space in this way, by defining it as a relation between a given point and space (a generalized neighborhood or open subspace) associated with it. This procedure enables one to specify a given space not

[17]Thus, Ferreirós's discussion of Riemann in [11, pp. 39–80] appears to me to displace Riemann's thinking into the axiomatic and set-theoretical register, dominant in the wake of Cantor, a displacement arguably due to Ferreirós's insufficient attention to the nature of Riemann's mathematical concepts, to Riemann's concept of mathematical concept. In fairness, Ferreirós does relate Riemanns view of axioms to his concept of "hypothesis" and distinguishes it from the understanding of axioms developed in the twentieth-century philosophy of mathematics and mathematical logic. It does not appear to me, however, that Riemann thinks either in terms of axioms or, especially, in terms of sets (of points), as Ferreirós contends, although it could be and subsequently has been translated into these terms (e.g. [22]). See also Note 12 above.

[18]It would be instructive on both counts, to consider, as part of this genealogy, Dedekind's and Noether's work in algebra, reflecting the impact of Riemann's work on modern algebra, and even apart from his work on the distribution of primes and his famous hypothesis concerning the ζ-function. See [19] on Noether's work in this connection.

in terms of its intrinsic structure (e.g., a set of points with relations among them) but sociological[ly], throughout its relationships with other spaces of the same category, say, that of Riemannian spaces as manifolds [17], p. 7]. Some among such spaces may play a special role in defining the initial space, *X*, and algebraic structures (such as homotopy and cohomology, as Riemann realized in the case of covering spaces over Riemann surfaces. Indeed, the concept of covering space was one of the main inspirations for Grothendieck's concept of topos. The so-called *étale* topos (of a scheme), one of the main motivations for the concept of topos, is directly linked to the concept of covering space, as the term *étale* suggests.

To make this scheme more rigorous and to explain (albeit still quite informally) the concept of topos, I need to explain in my own words category theory. It was introduced in as part of the cohomology theory in algebraic topology in 1940 and later extensively used by Grothendieck in his approach to algebraic geometry, leading to the concept of topos. Category theory considers multiplicities (which need not be sets) of mathematical objects conforming to a given concept, such as the category of Riemannian manifolds, and the arrows or morphisms, the mappings between these objects that preserve this structure. Studying morphisms allows one to learn about the individual objects involved, often to learn more than we would by considering them only or primarily individually. In a certain sense, by appealing to the conceptual determination of each manifold, Riemann already thinks categorically. Thus, one does not have to, and Riemann does not, start with a Euclidean space, whether seen in terms of sets of points or otherwise. Instead the latter is just one specifiable object of a large categorical multiplicity, here that of the category of Riemannian manifolds, an object marked by a particularly simple way we can measure the distance between any two points. Categories themselves may be viewed as such objects, and in this case one speaks of "functors" rather than "morphisms." Topology relates topological or geometrical objects, such as manifolds, to algebraic ones, especially, as in the case of homotopy and cohomology theories, groups, a concept, it is true, not used by Riemann, as against Poincaré, who made it central to his geometrical and topological thinking, which established his uniquely significant role in the rise of algebraic topology. Thus, in contrast to geometry (which relates its spaces to algebraic aspects of measurement), topology, almost by its nature, deals with functors between categories of topological objects, such as manifolds, and categories of algebraic objects, such as groups.

Now, a topos in Grothendieck's sense is a category of spaces and arrows over a given space, used especially for the purpose of allowing one to define richer algebraic structures associated with this space, as explained above. There are certain additional conditions such categories must satisfy, but this is not essential at the moment. To give a simple example, for any topological space *S*, the category of sheaves on *S* is a topos. The concept of topos is, however, very general, and extends far beyond spatial or space-like mathematical objects (thus, the category of finite sets is a topos); indeed it replaces the latter with a more algebraic structure of categorical and topos-theoretical relationships between objects. On the other hand, it derives from the properties of and (arrow-like) categorical relationships between properly topological objects, such as Riemann surfaces or manifolds. The conditions, mentioned above, that categories that form topoi must satisfy have to do with these connections. The concept of topos is especially suited to deal in the way we do with standard manifolds with objects, such as certain (discrete) algebraic varieties, which are solutions of polynomial equations and are space-like, that cannot be meaningfully defined otherwise sufficiently analogously to continuous spaces, specifically in order to define nontrivial cohomology or homotopy groups for them. This had been an outstanding problem of algebraic geometry, arising from the so-called Weil conjectures for algebraic varieties over finite fields, which was solved with the help of topos theory, specifically the concept of *étale* topos, mentioned above (e.g. [1]). What both types of objects now share are analogously defined topoi associated with them and, as a result, analogously defined algebraic structures associated with them, equally enabling the necessary functoriality in both cases.

