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K-Theory for Group C*-Algebras and Semigroup C*-Algebras

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Preface

This book grew out of the Oberwolfach Seminar "K-Theory for Group C^*-Algebras and Semigroup C^*-Algebras" organized by the authors in October 2014. In this seminar, we discussed several classes of C^*-algebras of current interest, including group C^*-algebras, crossed products, semigroup C^*-algebras and C^*-algebras of algebraic actions. Since K-theory is the most important tool to analyse and classify C^*-algebras, our main goal was to explain methods for explicit K-theory computations, both in a general framework and in the case of our particular classes of examples. Building on this, we showed how these K-theory tools help to shed some light on the structure of our C^*-algebras. On the one hand, we wanted to give a friendly introduction to these topics without assuming too much background, and on the other hand, our intention was to present an overview of recent developments.

This book essentially consists of expanded versions of our lecture notes for the seminar, together with additional material and some new results. As most of the topics of the seminar have not yet appeared in book form, this is an excellent opportunity to give an overview as well as a unified and systematic treatment of material which can only be found scattered across the literature.

We would like to thank the Mathematisches Forschungsinstitut Oberwolfach, its director Professor Dr. Gerhard Huisken and its highly professional staff for providing an excellent environment in which to work. It is also a pleasure to thank all the participants for their interest in the subject, their enthusiasm and many interesting discussions, which altogether provided a very inspiring atmosphere.

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Chapter 1

Introduction

The theory of operator algebras in general and C^*-algebras in particular has always benefited hugely and drawn a lot of inspiration from interactions with other areas of mathematics such as geometry, topology, group theory, dynamical systems or number theory, to mention just a few. The starting point for these connections is usually given by constructions of C^*-algebras which produce new examples of C^*-algebras on the one hand and lead to interesting invariants and applications on the other hand. The most powerful tool to extract important information out of C^*-algebras is given by K-theory. Actually, K-theory itself is already an excellent example for a fruitful exchange of ideas between C^*-algebras and other mathematical disciplines, namely topology and index theory. K-theory for C^*-algebras was initially defined as an extension of topological K-theory for spaces, in line with the philosophy viewing C^*-algebra theory as noncommutative topology. Nowadays K-theory plays an important role in the classification of C^*-algebras, a subject which has seen tremendous advances recently. It turns out that for a huge class of C^*-algebras, K-theoretic invariants provide a complete invariant. At the same time, K-theory leads to interesting applications of C^*-algebra theory to topology and geometry in the context of the Baum–Connes conjecture. Among many other consequences, this has an important impact on the Novikov conjecture in geometry.

This book focuses on C^*-algebras attached to groups, semigroups, dynamical systems in general and algebraic actions in particular. We describe the structure of these C^*-algebras and explain methods that allow for explicit K-theory computations.

We start with group C^*-algebras and crossed products in Chapter 2. These are constructions of C^*-algebras out of groups and group actions, and have been studied intensively. We discuss the basics of the theory and explain in detail the Mackey–Rieffel–Green machine which allows us to study the ideal structure of crossed products. The first six sections of Chapter 2 are needed for Chapter 3. To mention

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some examples of further references, we refer to [Bla06, Dav96, Dix77, Mur90] for more information on general C^*-algebra theory and to [Ped79, Wil07] for more about crossed products.

Chapter 3 gives an introduction to bivariant KK-theory and discusses the Baum–Connes conjecture. KK-theory generalizes at the same time K-theory and its companion homology theory, K-homology. It has established itself as a very powerful tool in the classification of C^*-algebras as well as in applications of C^*-algebras in geometry. For instance, particularly relevant for the latter point, it provides the technology to formulate and prove (at least particular cases of) the Baum–Connes conjecture. Roughly speaking, this conjecture gives a way to compute K-theory for group C^*-algebras and crossed products by using tools from algebraic topology. It has far-reaching consequences; for instance, it implies the Novikov conjecture and the Kadison–Kaplansky conjecture, to mention just a few. At the end of Chapter 3, we explain how the Baum–Connes conjecture (or rather its underlying principle) leads to a K-theory formula for certain crossed products. This will be important for Chapters 5 and 6.

Chapter 4 discusses another, though closely related, approach to K-theory computations: quantitative K-theory. This has been initiated in [Yu98] and further developed in [OOY15]. New applications have been recently worked out in [GWY16b, GWY16a]. The basic idea is to introduce a geometric structure on C^*-algebras, in analogy to geometric group theory, which allows for quantitative versions of K-theory, i.e., K-theory with scales. These quantitative versions have the advantage of being computable in typical situations by means of cutting-and-pasting techniques. In some sense, Chapter 4 provides a different angle on some of the aspects discussed in Chapter 3.

Chapter 5 discusses semigroup C^*-algebras. These are C^*-algebras generated by left regular representations of semigroups and provide a natural generalization of group C^*-algebras, as they were discussed in Chapter 2. These semigroup C^*-algebras provide interesting new examples of C^*-algebras which can have properties that are surprising and quite different from the group case. We introduce two important conditions (called independence and Toeplitz) that allow us to analyse the structure of these C^*-algebras. We also use the K-theory computations in Chapter 3 to classify some of them. In addition, we discuss important classes of examples coming from group theory and number theory. Parts of this chapter, for instance §5.9, are new and have not appeared before in the literature.

Chapter 6 discusses important examples of crossed products for actions of semigroups by endomorphisms. This includes examples of a single endomorphism of a compact abelian group, or of a semigroup of such endomorphisms. Such actions are commonly studied in ergodic theory. But, closely related, we also consider regular C^*-algebras of semigroups associated to rings of algebraic integers. These examples have been instrumental for an important part of the recent development on semigroup C^*-algebras described in Chapter 5, as well as for the design of new methods to compute their K-theory. The C^*-algebras arising that way have an intriguing structure and exhibit interesting new phenomena. The problem of computing their K-theory is challenging in each case and needs different, partly new, methods (one of these methods is the one described in Section 3.5.3 of Chapter 3).

The C^*-algebras for semigroups associated with an algebraic number field, which we consider, carry a natural one-parameter automorphism group. There is a striking parallel between the structure of the KMS-states for that automorphism group and the formula for the K-theory of the algebra that we obtain. Both split naturally over the ideal class group of the underlying number field. At the end of Chapter 6 we sketch an argument for the description of the KMS-states which uses an approach inspired by the method for the determination of the K-theory and thus explains the connection between K-theory and the KMS-structure in that case. This particular construction has not appeared in the literature before.

Finally, in Chapter 7 we consider semigroup C^*-algebras for finitely generated subsemigroups of $\mathbb{Z}^n$. These are important in algebraic geometry because their monoid rings define affine varieties with a torus action (so-called toric varieties). They have the interesting feature that they do not satisfy the independence condition which plays an important role in Chapters 3, 5 and 6. As explained in these chapters, this condition is a basis for the computation of the K-theory of the C^*-algebras for a large class of semigroups.

We study the case of subsemigroups of $\mathbb{Z}^2$ in detail. It turns out that a careful analysis of the structure of such a toric semigroup and of the projections in the canonical diagonal subalgebra of $C^*_\lambda S$, together with a comparison with the known K-theory of the Toeplitz algebra $C^*_\lambda \mathbb{N}^2$, leads to a simple formula that describes the K-theory in complete generality. The failure of the independence condition shows up in a torsion part of the K-theory.

The results in Chapter 7 are new and have not been published before.

Chapter 2

Crossed products and the Mackey–Rieffel–Green machine

Siegfried Echterhoff

2.1 Introduction

If a locally compact group G acts continuously via $*$-automorphisms on a C^*-algebra A, one can form the full and reduced crossed products $A \rtimes G$ and $A \rtimes_r G$ of A by G. The full crossed product should be thought of as a skew *maximal* tensor product of A with the full group C^*-algebra $C^*(G)$ of G and the reduced crossed product should be regarded as a skew *minimal* (or spacial) tensor product of A by the reduced group C^*-algebra $C^*_r(G)$ of G.

The crossed product construction provides a major source of examples in C^*-algebra theory, and it plays an important rôle in many applications of C^*-algebras in other fields of mathematics, such as group representation theory and topology – here in particular, in connection with the Baum–Connes conjecture, which we shall treat in Chapter 3 of this book. It is the purpose of this chapter to present in a concise way some of the most important constructions and features of crossed products with an emphasis on the Mackey–Rieffel–Green machine as a basic technique to investigate the ideal structure of crossed products. The contents of the first six sections of this chapter are also basic for the understanding of the contents of Chapter 3. Detailed proofs of most of the results on crossed products presented in this chapter (if not given here) can be found in the monograph [Wil07] by Dana Williams. Note that the material covered in this chapter is almost perpendicular to the material covered in Pedersen's book [Ped79]. Hence we recommend the interested reader to also have a look into [Ped79] to obtain a more complete and balanced picture of the theory. Pedersen's book also provides a good introduction

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into the general theory of C^*-algebras. An incomplete list of other good references on the general theory of C^*-algebas is [Bla06, Dav96, Dix77, Mur90]. The Morita (or correspondence) category has been studied in more detail in [EKQR06].

Some general notation: if X is a locally compact Hausdorff space and E is a normed linear space, then we denote by $C_b(X, E)$ the space of bounded continuous E-valued functions on X and by $C_c(X, E)$ and $C_0(X, E)$ those functions in $C_b(X, E)$ that have compact supports or that vanish at infinity. If $E = \mathbb{C}$, then we simply write $C_b(X), C_c(X)$ and $C_0(X)$, respectively. If E and F are two linear spaces, then $E \odot F$ always denotes the algebraic tensor product of E and F and we reserve the sign "$\otimes$" for certain kinds of topological tensor products.

2.2 Some preliminaries

We shall assume throughout this discussion that the reader is familiar with the basic concepts of C^*-algebras as can be found in any of the standard textbooks mentioned above. However, in order to make this treatment more self-contained we try to recall some basic facts and notation on C^*-algebras which will play an important rôle in this article.

2.2.1 C^*-algebras

A (complex) C^*-algebra is a complex Banach-algebra A together with an involution $a \mapsto a^*$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$. Note that we usually do not assume that A has a unit. Basic examples are given by the algebras $C_0(X)$ and $C_b(X)$ equipped with the supremum-norm and the involution $f \mapsto \bar{f}$. These algebras are clearly commutative, and a classical theorem of Gelfand and Naimark asserts that *all* commutative C^*-algebras are isomorphic to some $C_0(X)$ (see Section 2.2.3 below). Other examples are given by the algebras $\mathcal{B}(H)$ of bounded operators on a Hilbert space H with operator norm and involution given by taking the adjoint operators, and all closed $*$-subalgebras of $\mathcal{B}(H)$ (like the algebra $\mathcal{K}(H)$ of compact operators on H). Indeed, another classical result by Gelfand and Naimark shows that every C^*-algebra is isomorphic to a closed $*$-subalgebra of some $\mathcal{B}(H)$. If $S \subseteq A$ is any subset of a C^*-algebra A, we denote by $C^*(S)$ the smallest sub-C^*-algebra of A that contains S. A common way to construct C^*-algebras is by describing a certain set $S \subseteq \mathcal{B}(H)$ and forming the algebra $C^*(S) \subseteq \mathcal{B}(H)$. If $S = \{a_1, \dots, a_l\}$ is a finite set of elements of A, we shall also write $C^*(a_1, \dots, a_l)$ for $C^*(S)$. For example, if $U, V \in \mathcal{B}(H)$ are unitary operators such that $UV = e^{2\pi i\theta}VU$ for some irrational $\theta \in [0, 1]$, then $A_\theta := C^*(U, V)$ is the well-known *irrational rotation algebra* corresponding to θ, a standard example in C^*-algebra theory (in this example one can show that the isomorphism class of $C^*(U, V)$ does not depend on the particular choice of U and V).

C^*-algebras are very rigid objects: If A is a C^*-algebra, then every closed (two-sided) ideal of A is automatically self-adjoint and A/I, equipped with the obvious operations and the quotient norm is again a C^*-algebra. If B is any Banach $*$-algebra (i.e., a Banach algebra with isometric involution, which does not necessarily satisfy the C^*-relation $\|b^*b\| = \|b\|^2$), and if A is a C^*-algebra, then any $*$-homomorphism $\Phi : B \to A$ is automatically continuous with $\|\Phi(b)\| \le \|b\|$ for all $b \in B$. If B is also a C^*-algebra, then Φ factors through an isometric $*$-homomorphism $\tilde{\Phi} : B/(\ker\Phi) \to A$. In particular, if A and B are C^*-algebras and $\Phi : B \to A$ is an injective (resp. bijective) $*$-homomorphism, then Φ is automatically isometric (resp. an isometric isomorphism).

2.2.2 Multiplier algebras

The *multiplier algebra* $M(A)$ of a C^*-algebra A is the largest C^*-algebra that contains A as an essential ideal (an ideal J of a C^*-algebra B is called *essential* if for all $b \in B$ we have $bJ = \{0\} \Rightarrow b = 0$). If A is represented faithfully and nondegenerately on a Hilbert space H (i.e., $A \subseteq \mathcal{B}(H)$ with $AH = H$), then $M(A)$ can be realized as the idealizer

$$M(A) = \{T \in \mathcal{B}(H) : TA \cup AT \subseteq A\}$$

of A in $\mathcal{B}(H)$. In particular, we have $M(\mathcal{K}(H)) = \mathcal{B}(H)$, where $\mathcal{K}(H)$ denotes the algebra of compact operators on H.

The *strict topology* on $M(A)$ is the locally convex topology generated by the seminorms $m \mapsto \|am\|, \|ma\|$ with $a \in A$. Note that $M(A)$ is the strict completion of A. $M(A)$ is always unital and $M(A) = A$ if (and only if) A is unital. If $A = C_0(X)$ for some locally compact space X, then $M(A) \cong C_b(X) \cong C(\beta(X))$, where $\beta(X)$ denotes the Stone–Čech compactification of X. Hence $M(A)$ should be viewed as a noncommutative analogue of the Stone–Čech compactification. If A is any C^*-algebra, then the algebra $A_1 := C^*(A \cup \{1\}) \subseteq M(A)$ is called the *unitization* of A (note that $A_1 = A$ if A is unital). If $A = C_0(X)$ for some noncompact X, then $A_1 \cong C(X_+)$, where X_+ denotes the one-point compactification of X.

A $*$-homomorphism $\pi : A \to M(B)$ is called *nondegenerate* if $\pi(A)B = B$, which by Cohen's factorization theorem is equivalent to the weaker condition that $\mathrm{span}\{\pi(a)b : a \in A, b \in B\}$ is dense in B (e.g., see [RW98, Proposition 2.33]). If H is a Hilbert space, then $\pi : A \to M(\mathcal{K}(H)) = \mathcal{B}(H)$ is nondegenerate in the above sense iff $\pi(A)H = H$. If $\pi : A \to M(B)$ is nondegenerate, then there exists a unique $*$-homomorphism $\bar{\pi} : M(A) \to M(B)$ such that $\bar{\pi}|_A = \pi$. We shall usually make no notational difference between π and its extension $\bar{\pi}$.

2.2.3 Commutative C^*-algebras and functional calculus

If A is commutative, then we denote by $\Delta(A)$ the set of all nonzero algebra homomorphisms $\chi : A \to \mathbb{C}$ equipped with the weak-$*$ topology. Then $\Delta(A)$ is locally compact and it is compact if A is unital. If $a \in A$, then $\widehat{a} : \Delta(A) \to \mathbb{C}; \widehat{a}(\chi) := \chi(a)$ is an element of $C_0(\Delta(A))$, and the Gelfand–Naimark theorem asserts that $A \to C_0(\Delta(A)) : a \mapsto \widehat{a}$ is an (isometric) $*$-isomorphism.

If A is any C^*-algebra, then an element $a \in A$ is called *normal* if $a^*a = aa^*$. If $a \in A$ is normal, then $C^*(a, 1) \subseteq A_1$ is a commutative sub-C^*-algebra of A_1. Let $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } A_1\}$ denote the *spectrum* of a, a nonempty compact subset of $\mathbb{C}$. If $\lambda \in \sigma(a)$, then $a - \lambda 1$ generates a unique maximal ideal M_λ of $C^*(a, 1)$ and the quotient map $C^*(a, 1) \to C^*(a, 1)/M_\lambda \cong \mathbb{C}$ determines an element $\chi_\lambda \in \Delta(C^*(a, 1))$. One then checks that $\lambda \mapsto \chi_\lambda$ is a homeomorphism between $\sigma(a)$ and $\Delta(C^*(a, 1))$. Thus, the Gelfand–Naimark theorem provides a $*$-isomorphism $\Phi : C(\sigma(a)) \to C^*(a, 1)$. If $p(z) = \sum_{i,j=0}^{n} \alpha_{ij} z^i \bar{z}^j$ is a polynomial in z and $\bar{z}$ (which by the Stone–Weierstraß theorem form a dense subalgebra of $C(\sigma(a))$), then $\Phi(p) = \sum_{i,j=0}^{n} \alpha_{ij} a^i (a^*)^j$. In particular, we have $\Phi(1) = 1$ and $\Phi(\mathrm{id}_{\sigma(a)}) = a$. In what follows, we always write $f(a)$ for $\Phi(f)$. Note that $\sigma(f(a)) = f(\sigma(a))$ and if $g \in C\big(\sigma(f(a))\big)$, then $g(f(a)) = (g \circ f)(a)$, i.e., the functional calculus is compatible with composition of functions. If A is not unital, then $0 \in \sigma(a)$ and it is clear that for any polynomial p in z and $\bar{z}$ we have $p(a) \in A$ if and only if $p(0) = 0$. Approximating functions by polynomials, it follows that $f(a) \in A$ if and only if $f(0) = 0$ and we obtain an isomorphism $C_0(\sigma(a) \setminus \{0\}) \to C^*(a) \subseteq A; f \mapsto f(a)$.

Example 2.2.1. An element $a \in A$ is called *positive* if $a = b^*b$ for some $b \in A$. This is equivalent to saying that a is self-adjoint (i.e., $a = a^*$) and $\sigma(a) \subseteq [0, \infty)$. If $a \geq 0$, then the functional calculus provides the element $\sqrt{a} \in A$, which is the unique positive element of A such that $(\sqrt{a})^2 = a$. If $a \in A$ is self-adjoint, then $\sigma(a) \subseteq \mathbb{R}$ and the functional calculus allows a unique decomposition $a = a_+ - a_-$ with $a_+, a_- \geq 0$ such that $a_+ \cdot a_- = 0$. Simply take $a_+ = f(a)$ with $f(t) = \max\{t, 0\}$. Since we can write any $b \in A$ as a linear combination of two self-adjoint elements via $b = \frac{1}{2}(a + a^*) + i\frac{1}{2i}(a - a^*)$, we see that every element of A can be written as a linear combination of four positive elements. Since every positive element is a square, it follows that $A = A^2 := \mathrm{LH}\{ab : a, b \in A\}$ (Cohen's factorization theorem even implies that $A = \{ab : a, b \in A\}$).

Every C^*-algebra has an *approximate unit*, i.e., a net $(a_i)_{i \in I}$ in A such that $\|a_i a - a\|, \|a a_i - a\| \to 0$ for all $a \in A$. In fact, $(a_i)_{i \in I}$ can be chosen so that $a_i \geq 0$ and $\|a_i\| = 1$ for all $i \in I$. If A is separable (i.e., A contains a countable dense set), then one can find a sequence $(a_n)_{n \in \mathbb{N}}$ with these properties.

If A is a unital C^*-algebra, then $u \in A$ is called a *unitary*, if $uu^* = u^*u = 1$. If u is unitary, then $\sigma(u) \subseteq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and hence $C^*(u) = C^*(u, 1)$ is isomorphic to a quotient of $C(\mathbb{T})$. Note that if $u, v \in A$ are two unitaries such that $uv = e^{2\pi i\theta} vu$ for some irrational $\theta \in [0, 1]$, then one can show that $\sigma(u) = \sigma(v) = \mathbb{T}$, so that $C^*(u) \cong C^*(v) \cong C(\mathbb{T})$. It follows that the irrational rotation algebra $A_\theta = C^*(u, v)$ should be regarded as (the algebra of functions on) a "noncommutative product" of two tori which results in the expression of a *noncommutative* 2*-torus*.

2.2.4 Representation and ideal spaces of C^*-algebras

If A is a C^*-algebra, the *spectrum* $\widehat{A}$ is defined as the set of all unitary equivalence classes of irreducible representations $\pi : A \to \mathcal{B}(H)$ of A on Hilbert space.[1] We shall usually make no notational difference between an irreducible representation π and its equivalence class $[\pi] \in \widehat{A}$. The *primitive ideals* of A are the kernels of the irreducible representations of A, and we write $\mathrm{Prim}(A) := \{\ker \pi : \pi \in \widehat{A}\}$ for the set of all primitive ideals of A. Every closed two-sided ideal I of A is an intersection of primitive ideals. The spaces $\widehat{A}$ and $\mathrm{Prim}(A)$ are equipped with the *Jacobson topologies*, where the closure operations are given by $\pi \in \overline{R} :\Leftrightarrow \ker \pi \supseteq \cap\{\ker \rho : \rho \in R\}$ (resp. $P \in \overline{R} :\Leftrightarrow P \supseteq \cap\{Q : Q \in R\}$) for $R \subseteq \widehat{A}$ (resp. $R \subseteq \mathrm{Prim}(A)$). In general, the Jacobson topologies are far away from being Hausdorff. In fact, while $\mathrm{Prim}(A)$ is at least always a T_0-space (i.e., for any two different elements in $\mathrm{Prim}(A)$ at least one of them has an open neighborhood that does not contain the other), this very weak separation property often fails for the space $\widehat{A}$. If A is commutative, it follows from Schur's lemma that $\widehat{A} = \Delta(A)$ and the Jacobson topology coincides in this case with the weak-$*$ topology.

If I is a closed two-sided ideal of A, then $\widehat{A}$ can be identified with the disjoint union of $\widehat{I}$ with $\widehat{A/I}$, such that $\widehat{I}$ identifies with $\{\pi \in \widehat{A} : \pi(I) \neq \{0\}\} \subseteq \widehat{A}$ and $\widehat{A/I}$ identifies with $\{\pi \in \widehat{A} : \pi(I) = \{0\}\} \subseteq \widehat{A}$. It follows from the definition of the Jacobson topology that $\widehat{A/I}$ is closed and $\widehat{I}$ is open in $\widehat{A}$. The correspondence $I \leftrightarrow \widehat{I}$ (resp $I \leftrightarrow \widehat{A/I}$) is a one-to-one correspondence between the closed two-sided ideals of A and the open (resp. closed) subsets of $\widehat{A}$. Similar statements hold for the open or closed subsets of $\mathrm{Prim}(A)$.

A C^*-algebra is called *simple* if $\{0\}$ is the only proper closed two-sided ideal of A. Of course, this is equivalent to saying that $\mathrm{Prim}(A)$ has only one element (the zero ideal). Simple C^*-algebras are thought of as the basic "building blocks" of more general C^*-algebras. Examples of simple algebras are the algebras $\mathcal{K}(H)$ of compact operators on a Hilbert space H and the irrational rotation algebras A_θ. Note that while $\widehat{\mathcal{K}(H)}$ has also only one element (the equivalance class of its embedding into $\mathcal{B}(H)$), one can show that $\widehat{A}_\theta$ is an uncountable infinite set (this can actually be deduced from Proposition 2.7.40 below).

A C^*-algebra A is called *type I* (or *GCR*, or *postliminal*) if for every irreducible representation $\pi : A \to \mathcal{B}(H)$ we have $\pi(A) \supseteq \mathcal{K}(H)$. We refer to [Dix77, Chapter 12]

[1] A self-adjoint subset $S \subseteq \mathcal{B}(H)$ is called *irreducible* if there exists no proper nontrivial closed subspace $L \subseteq H$ with $SL \subseteq L$. By Schur's lemma, this is equivalent to saying that the commutator of S in $\mathcal{B}(H)$ is equal to $\mathbb{C} \cdot 1$. A representation $\pi : A \to \mathcal{B}(H)$ is *irreducible* if $\pi(A)$ is irreducible. Two representations π, ρ of A on H_π and H_ρ, respectively, are called unitarily equivalent, if there exists a unitary $V : H_\pi \to H_\rho$ such that $V \circ \pi(a) = \rho(a) \circ V$ for all $a \in A$.

for some important equivalent characterizations of type I algebras. A C^*-algebra A is called *CCR* (or *liminal*), if $\pi(A) = \mathcal{K}(H)$ for every irreducible representation $\pi \in \widehat{A}$. If A is type I, then the mapping $\widehat{A} \to \mathrm{Prim}(A) : \pi \mapsto \ker \pi$ is a homeomorphism, and the converse holds if A is separable (in the nonseparable case the question whether this converse holds leads to quite interesting logical implications, e.g. see [AW04]). Furthermore, if A is type I, then A is CCR if and only if $\widehat{A} \cong \mathrm{Prim}(A)$ is a T_1-space, i.e., points are closed.

A C^*-algebra is said to have *continuous trace* if there exists a dense ideal $\mathfrak{m} \subseteq A$ such that for all positve elements $a \in \mathfrak{m}$ the operator $\pi(a) \in \mathcal{B}(H_\pi)$ is trace-class and the resulting map $\widehat{A} \to [0, \infty); \pi \mapsto \mathrm{tr}(\pi(a))$ is continuous. Continuous-trace algebras are all CCR with Hausdorff spectrum $\widehat{A}$. Note that every type I C^*-algebra A contains a nonzero closed two-sided ideal I such that I is a continuous-trace algebra (see [Dix77, Chapter 4]).

2.2.5 Tensor products

The algebraic tensor product $A \odot B$ of two C^*-algebras A and B has a canonical structure as a $*$-algebra. To make it a C^*-algebra, we have to take completions with respect to suitable cross-norms $\| \cdot \|_\mu$ satisfying $\|a \otimes b\|_\mu = \|a\|\|b\|$. Among the possible choices of such norms there is a maximal cross-norm $\| \cdot \|_{\max}$ and a minimal cross-norm $\| \cdot \|_{\min}$ giving rise to the *maximal tensor product* $A \otimes_{\max} B$ and the *minimal tensor product* $A \otimes_{\min} B$ (which we shall always denote by $A \otimes B$).

The maximal tensor product is characterized by the universal property that any commuting pair of $*$-homomorphisms $\pi : A \to D, \rho : B \to D$ determines a $*$-homomorphism $\pi \times \rho : A \otimes_{\max} B \to D$ such that $\pi \times \rho(a \otimes b) = \pi(a)\rho(b)$ for all elementary tensors $a \otimes b \in A \odot B$. The minimal (or spatial) tensor product $A \otimes B$ is the completion of $A \odot B$ with respect to

$$\left\| \sum_{i=1}^{n} a_i \otimes b_i \right\|_{\min} = \left\| \sum_{i=1}^{n} \rho(a_i) \otimes \sigma(b_i) \right\|,$$

where $\rho : A \to \mathcal{B}(H_\rho), \sigma : B \to \mathcal{B}(H_\sigma)$ are faithful representations of A and B and the norm on the right is taken in $\mathcal{B}(H_\rho \otimes H_\sigma)$. It is a nontrivial fact (due to Takesaki) that $\| \cdot \|_{\min}$ is the smallest cross-norm on $A \odot B$ and that it does not depend on the choice of ρ and σ (e.g., see [RW98, Theorem B.38]).

A C^*-algebra A is called *nuclear*, if $A \otimes_{\max} B = A \otimes B$ for all B. Every type I C^*-algebra is nuclear (e.g. see [RW98, Corollary B.49]) as well as the irrational rotation algebra A_θ (which will follow from Theorem 2.4.7 below). In particular, all commutative C^*-algebras are nuclear and we have $C_0(X) \otimes B \cong C_0(X, B)$ for any locally compact space X. One can show that $\mathcal{B}(H)$ is not nuclear if H is an infinite-dimensional Hilbert space.

If H is an infinite-dimensional Hilbert space, then $\mathcal{K}(H) \otimes \mathcal{K}(H)$ is isomorphic to $\mathcal{K}(H)$ (which can be deduced from a unitary isomorphism $H \otimes H \cong H$). A C^*-algebra A is called *stable* if A is isomorphic to $A \otimes \mathcal{K}$, where we write $\mathcal{K} := \mathcal{K}(l^2(\mathbb{N}))$. It follows from the associativity of taking tensor products that $A \otimes \mathcal{K}$ is always stable and we call $A \otimes \mathcal{K}$ the *stabilisation* of A. Note that $A \otimes \mathcal{K}$ and A have isomorphic representation and ideal spaces. For example, the map $\pi \mapsto \pi \otimes \mathrm{id}_{\mathcal{K}}$ gives a homeomorphism between $\widehat{A} \to (A \otimes \mathcal{K})\widehat{\ }$. Moreover, A is type I (or CCR or continuous-trace or nuclear) if and only if $A \otimes \mathcal{K}$ is.

2.3 Actions and their crossed products

2.3.1 Haar measure and vector-valued integration on groups

If X is a locally compact space, we denote by $C_c(X)$ the set of all continuous functions with compact supports on X. A *positive integral* on $C_c(X)$ is a linear functional $\int : C_c(X) \to \mathbb{C}$ such that $\int_X f(x)\,dx := \int(f) \geq 0$ if $f \geq 0$. We refer to [Rud87] for a good treatment of the Riesz representation theorem which provides a one-to-one correspondence between integrals on $C_c(X)$ and positive Radon measures on X. If H is a Hilbert space and $f : X \to \mathcal{B}(H)$ is a weakly continuous function (i.e., $x \mapsto \langle f(x)\xi, \eta\rangle$ is continuous for all $\xi, \eta \in H$) with compact support, then there exists a unique operator $\int_X f(x)\,dx \in \mathcal{B}(H)$ such that

$$\left\langle \left(\int_X f(x)\,dx \right) \xi, \eta \right\rangle = \int_X \langle f(x)\xi, \eta\rangle\,dx \qquad \text{for all } \xi, \eta \in H.$$

If A is a C^*-algebra imbedded faithfully by a nondegenerate representation into some $\mathcal{B}(H)$ and $f \in C_c(X, A)$ is *norm-continuous*, then approximating f uniformly with controlled supports by elements in the algebraic tensor product $C_c(X) \odot A$ shows that $\int_X f(x)\,dx \in A$. Moreover, if $f : X \to M(A)$ is a strictly continuous function with compact support, then (via the canonical embedding $M(A) \subseteq \mathcal{B}(H)$) f is weakly continuous as a function into $\mathcal{B}(H)$, and since $\big(x \mapsto af(x), f(x)a\big) \in C_c(X, A)$ for all $a \in A$ it follows that $\int_X f(x)\,dx \in M(A)$.

If G is a locally compact group, then there exists a nonzero positive integral $\int : C_c(G) \to \mathbb{C}$, called the *Haar integral* on $C_c(G)$, such that $\int_G f(gx)\,dx = \int_G f(x)\,dx$ for all $f \in C_c(G)$ and $g \in G$. The Haar integral is unique up to multiplication with a positive number, which implies that for each $g \in G$ there exists a positive number $\Delta(g)$ such that $\int_G f(x)\,dx = \Delta(g) \int_G f(xg)\,dx$ for all $f \in C_c(G)$ (since the right-hand side of the equation defines a new Haar integral). One can show that $\Delta : G \to (0, \infty)$ is a continuous group homomorphism. A group G is called unimodular if $\Delta(g) = 1$ for all $g \in G$. All discrete, all compact and all abelian groups are unimodular, however, the $ax+b$-group, which is the semidirect product $\mathbb{R} \rtimes \mathbb{R}^*$ via the action of the multiplicative group $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ on the additive group $\mathbb{R}$ by dilation, is not unimodular. As a general reference for the Haar integral we refer to [DE14].

2.3.2 C^*-dynamical systems and their crossed products

An *action* of a locally compact group G on a C^*-algebra A is a homomorphism $\alpha : G \to \mathrm{Aut}(A); s \mapsto \alpha_s$ of G into the group $\mathrm{Aut}(A)$ of $*$-automorphisms of A such that $s \mapsto \alpha_s(a)$ is continuous for all $a \in A$ (we then say that α is *strongly continuous*). The triple (A, G, α) is then called a *C^*-dynamical system* (or *covariant system*). We also often say that A is a *G-algebra*, when A is equipped with a given G-action α.

Example 2.3.1 (Transformation groups)**.** If $G \times X \to X; (s, x) \mapsto s \cdot x$ is a continuous action of G on a locally compact Hausdorff space X, then G acts on $C_0(X)$ by $\big(\alpha_s(f)\big)(x) := f(s^{-1} \cdot x)$, and it is not difficult to see that every action on $C_0(X)$ arises in this way. Thus, general G-algebras are noncommutative analogues of locally compact G-spaces.

If A is a G-algebra, then $C_c(G, A)$ becomes a $*$-algebra with respect to convolution and involution defined by

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))\, dt \quad \text{and} \quad f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1}))^*. \tag{2.3.1}$$

A *covariant homomorphism* of (A, G, α) into the multiplier algebra $M(D)$ of some C^*-algebra D is a pair (π, U), where $\pi : A \to M(D)$ is a $*$-homomorphism and $U : G \to UM(D)$ is a strictly continuous homomorphism into the group $UM(D)$ of unitaries in $M(D)$ satisfying

$$\pi(\alpha_s(a)) = U_s\pi(a)U_{s^{-1}} \quad \text{for all } s \in G.$$

We say that (π, U) is *nondegenerate* if π is nondegenerate. A *covariant representation* of (A, G, α) on a Hilbert space H is a covariant homomorphism into $M(\mathcal{K}(H)) = \mathcal{B}(H)$. If (π, U) is a covariant homomorphism into $M(D)$, its *integrated form* $\pi \times U : C_c(G, A) \to M(D)$ is defined by

$$(\pi \times U)(f) := \int_G \pi(f(s))U_s\, ds \in M(D). \tag{2.3.2}$$

It is straightforward to check that $\pi \times U$ is a $*$-homomorphism.

Covariant homomorphisms do exist. Indeed, if $\rho : A \to M(D)$ is any $*$-homomorphism, then we can construct the induced covariant homomorphism $\mathrm{Ind}\,\rho := (\tilde{\rho}, 1 \otimes \lambda)$ of (A, G, α) into $M\big(D \otimes \mathcal{K}(L^2(G))\big)$ as follows: Let $\lambda : G \to U(L^2(G))$ denote the *left regular representation* of G given by $(\lambda_s\xi)(t) = \xi(s^{-1}t)$, and define $\tilde{\rho}$ as the composition

$$A \xrightarrow{\tilde{\alpha}} M\big(A \otimes C_0(G)\big) \xrightarrow{\rho\otimes M} M\big(D \otimes \mathcal{K}(L^2(G))\big),$$

where the $*$-homomorphism $\tilde{\alpha} : A \to C_b(G, A) \subseteq M\big(A \otimes C_0(G)\big)$ [2] is defined by $\tilde{\alpha}(a)(s) = \alpha_{s^{-1}}(a)$, and where $M : C_0(G) \to \mathcal{B}(L^2(G)) = M(\mathcal{K}(L^2(G)))$ denotes

[2] $C_b(G, A)$ is regarded as a subset of $M\big(A \otimes C_0(G)\big)$ via the identification $A \otimes C_0(G) \cong C_0(G, A)$ and taking pointwise products of functions.

the represention by multiplication operators. We call $\operatorname{Ind}\rho$ the *covariant homomorphism induced from* ρ, and we shall make no notational difference between $\operatorname{Ind}\rho$ and its integrated form $\tilde{\rho} \times (1 \otimes \lambda)$. $\operatorname{Ind}\rho$ is faithful on $C_c(G, A)$ whenever ρ is faithful on A. If $\rho = \mathrm{id}_A$, the identity on A, then we say that

$$\Lambda_A^G := \operatorname{Ind}(\mathrm{id}_A) : C_c(G, A) \to M\big(A \otimes \mathcal{K}(L^2(G))\big)$$

is *the regular representation* of (A, G, α). Note that

$$\operatorname{Ind}\rho = (\rho \otimes \mathrm{id}_{\mathcal{K}}) \circ \Lambda_A^G \tag{2.3.3}$$

for all $*$-homomorphisms $\rho : A \to M(D)$.[3]

Remark 2.3.2. If we start with a representation $\rho : A \to \mathcal{B}(H) = M(\mathcal{K}(H))$ of A on a Hilbert space H, then $\operatorname{Ind}\rho = (\tilde{\rho}, 1 \otimes \lambda)$ is the representation of (A, G, α) into $\mathcal{B}(H \otimes L^2(G))$ (which equals $M(\mathcal{K}(H) \otimes \mathcal{K}(L^2(G)))$) given by the formulas

$$\big(\tilde{\rho}(a)\xi\big)(t) = \rho(\alpha_{t^{-1}}(a))(\xi(t)) \quad \text{and} \quad \big((1 \otimes \lambda)(s)\xi\big)(t) = \xi(s^{-1}t),$$

for $a \in A$, $s \in G$ and $\xi \in L^2(G, H) \cong H \otimes L^2(G)$. Its integrated form is given by the convolution formula

$$f * \xi(t) := \big(\operatorname{Ind}\rho(f)\xi\big)(t) = \int_G \rho\big(\alpha_{t^{-1}}(f(s))\big)\xi(s^{-1}t)\, ds$$

for $f \in C_c(G, A)$ and $\xi \in L^2(G, H)$.

Definition 2.3.3. Let (A, G, α) be a C^*-dynamical system.

(i) The *full crossed product* $A \rtimes_\alpha G$ (or just $A \rtimes G$ if α is understood) is the completion of $C_c(G, A)$ with respect to

$$\|f\|_{\max} := \sup\{\|(\pi \times U)(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}.$$

(ii) The *reduced crossed product* $A \rtimes_{\alpha,r} G$ (or just $A \rtimes_r G$) is defined as

$$\overline{\Lambda_A^G\big(C_c(G, A)\big)} \subseteq M\big(A \otimes \mathcal{K}(L^2(G))\big).$$

Remark 2.3.4. (1) It follows directly from the above definition that every integrated form $\pi \times U : C_c(G, A) \to M(D)$ of a covariant homomorphism (π, U) extends to a $*$-homomorphism of $A \rtimes_\alpha G$ into $M(D)$. Conversely, every **nondegenerate** $*$-homomorphism $\Phi : A \rtimes_\alpha G \to M(D)$ is of the form $\Phi = \pi \times U$ for some

[3] This equation even makes sense if ρ is degenerate since $\rho \otimes \mathrm{id}_{\mathcal{K}}$ is well defined on the image of $C_b(G, A)$ in $M(A \otimes \mathcal{K}(L^2(G)))$.

nondegenerate covariant homomorphism (π, U). To see this, consider the canonical covariant homomorphism (i_A, i_G) of (A, G, α) into $M(A \rtimes_\alpha G)$ given by the formulas

$$(i_A(a)f)(s) = af(s) \qquad (i_G(t)f)(s) = \alpha_t(f(t^{-1}s))$$
$$(fi_A(a))(s) = f(s)\alpha_s(a) \qquad (fi_G(t))(s) = \Delta(t^{-1})f(st^{-1}),$$

$f \in C_c(G, A)$ (the given formulas extend to left and right multiplications of $i_A(a)$ and $i_G(s)$ with elements in $A \rtimes G$). It is then relatively easy to check that $\Phi = \pi \times U$ with

$$\pi = \Phi \circ i_A \quad \text{and} \quad U = \Phi \circ i_G.$$

Nondegeneracy of Φ is needed to have the compositions $\Phi \circ i_A$ and $\Phi \circ i_G$ well defined. In the definition of $\|\cdot\|_{\max}$ one could restrict to nondegenerate or even (topologically) irreducible representations of (A, G, α) on the Hilbert space. However, it is extremely useful to consider more general covariant homomorphisms into multiplier algebras.

(2) The above described correspondence between nondegenerate representations of (A, G, α) and $A \rtimes G$ induces a bijection between the set $(A, G, \alpha)\widehat{\ }$ of unitary equivalence classes of irreducible covariant Hilbert-space representations of (A, G, α) and $(A \rtimes G)\widehat{\ }$. We topologize $(A, G, \alpha)\widehat{\ }$ such that this bijection becomes a homeomorphism.

(3) The reduced crossed product $A \rtimes_r G$ does not enjoy the above-described universal properties, and therefore it is often more difficult to handle. However, it follows from (2.3.3) that whenever $\rho : A \to M(D)$ is a $*$-homomorphism, then $\operatorname{Ind}\rho$ factors through a representation of $A \rtimes_r G$ to $M(D \otimes \mathcal{K}(L^2(G)))$ which is faithful iff ρ is faithful. In particular, if $\rho : A \to \mathcal{B}(H)$ is a faithful representation of A, then $\operatorname{Ind}\rho$ is a faithful representation of $A \rtimes_r G$ into $\mathcal{B}(H \otimes L^2(G))$.

(4) By construction, the regular representation $\Lambda_A^G : C_c(G, A) \to A \rtimes_r G \subseteq M(A \otimes \mathcal{K}(L^2(G)))$ is the integrated form of the covariant homomorphism $(i_{A,r}, i_{G,r})$ of (A, G, α) into $M(A \otimes \mathcal{K}(L^2(G)))$ with

$$i_{A,r} = (\mathrm{id}_A \otimes M) \circ \tilde{\alpha} \quad \text{and} \quad i_{G,r} = 1_A \otimes \lambda_G.$$

Since both, $\tilde{\alpha} : A \to M(A \otimes C_0(G))$ and $\mathrm{id}_A \otimes M : A \otimes C_0(G) \to M(A \otimes \mathcal{K}(L^2(G)))$ are faithful, it follows that $i_{A,r}$ is faithful, too. Since $i_{A,r} = \Lambda_A^G \circ i_A$, where i_A denotes the embedding of A into $M(A \rtimes G)$, we see that i_A is injective, too.

(5) If G is discrete, then A embeds into $A \rtimes_{(r)} G$ via $a \mapsto \delta_e \otimes a \in C_c(G, A) \subseteq A \rtimes_{(r)} G$. If, in addition, A is unital, then G also embeds into $A \rtimes_{(r)} G$ via $g \mapsto \delta_g \otimes 1$. If we identify $a \in A$ and $g \in G$ with their images in $A \rtimes_{(r)} G$, we obtain the relations $ga = \alpha_g(a)g$ for all $a \in A$ and $g \in G$. The full crossed product is then the universal C^*-algebra generated by A and G (viewed as a group of unitaries) subject to the relation $ga = \alpha_g(a)g$.

(6) In the case $A = \mathbb{C}$, the maximal crossed product $C^*(G) := \mathbb{C} \rtimes G$ is called the *full group C^*-algebra* of G (note that $\mathbb{C}$ has only the trivial $*$-automorphism). The universal properties of $C^*(G)$ translate into a one-to-one correspondence between the unitary representations of G and the nondegenerate $*$-representations of $C^*(G)$ which induces a bijection between the set $\widehat{G}$ of equivalence classes of irreducible unitary Hilbert-space representations of G and $\widehat{C^*(G)}$. Again, we topologize $\widehat{G}$ so that this bijection becomes a homeomorphism.
The *reduced group C^*-algebra* $C_r^*(G) := \mathbb{C} \rtimes_r G$ is realized as the closure $\overline{\lambda(C_c(G))} \subseteq \mathcal{B}(L^2(G))$, where λ denotes the regular representation of G.

(7) If G is compact, then every irreducible representation of G is finite-dimensional and the Jacobson topology on $\widehat{G} = \widehat{C^*(G)}$ is the discrete topology. Moreover, it follows from the Peter–Weyl theorem (e.g., see [DE14, Fol95]) that $C^*(G)$ and $C_r^*(G)$ are isomorphic to the C^*-direct sum $\bigoplus_{U \in \widehat{G}} M_{\dim U}(\mathbb{C})$. In particular, we have $C^*(G) = C_r^*(G)$ if G is compact.

(8) The convolution algebra $C_c(G)$, and hence also its completion $C^*(G)$, is commutative if and only if G is abelian. In that case $\widehat{G}$ coincides with the set of continuous homomorphisms from G to the circle group $\mathbb{T}$, called *characters of* G, equipped with the compact-open topology. The Gelfand–Naimark theorem for commutative C^*-algebras then implies that $C^*(G) \cong C_0(\widehat{G})$ (which also coincides with $C_r^*(G)$ in this case). Note that $\widehat{G}$, equipped with the pointwise multiplication of characters, is again a locally compact abelian group and the Pontrjagin duality theorem asserts that $\widehat{\widehat{G}}$ is isomorphic to G via $g \mapsto \widehat{g} \in \widehat{\widehat{G}}$ defined by $\widehat{g}(\chi) = \chi(g)$. Note that the Gelfand isomorphism $C^*(G) \cong C_0(\widehat{G})$ extends the *Fourier transform*

$$\mathcal{F} : C_c(G) \to C_0(\widehat{G}); \mathcal{F}(f)(\chi) = \chi(f) = \int_G f(x)\chi(x)\, dx.$$

For the circle group $\mathbb{T}$ we have $\mathbb{Z} \cong \widehat{\mathbb{T}}$ via $n \mapsto \chi_n$ with $\chi_n(z) = z^n$, and one checks that the above Fourier transform coincides with the classical Fourier transform on $C(\mathbb{T})$. Similarly, if $G = \mathbb{R}$, then $\mathbb{R} \cong \widehat{\mathbb{R}}$ via $s \mapsto \chi_s$ with $\chi_s(t) = e^{2\pi i s t}$ and we recover the classical Fourier transform on $\mathbb{R}$. We refer to [DE14, Chapter 3] for a detailed treatment of Pontrjagin duality and its connection to the Gelfand isomorphism

Example 2.3.5 (Transformation group algebras)**.** If (X, G) is a topological dynamical system, then we can form the crossed products $C_0(X) \rtimes G$ and $C_0(X) \rtimes_r G$ with respect to the corresponding action of G on $C_0(X)$. These algebras are often called the (full and reduced) *transformation group algebras* of the dynamical system (X, G). Many important C^*-algebras are of this type. For instance if $X = \mathbb{T}$ is the circle group and $\mathbb{Z}$ acts on $\mathbb{T}$ via $n \cdot z = e^{i2\pi\theta n}z$, $\theta \in [0, 1]$, then $A_\theta = C(\mathbb{T}) \rtimes \mathbb{Z}$ is the (rational or irrational) rotation algebra corresponding to θ (compare with §2.2.1 above). Indeed, since $\mathbb{Z}$ is discrete and $C(\mathbb{T})$ is unital, we have canonical embeddings of $C(\mathbb{T})$ and $\mathbb{Z}$ into $C(\mathbb{T}) \rtimes \mathbb{Z}$. If we denote by v the image of $\mathrm{id}_{\mathbb{T}} \in C(\mathbb{T})$

and by u the image of $1 \in \mathbb{Z}$ under these embeddings, then the relations given in part (5) of the above remark show that u, v are unitaries that satisfy the basic commutation relation $uv = e^{2\pi i\theta}vu$. It is this realization as a crossed product of A_θ that motivates the notion "rotation algebra".

There is some quite interesting and deep work on crossed products by actions of $\mathbb{Z}$ (or $\mathbb{Z}^d$) on compact spaces, which we cannot cover in this article. We refer the interested reader to the article [GPS06] for a survey and for further references to this work.

Example 2.3.6 (Decomposition action)**.** Assume that $G = N \rtimes H$ is the semi-direct product ot two locally compact groups. If A is a G-algebra, then H acts canonically on $A \rtimes N$ (resp. $A \rtimes_r N$) via the extension of the action γ of H on $C_c(N, A)$ given by

$$\big(\gamma_h(f)\big)(n) = \delta(h)\alpha_h\big(f(h^{-1} \cdot n)\big),$$

where $\delta : H \to \mathbb{R}^+$ is determined by the equation $\int_N f(h \cdot n)\,dn = \delta(h)\int_N f(n)\,dn$ for all $f \in C_c(N)$. The inclusion $C_c(N, A) \subseteq A \rtimes_{(r)} N$ determines an inclusion $C_c(N \times H, A) \subseteq C_c(H, A \rtimes_{(r)} N)$ which extends to isomorphisms $A \rtimes (N \rtimes H) \cong (A \rtimes N) \rtimes H$ and $A \rtimes_r (N \rtimes H) \cong (A \rtimes_r N) \rtimes_r H$. In particular, if $A = \mathbb{C}$, we obtain canonical isomorphisms $C^*(N \rtimes H) \cong C^*(N) \rtimes H$ and $C_r^*(N \rtimes H) \cong C_r^*(N) \rtimes_r H$.

We shall later extend the notion of crossed products to allow also the decomposition of crossed products by group extensions that are not topologically split.