Topos theory allows for such esoteric constructions as non-trivial or non-punctual single-point "spaces" or, conversely, spaces (topoi) without points (first constructed by Pierre Deligne), sometimes slyly referred to by mathematicians as "pointless topology." Philosophically, this notion is far from pointless, especially if considered within the overall topos-theoretical framework. In particular, it amplifies a Riemannian idea that "space," especially is defined by its relation to other spaces, as a more primary object than a "point" or, again, a "set of points." Space becomes a Leibnizean, "monadological" concept, insofar as points in such a space (when it has points) may themselves be seen as a kind of monads, thus also giving a non-trivial structure to single-point spaces. These monads are certain elemental but structured entities, spaces, rather than structure-less entities (classical points), or at least as entities defined by (spatial) structures associated to and defining them [3]. Naturally, my appeal to monads here is qualified and metaphorical. Leibniz's monads are elemental souls, the atoms of soul-ness, as it were. But one might say that the space thus associated to a given point is the soul of this point, which defines its nature or structure, not unlike an infinite-dimensional Hilbert space associated with an elementary particle, such as an electron, in quantum mechanics and enabling us to predict its behavior. In other words, not all points are alike insofar as the mathematical (and possibly philosophical) nature of a given point may depend on the nature or structure of the space or topos to which it belongs or with which it is associated in the way just described. This approach also gives a much richer architecture to spaces with multiple points, such as Riemann's manifold (in which this architecture is inherent), and one might see (with caution) such spaces as analogous to Leibniz's universe composed by monads. It also allows for different (mathematical) universes associated with a given space, possibly a single-point one, in which case a monad and a universe would coincide. Grothendieck's topoi are such possible universes, possible worlds, or even com-possible worlds in Leibniz's sense, without assuming, like Leibniz (in dealing with the physical world), the existence of only one of them, the best possible one.

The outcome of Riemann's investigation into the foundations of geometry was, thus, a new mathematics of great generality, power, and potential, which involved not only new geometry, but also new topology and analysis (the tensor calculus on manifolds). Although it was Riemann's theory of continuous or, again, differentiable manifolds that had the greatest impact, the concept of discrete manifolds was important for Riemann's argument, and it is important for the modern understanding of both spatiality and geometry in mathematics, physics, and philosophy. While a discrete manifold has topological dimension zero, it may still be seen as multiply extended, if defined as forming a very fine lattice with very small intervals between points, which can be "filled," as it were, to form a continuous space of the corresponding topological dimensions. It is also possible to introduce metrical relations for discrete manifolds. This concept is important in the context of the relationships between physical, dynamical forces in nature and the nature of space or, to return to Riemann's terms, the physical "reality underlying space," although Riemann is cryptic on such metrical relations. Crucially, however, he does allow for the possibility that the physical "reality underlying space" might be "a discrete manifold" [31, p. 33]. This possibility has been entertained even before Riemann and has been even more often considered since, especially more recently. Also, mathematically, finite geometries were beginning to be developed, usually in more axiomatic ways, around Riemann's time as well, later on also under the impact of his geometrical thinking.[19] As indicated earlier, Riemann saw the relationships between continuity and discontinuity as foundationally central to mathematics, physics, and philosophy (e.g., [32, pp. 515–524]; [10, pp. 77–80]), a view confirmed by the subsequent developments in the foundations of mathematics, from Dedekind and Cantor on, and quantum physics. The latter uses continuous (technically, differential) mathematics to predict, in probabilistic terms, irreducibly discrete phenomena, that is, phenomena that are not, and that possibly cannot be, assumed to be connected to each other by a continuous physical process [27, pp. 232].

4 Physics: "The Reality Underlying Space"

As noted from the outset, Riemann's contribution to physics in his lecture was as important as his contribution to mathematics there. Riemann's terms space and geometry refer to the three-dimensional space and its geometry, in accordance with the use of these terms at the time, although this was soon to change. We now speak not only of manifolds of any dimensions, as Riemann does, but also of their geometry and refer to them as "spaces," or of discrete spaces and geometries, without necessarily assuming any connections between these objects and physical (or phenomenal) spatiality. While Riemann allows that "the reality underlying space" may prove to be discrete at a very small scale, this is not the same as extending, as was done subsequently, the terms "space" and "geometry" to finite entities defined by certain geometrical-like properties. On the other hand, closer to Riemann's view of what "the reality underlying space" could in principle be, some physical theories, dealing with such connections, suggest that the ultimate reality underlying space might be discrete or, as in superstring and brane theories, that physical space has a dimension

[19] For the discussion of some of these developments, see the chapter by V. Pambuccian, H. Struve, and R. Struve [21] and other chapters in the part of this volume that addresses later developments of Riemann's work.

higher than three (most commonly nine). While there is no experimental evidence thus far to support either claim, there are legitimate theoretical considerations in their favor.[20] Quantum theory also uses spaces of infinite dimensions, Hilbert spaces over complex numbers, although, as indicated above, for the purposes of predicting quantum events without representing physical space or physical processes in space and time. This is, admittedly, not the type of the relationships between mathematics and physics that Riemann appears to have entertained, but it may still be seen as in the spirit of Riemann's thinking concerning these relationships.[21] All this was to come later, however.