Remark 2.3.7. When working with crossed products, it is often useful to use the following concrete realization of an approximate unit in $A \rtimes G$ (resp. $A \rtimes_r G$) in terms of a given approximate unit $(a_i)_{i\in I}$ in A: Let $\mathcal{U}$ be any neighborhood basis of the identity e in G, and for each $U \in \mathcal{U}$ let $\varphi_U \in C_c(G)^+$ with $\operatorname{supp}\varphi_U \subseteq U$, $\varphi_U(s) = \varphi_U(s^{-1})$ for all $s \in G$, and such that $\int_G \varphi_U(t)\,dt = 1$. Let $\Lambda = I \times \mathcal{U}$ with $(i_1, U_1) \geq (i_2, U_2)$ if $i_1 \geq i_2$ and $U_1 \subseteq U_2$. Then a straightforward computation in the dense subalgebra $C_c(G, A)$ shows that $(\varphi_U \otimes a_i)_{(i,U)\in\Lambda}$ is an approximate unit of $A \rtimes G$ (resp. $A \rtimes_r G$), where we write $\varphi \otimes a$ for the function $(t \mapsto \varphi(t)a) \in C_c(G, A)$ if $\varphi \in C_c(G)$ and $a \in A$.

2.4 Crossed products versus tensor products

The following lemma indicates the conceptual similarity of full crossed products with maximal tensor products and of reduced crossed products with minimal tensor products of C^*-algebras.

Lemma 2.4.1. *Let (A, G, α) be a C^*-dynamical system and let B be a C^*-algebra. Let $\mathrm{id}\otimes_{\max}\alpha : G \to \mathrm{Aut}(B \otimes_{\max} A)$ be the diagonal action of G on $B \otimes_{\max} A$ (i.e., G acts trivially on B), and let $\mathrm{id}\otimes\alpha : G \to \mathrm{Aut}(B \otimes A)$ denote the diagonal action on $B \otimes A$. Then the obvious map $B \odot C_c(G, A) \to C_c(G, B \odot A)$ induces isomorphisms*

$$B\otimes_{\max}(A\rtimes_\alpha G) \cong (B\otimes_{\max}A)\rtimes_{\mathrm{id}\otimes\alpha}G \quad \textit{and} \quad B\otimes(A\rtimes_{\alpha,r}G) \cong (B\otimes A)\rtimes_{\mathrm{id}\otimes\alpha,r}G.$$

Sketch of proof. For the full crossed products, check that both sides have the same nondegenerate representations and use the universal properties of the full crossed products and the maximal tensor product. For the reduced crossed products, observe that the map $B \odot C_c(G, A) \to C_c(G, B \odot A)$ identifies $\mathrm{id}_B \otimes \Lambda_A^G$ with $\Lambda_{B\otimes A}^G$. □

Remark 2.4.2. As a special case of the above lemma (with $A = \mathbb{C}$) we see in particular that

$$B \rtimes_{\mathrm{id}} G \cong B \otimes_{\max} C^*(G) \quad \text{and} \quad B \rtimes_{\mathrm{id},r} G \cong B \otimes C_r^*(G).$$

We now want to study an important condition on G which implies that full and reduced crossed products by G always coincide.

Definition 2.4.3. Let $1_G : G \to \{1\} \subseteq \mathbb{C}$ denote the trivial representation of G. Then G is called *amenable* if $\ker 1_G \supseteq \ker \lambda$ in $C^*(G)$, i.e., if the integrated form of 1_G factors through a homomomorphism $1_G^r : C_r^*(G) \to \mathbb{C}$.[4]

Remark 2.4.4. The above definition is not the standard definition of amenability of groups, but it is one of the many equivalent formulations for amenability (e.g., see [Dix77, Pat88]), and it is best suited for our purposes. It is not hard to check (even using the above C^*-theoretic definition) that abelian groups and compact groups are amenable. Moreover, extensions, quotients, and closed subgroups of amenable groups are again amenable. In particular, all solvable groups are amenable.

On the other side, one can show that the nonabelian free group F_2 on two generators, and hence any group that contains F_2 as a closed subgroup, is not amenable. This shows that noncompact semi-simple Lie groups are never amenable. For extensive studies of amenability of groups (and groupoids) we refer the reader to [Pat88, ADR00].

If (π, U) is a covariant representation of (A, G, α) on some Hilbert space H, then the covariant representation $(\pi \otimes 1, U \otimes \lambda)$ of (A, G, α) on $H \otimes L^2(G) \cong L^2(G, H)$ is unitarily equivalent to $\mathrm{Ind}\,\pi$ via the unitary $W \in U(L^2(G, H))$ defined by $(W\xi)(s) = U_s\xi(s)$ (this simple fact is known as *Fell's trick*). Thus, if π is faithful on A, then $(\pi \otimes 1) \times (U \otimes \lambda)$ factors through a faithful representation of $A \rtimes_r G$. As an important application we get

Proposition 2.4.5. *If G is amenable, then $\Lambda_A^G : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ is an isomorphism.*

Proof. Choose any faithful representation $\pi \times U$ of $A \rtimes_\alpha G$ on some Hilbert space H. Regarding $(\pi\otimes 1, U\otimes\lambda)$ as a representation of (A, G, α) into $M(\mathcal{K}(H)\otimes C_r^*(G))$, we obtain the equation

$$(\mathrm{id} \otimes 1_G^r) \circ \big((\pi \otimes 1) \times (U \otimes \lambda)\big) = \pi \times U.$$

[4] In particular, it follows that $1_G^r(\lambda_s) = 1_G(s) = 1$ for all $s \in G$!

Since π is faithful, it follows that

$$\ker \Lambda_A^G = \ker(\operatorname{Ind}\pi) = \ker\big((\pi \otimes 1) \times (U \otimes \lambda)\big) \subseteq \ker(\pi \times U) = \{0\}. \qquad \square$$

The special case $A = \mathbb{C}$ gives

Corollary 2.4.6. *G is amenable if and only if $\lambda : C^*(G) \to C_r^*(G)$ is an isomorphism.*

A combination of Lemma 2.4.1 with Proposition 2.4.5 gives the following important result:

Theorem 2.4.7. *Let A be a nuclear G-algebra with G amenable. The $A \rtimes_\alpha G$ is nuclear.*

Proof. Using Lemma 2.4.1 and Proposition 2.4.5 we get

$$\begin{aligned} B \otimes_{\max} (A \rtimes_\alpha G) &\cong (B \otimes_{\max} A) \times_{\mathrm{id}\otimes\alpha} G \cong (B \otimes A) \times_{\mathrm{id}\otimes\alpha} G \\ &\cong (B \otimes A) \times_{\mathrm{id}\otimes\alpha,r} G \cong B \otimes (A \rtimes_{\alpha,r} G) \cong B \otimes (A \rtimes_\alpha G). \square \end{aligned}$$

If (A, G, α) and (B, G, β) are two systems, then a G-equivariant homomorphism $\phi : A \to M(B)$ [5] induces a $*$-homomorphism

$$\phi \rtimes G := (i_B \circ \phi) \times i_G : A \rtimes_\alpha G \to M(B \times_\beta G)$$

where (i_B, i_G) denote the canonical embeddings of (B, G) into $M(B \rtimes_\beta G)$, and a similar $*$-homomorphism

$$\phi \rtimes_r G := \operatorname{Ind}\phi : A \rtimes_{\alpha,r} G \to M(B \rtimes_{\beta,r} G) \subseteq M\big(B \otimes \mathcal{K}(L^2(G))\big).$$

Both maps are given on the level of functions by

$$\phi \rtimes_{(r)} G(f)(s) = \phi(f(s)), \quad f \in C_c(G, A).$$

If $\phi(A) \subseteq B$, then $\phi \rtimes G(A \rtimes_\alpha G) \subseteq B \rtimes_\beta G$ and similarly for the reduced crossed products. Moreover, $\phi \rtimes_r G = \operatorname{Ind}\phi$ is faithful if and only if ϕ is – a result that does **not** hold in general for $\phi \rtimes G$!

On the other hand, the following proposition shows that taking full crossed products gives an exact functor between the category of G-C^*-algebras and the category of C^*-algebras, which is not always true for the reduced crossed product functor!

Proposition 2.4.8. *Assume that $\alpha : G \to \operatorname{Aut}(A)$ is an action and I is a G-invariant closed ideal in A. Let $j : I \to A$ denote the inclusion and let $q : A \to A/I$ denote the quotient map. Then the sequence*

$$0 \to I \rtimes_\alpha G \xrightarrow{j \rtimes G} A \rtimes_\alpha G \xrightarrow{q \rtimes G} (A/I) \rtimes_\alpha G \to 0$$

is exact.

[5] where we uniquely extend β to an action of $M(B)$, which may fail to be strongly continuous.

Proof. If (π, U) is a nondegenerate representation of (I, G, α) into $M(D)$, then (π, U) has a canonical extension to a covariant homomorphism of (A, G, α) by defining $\pi(a)(\pi(b)d) = \pi(ab)d$ for $a \in A, b \in I$ and $d \in D$. By the definition of $\|\cdot\|_{\max}$, this implies that the inclusion $I \rtimes_\alpha G \to A \rtimes_\alpha G$ is isometric.

Assume now that $p : A \rtimes_\alpha G \to (A \rtimes_\alpha G)/(I \rtimes_\alpha G)$ is the quotient map. Then $p = \rho \times V$ for some covariant homomorphism (ρ, V) of (A, G, α) into $M\big((A \rtimes G)/(I \rtimes G)\big)$. Let $i_A : A \to M(A \rtimes G)$ denote the embedding. Then we have $i_A(I)C_c(G, A) = C_c(G, I) \subseteq I \rtimes G$ from which it follows that

$$\rho(I)\big(\rho \times V(C_c(G, A))\big) = \rho \times V\big(i_A(I)(A \rtimes G)\big) \subseteq \rho \times V(I \rtimes G) = \{0\}.$$

Since $\rho \times V(C_c(G, A))$ is dense in $A/I \rtimes G$, it follows that $\rho(I) = \{0\}$. Thus ρ factors through a representation of A/I and $p = \rho \times V$ factors through $A/I \rtimes_\alpha G$. This shows that the crossed product sequence is exact in the middle term. Since $C_c(G, A)$ clearly maps onto a dense subset in $A/I \rtimes_\alpha G$, $q \rtimes G$ is surjective and the result follows. □

For quite some time it was an open question whether the analogue of Proposition 2.4.8 also holds for the reduced crossed products. This problem led to

Definition 2.4.9 (Kirchberg–S. Wassermann)**.** A locally compact group G is called C^**-exact* (or simply exact) if for any system (A, G, α) and any G-invariant ideal $I \subseteq A$ the sequence

$$0 \to I \rtimes_{\alpha,r} G \xrightarrow{j \rtimes_r G} A \rtimes_{\alpha,r} G \xrightarrow{q \rtimes_r G} A/I \rtimes_{\alpha,r} G \to 0$$

is exact.

Let us remark that the only problem is exactness in the middle term, since $q \rtimes_r G$ is clearly surjective, and $j \rtimes_r G = \operatorname{Ind} j$ is injective since j is. We shall later report on Kirchberg's and S. Wassermann's permanence results on exact groups, which imply that the class of exact groups is indeed very large. However, a construction based on ideas of Gromov (see [Gro00, Ghy04, Osa14]) implies that there do exist finitely generated discrete groups that are not exact!

2.5 The correspondence categories

In this section we want to give some theoretical background for the discussion of imprimitivity theorems for crossed products and for the theory of induced representations on the one hand, and for the construction of Kasparov's bivariant K-theory groups on the other hand. The basic notion for this is the notion of the correspondence category in which the objects are C^*-algebras and the morphisms are unitary equivalence classes of Hilbert bimodules. Having this at hand, the theory of induced representations will reduce to taking compositions of morphisms in the correspondence category. All this is based on the fundamental idea of Rieffel (see [Rie74]) who first made a systematic approach to the theory of induced representations of C^*-algebras in terms of (pre-) Hilbert modules, and who showed how the theory of induced group representations can be seen as part of this more general theory. However, it seems that a systematic categorical treatment of this theory was first given in [EKQR00] and, in parallel work, by Landsman in [Lan01]. The standard reference for Hilbert modules is [Lan95].

2.5.1 Hilbert modules

If B is a C^*-algebra, then a (right) Hilbert B-module is a complex Banach space E equipped with a right B-module structure and a positive definite B-valued inner product (with respect to positivity in B) $\langle\cdot,\cdot\rangle_B : E \times E \to B$, which is linear in the second and antilinear in the first variable and satisfies

$$(\langle \xi, \eta\rangle_B)^* = \langle \eta, \xi\rangle_B, \quad \langle \xi, \eta\rangle_B b = \langle \xi, \eta \cdot b\rangle_B, \quad \text{and} \quad \|\xi\|^2 = \|\langle \xi, \xi\rangle_B\|$$

for all $\xi, \eta \in E$ and $b \in B$. With the obvious modifications we can also define *left*-Hilbert B-modules. The Hilbert $\mathbb{C}$-modules are precisely the Hilbert spaces. Moreover, every C^*-algebra B becomes a Hilbert B-module by defining $\langle b, c\rangle_B := b^*c$. We say that E is a *full* Hilbert B-module, if

$$B = \langle E, E\rangle_B := \overline{\text{span}}\{\langle \xi, \eta\rangle_B : \xi, \eta \in E\}.$$

In general, $\langle E, E\rangle_B$ is a closed two-sided ideal of B.

If E and F are Hilbert B-modules, then a linear map $T : E \to F$ is called *adjointable* if there exists a map $T^* : F \to E$ such that $\langle T\xi, \eta\rangle_B = \langle \xi, T^*\eta\rangle_B$ for all $\xi \in E$, $\eta \in F$.[6] Every adjointable operator from E to F is automatically bounded and B-linear. We write $\mathcal{L}_B(E, F)$ for the set of adjointable operators from E to F. Then

$$\mathcal{L}_B(E) := \mathcal{L}_B(E, E)$$

becomes a C^*-algebra with respect to the usual operator norm. Every pair ξ, η with $\xi \in F$, $\eta \in E$ determines an element $\Theta_{\xi,\eta} \in \mathcal{L}_B(E, F)$ given by

$$\Theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle_B \tag{2.5.1}$$

with adjoint $\Theta^*_{\xi,\eta} = \Theta_{\eta,\xi}$. The closed linear span of all such operators forms the set of compact operators $\mathcal{K}_B(E, F)$ in $\mathcal{L}_B(E, F)$. If $E = F$, then $\mathcal{K}_B(E) := \mathcal{K}_B(E, E)$ is a closed ideal in $\mathcal{L}_B(E)$. Note that there is an obvious $*$-isomorphism between the multiplier algebra $M(\mathcal{K}_B(E))$ and $\mathcal{L}_B(E)$, which is given by extending the action of $\mathcal{K}_B(E)$ on E to all of $M(\mathcal{K}_B(E))$ in the canonical way.

[6] Note that, different from the operators on Hilbert space, a bounded B-linear operator $T : E \to F$ is **not** automatically adjointable.

Example 2.5.1. **(1)** If $B = \mathbb{C}$ and H is a Hilbert space, then $\mathcal{L}_{\mathbb{C}}(H) = \mathcal{B}(H)$ and $\mathcal{K}_{\mathbb{C}}(H) = \mathcal{K}(H)$.

(2) If a C^*-algebra B is viewed as a Hilbert B-module with respect to the inner product $\langle b, c\rangle_B = b^*c$ and the obvious right module operation then $\mathcal{K}_B(B) = B$, where we let B act on itself via left multiplication, and we have $\mathcal{L}_B(B) = M(B)$.

It is important to note that, in the case $B \neq \mathbb{C}$, the notion of compact operators as given above does **not** coincide with the standard notion of compact operators on a Banach space (i.e., that the image of the unit ball has compact closure). For example, if B is unital, then $\mathcal{L}_B(B) = \mathcal{K}_B(B) = B$ and we see that the identity operator on B is a compact operator in the sense of the above definition. But if B is not finite-dimensional, the identity operator is not a compact operator in the usual sense of Banach-space operators.

There is a one-to-one correspondence between right and left Hibert B-modules given by the operation $E \mapsto E^* := \{\xi^* : \xi \in E\}$, with left action of B on E^* given by $b \cdot \xi^* := (\xi \cdot b^*)^*$ and with inner product ${}_B\langle \xi^*, \eta^*\rangle := \langle \xi, \eta\rangle_B$ (note that the inner product of a left Hilbert B-module is linear in the first and antilinear in the second variable). We call E^* the *adjoint module* of E. Of course, if F is a left Hilbert B-module, a similar construction yields an adjoint F^* – a right Hilbert B-module. Clearly, the notions of adjointable and compact operators also have their left analogues (thought of as acting on the right), and we have $\mathcal{L}_B(E) = \mathcal{L}_B(E^*)$ (resp. $\mathcal{K}_B(E) = \mathcal{K}_B(E^*)$) via $\xi^*T := (T^*\xi)^*$.

There are several important operations on Hilbert modules (such as taking the direct sum $E_1 \bigoplus E_2$ of two Hilbert B-modules E_1 and E_2 in the obvious way). But for our considerations the construction of the interior tensor products is most important. For this case assume that E is a (right) Hilbert A-module, F is a (right) Hilbert B-module, and $\Psi : A \to \mathcal{L}_B(F)$ is a $*$-homomorphism. Then the *interior tensor product* $E \otimes_A F$ is defined as the Hausdorff completion of $E \odot F$ with respect to the B-valued inner product

$$\langle \xi \otimes \eta, \xi' \otimes \eta'\rangle_B = \langle \eta, \Psi(\langle \xi, \xi'\rangle_A) \cdot \eta'\rangle_B,$$

where $\xi, \xi' \in E$ and $\eta, \eta' \in F$. With this inner product, $E \otimes_A F$ becomes a Hilbert B-module. Moreover, if C is a third C^*-algebra and if $\Phi : C \to \mathcal{L}_A(E)$ is a $*$-representation of C on $\mathcal{L}_A(E)$, then $\Phi\otimes 1 : C \to \mathcal{L}_B(E\otimes_A F)$ with $\Phi\otimes 1(c)(\xi\otimes\eta) = \Phi(c)\xi \otimes \eta$ becomes a $*$-representation of C on $E \otimes_A F$ (we refer to [Lan95, RW98] for more details). The construction of this representation is absolutely crucial in what follows below.

2.5.2 Morita equivalences

The notion of Morita equivalent C^*-algebras, which goes back to Rieffel [Rie74] is one of the most important tools in the study of crossed products.

Definition 2.5.2 (Rieffel)**.** Let A and B be C^*-algebras. An A-B *imprimitivity bimodule*[7] X is a Banach space X that carries the structure of both, a right Hilbert B-module and a left Hilbert A-module with commuting actions of A and B such that

(i) ${}_A\langle X, X\rangle = A$ and $\langle X, X\rangle_B = B$ (i.e., both inner products on X are full);

(ii) ${}_A\langle \xi, \eta\rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta\rangle_B$ for all $\xi, \eta, \zeta \in X$.

A and B are called *Morita equivalent* if such an A-B bimodule X exists.

Remark 2.5.3. (1) It follows from the above definition together with (2.5.1) that, if X is an A-B imprimitivity bimodule, then A canonically identifies with $\mathcal{K}_B(X)$ and B canonically identifies with $\mathcal{K}_A(X)$. Conversely, if E is any Hilbert B-module, then ${}_{\mathcal{K}(E)}\langle \xi, \eta\rangle := \Theta_{\xi,\eta}$ (see (2.5.1)) defines a full $\mathcal{K}_B(E)$-valued inner product on E, and E becomes a $\mathcal{K}_B(E)$-$\langle E, E\rangle_B$ imprimitivity bimodule. In particular, if E is a full Hilbert B-module (i.e., $\langle E, E\rangle_B = B$), then B is Morita equivalent to $\mathcal{K}_B(E)$.

(2) As a very special case of (1) we see that $\mathbb{C}$ is Morita equivalent to $\mathcal{K}(H)$ for every Hilbert space H.

(3) It is easily checked that Morita equivalence is an equivalence relation: If A is any C^*-algebra, then A becomes an A-A imprimitivity bimodule with respect to ${}_A\langle a, b\rangle = ab^*$ and $\langle a, b\rangle_A = a^*b$ for $a, b \in A$. If X is an A-B imprimitivity bimodule and Y is a B-C imprimitivity bimodule, then $X \otimes_B Y$ is an A-C imprimitivity bimodule. Finally, if X is an A-B-imprimitivity bimodule, then the adjoint module X^* is a B-A imprimitivity bimodule.

(4) Recall that a C^*-algebra A is a *full corner* of the C^*-algebra C, if there exists a full projection $p \in M(C)$ (i.e., $\overline{CpC} = C$) such that $A = pCp$. Then pC equipped with the canonical inner products and actions coming from multiplication and involution on C becomes an A-C imprimitivity bimodule. Thus, if A and B can be represented as full corners of a C^*-algebra C, they are Morita equivalent. Conversely, let X be an A-B imprimitivity bimodule. Let $L(X) = \left(\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right)$ with multiplication and involution defined by

$$\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix}\begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + {}_A\langle \xi_1, \eta_2\rangle & a_1 \cdot \xi_2 + \xi_1 \cdot b_2 \\ \eta_1^* \cdot a_2 + b_1 \cdot \eta_2^* & \langle \eta_1, \xi_2\rangle_B + b_1b_2 \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* = \begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix}.$$

Then $L(X)$ has a canonical embedding as a closed subalgebra of the adjointable operators on the Hilbert B-module $X \bigoplus B$ via

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}\begin{pmatrix} \zeta \\ d \end{pmatrix} = \begin{pmatrix} a\zeta + \xi d \\ \langle \eta, \zeta\rangle_B + bd \end{pmatrix}$$

[7] often called an A-B *equivalence* bimodule in the literature.

which makes $L(X)$ a C^*-algebra. If $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M(L(X))$, then p and $q := 1 - p$ are full projections such that $A = pL(X)p$, $B = qL(X)q$ and $X = pL(X)q$. The algebra $L(X)$ is called the *linking algebra* of X. It often serves as a valuable tool for the study of imprimitivity bimodules.

(5) It follows from **(4)** that A is Morita equivalent to $A \otimes \mathcal{K}(H)$ for any Hilbert space H (since A is a full corner of $A \otimes \mathcal{K}(H)$). Indeed, a deep theorem of Brown, Green and Rieffel (see [BGR77]) shows that if A and B are σ-unital[8], then A and B are Morita equivalent if and only if they are *stably isomorphic*, i.e., there exists an isomorphism between $A \otimes \mathcal{K}(H)$ and $B \otimes \mathcal{K}(H)$ with $H = l^2(\mathbb{N})$. A similar result does not hold if the σ-unitality assumption is dropped (see [BGR77]).

(6) The above results indicate that many important properties of C^*-algebras are preserved by Morita equivalences. Indeed, among these properties are: nuclearity, exactness, simplicity, the property of being a type I algebra (and many more). Moreover, Morita-equivalent C^*-algebras have homeomorphic primitive ideal spaces and isomorphic K-groups. Most of these properties will be discussed later in more detail (e.g., see Propositions 2.5.4, 2.5.11 and 2.5.12 below). The K-theoretic implications are discussed in Chapter 3 of this book.

A very important tool when working with imprimitivity bimodules is the Rieffel correspondence. To explain this correspondence suppose that X is an A-B imprimitivity bimodule and that I is a closed ideal of B. Then $X \cdot I$ is a closed A-B submodule of X and $\operatorname{Ind}^X I := {}_A\langle X \cdot I, X \cdot I\rangle$ (taking the closed span) is a closed ideal of A. The following proposition implies that Morita-equivalent C^*-algebras have equivalent ideal structures:

Proposition 2.5.4 (Rieffel correspondence)**.** *Assume notation as above. Then*

(i) *The assignments $I \mapsto X \cdot I$, $I \mapsto \operatorname{Ind}^X I$ and $I \mapsto J_I := \left(\begin{smallmatrix} \operatorname{Ind}^X I & X \cdot I \\ I \cdot X^* & I \end{smallmatrix}\right)$ provide inclusion-preserving bijective correspondences between the closed two-sided ideals of B, the closed A-B-submodules of X, the closed two-sided ideals of A, and the closed two-sided ideals of the linking algebra $L(X)$, respectively.*

(ii) *$X \cdot I$ is an $\operatorname{Ind}^X I$-I imprimitivity bimodule and $X/(X \cdot I)$, equipped with the obvious inner products and bimodule actions, becomes an $A/(\operatorname{Ind}^X I)$-$B/I$ imprimitivity bimodule. Moreover, we have $J_I = L(X \cdot I)$ and $L(X)/J_I \cong L(X/X \cdot I)$.*

Remark 2.5.5. Assume that X is an A-B imprimitivity bimodule and Y is a C-D imprimitivity bimodule. An *imprimitivity bimodule homomorphism* from X to Y is then a triple (ϕ_A, ϕ_X, ϕ_B) such that $\phi_A : A \to C$ and $\phi_B : B \to D$ are $*$-homomorphisms and $\phi_X : X \to Y$ is a linear map such that the triple (ϕ_A, ϕ_X, ϕ_B) satisfies the obvious compatibility conditions with respect to the inner products

[8] A C^*-algebra is called *σ-unital*, if it has a countable approximate unit. In particular, all separable and all unital C^*-algebras are σ-unital.

and module actions on X and Y (e.g. $\langle \phi_X(\xi), \phi_X(\eta)\rangle_D = \phi_B(\langle \xi, \eta\rangle_B)$, $\phi_X(\xi b) = \phi_X(\xi)\phi_B(b)$, etc.).

If (ϕ_A, ϕ_X, ϕ_B) is such an imprimitivity bimodule homomorphism, then one can check that $\ker \phi_A$, $\ker \phi_X$ and $\ker \phi_B$ all correspond to one another under the Rieffel correspondence for X (e.g., see [EKQR06, Chapter 1]).

As a simple application of the Rieffel correspondence and the above remark we now show:

Proposition 2.5.6. *Suppose that A and B are Morita-equivalent C^*-algebras. Then A is nuclear if and only if B is nuclear.*

Sketch of proof. Let X be an A-B imprimitivity bimodule. If C is any other C^*-algebra, we can equip $X \odot C$ with $A \odot C$- and $B \odot C$-valued inner products and an $A \odot C$-$B \odot C$ module structure in the obvious way. Then one can check that $X \odot C$ completes to an $A \otimes_{\max} C$-$B \otimes_{\max} C$ imprimitivity bimodule $X \otimes_{\max} C$ as well as to an $A \otimes C$-$B \otimes C$ imprimitivity bimodule $X \otimes C$. The identity map on $X \odot C$ then extends to a quotient map $X \otimes_{\max} C \to X \otimes C$ which together with the quotient maps $A \otimes_{\max} C \to A \otimes C$ and $B \otimes_{\max} C \to B \otimes C$ is an imprimitivity bimodule homomorphism. But then it follows from the above remark and the Rieffel correspondence that injectivity of any one of these quotient maps implies injectivity of all three of them. □

2.5.3 The correspondence categories

We now come to the definition of the correspondence categories. Suppose that A and B are C^*-algebras. A *(right) Hilbert A-B bimodule* is a pair (E, Φ) in which E is a Hilbert B-module and $\Phi : A \to \mathcal{L}_B(E)$ is a $*$-representation of A on E. We say that (E, Φ) is nondegenerate, if $\Phi(A)E = E$ (this is equivalent to $\Phi : A \to M(\mathcal{K}_B(E)) = \mathcal{L}_B(E)$ being nondegenerate in the usual sense). Two Hilbert A-B bimodules (E_i, Φ_i), $i = 1, 2$ are called *unitarily equivalent* if there exists an isomorphism $U : E_1 \to E_2$ preserving the B-valued inner products such that $U\Phi_1(a) = \Phi_2(a)U$ for all $a \in A$. Note that for any Hilbert A-B bimodule (E, Φ) the module $(\Phi(A)E, \Phi)$ is a nondegenerate A-B sub-bimodule of (E, Φ). Note that $\Phi(A)E = \{\Phi(a)\xi : a \in A, \xi \in E\}$ equals $\overline{\operatorname{span}}(\Phi(A)E)$ by Cohen's factorisation theorem.

Definition 2.5.7 (cf. [BEW14,EKQR00,EKQR06,Lan01])**.** The *correspondence category* (also called the *Morita category*) $\mathfrak{Corr}$ is the category whose objects are C^*-algebras and where the morphisms from A to B are given by equivalence classes $[E, \Phi]$ of Hilbert A-B bimodules (E, Φ) under the equivalence relation

$$(E_1, \Phi_i) \sim (E_2, \Phi_2) \Leftrightarrow \Phi_1(A)E_1 \cong \Phi_2(A)E_2,$$

where $\cong$ denotes unitary equivalence. The identity morphism from A to A is represented by the trivial A-A bimodule (A, id) and composition of two morphisms

$[E, \Phi] \in \mathrm{Mor}(A, B)$ and $[F, \Psi] \in \mathrm{Mor}(B, C)$ is given by taking the interior tensor product $[E \otimes_B F, \Phi \otimes 1]$.

The *compact correspondence category* $\mathfrak{Corr}_c$ is the subcategory of $\mathfrak{Corr}$ in which we additionally require $\Phi(A) \subseteq \mathcal{K}_B(E)$ for a morphism $[E, \Phi] \in \mathrm{Mor}_c(A, B)$.

Remark 2.5.8. (1) We should note that the correspondence category is not a category in the strong sense, since the morphisms $\mathrm{Mor}(A, B)$ from A to B do not form a set. This problem can be overcome by restricting the size of the objects and the underlying modules for the morphisms by assuming that they contain dense subsets of a certain maximal cardinality. But for most practical aspects this does not cause any problems.

(2) We should also note that in most places of the literature (e.g., in [EKQR06]) the correspondence category is defined as the category with objects the C^*-algebras and with morphism sets $\mathrm{Mor}(A, B)$ given by unitary equivalence classes of *nondegenerate* A-B bimodules. But the correspondence category $\mathfrak{Corr}$ defined above is equivalent to the one of [EKQR06] where the equivalence is given by the identity map on objects and by assigning a morphism $[E, \Phi] \in \mathrm{Mor}(A, B)$ to the unitary equivalence class $[\Phi(A)E, \Phi]$ in the morphism set as in [EKQR06].

(3) Note that every $*$-homomorphism $\Phi : A \to M(B)$ determines a morphism $[E, \Phi] \in \mathrm{Mor}(A, B)$ in $\mathfrak{Corr}$ with $E = B$, and $[E, \Phi]$ is a morphism in $\mathfrak{Corr}_c$ if and only if $\Phi(A) \subseteq B$.

(4) Taking direct sums of bimodules allows us to define sums of morphisms in the correspondence categories (and hence a semi-group structure with neutral element given by the zero-module). It is easy to check that this operation is commutative and satisfies the distributive law with respect to composition.

If X is an A-B imprimitivity bimodule, then the adjoint module X^* satisfies $X \otimes_B X^* \cong A$ as A-A bimodule (the isomorphism given on elementary tensors by $x \otimes y^* \mapsto {}_A\langle x, y\rangle$) and $X^* \otimes_A X \cong B$ as B-B bimodule, so X^* is an inverse of X in the correspondence categories. Indeed we have

Proposition 2.5.9 (cf [Lan01, EKQR06])**.** *The isomorphisms in the categories* $\mathfrak{Corr}$ *and* $\mathfrak{Corr}_c$ *are precisely the Morita equivalences.*

2.5.4 The equivariant correspondence categories

If G is a locally compact group, then the *G-equivariant correspondence category* $\mathfrak{Corr}(G)$ is the category in which the objects are systems (A, G, α) and morphisms from (A, G, α) to (B, G, β) are the equivalence classes (as in the nonequivariant case) of equivariant A-B Hilbert bimodules (E, Φ, u), i.e., E is equipped with a strongly continuous homomorphism $u : G \to \mathrm{Aut}(E)$ such that

$$\begin{gathered} \langle u_s(\xi), u_s(\eta)\rangle_B = \beta_s(\langle \xi, \eta\rangle_B), \quad u_s(\xi \cdot b) = u_s(\xi)\beta_s(b) \\ \text{and} \quad u_s(\Phi(a)\xi) = \Phi(\alpha_s(a))u_s(\xi). \end{gathered} \tag{2.5.2}$$

Of course, we require that a unitary equivalence $U : E_1 \to E_2$ between two G-equivariant Hilbert bimodules also intertwines with the actions of G on E_1, E_2. Again, composition of morphisms is given by taking interior tensor products equipped with the diagonal actions, and the isomorphisms in this category are just the equivariant Morita equivalences.

Note that the crossed product constructions $A \rtimes G$ and $A \rtimes_r G$ extend to descent functors

$$\rtimes_{(r)} : \mathfrak{Corr}(G) \to \mathfrak{Corr}.$$

In particular, Morita-equivalent systems have Morita-equivalent full (resp. reduced) crossed products. If $[E, \phi, u]$ is a morphism from (A, G, α) to (B, G, β), then the crossed product $[E \rtimes_{(r)} G, \Phi \rtimes_{(r)} G] \in \mathrm{Mor}(A \rtimes_{(r)} G, B \rtimes_{(r)} G)$ is given as the completion of $C_c(G, E)$ with respect to the $B \rtimes_{(r)} G$-valued inner product

$$\langle \xi, \eta \rangle_{B \rtimes_{(r)} G}(t) = \int_G \beta_{s^{-1}}(\langle \xi(s), \eta(st) \rangle_B)\, ds$$

(taking values in $C_c(G, B) \subseteq B \rtimes_{(r)} G$) and with left action of $C_c(G, A) \subseteq A \rtimes_{(r)} G$ on $E \rtimes_{(r)} G$ given by

$$\big(\Phi \rtimes_{(r)} G(f)\xi\big)(t) = \int_G \Phi(f(s)) u_s(\xi(s^{-1}t))\, ds.$$

The crossed product constructions for equivariant bimodules first appeared (to my knowledge) in Kasparov's famous Conspectus [Kas95], which circulated as a preprint from the early eighties. A more detailed study in the case of imprimitivity bimodules has been given in [Com84]. A very extensive study of the equivariant correspondence categories for actions and coactions of groups together with their relations to duality theory are given in [EKQR06].

2.5.5 Induced representations and ideals

If B is a C^*-algebra we denote by $\mathrm{Rep}(B)$ the collection of all unitary equivalence classes of nondegenerate $*$-representations of B on Hilbert space. In terms of the correspondence category, $\mathrm{Rep}(B)$ coincides with the collection $\mathrm{Mor}(B, \mathbb{C})$ of morphisms from B to $\mathbb{C}$ in $\mathfrak{Corr}$ (every morphism can be represented by a nondegenerate $*$-representation that is unique up to unitary equivalence). Thus, if A is any other C^*-algebra and if $[E, \Phi] \in \mathrm{Mor}(A, B)$, then composition with $[E, \Phi]$ determines a map

$$\mathrm{Ind}^{(E,\Phi)} : \mathrm{Rep}(B) \to \mathrm{Rep}(A); [H, \pi] \mapsto [H, \pi] \circ [E, \Phi] = [E \otimes_B H, \Phi \otimes 1].$$

If confusion seems unlikely, we will simply write π for the representation (H, π) and for its class $[H, \pi] \in \mathrm{Rep}(A)$ and we write $\mathrm{Ind}^E \pi$ for the representation $\Phi \otimes 1$ of A on $\mathrm{Ind}^E H := E \otimes_B H$. We call $\mathrm{Ind}^E \pi$ the *representation of A induced from π via E.*

Note that in the above, we did not require the action of A on E to be nondegenerate. If it fails to be nondegenerate, the representation $\Phi \otimes 1$ of A on $E \otimes_B H$ may also fail to be nondegenerate. We then pass to the restriction of $\Phi \otimes 1$ to $\Phi \otimes 1(A)(E \otimes_B H) \subseteq E \otimes_B H$ to obtain a nondegenerate representative of $[E \otimes_B H, \Phi \otimes 1] \in \mathrm{Mor}(A, \mathbb{C}) = \mathrm{Rep}(A)$.

Remark 2.5.10. **(1)** A special case of the above procedure is given in case when $\Phi : A \to M(B)$ is a nondegenerate $*$-homomorphism and $[B, \Phi] \in \mathrm{Mor}(A, B)$ is the corresponding morphism in $\mathfrak{Corr}$. Then the induction map $\mathrm{Ind}^B : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$ coincides with the obvious map

$$\Phi^* : \mathrm{Rep}(B) \to \mathrm{Rep}(A); \pi \mapsto \Phi^*(\pi) := \pi \circ \Phi.$$

(2) Induction in steps. If $[H, \pi] \in \mathrm{Rep}(B)$, $[E, \Phi] \in \mathrm{Mor}(A, B)$ and $[F, \Psi] \in \mathrm{Mor}(D, A)$ for some C^*-algebra D, then it follows directly from the associativity of composition in $\mathfrak{Corr}$ that (up to equivalence)

$$\mathrm{Ind}^F(\mathrm{Ind}^E \pi) = \mathrm{Ind}^{F \otimes_A E} \pi.$$

(3) If X is an A-B imprimitivity bimodule, then $\mathrm{Ind}^X : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$ gets inverted by $\mathrm{Ind}^{X^*} : \mathrm{Rep}(A) \to \mathrm{Rep}(B)$, where X^* denotes the adjoint of X (i.e., the inverse of $[X]$ in $\mathfrak{Corr}$). Since composition of morphisms in $\mathfrak{Corr}$ preserves direct sums, it follows from this that induction via X maps irreducible representations of B to irreducible representations of A and hence induces a bijection $\mathrm{Ind}^X : \widehat{B} \to \widehat{A}$ between the spectra.

It is useful to consider a similar induction map on the set $\mathcal{I}(B)$ of closed two-sided ideals of the C^*-algebra B. If (E, Φ) is any Hilbert A-B bimodule, we define

$$\mathrm{Ind}^E : \mathcal{I}(B) \to \mathcal{I}(A);\ \mathrm{Ind}^E I := \{a \in A : \langle \Phi(a)\xi, \eta\rangle_B \in I \text{ for all } \xi, \eta \in E\}.^9 \tag{2.5.3}$$

It is clear that induction preserves inclusion of ideals, and with a little more work one can check that

$$\mathrm{Ind}^E(\ker \pi) = \ker(\mathrm{Ind}^E \pi) \quad \text{for all } \pi \in \mathrm{Rep}(B). \tag{2.5.4}$$

Hence it follows from part (3) of Remark 2.5.10 that, if X is an A-B imprimitivity bimodule, then induction of ideals via X restricts to give a bijection $\mathrm{Ind}^X : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$ between the primitive ideal spaces of B and A. Since induction preserves inclusion of ideals, the next proposition follows directly from the description of the closure operations in $\widehat{A}$ and $\mathrm{Prim}(A)$ (see §2.2.4).

Proposition 2.5.11 (Rieffel). *Let X be an A-B imprimitivity bimodule. Then the bijections*

$$\mathrm{Ind}^X : \widehat{B} \to \widehat{A} \quad \textit{and} \quad \mathrm{Ind}^X : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$$

are homeomorphisms.

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[9] If X is an A-B imprimitivity bimodule, the induced ideal $\mathrm{Ind}^X I$ defined here coincides with the induced ideal $\mathrm{Ind}^X I = {}_A\langle X \cdot I, X \cdot I\rangle$ of the Rieffel correspondence (see Proposition 2.5.4).

Note that these homeomorphisms are compatible with the Rieffel-correspondence (see Proposition 2.5.4): If I is any closed ideal of B and if we identify $\widehat{B}$ with the disjoint union $\widehat{I} \cup \widehat{B/I}$ in the canonical way (see §2.2.4), then induction via X "decomposes" into induction via $Y := X \cdot I$ from $\widehat{I}$ to $(\operatorname{Ind}^X I)\widehat{\ }$ and induction via X/Y from $\widehat{B/I}$ to $(A/\operatorname{Ind}^X I)\widehat{\ }$. This helps to prove

Proposition 2.5.12. *Suppose that A and B are Morita-equivalent C^*-algebras. Then*

(i) *A is type I if and only if B is type I.*

(ii) *A is CCR if and only if B is CCR.*

(iii) *A has continuous trace if and only if B has continuous trace.*

Proof. Recall from §2.2.4 that a C^*-algebra B is type I if and only if for each $\pi \in \widehat{B}$ the image $\pi(B) \subseteq \mathcal{B}(H_\pi)$ contains $\mathcal{K}(H_\pi)$. Furthermore, B is CCR if and only if B is type I and points are closed in $\widehat{B}$.

If X is an A-B imprimitivity bimodule and $\pi \in \widehat{B}$, we may pass to $B/\ker\pi$ and $A/\ker(\operatorname{Ind}^X \pi)$ via the Rieffel correspondence to assume that π and $\operatorname{Ind}^X \pi$ are injective, and hence that $B \subseteq \mathcal{B}(H_\pi)$ and $A \subseteq \mathcal{B}(X \otimes_B H_\pi)$. If B is type I, it follows that $\mathcal{K} := \mathcal{K}(H_\pi)$ is an ideal of B. Let $Z := X \cdot \mathcal{K}$. Then Z is an $\operatorname{Ind}^X \mathcal{K} - \mathcal{K}$ imprimitivity bimodule and $Z \otimes_{\mathcal{K}} H_\pi$, the composition of Z with the $\mathcal{K} - \mathbb{C}$ imprimitivity bimodule H_π, is an $\operatorname{Ind}^X \mathcal{K} - \mathbb{C}$ imprimitivity bimodule. It follows that $\operatorname{Ind}^X \mathcal{K} \cong \mathcal{K}(Z \otimes_K H_\pi)$. Since $Z \otimes_{\mathcal{K}} H_\pi \cong X \otimes_B H_\pi$ via the identity map on both factors, we conclude that $\operatorname{Ind}^X \pi(A)$ contains the compact operators $\mathcal{K}(X \otimes_B H_\pi)$. This proves (i). Now (ii) follows from (i) since $\widehat{B}$ is homeomorphic to $\widehat{A}$. The proof of (iii) needs a bit more room and we refer the interested reader to [Wil07]. □

Of course, similar induction procedures as described above can be defined in the equivariant settings: If (A, G, α) is a system, then the morphisms from (A, G, α) to $(\mathbb{C}, G, \mathrm{id})$ in $\mathfrak{Corr}(G)$ are just the unitary equivalence classes of nondegenerate covariant representations of (A, G, α) on Hilbert space, which we shall denote by $\operatorname{Rep}(A, G)$ (surpressing the given action α in our notation). Composition with a fixed equivariant morphism $[E, \Phi, u]$ between two systems (A, G, α) and (B, G, β) gives an induction map

$$\operatorname{Ind}^E : \operatorname{Rep}(B, G) \to \operatorname{Rep}(A, G); [H, (\pi, U)] \mapsto [E, \Phi, u] \circ [H, \pi, U].$$

As above, we shall write

$$\operatorname{Ind}^E H := E \otimes_B H, \ \operatorname{Ind}^E \pi := \Phi \otimes 1, \text{ and } \operatorname{Ind}^E U := u \otimes U,$$

so that the composition $[E, \Phi, u] \circ [H, \pi, U]$ becomes the triple

$$[\operatorname{Ind}^E H, \operatorname{Ind}^E \pi, \operatorname{Ind}^E U].$$

Taking integrated forms allows to identify $\operatorname{Rep}(A, G)$ with $\operatorname{Rep}(A \rtimes G)$. A more or less straightforward computation gives:

Proposition 2.5.13. *Assume that* $[E, \Phi, u]$ *is a morphism from* (A, G, α) *to* (B, G, β) *in* $\mathfrak{Corr}(G)$ *and let* $[E \rtimes G, \Phi \rtimes G] \in \mathrm{Mor}(A \rtimes G, B \rtimes G)$ *denote its crossed product. Then, for each* $[H, (\pi, U)] \in \mathrm{Rep}(B, G)$ *we have*

$$[\mathrm{Ind}^E H, \mathrm{Ind}^E \pi \times \mathrm{Ind}^E U] = [\mathrm{Ind}^{E \rtimes G} H, \mathrm{Ind}^{E \rtimes G}(\pi \times U)] \quad \textit{in} \quad \mathrm{Rep}(A \rtimes G).$$

Hence induction from $\mathrm{Rep}(B, G)$ *to* $\mathrm{Rep}(A, G)$ *via* $[E, \Phi, u]$ *is equivalent to induction from* $\mathrm{Rep}(B \rtimes G)$ *to* $\mathrm{Rep}(A \rtimes G)$ *via* $[E \rtimes G, \Phi \rtimes G]$ *under the canonical identifications* $\mathrm{Rep}(A, G) \cong \mathrm{Rep}(A \rtimes G)$ *and* $\mathrm{Rep}(B, G) \cong \mathrm{Rep}(B \rtimes G)$.

Proof. Simply check that the map

$$W : C_c(G, E) \odot H \to E \otimes_B H; \quad W(\xi \otimes v) = \int_G \xi(s) \otimes U_s v \, ds$$

extends to a unitary from $(E \rtimes G) \otimes_{B \rtimes G} H$ to $E \otimes_B H$ which intertwines both representations (see [Ech94] or [EKQR06] for more details). □

We close this section with a brief discussion of corners: If A is a C*-algebra and $p \in M(A)$ is a projection, then Ap is a Hilbert pAp-module with inner product given by $\langle ap, bp \rangle_{pAp} = pa^* bp$, and multiplication from the left turns Ap into an A-pAp correspondence $[Ap, \phi]$. We then have $\mathcal{K}(Ap) \cong \overline{ApA}$, the ideal of A generated by p. In a similar way, we may regard pA as an pAp-A correspondence, with inner product given by $\langle pa, pb \rangle_A = a^* pb$. Note that pA is then isomorphic to the adjoint module $(Ap)^*$ with isomorphism given by $ap \mapsto pa^*$.

Recall that p is called full, iff $\overline{ApA} = A$. In this case Ap is an A-pAp imprimitivity bimodule and induction from $\mathrm{Rep}(pAp)$ to $\mathrm{Rep}(A)$ via Ap gives a bijection between $\mathrm{Rep}(pAp)$ and $\mathrm{Rep}(A)$ with inverse given by induction via the adjoint module $(Ap)^* = pA$.

In general, the induction map $\mathrm{Ind}^{Ap} : \mathrm{Rep}(pAp) \to \mathrm{Rep}(A)$ is split injective with converse given via *compression* by p: If $\pi : A \to \mathcal{B}(H_\pi)$ is a nondegenerate representation, we define $H_{\mathrm{comp}(\pi)} := \pi(p) H_\pi$ and

$$\mathrm{comp}(\pi) : pAp \to \mathcal{B}(H_{\mathrm{comp}(\pi)}) \quad \text{by} \quad \mathrm{comp}(\pi)(pap) = \pi(pap).$$

Note that in general, $\mathrm{comp}(\pi)$ could be the zero representation, which happens precisely if $\pi(p) = 0$. Since π is nondegenerate, this is equivalent to $\pi(ApA) = 0$.

Proposition 2.5.14. *Let* $p \in M(A)$ *be as above. Then the following are true:*

(i) *The compression map* $\mathrm{comp} : \mathrm{Rep}(A) \to \mathrm{Rep}(pAp)$ *coincides with the induction map* $\mathrm{Ind}^{pA} : \mathrm{Rep}(A) \to \mathrm{Rep}(pAp)$.

(ii) *For all* $\rho \in \mathrm{Rep}(pAp)$ *we have* $\mathrm{comp}(\mathrm{Ind}^{Ap} \rho) \cong \rho$.

(iii) p *is full if and only if* comp *is an inverse for* Ind^{Ap}.

(iv) p *is full if and only if* $\pi(p) \neq 0$ *for all* $\pi \in \mathrm{Rep}(A)$.

Proof. For (i) just check that for every nondegenerate representation $\pi : A \to \mathcal{B}(H_\pi)$ the map

$$pA \otimes_A H_\pi \to \pi(p)H_\pi; pa \otimes \xi \mapsto \pi(pa)\xi$$

is an isomorphism that intertwines $\mathrm{Ind}^{pA}\pi$ with $\mathrm{comp}(\pi)$.

For the proof of (ii) observe that $pA \otimes_A A_p \cong pAp$ as a pAp bimodule, hence by (i) we get $\mathrm{comp} \circ \mathrm{Ind}^{Ap} = \mathrm{Ind}^{pA} \circ \mathrm{Ind}^{Ap} = \mathrm{Ind}^{pAp} = \mathrm{id}_{\mathrm{Rep}(pAp)}$.

For (iii) we first observe that if p is full, then Ap is an equivalence bimodule and induction via $(Ap)^* \cong pA$ is inverse to Ind^{Ap}. Together with (i) this shows that comp is an inverse to Ind^{Ap}. Conversely, if $\overline{ApA}$ is a proper ideal of A, there exist nonzero, nondegenerate representations π of A that vanish on ApA, and hence on p. It is then clear that $\mathrm{comp}(\pi)$ is the zero representation, and then $\mathrm{Ind}^{Ap}(\mathrm{comp}(\pi))$ is the zero representation as well. Hence $\pi \ncong \mathrm{Ind}^{Ap}(\mathrm{comp}(\pi))$.