In Riemann's view, mathematically, one can define a general concept of manifold and the concept of metric relations in this manifold. These relations define a flat or curved nature of a given manifold, unless a manifold is discrete, in which case the metric relations, which could be defined for them, are no longer related to flatness or curvature of the manifold. This concept can then be used, suitably specified, to represent physical space and geometry there. Riemann considers this situation in more detail in the last chapter of the lecture, entitled "Applications to Space." As we have seen, however, he makes clear from the outset of the lecture that any such use can only be based on hypotheses that we form concerning space, and which we can then test. These hypotheses ground our thinking concerning space and geometry, although they may also reciprocally arise from this thinking from our previous hypotheses that we have tested or what we assumed, axiomatically, as self-evident. Hence, the fact "that a multiply extended magnitude is susceptible of various metric relations, so that space constitutes only a special case of a triply extended magnitude" implies that "those properties that distinguish space from other conceivable triply extended magnitudes [manifolds] can only be deduced from experience" [31, p. 23]. By "experience" Riemann means an experimental determination of the nature of physical space, rather than our phenomenal experience, although the latter may and even must play a role in this determination.

In other words, Riemann argues as follows, by both, in a very modern or even, *avant la lettre*, "modernist" way, separating mathematics from physics, making it independent, and then reconnecting them, an approach adopted by Einstein, expressly following Riemann, in creating general relativity [11, pp. 325–327]. There is math-

[20]The higher-dimensional spaces of superstring theory have been extensively discussed in literature and can be safely bypassed here, pertinent as their geometrical and topological features are. I would like, however, to mention a recent investigation, along quantum-informational lines, of the possibility that the reality underling space is discrete at the Planck scale, with a radical implication that the Lorentz invariance and hence special relativity is broken at the Planck scale as well [5] [27, pp. 259–262]. The article is also innovative mathematically in its use of geometric group theory, which emerged from Gromov's realization, Riemannian in spirit, that mathematical objects, such as groups, defined in algebraic terms, can be considered as geometric objects and studied with geometric techniques. This argument is still hypothetical, however, as, again, are all arguments thus far to the effect that the reality underlying space is discrete. If one accept what may be called the strong Copenhagen view, following Bohr, this "reality" may be beyond conception altogether and, hence, be neither continuous nor discontinuous [27, pp. 11–22]. I return to this possibility below.

[21]I have discussed the connections between Riemann's Habilitation lecture and quantum theory in [25].

ematics, which he introduced in the lecture, suitable for our description of physical space, via our phenomenal experience. This suitability allows one to have a geometry based on this mathematics. However, as grounded in the concept of manifold, this mathematics, the conceptual architecture of this mathematics, is sufficiently general both to be developed independently in mathematics itself quite apart from physics (it is has done subsequently as well) and to account for various possible forms of physical spatiality. This mathematics then needs to be adjusted in accordance with the hypotheses that we make concerning physical space, some of which may acquire the status of experimental evidence, possibly long-standing but not guaranteed to be permanent. Thus, such hypotheses may concern whether physical space is flat or curved, or whether this curvature is positive or negative, or (a question, again, never posed before Riemann) whether this curvature is constant or variable, or whether the ultimate (small-scale) "reality underlying space" is continuous or discrete, "beyond the bounds of observation." We can then "investigate the likelihood [of such hypotheses], which [in the case of Euclidean geometry] is certainly very great within the bounds of observation [in Riemann's time], and afterwards decide on the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large and in the direction of the immeasurably small."

Riemann's line of reasoning here both follows and goes beyond Kant, in part by adopting Johann Herbart's argument, which questioned Kant's view (e.g. [11], pp. 77–99). Riemann follows Kant insofar as he sees our observations, always defined by our phenomenal experiences, as the basis of possible hypotheses concerning nature, the truth of which is possible but is not assured. He goes beyond Kant insofar as he views these hypotheses as only having an established validity within the bounds of observation with one or another degree of certainty. These hypotheses may not be applicable at all if we extend our investigation of the nature of space arbitrarily far in either direction, that of the infinitely large and that of the infinitely small. Kant, by contrast, believes in the absolute validity of (the hypothesis of) Euclidean geometry or Newton's physics. On the other hand, as already noted, unlike Riemann or Herbart, Kant does not believe that space, or time, is an empirical concept, whose validity, either as a general concept or in any of its instantiation, is established by experience.[22] Kant sees it as an a priori given concept that we use to frame our experience, a claim persistently challenged from the time it was made, including by Herbart and Riemann. Nature may have a different form of spatiality from what our phenomenal concept of space tells us, or nature may not have spatial aspects to it at all. By the end of the lecture, in considering the question of space in the infinitely small, Riemann comes closer to a more Kantian (although not quite Kant's own) view that "the reality underlying space" may be different from our phenomenal intuition of spatiality or may not be spatial in our phenomenal sense. For example, this reality may be discrete or be beyond the reach of any concept, discrete or continuous, available to us. In this latter view, this reality, while still real, would be beyond any representation and thus