The proof of (iv) is left as an exercise for the reader. □

2.5.6 The Fell topologies and weak containment

For later use and for completeness it is necessary to discuss some more topological notions on the spaces $\mathrm{Rep}(B)$ and $\mathcal{I}(B)$: For $I \in \mathcal{I}(B)$ let $U(I) := \{J \in \mathcal{I}(B) : J \smallsetminus I \neq \emptyset\}$. Then $\{U(I) : I \in \mathcal{I}(B)\}$ is a sub-basis for the *Fell topology* on $\mathcal{I}(B)$. The Fell topology on $\mathrm{Rep}(B)$ is then defined as the inverse image topology with respect to the map $\ker : \mathrm{Rep}(B) \to \mathcal{I}(B); \pi \mapsto \ker\pi$.[10] The Fell topologies restrict to the Jacobson topologies on $\mathrm{Prim}(B)$ and $\widehat{B}$, respectively. Convergence of nets in $\mathrm{Rep}(B)$ (and hence also in $\mathcal{I}(B)$) can conveniently be described in terms of *weak containment*: If $\pi \in \mathrm{Rep}(B)$ and R is a subset of $\mathrm{Rep}(B)$, then π is said to be *weakly contained* in R (denoted $\pi \prec R$) if $\ker\pi \supseteq \cap\{\ker\rho : \rho \in R\}$. Two subsets S, R of $\mathrm{Rep}(A)$ are said to be *weakly equivalent* ($S \sim R$) if $\sigma \prec R$ for all $\sigma \in S$ and $\rho \prec S$ for all $\rho \in R$.

Lemma 2.5.15 (Fell). *Let $(\pi_j)_{j\in J}$ be a net in $\mathrm{Rep}(B)$ and let $\pi, \rho \in \mathrm{Rep}(B)$. Then*

(i) *$\pi_j \to \pi$ if and only if π is weakly contained in every subnet of $(\pi_j)_{j\in J}$.*

(ii) *If $\pi_j \to \pi$ and if $\rho \prec \pi$, then $\pi_j \to \rho$.*

For the proof see [Fel62, Propositions 1.2 and 1.3]. As a direct consequence of this and the fact that induction via bimodules preserves inclusion of ideals we get

Proposition 2.5.16. *Let $[E, \Phi] \in \mathrm{Mor}(A, B)$. Then induction via E preserves weak containment and the maps*

$$\mathrm{Ind}^E : \mathrm{Rep}(B) \to \mathrm{Rep}(A) \quad \textit{and} \quad \mathrm{Ind}^E : \mathcal{I}(B) \to \mathcal{I}(A)$$

are continuous with respect to the Fell topologies. Both maps are homeomorphisms if E is an imprimitivity bimodule.

[10] Recall that $\mathrm{Rep}(B)$ is a set only if we restrict the cardinality of the Hilbert spaces.

Another important observation is the fact that tensoring representations and ideals of C^*-algebras is continuous:

Proposition 2.5.17. *Suppose that A and B are C^*-algebras. For $\pi \in \operatorname{Rep}(A)$ and $\rho \in \operatorname{Rep}(B)$ let $\pi \otimes \rho \in \operatorname{Rep}(A \otimes B)$ denote the tensor product representation on the minimal tensor product $A \otimes B$. Moreover, if $I \in \mathcal{I}(A)$ and $J \in \mathcal{I}(B)$, define $I \circ J$ as the closed two-sided ideal of $A \otimes B$ generated by $I \otimes B + A \otimes J$. Then the maps*

$$\begin{aligned} &\operatorname{Rep}(A) \times \operatorname{Rep}(B) \to \operatorname{Rep}(A \otimes B);\ (\pi, \rho) \mapsto \pi \otimes \rho \\ \textit{and} \qquad &\mathcal{I}(A) \times \mathcal{I}(B) \to \mathcal{I}(A \otimes B);\ (I, J) \mapsto I \circ J \end{aligned}$$

are continuous with respect to the Fell topologies.

Proof. Note first that if $I = \ker \pi$ and $J = \ker \rho$, then $I \circ J = \ker(\pi \otimes \rho)$. Since tensoring ideals clearly preserves inclusion of ideals, the map $(\pi, \rho) \mapsto \pi \otimes \rho$ preserves weak containment in both variables. Hence the result follows from Lemma 2.5.15. □

It follows from deep work of Fell (e.g., see [Fel60, Dix77]) that weak containment (and hence the topologies on $\mathcal{I}(B)$ and $\operatorname{Rep}(B)$) can be described completely in terms of matrix coefficients of representations. In particular, if G is a locally compact group and if we identify the collection $\operatorname{Rep}(G)$ of equivalence classes of unitary representations of G with $\operatorname{Rep}(C^*(G))$ via integration, then it is shown in [Fel60, Dix77] that weak containment for representations of G can be described in terms of convergence of positive definite functions on G associated to the given representations.

2.6 Green's imprimitivity theorem and applications

2.6.1 The imprimitivity theorem

We are now presenting (a slight extension of) Phil Green's imprimitivity theorem as presented in [Gre78]. For this we start with the construction of an induction functor

$$\operatorname{Ind}_H^G : \mathfrak{Corr}(H) \to \mathfrak{Corr}(G); (A, H, \alpha) \mapsto \big(\operatorname{Ind}_H^G(A, \alpha), G, \operatorname{Ind}\alpha\big),$$

if H is a closed subgroup of G and $\alpha : H \to \operatorname{Aut}(A)$ an action of H on the C^*-algebra A. The induced C^*-algebra $\operatorname{Ind}_H^G(A, \alpha)$ (or just $\operatorname{Ind} A$ if all data are understood) is defined as

$$\operatorname{Ind}_H^G(A, \alpha) := \left\{ f \in C^b(G, A) : \begin{matrix} f(sh) = \alpha_{h^{-1}}(f(s)) \ \text{ for all } s \in G, h \in H \\ \text{and } (sH \mapsto \|f(s)\|) \in C_0(G/H) \end{matrix} \right\},$$

equipped with the pointwise operations and the supremum norm. The induced action $\operatorname{Ind}\alpha : G \to \operatorname{Aut}(\operatorname{Ind} A)$ is given by

$$\big(\operatorname{Ind}\alpha_s(f)\big)(t) := f(s^{-1}t) \qquad \text{for all} \quad s,t \in G.$$

A similar construction works for morphisms in $\mathfrak{Corr}(H)$, i.e., if $[E,\Phi,u]$ is a morphism from (A,H,α) to (B,H,β), then a fairly obvious extension of the above construction yields the induced morphism $[\operatorname{Ind}_H^G(E,u), \operatorname{Ind}\Phi, \operatorname{Ind} u]$ from $(\operatorname{Ind}_H^G(A,\alpha), G, \operatorname{Ind}\alpha)$ to $(\operatorname{Ind}_H^G(B,\beta), G, \operatorname{Ind}\beta)$. One then checks that induction preserves composition of morphisms, and hence gives a functor from $\mathfrak{Corr}(H)$ to $\mathfrak{Corr}(G)$ (see [EKQR00] for more details).

Remark 2.6.1. **(1)** If we start with an action $\alpha : G \to \operatorname{Aut}(A)$ and restrict this action to the closed subgroup H of G, then $\operatorname{Ind}_H^G(A,\alpha)$ is canonically G-isomorphic to $C_0(G/H, A) \cong C_0(G/H) \otimes A$ equipped with the diagonal action $l \otimes \alpha$, where l denotes the left-translation action of G on G/H. The isomorphism is given by

$$\Phi : \operatorname{Ind}_H^G(A,\alpha) \to C_0(G/H, A); \quad \Phi(f)(sH) = \alpha_s(f(s)).$$

(2) The construction of the induced algebra $\operatorname{Ind}_H^G(A,\alpha)$ is the C^*-analogue of the usual construction of the induced G-space $G \times_H Y$ of a topological H-space Y, which is defined as the quotient of $G \times Y$ by the H-action $h(g,y) = (gh^{-1}, hy)$ and which is equipped with the obvious G-action given by the left-translation action on the first factor. Indeed, if Y is locally compact, then $\operatorname{Ind}_H^G C_0(Y) \cong C_0(G \times_H Y)$.

A useful characterization of induced systems is given by the following result:

Theorem 2.6.2 (cf [Ech90, Theorem])**.** *Let (B,G,β) be a system and let H be a closed subgroup of G. Then (B,G,β) is isomorphic to an induced system $(\operatorname{Ind}_H^G(A,\alpha), G, \operatorname{Ind}\alpha)$ if and only if there exists a continuous G-equivariant map $\varphi : \operatorname{Prim}(B) \to G/H$, where G acts on $\operatorname{Prim}(B)$ via $s \cdot P := \beta_s(P)$.*

Indeed, we can always define a continuous G-map $\varphi : \operatorname{Prim}(\operatorname{Ind} A) \to G/H$ by sending a primitive ideal P to sH iff P contains the ideal $I_s := \{f \in \operatorname{Ind} A : f(s) = 0\}$. Conversely, if $\varphi : \operatorname{Prim}(B) \to G/H$ is given, define $A := B/I_e$ with

$$I_e := \cap\{P \in \operatorname{Prim}(B) : \varphi(P) = eH\}.$$

Since I_e is H-invariant, the action $\beta|_H$ induces an action α of H on A and (B,G,β) is isomorphic to $(\operatorname{Ind}_H^G A, G, \operatorname{Ind}\alpha)$ via $b \mapsto f_b \in \operatorname{Ind}_H^G A$; $f_b(s) := \beta_{s^{-1}}(b) + I_e$. We should remark at this point that a much more general result has been shown by Le Gall in [LG99] in the setting of Morita-equivalent groupoids. Applying Theorem 2.6.2 to commutative G-algebras, one gets:

Corollary 2.6.3. *Let X be a locally compact G-space and let H be a closed subgroup of G. Then X is G-homeomorphic to $G \times_H Y$ for some locally compact H-space Y if and only if there exists a continuous G-map $\varphi : X \to G/H$. If such a map is given, then Y can be chosen as $Y = \varphi^{-1}(\{eH\})$ and the homeomorphism $G \times_H Y \cong X$ is given by $[g,y] \mapsto gy$.*

In what follows, let $B_0 = C_c(H, A)$ and $D_0 = C_c(G, \operatorname{Ind} A)$, viewed as dense subalgebras of the full (resp. reduced) crossed products $A \rtimes_{(r)} H$ and $\operatorname{Ind} A \rtimes_{(r)} G$, respectively. Let $X_0(A) = C_c(G, A)$. We define left and right module actions of D_0 and B_0 on $X_0(A)$, and D_0- and B_0-valued inner products on $X_0(A)$ by the formulas

$$\begin{aligned} e \cdot x(s) &= \int_G e(t, s)x(t^{-1}s)\Delta_G(t)^{1/2}dt \\ x \cdot b(s) &= \int_H \alpha_h\big(x(sh)b(h^{-1})\big)\Delta_H(h)^{-1/2}dh \\ {}_{D_0}\langle x, y\rangle(s, t) &= \Delta_G(s)^{-1/2}\int_H \alpha_h\big(x(th)y(s^{-1}th)^*\big)\, dh \\ \langle x, y\rangle_{B_0}(h) &= \Delta_H(h)^{-1/2}\int_G x(t^{-1})^*\alpha_h(y(t^{-1}h))\, dt, \end{aligned} \tag{2.6.1}$$

for $e \in D_0, x, y \in X_0(A)$, and $b \in B_0$. The $C_c(H, A)$-valued inner product on $X_0(A)$ provides $X_0(A)$ with two different norms: $\|\xi\|^2_{\max} := \|\langle\xi, \xi\rangle_{B_0}\|_{\max}$ and $\|\xi\|^2_r := \|\langle\xi, \xi\rangle_{B_0}\|_r$, where $\|\cdot\|_{\max}$ and $\|\cdot\|_r$ denote the maximal and reduced norms on $C_c(G, A)$, respectively. Then Green's imprimitivity theorem reads as follows:

Theorem 2.6.4 (Green). *The actions and inner products on $X_0(A)$ extend to the completion $X^G_H(A)$ of $X_0(A)$ with respect to $\|\cdot\|_{\max}$ such that $X^G_H(A)$ becomes an $(\operatorname{Ind}^G_H A \rtimes G)$-$(A \rtimes H)$ imprimitivity bimodule.*

Similarly, the completion $X^G_H(A)_r$ of $X_0(A)$ with respect to $\|\cdot\|_r$ becomes an $(\operatorname{Ind}^G_H A \rtimes_r G)$-$(A \rtimes_r H)$ imprimitivity bimodule.

Remark 2.6.5. (1) Although the statement of Green's theorem looks quite straightforward, the proof requires a fair amount of work. The main problem is to show positivity of the inner products and continuiuty of the left and right actions of D_0 and B_0 on X_0 with respect to the appropriate norms. In [Gre78] Green only considered full crossed products. The reduced versions were obtained later by Kasparov ([Kas88]), by Quigg and Spielberg ([QS92]) and by Kirchberg and Wassermann ([KW00]).

The reduced module $X^G_H(A)_r$ can also be realized as the quotient of $X^G_H(A)$ by the submodule $Y := X^G_H(A) \cdot I$ with $I := \ker\big(A \rtimes H \to A \rtimes_r H\big)$. This follows from the fact that the ideal I corresponds to the ideal $J := \ker\big(\operatorname{Ind} A \rtimes G \to \operatorname{Ind} A \rtimes_r G\big)$ in $\operatorname{Ind} A \rtimes G$ via the Rieffel correspondence (see Proposition 2.5.4). We shall give an argument for this fact in Remark 2.7.14 below.

(2) In his original work [Gre78], Green first considered the special case where the action of H on A restricts from an action of G on A. In this case one obtains a Morita equivalence between $A \rtimes_{(r)} H$ and $C_0(G/H, A) \rtimes_{(r)} G$ (compare with Remark 2.6.1 above). Green then deduced from this a more general result (see

[Gre78, Theorem 17]), which by Theorem 2.6.2 is equivalent to the above formulation for full crossed products.

(3) In [EKQR00] it is shown that the construction of the equivalence bimodule $X(A)$, viewed as an isomorphism in the correspondence category $\mathfrak{Corr}$, provides a natural equivalence between the descent functor $\rtimes : \mathfrak{Corr}(H) \to \mathfrak{Corr}; (A, H, \alpha) \mapsto A \rtimes H$ with the composition $\rtimes \circ \mathrm{Ind}_H^G : \mathfrak{Corr}(H) \to \mathfrak{Corr}; (A, H, \alpha) \mapsto \mathrm{Ind}\, A \rtimes G$ (and similarly for the reduced descent functors $\rtimes_r$). This shows that the assignment $(A, H, \alpha) \mapsto X_H^G(A)$ is, in a very strong sense, natural in A.

Let us now present some basic examples:

Example 2.6.6. **(1)** Let H be a closed subgroup of G. Consider the trivial action of H on $\mathbb{C}$. Then $\mathrm{Ind}_H^G \mathbb{C} = C_0(G/H)$ and Green's theorem provides a Morita equivalence between $C^*(H)$ and $C_0(G/H) \rtimes G$, and similarly between $C_r^*(H)$ and $C_0(G/H) \rtimes_r G$. It follows then from Proposition 2.5.16 that induction via $X_H^G(\mathbb{C})$ identifies the representation spaces $\mathrm{Rep}(H)$ and $\mathrm{Rep}(C_0(G/H), G)$. This is a very strong version of Mackey's original imprimitivity theorem for groups (e.g., see [Mac51, Mac52, Bla61]).

(2) If $H = \{e\}$ is the trivial subgroup of G, we obtain a Morita equivalence between A and $C_0(G, A) \rtimes G$, where G acts on itself by left translation. Indeed, in this case we obtain a unitary isomorphism between Green's bimodule $X_{\{e\}}^G(A)$ and the Hilbert A-module $L^2(G, A) \cong A \otimes L^2(G)$ via the transformation

$$U : X_{\{e\}}^G(A) \to L^2(G, A); \big(U(x)\big)(s) = \Delta(s)^{-\frac{1}{2}} x(s).$$

It follows from this that $C_0(G, A) \rtimes G$ is isomorphic to $\mathcal{K}(A \otimes L^2(G)) \cong A \otimes \mathcal{K}(L^2(G))$. In particular, it follows that $C_0(G) \rtimes G$ is isomorphic to $\mathcal{K}(L^2(G))$ if G acts on itself by translation.

Since full and reduced crossed products by the trivial group coincide, it follows from part (2) of Remark 2.6.5 that $C_0(G, A) \rtimes_r G \cong C_0(G, A) \rtimes G$, and hence that $C_0(G, A) \rtimes_r G \cong A \otimes \mathcal{K}(L^2(G))$, too.

(3) Let H_3 denote the three-dimensional real Heisenberg group, i.e., $H_3 = \mathbb{R}^2 \rtimes \mathbb{R}$ with action of $\mathbb{R}$ on $\mathbb{R}^2$ given by $x \cdot (y, z) = (y, z + xy)$. We want to use Green's theorem to analyse the structure of $C^*(H_3) \cong C^*(\mathbb{R}^2) \rtimes \mathbb{R}$. We first identify $C^*(\mathbb{R}^2)$ with $C_0(\mathbb{R}^2)$ via Fourier transform. The transformed action of $\mathbb{R}$ on $\mathbb{R}^2$ is then given by $x \cdot (\eta, \zeta) = (\eta - x\zeta, \zeta)$. The short exact sequence

$$0 \to C_0(\mathbb{R} \times \mathbb{R}^*) \to C_0(\mathbb{R}^2) \to C_0(\mathbb{R} \times \{0\}) \to 0$$

determines a short exact sequence

$$0 \to C_0\big(\mathbb{R} \times \mathbb{R}^*\big) \rtimes \mathbb{R} \to C^*(H_3) \to C_0(\mathbb{R} \times \{0\}) \rtimes \mathbb{R} \to 0.$$

Since the action of $\mathbb{R}$ on the quotient $C_0(\mathbb{R}) \cong C_0(\mathbb{R}\times\{0\})$ is trivial, we see that $C_0(\mathbb{R}\times\{0\})\rtimes\mathbb{R} \cong C_0(\mathbb{R})\otimes C^*(\mathbb{R}) \cong C_0(\mathbb{R}^2)$. The homeomorphism $h:\mathbb{R}\times\mathbb{R}^*\to\mathbb{R}\times\mathbb{R}^*; h(\eta,\zeta) = (-\frac{\eta}{\zeta},\zeta)$ transforms the action of $\mathbb{R}$ on $C_0(\mathbb{R}\times\mathbb{R}^*) \cong C_0(\mathbb{R})\otimes C_0(\mathbb{R}^*)$ to the diagonal action $l\otimes\mathrm{id}$, where l denotes left translation. Thus, it follows from (2) and Lemma 2.4.1 that $C_0(\mathbb{R}\times\mathbb{R}^*)\rtimes\mathbb{R} \cong C_0(\mathbb{R}^*)\otimes\mathcal{K}(L^2(\mathbb{R}))$ and we obtain a short exact sequence

$$0 \to C_0(\mathbb{R}^*)\otimes\mathcal{K}(L^2(\mathbb{R})) \to C^*(H_3) \to C_0(\mathbb{R}^2) \to 0$$

for $C^*(H_3)$.

(4) Let $\mathbb{R}$ act on the two-torus $\mathbb{T}^2$ by an irrational flow, i.e., there exists an irrational number $\theta\in(0,1)$ such that $t\cdot(z_1,z_2) = (e^{2\pi i t}z_1, e^{2\pi i\theta t}z_2)$. Then $\mathbb{T}^2$ is $\mathbb{R}$-homeomorphic to the induced space $\mathbb{R}\times_{\mathbb{Z}}\mathbb{T}$, where $\mathbb{Z}$ acts on $\mathbb{T}$ by irrational rotation given by θ (compare with Example 2.3.5). Hence, it follows from Green's theorem that $C(\mathbb{T}^2)\rtimes_\theta\mathbb{R}$ is Morita equivalent to the irrational rotation algebra $A_\theta = C(\mathbb{T})\rtimes_\theta\mathbb{Z}$.

In [Gre80] Phil Green shows that for second countable G we always have a decomposition $\mathrm{Ind}_H^G A\rtimes_{\mathrm{Ind}\,\alpha} G \cong (A\rtimes_\alpha H)\otimes\mathcal{K}(L^2(G/H))$, where the L^2-space is taken with respect to some quasi invariant measure on G/H. In case where $H=\{e\}$ is the trivial group, this is part (2) of Example 2.6.6 above, but the general proof is more difficult because of some measure-theoretic technicalities. But if H is open in G, the proof of Green's structure theorem becomes quite easy:

Proposition 2.6.7. *Let H be an open subgroup of G and let $\alpha: H\to\mathrm{Aut}(A)$ be an action. Let $c: G/H\to G$ be a cross-section for the quotient map $q: G\to G/H$ such that $c(eH)=e$. Then there is an isomorphism $\Psi: X_H^G(A) \xrightarrow{\cong} (A\rtimes_\alpha H)\otimes\ell^2(G/H)$ of Hilbert $A\rtimes_\alpha H$ modules given on the dense subspace $X_0(A) = C_c(G,A)$ by*

$$\Psi(x) = \sum_{G/H} x_{sH}\otimes\delta_{sH},$$

where $x_{sH}\in C_c(H,A)$ is defined by $x_{sH}(h) = \Delta(c(s)h)^{-1/2}\alpha_h(x(c(s)h))$. A similar decomposition holds for the reduced module: $X_H^G(A)_r \cong (A\rtimes_r H)\otimes\ell^2(G/H)$. As a consequence, we get isomorphisms

$$\mathrm{Ind}_H^G A\rtimes G \cong (A\rtimes_\alpha H)\otimes\mathcal{K}(\ell^2(G/H)) \quad \textit{and}$$
$$\mathrm{Ind}_H^G A\rtimes_r G \cong (A\rtimes_{\alpha,r} H)\otimes\mathcal{K}(\ell^2(G/H)).$$

Proof. It is easy to check that the mapping $x\mapsto\Psi(x)$ is a bijection between $C_c(G,A)$ and the algebraic tensor product $C_c(H,A)\odot C_c(G/H)$, which is dense in $(A\rtimes_\alpha H)\otimes\ell^2(G/H)$. It therefore suffices to check that $\langle\Psi(x),\Psi(y)\rangle_{A\rtimes H} = \langle x,y\rangle_{A\rtimes H}$ for all $x,y\in C_c(G,A)$ and that Ψ intertwines the right action of $A\rtimes_\alpha H$ on both modules. We show the first and leave the second as an exercise for the reader: Note that the inner products on both dense subspaces take values in

$C_c(H, A)$. Since H is open in G we have $\Delta_G = \Delta_H$ and the formula $\int_G f(t)\,dt = \sum_{tH \in G/H} \int_H f(tl)\,dl$ for all $f \in C_c(G, A)$. We then compute for $x, y \in C_c(G, A)$ and $h \in H$:

$$\begin{aligned}
\langle \Psi(x), \Psi(y)\rangle(h) &= \sum_{G/H} (x_{sH}^* * y_{sH})(h) = \sum_{G/H} \int_H \alpha_l(x_{sH}(l)^* y_{sH}(lh))\,dl \\
&= \sum_{G/H} \Delta(h)^{-1/2} \int_H \Delta(c(s)l)^{-1} x(c(s)l)^* \alpha_h(y(c(s)lh))\,dl \\
&\overset{l \mapsto c(s)^{-1}sl}{=} \sum_{G/H} \Delta(h)^{-1/2} \int_H \Delta(sl)^{-1} x(sl)^* \alpha_h(y(slh))\,dl \\
&= \int_G \Delta(h)^{-1/2} \Delta(s)^{-1} x(s)^* \alpha_h(y(sh))\,ds \\
&= \Delta(h)^{-1/2} \int_G x(s^{-1})^* \alpha_h(y(s^{-1}h))\,ds = \langle x, y\rangle(h).
\end{aligned}$$

The decomposition of $\operatorname{Ind}_H^G A \rtimes G$ now follows from

$$\operatorname{Ind}_H^G A \rtimes G = \mathcal{K}(X_H^G(A)) \cong \mathcal{K}((A \rtimes H) \otimes \ell^2(G/H)) = (A \rtimes H) \otimes \mathcal{K}(\ell^2(G)),$$

and similarly for $\operatorname{Ind}_H^G A \rtimes_r G$. $\square$

We close this section with a proof of Green's imprimitivity theorem in the case where H is open in G. For this let $\alpha : H \to \operatorname{Aut}(A)$ be an action of H on a C*-algebra A. We shall see that in this case there exists a canonical full projection $p \in M(\operatorname{Ind}_H^G A \rtimes G)$ such that the crossed product $A \rtimes_\alpha H$ is isomorphic to the corner $p(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)p$. Of course, if we believe in the validity of Green's theorem, this follows basically from Proposition 2.6.7.

Consider the canonical embedding $\Psi : C_0(G/H) \to M(\operatorname{Ind}_H^G A)$ given by the (central) action

$$(\Psi(\varphi)F)(t) = \varphi(tH)F(t), \quad \varphi \in C_0(G/H), F \in \operatorname{Ind}_H^G A, t \in G.$$

In what follows we shall often write $\varphi \cdot F$ for $\Psi(\varphi)F$. Let $\tilde{p} = \varphi(\delta_{eH}) \in M(\operatorname{Ind}_H^G A)$ and let p be the image of $\tilde{p}$ under the extension to $M(\operatorname{Ind}_H^G A)$ of the embedding $i_{\operatorname{Ind}_H^G A} : \operatorname{Ind}_H^G A \to M(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)$.

Proposition 2.6.8. *Let $p \in M(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)$ be as above. Then p is a full projection in $M(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)$ such that there is a canonical isomorphism*

$$A \rtimes_\alpha H \cong p(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)p.$$

Moreover, the resulting $\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G$-$A \rtimes_\alpha H$ equivalence bimodule $(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)p$ is isomorphic to Green's equivalence bimodule $X_H^G(A)$ of Theorem 2.7.6.

A similar result holds for the reduced crossed products.

Proof. We first observe that A can be identified with the direct summand $\tilde{p}\operatorname{Ind}_H^G A$ of $\operatorname{Ind}_H^G A$ via the embedding $\psi : A \to \tilde{p}\operatorname{Ind}_H^G A; a \mapsto F_a$ with

$$F_a(s) = \begin{Bmatrix} \alpha_{s^{-1}}(a) & \text{if } s \in H \\ 0 & \text{else} \end{Bmatrix}. \tag{2.6.2}$$

If $F \in C_c(G, \operatorname{Ind}_H^G A)$, then using the formulas in Remark 2.3.4 we compute

$$(pFp)(s) = \tilde{p}F(s)\operatorname{Ind}\alpha_s(\tilde{p}) = \delta_{eH}F(s)\delta_{sH} = \begin{Bmatrix} \delta_{eH}F(s) & \text{if } s \in H \\ 0 & \text{else} \end{Bmatrix}. \tag{2.6.3}$$

Using (2.6.2) and (2.6.3) one easily checks that there is a canonical $*$-isomorphism

$$\Phi : C_c(H, A) \xrightarrow{\cong} pC_c(G, \operatorname{Ind}_H^G A)p \subseteq p(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)p$$

which maps a function $f \in C_c(H, A)$ to the function $F \in C_c(G, \operatorname{Ind}_H^G A)$ given by

$$F(s,t) = \begin{Bmatrix} \alpha_{t^{-1}}(f(s)) & \text{if } s, t \in H \\ 0 & \text{else} \end{Bmatrix}. \tag{2.6.4}$$

Indeed, by a straightforward but lengthy computation one checks that Φ coincides with the integrated form of the covariant homomorphism $(i_{\operatorname{Ind} A} \circ \psi, i_G|_H)$ of (A, H, α) into $M(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)$, where ψ is as above and $(i_{\operatorname{Ind} A}, i_G)$ is the canonical covariant homomorphism of $(\operatorname{Ind}_H^G A, G, \operatorname{Ind}\alpha)$ into $M(\operatorname{Ind}_H^G A \rtimes G)$. It follows from this that Φ extends to a surjective $*$-homomorphism $\Phi : A \rtimes_\alpha H \twoheadrightarrow p(\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G)p$.

To see that Φ is injective let (ρ, V) be any covariant representation of (A, H, α). Then we construct an induced representation $(\operatorname{Ind}\rho, \operatorname{Ind} V)$ of $(\operatorname{Ind}_H^G A, G, \operatorname{Ind}\alpha)$ as follows: We define

$$H_{\operatorname{Ind} V} = \left\{ \xi : G \to H_\rho : \begin{matrix} \xi(th) = V_{h^{-1}}\xi(t) \text{ for all } t \in G, h \in H \\ \text{and } \sum_{tH \in G/H} \|\xi(t)\|^2 < \infty. \end{matrix} \right\}$$

equipped with the inner product

$$\langle\langle \xi, \eta \rangle\rangle = \sum_{tH \in G/H} \langle \xi(t), \eta(t) \rangle.$$

Note that this sum is well defined since $\xi(th) = V_{h^{-1}}\xi(t)$ for all $t \in G, h \in H$. We then define $(\operatorname{Ind}\rho, \operatorname{Ind} V)$ by

$$(\operatorname{Ind}\rho(F)\xi)(t) = F(t)\xi(t) \quad \text{and} \quad (\operatorname{Ind} V_s \xi)(t) = \xi(s^{-1}t),$$

for $F \in \operatorname{Ind}_H^G A$ and $s \in G$. It is then straightforward to check the following items:

- $(\operatorname{Ind}\rho, \operatorname{Ind} V)$ is a covariant representation of $(\operatorname{Ind}_H^G A, G, \operatorname{Ind}\alpha)$.

- The composition of the compression $\mathrm{comp}(\mathrm{Ind}\,\rho \rtimes \mathrm{Ind}\,V)$ of $\mathrm{Ind}\,\rho \rtimes \mathrm{Ind}\,V$ to the corner $p(\mathrm{Ind}_H^G A \rtimes G)p$ with $\Phi : A \rtimes H \to p(\mathrm{Ind}_H^G A \rtimes G)p$ is equivalent to $\rho \rtimes V$.

Hence, if we choose $\rho \rtimes V$ to be faithful on $A \rtimes_\alpha H$, we see that Φ must be faithful as well.

To check that p is a full projection it suffices to show that no nonzero representation $\pi \rtimes U$ of $\mathrm{Ind}_H^G A \rtimes G$ vanishes on p. By definition of p we have $\pi \rtimes U(p) = \pi(\tilde{p}) = \pi(\delta_{eH})$. So assume to the contrary that $\pi(\delta_{eH}) = 0$, where we regard $C_0(G/H)$ as a subalgebra of $M(\mathrm{Ind}_H^G A)$ as described in the discussion preceding the proposition. Then for all $t \in G$ we have

$$\pi(\delta_{tH}) = \pi(\mathrm{Ind}\,\alpha_t \delta_{eH}) = U_t \pi(\delta_{eH}) U_t^* = 0$$

as well, and since $\sum_{tH \in G/H} \delta_{tH}$ converges strictly to 1 in $M(\mathrm{Ind}_H^G A)$, it follows from this that π is the zero-representation. But then $\pi \rtimes U$ is zero as well, which contradicts our assumption.

We now have seen that $A \rtimes_\alpha H$ is isomorphic to the full corner $p(\mathrm{Ind}_H^G A \rtimes G)p$. We want to compare Green's module $X_H^G(A)$ with the module $(\mathrm{Ind}_H^G A \rtimes G)p$. For this we first compute for $F \in C_c(G, \mathrm{Ind}_H^G A)$ that

$$(Fp)(s,t) = \big(F(s) \cdot \mathrm{Ind}\,\alpha_s(\delta_{eH})\big)(t) = \big(F(s) \cdot \delta_{sH}\big)(t) = \begin{Bmatrix} F(s,t) & \text{if } t \in sH \\ 0 & \text{else} \end{Bmatrix}.$$

Therefore, because of the condition $F(s,th) = \alpha_{h^{-1}}(F(s,t))$, it follows that Fp is completely determined by the values of F on the diagonal $\Delta_G = \{(s,s) : s \in G\}$. Recall that $X_H^G(A)$ is the completion of $X_0(A) = C_c(G,A)$ with respect to the $C_c(H,A)$ valued inner product

$$\langle x, y\rangle(h) = \Delta_H(h)^{-1/2} \int_G x(t^{-1})^* \alpha_h(y(t^{-1}h))\, dt.$$

We then obtain a well-defined bijective map

$$\Theta : C_c(G, \mathrm{Ind}_H^G A)p \to C_c(G,A);\, Fp \mapsto (s \mapsto \sqrt{\Delta_G(s)} F(s,s)).$$

To show that Θ preserves the $A \rtimes_\alpha H$-valued inner products, it follows from (2.6.4) that we need to check that for all $F_1, F_2 \in C_c(G, \mathrm{Ind}_H^G A)$ and all $(h,l) \in H$ we have

$$\begin{aligned} \alpha_{l^{-1}}(\langle \Theta(F_1), \Theta(F_2)\rangle(h)) &= F_1^* * F_2(h,l) \\ &= \int_G \Delta_G(t^{-1}) F_1(t^{-1}, t^{-1}l)^* F_2(t^{-1}h, t^{-1}l)\, dt. \end{aligned}$$

But this follows from a straightforward calculation using that $\Delta_G|_H = \Delta_H$, since H is open in G. One also easily checks that Θ intertwines the left action of $\mathrm{Ind}_H^G A \rtimes G$ on both modules.

In order to prove the analogue for the reduced case one checks that compression of a regular representation of $\operatorname{Ind}_H^G A \rtimes G$ gives a regular representation of $A \rtimes H$. We leave this as an exercise for the reader (or see Remark 2.7.14 below). $\square$

Remark 2.6.9. **(1)** If A is a G-algebra and H is a closed subgroup of G, then we saw in Remark 2.6.1 that $\operatorname{Ind}_H^G A$ is isomorphic to $C_0(G/H, A)$. Thus, if H is also open in G, it follows from Proposition 2.6.8 that $A \rtimes_\alpha H$ is a full corner in $C_0(G/H, A) \rtimes_{\tau\otimes\alpha} G$, and similarly for the reduced crossed products.

(2) Later, in Section 3.5.3, we need to investigate the structure of crossed products of the form $C_0(I, A) \rtimes G$ in which I is a *discrete* G-space, A is a G-algebra, and G act diagonally on $C_0(I, A) \cong C_0(I) \otimes A$. In this case we can decompose I as a disjoint union of G-orbits $Gi = \{si : s \in G\}$ which induces a direct sum decomposition

$$C_0(I, A) \rtimes G \cong \bigoplus_{G\backslash I} C_0(Gi, A) \rtimes G.$$

If $G_i = \{s \in G : si = i\}$ denotes the stabiliser of $i \in I$ for the action of G (which is open in G since I is discrete), we get G-equivariant bijections $G/G_i \cong Gi; sG_i \mapsto si$, and then the above decomposition becomes

$$C_0(I, A) \rtimes G \cong \bigoplus_{G\backslash I} C_0(G/G_i, A) \rtimes G.$$

Now by Green's imprimitivity theorem (or by Proposition 2.6.8) each summand $C_0(G/G_i, A) \rtimes G$ is Morita equivalent to $A \rtimes_\alpha G_i$, and hence we see that $C_0(I, A) \rtimes G$ is Morita equivalent to $\bigoplus_{G\backslash I} A \rtimes_\alpha G_i$. Indeed, if $p_i \in M(C_0(G/G_i, A) \rtimes G)$ is the full projection as in Proposition 2.6.8, we observe that the sum $\sum_{G\backslash I} p_i$ converges strictly in $M(C_0(I, A) \rtimes G)$ to a projection p and then $\bigoplus_{G\backslash I} A \rtimes_\alpha G_i$ is isomorphic to the full corner $p(C_0(I, A) \rtimes G)p$ in $C_0(I, A) \rtimes G$. All this goes through without change for the reduced crossed products.

2.6.2 The Takesaki–Takai duality theorem

From part (2) of Example 2.6.6 it is fairly easy to obtain the Takesaki–Takai duality theorem for crossed products by abelian groups. For this assume that (A, G, α) is a system with G abelian. The *dual action* $\widehat{\alpha} : \widehat{G} \to \operatorname{Aut}(A \rtimes G)$ of the dual group $\widehat{G}$ on the crossed product $A \rtimes G$ is defined by

$$\widehat{\alpha}_\chi(f)(s) := \chi(s)f(s) \quad \text{for } \chi \in \widehat{G} \text{ and } f \in C_c(G, A) \subseteq A \rtimes G.$$

With a similar action of $\widehat{G}$ on crossed products $E \rtimes G$ for an equivariant bimodule (E, Φ, u) we obtain from this a descent functor

$$\rtimes : \mathfrak{Corr}(G) \to \mathfrak{Corr}(\widehat{G}).$$

The double dual crossed product $A \rtimes G \rtimes \widehat{G}$ is isomorphic to $C_0(G, A) \rtimes G$ with respect to the diagonal action $l \otimes \alpha$ of G on $C_0(G, A) \cong C_0(G) \otimes A$. Indeed, we have canonical (covariant) representations $(k_A, k_G, k_{\widehat{G}})$ of the triple $(A, G, \widehat{G})$ into $M\big(C_0(G, A) \rtimes G\big)$ given by the formulas

$$\big(k_A(a) \cdot f\big)(s,t) = a\big(f(s,t)\big), \quad \big(k_G(r) \cdot f\big)(s,t) = \alpha_r\big(f(r^{-1}s, r^{-1}t)\big), \quad \text{and}$$
$$\big(k_{\widehat{G}}(\chi) \cdot f\big)(s,t) = \chi(t) f(s,t),$$

for f in the dense subalgebra $C_c(G, C_0(G, A))$ of $C_0(G, A) \rtimes G$. Making extensive use of the universal properties, one checks that the integrated form

$$(k_A \times k_G) \times k_{\widehat{G}} : (A \rtimes G) \rtimes \widehat{G} \to M(C_0(G, A) \rtimes G)$$

gives the desired isomorphism $A \rtimes G \rtimes \widehat{G} \cong C_0(G, A) \rtimes G$. Using the isomorphism $C_0(G, A) \rtimes G \cong A \otimes \mathcal{K}(L^2(G))$ of Example 2.6.6 (2) and checking what this isomorphism does on the double-dual action $\widehat{\widehat{\alpha}}$ we arrive at

Theorem 2.6.10 (Takesaki–Takai). *Suppose that (A, G, α) is a system with G abelian. Then the double dual system $(A \rtimes G \rtimes \widehat{G}, G, \widehat{\widehat{\alpha}})$ is equivariantly isomorphic to the system $(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \operatorname{Ad} \rho)$, where $\rho : G \to U(L^2(G))$ denotes the right regular representation of G on $L^2(G)$.*

Recall that the *right regular representation* $\rho : G \to U(L^2(G))$ is defined by $(\rho_t \xi)(s) = \sqrt{\Delta(t)}\xi(st)$ for $\xi \in L^2(G)$ (but if G is abelian, the modular function Δ dissapears). Note that the system $(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \operatorname{Ad} \rho)$ in the Takesaki–Takai theorem is Morita equivalent to the original system (A, G, α) via the equivariant imprimitivity bimodule $(A \otimes L^2(G), \alpha \otimes \rho)$. In fact, the assignment $(A, G, \alpha) \mapsto (A \otimes L^2(G), \alpha \otimes \rho)$ is easily seen to give a natural equivalence between the identity functor on $\mathfrak{Corr}(G)$ and the composition

$$\mathfrak{Corr}(G) \xrightarrow{\ \rtimes\ } \mathfrak{Corr}(\widehat{G}) \xrightarrow{\ \rtimes\ } \mathfrak{Corr}(G).$$

In general, if G is not abelian, one can obtain similar duality theorems by replacing the dual action of $\widehat{G}$ by a dual coaction of the group algebra $C^*(G)$ on $A \rtimes G$. A fairly complete account of that theory in the group case is given in the appendix of [EKQR06]; however a much more general duality theory for Hopf-C^*-algebras was developed by Baaj and Skandalis in [BS93] and Kustermans and Vaes in [KV00].

2.6.3 Permanence properties of exact groups

As a further application of Green's imprimitivity theorem we now present some of Kirchberg's and Wassermann's permanence results for C^*-exact groups. Recall from Definition 2.4.9 that a group G is called C^*-exact (or just exact) if for every system (A, G, α) and for each G-invariant ideal $I \subseteq A$ the sequence

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

is exact (which is equivalent to exactness of the sequence in the middle term). Recall from Proposition 2.4.8 that the corresponding sequence of full crossed products is always exact. Using Proposition 2.4.5, this implies that all amenable groups are exact.

In what follows we want to relate exactness of G with exactnesss of a closed subgroup H of G. For this we start with a system (A, H, α) and a closed H-invariant ideal I of A. Recall that Green's $\operatorname{Ind} A \rtimes_r G - A \rtimes_r H$ imprimitivity bimodule $X_H^G(A)_r$ is a completion of $C_c(G, A)$. Using the formulas for the actions and inner products as given in (2.6.1) one observes that $X_H^G(I)_r$ can be identified with the closure of $C_c(G, I) \subseteq C_c(G, A)$ in $X_H^G(A)_r$. It follows that the ideals $\operatorname{Ind} I \rtimes_r G$ and $I \rtimes_r H$ are linked via the Rieffel correspondence with respect to $X_H^G(A)_r$ (see Proposition 2.5.4). Similarly, the imprimitivity bimodule $X_H^G(A/I)_r$ is isomorphic to the quotient $X_H^G(A)_r/Y$ with $Y := X_H^G(A)_r \cdot \ker\big(A \rtimes_r H \to A/I \rtimes_r H\big)$, which implies that the ideals

$$\ker\big(A \rtimes_r H \to A/I \rtimes_r H\big) \quad \text{and} \quad \ker\Big(\operatorname{Ind} A \rtimes_r G \to \operatorname{Ind}(A/I) \rtimes_r G\Big)$$

are also linked via the Rieffel correspondence. Since the Rieffel correspondence is one-to-one, we obtain

$$\begin{aligned} & I \rtimes_r H = \ker\big(A \rtimes_r H \to A/I \rtimes_r H\big) \\ \Longleftrightarrow\quad & \operatorname{Ind} I \rtimes_r G = \ker\Big(\operatorname{Ind} A \rtimes_r G \to \operatorname{Ind}(A/I) \rtimes_r G\Big). \end{aligned} \tag{2.6.5}$$

Using this, we now give proofs of two of the main results of [KW00].

Theorem 2.6.11 (Kirchberg and Wassermann)**.** *Let G be a locally compact group. Then the following are true:*

(i) *If G is exact and H is a closed subgroup of G, then H is exact.*

(ii) *Let H be a closed subgroup of G such that G/H is compact. Then H exact implies G exact.*

Proof. Suppose that I is an H-invariant ideal of the H-algebra A. If G is exact, then $\operatorname{Ind} I \rtimes_r G = \ker\big(\operatorname{Ind} A \rtimes_r G \to \operatorname{Ind}(A/I) \rtimes_r G\big)$ and hence $I \rtimes_r H = \ker\big(A \rtimes_r H \to A/I \rtimes_r H\big)$ by (2.6.5). This proves (i).

To see (ii) we start with an arbitrary G-algebra A and a G-invariant ideal I of A. Since A, I, and A/I are G-algebras and G/H is compact, we have $\operatorname{Ind}_H^G A \cong C(G/H, A)$ and similar statements hold for I and A/I. Since H is exact we see that the lower row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \rtimes_r G & \longrightarrow & A \rtimes_r G & \longrightarrow & (A/I) \rtimes_r G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(G/H, I) \rtimes_r G & \longrightarrow & C(G/H, A) \rtimes_r G & \longrightarrow & C(G, A/I) \rtimes_r G & \longrightarrow & 0, \end{array}$$

is exact, where the vertical maps are induced by the canonical inclusions of I, A, and A/I into $C(G/H, I), C(G/H, A)$ and $C(G/H, A/I)$, respectively. Since these inclusions are injective, all vertical maps are injective, too (see the remarks preceeding Proposition 2.4.8). This and the exactness of the lower horizontal row imply that

$$\ker\big(A \rtimes_r G \to (A/I) \rtimes_r G\big) =: J = \big(A \rtimes_r G\big) \cap \big(C(G/H, I) \rtimes_r G\big).$$

Let $(x_i)_i$ be a bounded approximate unit of I and let $(\varphi_j)_j$ be an approximate unit of $C_c(G)$ (compare with Remark 2.3.7). Then $z_{i,j} := \varphi_j \otimes x_i \in C_c(G, I)$ serves as an approximate unit of $I \rtimes_r G$ and of $J := \big(A \rtimes_r G\big) \cap \big(C(G/H, I) \rtimes_r G\big)$. Thus, if $y \in J$, then $z_{i,j} \cdot y \in I \rtimes_r G$ and $z_{i,j} \cdot y$ converges to y. Hence $J \subseteq I \rtimes_r G$. $\square$

Corollary 2.6.12. *Every closed subgroup of an almost connected group is exact (in particular, every free group in countably many generators is exact). Also, every closed subgroup of* $\mathrm{GL}(n, \mathbb{Q}_p)$*, where* $\mathbb{Q}_p$ *denotes the field of* p*-adic rational numbers equipped with the Hausdorff topology is exact.*

Proof. Recall first that a locally compact group is called *almost connected* if the component G_0 of the identity in G is cocompact. By part (i) of Theorem 2.6.11 it is enough to show that every almost connected group G is exact and that $\mathrm{GL}(n, \mathbb{Q}_p)$ is exact for all $n \in \mathbb{N}$. But structure theory for those groups implies that in both cases one can find an amenable cocompact subgroup. Since amenable groups are exact (by Propositions 2.4.5 and 2.4.8), the result then follows from part (ii) of the theorem. $\square$

Remark 2.6.13. We should mention that Kirchberg and Wassermann proved some further permanence results: If H is a closed subgroup of G such that G/H carries a finite invariant measure, then H exact implies G exact. Another important result is the extension result: If N is a closed normal subgroup of G such that N and G/N are exact, then G is exact. The proof of this result needs the notion of twisted actions and twisted crossed products. We shall present that theory and the proof of the extension result for exact groups in §2.8.2 below. We should also mention that the proof of part (ii) of Theorem 2.6.11, and hence of Corollary 2.6.12 followed some ideas of Skandalis (see also the discussion at the end of [KW99]).

From the work of Ozawa and others (e.g., see [Oza06] for a general discussion), the class of discrete exact groups is known to be identical to the class of all discrete groups which can act amenably on some compact Hausdorff space X (we refer to [ADR00] for a quite complete exposition of amenable actions). An analogous result for general second countable locally compact groups has been shown very recently by Brodzki, Cave and Li in [BCL16]. This implies a new proof that exactness passes to closed subgroups, since the restriction of an amenable action to a closed subgroup is amenable. If we apply the exactness condition of a group G to trivial actions, it follows from Remark 2.4.2 that $C_r^*(G)$ is an exact C^*-algebra if G is exact; the converse is known for discrete groups by [KW99] but is still open in the general case. As mentioned at the end of §2.4, it is now known that there exist nonexact finitely generated discrete groups.

2.7 Induced representations and the ideal structure of crossed products

In this section we use Green's imprimitivity theorem as a basis for computing the representation theory and/or the ideal structure of C^*-group algebras and crossed products. The first ideas towards this theory appeared in the work of Frobenius and Schur on representations of finite groups. In the 1940s George Mackey introduced the theory of induced representations of second countable locally compact groups together with a procedure (now known as the Mackey machine) to compute the irreducible representations of a second countable locally compact group G in terms of representations of a (nice!) normal subgroup N and projective representations of the stabilisers for the action of G on $\widehat{N}$ (see [Mac51, Mac52, Mac53, Mac57, Mac58].) For most of the theory, the separability assumption has been eliminated by Blattner in [Bla61, Bla62]. An extension of this theory to crossed products was first worked out by Takesaki in [Tak67]. In the 1970s Marc Rieffel first showed that the theory of induced representations of groups could be embedded into his more algebraic theory of induced representations of C^*-algebras as introduced in Section 2.5.5 ([Rie74, Rie79]).

The full power of this theory became evident with the fundamental work of Phil Green on twisted crossed products ([Gre78]). In what follows we will try to explain the basics of Green's theory by first restricting to ordinary crossed products. The twisted crossed products will be studied later in Section 2.8. We will also report on the important work of Sauvageot, Gootmann, and Rosenberg ([Sau77, Sau79, GR79]) on the generalised Effros–Hahn conjecture, i.e., on the ideal structure of (twisted) crossed products $A \rtimes_\alpha G$ in which the action of G on $\mathrm{Prim}(A)$ does not have very good properties.

Many of the results explained in this section also carry over to groupoids and to crossed products by (twisted) actions of groupoids on C^*-algebras (e.g., see [Ren80, Ren87, IW09]), but we shall stick to (twisted) crossed products by group actions in these notes.

2.7.1 Induced representations of groups and crossed products

If (A, G, α) is a system and H is a closed subgroup of G, then Green's imprimitivity theorem provides an imprimitivity bimodule $X_H^G(A)$ between $C_0(G/H, A) \rtimes G \cong \mathrm{Ind}_H^G A \rtimes G$ and $A \rtimes H$. In particular, $C_0(G/H, A) \rtimes G$ identifies with the compact operators $\mathcal{K}(X_H^G(A))$ on $X_H^G(A)$. There is a canonical covariant homomorphism

$$(k_A, k_G) : (A, G) \to M(C_0(G/H, A) \rtimes G) \cong \mathcal{L}(X_H^G(A)),$$

where $k_A = i_{C_0(G/H,A)} \circ j_A$ denotes the composition of the inclusion $j_A : A \to M\big(C_0(G/H,A)\big)$ with the inclusion $i_{C_0(G/H,A)} : C_0(G/H,A) \to M\big(C_0(G/H,A) \rtimes G\big)$ and k_G denotes the canonical inclusion of G into $M\big(C_0(G/H,A) \rtimes G\big)$. The integrated form

$$k_A \times k_G : A \rtimes G \to M\big(C_0(G/H,A) \rtimes G\big) \cong \mathcal{L}(X_H^G(A))$$

determines a left action of $A \rtimes G$ on $X_H^G(A)$ and we obtain a canonical element $[X_H^G(A), k_A \times k_G] \in \mathrm{Mor}(A \rtimes G, A \rtimes H)$ – a morphism from $A \rtimes G$ to $A \rtimes H$ in the correspondence category. Using the techniques of §2.5.5, we can define induced representations of $A \rtimes G$ as follows:

Definition 2.7.1. For $\rho \times V \in \mathrm{Rep}(A \rtimes H)$ we define the induced representation $\mathrm{ind}_H^G(\rho \times V) \in \mathrm{Rep}(A \rtimes G)$ as the representation induced from $\rho \times V$ via $[X_H^G(A), k_A \times k_G] \in \mathrm{Mor}(A \rtimes G, A \rtimes H)$.

Similarly, for $J \in \mathcal{I}(A \rtimes H)$, we define the induced ideal $\mathrm{ind}_H^G J \in \mathcal{I}(A \rtimes G)$ as the ideal induced from J via $[X_H^G(A), k_A \times k_G]$.

On the other hand, if we restrict the canonical embedding $i_G : G \to M(A \rtimes G)$ to H, we obtain a nondegenerate homomorphism $i_A \times i_G|_H : A \rtimes H \to M(A \rtimes G)$ which induces a morphism $[A \rtimes G, i_A \times i_G|_H] \in \mathrm{Mor}(A \rtimes H, A \rtimes G)$. This leads to:

Definition 2.7.2. For $\pi \times U \in \mathrm{Rep}(A \rtimes G)$ we define the *restriction* $\mathrm{res}_H^G(\pi \times U) \in \mathrm{Rep}(A \rtimes H)$ as the representation induced from $\pi \times U$ via $[A \rtimes G, i_A \times i_G|_H] \in \mathrm{Mor}(A \rtimes H, A \rtimes G)$.

Similarly, for $I \in \mathcal{I}(A \rtimes G)$, we define the restricted ideal $\mathrm{res}_H^G I \in \mathcal{I}(A \rtimes H)$ as the ideal induced from I via $[A \rtimes G, i_A \times i_G|_H]$.

Remark 2.7.3. It is a good exercise to show that for any $\pi \times U \in \mathrm{Rep}(A \rtimes G)$ we have $\mathrm{res}_H^G(\pi \times U) = \pi \times U|_H$ – the integrated form of the restriction of (π, U) to (A, H, α).

As a consequence of Definitions 2.7.1 and 2.7.2 and Proposition 2.5.16 we get:

Proposition 2.7.4. *The maps* $\mathrm{ind}_H^G : \mathrm{Rep}(A \rtimes H) \to \mathrm{Rep}(A \rtimes G)$ *and* $\mathrm{ind}_H^G : \mathcal{I}(A \rtimes H) \to \mathcal{I}(A \rtimes G)$ *as well as the maps* $\mathrm{res}_H^G : \mathrm{Rep}(A \rtimes G) \to \mathrm{Rep}(A \rtimes H)$ *and* $\mathrm{res}_H^G : \mathcal{I}(A \rtimes G) \to \mathcal{I}(A \rtimes H)$ *are continuous with respect to the Fell topologies.*

Remark 2.7.5. **(1)** Note that the left action of $A \rtimes G$ on $X_H^G(A)$ can be described conveniently on the level of $C_c(G,A)$ via convolution: If $f \in C_c(G,A) \subseteq A \rtimes G$ and $\xi \in C_c(G,A) \subseteq X_H^G(A)$, then $k_A \times k_G(f)\xi = f * \xi$.

(2) For $A = \mathbb{C}$ we obtain, after identifying unitary representations of G (resp. H) with $*$-representations of $C^*(G)$ (resp. $C^*(H)$) an induction map $\mathrm{ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. With a bit of work one can check that $\mathrm{ind}_H^G U$ for $U \in \mathrm{Rep}(H)$ coincides (up to equivalence) with the induced representation defined by Mackey in [Mac51] or Blattner in [Bla61]. Similarly, the induced representations for C^*-dynamical systems as defined above coincide up to equivalence with the induced representations as constructed by Takesaki in [Tak67]. We will present some more details on these facts in Proposition 2.7.7 and Corollary 2.7.8 below.

(3) If ρ is a nondegenerate representation of A on a Hilbert space H_ρ, then $\operatorname{ind}_{\{e\}}^G \rho$ is equivalent to the regular representation $\operatorname{Ind}\rho$ of $A \rtimes G$ on $L^2(G, H_\rho)$ (see Remark 2.3.2). The intertwining unitary $V : X_H^G(A) \otimes_A H_\rho \to L^2(G, H_\rho)$ is given by

$$\big(V(\xi \otimes v)\big)(s) = \rho\big(\alpha_{s^{-1}}(\xi(s))\big)v$$

for $\xi \in C_c(G, A) \subseteq X_H^G(A)$ and $v \in H_\rho$.

The construction of $[X_H^G(A), k_A \times k_G]$ shows that we have a decomposition

$$[X_H^G(A), k_A \times k_G] = [C_0(G/H, A) \rtimes G, k_A \times k_G] \circ [X_H^G(A)]$$

as morphisms in the correspondence category. Hence the induction map ind_H^G : $\operatorname{Rep}(A \rtimes H) \to \operatorname{Rep}(A \rtimes G)$ factors as the composition

$$\operatorname{Rep}(A \rtimes H) \xrightarrow[\cong]{\operatorname{Ind}^{X_H^G(A)}} \operatorname{Rep}\big(C_0(G/H, A) \rtimes G\big) \xrightarrow{(k_A \times k_G)^*} \operatorname{Rep}(A \rtimes G)$$

(see Remark 2.5.10 for the meaning of $(k_A \times k_G)^*$). The representations of $C_0(G/H, A) \rtimes G$ are of the form $(P \otimes \pi) \times U$, where P and π are commuting representations of $C_0(G/H)$ and A, respectively (we use the identification $C_0(G/H, A) \cong C_0(G/H) \otimes A$). The covariance condition for $(P \otimes \pi, U)$ is equivalent to (π, U) and (P, U) being covariant representations of (A, G, α) and $(C_0(G/H), G, l)$, respectively (where $l : G \to \operatorname{Aut}(C_0(G/H))$ is the left translation action). One then checks that

$$\big((P \otimes \pi) \times U\big) \circ (k_A \times k_G) = \pi \times U.$$

Since induction from $\operatorname{Rep}(A \rtimes H)$ to $\operatorname{Rep}\big(C_0(G/H, A) \rtimes G\big)$ via $X_H^G(A)$ is a bijection, we obtain the following general version of Mackey's classical imprimitivity theorem for group representations (see [Mac51] and [Tak67]):

Theorem 2.7.6 (Mackey–Takesaki–Rieffel–Green)**.** *Suppose that (A, G, α) is a system and let H be a closed subgroup of G. Then:*

(i) *A representation $\pi \times U \in \operatorname{Rep}(A \rtimes G)$ on a Hilbert space H_π is induced from a representation $\sigma \times V \in \operatorname{Rep}(A \rtimes H)$ if and only if there exists a nondegenerate representation $P : C_0(G/H) \to \mathcal{B}(H_\pi)$ which commutes with π such that (P, U) is a covariant representation of $(C_0(G/H), G, l)$.*

(ii) *If $\pi \times U \in \operatorname{Rep}(A \rtimes G)$ is induced from the irreducible representation $\sigma \times V \in \operatorname{Rep}(A \rtimes H)$, and if $P : C_0(G/H) \to \mathcal{B}(H_\pi)$ is the corresponding representation such that $(P \otimes \pi) \times U \cong \operatorname{Ind}^{X_H^G(A)}(\sigma \times V)$, then $\pi \times U$ is irreducible if and only if every $W \in \mathcal{B}(H_\pi)$ that intertwines with π and U (and hence with $\pi \times U$) also intertwines with P.*

Proof. The first assertion follows directly from the above discussions. The second statement follows from Schur's irreducibilty criterion (a representation is irreducible iff every intertwiner is a multiple of the identity) together with the fact that induction via imprimitivity bimodules preserves irreducibility of representations in both directions (see Proposition 2.5.11). □

In many situations it is convenient to have a more concrete realization of the induced representations. The following construction follows Blattner's construction of induced group representations of groups (see [Bla61, Fol95]). We start with the situation of an induced system: Assume that H is a closed subgroup of G and that $\alpha : H \to \mathrm{Aut}(A)$ is an H-action. If $\rho \times V \in \mathrm{Rep}(A \rtimes H)$ is a representation on the Hilbert space H_ρ we put

$$\mathcal{F}_{\rho\times V} := \left\{ \xi : G \to H_\rho : \begin{array}{r} \xi(sh) = \sqrt{\Delta_H(h)/\Delta_G(h)}V_{h^{-1}}\xi(s) \text{ for all } s \in G, h \in H \\ \text{and } \xi \text{ is continuous with compact support modulo } H \end{array} \right\}.$$

Let $c : G \to [0,\infty)$ be a Bruhat section for H, i.e., c is continuous with $\operatorname{supp} c \cap C\cdot H$ compact for all compact $C \subseteq G$ and such that $\int_H c(sh)\,dh = 1 \quad$ for all $s \in G$ (for the existence of such c see [Bou71]). Then

$$\langle \xi, \eta\rangle := \int_G c(s)\langle \xi(s), \eta(s)\rangle\, ds$$

determines a well-defined inner product on $\mathcal{F}_{\rho\times V}$ and we let $H_{\mathrm{ind}(\rho\times V)}$ denote its Hilbert space completion. We can now define representations σ and U of $\mathrm{Ind}_H^G A$ and G on $H_{\mathrm{ind}(\rho\times V)}$, respectively, by

$$(\sigma(f)\xi)(s) := \rho(f(s))\xi(s) \quad \text{and} \quad (U_t\xi)(s) := \xi(t^{-1}s). \tag{2.7.1}$$

Then $\sigma \times U$ is a representation of $\mathrm{Ind}_H^G A \rtimes G$ on $H_{\mathrm{ind}(\rho\times V)}$ and a straightforward but lengthy computation gives:

Proposition 2.7.7. *Let $X := X_H^G(A)$ denote Green's $\mathrm{Ind}_H^G A \rtimes G - A \rtimes H$ imprimitivity bimodule and let $\rho \times V$ be a representation of $A \rtimes H$ on H_ρ. Then there is a unitary $W : X \otimes_{A\rtimes H} H_\rho \to H_{\mathrm{ind}(\rho\times V)}$, given on elementary tensors $x \otimes v \in X \odot H_\rho$ by*

$$W(x\otimes v)(s) = \Delta_G(s)^{-\frac12}\int_H \Delta_H(h)^{-\frac12} V_h\rho(x(sh))v\, dh,$$

which implements a unitary equivalence between $\mathrm{Ind}^X(\rho\times V)$ and the representation $\sigma \times U$ defined above.

Observe that in the case where H is open in G, the representation (σ, U) constructed above coincides with the representation $(\mathrm{Ind}\,\rho, \mathrm{Ind}\,V)$ as constructed in the proof of Proposition 2.6.8.

In the special case where A is a G-algebra we identify $\operatorname{Ind}_H^G A$ with $C_0(G/H, A)$ via the isomorphism Φ of Remark 2.6.1 (1). It is then easy to check that the representation σ defined above corresponds to the representation $P \otimes \pi$ of $C_0(G/H, A) \cong C_0(G/H) \otimes A$ on $H_{\operatorname{ind}(\rho \times V)}$ given by the formula

$$(P(\varphi)\xi)(s) := \varphi(sH)\xi(s) \quad \text{and} \quad (\pi(a)\xi)(s) = \rho(\alpha_{s^{-1}}(a))\xi(s). \tag{2.7.2}$$

Hence, as a direct corollary of the above proposition we get:

Corollary 2.7.8. *Let (A, G, α) be a system and let $\rho \times V \in \operatorname{Rep}(A \rtimes H)$ for some closed subgroup H of G. Then $\operatorname{ind}_H^G(\rho \times V)$, as defined in Definition 2.7.1, is unitarily equivalent to the representation $\pi \times U$ of $A \rtimes G$ on $H_{\operatorname{ind}(\rho \times V)}$ with π and U as in Equations* (2.7.2) *and* (2.7.1) *respectively.*

Another corollary that we can easily obtain from Blattner's realisation is the following useful observation: Assume that H is a closed subgroup of G and that A is an H-algebra. Let $\epsilon_e : \operatorname{Ind}_H^G A \to A$ be the H-equivariant surjection defined by evaluation of functions $f \in \operatorname{Ind}_H^G A$ at the unit $e \in G$. If $\rho \times V \in \operatorname{Rep}(A \rtimes H)$ then $(\rho \circ \epsilon_e) \times V$ is a representation of $\operatorname{Ind}_H^G A \rtimes H$. We then get

Corollary 2.7.9. *The induced representation $\operatorname{ind}_H^G\big((\rho \circ \epsilon_e) \times V\big)$ (induction from H to G for the system $(\operatorname{Ind}_H^G A, G, \operatorname{Ind} \alpha)$) is unitarily equivalent to $\operatorname{ind}^{X_H^G(A)}(\rho \times V)$ (induction via Green's $\operatorname{Ind}_H^G A \rtimes G - A \rtimes H$ imprimitivity bimodule $X_H^G(A)$).*

Proof. By Proposition 2.7.7 and Corollary 2.7.8, both representations can be realized on the Hilbert space $H_{\operatorname{ind}(\rho \times V)}$ whose construction only depends on G and the unitary representation V of H. Applying the formula for π in (2.7.2) to the present situation, we see that the $\operatorname{Ind}_H^G A$-part of $\operatorname{ind}_H^G\big((\rho \circ \epsilon_e) \times V\big)$ is given by the formula

$$(\pi(f)\xi)(s) = \rho\big(\operatorname{ind} \alpha_{s^{-1}}(f)(e)\big)\xi(s) = \rho(f(s))\xi(s) = (\sigma(f)\xi)(s)$$

with σ as in (2.7.1). $\square$

We now turn to some further properties of induced representations. To obtain those properties we shall pass from Green's to Blattner's realizations of the induced representations and back whenever it seems convenient. We start the discussion with the theorem of induction in steps. For this suppose that $L \subseteq H$ are closed subgroups of G. To avoid confusion, we write Φ_H^G for the left action of $A \rtimes G$ on $X_H^G(A)$ (i.e., $\Phi_H^G = k_A \times k_G$ in the notation used above) and we write Φ_L^G and Φ_L^H for the left actions of $A \rtimes G$ and $A \rtimes H$ on $X_L^G(A)$ and $X_L^H(A)$, respectively. Then the theorem of induction in steps reads as

Theorem 2.7.10 (Green)**.** *Let (A, G, α) and $L \subseteq H$ be as above. Then*

$$[X_H^G(A), \Phi_H^G] \circ [X_L^H(A), \Phi_L^H] = [X_L^G(A), \Phi_L^G]$$

as morphisms from $A \rtimes G$ to $A \rtimes L$ in the correspondence category $\mathfrak{Corr}$. As a consequence, we have

$$\operatorname{ind}_H^G\left(\operatorname{ind}_L^H(\rho \times V)\right) = \operatorname{ind}_L^G(\rho \times V)$$

for all $\rho \times V \in \operatorname{Rep}(A \rtimes L)$.

Proof. For the proof one has to check that $X_H^G \otimes_{A \rtimes H} X_L^H \cong X_L^G(A)$ as Hilbert $A \rtimes G - A \rtimes L$ bimodule. Indeed, one can check that such isomorphism is given on the level functions by the pairing $C_c(G, A) \otimes C_c(H, A) \to C_c(G, A)$ as given by the second formula in (2.6.1). We refer to [Gre78] and [Wil07, Theorem 5.9] for more details. □

By an *automorphism* γ of a system (A, G, α) we understand a pair $\gamma = (\gamma_A, \gamma_G)$, where γ_A is a $*$-automorphism of A and $\gamma_G : G \to G$ is an automorphism of G such that $\alpha_{\gamma_G(t)} = \gamma_A \circ \alpha_t \circ \gamma_A^{-1}$ for all $t \in G$. An *inner automorphism* of (A, G, α) is an automorphism of the form (α_s, C_s), $s \in G$, with $C_s(t) = sts^{-1}$. If $\gamma = (\gamma_A, \gamma_G)$ is an automorphism of (A, G, α) and if H is a closed subgroup of G, then γ induces an isomorphism $\gamma_{A \rtimes H} : A \rtimes H \to A \rtimes H_\gamma$ with $H_\gamma := \gamma_G(H)$ via

$$\gamma_{A \rtimes H}(f)(h) := \gamma_A\big(f(\gamma_G^{-1}(h))\big) \quad \text{for } h \in H_\gamma \text{ and } f \in C_c(H, A),$$

where we adjust Haar measures on H and H_γ such that $\int_H f(\gamma_G(h))\, dh = \int_{H_\gamma} f(h')\, dh'$ for $f \in C_c(H_\gamma)$. Note that if $(\rho, V) \in \operatorname{Rep}(A, H_\gamma)$, then $(\rho \circ \gamma_A, V \circ \gamma_G) \in \operatorname{Rep}(A, H)$ and we have

$$(\rho \times V) \circ \gamma_{A \rtimes H} \cong (\rho \circ \gamma_A) \times (V \circ \gamma_G)$$

for their integrated forms.

Remark 2.7.11. If $H = N$ is normal in G and if $\gamma_s = (\alpha_s, C_s)$ is an inner automorphism of (A, G, α), then we will write α_s^N for the corresponding automorphism of $A \rtimes N$. Then $s \mapsto \alpha_s^N$ is an action of G on $A \rtimes N$. This action will serve as a starting point for the study of twisted actions in §2.8 below.

Proposition 2.7.12. *Suppose that $\gamma = (\gamma_A, \gamma_G)$ is an automorphism of (A, G, α) and let $H \subseteq L$ be two closed subgroups of G. Then*

$$\operatorname{ind}_H^L\left((\rho \times V) \circ \gamma_{A \rtimes H}\right) \cong \left(\operatorname{ind}_{H_\gamma}^{L_\gamma}(\rho \times V)\right) \circ \gamma_{A \rtimes L}$$

for all $\rho \times V \in \operatorname{Rep}(A \rtimes H_\gamma)$, where "$\cong$" denotes unitary equivalence. In particular, if $\rho \times V \in \operatorname{Rep}(A, H)$ and (α_s, C_s) is an inner automorphism of (A, G, α) then

$$\operatorname{ind}_H^G(\rho \times V) \cong \operatorname{ind}_{sHs^{-1}}^G\left(s \cdot (\rho \times V)\right),$$

where we put $s \cdot (\rho \times V) := (\rho \circ \alpha_{s^{-1}}) \times (V \circ C_{s^{-1}}) \in \operatorname{Rep}(A, sHs^{-1})$.

Proof. Simply check that the map $\gamma_{A\rtimes L} : C_c(L, A) \to C_c(L_\gamma, A)$ as defined above also extends to a bijection $\Phi_L : X_H^L(A) \to X_{H_\gamma}^{L_\gamma}(A)$ which is compatible with the isomorphisms $\gamma_{A\rtimes L} : A \rtimes L \to A \rtimes L_\gamma$ and $\gamma_{A\rtimes H} : A \rtimes H \to A \rtimes H_\gamma$ on the left and right. This implies that $\gamma^*_{A\rtimes L} \circ [X_{H_\gamma}^{L_\gamma}(A), k_A \times k_{L_\gamma}] = [X_H^L(A), k_A \times k_L] \circ \gamma^*_{A\rtimes H}$ in $\mathfrak{Corr}$ and the first statement follows. The second statement follows from the first applied to $L = G$ and $\gamma = (\alpha_s, C_s)$ together with the fact that for any $\pi \times U \in \mathrm{Rep}(A, G)$ the unitary $U_s \in \mathcal{U}(H_\pi)$ implements a unitary equivalence between $s \cdot (\pi \times U) = (\pi \circ \alpha_{s^{-1}}) \times (U \circ C_{s^{-1}})$ and $\pi \times U$. □

As a direct consequence we get:

Corollary 2.7.13. *Let (A, G, α) be a system. For $J \in \mathcal{I}(A)$ let*

$$J^G := \cap\{\alpha_s(J) : s \in G\}.$$

Then $\mathrm{ind}_{\{e\}}^G J^G = \mathrm{ind}_{\{e\}}^G J$ in $A \rtimes G$. As a consequence, if $\rho \in \mathrm{Rep}(A)$ such that $\cap\{\ker(\rho\circ\alpha_s) : s \in G\} = \{0\}$, then $\mathrm{ind}_{\{e\}}^G \rho$ factors through a faithful representation of the reduced crossed product $A \rtimes_r G$.

Proof. Let $J = \ker \rho$ for some $\rho \in \mathrm{Rep}(A)$ and let $\rho^G := \bigoplus_{s\in G} \rho \circ \alpha_s$. Then $J^G = \ker \rho^G$. It follows from Proposition 2.7.12 that $\mathrm{ind}_{\{e\}}^G \rho \circ \alpha_s \cong \mathrm{ind}_{\{e\}}^G \rho$ for all $s \in G$. Since induction preserves direct sums, it follows that

$$\mathrm{ind}_{\{e\}}^G J = \ker(\mathrm{ind}_{\{e\}}^G \rho) = \ker(\mathrm{ind}_{\{e\}}^G \rho^G) = \mathrm{ind}_{\{e\}}^G J^G.$$

If $\cap\{\ker(\rho \circ \alpha_s) : s \in G\} = \{0\}$, then ρ^G is faithful and it follows from Remark 2.3.4 (3) and Remark 2.7.5 (3) that $\ker \Lambda_A^G = \ker(\mathrm{ind}_{\{e\}}^G \rho^G) = \ker(\mathrm{ind}_{\{e\}}^G \rho)$. □

Remark 2.7.14. From the previous results it is now possible to obtain a fairly easy proof of the fact that Green's $\mathrm{Ind}_H^G A \rtimes G - A \rtimes H$ imprimitivity bimodule $X_H^G(A)$ factors to give a $\mathrm{Ind}_H^G A \rtimes_r G - A \rtimes_r H$ imprimitivity bimodule for the reduced crossed products (compare with Remark 2.6.5). Indeed, if ρ is any faithful representation of A, and if $\epsilon_e : \mathrm{Ind}\, A \to A$ denotes evaluation at the unit e, it follows from Corollary 2.7.13, that $\ker(\Lambda^G_{\mathrm{Ind}\, A}) = \ker\big(\mathrm{ind}_{\{e\}}^G(\rho \circ \epsilon_e)\big)$. The latter coincides with $\ker(\mathrm{ind}_H^G(\mathrm{ind}_{\{e\}}^H(\rho \circ \epsilon_e)))$ by Theorem 2.7.10. If $\sigma \times V$ denotes the representation $\mathrm{ind}_{\{e\}}^H \rho \in \mathrm{Rep}(A \rtimes H)$, then $\ker(\sigma \times V) = \ker \Lambda_A^H$ since ρ is faithful on A and a short computation shows that $\mathrm{ind}_{\{e\}}^H(\rho \circ \epsilon_e) = (\sigma \circ \epsilon_e) \times V$, where on the left-hand side we use induction in the system $(\mathrm{Ind}\, A, H, \mathrm{Ind}\, \alpha)$. Putting all this together we get

$$\begin{aligned}
\ker(\Lambda^G_{\mathrm{Ind}\, A}) &= \ker\big(\mathrm{ind}_{\{e\}}^G(\rho \circ \epsilon_e)\big) = \ker\big(\mathrm{ind}_H^G(\mathrm{ind}_{\{e\}}^H(\rho \circ \epsilon_e))\big)\\
&= \ker\big(\mathrm{ind}_H^G\big((\sigma \circ \epsilon_e) \times V\big)\big) \overset{*}{=} \ker\big(\mathrm{ind}^{X_H^G(A)}(\sigma \times V)\big)\\
&= \ker\big(\mathrm{ind}^{X_H^G(A)} \Lambda_A^H\big) \overset{**}{=} \mathrm{ind}^{X_H^G(A)}\big(\ker \Lambda_A^H\big),
\end{aligned}$$

where * follows from Corollary 2.7.9 and ** follows from Equation (2.5.4). The desired result then follows from the Rieffel correspondence (Proposition 2.5.4).

We now come to some important results concerning the relation between induction and restriction of representations and ideals (see Definition 2.7.2 for the definition of the restriction maps). We start with:

Proposition 2.7.15. *Suppose that (A, G, α) is a system and let $N \subseteq H$ be closed subgroups of G such that N is normal in G. Let $\mathcal{F}_{\rho\times V}$ be the dense subspace of Blattner's induced Hilbert space $H_{\mathrm{ind}(\rho\times V)}$ as constructed above. Then*

$$\big(\mathrm{res}_N^G(\mathrm{ind}_H^G(\rho \times V))(f)\xi\big)(s) = \mathrm{res}_N^H(\rho \times V)(\alpha^N_{s^{-1}}(f))\xi(s) \tag{2.7.3}$$

for all $f \in A \rtimes N$, $\xi \in \mathcal{F}_{\rho\times V}$ and $s \in G$, where $\alpha^N : G \to \mathrm{Aut}(A \rtimes N)$ is the canonical action of G on $A \rtimes N$ (see Remark 2.7.11 above). As a consequence, if $J \in \mathcal{I}(A \rtimes H)$, we get

$$\mathrm{res}_N^G\left(\mathrm{ind}_H^G J\right) = \cap\{\alpha^N_s(\mathrm{res}_N^H(J)) : s \in G\}. \tag{2.7.4}$$

Proof. Define $\sigma : A \rtimes N \to \mathcal{B}(H_{\mathrm{ind}(\rho\times V)})$ by $(\sigma(f)\xi)(s) = \rho\times V|_N(\alpha^N_{s^{-1}}(f))\xi(s)$ for $f \in A \rtimes N$ and $\xi \in \mathcal{F}_{\rho\times V}$. Then σ is a nondegenerate $*$-representation and hence it suffices to check that the left-hand side of (2.7.3) coincides with $(\sigma(f)\xi)(s)$ for $f \in C_c(N, A)$. But using (2.7.2) together with the transformation $n \mapsto sns^{-1}$ and the equation $\xi(sn^{-1}) = V_n\xi(s)$ for $s \in G, n \in N$, the left-hand side becomes

$$\begin{aligned}\big(\mathrm{res}_N^G\left(\mathrm{ind}_H^G(\rho \times V))(f)\xi\right)\big)(s) &= \int_N \rho(\alpha_{s^{-1}}(f(n)))\xi(n^{-1}s)\,dn\\ &= \delta(s^{-1})\int_N \rho(\alpha_{s^{-1}}(f(sns^{-1})))\xi(sn^{-1})\,dn\\ &= \int_N \rho(\alpha^N_{s^{-1}}(f)(n))V_n\xi(s)\,dn = (\sigma(f)\xi)(s). \quad\square\end{aligned}$$

Remark 2.7.16. Suppose that (A, G, α) is a system, H is a closed subgroup of G, and $J \subseteq A$ is a G-invariant ideal of A. If $\rho \times V$ is a representation of $A \rtimes H$ and if we put $\pi \times U := \mathrm{ind}_H^G(\rho \times V)$, then it follows from the above proposition that

$$J \subseteq \ker\rho \quad\Longleftrightarrow\quad J \subseteq \ker\pi.$$

Hence, we see that the induction map for the system (A, G, α) determines a map from $\mathrm{Rep}(A/J \rtimes H)$ to $\mathrm{Rep}(A/J \rtimes G)$ if we identify representations of $A/J \rtimes H$ with the representations $\rho \times V$ of $A \rtimes H$ that satisfy $J \subseteq \ker\rho$ (and similarly for G). It is easy to check (e.g., by using Blattner's construction of the induced representations) that this map coincides with the induction map for the system $(A/J, G, \alpha)$.

Also, if $\rho \times V$ is a representation of $A \rtimes H$ such that ρ restricts to a nondegenerate representation of J, then one can check that the restriction of $\mathrm{ind}_H^G(\rho \times V)$ to $J \rtimes G$ conicides with the induced representation $\mathrm{Ind}_H^G(\rho|_J \times V)$ where the latter representation is induced from $J \rtimes H$ to $J \rtimes G$ via $X_H^G(J)$.[11] We shall use these facts quite frequently below.

We close this section with some useful results on tensor products of representations. If (π, U) is a covariant representation of the system (A, G, α) on H_π and if V is a unitary representation of G on H_V, then $(\pi \otimes 1_{H_V}, U \otimes V)$ is a covariant representation of (A, G, α) on $H_\pi \otimes H_V$ and we obtain a pairing

$$\begin{aligned}\otimes : &\operatorname{Rep}(A \rtimes G) \times \operatorname{Rep}(G) \to \operatorname{Rep}(A \rtimes G);\\ &\big((\pi \times U), V\big) \mapsto (\pi \times U) \otimes V := (\pi \otimes 1_{H_V}) \times (U \otimes V).\end{aligned}$$

Identifying $\operatorname{Rep}(G) \cong \operatorname{Rep}(C^*(G))$, this map can also be obtained via the composition

$$\operatorname{Rep}(A \rtimes G) \times \operatorname{Rep}(C^*(G)) \to \operatorname{Rep}\big((A \rtimes G) \otimes C^*(G)\big) \xrightarrow{\widehat{\alpha}^*} \operatorname{Rep}(A \rtimes G),$$

where the first map sends a pair $(\pi \times U, V)$ to the external tensor-product representation $(\pi \times U)\hat{\otimes} V$ of $(A \rtimes G) \otimes C^*(G)$ and $\widehat{\alpha} : A \rtimes G \to M\big((A \rtimes G) \otimes C^*(G)\big)$ denotes the integrated form of the tensor product $(i_A \otimes 1_{C^*(G)}, i_G \otimes i_G)$ of the canonical inclusions $(i_A, i_G) : (A, G) \to M(A \rtimes G)$ with the inclusion $i_G : G \to M(C^*(G))$ ($\widehat{\alpha}$ is the dual *coaction* of G on $A \rtimes G$). Thus, from Propositions 2.5.16 and 2.5.17 we get

Proposition 2.7.17. *The map* $\otimes : \operatorname{Rep}(A \rtimes G) \times \operatorname{Rep}(G) \to \operatorname{Rep}(A \rtimes G)$ *preserves weak containment in both variables and is jointly continuous with respect to the Fell topologies.*

Proposition 2.7.18. *Let (A, G, α) be a system and let H be a closed subgroup of G. Then*

(i) $\operatorname{ind}_H^G\big((\rho \times V) \otimes U|_H\big) \cong \big(\operatorname{ind}_H^G(\rho \times V)\big) \otimes U$ *for all* $\rho \times V \in \operatorname{Rep}(A \rtimes H)$ *and* $U \in \operatorname{Rep}(G)$*;*

(ii) $\operatorname{ind}_H^G\big((\pi \times U|_H) \otimes V\big) \cong (\pi \times U) \otimes \operatorname{ind}_H^G V$ *for all* $V \in \operatorname{Rep}(H)$ *and* $\pi \times U \in \operatorname{Rep}(A \rtimes G)$.

In particular, if $\pi \times U \in \operatorname{Rep}(A \rtimes G)$ *and N is a normal subgroup of G, then*

$$\operatorname{ind}_N^G(\pi \times U|_N) \cong (\pi \times U) \otimes \lambda_{G/N},$$

where $\lambda_{G/N}$ denotes the regular representation of G/N, viewed as a representation of G.

Proof. This result can be most easily shown using Blattner's realization of the induced representations: In the first case define

$$W : \mathcal{F}_{\rho \times V} \otimes H_U \to \mathcal{F}_{(\rho \times V) \otimes U|_H};\ W(\xi \otimes v)(s) = \xi(s) \otimes U_{s^{-1}} v.$$

[11] Of course, these results are also consequences of the naturality of the assignment $A \mapsto X_H^G(A)$ as stated in Remark 2.6.5 (3).

Then a short computation shows that W is a unitary intertwiner of $\big(\mathrm{ind}_H^G(\rho \times V)\big)\otimes U$ and $\mathrm{ind}_H^G\big((\rho\times V)\otimes U|_H)\big)$. A similar map works for the second equivalence. Since $\lambda_{G/N} = \mathrm{ind}_N^G 1_N$, the last assertion follows from (ii) for the case $V = 1_N$. $\square$

Corollary 2.7.19. *Suppose that (A, G, α) is a system and that N is a normal subgroup of G such that G/N is amenable. Then $\pi \times U$ is weakly contained in $\mathrm{ind}_N^G(\mathrm{res}_N^G(\pi\times U))$ for all $\pi\times U \in \mathrm{Rep}(A\rtimes G)$. As a consequence, $\mathrm{ind}_N^G(\mathrm{res}_N^G I) \subseteq I$ for all $I \in \mathcal{I}(A \rtimes G)$.*

Proof. Since G/N is amenable if and only if $1_{G/N} \prec \lambda_{G/N}$ we obtain from Proposition 2.7.18

$$\pi \times U = (\pi \times U) \otimes 1_{G/N} \prec (\pi \times U) \otimes \lambda_{G/N} = \mathrm{ind}_N^G(\pi \times U|_N),$$

which proves the first statement. The second statement follows from the first by choosing $\pi \times U \in \mathrm{Rep}(A \rtimes G)$ such that $I = \ker(\pi \times U)$. $\square$

2.7.2 The ideal structure of crossed products

In this section we come to the main results on the Mackey–Rieffel–Green machine, namely, the description of the spectrum $(A \rtimes G)\widehat{\ }$ and the primitive ideal space $\mathrm{Prim}(A\rtimes G)$ in terms of induced representations (resp. ideals) under some favorable circumstances. We start with some topological notations:

Definition 2.7.20. Let Y be a topological space.

(i) We say that Y is *almost Hausdorff* if every nonempty closed subset F of Y contains a nonempty relatively open Hausdorff subset U (which can then be chosen to be dense in F).

(ii) A subset $C \subseteq Y$ is called *locally closed* if C is relatively open in its closure $\overline{C}$, i.e., $\overline{C} \setminus C$ is closed in Y.

It is important to note that if A is a type I algebra, then the spectrum $\widehat{A}$ (and then also $\mathrm{Prim}(A) \cong \widehat{A}$) is almost Hausdorff with respect to the Jacobson topology. This follows from the fact that every quotient of a type I algebra is type I and that every nonzero type I algebra contains a nonzero continuous-trace ideal, and hence its spectrum contains a nonempty Hausdorff subset U (see [Dix77, Chapter 4] and §2.2.4). Note also that if Y is almost Hausdoff, then the one-point sets $\{y\}$ are locally closed for all $y \in Y$.

If A is a C^*-algebra and if $J \subseteq I$ are two closed two-sided ideals of A, then we may view $\widehat{I/J}$ (resp. $\mathrm{Prim}(I/J)$) as a locally closed subset of $\widehat{A}$ (resp. $\mathrm{Prim}(A)$). Indeed, we first identify $\widehat{A/J}$ with the closed subset $\{\pi \in \widehat{A} : J \subseteq \ker \pi\}$ of $\widehat{A}$ and then we identify $\widehat{I/J}$ with the open subset $\{\pi \in \widehat{A/J} : \pi(I) \neq \{0\}\}$ (and similarly for $\mathrm{Prim}(I/J)$; compare with §2.2.4).

Conversely, if C is a locally closed subset of $\widehat{A}$, then C is canonically homeomorphic to $\widehat{I_C/J_C}$ if we take $J_C := \ker(C)$ and $I_C := \ker\left(\overline{C} \smallsetminus C\right)$ (we write $\ker(E) := \cap\{\ker\pi : \pi \in E\}$ if $E \subseteq \widehat{A}$ and similarly $\ker(D) := \cap\{P : P \in D\}$ for $D \subseteq \mathrm{Prim}(A)$). If we apply this observation to commutative C^*-algebras, we recover the well-known fact that the locally closed subsets of a locally compact Hausdorff space Y are precisely those subsets of Y that are locally compact in the relative topology.

Definition 2.7.21. Let A be a C^*-algebra and let C be a locally closed subset of $\widehat{A}$ (resp. $\mathrm{Prim}(A)$). Then $A_C := I_C/J_C$ with I_C, J_C as above is called the *restriction of A to C*. In the same way, we define the restriction A_D of A to D for a locally closed subset D of $\mathrm{Prim}(A)$.

In what follows, we shall use the following notations:

Notations 2.7.22. If (A, G, α) is a system, we consider $\mathrm{Prim}(A)$ as a G-space via the continuous action $G \times \mathrm{Prim}(A) \to \mathrm{Prim}(A)$; $(s, P) \mapsto s \cdot P := \alpha_s(P)$. We write

$$G_P := \{s \in G : s \cdot P = P\} \quad \text{and} \quad G(P) := \{s \cdot P : s \in G\}$$

for the *stabiliser* and the *G-orbit* of $P \in \mathrm{Prim}(A)$, respectively. Moreover, we put

$$P^G := \ker G(P) = \cap\{s \cdot P : s \in G\}.$$

Note that the stabilisers G_P are closed subgroups of G for all $P \in \mathrm{Prim}(A)$.[12]

Remark 2.7.23. Similarly, we may consider the G-space $\widehat{A}$ with G-action $(s, \pi) \mapsto s \cdot \pi := \pi \circ \alpha_{s^{-1}}$ (identifying representations with their equivalence classes) and we then write G_π and $G(\pi)$ for the stabilisers and the G-orbits, respectively. However, the stabilisers G_π are not necessarily closed in G if A is not a type I algebra. If A is type I, then $\pi \mapsto \ker\pi$ is a G-equivariant homeomorphism from $\widehat{A}$ to $\mathrm{Prim}(A)$.

The following theorem is due to Glimm [Gli61]:

Theorem 2.7.24 (Glimm's theorem)**.** *Suppose that (A, G, α) is a separable type I system (i.e., A is a separable type I algebra and G is second countable). Then the following are equivalent:*

(i) *The quotient space $G\backslash\mathrm{Prim}(A)$ is almost Hausdorff.*

(ii) *$G\backslash\mathrm{Prim}(A)$ is a T_0-space.*

(iii) *All points in $G\backslash\mathrm{Prim}(A)$ are locally closed.*

(iv) *For all $P \in \mathrm{Prim}(A)$ the quotient G/G_P is homeomorphic to $G(P)$ via $s \cdot G_P \mapsto s \cdot P$.*

[12] The fact that G_P is closed in G follows from the fact that $\mathrm{Prim}(A)$ is a T_0-space. Indeed, if $\{s_i\}$ is a net in G_P that converges to some $s \in G$, then $P = s_i \cdot P \to s \cdot P$, so $s \cdot P$ is in the closure of $\{P\}$. Conversely, we have $s \cdot P = ss_i^{-1} \cdot P \to P$, and hence $\{P\} \in \overline{\{sP\}}$. Since $\mathrm{Prim}(A)$ is a T_0-space it follows that $P = s \cdot P$.

(v) *There exists an ordinal number μ and an increasing sequence $\{I_\nu\}_{\nu\leq\mu}$ of G-invariant ideals of A such that $I_0 = \{0\}$, $I_\mu = A$ and $G\backslash\mathrm{Prim}(I_{\nu+1}/I_\nu)$ is Hausdorff for all $\nu < \mu$.*

Hence, if (A, G, α) is a separable type I system satisfying one of the equivalent conditions above, then (A, G, α) is smooth in the sense of:

Definition 2.7.25. A system (A, G, α) is called *smooth* if the following two conditions are satisfied:

(i) The map $G/G_P \to G(P)$; $s \cdot G_P \to s \cdot P$ is a homeomorphism for all $P \in \mathrm{Prim}(A)$.

(ii) The quotient $G\backslash\mathrm{Prim}(A)$ is almost Hausdorff, or A is separable and all orbits $G(P)$ are locally closed in $\mathrm{Prim}(A)$.

If $G(P)$ is a locally closed orbit of $\mathrm{Prim}(A)$, then we may identify $G(P)$ with $\mathrm{Prim}(A_{G(P)})$, where $A_{G(P)} = I_{G(P)}/J_{G(P)}$ denotes the restriction of A to $G(P)$ as in Definition 2.7.21 (note that $J_{G(P)} = P^G = \cap\{\alpha_s(P) : s \in G\}$). Since the ideals $I_{G(P)}$ and $J_{G(P)}$ are G-invariant, the action of G on A restricts to an action of G on $A_{G(P)}$. Using exactness of the full crossed-product functor, we get

$$A_{G(P)} \rtimes G = (I_{G(P)}/J_{G(P)}) \rtimes G \cong (I_{G(P)} \rtimes G)/(J_{G(P)} \rtimes G). \tag{2.7.5}$$

If G is exact, a similar statement holds for the reduced crossed products.

Proposition 2.7.26. *Suppose that (A, G, α) is a system such that*

(i) *$G\backslash\mathrm{Prim}(A)$ is almost Hausdorff, or*

(ii) *A is separable.*

Then, for each $\pi \times U \in (A \rtimes G)\widehat{\ }$, there exists an orbit $G(P) \subseteq \mathrm{Prim}(A)$ such that $\ker\pi = P^G$. If, in addition, all orbits in $\mathrm{Prim}(A)$ are locally closed (which is automatic in the case of (i)), then $G(P)$ is uniquely determined by $\pi \times U$.

Proof. (Following ideas from [Rie79]) Let $J = \ker\pi$. By passing from A to A/J we may assume without loss of generality that π is faithful. We then have to show that there exists a $P \in \mathrm{Prim}(A)$ such that $G(P)$ is dense in $\mathrm{Prim}(A)$.

We first show that under these assumptions every open subset $W \subseteq G\backslash\mathrm{Prim}(A)$ is dense. Indeed, since π is faithful, it follows that $\pi \times U$ restricts to a nonzero, and hence irreducible representation of $I \rtimes G$, whenever I is a nonzero G-invariant ideal of A. In particular, $\pi(I)H_\pi = H_\pi$ for all such ideals I. Assume now that there are two nonempty G-invariant open sets $U_1, U_2 \subseteq \mathrm{Prim}(A)$ with $U_1 \cap U_2 = \emptyset$. Put $I_i = \ker(\mathrm{Prim}(A) \smallsetminus U_i)$, $i = 1, 2$. Then I_1, I_2 would be nonzero G-invariant ideals such that $I_1 \cdot I_2 = I_1 \cap I_2 = \{0\}$, and then

$$H_\pi = \pi(I_1)H_\pi = \pi(I_1)\big(\pi(I_2)H_\pi\big) = \pi(I_1 \cdot I_2)H_\pi = \{0\},$$

which is a contradiction.

Assume that $G\backslash\mathrm{Prim}(A)$ is almost Hausdorff. If there is no dense orbit $G(P)$ in $\mathrm{Prim}(A)$, then $G\backslash\mathrm{Prim}(A)$ contains an open dense Hausdorff subset that contains at least two different points. But then there exist nonempty G-invariant open subsets U_1, U_2 of $\mathrm{Prim}(A)$ with $U_1 \cap U_2 = \emptyset$, which is impossible.

If A is separable, then $G\backslash\mathrm{Prim}(A)$ is second countable (see [Dix77, Chapter 3]) and we find a countable basis $\{U_n : n \in \mathbb{N}\}$ for its topology with $U_n \neq \emptyset$ for all $n \in \mathbb{N}$. By the first part of this proof we know that all U_n are dense in $G\backslash \mathrm{Prim}(A)$. Since $G\backslash\mathrm{Prim}(A)$ is a Baire space by [Dix77, Chapter 3], it follows that $D := \cap_{n\in\mathbb{N}} U_n$ is also dense in $G\backslash \mathrm{Prim}(A)$. Note that every open subset of $G\backslash\mathrm{Prim}(A)$ contains D. Hence, if we pick any orbit $G(P) \in D$ then $G(P)$ is dense in $\mathrm{Prim}(A)$, since otherwise D would be a subset of the nonempty open set $G\backslash\big(\mathrm{Prim}(A) \smallsetminus \overline{G(P)}\big)$, which is impossible.

If the dense orbit $G(P)$ is locally closed then $G(P)$ is open in its closure $\mathrm{Prim}(A)$, which implies that $G(P)$ is the unique dense orbit in $\mathrm{Prim}(A)$. This gives the uniqueness assertion of the proposition. □

Recall that if $G(P)$ is a locally closed orbit in $\mathrm{Prim}(A)$, we get $A_{G(P)} \rtimes G \cong (I_{G(P)} \rtimes G)/(J_{G(P)} \rtimes G)$ and similarly $A_{G(P)} \rtimes_r G \cong (I_{G(P)} \rtimes_r G)/(J_{G(P)} \rtimes_r G)$ if G is exact. Using this we get

Corollary 2.7.27. *Suppose that (A, G, α) is smooth. Then we obtain a decomposition of $(A \rtimes G)\widehat{\ }$ (resp. $\mathrm{Prim}(A \rtimes G)$) as the disjoint union of the locally closed subsets $(A_{G(P)} \rtimes G)\widehat{\ }$ (resp. $\mathrm{Prim}(A_{G(P)} \rtimes G)$), where $G(P)$ runs through all G-orbits in $\mathrm{Prim}(A)$. If G is exact, similar statements hold for the reduced crossed products.*

Proof. It follows from Proposition 2.7.26 that for each $\pi \times U \in (A \rtimes G)\widehat{\ }$, there exists a unique orbit $G(P)$ such that $\ker \pi = P^G = J_{G(P)}$ and then $\pi \times U$ restricts to an irreducible representation of $A_{G(P)} \rtimes G$. Hence

$$(A \rtimes G)\widehat{\ } = \cup\{(A_{G(P)} \rtimes G)\widehat{\ } : G(P) \in G\backslash \mathrm{Prim}(A)\}.$$

To see that this union is disjoint, assume that there exists an element $\rho \times V \in (A_{G(P)} \rtimes G)\widehat{\ }$ (viewed as a representation of $A \rtimes G$) such that $\ker \rho \neq P^G$. Since ρ is a representation of $A_{G(P)} = I_{G(P)}/P^G$ we have $\ker \rho \supseteq P^G$. By Proposition 2.7.26 there exists a $Q \in \mathrm{Prim}(A)$ such that $\ker \rho = Q^G$. Then $Q^G \supseteq P^G$, which implies that $G(Q) \subseteq \big(\overline{G(P)} \smallsetminus G(P)\big)$. But then

$$\ker \rho = Q^G = \ker G(Q) \supseteq \ker \big(\overline{G(P)} \smallsetminus G(P)\big) = I_{G(P)},$$

which contradicts the assumption that $\rho \times V \in (A_{G(P)} \rtimes G)\widehat{\ }$. □

It is now easy to give a proof of the Mackey–Green–Rieffel theorem, which is the main result of this section. If (A, G, α) is smooth, one can easily check that points in $\mathrm{Prim}(A)$ are automatically locally closed (since they are closed in their orbits). Hence, for each $P \in \operatorname{Prim}(A)$ the restriction $A_P := I_P/P$ of A to $\{P\}$ is a simple subquotient of A. Since I_P and P are invariant under the action of the stabiliser G_P, the action of G_P on A factors through an action of G_P on A_P. It is then straightforward to check (using the same arguments as given in the proof of Corollary 2.7.27) that there is a canonical one-to-one correspondence between the irreducible representations of $A_P \rtimes G_P$ and the set of all irreducible representations $\rho \times V$ of $A \rtimes G_P$ satisfying $\ker \rho = P$. In case where $A = C_0(X)$ is commutative, we will study this problem in §2.7.3 below, and the case where A is type I will be studied in §2.8.6.

Remark 2.7.28. If A is type I, then $A_P \cong \mathcal{K}(H_\pi)$, the compact operators on the Hilbert space H_π, where $\pi : A \to \mathcal{B}(H_\pi)$ is the unique (up to equivalence) irreducible representation of A with $\ker \pi = P$. To see this we first pass to $A/P \cong \pi(A)$. Since A is type I we know that $\mathcal{K}(H_\pi) \subseteq \pi(A)$. Hence, if we identify $\mathcal{K}(H_\pi)$ with an ideal of A/P, we see (since π does not vanish on this ideal) that this ideal must correspond to the open set $\{\pi\}$ (resp. $\{P\}$) in its closure $\widehat{A/P}$ (resp. $\operatorname{Prim}(A/P)$).