[22] How our phenomenal experience of space emerges is separate question, psychological, physiological, or now neurological. Remarkably, Riemannian geometry is used in recent neurological research, as in the work of Jean Petitot (neurogeometry).

beyond realism [27, pp. 11–22]. It is, as indicated earlier, doubtful that Riemann entertained, anymore than did Kant [27, pp. 17–21], so radical a hypothesis, which emerged only in the wake of quantum mechanics. This hypothesis may, nevertheless, be seen as an implication of Riemann's closing reflections in the lecture and possibly Kant's epistemology [25].[23]

Thus far, Riemann only spoke of *physical space* rather than of *physics* in the sense of material forces, bodies, and motion. In closing, however, he brings physics into consideration. He argues that it is physics that defines the nature of space in the immeasurably small. Thus, while space may be assumed—this was a plausible hypothesis at the time and still is now–to be a three-dimensional manifold, what kind of manifold it is will be defined by physics. According to Weyl: "Riemann rejects the opinion that had prevailed up to his own time, namely, that the metrical structure of space is fixed and is inherently independent of the physical phenomena for which it serves as a background, and that the real [physical] content takes possession of it as a residential flat" [36, p. 98]. This was a revolutionary move on Riemann's part, later furthered by Einstein, who rigorously connected Riemannian geometry to the physics of gravity. For Riemann and Einstein, on this point following Leibniz (who, it is true, did not appear to have contemplated non-Euclidean geometry), matter defines the character of space, say, as flat or curved, while for Newton, space preexists matter, as an absolute space, a *flat* residential flat. Earlier arguments for non-Euclidean geometry had only changed the Newtonian view of space insofar as they imply that space might not be flat, which, however, still leaves open whether or not the Euclidean or non-Euclidean nature of space is defined by matter. Weyl adds: "*[Riemann] asserts, on the contrary, that space is itself nothing more than a three-dimensional manifold devoid of all form; it acquires a definite form only through the advent of the material content filling it and determining its metric relations*" [36, p. 98; Weyl's italics]. This is not quite what Riemann says. Weyl's statement may suggest (although Weyl does not appear to intend this) that space, as "a manifold, devoid of all form," preexists a given form, which form is then determined by matter, "*through the advent of the material content filling it and determining its metric relations*." For Riemann, as for Leibniz and Einstein, and ultimately for Weyl, matter preexists space, or, more accurately, it reciprocally co-exists with space and defines its character as a manifold. In addition, to return to Riemann's more precise language, "the reality underlying space" may be different from our phenomenal or, depending on scale, even physical experience of space, in particular, as flat, Euclidean space [31, p. 33]. It may reveal itself to be curved and have a varying curvature (which is to say, to be assumed to conform to the corresponding hypothesis), as in general relativity, which proved Riemann's insights especially prescient, and his concept of manifold especially capacious. Physics may also find that this reality requires manifolds of different types (including possibly, discrete) on different scales. One may, accordingly, modify Weyl's statement by saying that a general form of space may be assumed, hypothesized, to be, say, a three-dimensional continuous manifold

[23] Cf. [20], on the epistemological differences between Kant and Riemann.

or a three-dimensional discrete lattice, while its specific form (local or global) is determined by matter and forces acting upon it.

In approaching the subject, Riemann first states that "the questions about the immeasurably large [*Unmessbargrosse*] are idle questions for the explanation of nature [*die Naturerklärung*]," an assessment, for which Riemann offers no further justification and which one might question now, at least insofar as very large scales as concerned [31, p. 32]. From the present-day perspective, the question of the character of space on a very large cosmic scale is far from idle, although the idea, the hypothesis, of the infinite cosmic space poses conceptual difficulties, and it is possible that Riemann sensed some of them in making his assessment.[24] Be it as it may, the subject could be put aside, given that these are Riemann's reflections concerning "the questions about the immeasurably small [*Unmessbarkleine*]" that are most important for the present argument. These questions, Riemann argues, are "not idle ones:"

> Upon the exactness with which we pursue phenomena into the infinitely small [*Unendlichkleine*] does our knowledge of their causal connections essentially depends. The progress of recent centuries in understanding the mechanisms of Nature depends almost entirely on the exactness of construction which has become possible through the invention of the analysis of the infinite and through the simple principles discovered by Archimedes, Galileo, and Newton, which modern physics makes the use of. By contrast, in the natural sciences where the simple principles for such constructions are still lacking, to discover causal connections one follows phenomenon into the spatially small, just so far as the microscope permits. Questions about the metric relations of space in the immeasurably small are thus not idle ones [31, p. 32].

Riemann, thus, sees the mathematical representation of space, or time, or physical processes in space and time, offered by classical physics, as defined by the kinematical and dynamical principles established by the figures he mentions here. Riemann also sees physics as based, mathematically, on the principles of differential calculus, which is an analysis of the infinitely small. This is not the same as the immeasurably small [*Unmessbarkleine*], but it provides the proper mathematical representation of the physical concepts just mentioned, which explains Riemann's shift in this paragraph from "the immeasurably small" [*Unmessbarkleine*] to "the infinitely small" [*Unendlichkleine*]. More generally, as Weyl noted, "The principle of gaining knowledge of the external world from the behavior of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry, and, indeed, the mainspring of all the eminent work of Riemann" [36, p. 92]. As the mathematics of the infinitely small, differential calculus also allows one to relate classical physics to causality, indeed is correlative to causality (which is one of the physical principles in question, defined by the fact that the state of a given system at a given moment of time determines its state at any other moment of time).[25] Riemann was, again, aware, as was Weyl, that the reality underlying space, in the immeasurable small, may be discrete and hence, at that scale, no longer subject to a continuous analysis. Hence, again, there is the difference between the immeasurably

[24] On some of these difficulties, see [12, pp. 31–42].

[25] Riemann offered important reflections on causality, which he linked to continuity [32, p. 522].

small [*Unmessbarkleine*], also in its direct sense of that which cannot be measured, and the infinitely small [*Unendlichkleine*], which is an important point, especially, as became apparent later, in the context of quantum theory. As noted above, however, quantum theory, while dealing with discrete phenomena, does not generally assume or imply that "the reality underlying space" is discrete, and if anything, suggests that the ultimate reality of nature may be beyond any possible representation of even conception (discrete or continuous) [27, pp. 11–22].[26] By the same token, the theory is no longer causal, but is irreducibly probabilistic even in dealing with elemental individual quantum processes (always assumed to be causal in classical physics or relativity). This fact is reflected in Heisenberg's uncertainty relations, which prevent us from ever simultaneously determining both the position and the momentum of a quantum object, which is necessary in order to maintain causality. In any event, it is clear that "questions about the metric relations of space in the immeasurably small are not idle ones." They connect the hypotheses that lie at the foundations of geometry to those that lie at the foundations of physics. Riemann says next:

> If one assumes that bodies exist independently of position, then the curvature is everywhere constant, and it then follows from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes can be neglected. But if such an independence of bodies from position does not exist, then one cannot draw conclusions from metric relations in the infinitely small from those in the large; at every point the curvature can have arbitrary values in three directions, provided only that the total curvature of every measurable portion of space is not perceptibly different from zero [31, p. 32].

It took Einstein's general relativity to give, for the first time, a rigorous physical content to these insights by bringing together the physics of gravitation and Riemannian geometry. The curvature of a manifold representing the physical reality underlying space not only may not be zero, but may also not be constant, which is, again, a powerful new mathematical concept and, as possible physics is concerned, a tremendous physical insight of Riemann. It is generally not constant in a gravitational field, and establishing this fact in rigorous terms is an equally tremendous contribution of Einstein. We do know now that the hypothesis of Euclidean geometry or even non-Euclidean geometry of constant curvature, do not apply to the ultimate nature of space, or again, the physical reality underlying space, except perhaps on average on a very large scale, as current observations suggest. Nor, in part correlatively, do the hypotheses of classical physics, specifically those that ground Newton's law of gravity, apply at any scale, except as an approximation, workable within very large limits as this approximation is. Newton's law of gravity is incorrect even within its proper scope, as was first exemplified by the aberrant precession of the perihelion of

---

[26] We cannot conceive of entities that are simultaneously continuous and discontinuous, the difficulty handled in quantum mechanics by means of Bohr's concept of complementarity. Complementarity reflects the fact that continuous and discontinuous *quantum phenomena* (defined by what is observed in measuring instruments) are always mutually exclusive, while *quantum objects* themselves, responsible for these phenomena through their impacts on measuring instruments, are, again, assumed to be beyond any representation or even conception, continuous or discontinuous. For a full treatment, see [27, pp. 107–172].