Theorem 2.7.29 (Mackey–Rieffel–Green)**.** *Suppose that (A, G, α) is smooth. Let $\mathcal{S} \subseteq \operatorname{Prim}(A)$ be a cross-section for the orbit space $G\backslash \operatorname{Prim}(A)$, i.e., $\mathcal{S}$ intersects each orbit $G(P)$ in exactly one point. Then induction of representations and ideals induces bijections*

$$\operatorname{Ind} : \cup_{P\in\mathcal{S}} (A_P \rtimes G_P)\widehat{\ } \to (A \rtimes G)\widehat{\ }; \rho \times V \mapsto \operatorname{ind}_{G_P}^G(\rho \times V) \quad \textit{and}$$

$$\operatorname{Ind} : \cup_{P\in\mathcal{S}} \operatorname{Prim}(A_P \rtimes G_P) \to \operatorname{Prim}(A \rtimes G);\ Q \mapsto \operatorname{ind}_{G_P}^G Q.$$

If G is exact, these maps restrict to similar bijections

$$\cup_{P\in\mathcal{S}} (A_P \rtimes_r G_P)\widehat{\ } \overset{\operatorname{Ind}}{\to} (A \rtimes_r G)\widehat{\ } \quad \textit{and} \quad \cup_{P\in\mathcal{S}} \operatorname{Prim}(A_P \rtimes_r G_P) \overset{\operatorname{Ind}}{\to} \operatorname{Prim}(A \rtimes_r G)$$

for the reduced crossed products.

Proof. We show that $\operatorname{Ind} : \cup_{P\in\mathcal{S}} (A_P \rtimes G_P)\widehat{\ } \to (A \rtimes G)\widehat{\ }$ is a bijection. Bijectivity of the other maps follows similarly.

By Corollary 2.7.27 it suffices to show that $\operatorname{Ind} : (A_P \rtimes G_P)\widehat{\ } \to (A_{G(P)} \rtimes G)\widehat{\ }$ is a bijection for all $P \in \mathcal{S}$. By definition of $A_{G(P)}$ we have $\operatorname{Prim}(A_{G(P)}) \cong G(P)$ and by the smoothness of the action we have $G(P) \cong G/G_P$ as G-spaces. Hence, it follows from Theorem 2.6.2 that $A_{G(P)} \cong \operatorname{Ind}_{G_P}^G A_P$. Hence induction via Green's $A_{G(P)} \rtimes G - A_P \rtimes G_P$ imprimitivity bimodule $X_P := X_{G_P}^G(A_P)$ gives the desired bijection $\operatorname{ind}^{X_P} : (A_P \rtimes G_P)\widehat{\ } \to (A_{G(P)} \rtimes G)\widehat{\ }$. By Corollary 2.7.9, induction via X_P coincides with the usual induction for the system $(A_{G(P)}, G, \alpha)$, which by Remark 2.7.16 is compatible with inducing the corresponding representations for (A, G, α). $\square$

The above result shows that for smooth systems, all representations are induced from the stabilisers for the corresponding action of G on $\mathrm{Prim}(A)$. In fact the above result is much stronger, since it shows that $A \rtimes G$ has a "fibration" over $G\backslash \mathrm{Prim}(A)$ such that the fiber $A_{G(P)} \rtimes G$ over an orbit $G(P)$ is Morita equivalent to $A_P \rtimes G_P$, hence, up to the global structure of the fibration, the study of $A \rtimes G$ reduces to the study of the fibers $A_P \rtimes G_P$. Note that under the assumptions of Theorem 2.7.29 the algebra A_P is always simple. We shall give a more detailed study of the crossed products $A_P \rtimes G_P$ in the important special case where A is type I in §2.8.6 below. The easier situation where $A = C_0(X)$ is treated in §2.7.3 below.

Note that the study of the global structure of $A \rtimes G$, i.e., of the global structure of the fibration over $G\backslash \mathrm{Prim}(A)$ is in general quite complicated, even in the situation where $G\backslash \mathrm{Prim}(A)$ is Hausdorff. In general, it is also very difficult (if not impossible) to describe the global topology of $\mathrm{Prim}(A \rtimes G)$ in terms of the bijection of Theorem 2.7.29. Some progress has been made in the case where A is a continuous-trace C^*-algebra and/or where the stabilisers are assumed to vary continuously, and we refer to [CKRW97, Ech96, EE11, ER96, EN01, EW98, EW14, RW98] and the references given in those papers and books for more information on this problem. If $A = C_0(X)$ is commutative and G is abelian, a very satisfying description of the topology of $\mathrm{Prim}(C_0(X) \rtimes G)$ has been obtained by Dana Williams in [Wil81]. We shall discuss this situation in §2.7.3 below.

Even worse, the assumption of having a smooth action is a very strong one and for arbitrary systems one cannot expect that one can compute all irreducible representations via induction from stabilisers. Indeed, in general it is not possible to classify all irreducible representations of a non–type I C^*-algebra, and a similar problem occurs for crossed products $A \rtimes G$ if the action of G on $\mathrm{Prim}(A)$ fails to be smooth. However, at least if (A, G, α) is separable and G is amenable, there is a positive result towards the description of $\mathrm{Prim}(A \rtimes G)$ which was obtained by work of Sauvageot and Gootman–Rosenberg, thus giving a positive answer to an earlier formulated conjecture by Effros and Hahn (see [EH67]). To give precise statements, we need

Definition 2.7.30. A nondegenerate representation ρ of a C^*-algebra A is called *homogeneous* if all nontrivial subrepresentations of ρ have the same kernel as ρ.

It is clear that every irreducible representation is homogeneous and one can show that the kernel of any homogeneous representation is a prime ideal, and hence it is primitive if A is second countable. We refer to [Wil07] for a discussion on this and for very detailed proofs of Theorems 2.7.31 and 2.7.32 stated below:

Theorem 2.7.31 (Sauvageot ([Sau79])). *Suppose that (A, G, α) is a separable system (i.e., A is separable and G is second countable). Let $P \in \mathrm{Prim}(A)$ and let G_P denote the stabiliser of P in G. Suppose that $\rho \times V$ is a homogeneous representation of $A \rtimes G_P$ such that ρ is a homogeneous representation of A with* $\ker \rho = P$. *Then*

$\mathrm{ind}_{G_P}^G(\rho \times V)$ *is a homogeneous representation of* $A \rtimes G$ *and* $\ker\big(\mathrm{ind}_{G_P}^G(\rho \times V)\big)$ *is a primitive ideal of* $A \rtimes G$.

We say that a primitive ideal of $A \rtimes G$ is *induced* if it is obtained as in the above theorem. Note that Sauvageot already showed in [Sau79] that in the case where G is amenable, every primitive ideal of $A \rtimes G$ contains an induced primitive ideal and in case where G is discrete every primitive ideal is contained in an induced primitive ideal. Together, this shows that for actions of discrete amenable groups all primitive ideals of $A \rtimes G$ are induced from the stabilisers. Sauvageot's result was generalized by Gootman and Rosenberg in [GR79, Theorem 3.1]:

Theorem 2.7.32 (Gootman–Rosenberg)**.** *Suppose that* (A, G, α) *is a separable system. Then every primitive ideal of* $A \rtimes G$ *is contained in an induced ideal. As a consequence, if* G *is amenable, then every primitive ideal of* $A \rtimes G$ *is induced.*

The condition in Theorem 2.7.31 that the representations $\rho \times V$ and ρ are homogeneous is a little bit unfortunate. In fact, a somehow more natural formulation of Sauvageot's theorem (using the notion of induced ideals) would be to state that whenever $Q \in \mathrm{Prim}(A \rtimes G_P)$ such that $\mathrm{res}_{\{e\}}^{G_P}(Q) = P$, then $\mathrm{ind}_{G_P}^G(Q)$ is a primitive ideal of $A \rtimes G$. Note that if $\rho \times V$ is as in Theorem 2.7.31, then $Q = \ker(\rho \times V)$ is an element of $\mathrm{Prim}(A \rtimes G_P)$, which satisfies the above conditions. At present time, we do not know whether this more general statement is true, and we want to take this opportunity to point out that the statement of [Ech96, Theorem 1.4.14] is not correct (or at least not known) as it stands. We are very grateful to Dana Williams for pointing out this error and we refer to the paper [EW08] for a more elaborate discussion of this problem. But let us indicate here that the problem vanishes if all points in $\mathrm{Prim}(A)$ are locally closed (which is particularly true if A is type I).

Proposition 2.7.33. *Suppose that* (A, G, α) *is a separable system such that one of the following conditions is satisfied:*

(i) *All points in* $\mathrm{Prim}(A)$ *are locally closed (which is automatic if* A *is type I).*

(ii) *All stabilisers* G_P *for* $P \in \mathrm{Prim}(A)$ *are normal subgroups of* G *(which is automatic if* G *is abelian).*

Then $\mathrm{ind}_{G_P}^G Q \in \mathrm{Prim}(A \rtimes G)$ *for all* $P \in \mathrm{Prim}(A)$ *and* $Q \in \mathrm{Prim}(A \rtimes G_P)$ *such that* $\mathrm{res}_{\{e\}}^{G_P} Q = P$. *If, in addition,* G *is amenable, then all primitive ideals of* $A \rtimes G$ *are induced in this way.*

Proof. Let us first assume condition (i). Choose $\rho \times V \in (A \rtimes G_P)\widehat{\ }$ such that $\ker(\rho \times V) = Q$ and $\ker \rho = P$. Then we may regard ρ as a representation of A_P, the simple subquotient of A corresponding to the locally closed subset $\{P\}$ of $\mathrm{Prim}(A)$. Since A_P is simple, all nontrivial subrepresentations of ρ have kernel $\{0\}$ in A_P (and hence they have kernel P in A). Hence ρ is homogeneous and the result follows from Theorems 2.7.31 and 2.7.32.

Let us now assume (ii). If $N := G_P$ is normal, we may use the theory of twisted actions, which we shall present in §2.8 below, to pass to the system $((A \rtimes N) \otimes \mathcal{K}, G/N, \beta)$. If $\rho \times V \in (A \rtimes N)^\frown$ with $\ker(\rho \times V) = P$, then the corresponding representation of $(A \rtimes N) \otimes \mathcal{K}$ has trivial stabiliser in G/N, and therefore the induced representation has primitive kernel in $A \rtimes G \sim_M \big((A \rtimes N) \otimes \mathcal{K}\big) \rtimes G/N$ by Theorem 2.7.31. $\square$

Recall that if M is a topological G-space, then two elements $m_1, m_2 \in M$ are said to be in the same *quasi-orbit* if $m_1 \in \overline{G(m_2)}$ and $m_2 \in \overline{G(m_1)}$. Being in the same quasi-orbit is clearly an equivalence relation on M and we denote by $G_q(m)$ the quasi-orbit (i.e., the equivalence class) of m and by $\mathcal{Q}_G(M)$ the set of all quasi-orbits in M equipped with the quotient topology. Note that $\mathcal{Q}_G(M)$ is always a T_0-space. If $G\backslash M$ is a T_0-space, then $\mathcal{Q}_G(M)$ coincides with $G\backslash M$.

If (A, G, α) is a system, it follows from the definition of the Jacobson topology that two elements $P, Q \in \mathrm{Prim}(A)$ are in the same quasi-orbit if and only if $P^G = Q^G$. If the action of G on A is smooth, then all points in $G\backslash \mathrm{Prim}(A)$ are locally closed, which implies in particular that $G\backslash \mathrm{Prim}(A)$ is a T_0-space. Hence in this case we have $\mathcal{Q}_G(\mathrm{Prim}(A)) = G\backslash \mathrm{Prim}(A)$. In what follows, we let

$$\mathrm{Prim}^G(A) := \{P^G : P \in \mathrm{Prim}(A)\} \subseteq \mathcal{I}(A)$$

equipped with the relative Fell topology. Then [Gre78, Lemma on p. 221] gives

Lemma 2.7.34. *Let (A, G, α) be a system. Then the map*

$$q : \mathrm{Prim}(A) \to \mathrm{Prim}^G(A) : P \mapsto P^G$$

is a continuous and open surjection and therefore factors through a homeomorphism between $\mathcal{Q}_G(\mathrm{Prim}(A))$ and $\mathrm{Prim}^G(A)$.

As a consequence of the previous results we get

Corollary 2.7.35. *Suppose that (A, G, α) is smooth or that (A, G, α) is separable and G is amenable. Suppose further that the action of G on $\mathrm{Prim}(A)$ is free (i.e., all stabilisers are trivial). Then the map*

$$\mathrm{Ind} : \mathrm{Prim}^G(A) \cong \mathcal{Q}_G(\mathrm{Prim}(A)) \to \mathrm{Prim}(A \rtimes G); P \mapsto \mathrm{ind}_{\{e\}}^G P^G$$

is a homeomorphism. In particular, $A \rtimes G$ is simple if and only if every G-orbit is dense in $\mathrm{Prim}(A)$, and $A \rtimes G$ is primitive (i.e., $\{0\}$ is a primitive ideal of $A \rtimes G$) if and only if there exists a dense G-orbit in $\mathrm{Prim}(A)$.

Proof. It follows from Theorem 2.7.29 and Theorem 2.7.32 that the map $\mathrm{ind}_{\{e\}}^G : \mathrm{Prim}(A) \to \mathrm{Prim}(A \rtimes G); P \mapsto \mathrm{ind}_{\{e\}}^G P$ is well defined and surjective. By Corollary 2.7.13 we know that $\mathrm{ind}_{\{e\}}^G P = \mathrm{ind}_{\{e\}}^G P^G$, so the induction map $\mathrm{Ind} : \mathrm{Prim}^G(A) \to \mathrm{Prim}(A \rtimes G)$ is also well defined and surjective. Equation (2.7.4) applied to $H = \{e\}$ gives $\mathrm{res}^G_{\{e\}}(\mathrm{ind}^G_{\{e\}} P) = P^G$, which shows that $\mathrm{res}^G_{\{e\}} : \mathrm{Prim}(A \rtimes G) \to \mathrm{Prim}^G(A)$ is the inverse of Ind. Since induction and restriction are continuous by Proposition 2.7.4 the result follows. □

A quite recent result of Sierakowski (see [Sie10, Proposition 1.3 and Theorem 1.20] and [EL13, Corollary 2.7]) shows that for countable discrete groups G, the assumptions for the action of G on $\mathrm{Prim}(A)$ and of amenability of G can be weakened considerably. We need

Definition 2.7.36. An action of a group G on a topological space X is called *essentially free* if every G-invariant closed subset $C \subseteq X$ contains a dense G-invariant subset D such that G acts freely on D.

Theorem 2.7.37 (Sierakowski)**.** *Suppose that (A, G, α) is a C^*-dynamical system with A separable and G countable (hence discrete) and exact. Suppose further that the action of G on $\widehat{A}$ is essentially free (which is true if the action of G on $\mathrm{Prim}(A)$ is essentially free). Then the map*

$$\mathrm{Ind} : \mathrm{Prim}^G(A) \cong \mathcal{Q}_G(\mathrm{Prim}(A)) \to \mathrm{Prim}(A \rtimes_r G); P \mapsto \mathrm{Ind}_e^G P$$

is a well-defined homeomorphism.

Since the induced ideals in the above theorem clearly contain the kernel of the regular representation of $A \rtimes G$, it is clear that a similar statement cannot hold for the full crossed product $A \rtimes G$ if it differs from $A \rtimes_r G$.

We should note that Sierakowski's original results [Sie10, Proposition 1.3 and Theorem 1.20] show, that under the assumptions of the theorem the map $\mathrm{res} : \mathcal{I}(A \rtimes_r G) \to \mathcal{I}^G(A); J \mapsto J \cap A$ is a bijection between the set of closed two-sided ideals in $A \rtimes_r G$ and the set of G-invariant closed two-sided ideals in A, with the inverse given by $I \mapsto I \rtimes_r G$. The straightforward translation of this into the statement of the above theorem has been given in [EL13]. We should also mention that Sierakowski's result still holds under some slightly weaker assumptions, which he calls the *residual Rokhlin* property*. We refer the interested reader to [Sie10] for more details on this property.

Remark 2.7.38. If (A, G, α) is a system with constant stabiliser N for the action of G on $\mathrm{Prim}(A)$, then N is normal in G and one can pass to the iterated twisted system $(A \rtimes N, G, N, \alpha^N, \tau^N)$ (see §2.8 below), and then to an equivariantly Morita equivalent system $(B, G/N, \beta)$ (see Proposition 2.8.8) to see that induction of primitive ideals gives a homeomorphism between $\mathcal{Q}_{G/N}(\mathrm{Prim}(A \rtimes N))$ and $\mathrm{Prim}(A \rtimes G)$ if one of the following conditions are satisfied:

(i) (A, G, α) is smooth.

(ii) $(A \rtimes N, G, N, \alpha^N, \tau^N)$ is smooth (i.e., the action of G/N on $\mathrm{Prim}(A \rtimes N)$ via α^N satisfies the conditions of Definition 2.7.25).

(iii) (A, G, α) is separable and G/N is amenable.

A similar result can be obtained for systems with continuously varying stabilisers (see [Ech92]). In the case of constant stabilisers, the problem of describing the topology of $\mathrm{Prim}(A \rtimes G)$ now reduces to the description the topology of $\mathrm{Prim}(A \rtimes N)$ and the action of G/N on $\mathrm{Prim}(A \rtimes N)$. In general, both parts can be quite difficult to perform, but in some interesting special cases, e.g., if A has continuous trace, some good progress has been made for the description of $\mathrm{Prim}(A \rtimes N)$ (e.g. see [EW01, EW98, EN01] and the references given there). Of course, if $A = C_0(X)$ is abelian, and N is the constant stabiliser of the elements of $\mathrm{Prim}(A) = X$, then N acts trivially on X and $\mathrm{Prim}(C_0(X) \rtimes N) = \mathrm{Prim}(C_0(X) \otimes C^*(N)) = X \times \mathrm{Prim}(C^*(N))$.

Example 2.7.39. As an easy application of Corollary 2.7.35 we get the simplicity of the irrational rotation algebra A_θ, for θ an irrational number in $(0,1)$. Recall that $A_\theta = C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ where $n \in \mathbb{Z}$ acts on $z \in \mathbb{T}$ via $n \cdot z := e^{2\pi i \theta n} z$. Since θ is irrational, the action of $\mathbb{Z}$ on $\mathrm{Prim}(C(\mathbb{T})) = \mathbb{T}$ is free and all $\mathbb{Z}$-orbits are dense in $\mathbb{T}$. Hence, there exists only one quasi-orbit in $\mathbb{T}$ and the crossed product is simple. Of course, there are other more elementary proofs for the simplicity of A_θ which do not use such heavy machinery, but this example illustrates quite well how one can use the above results.

2.7.3 The Mackey machine for transformation groups

Suppose that X is a locally compact G-space and consider the corresponding action of G on $A = C_0(X)$ given by $(s \cdot \varphi)(x) = \varphi(s^{-1}x)$ for $s \in G$, $\varphi \in C_0(X)$. Then $\mathrm{Prim}(A) = X$ and $A_x \cong \mathbb{C}$ for all $x \in X$, so that $A_x \rtimes G_x \cong C^*(G_x)$ for all $x \in X$, where G_x denotes the stabiliser of x. Hence, if the action of G on X is smooth in the sense of Definition 2.7.25, then it follows from Theorem 2.7.29 that $C_0(X) \rtimes G$ is "fibred" over $G\backslash X$ with fibres $C_0(G(x)) \rtimes G \sim_M C^*(G_x)$ (compare the discussion following Theorem 2.7.29).

If $V \in \widehat{G_x}$ and if $\epsilon_x : C_0(X) \to \mathbb{C}$ denotes evaluation at x, then $\epsilon_x \times V$ is the representation of $C_0(X) \rtimes G_x$ which corresponds to V by regarding $\widehat{G_x} \cong (A_x \rtimes G_x)\widehat{\ }$ as a subset of $(A \rtimes G_x)\widehat{\ }$ as described in the discussion preceeding Theorem 2.7.29. In this situation, the result of Theorem 2.7.31 can be improved by showing:

Proposition 2.7.40 (cf. [Wil81, Proposition 4.2])**.** *Let $\epsilon_x \times V \in (C_0(X) \rtimes G_x)\widehat{\ }$ be as above. Then $\mathrm{ind}_{G_x}^G(\epsilon_x \rtimes V)$ is irreducible. Moreover, if $V, W \in \widehat{G_x}$, then*

$$\mathrm{ind}_{G_x}^G(\epsilon_x \times V) \cong \mathrm{ind}_{G_x}^G(\epsilon_x \times W) \quad \Longleftrightarrow \quad V \cong W.$$

Combining this with Theorem 2.7.29 and Theorem 2.7.32 gives:

Theorem 2.7.41. *Suppose that X is a locally compact G-space.*

(i) *If G acts smoothly on X, and if $\mathcal{S} \subseteq X$ is a section for $G\backslash X$, then we get a bijection*

$$\mathrm{Ind} : \cup_{x \in \mathcal{S}} \widehat{G_x} \to \big(C_0(X) \rtimes G\big)\widehat{\ }; V \mapsto \mathrm{ind}_{G_x}^G(\epsilon_x \times V).$$

(ii) *If X and G are second countable and if G is amenable, then every primitive ideal of $C_0(X) \rtimes G$ is the kernel of some induced irreducible representation* $\mathrm{ind}_{G_x}^G(\epsilon_x \times V)$.

If G is abelian, then so are the stabilisers G_x for all $x \in X$. Then $\widehat{G}_x$ is the Pontrjagin dual group of G_x and we get a short exact sequence

$$0 \to \widehat{G/G_x} \to \widehat{G} \stackrel{\mathrm{res}}{\to} \widehat{G}_x \to 0$$

for all $x \in X$. Moreover, since G_x is normal in G, it follows that the stabilisers are constant on quasi-orbits $G_q(x)$ in X. We can then consider an equivalence relation on $X \times \widehat{G}$ by

$$(x,\chi) \sim (y,\mu) \Leftrightarrow G_q(x) = G_q(y) \text{ and } \chi|_{G_x} = \mu|_{G_y}.$$

Then the following result is [Wil81, Theorem 5.3]:

Theorem 2.7.42 (Williams)**.** *Suppose that G is abelian and the action of G on X is smooth or G and X are second countable. Then the map*

$$\mathrm{Ind} : (X \times \widehat{G})/\sim \to \mathrm{Prim}(C_0(X) \rtimes G); [(x,\chi)] \mapsto \ker(\mathrm{Ind}_{G_x}^G(\epsilon_x \rtimes \chi|_{G_x}))$$

is a homeomorphism.

Note that in the case where the action of G on X is smooth and G is abelian, the crossed product $C_0(X) \rtimes G$ is type I, since $C^*(G_x)$ is type I. Hence, in this case we get a homeomorphism between $(X \times \widehat{G})/\sim$ and $(C_0(X) \rtimes G)\widehat{\ }$. We now want to present some applications to group representation theory:

Example 2.7.43. Suppose that $G = N \rtimes H$ is the semi-direct product of the abelian group N by the group H. Then, as seen in Example 2.3.6, we have

$$C^*(N \rtimes H) \cong C^*(N) \rtimes H \cong C_0(\widehat{N}) \rtimes H,$$

where the last isomorphism is given via the Gelfand-transform $C^*(N) \cong C_0(\widehat{N})$. The corresponding action of H on $C_0(\widehat{N})$ is induced by the action of H on $\widehat{N}$ given by $\big(h\cdot\chi\big)(n) := \chi(h^{-1}\cdot n)$ if $h \in H, \chi \in \widehat{N}$ and $n \in N$. Thus, if the action of H on $\widehat{N}$ is smooth, we obtain every irreducible representation of $C^*(N \rtimes H) \cong C_0(\widehat{N}) \rtimes H$ as an induced representation $\mathrm{ind}_{H_\chi}^H(\epsilon_\chi \times V)$ for some $\chi \in \widehat{N}$ and $V \in \widehat{H}_\chi$. The isomorphism $C_0(\widehat{N}) \rtimes H_\chi \cong C^*(N \rtimes H_\chi)$, transforms the representation $\epsilon_\chi \times V$ to the representation $\chi \times V$ of $N \rtimes H_\chi$ defined by $\chi \times V(n,h) = \chi(n)V(h)$, and one can show that $\mathrm{ind}_{N \rtimes H_\chi}^{N \rtimes H}(\chi \times V)$ corresponds to the representation $\mathrm{ind}_{H_\chi}^H(\epsilon_\chi \times V)$ under the isomorphism $C^*(N \rtimes H) \cong C_0(\widehat{N}) \rtimes H$. Thus, choosing a cross-section $\mathcal{S} \subseteq \widehat{N}$ for $H\backslash\widehat{N}$, it follows from Theorem 2.7.29 that

$$\mathrm{Ind} : \cup\{\widehat{H}_\chi : \chi \in \mathcal{S}\} \to \widehat{N \rtimes H}; V \mapsto \mathrm{ind}_{N \rtimes H_\chi}^{N \rtimes H}(\chi \times V)$$

is a bijection.

If the action of H on $\widehat{N}$ is not smooth, but $N \rtimes H$ is second countable and amenable, then we get at least all primitive ideals of $C^*(N \rtimes H)$ as kernels of the induced representations $\mathrm{ind}_{N \rtimes H_\chi}^{N \rtimes H}(\chi \times V)$.

Let us now discuss some explicit examples:

(1) Let $G = \mathbb{R} \rtimes \mathbb{R}^*$ denote the $ax+b$-group, i.e., G is the semi-direct product for the action of the multiplicative group $\mathbb{R}^*$ on $\mathbb{R}$ via dilation. Identifying $\mathbb{R}$ with $\widehat{\mathbb{R}}$ via $t \mapsto \chi_t$ with $\chi_t(s) = e^{2\pi i t s}$, we see easily that the action of $\mathbb{R}^*$ on $\widehat{\mathbb{R}}$ is also given by dilation. Hence there are precisely two orbits in $\widehat{\mathbb{R}}$: $\{\chi_0\}$ and $\widehat{\mathbb{R}} \smallsetminus \{\chi_0\}$. Let $\mathcal{S} = \{\chi_0, \chi_1\} \subseteq \widehat{\mathbb{R}}$. Then $\mathcal{S}$ is a cross-section for $\mathbb{R}^* \backslash \widehat{\mathbb{R}}$, the stabiliser of χ_1 in $\mathbb{R}^*$ is $\{1\}$ and the stabiliser of χ_0 is all of $\mathbb{R}^*$. Thus, we see that

$$\widehat{G} = \{\chi_0 \times \mu : \mu \in \widehat{\mathbb{R}^*}\} \cup \{\mathrm{ind}_{\mathbb{R}}^{\mathbb{R} \rtimes \mathbb{R}^*} \chi_1\}.$$

It follows from Theorem 2.7.42 that the single representation $\pi := \mathrm{ind}_{\mathbb{R}}^{\mathbb{R} \rtimes \mathbb{R}^*} \chi_1$ is dense in $\widehat{G}$ and that the set $\{\chi_0 \times \mu : \mu \in \widehat{\mathbb{R}^*}\} \subseteq \widehat{G}$ is homeomorphic to $\widehat{\mathbb{R}^*} \cong \mathbb{R}^*$.

Note that we could also consider the C^*-algebra $C^*(G)$ as "fibred" over $\mathbb{R}^* \backslash \widehat{\mathbb{R}}$: The open orbit $\widehat{\mathbb{R}} \smallsetminus \{\chi_0\} \cong \mathbb{R}^*$ corresponds to the ideal $C_0(\mathbb{R}^*) \rtimes \mathbb{R}^* \cong \mathcal{K}(L^2(\mathbb{R}^*))$ and the closed orbit $\{\chi_0\}$ corresponds to the quotient $C_0(\widehat{\mathbb{R}^*})$ of $C^*(G)$, so that this picture yields the short exact sequence

$$0 \to \mathcal{K}(L^2(\mathbb{R}^*)) \to C^*(G) \to C_0(\widehat{\mathbb{R}^*}) \to 0$$

(compare also with Example 2.6.6).

(2) A more complicated example is given by the Mautner group. This group is the semi-direct product $G = \mathbb{C}^2 \rtimes \mathbb{R}$ with action given by

$$t \cdot (z, w) = (e^{-2\pi i t} z, e^{-2\pi i \theta t} w),$$

where $\theta \in (0,1)$ is a fixed irrational number. Identifying $\mathbb{C}^2$ with the dual group $\widehat{\mathbb{C}^2}$ via $(u,v) \mapsto \chi_{(u,v)}$ such that

$$\chi_{(u,v)}(z,w) = \exp(2\pi i \mathrm{Re}(z\bar{u} + w\bar{v})),$$

we get $t \cdot \chi_{(u,v)} = \chi_{(e^{2\pi i t}u, e^{2\pi i \theta t}z)}$. The quasi-orbit space for the action of $\mathbb{R}$ on $\widehat{\mathbb{C}^2}$ can then be parametrized by the set $[0,\infty) \times [0,\infty)$: If $(r,s) \in [0,\infty)^2$, then the corresponding quasi-orbit $\mathcal{O}_{(r,s)}$ consists of all $(u,v) \in \mathbb{C}^2$ such that $|u| = r$ and $|v| = s$. Hence, if $r, s > 0$, then $\mathcal{O}_{(r,s)}$ is homeomorphic to $\mathbb{T}^2$ and this homeomorphism carries the action of $\mathbb{R}$ on $\mathcal{O}_{(r,s)}$ to the irrational flow of $\mathbb{R}$ on $\mathbb{T}^2$ corresponding to θ as considered in part (4) of Example 2.6.6. In particular, $\mathbb{R}$ acts freely but not smoothly on those quasi-orbits. If $r \neq 0$ and $s = 0$, the quasi-orbit

$\mathcal{O}_{(r,s)}$ is homeomorphic to $\mathbb{T}$ with action $t \cdot u := e^{2\pi i t} u$ and constant stabiliser $\mathbb{Z}$. In particular, all those quasi-orbits are orbits. Similarly, if $r = 0$ and $s \neq 0$, the quasi-orbit $\mathcal{O}_{(r,s)}$ is homeomorphic to $\mathbb{T}$ with action $t \cdot v = e^{2\pi i \theta t} v$ and stabiliser $\frac{1}{\theta}\mathbb{Z}$. Finally, the quasi-orbit corresponding to $(0,0)$ is the point-set $\{(0,0)\}$ with stabiliser $\mathbb{R}$.

Since G is second countable and amenable, we can therefore parametrize $\mathrm{Prim}(C^*(G))$ by the set

$$\{(r,s) : r,s > 0\} \cup \big((0,\infty) \times \widehat{\mathbb{Z}}\big) \cup \big((0,\infty) \times \widehat{\tfrac{1}{\theta}\mathbb{Z}}\big) \cup \widehat{\mathbb{R}}.$$

In fact, we can also view $C^*(G)$ as “fibred” over $[0,\infty)^2$ with fibers

$$C^*(G)_{(r,s)} \cong C(\mathbb{T}^2) \rtimes_\theta \mathbb{R} \sim_M A_\theta \qquad \text{for } r,s > 0,$$

where A_θ denotes the irrational rotation algebra,

$$C^*(G)_{(r,0)} \cong C(\mathbb{T}) \rtimes \mathbb{R} \sim_M C(\widehat{\mathbb{Z}}) \cong C(\mathbb{T}) \qquad \text{for } r > 0,$$

$$C^*(G)_{(0,s)} \cong C(\mathbb{T}) \rtimes_\theta \mathbb{R} \sim_M C(\widehat{\tfrac{1}{\theta}\mathbb{Z}}) \cong C(\mathbb{T}) \qquad \text{for } s > 0,$$

and $C^*(G)_{(0,0)} \cong C_0(\mathbb{R})$. Using Theorem 2.7.42, it is also possible to describe the topology of $\mathrm{Prim}(G)$, but we do not go into the details here. We should mention that the Mautner group is the lowest-dimensional example of a connected Lie-group G with a non–type I group algebra $C^*(G)$.

Remark 2.7.44. It follows from Theorems 2.7.29 and 2.7.32 that for understanding the ideal structure of $A \rtimes G$, it is necessary to understand the structure of $A_P \rtimes G_P$ for $P \in \mathrm{Prim}(A)$. We saw in this section that this is the same as understanding the group algebras $C^*(G_x)$ for the stabilisers G_x if $A = C_0(X)$ is abelian. In general, the problem becomes much more difficult. However, at least in the important special case where A is type I, one can still give a quite satisfactory description of $A_P \rtimes G_P$ in terms of the stabilisers. Since an elegant treatment of that case uses the theory of twisted actions and crossed products, we postpone the discussion of this case to §2.8.6 below.

2.8 The Mackey–Rieffel–Green machine for twisted crossed products

2.8.1 Twisted actions and twisted crossed products

One drawback of the theory of crossed products by ordinary actions is the fact that crossed products $A \rtimes G$ (and their reduced analogues) cannot be written as iterated crossed products $(A \rtimes N) \rtimes G/N$ if N is a normal subgroup such that the extension

$$1 \to N \to G \to G/N \to 0$$

is not topologically split (compare with Example 2.3.6). In order to close this gap, we now introduce twisted actions and twisted crossed products following Phil Green's approach of [Gre78]. Note that there is an alternative approach due to Leptin and Busby-Smith (see [Lep65, BS70, PR89] for the construction of twisted crossed products within this theory), but Green's theory seems to be better suited for our purposes.

As a motivation, consider a closed normal subgroup N of the locally compact group G, and assume that $\alpha : G \to \mathrm{Aut}(A)$ is an action. Let $A \rtimes N$ be the crossed product of A by N. Let $\delta : G \to \mathbb{R}^+$ be the module for the conjugation action of G on N, i.e., $\delta(s)\int_N f(s^{-1}ns)\,dn = \int_N f(n)\,dn$ for all $f \in C_c(N)$. A short computation using the formula

$$\int_G g(s)\,ds = \int_{G/N}\left(\int_N g(sn)\,dn\right)\,dsN \tag{2.8.1}$$

(with respect to suitable choices of Haar measures) shows that $\delta(s) = \Delta_G(s)\Delta_{G/N}(s^{-1})$ for all $s \in G$. Similar to Example 2.3.6 we define an action $\alpha^N : G \to \mathrm{Aut}(A \rtimes N)$ by

$$\big(\alpha_s^N(f)\big)(n) = \delta(s)\alpha_s\big(f(s^{-1}ns)\big) \tag{2.8.2}$$

for f in the dense subalgebra $C_c(N, A) \subseteq A \rtimes N$. If we denote by $\tau^N := i_N : N \to UM(A \rtimes N)$ the canonical embedding as defined in part (1) of Remark 2.3.4, then the pair (α^N, τ^N) satisfies the equations

$$\tau_n^N x \tau_{n^{-1}}^N = \alpha_n^N(x) \quad \text{and} \quad \alpha_s^N(\tau_n^N) = \tau_{sns^{-1}}^N \tag{2.8.3}$$

for all $x \in A \rtimes N$, $n \in N$ and $s \in G$, where in the second formula we extended the automorphism α_s^N of $A \rtimes N$ to $M(A \rtimes N)$. Suppose now that (π, U) is a covariant homomorphism of (A, G, α) into some $M(D)$. Let $(\pi, U|_N)$ denote its restriction to (A, N, α) and let $\pi \times U|_N : A \rtimes N \to M(D)$ be its integrated form. Then $(\pi \times U|_N, U)$ is a nondegenerate covariant homomorphism of $(A \rtimes N, G, \alpha^N)$ that satisfies

$$\pi \times U|_N(\tau_n^N) = U_n$$

for all $n \in N$ (see Remark 2.3.4). The pair (α^N, τ^N) is the prototype for a twisted action (which we denote the *decomposition twisted action*) and $(\pi \times U|_N, U)$ is the prototype of a twisted covariant homomorphism as in

Definition 2.8.1 (Green)**.** Let N be a closed normal subgroup of G. A *twisted action* of (G, N) on a C^*-algebra A is a pair (α, τ) such that $\alpha : G \to \mathrm{Aut}(A)$ is an action and $\tau : N \to UM(A)$ is a strictly continuous homomorphism such that

$$\tau_n a \tau_{n^{-1}} = \alpha_n(a) \quad \text{and} \quad \alpha_s(\tau_n) = \tau_{sns^{-1}}$$

for all $a \in A$, $n \in N$ and $s \in G$. We then say that (A, G, N, α, τ) is a twisted system. A *(twisted) covariant homomorphism* of (A, G, N, α, τ) into some $M(D)$

is a covariant homomorphism (ρ, V) of (A, G, α) into $M(D)$ which *preserves* τ in the sense that $\rho(\tau_n a) = V_n \rho(a)$ for all $n \in N, a \in A$.[13]

Remark 2.8.2. Note that the kernel of the regular representation $\Lambda_A^N : A \rtimes N \to A \rtimes_r N$ is easily seen to be invariant under the decomposition twisted action (α^N, τ^N) (which just means that it is invariant under α^N), so that (α^N, τ^N) induces a twisted action on the quotient $A \rtimes_r N$. In what follows, we shall make no notational difference between the decomposition twisted actions on the full or the reduced crossed products.

Let $C_c(G, A, \tau)$ denote the set of all continuous A-valued functions on G with compact support mod N and that satisfy

$$f(ns) = f(s)\tau_{n^{-1}} \qquad \text{for all } n \in N,\ s \in G.$$

Then $C_c(G, A, \tau)$ becomes a $*$-algebra with convolution and involution defined by

$$f * g(s) = \int_{G/N} f(t)\alpha_t(g(t^{-1}s))\, dtN \quad \text{and} \quad f^*(s) = \Delta_{G/N}(s^{-1})\alpha_s\big(f(s^{-1})^*\big).$$

If (ρ, V) is a covariant representation of (A, G, N, α, τ), then the equation

$$\rho \times V(f) = \int_{G/N} \rho(f(s))V_s\, dsN$$

defines a $*$-homomorphism $\rho \times V : C_c(G, A, \tau) \to M(D)$, and the *full twisted crossed product* $A \rtimes_{\alpha,\tau} (G, N)$ (or just $A \rtimes (G, N)$ if (α, τ) is understood) is defined as the completion of $C_c(G, A, \tau)$ with respect to

$$\|f\|_{\max} := \sup\{\|\rho\times V(f)\| : (\rho, V) \text{ is a covariant homomorphism of } (A, G, N, \alpha, \tau)\}.$$

Note that the same formulas as given in Remark 2.3.4 define a twisted covariant homomorphism (i_A, i_G) of (A, G, N, α, τ) into $M(A \rtimes (G, N))$ such that any nondegenerate homomorphism $\Phi : A \rtimes (G, N) \to M(D)$ is the integrated form $\rho \times V$ with $\rho = \Phi \circ i_A$ and $V = \Phi \circ i_G$.

Remark 2.8.3. It is important to note that for any twisted action (α, τ) of (G, N) the map

$$\Phi : C_c(G, A) \to C_c(G, A, \tau); \Phi(f)(s) = \int_N f(sn)\tau_{sns^{-1}}\, dn$$

extends to a quotient map $\Phi : A \rtimes G \to A \rtimes (G, N)$ of the full crossed products, such that $\ker \Phi = \cap\{\ker(\pi \times U) : (\pi, U) \text{ preserves } \tau\}$. The ideal $I_\tau := \ker \Phi$ is called the *twisting ideal* of $A \rtimes G$. Note that if $G = N$, then $A \rtimes (N, N) \cong A$ via $f \mapsto f(e); C_c(N, A, \tau) \to A$.

[13] The latter condition becomes $\rho(\tau_n) = V_n$ if (ρ, V) is nondegenerate.

For the definition of the *reduced twisted crossed products* $A \rtimes_{\alpha,\tau,r} (G,N)$ (or just $A \rtimes_r (G,N)$) we define a Hilbert A-module $L^2(G,A,\tau)$ by taking the completion of $C_c(G,A,\tau)$ with respect to the A-valued inner product

$$\langle \xi, \eta \rangle_A := \xi^* * \eta(e) = \int_{G/N} \alpha_{s^{-1}}\big(\xi(s)^*\eta(s)\big)\, dsN.$$

The *regular representation*

$$\Lambda_A^{G,N} : C_c(G,A,\tau) \to \mathcal{L}_A(L^2(G,A,\tau)); \;\; \Lambda_A^{G,N}(f)\xi = f * \xi$$

embeds $C_c(G,A,\tau)$ into the algebra of adjointable operators on $L^2(G,A,\tau)$ and then $A \rtimes_r (G,N) := \overline{\Lambda_A^{G,N}\big(C_c(G,A,\tau)\big)} \subseteq \mathcal{L}_A(L^2(G,A,\tau))$. If $N = \{e\}$ is trivial, then $\mathcal{L}_A(L^2(G,A))$ identifies naturally with $M(A \otimes \mathcal{K}(L^2(G)))$, and we recover the original definition of the regular representation Λ_A^G of (A,G,α) and of the reduced crossed product $A \rtimes_r G$ of A by G.

Remark 2.8.4. The analogue of Remark 2.8.3 **does not hold** in general for the reduced crossed products, i.e., $A \rtimes_r (G,N)$ is in general not a quotient of $A \rtimes_r G$. For example, if N is not amenable, the algebra $C_r^*(G/N) = \mathbb{C} \rtimes_{\mathrm{id},1,r} (G,N)$ is not a quotient of $C_r^*(G) = \mathbb{C} \rtimes_{\mathrm{id},r} G$ – at least not in a canonical way.

We are now coming back to the decomposition problem

Proposition 2.8.5 (Green)**.** *Let $\alpha : G \to \mathrm{Aut}(A)$ be an action, let N be a closed normal subgroup of G, and let (α^N, τ^N) be the decomposition twisted action of (G,N) on $A \rtimes N$. Then the map*

$$\Psi : C_c(G,A) \to C_c(G, C_c(N,A), \tau^N); \quad \Psi(f)(s)(n) = \delta(s) f(ns) \qquad (2.8.4)$$

extends to isomorphisms $A \rtimes G \cong (A \rtimes N) \rtimes (G,N)$ and $A \rtimes_r G \cong (A \rtimes_r N) \rtimes_r (G,N)$. In particular, if $A = \mathbb{C}$ we obtain isomorphisms $C^*(G) \cong C^*(N) \rtimes (G,N)$ and $C_r^*(G) \cong C_r^*(N) \rtimes_r (G,N)$. Under the isomorphism of the full crossed products, a representation $\pi \times U$ of $A \rtimes G$ corresponds to the representation $(\pi \times U|_N) \times U$ of $(A \rtimes N) \rtimes (G,N)$.

A similar result holds if we start with a twisted action of (G,M) on A with $M \subseteq N$ (see [Gre78, Proposition 1] and [CE01b]). We should note that Green only considered full crossed products in [Gre78]. The above decomposition of reduced crossed products was first shown by Kirchberg and Wassermann in [KW00]. Note that all results stated in §2.3 for ordinary crossed products have their complete analogues in the twisted case, where G/N plays the rôle of G. In particular, the full and reduced crossed products coincide if G/N is amenable. Indeed, we shall see in §2.8.2 below that there is a convenient way to extend results known for ordinary actions to the twisted case via Morita equivalence (see Theorem 2.8.9 below).

2.8.2 The twisted equivariant correspondence category and the stabilisation trick

As done for ordinary actions in §2.5 we may consider the *twisted equivariant correspondence category* $\mathfrak{Corr}(G,N)$ (resp. the *compact twisted equivariant correspondence category* $\mathfrak{Corr}_c(G,N)$) in which the objects are twisted systems (A,G,N,α,τ) and in which the morphism from (A,G,N,α,τ) and (B,G,N,β,σ) are given by morphisms $[E,\Phi,u]$ from (A,G,α) to (B,G,β) in $\mathfrak{Corr}(G)$ (resp. $\mathfrak{Corr}_c(G)$) which preserve the twists in the sense that

$$\Phi(\tau_n)\xi = u_n(\xi)\sigma_n \quad \text{for all } n \in N. \tag{2.8.5}$$

As for ordinary actions, the crossed product construction $(A,G,N,\alpha,\tau) \mapsto A \rtimes_{(r)} (G,N)$ extend to (full and reduced) descent functors

$$\rtimes_{(r)} : \mathfrak{Corr}(G,N) \to \mathfrak{Corr}.$$

If $[E,\Phi,u] \in \mathrm{Mor}(G,N)$ is a morphism from (A,G,N,α,τ) to (B,G,N,β,σ), then the descent $[E \rtimes_{(r)} (G,N), \Phi \rtimes_{(r)} (G,N)]$ can be defined by setting $E \rtimes_{(r)} (G,N) := (E \rtimes G)/\big((E \rtimes G) \cdot I_{(r)}\big)$ with $I_{(r)} := \ker\big(B \rtimes G \to B \rtimes_{(r)} (G,N)\big)$. Alternatively, one can construct $E \rtimes_{(r)} G$ as the closure of $C_c(G,E,\sigma)$, the continuous E-valued functions ξ on G with compact support modulo N satisfying $\xi(ns) = \xi(s)\sigma_{n^{-1}}$ for $s \in G$, $n \in N$, with respect to the $B \rtimes_{(r)} (G,N)$-valued inner product given by

$$\langle \xi, \eta \rangle_{B \rtimes_{(r)} (G,N)}(t) = \int_{G/N} \beta_{s^{-1}}(\langle \xi(s), \eta(ts) \rangle_B)\, dsN$$

(compare with the formulas given in §2.5.4).

There is a natural inclusion functor $\inf : \mathfrak{Corr}(G/N) \to \mathfrak{Corr}(G,N)$ given as follows: If $(A,G/N,\alpha)$ is an action of G/N, we let $\inf\alpha : G \to \mathrm{Aut}(A)$ denote the inflation of α from G/N to G and we let $1_N : N \to U(A)$ denote the trivial homomorphism $1_N(s) = 1$. Then $(\inf\alpha, 1_N)$ is a twisted action of (G,N) on A and we set

$$\inf\big((A,G/N,\alpha)\big) := (A,G,N,\inf\alpha,1_N).$$

Similarly, on morphisms we set $\inf\big([E,\Phi,u]\big) := [E,\Phi,\inf u]$, where $\inf u$ denotes the inflation of u from G/N to G. The dense subalgebra $C_c(G,A,1_N)$ of the crossed product $A \rtimes_{(r)} (G,N)$ for $(\inf\alpha, 1_N)$ consists of functions that are constant on N-cosets and that have compact supports in G/N, hence it coincides with $C_c(G/N,A)$ (even as a $*$-algebra). The identification $C_c(G,A,1_N) \cong C_c(G/N,A)$ extends to the crossed products, and we obtain canonical isomorphisms $A \rtimes_{(r)} G/N \cong A \rtimes_{(r)} (G,N)$. A similar observation can be made for the crossed products of morphism and we see that the inclusion $\inf : \mathfrak{Corr}(G/N) \to \mathfrak{Corr}(G,N)$ is compatible with the crossed product functor in the sense that the diagram

$$\begin{tikzcd}
\mathfrak{Corr}(G/N) \arrow[r, "\inf"] \arrow[d, "\rtimes"'] & \mathfrak{Corr}(G,N) \arrow[d, "\rtimes"] \\
\mathfrak{Corr} \arrow[r, equal] & \mathfrak{Corr}
\end{tikzcd}$$

commutes.

In what follows next we want to see that every twisted action is Morita equivalent (and hence isomorphic in $\mathfrak{Corr}(G,N)$) to some inflated twisted action as above. This will allow us to pass to an untwisted system whenever a theory (such as the theory of induced representations, or K-theory of crossed products, etc.) only depends on the Morita equivalence class of a given twisted action.

To do this, we first note that Green's imprimitivity theorem (see Theorem 2.6.4) extends easily to crossed products by twisted actions: If N is a closed normal subgroup of G such that $N \subseteq H$ for some closed subgroup H of G, and if (α,τ) is a twisted action of (H,N) on A, then we obtain a twisted action $(\operatorname{Ind}\alpha, \operatorname{Ind}\tau)$ of (G,N) on $\operatorname{Ind}_H^G(A,\alpha)$ by defining

$$(\operatorname{Ind}\tau_n f)(s) = \tau_{s^{-1}ns} f(s) \quad \text{for } f \in \operatorname{Ind} A, s \in G \text{ and } n \in N.$$

One can check that the twisting ideals $I_\tau \subseteq A \rtimes H$ and $I_{\operatorname{Ind}\tau} \subseteq \operatorname{Ind} A \rtimes G$ (see Remark 2.8.3) are linked via the Rieffel correspondence of the $\operatorname{Ind} A \rtimes G - A \rtimes H$ imprimitivity bimodule $X_H^G(A)$. Similarly, the kernels $I_{\tau,r} := \ker\big(A \rtimes H \to A \rtimes_r (H,N)\big)$ and $I_{\operatorname{Ind}\tau,r} := \ker\big(\operatorname{Ind} A \rtimes G \to \operatorname{Ind} A \rtimes_r (G,N)\big)$ are linked via the Rieffel correspondence (we refer to [Gre78] and [KW00] for the details). Thus, from Proposition 2.5.4 it follows:

Theorem 2.8.6. *The quotient* $Y_H^G(A) := X_H^G(A)/(X_H^G(A) \cdot I_\tau)$ *(resp.* $Y_H^G(A)_r := X_H^G(A)/(X_H^G(A) \cdot I_{\tau,r})$*) becomes an* $\operatorname{Ind}_H^G(A,\alpha) \rtimes (G,N)$ *-* $A \rtimes (H,N)$ *(resp.* $\operatorname{Ind}_H^G(A,\alpha) \rtimes_r (G,N)$ *-* $A \rtimes_r (H,N)$*) imprimitivity bimodule.*

Remark 2.8.7. (1) Alternatively, one can construct the modules $Y_H^G(A)$ and $Y_H^G(A)_r$ by taking completions of $Y_0(A) := C_c(G,A,\tau)$ with respect to suitable $C_c(G,\operatorname{Ind} A,\operatorname{Ind}\tau)$- and $C_c(N,A,\tau)$-valued inner products. The formulas are precisely those of (2.6.1) if we integrate over G/N and H/N, respectively (compare with the formula for convolution in $C_c(G,A,\tau)$ as given in §2.8).

(2) If we start with a twisted action (α,τ) of (G,N) on A and restrict this to (H,N), then the induced algebra $\operatorname{Ind}_H^G(A,\alpha)$ is isomorphic to $C_0(G/H,A) \cong C_0(G/H) \otimes A$ as in Remark 2.6.1. The isomorphism transforms the action $\operatorname{Ind}\alpha$ to the action $l \otimes \alpha : G \to \operatorname{Aut}(C_0(G/H,A))$, with $l : G \to \operatorname{Aut}(C_0(G/H))$ being left-translation action, and the twist $\operatorname{Ind}\tau$ is transformed to the twist $1 \otimes \tau : N \to U(C_0(G/H) \otimes A)$. Hence, in this setting, the above theorem provides Morita equivalences

$$A \rtimes_{(r)} (H,N) \sim_M C_0(G/H,A) \rtimes_{(r)} (G,N)$$

for the above described twisted action $(l \otimes \alpha, 1 \otimes \tau)$ of (G, N).

We want to use Theorem 2.8.6 to construct a functor

$$\mathcal{F} : \mathfrak{Corr}(G, N) \to \mathfrak{Corr}(G/N)$$

which, up to a natural equivalence, inverts the inflation functor $\inf : \mathfrak{Corr}(G/N) \to \mathfrak{Corr}(G, N)$. We start with the special case of the decomposition twisted actions (α^N, τ^N) of (G, N) on $A \rtimes N$ with respect to a given system (A, G, α) and a normal subgroup N of G (see §2.8 for the construction). Since A is a G-algebra, it follows from Remark 2.6.1 that $\mathrm{Ind}_N^G(A, \alpha)$ is isomorphic to $C_0(G/N, A)$ as a G-algebra. Let $X_N^G(A)$ be Green's $C_0(G/N, A) \rtimes G - A \rtimes N$ imprimitivity bimodule. Since right translation of G/N on $C_0(G/N, A)$ commutes with $\mathrm{Ind}\,\alpha$, it induces an action

$$\beta^N : G/N \to \mathrm{Aut}\big(C_0(G/N, A) \rtimes G\big)$$

on the crossed product. For $s \in G$ and $\xi \in C_c(G, A) \subseteq X_N^G(A)$ let

$$u_s^N(\xi)(t) := \sqrt{\delta(s)}\alpha_s(\xi(ts)), \quad \xi \in C_c(G, A)$$

where $\delta(s) = \Delta_G(s)\Delta_{G/N}(s^{-1})$. This formula determines an action $u^N : G \to \mathrm{Aut}(X_N^G(A))$ such that $(X_N^G(A), u^N)$ becomes a (G, N)-equivariant $C_0(G/N, A) \rtimes G$-$A \rtimes N$ Morita equivalence with respect to the twisted actions $(\inf \beta^N, 1_N)$ and (α^N, τ^N), respectively. All these twisted actions pass to the quotients to give also a (G, N)-equivariant equivalence $(X_N^G(A)_r, u^N)$ for the reduced crossed products. Thus we get

Proposition 2.8.8 (cf. [Ech94, Theorem 1])**.** *The decomposition action (α^N, τ^N) of (G, N) on $A \rtimes_{(r)} N$ is canonically Morita equivalent to the (untwisted) action β^N of G/N on $C_0(G/N, A) \rtimes_{(r)} G$ as described above.*

If one starts with an arbitrary twisted action (α, τ) of (G, N) on A, one checks that the twisting ideals $I_\tau \subseteq A \rtimes N$ and $I_{\mathrm{Ind}\,\tau} \subseteq C_0(G/N, A) \rtimes G$ are (G, N)-invariant and that the twisted action on $A \cong (A \rtimes N)/I_\tau$ (cf. Remark 2.8.3) induced from (α^N, τ^N) is equal to (α, τ). Hence, if β denotes the action of G/N on $C_0(G/N, A) \rtimes (G, N) \cong \big(C_0(G/N, A) \rtimes G\big)/I_{\mathrm{Ind}\,\tau}$ induced from β^N, then u^N factors through an action u of G on $Y_N^G(A) = X_N^G(A)/(X_N^G(A) \cdot I_\tau)$ such that $(Y_N^G(A), u)$ becomes a (G, N)-equivariant $C_0(G/N, A) \rtimes (G, N)$-$A$ Morita equivalence with respect to the twisted actions $(\inf \beta, 1_N)$ and (α, τ), respectively. Following the arguments given in [EKQR00] one can show that there is a functor $\mathcal{F} : \mathfrak{Corr}(G, N) \to \mathfrak{Corr}(G/N)$ given on objects by the assignment

$$(A, G, N, \alpha, \tau) \overset{\mathcal{F}}{\mapsto} (C_0(G/N, A) \rtimes (G, N),\ G/N,\ \beta)$$

(and a similar crossed product construction on the morphisms) such that

Theorem 2.8.9 (cf. [Ech94, Theorem 1] and [EKQR00, Theorem 4.1])**.** *The assignment*

$$(A, G, N, \alpha, \tau) \mapsto (Y_N^G(A), u)$$

is a natural equivalence between the identity functor on $\mathfrak{Corr}(G, N)$ *and the functor* $\inf \circ \mathcal{F} : \mathfrak{Corr}(G, N) \to \mathfrak{Corr}(G, N)$*, where* $\inf : \mathfrak{Corr}(G/N) \to \mathfrak{Corr}(G, N)$ *denotes the inflation functor. In particular, every twisted action of* (G, N) *is Morita equivalent to an ordinary action of* G/N *(viewed as a twisted action via inflation).*

Note that a first version of the above theorem was obtained by Packer and Raeburn in the setting of Busby–Smith twisted actions ([PR89]). We therefore call it the *Packer–Raeburn stabilisation trick.* As mentioned before, it allows us to extend results known for ordinary actions to the twisted case as soon as they are invariant under Morita equivalence. If A is separable and G is second countable, the algebra $B = C_0(G/N, A) \rtimes (G, N)$ is separable, too. Thus, it follows from a theorem of Brown, Green, and Rieffel (see [BGR77]) that A and B are stably isomorphic (a direct isomorphism $B \cong A \otimes \mathcal{K}(L^2(G/N))$ is obtained in [Gre80] but see also Proposition 2.6.7). Hence, as a consequence of Theorem 2.8.9 we get

Corollary 2.8.10. *If* G *is second countable and* A *is separable, then every twisted action of* (G, N) *on* A *is Morita equivalent to some action* β *of* G/N *on* $A \otimes \mathcal{K}$*.*

We want to discuss some further consequences of Theorem 2.8.9:

2.8.3 Twisted Takesaki–Takai duality

If (A, G, N, α, τ) is a twisted system with G/N abelian, then we define the dual action

$$\widehat{(\alpha, \tau)} : \widehat{G/N} \to \operatorname{Aut}\big(A \rtimes (G, N)\big)$$

as in the previous section by pointwise multiplying characters of G/N with functions in the dense subalgebra $C_c(G, A, \tau)$. Similarly, we can define actions of $\widehat{G/N}$ on (twisted) crossed products of Hilbert bimodules, so that taking dual actions gives a descent functor $\rtimes : \mathfrak{Corr}(G, N) \to \mathfrak{Corr}(\widehat{G/N})$. The Takesaki–Takai duality theorem shows that on $\mathfrak{Corr}(G/N) \subseteq \mathfrak{Corr}(G, N)$ this functor is inverted, up to a natural equivalence, by the functor $\rtimes : \mathfrak{Corr}(\widehat{G/N}) \to \mathfrak{Corr}(G/N)$. Using Theorem 2.8.9, this directly extends to the twisted case.

2.8.4 Stability of exactness under group extensions

Recall from §2.6.3 that a group is called exact if for every short exact sequence $0 \to I \to A \to A/I \to 0$ of G-algebras the resulting sequence

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

of reduced crossed products is exact. We want to use Theorem 2.8.9 to give a proof of the following result of Kirchberg and Wassermann:

Theorem 2.8.11 (Kirchberg and Wassermann [KW00]). *Suppose that N is a closed normal subgroup of the locally compact group G such that N and G/N are exact. Then G is exact.*

The result will follow from:

Lemma 2.8.12. *Suppose that N is a closed normal subgroup of G and that (X, u) is a (G, N)-equivariant Morita equivalence for the twisted actions (β, σ) and (α, τ) of G on B and A, respectively. Let $I \subseteq A$ be a (G, N)-invariant ideal of A, and let $J := \operatorname{Ind}^X I \subseteq B$ denote the ideal of B induced from I via X (which is a (G, N)-equivariant ideal of B).*

Then $J \rtimes_{(r)} (G, N)$ (resp. $(B/J) \rtimes_{(r)} (G, N)$) corresponds to $I \rtimes_{(r)} (G, N)$ (resp. $(A/I) \rtimes_{(r)} (G, N)$) under the Rieffel correspondence for $X \rtimes_{(r)} (G, N)$.

Proof. Let $Y := X \cdot I \subseteq X$. Then the closure $C_c(G, Y, \tau) \subseteq C_c(G, X, \tau)$ is a $B \rtimes_{(r)} (G, N) - A \rtimes_{(r)} (G, N)$ submodule of $X \rtimes_{(r)} (G, N)$ which corresponds to the ideals $J \rtimes_{(r)} (G, N)$ and $I \rtimes_{(r)} (G, N)$ under the Rieffel correspondence. For the quotients observe that the obvious quotient map $C_c(G, X, \tau) \to C_c(G, X/X \cdot I, \tau)$ extends to an imprimitivity bimodule quotient map $X \rtimes_{(r)} (G, N) \to (X/X \cdot I) \rtimes_{(r)} (G, N)$, whose kernel corresponds to the ideals $K_B := \ker\big(B \rtimes_{(r)} (G, N) \to (B/J) \rtimes_{(r)} (G, N)\big)$ and $K_A := \ker\big(A \rtimes_{(r)} (G, N) \to (A/I) \rtimes_{(r)} (G, N)\big)$ under the Rieffel correspondence (see Remark 2.5.5). □

As a consequence we get:

Lemma 2.8.13. *Suppose that N is a closed normal subgroup of G such that G/N is exact. Suppose further that $0 \to I \to A \to A/I \to 0$ is a short exact sequence of (G, N)-algebras. Then the sequence*

$$0 \to I \rtimes_r (G, N) \to A \rtimes_r (G, N) \to (A/I) \rtimes_r (G, N) \to 0$$

is exact.

Proof. By Theorem 2.8.9 there exists a system $(B, G/N, \beta)$ such that $(B, G, N, \inf \beta, 1_N)$ is Morita equivalent to the given twisted system (A, G, N, α, τ) via some equivalence (X, u). If I is a (G, N)-invariant ideal of A, let $J := \operatorname{Ind}^X I \subseteq B$. It follows then from Lemma 2.8.12 and the Rieffel correspondence, that

$$0 \to I \rtimes_r (G, N) \to A \rtimes_r (G, N) \to (A/I) \rtimes_r (G, N) \to 0$$

is exact if and only if

$$0 \to J \rtimes_r (G, N) \to B \rtimes_r (G, N) \to (B/J) \rtimes_r (G, N) \to 0$$

is exact. But the latter sequence is equal to the sequence

$$0 \to J \rtimes_r G/N \to B \rtimes_r G/N \to (B/J) \rtimes_r G/N \to 0,$$

which is exact since G/N is exact. □

Proof of Theorem 2.8.11. Suppose that $0 \to I \to A \to A/I \to 0$ is an exact sequence of G-algebras and consider the decomposition twisted action (α^N, τ^N) of (G, N) on $A \rtimes_r N$. Since N is exact, the sequence

$$0 \to I \rtimes_r N \to A \rtimes_r N \to (A/I) \rtimes_r N \to 0$$

is a short exact sequence of (G, N)-algebras. Since G/N is exact, it follows therefore from Lemma 2.8.13 that

$$0 \to (I \rtimes_r N) \rtimes_r (G, N) \to (A \rtimes_r N) \rtimes_r (G, N) \to \big((A/I) \rtimes_r N\big) \rtimes_r (G, N) \to 0$$

is exact. But it follows from Proposition 2.8.5 that this sequence equals

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0. \qquad \square$$

2.8.5 Induced representations of twisted crossed products

Using Green's imprimitivity theorem for twisted systems, we can define induced representations and ideals for twisted crossed products $A \rtimes (G, N)$ as in the untwisted case, using the spaces $C_c(G, A, \tau)$ and $C_c(G, \operatorname{Ind} A, \operatorname{Ind} \tau)$, etc. (e.g., see [Ech96, Chapter 1] for this approach). An alternative but equivalent way, as followed in Green's original paper [Gre78], is to define induced representations via the untwisted crossed products: Suppose that (α, τ) is a twisted action of (G, N) on A and let $H \subseteq G$ be a closed subgroup of G such that $N \subseteq H$. Since $A \rtimes (H, N)$ is a quotient of $A \rtimes H$ we can regard every representation of $A \rtimes (H, N)$ as a representation of $A \rtimes H$. We can use the untwisted theory to induce the representation to $A \rtimes G$. But then we have to check that this representation factors through the quotient $A \rtimes (G, N)$ to have a satisfying theory. This has been done in [Gre78, Corollary 5], but one can also obtain it as an easy consequence of Proposition 2.7.15: Let $I_\tau^N \subset A \rtimes N$ denote the twisting ideal for $(\alpha|_N, \tau)$. It is then clear from the definition of representations $\pi \times U$ of $A \rtimes H$ (resp. $A \rtimes G$) which preserve τ, that $\pi \times U$ preserves τ iff $\pi \times U|_N$ preserves τ as a representation of $A \rtimes N$. Hence, $\pi \times U$ is a representation of $A \rtimes (H, N)$ (resp. $A \rtimes (G, N)$) iff $I_\tau^N \subseteq \ker(\pi \times U|_N)$. Since I_τ^N is easily seen to be a G-invariant ideal of $A \rtimes N$, this property is preserved under induction by Proposition 2.7.15.

The procedure of inducing representations is compatible with passing to Morita-equivalent systems. To be more precise: Suppose that (X, u) is a Morita equivalence for the systems (A, G, α) and (B, G, β). If H is a closed subgroup of G and $\pi \times U$ is a representation of $B \rtimes H$, then we get an equivalence

$$\operatorname{Ind}_H^G\big(\operatorname{Ind}^{X \rtimes H}(\pi \times U)\big) \cong \operatorname{Ind}^{X \rtimes G}\big(\operatorname{Ind}_H^G(\pi \times U)\big).$$

This result follows from an isomorphism of $A \rtimes G$-$B \rtimes H$ bimodules

$$X_H^G(A) \otimes_{A \times H} (X \rtimes H) \cong (X \rtimes G) \otimes_{B \rtimes G} X_H^G(B),$$

which just means that the respective compositions in the correspondence categories coincide. A similar result can be shown for the reduction of representations to subgroups. Both results will follow from a linking algebra trick as introduced in [ER96, §4]. Similar statements holds for twisted systems.

2.8.6 Twisted group algebras, actions on $\mathcal{K}$ and Mackey's little group method

In this section we want to study crossed products of the form $\mathcal{K} \rtimes_{(r)} G$, where $\mathcal{K} = \mathcal{K}(H)$ is the algebra of compact operators on some Hilbert space H. As we shall see below, such actions are strongly related to twisted actions on the algebra $\mathbb{C}$ of complex numbers. While there are only trivial actions of groups on $\mathbb{C}$, there are usually many nontrivial twisted actions of pairs (G, N) on $\mathbb{C}$. However, in a certain sense they are all equivalent to twisted actions of the following type:

Example 2.8.14. Assume that $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$ is a central extension of the locally compact group G by the circle group $\mathbb{T}$. Let $\iota : \mathbb{T} \to \mathbb{T}; \iota(z) = z$ denote the identity character on $\mathbb{T}$. Then (id, ι) is a twisted action of $(\tilde{G}, \mathbb{T})$ on $\mathbb{C}$. A (covariant) representation of the twisted system $(\mathbb{C}, \tilde{G}, \mathbb{T}, \mathrm{id}, \iota)$ on a Hilbert space H consists of the representation $\lambda \mapsto \lambda 1_H$ of $\mathbb{C}$ together with a unitary representation $U : \tilde{G} \to \mathcal{U}(H)$ satisfying $U_z = z \cdot 1_H$ for all $z \in \mathbb{T} \subseteq \tilde{G}$, i.e., of those representations of $\tilde{G}$ which restrict to a multiple of ι on the central subgroup $\mathbb{T}$ of $\tilde{G}$. Hence, the twisted crossed product $\mathbb{C} \rtimes (\tilde{G}, \mathbb{T})$ is the quotient of $C^*(\tilde{G})$ by the ideal $I_\iota = \cap\{\ker U : U \in \mathrm{Rep}(\tilde{G}) \text{ and } U|_{\mathbb{T}} = \iota \cdot 1_H\}$. Note that the isomorphism class of $\mathbb{C} \rtimes (\tilde{G}, \mathbb{T})$ only depends on the isomorphism class of the extension $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$.

If G is second countable[14], we can choose a Borel section $c : G \to \tilde{G}$ in the above extension, and we then obtain a Borel map $\omega : G \times G \to \mathbb{T}$ by

$$\omega(s, t) := c(s)c(t)c(st)^{-1} \in \mathbb{T}.$$

A short computation then shows that ω satisfies the cocycle conditions $\omega(s, e) = \omega(e, s) = 1$ and $\omega(s, t)\omega(st, r) = \omega(s, tr)\omega(t, r)$ for all $s, t, r \in G$. Hence it is a 2-cocycle in $Z^2(G, \mathbb{T})$ of Moore's group cohomology with Borel cochains (see [Moo64a, Moo64b, Moo76a, Moo76b]). The cohomology class $[\omega] \in H^2(G, \mathbb{T})$ then only depends on the isomorphism class of the given extension $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$.[15] Conversely, if $\omega : G \times G \to \mathbb{T}$ is any Borel 2-cocycle on G, let G_ω denote the cartesian product $G \times \mathbb{T}$ with multiplication given by

$$(s, z) \cdot (t, w) = (st, \omega(s, t)zw).$$

[14] This assumptions is made to avoid measurability problems. With some extra care, much of the following discussion also works in the nonseparable case (e.g. see [Kle65])

[15] Two cocycles ω and ω' are in the same class in $H^2(G, \mathbb{T})$ iff they differ by a boundary $\partial f(s, t) := f(s)f(t)\overline{f(st)}$ of some Borel function $f : G \to \mathbb{T}$.

By [Mac57] there exists a unique locally compact topology on G_ω whose Borel structure coincides with the product Borel structure. Then G_ω is a central extension of G by $\mathbb{T}$ corresponding to ω (just consider the section $c : G \to G_\omega; c(s) = (s, 1)$) and we obtain a complete classification of the (isomorphism classes of) central extensions of G by $\mathbb{T}$ in terms of $H^2(G, \mathbb{T})$. We then write $C^*_{(r)}(G, \omega) := \mathbb{C} \rtimes_{(r)} (G_\omega, \mathbb{T})$ for the corresponding full (resp. reduced) twisted crossed products, which we now call the (full or reduced) *twisted group algebra of G corresponding to ω*.

There is a canonical one-to-one correspondence between the (nondegenerate) covariant representations of the twisted system $(\mathbb{C}, G_\omega, \mathbb{T}, \mathrm{id}, \iota)$ on a Hilbert space H and the *projective ω-representations* of G on H, which are defined as Borel maps

$$V : G \to \mathcal{U}(H) \quad \text{satisfying} \quad V_s V_t = \omega(s, t) V_{st} \quad s, t \in G.$$

Indeed, if $\tilde{V} : G_\omega \to U(H)$ is a unitary representation of G_ω which restricts to a multiple of ι on $\mathbb{T}$, then $V_s := \tilde{V}(s, 1)$ is the corresponding ω-representation of G.

A convenient alternative realization of the twisted group algebra $C^*(G, \omega)$ is obtained by taking a completion of the convolution algebra $L^1(G, \omega)$, where $L^1(G, \omega)$ denotes the algebra of all L^1-functions on G with convolution and involution given by

$$f * g(s) = \int_G f(t) g(t^{-1}s) \omega(t, t^{-1}s)\, dt \quad \text{and} \quad f^*(s) = \Delta_G(s^{-1}) \overline{\omega(s, s^{-1}) f(s^{-1})}.$$

One checks that the $*$-representations of $L^1(G, \omega)$ are given by integrating projective ω-representations and hence the corresponding C^*-norm for completing $L^1(G, \omega)$ to $C^*(G, \omega)$ is given by

$$\|f\|_{\max} = \sup\{\|V(f)\| : V \text{ is an } \omega\text{-representation of } G\}.$$

The map

$$\Phi : C_c(G_\omega, \mathbb{C}, \iota) \to L^1(G, \omega); \Phi(f)(s) := f(s, 1)$$

then extends to an isomorphism between the two pictures of $C^*(G, \omega)$.[16] Similarly, we can define a *left ω-regular representation* L_ω of G on $L^2(G)$ by setting

$$\big(L_\omega(s)\xi\big)(t) = \omega(s, s^{-1}t)\xi(s^{-1}t), \quad \xi \in L^2(G),$$

and then realize $C^*_r(G, \omega)$ as $\overline{L_\omega\big(L^1(G, \omega)\big)} \subseteq \mathcal{B}(L^2(G))$.

Example 2.8.15. Twisted group algebras appear quite often in C^*-algebra theory. For instance the rational and irrational rotation algebras A_θ for $\theta \in [0, 1)$

[16] Use the identity $\overline{\omega(t, t^{-1})}\omega(t^{-1}, s)\omega(t, t^{-1}s) = 1$ in order to check that Φ preserves multiplication.

are isomorphic to the twisted group algebras $C^*(\mathbb{Z}^2, \omega_\theta)$ with $\omega_\theta\big((n,m),(k,l)\big) = e^{i2\pi\theta mk}$. Note that every cocycle on $\mathbb{Z}^2$ is equivalent to ω_θ for some $\theta \in [0,1)$. If $\theta = 0$ we simply get $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, the classical commutative 2-torus. For this reason the A_θ are often denoted as noncommutative 2-tori.

More generally, a *noncommutative n-torus* is a twisted group algebra $C^*(\mathbb{Z}^n, \omega)$ for some cohomology class $[\omega] \in H^2(\mathbb{Z}^n, \mathbb{T})$.

An extensive study of 2-cocycles on abelian groups is given by Kleppner in [Kle65]. In particular, for $G = \mathbb{R}^n$, every cocycle is similar to a cocycle of the form $\omega(x,y) = e^{\pi i\langle Ax,y\rangle}$, where A is a skew-symmetric real $n \times n$-matrix, and every cocycle of $\mathbb{Z}^n$ is similar to a restiction to $\mathbb{Z}^n$ of some cocycle on $\mathbb{R}^n$. The general structure of the twisted group algebras $C^*(G,\omega)$ for abelian G is studied extensively in [ER95] in the type I case and in [Pog97] in the general case. If G is abelian, then the *symmetry group* S_ω of ω is defined by

$$S_\omega := \{s \in G : \omega(s,t) = \omega(t,s) \text{ for all } t \in G\}.$$

Poguntke shows in [Pog97] (in case G satisfies some mild extra conditions, which are always satisfied if G is compactly generated) that $C^*(G,\omega)$ is stably isomorphic to an algebra of the form $C_0(\widehat{S}_\omega) \otimes C^*(\mathbb{Z}^n, \mu)$, where $C^*(\mathbb{Z}^n,\mu)$ is some simple noncommutative n-torus (here we allow $n = 0$ in which case we put $C^*(\mathbb{Z}^n,\mu) := \mathbb{C}$).[17]

It follows from Theorems 2.7.29 and 2.7.32 that for understanding the ideal structure of $A \rtimes G$, it is necessary to understand the structure of $A_P \rtimes G_P$ for $P \in \mathrm{Prim}(A)$. In the special case $A = C_0(X)$, we saw in the previous section that this is the same as understanding the group algebras $C^*(G_x)$ for the stabilisers G_x, $x \in X$. In general, the problem becomes much more difficult. However, at least in the important special case where A is type I, one can still give a quite satisfactory description of $A_P \rtimes G_P$ in terms of the stabilisers. If A is type I, we have $\widehat{A} \cong \mathrm{Prim}(A)$ via $\sigma \mapsto \ker\sigma$ and if $P = \ker\sigma$ for some $\sigma \in \widehat{A}$, then the simple subquotient A_P of A corresponding to P is isomorphic to $\mathcal{K}(H_\sigma)$ (see Remark 2.7.28). Thus, we have to understand the structure of the crossed products $\mathcal{K}(H_\sigma) \rtimes G_\sigma$, where G_σ denotes the stabiliser of $\sigma \in \widehat{A}$.

Hence, in what follows we shall always assume that G is a locally compact group acting on the algeba $\mathcal{K}(H)$ of compact operators on some Hilbert space H. In order to avoid measerability problems, we shall always assume that G is second countable and that H is separable (see Remark 2.8.18 for a brief discussion of the general case). Since every automorphism of $\mathcal{K}(H)$ is given by conjugation with some unitary $U \in \mathcal{B}(H)$, it follows that the automorphism group of $\mathcal{K}(H)$ is isomorphic (as topological groups) to the group $\mathcal{P}U := \mathcal{U}/\mathbb{T}\cdot 1$, where $\mathcal{U} = \mathcal{U}(H)$ denotes the group of unitary operators on H equipped with the strong operator topology.

[17] Two C^*-algebras A and B are called *stably isomorphic* if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(l^2(\mathbb{N}))$.

Choose a Borel section $c : \mathcal{P}U \to \mathcal{U}$. If $\alpha : G \to \mathcal{P}U$ is a continuous homomorphism, let $V_\alpha := c \circ \alpha : G \to \mathcal{U}$. Since $V_\alpha(s)V_\alpha(t)$ and $V_\alpha(st)$ both implement the automorphism α_{st}, there exists a number $\omega_\alpha(s,t) \in \mathbb{T}$ with

$$\omega_\alpha(s,t) \cdot 1 = V_\alpha(st)V_\alpha(t)^* V_\alpha(s)^*.$$

A short computation using the identity $\mathrm{Ad}(V_\alpha(s)V_\alpha(tr)) = \mathrm{Ad}(V_\alpha(s)V_\alpha(t)V_\alpha(r)) = \mathrm{Ad}(V_\alpha(st)V_\alpha(r))$ for $s,t,r \in G$ shows that ω_α is a Borel 2-cocycle on G as in Example 2.8.14 and that V_α is a projective $\bar{\omega}_\alpha$-representation of G on H.

The class $[\omega_\alpha] \in H^2(G,\mathbb{T})$ only depends on α and it vanishes if and only if α is *unitary* in the sense that α is implemented by a strongly continuous homomorphism $V : G \to \mathcal{U}$.[18] Therefore, the class $[\omega_\alpha] \in H^2(G,\mathbb{T})$ is called the *Mackey obstruction for α being unitary.* An easy computation gives:

Lemma 2.8.16. *Let $\alpha : G \to \mathrm{Aut}(\mathcal{K}(H))$, $V_\alpha : G \to \mathcal{U}(H)$ and ω_α be as above. Let G_{ω_α} denote the central extension of G by $\mathbb{T}$ corresponding to ω_α as described in Example* 2.8.14 *and let $\iota : \mathbb{T} \to \mathbb{C}$ denote the inclusion. Let*

$$\tilde{V}_\alpha : G_{\omega_\alpha} \to \mathcal{U}(H); \quad \tilde{V}_\alpha(s,z) = \bar{z}V_\alpha(s).$$

Then $(H, \tilde{V}_\alpha)$ is a $(G_{\omega_\alpha}, \mathbb{T})$-equivariant Morita equivalence between the action α of $G \cong G_{\omega_\alpha}/\mathbb{T}$ on $\mathcal{K}(H)$ and the twisted action (id, ι) of $(G_{\omega_\alpha}, \mathbb{T})$ on $\mathbb{C}$.

We refer to §2.8.2 for the definition of twisted equivariant Morita equivalences. Since Morita-equivalent twisted systems have Morita-equivalent full and reduced crossed products, it follows that $\mathcal{K}(H) \rtimes_\alpha G$ is Morita equivalent to the twisted group algebra $C^*(G, \omega_\alpha)$ (and similarly for $\mathcal{K}(H) \rtimes_r G$ and $C^*_r(G,\omega_\alpha)$). Recall from Example 2.8.14 that there is a one-to-one correspondence between the representations of $C^*(G,\omega_\alpha)$ (or the covariant representations of $(\mathbb{C}, G_{\omega_\alpha}, \mathbb{T}, \mathrm{id}, \iota)$) and the projective ω_α-representations of G. Using the above lemma and induction of covariant representations via the bimodule $(H, \tilde{V}_\alpha)$ then gives:

Theorem 2.8.17. *Let $\alpha : G \to \mathrm{Aut}(\mathcal{K}(H))$ be an action and let ω_α and $V_\alpha : G \to \mathcal{U}(H)$ be as above. Then the assignment*

$$L \mapsto (\mathrm{id} \otimes 1, V_\alpha \otimes L)$$

gives a homeomorphic bijection between the (irreducible) ω_α-projective representations of G and the (irreducible) nondegenerate covariant representations of $(\mathcal{K}(H), G, \alpha)$.

Remark 2.8.18. (1) It is actually quite easy to give a direct isomorphism between $C^*(G,\omega_\alpha) \otimes \mathcal{K}$ and the crossed product $\mathcal{K} \rtimes_\alpha G$, where we write $\mathcal{K} = \mathcal{K}(H)$. If $V_\alpha : G \to \mathcal{U}(H)$ is as above, then one easily checks that

$$\Phi : L^1(G,\omega_\alpha) \odot \mathcal{K} \to L^1(G,\mathcal{K}); \Phi(f \otimes k)(s) = f(s)kV_s^*.$$

[18] To see this, one should use the fact that any measurable homomorphism between polish groups is automatically continuous by [Moo76a, Proposition 5].

is a $*$-homomorphism with dense range such that

$$(\mathrm{id}\otimes 1)\times(V_\alpha\otimes L)\big(\Phi(f\otimes k)\big)=L(f)\otimes k$$

for all $f\in L^1(G,\omega)$ and $k\in\mathcal{K}$, and hence the above theorem implies that Φ is isometric with respect to the C^*-norms. A similar argument also shows that $\mathcal{K}\rtimes_r G\cong C_r^*(G,\omega_\alpha)\otimes\mathcal{K}$.

(2) The separability assumptions made above are not really necessary: Indeed, if $\alpha: G\to \mathrm{Aut}(\mathcal{K}(H))\cong \mathcal{P}U(H)$ is an action of any locally compact group on the algebra of compact operators on any Hilbert space H, then

$$\tilde{G}:=\{(s,U)\in G\times\mathcal{U}(H):\alpha_s=\mathrm{Ad}(U)\}$$

fits into the central extension

$$1\longrightarrow\mathbb{T}\xrightarrow{z\mapsto(e,z\cdot 1)}\tilde{G}\xrightarrow{(s,U)\mapsto s}G\longrightarrow 1.$$

If we define $u:\tilde{G}\to\mathcal{U}(H); u(s,U)=U$, then it is easy to check (H,u) implements a Morita equivalence between $(\mathcal{K}(H),G,\alpha)$ and the twisted system $(\mathbb{C},\tilde{G},\mathbb{T},\mathrm{id},\iota)$. Thus we obtain a one-to-one correspondence between the representations of $\mathcal{K}(H)\rtimes_\alpha G$ and the representations of $\mathbb{C}\rtimes_{\mathrm{id},\iota}(\tilde{G},\mathbb{T})$. We refer to [Gre78, Theorem 18] for more details.

Combining the previous results (and using the identification $\widehat{A}\cong\mathrm{Prim}(A)$ if A is type I) with Theorem 2.7.29 now gives:

Theorem 2.8.19 (Mackey's little group method)**.** *Suppose that (A,G,α) is a smooth separable system such that A is type I. Let $S\subseteq\widehat{A}$ be a section for the quotient space $G\backslash\widehat{A}$ and for each $\pi\in S$ let $V_\pi: G_\pi\to\mathcal{U}(H_\pi)$ be a measurable map such that $\pi(\alpha_s(a))=V_\pi(s)\pi(a)V_\pi(s)^*$ for all $a\in A$ and $s\in G_\pi$ (such map always exists). Let $\omega_\pi\in Z^2(G_\pi,\mathbb{T})$ be the 2-cocycle satisfying*

$$\omega_\pi(s,t)\cdot 1_{H_\pi}:=V_\pi(st)V_\pi(t)^*V_\pi(s)^*.$$

Then

$$\mathrm{IND}:\cup_{\pi\in S}C^*(G_\pi,\omega_\pi)\widehat{\ }\to(A\rtimes G)\widehat{\ };\mathrm{IND}(L)=\mathrm{ind}_{G_\pi}^G(\pi\otimes 1)\times(V_\pi\otimes L).$$

is a bijection, which restricts to homeomorphisms between $C^*(G_\pi,\omega_\pi)\widehat{\ }$ and its image $(A_{G_\pi}\rtimes G)\widehat{\ }$ for each $\pi\in S$.

Remark 2.8.20. **(1)** If G is exact, then a similar result holds for the reduced crossed product $A\rtimes_r G$, if we also use the reduced twisted group algebras $C_r^*(G_\pi,\omega_\pi)$ of the stabilisers.

(2) If (A,G,α) is a type I smooth system that is not separable, then a similar result can be formulated using the approach described in part (2) of Remark 2.8.18.

Note that the above result in particular applies to all systems (A, G, α) with A type I and G compact, since actions of compact groups on type I algebras are always smooth in the sense of Definition 2.7.25. Since the central extensions G_ω of a compact group G by $\mathbb{T}$ are compact, and since $C^*(G, \omega)$ is a quotient of $C^*(G_\omega)$ (see Example 2.8.14), it follows that the twisted group algebras $C^*(G, \omega)$ are direct sums of matrix algebras if G is compact. Using this, we easily get from Theorem 2.8.19:

Corollary 2.8.21. *Suppose that (A, G, α) is a system with A type I and G compact. Then $A \rtimes G$ is type I. If, moreover, A is CCR, then $A \rtimes G$ is CCR, too.*

Proof. Since the locally closed subset $(A_{G_\pi} \rtimes G)\widehat{\ }$ corresponding to some orbit $G(\pi) \subseteq \widehat{A}$ is homeomorphic (via Morita equivalence) to $(\mathcal{K}(H_\pi) \rtimes G)\widehat{\ } \cong C^*(G_\pi, \omega_\pi)\widehat{\ }$, it follows that $(A_{G(\pi)} \rtimes G)\widehat{\ }$ is a discrete set in the induced topology. This implies that all points in $(A \rtimes G)\widehat{\ }$ are locally closed. Moreover, if A is CCR, then the points in $\widehat{A}$ are closed. Since G is compact, it follows then that the G-orbits in $\widehat{A}$ are closed, too. But then the discrete set $(A_{G_\pi} \rtimes G)\widehat{\ }$ is closed in $(A \rtimes G)\widehat{\ }$, which implies that the points in $(A \rtimes G)\widehat{\ }$ are closed. $\square$

Chapter 3

Bivariant KK-Theory and the Baum–Connes conjecure

Siegfried Echterhoff

3.1 Introduction

The extension of K-theory from topological spaces to operator algebras provides the most powerful tool for the study of C^*-algebras. On one side there now exist far reaching classification results in which certain classes of C^*-algebras can be classified by their K-theoretic data. This started with the early work of Elliott [Ell76] on the classification of AF-algebras – inductive limits of finite-dimensional C^*-algebras. It went on with the classification of simple, separable, nuclear, purely infinite C^*-algebras by Kirchberg and Phillips [KP00, Phi00]. Presently, due to the work of many authors (e.g., see [Win16] for a survey on the most recent developments) the classification program covers a very large class of nuclear algebras.

On the other hand, the K-theory groups of group algebras $C^*(G)$ and $C_r^*(G)$ serve as recipients of indices of G-invariant elliptic operators, and the study of such indices has an important impact in modern topology and geometry. To get a rough idea, the Baum–Connes conjecture implies that every element in the K-theory groups of the reduced C^*-group algebra $C_r^*(G)$ of a locally compact group G appears as an index of some generalised G-invariant elliptic operator. To be more precise, these generalised elliptic operators form the cycles of the G-equivariant K-homology (with G-compact supports) $K_*^G(\underline{EG})$, in which $\underline{EG}$ is a certain classifying space for proper actions of G (often realised as a G-manifold) and the index map

$$\mu_G : K_*^G(\underline{EG}) \to K_*(C_r^*(G))$$

is then a well-defined group homomorphism. It is called the *assembly map* for G.

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The *Baum–Connes conjecture (with trivial coefficients)* asserts that the assembly map is an isomorphism for all G.

The construction of the assembly map naturally extends to crossed products and provides a map

$$\mu_{(G,A)} : K_*^G(\underline{EG}, A) \to K_*(A \rtimes_r G).$$

The *Baum–Connes conjecture with coefficients* asserts that this more general assembly map should be an isomorphism as well. Although this general version of the conjecture is now known to be false in general (e.g., see [HLS02]), it is known to be true for a large class of groups, including the class of all amenable groups, and it appears to be an extremely useful tool for the computation of K-theory groups in several important applications.

In this chapter we give a concise introduction to the Baum–Connes conjecture and to some of the applications that allow the explicit computation of K-theory groups with the help of the conjecture. We start with a very short reminder of the basic properties of C^*-algebra K-theory before we give an introduction of Kasparov's bivariant K-theory functor which assigns to each pair of G-C^*-algebras A, B a pair of abelian groups $KK_*^G(A, B)$, $* = 0, 1$. Kasparov's theory is not only fundamental for the definition of the groups $K_*^G(\underline{EG})$ and $K_*^G(\underline{EG}, A)$ and the construction of the assembly map, but it also provides the most powerful tools for proving the Baum–Connes conjecture for certain classes of groups. In this chapter we will restrict ourselves to Kasparov's picture of KK-theory and we will not touch on other descriptions or variants such as the Cuntz picture of KK-theory or E-theory as introduced by Connes and Higson. We refer to [Bla86] for a treatment of these and their connections to Kasparov's theory.

As part of our introduction to KK-theory we will give a detailed and a fairly elementary proof of Kasparov's Bott-periodicity theorem in one dimension by constructing Dirac and dual Dirac elements that implement a KK-equivalence between $C_0(\mathbb{R})$ and the first complex Clifford algebra Cl_1. We shall later use these computations to give a complete proof of the Baum–Connes conjecture for $\mathbb{R}$ and $\mathbb{Z}$ with the help of Kasparov's Dirac-dual Dirac method. This method is the most powerful tool for proving the conjecture and has been successfully applied to a very large class of groups including all amenable groups. As corollaries of our proof of the conjecture for $\mathbb{R}$ and $\mathbb{Z}$, we shall also present proofs of Connes's Thom isomorphism for crossed products by $\mathbb{R}$ and the Pimsner–Voiculescu six-term exact sequence for crossed products by $\mathbb{Z}$.

In the last part of this chapter we shall present the "going-down" principle, which roughly says the following: Suppose G satisfies the Baum–Connes conjecture with coefficients. Then any G-equivariant $*$-homomorphism (or KK-class) between two G-algebras A and B that induces isomorphisms between the K-theory groups of $A \rtimes K$ and $B \rtimes K$ for **all** compact subgroups K of G also induces an isomorphism between the K-groups of $A \rtimes_r G$ and $B \rtimes_r G$. We shall give a complete proof of this principle if G is discrete and we present a number of applications of this result. In particular, as one application we shall present a theorem about possible explicit computations of the K-theory of crossed products $C_0(\Omega) \rtimes_r G$ in which a discrete group G acts on a totally disconnected space Ω with some additional "good" properties which we shall explain in detail. This result is basic for the K-theory computations of the reduced semi-group C^*-algebras as presented in Chapters 5 and 6 of this book.

There are many other surveys on the Baum–Connes conjecture that look at the conjecture from quite different angles. The reader should definitely have a look at the paper [BCH94] of Baum, Connes, and Higson, where a broad discussion of various applications of the conjecture is given. The survey [Val03] by Alain Valette restricts itself to a discussion of the Baum–Connes conjecture for discrete groups, but also provides a good discussion of applications to other important conjectures. The survey [MV03] by Mislin discusses the conjecture from the topologist's point of view, where the left-hand side (the topological K-theory of G) is defined in terms of the Bredon cohomology – a picture of the Baum–Connes conjecture first given by Davis and Lück [DL98]. We also want to mention the paper [HG04] of Higson and Guentner, which gives an introduction to the Baum–Connes conjecture based on E-theory. Last, but not least, we suggest to the interested reader to study the book [HR00] by Higson and Roe, where many of the relevant techniques for producing important KK-classes by elliptic operators (such as the Dirac-class in K-homology) are treated in a very nice way.

Throughout this chapter we assume that the reader is familiar with the basics on C^*-algebras, the basic constructions and properties of full and reduced crossed products and the notion of Morita equivalence and Hilbert C^*-modules. A detailed introduction to these topics is given in the first six sections of Chapter 2.

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3.2 Operator K-Theory

In this section we give a very brief overview of the definition and some basic properties of the K-theory groups of C^*-algebras. We urge the reader to have a look at one of the standard books on operator K-theory (e.g., [Bla86, RLL00, WO93]) for more detailed expositions of this theory.

Let us fix some notation: If A is a C^*-algebra, we denote by $M_n(A)$ the C^*-algebra of all $n \times n$-matrices over A and by $A[0,1]$ the C^*-algebra of continuous functions $f : [0,1] \to A$. Moreover, we denote by $A^1 = A \oplus \mathbb{C}1$ the unique C^*-algebra with underlying vector space $A \oplus \mathbb{C}1$ and with multiplication and involution given by

$$(a + \lambda 1)(b + \mu 1) = ab + \lambda b + \mu a + \lambda\mu 1 \quad \text{and} \quad (a + \lambda 1)^* = a^* + \bar{\lambda} 1,$$

for $a + \lambda 1, b + \mu 1 \in A + \mathbb{C}1$. Let $\epsilon : A_1 \to \mathbb{C}; \epsilon(a + \lambda 1) = \lambda$. We call A^1 the *unitisation* of A (even if A already has a unit). We write

$$M_\infty(A) := \cup_{n \in \mathbb{N}} M_n(A)$$

where we regard $M_n(A)$ as a subalgebra of $M_{n+1}(A)$ via $T \mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$. We denote by $\mathcal{P}(A)$ the set of projections $p \in M_\infty(A)$, i.e., $p = p^* = p^2$.

Definition 3.2.1. Let A be a unital C^*-algebra and let $p, q \in \mathcal{P}(A)$. Then p, q are called

- *Murray–von Neumann equivalent* (denoted $p \sim q$) if there exist $x, y \in M_\infty(A)$ such that $p = xy$ and $q = yx$;
- *unitarily equivalent* (denoted $p \sim_u q$) if there exists some $n \in \mathbb{N}$ and a unitary $u \in U(M_n(A))$ such that $p, q \in M_n(A)$ and $q = upu^*$ in $M_n(A)$;
- *homotopic* (denoted $p \sim_h q$), if there exists a projection $r \in \mathcal{P}(A[0,1])$ such that $p = r(0)$ and $q = r(1)$.

All three equivalence relations coincide on $\mathcal{P}(A)$ (but not on the level of $M_n(A)$ for fixed $n \in \mathbb{N}$). If A is unital and if $p, q \in M_n(A)$, then $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ and $\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$ are Murray–von Neumann equivalent in $M_{2n}(A)$ with $x = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$. This allows us to define an abelian semigroup structure on $\mathcal{P}(A)/\sim$ with addition given by $[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$. For each unital C^*-algebra A, we define $K_0(A)$ as the Grothendieck group of the semigroup $\mathcal{P}/\sim$, i.e.,

$$K_0(A) = \left\{ \big[[p] - [q]\big] : [p], [q] \in \mathcal{P}(A)/\sim \right\}$$

where we write $\big[[p] - [q]\big] = \big[[p'] - [q']\big]$ if and only if there exists $h \in \mathcal{P}(A)$ such that

$$[p] + [q'] + [h] = [p'] + [q] + [h] \text{ in } \mathcal{P}(A)/\sim .$$

Example 3.2.2. If $A = \mathbb{C}$, then two projections $p, q \in \mathcal{P}(\mathbb{C})$ are homotopic, if and only if they have the same rank. It follows from this that $\mathcal{P}(\mathbb{C})/\sim \cong \mathbb{N}$ as semigroup and hence we get $K_0(\mathbb{C}) \cong \mathbb{Z}$.

If $\Phi : A \to B$ is a unital $*$-homomorphism between the unital C^*-algebras A and B, then there exists a unique group homomorphism $\Phi_0 : K_0(A) \to K_0(B)$ such that $\Phi_0([p]) = [\Phi(p)]$.

We then define

$$K_0(A) := \ker \left(K_0(A^1) \overset{\epsilon_0}{\to} K_0(\mathbb{C}) \cong \mathbb{Z} \right)$$

for any C^*-algebra A. If A is unital, then $A^1 \cong A \oplus \mathbb{C}$ as a direct sum of the C^*-algebras A and $\mathbb{C}$ (the isomorphism is given by $a + \lambda 1 \mapsto (a - \lambda 1_A, \lambda)$) and it is not difficult to check that in this case both definitions of $K_0(A)$ coincide. Any $*$-homomorphism $\Phi : A \to B$ extends to a unital $*$-homomorphism $\Phi^1 : A^1 \to B^1; \Phi^1(a+\lambda 1) = \Phi(a) + \lambda 1$ and the resulting map $\Phi_0^1 : K_0(A^1) \to K_0(B^1)$ factors through a well-defined homomorphism $\Phi_0 : K_0(A) \to K_0(B)$.

For the construction of $K_1(A)$ let $U_n(A^1)$ denote the group of unitary elements of $M_n(A^1)$. We embed $U_n(A^1)$ into $U_{n+1}(A^1)$ via $U \mapsto \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$, and we define $U_\infty(A^1) = \cup_{n\in\mathbb{N}} U_n(A)$. Let $U_\infty(A^1)_0$ denote the path-connected component of $U_\infty(A^1)$, where we say that two unitaries can be joined by a path in $U_\infty(A^1)$ if and only if they can be joined by a continuous path in $U_n(A^1)$ for some $n \in \mathbb{N}$. For $u, v \in U_n(A^1)$, one can check that

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \tag{3.2.1}$$

in $U_{2n}(A^1) \subseteq U_\infty(A^1)$. Therefore, if we define

$$K_1(A) := U_\infty(A^1)/U_\infty(A^1)_0$$

with addition given by

$$[u] + [v] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right],$$

(which by (3.2.1) is equal to $[uv]$) we see that $K_1(A)$ is an abelian group. As for K_0, for unital A we can alternatively construct $K_1(A)$ without passing to the unitization A^1 as $K_1(A) = U_\infty(A)/U_\infty(A)_0$.

Example 3.2.3. Since $U_n(\mathbb{C})$ is path connected for all $n \in \mathbb{N}$, we have $K_1(\mathbb{C}) = \{0\}$.

If $\Phi : A \to B$ is a $*$-homomorphism, there is a well-defined group homomorphism

$$\Phi_1 : K_1(A) \to K_1(B); \Phi_1([u]) = \big[\Phi^1(u)\big],$$

where, as before, $\Phi^1 : A^1 \to B^1$ denotes the unique unital extension of Φ to A^1.