Mercury. The principles of calculus, used in tensor calculus, still apply in the infinitely small in general relativity as a way of providing the mathematics, the mathematical model, of space as defined by gravity. Riemann then adds:

> Still more complicated relations can occur if the line element cannot be represented, as was presupposed, as the square root of a differential expression of the second degree. Now it seems that the empirical notions on which the metrical determinations of space are based, the concept of a solid body and that of a ray of light, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena [31, p. 32].

Riemann, thus, envisions not only that space in the infinitely small, or, as it would, again, be more accurate to say, the immeasurably small may not conform to the hypothesis of Euclidean geometry, but also that it may not conform even to Riemannian geometry, used by Einstein in general relativity. In the latter case, the concept of metric relations still applies, although they are non-Euclidean and allow for a variable curvature. Relativity merely modifies, albeit radically, Euclidean concepts in view of the relativistic contraction of bodies and of the curving of light in the vicinity of a heavy body, such as the Sun. Bringing together gravity and quantum theory (a still outstanding problem) may change this. Riemann is about to suggest that "the reality underlying space" may be "a discrete manifold:"

> The question of the validity of the hypotheses of geometry in the infinitely [the immeasurably?] small is bound up with the question of the basis for the metric relations of space. In connection with this question, which may indeed still be ranked as part of the study of space, the above remark is applicable, that in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces that act upon it.
>
> An answer to these questions can be found only by starting from the conception of phenomena which has hitherto been approved by experience, and for which Newton laid the foundation, and gradually modifying it under the compulsion of facts that cannot be explained by it. Investigations like the one just made here, which begin from general concepts, can *only* serve to insure that this work is not hindered by unduly restricted concepts and that progress in comprehending the connection of things is not obstructed by traditional prejudices. This leads us away into the domain of another science, the realm of physics, into which the nature of the present occasion does not allow us to enter. [31, p. 33]

Riemann, again, differs from Kant, insofar as Riemann assumes that this reality must at least be established by physical experiments, even if not perceived phenomenally, rather than is given a priori, indeed by definition because we do not phenomenally perceive space as discrete. On the other hand, as explained earlier, Kant, who is often misunderstood on this point, does not assume that the physical reality underlying space is given a priori or is phenomenal otherwise. With this qualification in mind, Riemann's formulation becomes close to Kant, except perhaps that Riemann believes that our hypotheses concerning the character of "the reality underlying space" could be tested so as to bring us closer to knowing this reality. I qualify by "perhaps," because Kant might have even agreed on this point as well. In addition, as also explained earlier, just as Riemann, Kant would likely have been hesitant to call this reality space, if it is different from the continuous three-dimensional space of our phenomenal experiences. The main point here is that one needs physics, as an experimental-mathematical science of nature, to establish the facts that would enable us to test and, to begin with, to form hypotheses concerning the reality underlying space, and have geometry of this space, possibly a higher-dimensional or discrete geometry.

Einstein's relativity justified Riemann's view that we must proceed "by starting from the conception of phenomena which has hitherto been approved by experience, and for which Newton laid the foundation, and gradually modifying this conception under the compulsion of facts that cannot be explained by it." The facts at stake in relativity can no longer be explained by the conception of physical phenomena provided by Newton's physics; and, as is clear from this elaboration, Riemann saw this conception as likely to be insufficient. This is not surprising given his investigations into electromagnetism and the contemporary development of this field and of physics in general, even though Maxwell's electromagnetic theory was not yet in place at the time of the lecture [22]. Riemann's subsequent work on electromagnetism suggests intriguing affinities with that of Maxwell and then that of Einstein (e.g. [16, pp. 257–271]). The subject would require a separate treatment. It may, however, be fitting to note that, extending its role in general relativity, Riemannian geometry also served as the basis for several early projects of establishing a unification of gravity and electromagnetism, the first form of the unified field theory, pursued, in particular, by Einstein, Hilbert, and Weyl. While they set into motion the program that still dominates fundamental physics, these attempts, all essentially along the lines of classical-like field theory (on the model of Maxwell's electromagnetic theory), were unsuccessful. This was in part because such a theory appears unlikely to be developed without taking into account quantum aspects of electromagnetism or, by now, of other strong and weak forces, covered by quantum field theory, within the so-called standard model of all known forces of nature, except for gravity, with which the standard model is incompatible. Both Einstein and Weyl made attempts, again, unsuccessful, to incorporate Dirac's 1928 relativistic theory of the electron into their unified-field-theoretical schemes, still governed, however, by a classical-like field theoretical thinking, which was in a manifested conflict with the principles behind Dirac's theory [27, pp. 207-226].[27] It is, accordingly, not surprising, at least in retrospect, that these attempts did not succeed. Unsuccessful as they have been, they, nevertheless, showed the fruitfulness of Riemann's thinking in geometry for foundational thinking in physics, and Riemann's foundational thinking in geometry was, again, also a foundational thinking in physics.

[27] Dirac's famous equation also introduced spinors into physics. Although the name itself was coined (by Paul Ehrenfest) in 1929, following Dirac's theory, the concept, still enigmatic and uneasily suspended between geometry and algebra, existed in mathematics previously and was extensively studied by Cartan, for example. It belongs to the post-Riemannian evolution of geometrical thinking, also as extending beyond geometry, for example, to algebra, although spinors are important for geometry as well.

The search for such a unified theory is still ongoing, now in attempting to unify all forces of nature, although, thus far, even within the standard model, we only have the electroweak unification, which is, besides, quite different in nature from the way such a unification was envisioned previously. As indicated above, the incompatibility between the standard model, as quantum theory, and general relativity is one of the great outstanding problems of the present-day fundamental physics, perhaps the greatest one. Superstring and brane theories, which have been around for quite a while now, are still generally seen as the best candidate, although the skepticism that has always shadowed them has become, for both mathematical and physical reasons, even more pronounced more recently. But then, that currently available alternatives, such as loop quantum gravity, will succeed appears no more likely. Both programs, that of superstring and brane and that of loop quantum gravity, have Riemannian genealogies, loop quantum gravity more immediately, via Einstein's general relativity, and superstring and brane theories, which originated in quantum field theory, via a more complex history of development, by virtue of using the so-called Calabi-Yau manifolds.[28] Which among these or other currently available programs, such as those along the lines of quantum information theory, are likely to succeed in approaching the reality underlying, to borrow Weyl's famous title, "space, time, and matter" is impossible to predict, perhaps none of them, given, thus far, the immense physical and mathematical difficulties they pose. These difficulties, however, also open new possibilities, both inside these programs, which might succeed after all, and for new, possibly as yet unimaginable, alternatives. Whatever the future holds, just as does mathematics, fundamental physics continues to return us to Riemann and to show that the manifold of connection[s] of things, mathematical, physical, and philosophical, that Riemann's thought brought into existence is inexhaustible.

5 Conclusion

I return, in closing, to Riemann's assessment of his own project in his lecture: "Investigations like the one just made here, which begin with general concepts, can *only* serve to insure that this work [of developing new physics] is not hindered by unduly restricted concepts and that progress in comprehending the connection of things is not obstructed by traditional prejudices" (emphasis added). "Only" is hardly necessary here. We do need physics to test and indeed to form our hypotheses concerning space. But we also need—physics proves that we do!—mathematical and philosophical plane of thought and new, richer concepts to counteract "unduly restricted concepts" and to be able "in comprehending the connection of things," which is the aim of thought in its cooperative confrontation with chaos, and not to be obstructed by traditional prejudices or rigid, dogmatic opinions, the danger of which is, Deleuze and Guattari warn us, as Riemann does here, a constant threat to thought. Riemann created such planes and such concepts. This does not mean that we need to stop with Riemann, who never stopped. The subsequent history of mathematics and physics

[28] On these connections, see [22].

has proven that we need to go beyond Riemann. Otherwise, his concepts cannot continue to live on, to remain "always new," to be concepts of the future.

Acknowledgements I am grateful to Franck Jedrzejewski, Ken'ichi Ohshika, and Athanase Papadopoulos for reading the original draft of the article and helpful suggestions for improving it.