Proposition 3.2.4. *The assignments $A \mapsto K_0(A), K_1(A)$ are homotopy invariant covariant functors from the category of C^*-algebras to the category of abelian groups.*

Homotopy invariance means that if $\Phi, \Psi : A \to B$ are two homotopic $*$-homomorphisms, then $\Phi_* = \Psi_* : K_*(A) \to K_*(B)$, $* = 0, 1$. Here a homotopy between Φ and Ψ is a $*$-homomorphism $\Theta : A \to B[0,1]$ such that $\Phi = \epsilon_0 \circ \Theta$ and $\Psi = \epsilon_1 \circ \Theta$, where for all $t \in [0,1]$, $\epsilon_t : B[0,1] \to B$ denotes evaluation at t. Of course, the homotopy invariance is a direct consequence of the fact that "$\sim$" coincides with "$\sim_h$" on $\mathcal{P}(B^1)$.

Recall that a C^*-algebra is called *contractible*, if the identity id $: A \to A$ is homotopic to the zero map $0 : A \to A$. As an example, let A be any C^*-algebra, then $A(0,1] := \{a \in A[0,1] : a(0) = 0\}$ is contractible. A homotopy between $\mathrm{id} : A(0,1] \to A(0,1]$ and 0 is given by the path of $*$-homomorphism $\Phi_t : A(0,1] \to A(0,1]$; $\big(\Phi_t(a)\big)(s) = a(ts)$.

Corollary 3.2.5. *Suppose that A is a contractible C^*-algebra. Then $K_0(A) = \{0\} = K_1(A)$.*

If $0 \to I \overset{\iota}{\to} A \overset{q}{\to} B \to 0$ is a short exact sequence of C^*-algebras, then the functoriality of K_0 and K_1 gives two sequences

$$K_0(I) \overset{\iota_0}{\to} K_0(A) \overset{q_0}{\to} K_0(B) \quad \text{and} \quad K_1(I) \overset{\iota_1}{\to} K_1(A) \overset{q_1}{\to} K_1(B) \tag{3.2.2}$$

which both can be shown to be exact in the middle. If there exists a splitting homomorphism $s : B \to A$ for the quotient map q, it induces a splitting homomorphism $s_* : K_*(B) \to K_*(A)$, $* = 0, 1$, and in this case the groups $K_0(A)$ and $K_1(A)$ decompose as direct sums

$$K_0(A) = K_0(I) \oplus K_0(B) \quad \text{and} \quad K_1(A) = K_1(I) \oplus K_1(B).$$

In particular, in this special case the sequences in (3.2.2) become short exact sequences of abelian groups.

Six-term exact sequence. In general, the sequences in (3.2.2) can be joined into a six-term exact sequence

$$\begin{tikzcd}
K_0(I) \arrow[r, "\iota_0"] & K_0(A) \arrow[r, "q_0"] & K_0(B) \arrow[d, "\exp"] \\
K_1(B) \arrow[u, "\partial"] & K_1(A) \arrow[l, "q_1"] & K_1(I) \arrow[l, "\iota_1"]
\end{tikzcd}$$

which serves as a very important tool for explicit computations as well as for proving theorems on K-theory. We refer to [Bla86, RLL00] for a precise description of the boundary maps exp and ∂. Note that the six-term sequence is natural in the sense that if we have a morphism between two short exact sequences, i.e., we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\psi"] & B \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & J \arrow[r] & C \arrow[r] & D \arrow[r] & 0
\end{tikzcd}$$

in which the horizontal lines are exact sequences of C^*-algebras, then we have corresponding commutative diagrams

$$\begin{tikzcd}
{} \arrow[r] & K_i(A) \arrow[r] \arrow[d, "\psi_i"] & K_i(B) \arrow[r] \arrow[d, "\theta_i"] & K_{i+1}(I) \arrow[r] \arrow[d, "\varphi_{i+1}"] & K_{i+1}(A) \arrow[r] \arrow[d, "\psi_{i+1}"] & {} \\
{} \arrow[r] & K_i(C) \arrow[r] & K_i(D) \arrow[r] & K_{i+1}(J) \arrow[r] & K_{i+1}(C) \arrow[r] & {}
\end{tikzcd}$$

Aside from theoretical importance, this fact can often be used quite effectively for explicit computations of the boundary maps in the six-term sequence.

If we apply the six-term sequence to the short exact sequence

$$0 \to C_0(0,1) \otimes A \xrightarrow{\iota} A(0,1] \xrightarrow{q} A \to 0$$

in which the quotient map $q : A(0,1] \to A$ is given by evaluation at 1, we get the six-term sequence

$$\begin{array}{ccccc} K_0(C_0(0,1)\otimes A) & \xrightarrow{\iota_0} & 0 & \xrightarrow{q_0} & K_0(A) \\ \partial \big\uparrow & & & & \big\downarrow \exp \\ K_1(A) & \xleftarrow[q_1]{} & 0 & \xleftarrow[\iota_1]{} & K_1(C_0(0,1)\otimes A) \end{array}$$

which shows that the connecting maps $\exp : K_0(A) \to K_1(C_0(0,1) \otimes A)$ and $\partial : K_1(A) \to K_0(C_0(0,1) \otimes A)$ are isomorphisms which are natural in A. Hence, by identifying $(0,1)$ with $\mathbb{R}$ and $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ with $C_0(\mathbb{R}^2)$, we can deduce the following important results from the above six-term sequence

Theorem 3.2.6 (Bott periodicity). *For each C^*-algebra A there are natural isomorphisms*

$$K_1(A) \cong K_0(C_0(\mathbb{R}) \otimes A) \quad \text{and} \quad K_1(C_0(\mathbb{R}) \otimes A) \cong K_0(A).$$

Moreover, if we apply the first isomorphism to $B = C_0(\mathbb{R}) \otimes A$, we obtain isomorphisms

$$K_0(A) \cong K_1(C_0(\mathbb{R}) \otimes A) \cong K_0(C_0(\mathbb{R}^2) \otimes A).$$

We should note that the proof of the six-term sequence usually uses the Bott periodicity theorem, so we do present the results in the wrong order. In any case, the proofs of the six-term sequence and of the Bott periodicity theorem are quite deep and we refer to the standard literature on K-theory (e.g., [Bla86]) for the details. We shall later provide a proof of Bott periodicity in KK-theory, which does imply Theorem 3.2.6. We close this short section with two more important features of K-theory:

Continuity. If $A = \lim_i A_i$ is the inductive limit of an inductive system $\{A_i, \Phi_{ij}\}$ of C^*-algebras, then

$$K_*(A) = \lim_i K_*(A_i), \quad * = 0, 1.$$

Morita invariance. If $e \in M(n, \mathbb{C})$ is any rank-one projection, then $\Phi_e : A \to M_n(A); \Phi_e(a) = e \otimes a$ induces an isomorphism between $K_*(A)$ and $K_*(M_n(A))$, $* = 0, 1$. More generally, if e is any rank-one projection in $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$, then the homomorphism $a \mapsto e \otimes a$ induces isomorphisms $K_*(A) \cong K_*(\mathcal{K} \otimes A)$, $* = 0, 1$. Since, by [BGR77] two σ-unital C*-algebras A and B are stably isomorphic if and only if they are Morita equivalent, it follows that Morita equivalent σ-unital C*-algebras have isomorphic K-theory groups.

3.3 Kasparov's equivariant KK-theory

We now come to Kasparov's construction of the G-equivariant bivariant K-theory, which in some sense is built on the correspondence category as described in Section 2.5. Readers who are not familiar with the notion of Hilbert modules and correspondences are advised to read that section before going on here. Since some of the constructions require that the C^*-algebras are separable and that Hilbert B-modules $\mathcal{E}$ are countably generated (which means that there is a countable subset $\mathcal{C} \subseteq \mathcal{E}$ such that $\mathcal{C} \cdot B$ is dense in $\mathcal{E}$), we shall from now on assume that these conditions will hold throughout, except for the multiplier algebras of separable C^*-algebras and the algebras of adjointable operators on a countably generated Hilbert module. We refer to [Bla86] or Kasparov's original paper [Kas88] for a more detailed account on where these conditions can be relaxed.

3.3.1 Graded C^*-algebras and Hilbert modules

We write $\mathbb{Z}_2$ for the group with two elements. A $\mathbb{Z}_2$ grading of a G-C^*-algebra (A, α) is given by an action $\epsilon_A : \mathbb{Z}_2 \to \mathrm{Aut}(A)$ which commutes with α. We then might consider A as a $G \times \mathbb{Z}_2$-C^*-algebra with action $\alpha \times \epsilon_A$ and a graded G-equivariant correspondence between the graded G-C^*-algebras $(A, \alpha \times \epsilon_A)$ and $(B, \beta \times \epsilon_B)$ is just a $G \times \mathbb{Z}_2$-equivariant correspondence $(\mathcal{E}, u \times \epsilon_{\mathcal{E}}, \Phi)$ between these algebras.

Moreover, if ϵ_A is a grading of A, we write $A_0 := \{a \in A : \epsilon_A(a) = a\}$ and $A_1 = \{a \in A : \epsilon_A(a) = -a\}$, for the eigenspaces of the eigenvalues 1 and -1 for ϵ_A, and similar for gradings on Hilbert modules. The elements in A_0 and A_1 are called the *homogeneous elements* of A. We write $\deg(a) = 0$ if $a \in A_0$ and $\deg(a) = 1$ if $a \in A_1$. The expression $\deg(a)$ is called the *degree* of the homogeneous element a. Note that A_0 is a C^*-subalgebra of A and every element $a \in A$ has a unique decomposition $a = a_0 + a_1$ with $a_0 \in A_0, a_1 \in A_1$. If A is a $\mathbb{Z}_2$ graded C^*-algebra, the graded commutator $[a, b]$ is defined as

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba$$

for homogeneous elements $a, b \in A$ and it is defined on all of A by bilinear continuation.

If A and B are two graded C^*-algebras, we define the *graded algebraic tensor product* $A \odot_{gr} B$ as the usual algebraic tensor product with graded multiplication and involution given on elementary tensors of homogeneous elements by

$$\begin{aligned}(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) &= (-1)^{\deg(b_1)\deg(a_2)}(a_1 a_2 \otimes b_1 b_2)\\ (a \otimes b)^* &= (-1)^{\deg(a)\deg(b)}(a^* \otimes b^*).\end{aligned}$$

In what follows we write $A \hat{\otimes} B$ for the minimal (or spatial) completion of $A \odot_{gr} B$. We refer to [Bla86, 14.4] for more details of this construction.

Example 3.3.1. **(a)** For any C^*-algebra A there is a grading on $M_2(A)$ given by conjugation with the symmetry $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This grading is called the *standard even* grading on $M_2(A)$. We then have

$$M_2(A)_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad M_2(A)_1 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

(b) If A is a C^*-algebra, then the direct sum $A \oplus A$ carries a grading given by $(a, b) \mapsto (b, a)$, which is called the *standard odd* grading. We then have

$$(A \oplus A)_0 = \{(a, a) : a \in A\} \quad \text{and} \quad (A \oplus A)_1 = \{(a, -a) : a \in A\}.$$

(c) Examples of nontrivially graded C^*-algebras, which play an important rôle in the theory, are the Clifford algebras $Cl(V, q)$ where $q : V \times V \to \mathbb{R}$ is a (possibly degenerate) symmetric bilinear form on a finite-dimensional real vector space V. $Cl(V, q)$ is defined as the universal C^*-algebra generated by the elements $v \in V$ subject to the relations

$$v^2 = q(v, v)1 \quad \forall v \in V$$

and such that the embedding $\iota : V \hookrightarrow Cl(V, q)$ is $\mathbb{R}$-linear. Using the equation $(v + v')^2 - (v - v')^2 = 2(vv' + v'v)$ we obtain the relations

$$vv' + v'v = 2q(v, v')1 \quad \forall v, v' \in V.$$

If $\dim(V) = n$, then $\dim(Cl(V, q)) = 2^n$. The grading on $Cl(V, q)$ is given as follows: the linear span of all products of the form $v_1 v_2 \cdots v_m$ with $m = 2k$ even is the set of homogeneous elements of degree 0 and the linear span of all such products with $m = 2k - 1$ odd is the set of homogeneous elements of degree 1.

For all $n \in \mathbb{N}_0$ we write Cl_n for $Cl(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Then $Cl_0 \cong \mathbb{C}$ and $Cl_1 = \mathbb{C}1 + \mathbb{C}e_1 \cong \mathbb{C} \oplus \mathbb{C}$ with the standard odd grading (sending $\lambda 1 + \mu e_1$ to $(\lambda + \mu, \lambda - \mu) \in \mathbb{C}^2$). If $n = 2$ and if $\{e_1, e_2\}$ is the standard orthonormal basis of $\mathbb{R}^2$, then there is an isomorphism of $Cl_2 \cong M_2(\mathbb{C})$, equipped with the standard even grading, given by sending the generator e_1 to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and e_2 to $i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

In general, we have the formula $Cl_n \hat{\otimes} Cl_m \cong Cl_{n+m}$ as graded C^*-algebras, where we use the *graded tensor product* and the diagonal grading on the left-hand side of this equation. Note that the isomorphism is given on the generators $\{v : v \in \mathbb{R}^n\}$ and $\{w : w \in \mathbb{R}^m\}$ by sending $v \otimes w$ to $(v, 0) \cdot (0, w) \in Cl_{n+m}$. In particular, Cl_n can be constructed as the nth graded tensor product of Cl_1 with itself.

Note that for any C^*-algebra A, it is an easy exercise to show that the graded tensor product $(A \oplus A) \hat{\otimes} Cl_1$, where $A \oplus A$ is equipped with the standard odd grading, is isomorphic to $M_2(A)$ equipped with the standard even grading. As a consequence, it follows that

$$Cl_{2n} \cong M_{2^n}(\mathbb{C}) \quad \text{and} \quad Cl_{2n+1} \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$$

where the grading in the even case is given by conjugation with a symmetry $J \in M_{2^n}(\mathbb{C})$ (i.e., an isometry with $J^2 = 1$) and the standard odd grading in the odd case (e.g., see [Bla86, §14.4] for more details).

Note that the grading $\epsilon_{\mathcal{E}}$ of a Hilbert B-module induces a grading $\operatorname{Ad} \epsilon_{\mathcal{E}}$ on $\mathcal{L}_B(\mathcal{E})$ and $\mathcal{K}(\mathcal{E})$ in a canonical way and the morphism $\Phi : A \to \mathcal{L}_B(\mathcal{E})$ in a $\mathbb{Z}_2$-graded correspondence has to be equivariant for the given grading on A and this grading on $\mathcal{L}_B(\mathcal{E})$. In what follows we want to suppress the grading in our notation and just keep in mind that everything in sight will be $\mathbb{Z}_2$ graded. In most cases (except for Clifford algebras), we shall consider the trivial grading $\epsilon_A = \mathrm{id}_A$ for our C^*-algebras A, but we shall usually have nontrivial gradings on our Hilbert modules.

3.3.2 Kasparov's bivariant K-groups

In this section we give the definition of Kasparovs's G-equivariant bivariant K-groups. We refer to [Kas88] for the details (but see also [Bla86, Ska84]). We start with the definition of the underlying KK-cycles:

Definition 3.3.2. Suppose that (A, α) and (B, β) are $\mathbb{Z}_2$-graded G-C^*-algebras. A *G-equivariant A-B Kasparov cycle* is a quadruple $(\mathcal{E}, u, \Phi, T)$ in which $(\mathcal{E}, u, \Phi)$ is a $\mathbb{Z}_2$-graded (A, α)-(B, β) correspondence and $T \in \mathcal{L}_B(\mathcal{E})$ is a homogeneous element with $\deg(T) = 1$ such that

(i) $g \mapsto \operatorname{Ad} u_g(\Phi(a)T); G \to \mathcal{L}_B(\mathcal{E})$ is continuous for all $a \in A$;

(ii) for all $a \in A$ and $g \in G$ we have

$$(T - T^*)\Phi(a),\ (T^2 - 1)\Phi(a),\ (\operatorname{Ad} u_g(T) - T)\Phi(a),\ [T, \Phi(a)] \in \mathcal{K}(\mathcal{E})$$

(where $[\cdot, \cdot]$ denotes the graded commutator). Two Kasparov cycles $(\mathcal{E}, u, \Phi, T)$ and $(\mathcal{E}', u', \Phi', T')$ are called *isomorphic*, if there exists an isomorphism $W : \mathcal{E} \to \mathcal{E}'$ of the correspondences $(\mathcal{E}, u, \Phi)$ and $(\mathcal{E}', u', \Phi')$ such that $T' = W \circ T \circ W^{-1}$. A Kasparov cycle is called *degenerate* if

$$(T - T^*)\Phi(a),\ (T^2 - 1)\Phi(a),\ (\operatorname{Ad} u_g(T) - T)\Phi(a),\ [T, \Phi(a)] = 0$$

for all $a \in A$ and $g \in G$. We write $\mathbb{E}_G(A, B)$ for the set of isomorphism classes of all G-equivariant A-B Kasparov cycles and we write $\mathbb{D}_G(A, B)$ for the equivalence classes of degenerate Kasparov cycles.

Example 3.3.3. Every G-equivariant $*$-homomorphism $\Phi : A \to B$ determines a G-equivariant A-B Kasparov cycle $(B, \beta, \Phi, 0)$, where B is considered as a Hilbert

B-module in the obvious way. More generally, if $(\mathcal{E}, u, \Phi)$ is any $\mathbb{Z}_2$ graded (A, α)-(B, β) correspondence such that $\Phi(A) \subseteq \mathcal{K}(\mathcal{E})$ (i.e., $(\mathcal{E}, u, \Phi)$ is a morphism in the compact correspondence category in the sense of Definition 2.5.7), then it is an easy exercise to check that $(\mathcal{E}, u, \Phi, 0)$ is a G-equivariant A-B Kasparov cycle as well. Note that the condition $(T^2 - 1)\Phi(a) \in \mathcal{K}(\mathcal{E})$ for a Kasparov cycle implies that conversely, if $(\mathcal{E}, u, \Phi, 0)$ is a Kasparov cycle, then $\Phi(A) \subseteq \mathcal{K}(\mathcal{E})$. A special situation of the above is the case in which $A = B$ and $\Phi = \mathrm{id}_B$ which gives us the Kasparov cycle $(B, \beta, \mathrm{id}_B, 0)$. It will play an important role when looking at Kasparov products below.

In what follows, if (B, β) is a $\mathbb{Z}_2$-graded G-C^*-algebra, then we denote by $B[0,1]$ the algebra $C([0,1], B)$ with pointwise G-action and grading. Suppose now that $(\mathcal{E}, u, \Phi, T)$ is a G-equivariant A-$B[0,1]$ Kasparov cycle. For each $t \in [0,1]$ let $\delta_t : B[0,1] \to B; \delta_t(f) = f(t)$ be evaluation at t. Then we obtain a G-equivariant A-B Kasparov cycle $(\mathcal{E}_t, u_t, \Phi_t, T_t)$ by putting

$$\mathcal{E}_t = \mathcal{E} \otimes_{B[0,1],\delta_t} B,\ u_t = u \otimes \beta,\ \Phi_t = \Phi \otimes 1, \text{ and } T_t = T \otimes 1.$$

Alternatively, we could define a B-valued inner product on $\mathcal{E}$ by

$$\langle e, f \rangle_B := \langle e, f \rangle_{B[0,1]}(t)$$

which factors through $\mathcal{E}_t := \mathcal{E}/(\mathcal{E} \cdot \ker \delta_t)$. Then u, Φ, T factor uniquely through some action u_t of G, a $*$-homomorphism $\Phi_t : A \to \mathcal{L}_B(\mathcal{E}_t)$ and an operator $T_t \in \mathcal{L}_B(\mathcal{E}_t)$ such that $(\mathcal{E}_t, u_t, \Phi_t, T_t)$ is a G-equivariant A-B Kasparov cycle. It is isomorphic to the one constructed above. We call $(\mathcal{E}_t, u_t, \Phi_t, T_t)$ the *evaluation* of $(\mathcal{E}, u, \Phi, T)$ at $t \in [0,1]$.

Definition 3.3.4 (Homotopy)**.** Two Kasparov cycles $(\mathcal{E}_0, u_0, \Phi_0, T_0)$ and $(\mathcal{E}_1, u_1, \Phi_1, T_1)$ in $\mathbb{E}_G(A, B)$ are said to be *homotopic* if there exists a G-equivariant A-$B[0,1]$ Kasparov cycle $(\mathcal{E}, u, \Phi, T)$ such that $(\mathcal{E}_0, u_0, \Phi_0, T_0)$ is isomorphic to the evaluation of $(\mathcal{E}, u, \Phi, T)$ at 0 and $(\mathcal{E}_1, u_1, \Phi_1, T_1)$ is isomorphic to the evaluation of $(\mathcal{E}, u, \Phi, T)$ at 1. We then write $(\mathcal{E}_0, u_0, \Phi_0, T_0) \sim_h (\mathcal{E}_1, u_1, \Phi_1, T_1)$. We define

$$KK^G(A, B) := \mathbb{E}_G(A, B)/\sim_h .$$

Remark 3.3.5. Every degenerate Kasparov cycle $(\mathcal{E}, u, \Phi, T)$ is homotopic to the zero-cycle $(0,0,0,0)$. To see this, consider the quadruple $(\mathcal{E} \otimes C_0([0,1)), u \otimes \mathrm{id}, \Phi \otimes 1, T \otimes 1)$ where we view $\mathcal{E} \otimes C_0([0,1)) \cong C_0([0,1), \mathcal{E})$ as a $B[0,1]$-Hilbert module in the obvious way. It follows from degeneracy of $(\mathcal{E}, u, \Phi, T)$ that $(\mathcal{E} \otimes C_0([0,1)), u \otimes \mathrm{id}, \Phi \otimes 1, T \otimes 1)$ is an A-$B[0,1]$ Kasparov cycle and it is straightforward to check that its evaluation at 0 coincides with $(\mathcal{E}, u, \Phi, T)$ while its evaluation at 1 is the zero-cycle.

Remark 3.3.6. A special kind of homotopy is the *operator homotopy* which is defined as follows: Assume that $(\mathcal{E}, u, \Phi, T_0)$ and $(\mathcal{E}, u, \Phi, T_1)$ are two A-B Kasparov cycles such that the underlying correspondence $(\mathcal{E}, u, \Phi)$ coincides for both cycles.

An operator homotopy between $(\mathcal{E}, u, \Phi, T_0)$ and $(\mathcal{E}, u, \Phi, T_1)$ is a family of A-B Kasparov cycles $(\mathcal{E}, u, \Phi, T_t)$, $t \in [0,1]$, such that the path of operators $(T_t)_{t\in[0,1]}$ is norm continuous and connects T_0 with T_1. Such operator homotopy determines a homotopy between $(\mathcal{E}, u, \Phi, T_0)$ and $(\mathcal{E}, u, \Phi, T_1)$ in which the A-$B[0,1]$ Kasparov cycle is given by $(\mathcal{E} \otimes C[0,1], u \otimes \mathrm{id}, \Phi \otimes 1, \tilde{T})$ with $\big(\tilde{T}(e)\big)(t) = T_t e(t)$ for $e \in \mathcal{E} \otimes C[0,1] = C([0,1], \mathcal{E})$.

Example 3.3.7. Assume that $(\mathcal{E}, u, \Phi, T_0)$ and $(\mathcal{E}, u, \Phi, T_1)$ are two A-B Kasparov cycles. We then say that T_1 is a *compact perturbation* of T_0 if $(T_1 - T_0)\Phi(a) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$. In this case, the path $T_t = (1-t)T_0 + tT_1$ gives an operator homotopy between T_0 and T_1, so both Kasparov cycles are homotopic.

The following observation is due to Skandalis (see [Ska84]):

Proposition 3.3.8. *The equivalence relation $\sim_h$ on $\mathbb{E}_G(A, B)$ coincides with the equivalence relation generated by operator homotopy together with adding degenerate Kasparov cycles.*

Theorem 3.3.9 (Kasparov)**.** *$KK^G(A, B)$ is an abelian group with addition defined by direct sum of Kasparov cycles:*

$$[\mathcal{E}_1, u_1, \Phi_1, T_1] + [\mathcal{E}_2, u_2, \Phi_2, T_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2, u_1 \oplus u_2, \Phi_1 \oplus \Phi_2, T_1 \oplus T_2],$$

where $[\mathcal{E}, u, \Phi, T]$ denotes the homotopy class of the Kasparov cycle $(\mathcal{E}, u, \Phi, T)$. The inverse of a class $[\mathcal{E}, u, \Phi, T]$ is given by the class $[\mathcal{E}^{\mathrm{op}}, u, \Phi \circ \epsilon_A, -T]$ in which $\mathcal{E}^{\mathrm{op}}$ denotes the module $\mathcal{E}$ with the opposite grading $\epsilon_{\mathcal{E}^{\mathrm{op}}} = -\epsilon_{\mathcal{E}}$.

Functoriality. Every G-equivariant $*$-homomorphism $\Psi : A_1 \to A_2$ induces a group homomorphism

$$\Psi^* : KK^G(A_2, B) \to KK^G(A_1, B)\ ;\ [\mathcal{E}, u, \Phi, T] \mapsto [\mathcal{E}, u, \Phi \circ \Psi, T]$$

and every G-equivariant $*$-homomorphism $\Psi : B_1 \to B_2$ induces a group homomorphism

$$\Psi_* : KK^G(A, B_1) \to KK^G(A, B_2)\ ;\ [\mathcal{E}, u, \Phi, T] \mapsto [\mathcal{E} \otimes_{B_1} B_2, u \otimes \beta_2, \Phi \otimes 1, T \otimes 1].$$

Hence, KK^G is contravariant in the first variable and covariant in the second variable.

Direct sums. If $A = \bigoplus_{i=1}^{l} A_i$ is a finite direct sum, then $KK^G(B, A) \cong \bigoplus_{i=1}^{l} KK^G(B, A_i)$. The formula does not hold in general for (countable) infinite direct sums (see [Bla86, 19.7.2]). On the other side, if $A = \bigoplus_{i\in I} A_i$ is a countable direct sum of G-C^*-algebras, then $KK^G(A, B) \cong \prod_{i\in I} KK^G(A_i, B)$ for every G-C^*-algebra B. We leave it to the reader to construct these isomorphisms.

The ordinary K-theory groups. Recall that for a trivially graded unital C^*-algebra B the ordinary K_0-group can be defined as the Grothendieck group generated by the semigroup of all homotopy classes $[p]$ of projections $p \in M_\infty(B) = \cup_{n\in\mathbb{N}} M_n(B)$ with direct sum $[p] + [q] = [p \oplus q]$ as addition. If $p \in M_n(B)$, then p determines a $*$-homomorphism $\Phi_p : \mathbb{C} \to M_n(B) \cong \mathcal{K}(B^n); \lambda \mapsto \lambda p$, and hence a class $[B^n, \Phi_p, 0] \in KK_0(\mathbb{C}, B)$. Note that all elements of the module B^n are homogeneous of degree 0. This construction preserves homotopy and direct sums and therefore induces a homomorphism of $K_0(B)$ into $KK(\mathbb{C}, B)$, which, as shown by Kasparov, is actually an isomorphism of abelian groups. Thus we get

$$KK(\mathbb{C}, B) \cong K_0(B).$$

If B is not unital, we may first apply the above to the unitisation B^1 and for $\mathbb{C}$ and then use split-exactness to get the general case. If $\Phi : A \to B$ is a $*$-homomorphism, then the isomorphisms $KK(\mathbb{C}, A) \cong K_0(A)$ and $KK(\mathbb{C}, B) \cong K_0(B)$ intertwines the induced homomorphism $\Phi_* : KK(\mathbb{C}, A) \to KK(\mathbb{C}, B)$ in KK-theory with the morphism of K-theory groups sending $[p]$ to $[\Phi(p)]$.

If we put the complex numbers $\mathbb{C}$ into the second variable, we obtain Kasparov's operator theoretic K-homology functor $K^0(A) := KK(A, \mathbb{C})$.

The group $KK(\mathbb{C}, \mathbb{C})$. Each element in $KK(\mathbb{C}, \mathbb{C})$ can be represented by a Kasparov cycle of the form $(\mathcal{H}, \mathbf{1}, T)$ in which $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a graded Hilbert space, $\mathbf{1} : \mathbb{C} \to \mathcal{L}(\mathcal{H})$ is the action $\mathbf{1}(\lambda)\xi = \lambda\xi$ and T is a self-adjoint operator satisfying $T^2 - 1 \in \mathcal{K}(H)$. This follows from the standard simplifications as described in detail in [Bla86, §17.4]. If $T = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}$, then $T^2 = \begin{pmatrix} P^*P & 0 \\ 0 & PP^* \end{pmatrix}$ and the condition $T^2 - 1 \in \mathcal{K}(H)$ then implies that P is a Fredholm operator. We then obtain a well-defined map

$$\mathrm{index} : KK(\mathbb{C}, \mathbb{C}) \to \mathbb{Z}; [\mathcal{H}, \mathbf{1}, T] \mapsto \mathrm{index}(\mathcal{H}, \mathbf{1}, T) := \mathrm{index}(P),$$

where $\mathrm{index}(P) = \dim(\ker(P)) - \dim(\ker(P^*))$ denotes the Fredholmindex of P. The index map induces the isomorphism $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z} = K_0(\mathbb{C})$. (Compare this with the above isomorphism $KK(\mathbb{C}, B) \cong K_0(B)$ in the case $B = \mathbb{C}$.)

3.3.3 The Kasparov product

We are now coming to the main feature of Kasparov's KK-theory, namely, the Kasparov product, which is a pairing

$$KK^G(A, B) \times KK^G(B, C) \to KK^G(A, C),$$

where A, B, C are G-C^*-algebras. Starting with an A-B Kasparov cycle $(\mathcal{E}, u, \Phi, T)$ and a B-C Kasparov cycle $(\mathcal{F}, v, \Psi, S)$, the Kasparov product will be represented by a Kasparov cycle of the form $(\mathcal{E} \otimes_B \mathcal{F}, u \otimes v, \Phi \otimes 1, R)$, where all the ingredients with the exception of the operator R are well-known objects by now: $\mathcal{E} \otimes_B \mathcal{F}$ denotes the internal tensor product of $\mathcal{E}$ with $\mathcal{F}$ over B (with diagonal grading),

$u\otimes v : G \to \mathrm{Aut}(\mathcal{E}\otimes_B \mathcal{F})$ denotes the diagonal action, and $\Phi\otimes 1 : A \to \mathcal{L}(\mathcal{E}\otimes_B \mathcal{F})$ sends $a \in A$ to the operator $\Phi(a)\otimes 1$ of $\mathcal{E}\otimes_B \mathcal{F}$. But the construction of the operator R is, unfortunately, quite complicated and reflects the high complexity of Kasparov's theory.

As a first attempt one would look at the operator

$$R = T\otimes 1 + 1\otimes S.$$

But there are several problems with this choice. First of all, the operator $1\otimes S$ on the internal tensor product is not well defined in general (it only makes sense, if S commutes with $\Psi(B)\subseteq \mathcal{L}(\mathcal{F})$). To resolve this, we need to replace $1\otimes S$ with a so-called S-*connection*, which we shall explain below. But even if $1\otimes S$ is well-defined, the triple $(\mathcal{E}\otimes_B \mathcal{F}, u\otimes v, \Phi\otimes 1, T\otimes 1 + 1\otimes S)$ will usually fail to be a Kasparov triple unless $S = 0$, and one needs to replace $T\otimes 1 + 1\otimes S$ by a combination

$$M^{1/2}(T\otimes 1) + N^{1/2}(1\otimes S)$$

where $M, N \geq 1$ are suitable operators with $M + N = 1$ which can be obtained by an application of Kasparov's technical theorem [Kas88, Theorem 1.4].

S-connection. For any $\xi \in \mathcal{E}$ define

$$\theta_\xi : \mathcal{F} \to \mathcal{E}\otimes_B \mathcal{F}; \theta_\xi(\eta) = \xi\otimes\eta.$$

Then $\theta_\xi \in \mathcal{K}(\mathcal{F}, \mathcal{F}\otimes_B \mathcal{E})$ with adjoint given by

$$\theta_\xi^*(\zeta\otimes\eta) = \Psi(\langle\xi,\zeta\rangle_B)\eta.$$

An operator $S_{12} \in \mathcal{L}(\mathcal{E}\otimes_B \mathcal{F})$ is then called an S-connection, if for all homogeneous elements $\xi \in \mathcal{E}$ we have

$$\theta_\xi S - (-1)^{\deg(\xi)\deg S} S_{12}\theta_\xi,\;\; \theta_\xi S^* - (-1)^{\deg(\xi)\deg S} S_{12}^*\theta_\xi \in \mathcal{K}(\mathcal{F}, \mathcal{E}\otimes_B \mathcal{F}) \quad (3.3.1)$$

It is a good exercise to check that if S commutes with $\Psi(B)\subseteq \mathcal{L}(\mathcal{F})$, then $S_{12} = 1\otimes S$ makes sense and is an S-connection in the above sense.

Definition 3.3.10 (Kasparov product)**.** Suppose that A, B, C are G-C^*-algebras, and that $(\mathcal{E}, u, \Phi, T)$ is an A-B Kasparov cycle and $(\mathcal{F}, v, \Psi, S)$ is a B-C Kasparov cycle. Let $S_{12} \in \mathcal{L}(\mathcal{E}\otimes_B \mathcal{F})$ be an S-connection as explained above. Then the quadruple $(\mathcal{E}\otimes_B \mathcal{F}, u\otimes v, \Phi\otimes 1, S_{12})$ will be a *Kasparov product* for $(\mathcal{E}, u, \Phi, T)$ and $(\mathcal{F}, v, \Psi, S)$ if the following two conditions hold:

(i) $(\mathcal{E}\otimes_B \mathcal{F}, u\otimes v, \Phi\otimes 1, S_{12})$ is an A-C Kasparov cycle.

(ii) For all $a \in A$ we have $(\Phi(a)\otimes 1)\,[T\otimes 1, S_{12}]\,(\Phi(a^*)\otimes 1) \geq 0$ mod. $\mathcal{K}(\mathcal{E}\otimes_B \mathcal{F})$.

In this case the class $[\mathcal{E}\otimes_B \mathcal{F}, u\otimes v, \Phi\otimes 1, S_{12}] \in KK^G(A, C)$ is called a Kasparov product of $[\mathcal{E}, u, \Phi, T] \in KK^G(A, B)$ with $[\mathcal{F}, v, \Psi, S] \in KK^G(B, C)$.

We should note that the existence of an S-connection S_{12} which satisfies the conditions of the above definition follows from an application of Kasparov's technical theorem [Kas88, Theorem 14]. The proof is quite technical and we refer to the literature (see one of the references [Ska84, Kas88, Bla86]).

Remark 3.3.11. **(a)** One can show that the operator S_{12} in a Kasparov product is unique up to operator homotopy.

(b) The following easy case is often very useful: Suppose that B acts on $\mathcal{F}$ by compact operators, i.e., $\Psi : B \to \mathcal{L}(\mathcal{F})$ takes its values in $\mathcal{K}(\mathcal{F})$. Then $(\mathcal{F}, v, \Psi, 0)$ is a B-C Kasparov cycle and the 0-operator on $\mathcal{E} \otimes_B \mathcal{F}$ is then clearly a 0-connection. Now if $\Phi : A \to \mathcal{L}(\mathcal{E})$ also takes values in $\mathcal{K}(\mathcal{E})$ and if $T = 0$, it follows that $[\mathcal{E} \otimes_B \mathcal{F}, u \otimes v, \Phi \otimes 1, 0]$ is a Kasparov product for $[\mathcal{E}, u, \Phi, 0]$ and $[\mathcal{F}, v, \Psi, 0]$. We therefore obtain a functor from the compact correspondence category $\mathfrak{Corr}_c(G)$ (see Section 2.5.3; here we use countably generated $\mathbb{Z}_2$-graded Hilbert modules) into KK^G given by $(\mathcal{E}, u, \Phi) \mapsto [\mathcal{E}, u, \Phi, 0]$ which preserves multiplication.

The details of the following theorem can be found in [Bla86] or [Kas88].

Theorem 3.3.12 (Kasparov). *Suppose that A, B, C are separable G-C^*-algebras and let $x = [\mathcal{E}, u, \Phi, T] \in KK^G(A, B)$ and $y = [\mathcal{F}, v, \Psi, S] \in KK^G(B, C)$. Then the Kasparov product*

$$x \otimes_B y := [\mathcal{E} \otimes_B \mathcal{F}, u \otimes v, \Phi \otimes 1, S_{12}]$$

exists and induces a well-defined bilinear pairing

$$\otimes_B : KK^G(A, B) \times KK^G(B, C) \to KK^G(A, C).$$

Moreover, the Kasparov product is associative: If D is another G-C^*-algebra and $z \in KK^G(C, D)$, then we have

$$(x \otimes_B y) \otimes_C z = x \otimes_B (y \otimes_C z) \in KK^G(A, D).$$

The elements $1_A = [A, \alpha, \mathrm{id}_A, 0] \in KK^G(A, A)$ and $1_B = [B, \beta, \mathrm{id}_B, 0] \in KK^G(B, B)$ act as identities from the left and right on $KK^G(A, B)$, i.e., we have

$$1_A \otimes_A x = x = x \otimes_B 1_B \in KK^G(A, B)$$

for all $x \in KK^G(A, B)$. In particular, $KK^G(A, A)$ equipped with the Kasparov product is a unital ring.

The following result is helpful for the computation of Kasparov products in some important special cases. For the proof we refer to [Bla86, 8.10.1].

Proposition 3.3.13. *Suppose that A, B, C are G-C^*-algebras, $(\mathcal{E}, u, \Phi, T)$ is an A-B Kasparov cycle and $(\mathcal{F}, v, \Psi, S)$ is a B-C Kasparov cycle. Suppose further that $T = T^*$ and $\|T\| \leq 1$. Let $S_{12} \in \mathcal{L}(\mathcal{E} \otimes_B \mathcal{F})$ be a G-invariant S-connection of degree one and let*

$$R := T \otimes 1 + (\sqrt{1 - T^2} \otimes 1) S_{12}.$$

If $[R, \Phi(A) \otimes 1] \in \mathcal{K}(\mathcal{E} \otimes_B \mathcal{F})$, *then* $(\mathcal{E} \otimes_B \mathcal{F}, u \otimes v, \Phi \otimes 1, R) \in \mathbb{E}^G(A, C)$ *and represents the Kasparov product of* $[\mathcal{E}, u, \Phi, T]$ *with* $[\mathcal{F}, v, \Psi, S]$

We should note that the conditions $T = T^*$ and $\|T\| \leq 1$ can always be fulfilled by choosing an appropriate Kasparov cycle representing the given class $x \in KK^G(A, B)$.

Associativity and the existence of neutral elements for the Kasparov product gives rise to an easy notion of KK^G-equivalence for two G-C^*-algebras A and B: Assume that there are elements $x \in KK^G(A, B)$ and $y \in KK^G(B, A)$ such that

$$x \otimes_B y = 1_A \in KK^G(A, A) \quad \text{and} \quad y \otimes_A x = 1_B \in KK^G(B, B).$$

Then taking products with x from the left induces an isomorphism

$$x \otimes_B \cdot : KK^G(B, C) \to KK^G(A, C); z \mapsto x \otimes_B z$$

with inverse given by

$$y \otimes_A \cdot : KK^G(A, C) \to KK^G(B, C); w \mapsto y \otimes_A w.$$

This follows from the simple identities

$$y \otimes_A (x \otimes_B z) = (y \otimes_A x) \otimes_B z = 1_B \otimes_B z = z$$

for all $z \in KK^G(B, C)$ and similarly we have $x \otimes_B (y \otimes_A w) = w$ for all $w \in KK^G(A, C)$. Of course, taking products from the right by x and y will give us an isomorphism $\cdot \otimes_A x : KK^G(C, A) \to KK^G(C, B)$ with inverse $\cdot \otimes_B y : KK^G(C, B) \to KK^G(C, A)$.

Definition 3.3.14. Suppose that $x \in KK^G(A, B)$ and $y \in KK^G(B, A)$ are as above. Then we say that x is a *KK^G-equivalence from A to B with inverse y*. Two G-C^*-algebras A and B are called *KK^G-equivalent*, if such elements x and y exist.

Lemma 3.3.15. *Suppose that* $(\mathcal{E}, u, \Phi)$ *is a G-equivariant A-B Morita equivalence for the G-C^*-algebras* (A, α) *and* (B, β) *and let* $(\mathcal{E}^*, u^*, \Phi^*)$ *denote its inverse. Then* $x = [\mathcal{E}, u, \Phi, 0] \in KK^G(A, B)$ *is a KK^G-equivalence with inverse* $y = [\mathcal{E}^*, u^*, \Phi^*, 0]$.

Proof. It follows from Remark 3.3.11 that the Kasparov product $x \otimes_B y$ is represented by the Kasparov cycle $[\mathcal{E} \otimes_B \mathcal{E}^*, u \otimes u^*, \Phi \otimes 1, 0]$. But since the correspondence $(\mathcal{E} \otimes_B \mathcal{E}^*, u \otimes u^*, \Phi \otimes 1)$ is isomorphic to $(A, \alpha, \mathrm{id}_A)$, it follows that $x \otimes_B y = 1_A$. Similarly, we have $y \otimes_B x = 1_B$. □

Remark 3.3.16. If $p \in M(A)$ is a full projection in a C^*-algebra A, then pA is a pAp-A Morita equivalence and hence, if $\varphi : pAp \to \mathcal{L}(pA)$ denotes the canonical morphism, the element $[pA, \varphi, 0] \in KK(pAp, A)$ is a KK-equivalence (it is a KK^G-equivalence if A is a G-C^*-algebra and $p \in M(A)$ is G-invariant). On the other hand, if $\psi : pAp \to A$ denotes the canonical inclusion, then $[A, \psi, 0]$ determines the class of the $*$-homomorphism ψ. Both classes actually coincide, which follows from the simple fact that we can decompose the KK-cycle $(A, \psi, 0)$ as the direct sum $(pA, \varphi, 0) \oplus ((1-p)A, 0, 0)$ where the second summand is degenerate. In particular, it follows from this that $\psi_* : K_*(pAp) \to K_*(A)$ is an isomorphism.

We now describe a more general version of the Kasparov product. For this we first need to introduce a homomorphism

$$\cdot\hat{\otimes}1_D : KK^G(A, B) \to KK^G(A\hat{\otimes}D, B\hat{\otimes}D)$$

which is defined for any G-C^*-algebra (D, δ) by

$$[\mathcal{E}, u, \Phi, T]\hat{\otimes}1_D := [\mathcal{E}\hat{\otimes}D, u\hat{\otimes}\delta, \Phi\hat{\otimes}1, T\hat{\otimes}1].$$

One can check that $\cdot\hat{\otimes}1_D$ is compatible with Kasparov products in the sense that

$$(x \otimes_B y)\hat{\otimes}1_D = (x\hat{\otimes}1_D) \otimes_{B\hat{\otimes}D} (y\hat{\otimes}1_D)$$

and it follows directly from the definition that $1_A\hat{\otimes}1_D = 1_{A\hat{\otimes}D}$. In particular, it follows that $\cdot\hat{\otimes}1_D$ sends KK^G-equivalences to KK^G-equivalences. Of course, in a similar way we can define a homomorphism

$$1_D \otimes \cdot : KK^G(A, B) \to KK^G(D\hat{\otimes}A, D\hat{\otimes}B).$$

Remark 3.3.17. By our conventions "$\hat{\otimes}$" denotes the minimal graded tensor product of the C^*-algebras A and B. But a similar map $\cdot\hat{\otimes}_{\max}D : KK^G(A, B) \to KK^G(A\hat{\otimes}_{\max}D, B\hat{\otimes}_{\max}D)$ exists for the maximal graded tensor product.

Theorem 3.3.18 (Generalized Kasparov product)**.** *Suppose that (A_1, α_1), (A_2, α_2), (B_1, β_1), (B_2, β_2) and (D, δ) are G-C^*-algebras. Then there is a pairing*

$$\otimes_D : KK^G(A_1, B_1\hat{\otimes}D) \times KK^G(D\hat{\otimes}A_2, B_2) \to KK^G(A_1\hat{\otimes}A_2, B_1\hat{\otimes}B_2)$$

given by

$$(x, y) \mapsto x \otimes_D y := (x\hat{\otimes}1_{A_2}) \otimes_{B_1\hat{\otimes}D\hat{\otimes}A_2} (1_{B_1}\hat{\otimes}y).$$

This pairing is associative (in a suitable sense) and coincides with the ordinary Kasparov product if $B_1 = \mathbb{C} = A_2$. Moreover, in the case $D = \mathbb{C}$, the product

$$\otimes_{\mathbb{C}} : KK^G(A_1, B_1) \times KK^G(A_2, B_2) \to KK^G(A_1\hat{\otimes}A_2, B_1\hat{\otimes}B_2)$$

is commutative.

Note that there are several other important properties of the generalized Kasparov product, which we do not state here. We refer to [Kas88, Theorem 2.14] for the complete list and their proofs. We close this section with a useful description of Kasparov cycles in terms of unbounded operators due to Baaj and Julg (see [BJ83]). For our purposes it suffices to restrict to the case where $B = \mathbb{C}$, in which case we may rely on the classical theory of unbounded operators on Hilbert spaces. But the picture extends to the general case using a suitable theory of regular unbounded operators on Hilbert modules.

Lemma 3.3.19 (Baaj-Julg)**.** *Suppose that A is a graded C^*-algebra and $\Phi : A \to \mathcal{B}(\mathcal{H})$ is a graded $*$-representation of A on the graded separable Hilbert space $\mathcal{H}$. Suppose further that $D = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix}$ is an unbounded self-adjoint operator on $\mathcal{H}$ of degree one such that*

(i) $(1 + D^2)^{-1}\Phi(a) \in \mathcal{K}(\mathcal{H})$ *for all $a \in A$, and*

(ii) *the set of all $a \in A$ such that $\Phi(a)$ maps the domain* $\mathrm{Dom}(D)$ *of D into itself and such that $[D, \Phi(a)]$ is bounded on* $\mathrm{Dom}(D)$ *is dense in A.*

Then $(\mathcal{H}, \Phi, T = \frac{D}{\sqrt{1+D^2}})$ is an A-$\mathbb{C}$ Kasparov cycle.

For the proof of this lemma, even in the more general context of A-B Kasparov cycles, we refer to [BJ83, Proposition 2.2][1]. Note that the operator $T = \frac{D}{\sqrt{1+D^2}}$ is constructed via functional calculus for unbounded self-adjoint operators.

3.3.4 Higher KK-groups and Bott-periodicity

Definition 3.3.20. Suppose that (A, α) and (B, β) are G-C^*-algebras. For each $n \in \mathbb{N}_0$ we define the (higher) KK^G-group as

$$KK^G_n(A, B) := KK^G(A, B \hat{\otimes}\, Cl_n) \quad \text{and} \quad KK^G_{-n}(A, B) := KK^G(A \hat{\otimes}\, Cl_n, B),$$

where Cl_n denotes the nth complex Clifford algebra with trivial G-action and grading as defined in Section 3.3.

With this definition of higher KK-groups it is easy to prove a (formal) version of Bott-periodicity. We need the following easy lemma:

Lemma 3.3.21. *If $n \in \mathbb{N}$ is even, then Cl_n is Morita equivalent to $\mathbb{C}$ as graded C^*-algebras. If n is odd, then Cl_n is Morita equivalent to Cl_1 as graded C^*-algebras.*

Proof. Let $n \in \mathbb{N}_0$. We know from Section 3.3 that $Cl_{2n} \cong M_{2^n}(\mathbb{C})$ with grading given by cunjugation with a symmetry $J \in M_{2^n}(\mathbb{C})$. It is then easy to check that the Hilbert space $\mathbb{C}^{2^n}$ equipped with the grading operator J and the canonical left action of Cl_n on $\mathbb{C}^{2^n}$ gives the desired Morita equivalence. Similarly, we have $Cl_{2n+1} \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}) \cong M_{2^n} \hat{\otimes}\, Cl_1$ as graded C^*-algebras, which is Morita equivalent to $\mathbb{C} \hat{\otimes}\, Cl_1 = Cl_1$. □

Notations 3.3.22. In what follows we shall denote by $x_{2n} \in KK(Cl_{2n}, \mathbb{C})$ the (invertible) class of the Morita equivalence between Cl_{2n} and $\mathbb{C}$ as in the above lemma and by $x_{2n+1} \in KK(Cl_{2n+1}, Cl_1)$ the class of the Morita equivalence between Cl_{2n+1} and Cl_1.

[1]We are grateful to Adam Rennie for pointing out to us that the weaker formulation of the above result as stated in [Bla86, Proposition 17.11.3] is not correct. We refer to [FMR14] for a discussion and for counterexamples.

Proposition 3.3.23 (Formal Bott-periodicity). *For each $n \in \mathbb{N}_0$ there are canonical isomorphisms*

$$KK^G(A\hat{\otimes}\, Cl_{2n}, B) \cong KK_0^G(A, B) \cong KK^G(A, B\hat{\otimes}\, Cl_{2n})$$

and

$$\begin{aligned} KK^G(A\hat{\otimes}\, Cl_{2n+1}, B) &\cong KK^G(A\hat{\otimes}\, Cl_1, B) \\ &\cong KK^G(A, B\hat{\otimes}\, Cl_1) \cong KK^G(A, B\hat{\otimes}\, Cl_{2n+1}). \end{aligned}$$

As a consequence, we have $KK_l^G(A, B) \cong KK_{l+2}^G(A, B)$ for all $l \in \mathbb{Z}$.