References

1. M.A. Artin, A. Grothendieck, J.-L. Verdier, *Séminaire de Géométrie Algébrique du Bois Marie - 1963–64 - Théorie des topos et cohomologie étale des schémas - (SGA 4)*, vol. 1 (Springer, Berlin, New York, 1972)
2. E. Borel, La logique et lintuition en mathématique. Revue de Métaphysique et de Morale **15**(3), 273–283 (1907)
3. P. Cartier, A Mad Days Work: From Grothendieck to Connes and Kontsevitch. The Evolution of Concepts of Space and Symmetry. B. Am. Math. Soc. **38**(4), 389–408 (2001)
4. L. Corry, Axiomatics, empiricism, and anschauung in hilberts conception of geometry: between arithmetic and general relativity, in *The Architecture of Modern Mathematics: Essays in History and Philosophy*, ed. by J. Ferreirós, J.J. Gray (Oxford University Press, Oxford, 2006), pp. 133–156
5. G.M. D'Ariano, P. Perinotti, Derivation of the Dirac equation from principles of information processing. Phys. Rev. A **90**, 062106 (2014)
6. G. Deleuze, *Difference and Repetition* (Columbia University Press, New York, 1994). Translated by P. Patton
7. G. Deleuze, F. Guattari, *A Thousand Plateaus* (University of Minnesota Press, Minneapolis, MN, 1987). Translated by B. Massumi
8. G. Deleuze, F. Guattari, *What Is Philosophy?* (Columbia University Press, New York, 1994). Translated by H. Tomlinson and G. Burchell
9. J. Ferreirós, Riemann's habilitationsvortrag at the crossroads of mathematics, physics, and philosophy, in *The Architecture of Modern Mathematics: Essays in History and Philosophy*, ed. by J. Ferreirós, J.J. Gray (Oxford, Oxford University Press, 2006), pp. 67–96
10. J. Ferreirós, *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics* (Birkhäuser, Boston, MA, 2010)
11. J. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics* (Princeton University Press, Princeton, 2008)
12. B. Greene, *The Hidden Reality: Parallel Universes and the Deep Laws of the Cosmos* (Vintage, New York, 2011)
13. D. Hilbert, *Foundations of Geometry*, (10th ed.) (Translated by L. Unger. Open Court, La Salle, IL, 1979)
14. F. Jedrzejewski, Deleuze et la géométrie Riemannienne: Une topologie des multiplicitités, this volume, pp. 311–327
15. I. Kant, *Critique of Pure Reason* (Cambridge University Press, Cambridge, 1997). Translated by P. Guyer and A. W. Wood
16. D. Laugwitz, *Bernhard Riemann: Turnings Points in the Conception of Mathematics* (Birkhäuser, Boston, MA, 1999). Translated by A. Shenitzer
17. Y. I. Manin, Georg Cantor and His Heritage, arXiv.math.AG/0209244 v1. Accessed 19 Sept 2002
18. C. McLarty, *Elementary Categories* (Elementary Toposes, Clarendon, Oxford, 1992)
19. C. McLarty, Emmy noether's set-theoretical topology: From Dedekind to the rise of functors, in *The Architecture of Modern Mathematics: Essays in History and Philosophy*, ed. by J. Ferreirós, J.J. Gray (Oxford University Press, Oxford, 2006), pp. 211–236

20. K. Ohshika, The origin of the notion of manifold: Riemann's Habilitations vortrag onward, this volume, pp. 295–308
21. V. Pambuccian, H. Struve, R. Struve, Metric Geometries in Axiomatic Perspective, this volume, pp. 413–451
22. A. Papadopoulos, Physics in Riemann's Mathematical Papers, this volume, pp. 151–199
23. P. Pesic, Introduction, in *Beyond Geometry: Classic Papers from Riemann to Einstein*, ed. by P. Pesic (Dover, Mineola, NY, 2007), pp. 1–22
24. A. Plotnitsky, Chaosmologies: Quantum field theory, chaos, and thought in deleuze and guattaris what is philosophy? Paragraph **29**(2), 40–56 (2006)
25. A. Plotnitsky, On foundational thinking in fundamental physics: From Riemann to Heisenberg. in G. M. D'Ariano et al. (ed). Foundations of Probability in Physics 6. Melville, NY: American Institute of Physics, Melville, NY, 2012, 282–304
26. A. Plotnitsky, The spacetimes of nympheas: Matter and multiplicity in Einstein, Monet, and Deleuze and Guattari, in *Revisiting Normativity with Deleuze*, ed. by R. Braidotti, P. Pisters (Bloomsbury, London, 2012), pp. 37–50
27. A. Plotnitsky, *The Principles of Quantum Theory, From Planck's Quantum to the Higgs Boson: The Nature of Quantum Theory and the Spirit of Copenhagen* (Springer, New York, 2016)
28. H. Poincaré, *La Science et l'hypothèse* (Flammarion, Paris, 1902)
29. H. Poincaré, *Science et methodé* (Flammarion, Paris, 2008)
30. G. Polya, *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics* (Princeton, Princeton University Press, 1958, rpt. 1990)
31. B. Riemann, B. On the Hypotheses That Lie at the Foundations of Geometry (1854). In P. Pesic (ed.) *Beyond Geometry: Classic Papers from Riemann to Einstein* (Dover, Mineola, NY, Dover, 2007), pp. 23–40
32. B. Riemann, *Gesammelte mathematische Werke und wissenschaftlicher* Nachlass. ed. by R. Narasimhan (Springer, New York, 1991)
33. B. Russell, *An Essay on the Foundations of Geometry*. (Cambridge University Press, 1997). (rpt. 2012)
34. A. Weir, Formalism in the Philosophy of Mathematics. The Stanford Encyclopedia of Philosophy (Spring 2015 Edition), E. N. Zalta (ed.), http://plato.stanford.edu/archives/spr2015/entries/formalism-mathematics/
35. H. Weyl, *The Concept of a Riemann Surface*. Translated G. L. MacLane. Dover, Mineola, NY, 1913. (rpt. 2013)
36. H. Weyl, *Space-Time-Matter* (Dover, Mineola, NY, 1952). Translated H. L. Brose