Proof. In the even case, the isomorphisms follow by taking Kasparov products with the KK-equivalences $1_A \otimes x_{2n}^{-1} \in KK^G(A, A\hat{\otimes}\, Cl_{2n})$ and $1_B \otimes x_{2n}^{-1} \in KK^G(B, B\hat{\otimes}\, Cl_{2n})$ from the left and right, respectively (where we consider the trivial action of G on the Clifford algebras). The same argument will provide isomorphisms $KK^G(A\hat{\otimes}\, Cl_{2n+1}, B) \cong KK^G(A\hat{\otimes}\, Cl_1, B)$ and $KK^G(A, B\hat{\otimes}\, Cl_1) \cong KK^G(A, B\hat{\otimes}\, Cl_{2n+1})$, respectively.

To finish off, one checks that the composition

$$KK^G(A\hat{\otimes}\, Cl_1, B) \overset{y\mapsto y\otimes 1_{Cl_1}}{\longrightarrow} KK^G(A\hat{\otimes}\, Cl_1 \hat{\otimes}\, Cl_1, B\hat{\otimes}\, Cl_1) \overset{(1_A\otimes x_2^{-1})\otimes\cdot}{\cong} KK^G(A, B\hat{\otimes}\, Cl_1)$$

is an isomorphism with the inverse given by the composition

$$KK^G(A, B\hat{\otimes}\, Cl_1) \overset{y\mapsto y\otimes 1_{Cl_1}}{\longrightarrow} KK^G(A\hat{\otimes}\, Cl_1, B\hat{\otimes}\, Cl_1 \hat{\otimes}\, Cl_1) \overset{\cdot\otimes(1_B\otimes x_2)}{\cong} KK^G(A\hat{\otimes}\, Cl_1, B),$$

where we use that $Cl_1 \hat{\otimes}\, Cl_1 \cong Cl_2 \sim_M \mathbb{C}$. □

Of course, we would like to have a version of Bott-periodicity showing that, alternatively, we could define the higher KK-groups via suspension. For this we are going to construct a KK-equivalence between Cl_1 and $C_0(\mathbb{R})$. Indeed, we shall do this by first constructing a KK-equivalence between $C_0(\mathbb{R})\hat{\otimes}\, Cl_1 \cong C_0(\mathbb{R}, Cl_1)$ with $\mathbb{C}$. Since we consider the trivial G-action on $C_0(\mathbb{R})$ and Cl_1 it suffices to do this for the trivial group $G = \{e\}$. In what follows next we write $\mathcal{D} := C_0(\mathbb{R})\hat{\otimes}\, Cl_1$ and $\mathcal{D}_0 := C_c^\infty(\mathbb{R})\hat{\otimes}\, Cl_1 \subseteq \mathcal{D}$, where $C_c^\infty(\mathbb{R})$ denotes the dense subalgebra of $C_0(\mathbb{R})$ consisting of smooth functions with compact supports. A typical element of $\mathcal{D}$ can be written as $f_1 + f_2 e$ with $f_1, f_2 \in C_0(\mathbb{R})$, where we identify f_1 with $f_1 1_{Cl_1}$ and where $e = e_1$ denotes the generator of Cl_1 with $e^2 = 1$.

The Dirac element. We define an element

$$\alpha = [\mathcal{H}, \Phi, T] \in KK(\mathcal{D}, \mathbb{C})$$

as follows: We let $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ be equipped with the grading induced by the operator $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We define $T = \frac{D}{\sqrt{1+D^2}}$ with $D = \begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix}$, where

$d : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the densely defined operator $d = \frac{d}{dt}$. Then D is an essentially self-adjoint operator on the dense subspace $C_c^\infty(\mathbb{R}) \oplus C_c^\infty(\mathbb{R})$ of $\mathcal{H}$ and therefore extends to a densely defined self-adjoint operator on $\mathcal{H}$ (we refer to [HK01, Chapter 10] for details). Let $\Phi : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ be given by

$$\Phi(f_1 + f_2 e) \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} f_1 \cdot \xi_1 + f_2 \cdot \xi_2 \\ f_2 \cdot \xi_1 + f_1 \cdot \xi_2 \end{pmatrix}. \tag{3.3.2}$$

In order to check that $(\mathcal{H}, \Phi, T)$ is a $\mathcal{D}$-$\mathbb{C}$ Kasparov cycle we need to check the conditions of Lemma 3.3.19 for the triple $(\mathcal{H}, \Phi, D)$. Note that $D^2 = \operatorname{diag}(\Delta, \Delta)$, where $\Delta = -\frac{d^2}{dt^2}$ denotes the (positive) Laplace operator on $\mathbb{R}$. By [RS78, XIII.4 Example 6] (or [HK01, 10.5.1]) the operator $(1+\Delta)^{-1}M(f)$ is a compact operator for all $f \in C_c^\infty(\mathbb{R})$ (where $M : C_b(\mathbb{R}) \to \mathcal{B}(L^2(\mathbb{R}))$ denotes the representation as multiplication operators). Hence

$$(1 + D^2)^{-1} \circ \Phi(f_1 + f_2 e) = \begin{pmatrix} (1+\Delta)^{-1} M(f_1) & (1+\Delta)^{-1} M(f_2) \\ (1+\Delta)^{-1} M(f_2) & (1+\Delta)^{-1} M(f_1) \end{pmatrix} \in \mathcal{K}(\mathcal{H})$$

for all $f_1 + f_2 e \in \mathcal{D}_0 = C_c^\infty(\mathbb{R}) \oplus C_c^\infty(\mathbb{R})$. Since $\mathcal{D}_0$ is dense in $\mathcal{D}$ and since $(1 + D^2)^{-1}\Phi(f_1 + f_2 e)$ depends continuously on (f_1, f_2), this proves condition (i) of Lemma 3.3.19. To see condition (ii) we first observe that for all $f \in C_c^\infty(\mathbb{R})$ the operator $[d, M(f)]$ is defined for all $\xi \in C_c^\infty(\mathbb{R}) \subseteq L^2(\mathbb{R})$ and we have

$$[d, M(f)]\xi = \frac{d}{dt}(f\xi) - f \cdot \left(\frac{d}{dt}\xi\right) = \left(\frac{d}{dt} f\right) \cdot \xi,$$

Hence $[d, M(f)]$ extends to a bounded operator on $L^2(\mathbb{R})$ and

$$[D, \Phi(f_1 + f_2 e)] = \begin{pmatrix} -[d, M(f_2)] & -[d, M(f_1)] \\ [d, M(f_1)] & [d, M(f_2)] \end{pmatrix}$$

is densely defined and bounded for all $f_1 + f_2 e \in \mathcal{D}_0$.

The dual-Dirac element. Choose any odd continuous function $\varphi : \mathbb{R} \to [-1, 1]$ such that $\varphi(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \varphi(x) = 1$. For instance, we could take

$$\varphi(x) = \frac{x}{\sqrt{1 + x^2}} \quad \text{or} \quad \varphi(x) = \left\{ \begin{matrix} \sin(x/2) & |x| \le \pi \\ \frac{x}{|x|} & |x| \ge \pi \end{matrix} \right\}.$$

We then define an element $\beta = [\mathcal{D}, 1, S] \in KK(\mathbb{C}, \mathcal{D})$ as follows: We consider $\mathcal{D}$ as a graded Hilbert $\mathcal{D}$-module in the canonical way, and we put $1(\lambda)a = \lambda a$ for all $\lambda \in \mathbb{C}$ and $a \in \mathcal{D}$. The operator $S \in \mathcal{M}(\mathcal{D})$ is defined via multiplication with the element $\varphi e \in C_b(\mathbb{R}) \hat{\otimes} Cl_1 \subseteq \mathcal{M}(\mathcal{D})$. To check that $(\mathcal{D}, \Phi, S)$ is a Kasparov cycle it suffices to check that $S^2 - 1 \in \mathcal{K}(\mathcal{D}) = \mathcal{D}$. But this follows from the fact that $S^2 - 1$ is given by pointwise multiplication with the function $\varphi^2 - 1$ which lies in $\mathcal{D}$ since $\lim_{\pm x\to\infty} \varphi^2(x) = 1$ by conditions (i) and (ii) for φ. Since $S = S^*$ all other conditions of Definition 3.3.2 are trivial.

Note that the class β does not depend on the particular choice of the function $\varphi : \mathbb{R} \to \mathbb{R}$. Indeed, if two functions φ_0, φ_1 are given that satisfy conditions (i) and (ii), we can define for each $t \in [0,1]$ a function $\varphi_t : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_t(x) = t\varphi_1(x) + (1-t)\varphi_0(x).$$

Then each φ_t satisfies the requirements (i) and (ii) and if S_t denotes the corresponding operators it follows that $t \mapsto S_t$ is an operator homotopy joining S_0 with S_1.

Notations 3.3.24. The element $\beta \in KK(\mathbb{C}, C_0(\mathbb{R})\hat{\otimes}\, Cl_1) = KK_1(\mathbb{C}, C_0(\mathbb{R}))$ constructed above is called the *Bott class*.

The Kasparov product $\beta \otimes_{\mathcal{D}} \alpha \in KK(\mathbb{C},\mathbb{C})$. We are now going to show that $\beta \otimes_{\mathcal{D}} \alpha = 1_{\mathbb{C}} \in KK(\mathbb{C},\mathbb{C})$. For this we first claim that $\beta \otimes_{\mathcal{D}} \alpha$ is represented by the triple $(\mathcal{H}, \mathbf{1}, T')$ with

$$T' = \begin{pmatrix} 0 & M(\varphi) \\ M(\varphi) & 0 \end{pmatrix} - M(\sqrt{1-\varphi^2}) \begin{pmatrix} 0 & -\frac{d}{\sqrt{1+\Delta}} \\ \frac{d}{\sqrt{1+\Delta}} & 0 \end{pmatrix}, \tag{3.3.3}$$

with $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ as above. Indeed, since $\Phi : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ is a non-degenerate representation we obtain an isomorphism

$$\mathcal{D} \otimes_{\mathcal{D}} \mathcal{H} \cong \mathcal{H}; a \otimes \xi \mapsto \Phi(a)\xi. \tag{3.3.4}$$

Let T_{12} denote the operator on $\mathcal{D} \otimes_{\mathcal{D}} \mathcal{H}$ corresponding to $T = \begin{pmatrix} 0 & -\frac{d}{\sqrt{1+\Delta}} \\ \frac{d}{\sqrt{1+\Delta}} & 0 \end{pmatrix}$ under this isomorphism. We claim that T_{12} is a T-connection. Recall that for any $a \in \mathcal{D}$ the operator $\theta_a : \mathcal{H} \to \mathcal{D} \otimes_{\mathcal{D}} \mathcal{H}$ is given by $\xi \mapsto a \otimes \xi$. Composed with the above isomorphism we get the operator $\xi \mapsto \Phi(a)\xi$ on $\mathcal{H}$. Condition (3.3.1) follows then for T_{12} from the fact that $[T, \Phi(a)] \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{D}$. We now use Proposition 3.3.13 to see that the Kasparov product $\beta \otimes_{\mathcal{D}} \alpha$ is represented by the triple

$$(\mathcal{D} \otimes_{\mathcal{D}} \mathcal{H}, 1 \otimes 1, S \otimes 1 + (\sqrt{1-S^2} \otimes 1)T_{12}).$$

We leave it as an exercise to check that this operator corresponds to the operator R of (3.3.3) under the isomorphism (3.3.4).

Hence to see that $\beta \otimes_{\mathcal{D}} \alpha = 1_{\mathbb{C}} \in KK(\mathbb{C},\mathbb{C})$ we only need to show that the Fredholm index of the operator

$$F := M(\varphi) + M\left(\sqrt{1-\varphi^2}\right) \frac{d}{\sqrt{1+\Delta}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is one (see the discussion at the end of Section 3.3.2).

Recall that $\varphi : \mathbb{R} \to \mathbb{R}$ can be any function satisfying the conditions (i) and (ii) as stated in the construction of β. Thus we may choose

$$\varphi(x) = \begin{Bmatrix} \sin(x/2) & \text{if } x \in [-\pi, \pi] \\ \frac{x}{|x|} & \text{if } |x| > \pi \end{Bmatrix}.$$

To do the computation we want to restrict the operator to the interval $[-\pi, \pi]$. For this consider the orthogonal projection $Q : L^2(\mathbb{R}) \to L^2[-\pi, \pi]$. Since Q commutes with $M(1 - \varphi^2)$ and since $[\frac{d}{\sqrt{1+\Delta}}, M(\psi)] \in \mathcal{K}(L^2(\mathbb{R}))$ for all $\psi \in C_0(\mathbb{R})$ (which follows from $[T, \Phi(a)] \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{D}$), we have

$$\begin{aligned} M(1-\varphi^2)\frac{d}{\sqrt{1+\Delta}} &\sim M(\sqrt{1-\varphi^2})\frac{d}{\sqrt{1+\Delta}}M(\sqrt{1-\varphi^2}) \\ = M(\sqrt{1-\varphi^2})Q\frac{d}{\sqrt{1+\Delta}}M(\sqrt{1-\varphi^2})Q &\sim M(1-\varphi^2)\frac{d}{\sqrt{1+\Delta}}Q, \end{aligned}$$

where $\sim$ denotes equality up to compact operators. Thus we may replace F by the operator

$$F_1 := M(\varphi) + M\left(\sqrt{1-\varphi^2}\right)\frac{d}{\sqrt{1+\Delta}}Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

Decomposing $L^2(\mathbb{R})$ as the direct sum $L^2[-\pi, \pi] \oplus L^2((-\infty, -\pi) \cup (\pi, \infty))$, we see that the operator F_1 fixes both summands and acts as the identity on the second summand. Hence for computing the index we may restrict our operator to the summand $L^2[-\pi, \pi]$ on which it acts by

$$F_2 := M(\sin(x/2)) + M(\cos(x/2))\frac{d}{\sqrt{1+\Delta}}.$$

Now there comes a slightly tricky point and we need to appeal to some computations given in [HR00, Chapter 10]. We want to replace the operator $\frac{d}{\sqrt{1+\Delta}}$ with the operator $\frac{\tilde{d}}{\sqrt{1-\Delta^{\mathbb{T}}}} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ (identifying $L^2[-\pi, \pi]$ with $L^2(\mathbb{T})$), where $\tilde{d} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the operator given by the differential $\frac{d}{dt}$ on the smooth 2π-periodic functions on $\mathbb{R}$ and where $\Delta^{\mathbb{T}} = -\tilde{d}^2$ denotes the corresponding Laplace operator. Although the operators d and $\tilde{d}$ clearly coincide on $C_c^\infty(-\pi, \pi)$ the functional calculus that has been applied for producing the operators $\frac{d}{\sqrt{1+\Delta}}$ and $\frac{\tilde{d}}{\sqrt{1-\Delta^{\mathbb{T}}}}$ depends on the full domains of the self-adjoint extensions of these operators, which clearly differ. The solution of this problem is implicitly given in [HR00, Lemma 10.8.4]: Recall that $\frac{d}{\sqrt{1+\Delta}}$ is equal to $-i\chi(id)$ where we apply the functional calculus for unbounded self-adjoint operators for the function $\chi(x) = \frac{x}{\sqrt{1+x^2}}$ to the (unique) self-adjoint extension of id. Let $\psi \in C_c^\infty(-\pi, \pi)$ be any fixed function. Choose a positive function $\mu \in C_c^\infty(\mathbb{R})$

with $\operatorname{supp}\mu \in (-\pi, \pi)$ such that $\mu \equiv 1$ on $U := \operatorname{supp}\psi + (-\delta, \delta)$ for some suitable $\delta > 0$. Let $d_\mu = M(\mu) \circ d \circ M(\mu)$. It follows then from [HR00, Corollary 10.2.6] that id_μ is an essentially self-adjoint operator that coincides with id on U. It then follows from [HR00, Lemma 10.8.4] that $M(\psi)\chi(id) \sim M(\psi)\chi(id_\mu)$ on $L^2[-\pi, \pi]$, where, as above, $\sim$ denotes equality up to compact operators. Applying the same argument to the canonical inclusion of the interval $(-\pi, \pi)$ into $\mathbb{T}$ shows that $M(\psi)\chi(id_\mu) \sim M(\psi)\chi(i\tilde{d})$. Together we see that $M(\psi)\chi(id) \sim M(\psi)\chi(i\tilde{d})$ on $L^2[-\pi, \pi]$ for all $\psi \in C_c^\infty(-\pi, \pi)$ and then also for all $\psi \in C_0(-\pi, \pi)$. Applying this to $\psi(x) = \cos(x/2)$ gives the desired result.

Thus we may replace the operator F_2 by the operator

$$F_3 := M(\sin(x/2)) + M(\cos(x/2))\frac{\tilde{d}}{\sqrt{1 + \Delta^{\mathbb{T}}}}.$$

Multiplying F_3 from the left with the invertible operator $M(2ie^{i\frac{x}{2}})$ does not change the Fredholm index, so we compute the index of the operator

$$\begin{aligned} F_4 &= M(2ie^{i\frac{x}{2}}\sin(x/2)) + M(2ie^{i\frac{x}{2}}\cos(x/2))\frac{\tilde{d}}{\sqrt{1 + \Delta^{\mathbb{T}}}} \\ &= M(e^{ix} - 1) + iM(e^{ix} + 1)\frac{\tilde{d}}{\sqrt{1 + \Delta^{\mathbb{T}}}}. \end{aligned}$$

In what follows let $\{e_n : n \in \mathbb{Z}\}$ denote the standard othonormal basis of $\ell^2(\mathbb{Z})$ and let $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denote the bilateral shift operator $U(e_n) = e_{n+1}$. Using Fourier transform and the Plancherel isomorphism $L^2[-\pi, \pi] \cong \ell^2(\mathbb{Z})$ the operator F_4 transforms to the operator $\widehat{F_4} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by

$$\widehat{F_4} = (U - \mathrm{id}) + i(U + \mathrm{id})R$$

where $R : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is given by $R(e_n) = \frac{in}{\sqrt{1+n^2}} e_n$. Let

$$\operatorname{sign}(n) = \begin{Bmatrix} \frac{n}{|n|} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{Bmatrix}$$

and let $R'(e_n) = i \operatorname{sign}(n) e_n$ for $n \in \mathbb{Z}$ and let us write $\widehat{F_5} := (U - \mathrm{id}) + i(U + \mathrm{id})R'$. Since $\left| \frac{in}{\sqrt{1+n^2}} - i\operatorname{sign}(n) \right| \to 0$ for $|n| \to \infty$ we have $R - R' \in \mathcal{K}(\ell^2(\mathbb{Z}))$, which implies that $\widehat{F_5} - \widehat{F_4} \in \mathcal{K}(\ell^2(\mathbb{Z}))$ and hence $\operatorname{index}(\widehat{F_5}) = \operatorname{index}(\widehat{F_4})$. Applying $\widehat{F_5}$ to some basis element e_n gives

$$\begin{aligned} \widehat{F_5}(e_n) &= (U - \mathrm{id}) + i(U + \mathrm{id})R'(e_n) \\ &= (e_{n+1} - e_n) - \operatorname{sign}(n)(e_{n+1} + e_n) = \begin{Bmatrix} 2e_{n+1} & \text{if } n < 0 \\ e_1 - e_0 & \text{if } n = 0 \\ -2e_n & \text{if } n > 0 \end{Bmatrix}. \end{aligned}$$

It follows from this that $\widehat{F_5}$ is surjective and $\ker(\widehat{F_5}) = \mathbb{C}(e_1 + 2e_0 + e_{-1})$. Hence

$$\operatorname{index}(\beta \otimes_{\mathcal{D}} \alpha) = \operatorname{index}(\widehat{F_5}) = 1.$$

Let us state as a lemma what we have proved so far:

Lemma 3.3.25. *Let $\mathcal{D} = C_0(\mathbb{R})\hat{\otimes}\, Cl_1 = C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$ with the standard odd grading and let $\alpha \in KK(\mathcal{D}, \mathbb{C})$ and $\beta \in KK(\mathbb{C}, \mathcal{D})$ as above. Then*

$$\beta \otimes_{\mathcal{D}} \alpha = 1_{\mathbb{C}} \in KK(\mathbb{C}, \mathbb{C}).$$

In order to show that α and β are KK-equivalences, we also need to check that the product $\alpha \otimes_{\mathbb{C}} \beta = 1_{\mathcal{D}} \in KK(\mathcal{D}, \mathcal{D})$. For this we shall use a rotation trick that originally goes back to Atiyah, and which has been adapted very successfully to this situation by Kasparov. Recall that by Theorem 3.3.18 the Kasparov product over $\mathbb{C}$ is commutative, i.e., we have

$$\begin{aligned} \alpha \otimes_{\mathbb{C}} \beta = \beta \otimes_{\mathbb{C}} \alpha &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (1_{\mathcal{D}} \otimes \alpha) \\ &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} \big(\Sigma_{\mathcal{D},\mathcal{D}} \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (\alpha \otimes 1_{\mathcal{D}})\big), \end{aligned}$$

where $\Sigma_{\mathcal{D},\mathcal{D}} : \mathcal{D}\hat{\otimes}\mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{D}$ denotes (the KK-class of) the flip homomorphism $x \otimes y \mapsto (-1)^{\deg(x)\deg(y)} y \otimes x$. If we can show that there is an invertible class $\eta \in KK(\mathcal{D}, \mathcal{D})$ such that $\Sigma_{\mathcal{D},\mathcal{D}} = 1_{\mathcal{D}}\hat{\otimes}\eta \in KK(\mathcal{D}\hat{\otimes}\mathcal{D}, \mathcal{D}\hat{\otimes}\mathcal{D})$ the result will follow from the following reasoning:

$$\begin{aligned} \alpha \otimes_{\mathbb{C}} \beta &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} \big(\Sigma_{\mathcal{D},\mathcal{D}} \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (\alpha \otimes 1_{\mathcal{D}})\big) \\ &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} \big((1_{\mathcal{D}} \otimes \eta) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (\alpha \otimes 1_{\mathcal{D}})\big) \\ &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} \big(\Sigma_{\mathcal{D},\mathcal{D}} \circ (\eta \otimes 1_{\mathcal{D}}) \circ \Sigma_{\mathcal{D},\mathcal{D}}^{-1}\big) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} \big(\Sigma_{\mathcal{D},\mathcal{D}} \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (1_{\mathcal{D}} \otimes \alpha)\big) \\ &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} \big(\Sigma_{\mathcal{D},\mathcal{D}} \circ (\eta \otimes 1_{\mathcal{D}})\big) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (1_{\mathcal{D}} \otimes \alpha) \\ &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} \Sigma_{\mathcal{D},\mathcal{D}} \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (\eta \otimes_{\mathbb{C}} \alpha) \\ &= (\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (\alpha \otimes_{\mathbb{C}} \eta) \\ &= \big((\beta \otimes 1_{\mathcal{D}}) \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (\alpha \otimes 1_{\mathcal{D}})\big) \otimes_{\mathbb{C}\hat{\otimes}\mathcal{D}} (1_{\mathbb{C}} \otimes \eta) \\ &= \eta. \end{aligned}$$

Since η is invertible in $KK(\mathcal{D}, \mathcal{D})$ this implies that α has a right KK-inverse γ, say. But then $\gamma = \beta$, since

$$\beta = \beta \otimes_{\mathcal{D}} (\alpha \otimes_{\mathbb{C}} \gamma) = (\beta \otimes_{\mathcal{D}} \alpha) \otimes_{\mathbb{C}} \gamma = \gamma.$$

To see that $\Sigma_{\mathcal{D},\mathcal{D}} = 1_{\mathcal{D}}\hat{\otimes}\eta \in KK(\mathcal{D}\hat{\otimes}\mathcal{D}, \mathcal{D}\hat{\otimes}\mathcal{D})$ for a suitable invertible KK-class η we consider the isomorphism

$$\mathcal{D}\hat{\otimes}\mathcal{D} = \big(C_0(\mathbb{R})\hat{\otimes}\, Cl_1\big)\hat{\otimes}\big(C_0(\mathbb{R})\hat{\otimes}\, Cl_1\big) \cong C_0(\mathbb{R}^2)\hat{\otimes}\, Cl_2\,.$$

If $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is any orthogonal transformation, it induces an automorphism $\tau^* : C_0(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ by $\tau^*(f)(x) = f(\tau^{-1}(x))$ and an automorphism $\tilde{\tau}$ of Cl_n by extending, via the universal property of Cl_n, the map $\tilde{\tau} : \mathbb{R}^n \to Cl_n; v \mapsto \iota \circ \tau(v)$ to all of Cl_n, where $\iota : \mathbb{R}^n \to Cl_n$ denotes the canonical inclusion. We then get an automorphism

$$\Phi_\tau := \tau^* \hat{\otimes} \tilde{\tau} : C_0(\mathbb{R}^n) \hat{\otimes}\, Cl_n \to C_0(\mathbb{R}^n) \hat{\otimes}\, Cl_n\,.$$

Moreover, a homotopy of orthogonal transformations of $\mathbb{R}^n$ between τ_0 and τ_1 clearly induces a homotopy between the automorphisms Φ_{τ_0} and Φ_{τ_1}. In particular, for any orthogonal transformation that is homotopic to $\mathrm{id}_{\mathbb{R}^n}$ we get

$$[\Phi_\tau] = 1_{C_0(\mathbb{R}^n, Cl_n)} \in KK(C_0(\mathbb{R}^n, Cl_n), C_0(\mathbb{R}^n, Cl_n)).$$

It is not difficult to check that under the isomorphism $\mathcal{D}\hat{\otimes}\mathcal{D} \cong C_0(\mathbb{R}^2)\hat{\otimes}\, Cl_2$ the flip automorphism $\Sigma_{\mathcal{D},\mathcal{D}}$ corresponds to $\Phi_\sigma : C_0(\mathbb{R}^2)\hat{\otimes}\, Cl_2 \to C_0(\mathbb{R}^2)\hat{\otimes}\, Cl_2$ with

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^2; \sigma(x, y) = (y, x).$$

Since $\det(\sigma) = -1$, it is, unfortunately, not homotopic to $\mathrm{id}_{\mathbb{R}^2}$. But the orthogonal transformation $\rho : \mathbb{R}^2 \to \mathbb{R}^2; \rho(x, y) = (-y, x)$ is homotopic to $\mathrm{id}_{\mathbb{R}^2}$ via the path of transformations ρ_t, $t \in [0, \pi/2]$ with

$$\rho_t(x, y) = (\cos(t)x - \sin(t)y, \sin(t)x + \cos(t)y).$$

One checks that Φ_ρ corresponds to $\Sigma_{\mathcal{D},\mathcal{D}} \circ (\mathrm{id}_{\mathcal{D}} \otimes \Phi_{-\mathrm{id}})$, where $-\mathrm{id} : \mathbb{R} \to \mathbb{R}, x \mapsto -x$ is the flip on $\mathbb{R}$. Hence, if $\eta = [\Phi_{-\mathrm{id}}] \in KK(\mathcal{D}, \mathcal{D})$, we have

$$\Sigma_{\mathcal{D},\mathcal{D}} \otimes_{\mathcal{D}\hat{\otimes}\mathcal{D}} (1_{\mathcal{D}} \otimes \eta) = [\Sigma_{\mathcal{D},\mathcal{D}} \circ (\mathrm{id}_{\mathcal{D}} \otimes \Phi_{-\mathrm{id}})] = [\Phi_\rho] = 1_{\mathcal{D}\hat{\otimes}\mathcal{D}},$$

where, by abuse of notation, we identify Φ_ρ with the corresponding automorphism of $\mathcal{D}\hat{\otimes}\mathcal{D}$. Since $\Sigma_{\mathcal{D},\mathcal{D}} = \Sigma_{\mathcal{D},\mathcal{D}}^{-1}$ it follows that $1_{\mathcal{D}} \otimes \eta = \Sigma_{\mathcal{D},\mathcal{D}} \in KK(\mathcal{D}\hat{\otimes}\mathcal{D}, \mathcal{D}\hat{\otimes}\mathcal{D})$ and we are done.

Corollary 3.3.26. *Let $\alpha_1 \in KK(C_0(\mathbb{R}), Cl_1)$ and $\beta_1 \in KK(Cl_1, C_0(\mathbb{R}))$ be the images of α and β under the isomorphisms $KK(C_0(\mathbb{R}) \otimes Cl_1, \mathbb{C}) \cong KK(\mathcal{C}_0(\mathbb{R}), Cl_1)$ and $KK(\mathbb{C}, C_0(\mathbb{R}) \otimes Cl_1) \cong KK(Cl_1, C_0(\mathbb{R}))$ of Proposition 3.3.23. Then α_1 is a KK-equivalence with inverse β_1. As a consequence, for all G-algebras A and B, there are canonical Bott-isomorphisms*

$$KK^G(A\hat{\otimes}C_0(\mathbb{R}), B) \cong KK_1^G(A, B) \cong KK^G(A, B\hat{\otimes}C_0(\mathbb{R})).$$

More generally, for all $n, m \in \mathbb{N}_0$ we get Bott-isomorphisms

$$KK^G(A\hat{\otimes}C_0(\mathbb{R}^n), B\hat{\otimes}C_0(\mathbb{R}^m)) \cong KK_{n+m}^G(A, B).$$

Proof. The proof is a straightforward consequence of the results and techniques explained above and is left to the reader.[2] □

[2]The above proof of Bott-periodicity follows in part some unpublished notes of Walter Paravicini. See http://wwwmath.uni-muenster.de/u/echters/Focused-Semester/downloads.html.

Kasparov actually proved a more general version of the above KK-theoretic Bott-periodicity theorem, which provides a KK^G-equivalence between $C_0(V)$ and the Clifford algebra $Cl(V, \langle\cdot,\cdot\rangle)$, in which G is a (locally) compact group that acts by a continuous orthogonal representation $\rho : G \to \mathrm{O}(V)$ on the finite-dimensional euclidean vector space V, and $\langle\cdot,\cdot\rangle$ is any G-invariant inner product on V. The action of G on $Cl(V)$ is the unique action that extends the given action of G on $V \subseteq Cl(V)$. Identifying V with $\mathbb{R}^n$ equipped with the standard inner product, this KK-equivalence is constructed as in the above special case where $n = 1$ via a KK^G-equivalence between $C_0(\mathbb{R}^n)\hat{\otimes}\, Cl_n$ and $\mathbb{C}$. In the case of trivial actions, this result follows from the case $n = 1$, using the fact that the n-fold graded tensor product of $\mathcal{C}_0(\mathbb{R})\hat{\otimes}\, Cl_1$ with itself is isomorphic to $C_0(\mathbb{R}^n)\hat{\otimes}\, Cl_n$. We refer to Kasparov's original papers [Kas75, Kas81] for the proof of the general case.

3.3.5 Excision in KK-theory

Recall that every short exact sequence $0 \to I \xrightarrow{\iota} A \xrightarrow{q} A/I \to 0$ of C^*-algebras induces a six-term exact sequence in K-theory

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{\iota_0} & K_0(A) & \xrightarrow{q_0} & K_0(A/I) \\ \delta\uparrow & & & & \downarrow\exp \\ K_1(A/I) & \xleftarrow[\iota_1]{} & K_1(A) & \xleftarrow[q_1]{} & K_1(I) \end{array}$$

which happens to be extremely helpful for the computation of K-theory groups. To some extend we get similar six-term sequences in KK-theory, but one has to impose some extra conditions on the short exact sequence:

Definition 3.3.27. Let G be a locally compact group. A short exact sequence of graded G-C^*-algebras

$$0 \to I \xrightarrow{\iota} A \xrightarrow{q} A/I \to 0$$

is called *G-equivariantly semisplit* if there exists a G-equivariant completely positive, normdecreasing, grading preserving cross setion $\phi : A/I \to A$ for the quotient map $q : A \to A/I$. We then also say that A is a *G-semisplit extension* of A/I by I.

By an important result of Choi–Effros [CE76] a short exact sequence $0 \to I \xrightarrow{\iota} A \xrightarrow{q} A/I \to 0$ (with trivial G-action) is always semisplit if A is nuclear. But there are many other important cases of semisplit extensions.

Every G-semisplit extension determines a unique class in $KK_1^G(A/I, I)$ which plays an important rôle in the construction of the six-term exact sequences in KK-theory. The nonequivariant version is well documented (e.g., see [Kas81, Ska85, CS86] and [Bla86, Section 19.5]). But the details of the equivariant version, which we shall need below, are somewhat scattered in the literature. The main ingredients are explained in [BS89, Remarques 7.5] (see also the proof of [CE01a, Lemma 5.17]). We summarise the important steps as follows:

(i) If $0 \to I \xrightarrow{\iota} A \xrightarrow{q} A/I \to 0$ is a G-equivariant semisplit extension, then the canonical embedding $e : I \to C_q := C_0([0,1), A)/C_0((0,1), I)$, which sends $a \in I$ to the equivalence class of $(1-t)a$ in C_q, determines a KK^G-equivalence $[e] \in KK^G(I, C_q)$.

(ii) View the Bott-class $\beta \in KK_1(\mathbb{C}, C_0(0,1))$ as an element in $KK_1^G(\mathbb{C}, C_0(0,1))$ with trivial G-actions everywhere and let $i : C_0((0,1), A/I) \to C_q$ denote the canonical map. Let $c \in KK_1^G(A/I, I)$ be the class defined via the Kasparov product

$$c = (\beta \otimes 1_{A/I}) \otimes_{C_0((0,1),A/I)} [i] \otimes_{C_q} [e]^{-1},$$

where $[e]^{-1} \in KK^G(C_q, I)$ denotes the inverse of $[e]$. We call $c \in KK_1^G(A/I, I)$ the class attached to the equivariant semisplit extension $0 \to I \xrightarrow{\iota} A \xrightarrow{q} A/I \to 0$.

We then have the following theorem, which can be proved basically along the lines of the nonequivariant case using G-equivariant versions of Stinespring's theorem and of Kasparov's stabilisation theorem ([MP84]).

Theorem 3.3.28. *Suppose that $0 \to I \xrightarrow{\iota} A \xrightarrow{q} A/I \to 0$ is a G-equivariant semisplit short exact sequence of C^*-algebras. Then for every G-C^*-algebra B, we have the following two six-term exact sequences:*

$$\begin{array}{ccccc}
KK_0^G(B,I) & \xrightarrow{\iota_*} & KK_0^G(B,A) & \xrightarrow{q_*} & KK_0^G(B,A/I) \\
\partial \uparrow & & & & \downarrow \partial \\
KK_1^G(B,A/I) & \xleftarrow[q_*]{} & KK_1^G(B,A) & \xleftarrow[\iota_*]{} & KK_1^G(B,I)
\end{array}$$

and

$$\begin{array}{ccccc}
KK_0^G(A/I,B) & \xrightarrow{q^*} & KK_0^G(A,B) & \xrightarrow{\iota^*} & KK_0^G(I,B) \\
\partial \uparrow & & & & \downarrow \partial \\
KK_1^G(I,B) & \xleftarrow[\iota^*]{} & KK_1^G(A,B) & \xleftarrow[q^*]{} & KK_1^G(A/I,B),
\end{array}$$

where the boundary maps are all given by taking the Kasparov product with the class $c \in KK_1^G(A/I, I)$ of the given extension.

3.4 The Baum–Connes conjecture

3.4.1 The universal proper G-space

In what follows, for a locally compact group G, a *G-space* will mean a locally compact space X together with a homomorphism $h : G \to \mathrm{Homeo}(X)$ such that the map

$$G \times X \to X; (s,x) \mapsto s \cdot x := h(s)(x)$$

is continuous. A G-space X is called *proper*, if the map

$$\varphi : G \times X \to X \times X; (s, x) \mapsto (s \cdot x, x)$$

is proper in the sense that inverse images of compact sets are compact. Equivalently, X is a proper G-space, if every net (s_i, x_i) in $G \times X$ such that $(s_i \cdot x_i, x_i) \to (y, x)$ for some $(y, x) \in X \times X$ has a convergent subnet. We also say that G *acts properly* on X. Proper G-spaces have an extremely nice behaviour and they are very closely connected to actions by compact groups.

Let us state some important properties:

Lemma 3.4.1. *Suppose that X is a proper G-space. Then the following hold:*

(i) *For every $x \in X$ the stabiliser $G_x = \{s \in G : s \cdot x = x\}$ is compact.*

(ii) *The orbit space $G\backslash X$ equipped with the quotient topology is a locally compact Hausdorff space.*

(iii) *If X is a G-space, Y is a proper G-space and $\phi : X \to Y$ is a G-equivariant continuous map, then X is a proper G-space as well.*

Proof. The first assertion follows from $G_x \times \{x\} = \varphi^{-1}(\{(x, x)\})$, if $\varphi : G \times X \to X \times X$ is the structure map.

For the second assertion, we first observe that the quotient map $q : X \to G\backslash X$ is open since for any open subset $U \subseteq X$ we have $q^{-1}(q(U)) = G \cdot U$ is open in X. This then easily implies that $G\backslash X$ is locally quasi-compact. We need to show that $G\backslash X$ is Hausdorff. For this, assume that there is net (x_i) such that the net of orbits $(G(x_i))$ converges to two orbits $G(x)$, $G(y)$. We need to show that $y = s \cdot x$ for some $s \in G$. Since the quotient map $q : X \to G\backslash X$ is open, we may assume, after passing to a subnet if necessary, that $x_i \to x$ and $s_i \cdot x_i \to y$ for some suitable net (s_i) in G. Hence $(s_i \cdot x_i, x_i) \to (y, x)$, and by properness we may assume, after passing to a subnet if necessary, that $(s_i, x_i) \to (s, x)$ in $G \times X$ for some $s \in G$. But then $s_i \cdot x_i \to s \cdot x$, which implies $y = s \cdot x$.

For the third assertion let $K \subseteq X$ be compact. If $(s \cdot x, x) \in K \times K$ it follows that $\phi \times \phi(s \cdot x, x) \in \phi(K) \times \phi(K)$; hence, by properness of Y, $(s, \phi(x))$ lies in the compact set $\varphi_Y^{-1}(\phi(K) \times \phi(K))$ of $G \times Y$. If $C \subseteq G$ denotes the compact projection of this set in G, we see that $\varphi_X^{-1}(K \times K) \subseteq C \times K$ is compact as well. □

Example 3.4.2. **(a)** If G is compact, then every G-space X is proper, since for all $C \subseteq X$ compact, we have that $\varphi^{-1}(C \times C) \subseteq G \times C$ is compact.

(b) Suppose that $H \subseteq G$ is a closed subgroup of G and assume that Y is a proper H-space. The induced G-space $G \times_H Y$ is defined as the quotient $H\backslash(G \times Y)$ with respect to the H-action $h \cdot (s, y) = (sh^{-1}, h \cdot y)$. This action is proper by part (iii) of the above lemma, hence $G \times_H Y$ is a locally compact Hausdorff space. We let G act on $G \times_H Y$ by $s \cdot [t, y] = [st, y]$. We leave it as an exercise to check that this action is proper as well.

(c) It follows as a special case of **(b)** that whenever $K \subseteq G$ is a compact subgroup of G and Y is a K-space, then the induced G-space $G \times_K Y$ is a proper G-space. Indeed, by a theorem of Abels (see [Abe78, Theorem 3.3]) every proper G-space is locally induced from compact subgroups. More precise, if X is a proper G-space and $x \in X$, then there exists a G-invariant open neighbourhood U of x such that $U \cong G \times_K Y$ as G-space for some compact subgroup K of G (depending on U) and some K-space Y. In particular, if G does not have any compact subgroup, then every proper G-space is a principal G-bundle.

(d) If M is a finite-dimensional manifold, then the action of the fundamental group $G = \pi_1(M)$ on the universal covering $\widetilde{M}$ by deck transformations is a (free and) proper action.

Definition 3.4.3. A proper G-space Z is called a *universal proper G-space* if for every proper G-space X there exists a continuous G-map $\phi : X \to Z$ that is unique up to G-homotopy. We then write $Z =: \underline{EG}$. Note that $\underline{EG}$ is unique up to G-homotopy equivalence.

The following result is due to Kasparov and Skandalis (see [KS91, Lemma 4.1]):

Proposition 3.4.4. *Let X be a proper G-space and let $\mathfrak{M}(X)$ denote the set of finite Radon measures on X with total mass in $(\frac{1}{2}, 1]$ equipped with the weak-* topology as a subset of the dual $C_0(X)'$ of $C_0(X)$ and equipped with the action induced by the action of G on $C_0(X)$. Then $\mathfrak{M}(X)$ is a universal proper G-space.*

Since the restriction of a proper G-action to a closed subgroup H is again proper, it follows that the restriction of the G-space $\mathfrak{M}(X)$ of the above proposition to any closed subgroup H is a universal proper H-space as well. By uniqueness of $\underline{EG}$ up to G-homotopy we get

Corollary 3.4.5. *Suppose that H is a closed subgroup of G and let Z be a universal proper G-space. Then Z is also a universal proper H-space if we restrict the given G-action to H.*

Example 3.4.6. (a) If G is compact, then the one-point space $\{\mathrm{pt}\}$ with the trivial G-action is a universal proper G-space. Similarly, every contractible space Z with trivial G-action is universal. Hence $\underline{EG} = \{\mathrm{pt}\}$ and $\underline{EG} = Z$.

(b) We have $\underline{E\mathbb{R}^n} = \mathbb{R}^n$: Since $\mathbb{R}^n$ has no compact subgroups it follows that every proper G-space is a principal $\mathbb{R}^n$-bundle. On the other hand, since $\mathbb{R}^n$ is contractible, it follows that every principal $\mathbb{R}^n$-bundle is trivial. It follows that every proper $\mathbb{R}^n$-space X is isomorphic to $\mathbb{R}^n \times Y$ with trivial action on Y and translation action on $\mathbb{R}^n$. Hence the projection $p : X \cong \mathbb{R}^n \times Y \to \mathbb{R}^n$ maps X equivariantly into $\mathbb{R}^n$. If $\phi_0, \phi_1 : X \to \mathbb{R}^n$ are two such maps, then

$$\phi_t : X \to \mathbb{R}^n : \phi_t(x) = t\phi_1(x) + (1-t)\phi_0(x)$$

is a G-homotopy of equivariant maps between them. Thus $\mathbb{R}^n$ is universal.

(c) It follows from **(b)** together with Corollary 3.4.5 that $\underline{E\mathbb{Z}^n} = \mathbb{R}^n$.

(d) If G is a torsion-free discrete group, then, as explained above, every proper G-space is a principal G-bundle. It follows that $\underline{EG} = EG$, the universal principal G-bundle.

(e) If G is an almost connected group (i.e., the quotient G/G_0 of G by the connected component G_0 of the identity in G is compact), then G has a maximal compact subgroup $K \subseteq G$. It is then shown by Abels in [Abe74] that G/K is a universal proper G-space.

(f) It follows from **(e)** and Corollary 3.4.5 that for every closed subgroup H of an almost connected group G, we have $\underline{EH} = G/K$, with K a maximal compact subgroup of G. In particular, we have $\underline{E\,\mathrm{SL}(n,\mathbb{Z})} = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$.

3.4.2 The Baum–Connes assembly map

The *Baum–Connes conjecture with coefficients in a C^*-algebra A* (denoted BCC for short) states, that for every G-C^*-algebra A a certain assembly map

$$\mu_{(G,A)} : K_*^G(\underline{EG}; A) \to K_*(A \rtimes_\alpha G)$$

is an isomorphism of abelian groups. Here $K_*^G(\underline{EG}; A)$, often called the *topological K-theory of G with coefficients in A*, can be regarded as the equivariant K-homology of $\underline{EG}$ with coefficients in A. We give a precise definition of this group and of the assembly map below. In case $A = \mathbb{C}$ we get the *Baum–Connes conjecture with trivial coefficients* (BC for short), which relates the equivariant K-homology $K_*^G(\underline{EG}) := K_*^G(\underline{EG}; \mathbb{C})$ with $K_*(C_r^*(G))$, the K-theory of the reduced group algebra of G.

It is well known from the work of Higson, Lafforgue and Skandalis ([HLS02]) that the now often-called *Gromov Monster group* G fails the conjecture with coefficients. But there is still no counterexample for the conjecture with trivial coefficients. On the other hand, we know from the work of Higson and Kasparov [HK01] that (even a very strong version of) BCC holds for all a-T-menable groups – a large class of groups that contains all amenable groups. We give a more detailed discussion of this in Section 3.4.3 below. The relevance of the Baum–Connes conjecture comes from a number of facts:

(i) It implies many other important conjectures, such as the Novikov conjecture in topology, the Kaplansky conjecture on idempotents in group algebras, and the Gromov–Lawson conjecture on positive scalar curvatures in differential geometry. So the validity of the conjecture for a given group G has many positive consequences. We refer to [BCH94, Val02] for more detailed discussions on these applications.

(ii) At least in the case of trivial coefficients the left-hand side $K_*^G(\underline{EG})$ is computable (at least in principle) by classical techniques from algebraic topology such as excision, taking direct limits, and such. These methods are usually not available for the computation of $K_*(C_r^*(G))$ (or $K_*(A \rtimes_r G)$).

(iii) As we shall see further down the line, the conjecture allows a certain flexibility for the coefficients in a number of interesting cases, which makes it possible to perform explicit K-theory computations for certain crossed products and (twisted) group algebras.

Before we go on with this general discussion, we now want to explain the ingredients of the conjecture. For this we let G be a locally compact group and X a proper G-space. Let us further assume that X is *G-compact*, which means that $G\backslash X$ is compact. Then there exists a continuous function $c : X \to [0, \infty)$ with compact support such that for all $x \in X$ we have

$$\int_G c(s^{-1} \cdot x)^2 \, ds = 1.$$

For the construction, just choose any compactly supported positive function $\tilde{c}$ on X such that for each $x \in X$ there exists $s \in G$ with $\tilde{c}(s \cdot x) \neq 0$, divide this function by the strictly positive function $d(x) := \int_G \tilde{c}(s^{-1} \cdot x) \, ds$ and then put $c = \sqrt{\frac{\tilde{c}}{d}}$. We shall call such function $c : X \to [0, \infty)$ a *cut-off function* for (X, G). For such c consider the function

$$p_c : G \times X \to [0, \infty); p_c(s, x) = \Delta_G(s)^{-1/2} c(x) c(s^{-1} \cdot x), \quad \forall (s, x) \in G \times X.$$

It follows from the properness of the action and the fact that c has compact support that $p_c \in C_c(G \times X) \subseteq C_c(G, C_0(X))$. Thus p_c can be regarded as an element of the reduced (or full) crossed product $C_0(X) \rtimes_r G = \overline{C_c(G, C_0(X))}$. In fact, p_c is a projection in $C_0(X) \rtimes_r G$: For every $(s, x) \in G \times X$ we have

$$\begin{aligned} p_c * p_c(s, x) &= \int_G p_c(t, x) p_c(t^{-1}s, t^{-1} \cdot x) \, dt \\ &= \int_G \Delta_G(s)^{-1/2} c(x) c(t^{-1} \cdot x)^2 c(s^{-1} \cdot x) \, dt \\ &= p_c(s, x) \cdot \int_G c(t^{-1} \cdot x)^2 \, dt = p_c(s, x) \end{aligned}$$

and it is trivial to check that $p_c^* = p_c$. Thus p_c determines a class $[p_c] \in K_0(C_0(X) \rtimes_r G) = KK_0(\mathbb{C}, C_0(X) \rtimes_r G)$. Note that this class does not depend on the particular choice of the cut-off function c, for if $\tilde{c}$ is another cut-off function, then

$$c_t = \sqrt{tc^2 + (1 - t)\tilde{c}^2}, \quad t \in [0, 1]$$

is a path of cut-off functions joining c with $\tilde{c}$, and then p_{c_t} is a path of projections joining p_c with $p_{\tilde{c}}$. We call $[p_c]$ the *fundamental K-theory class of* $C_0(X) \rtimes_r G$.

Recall that Kasparov's descent homomorphism

$$J_G : KK_*^G(A,B) \to KK_*(A \rtimes_r G, B \rtimes_r G)$$

is defined by sending a class $x = [\mathcal{E}, \Phi, \gamma, T] \in KK^G(A,B)$ to the class $J_G(x) = [\mathcal{E} \rtimes_r G, \Phi \rtimes_r G, \tilde{T}] \in KK(A \rtimes_r G, B \rtimes_r G)$, where $[\mathcal{E}, \Phi, \gamma] \mapsto [\mathcal{E} \rtimes_r G, \Phi \rtimes_r G]$ is the descent in the correspondence categories as described in Section 2.5.4, and the operator $\tilde{T}$ on $\mathcal{E} \rtimes_r G$ is given on the dense subspace $C_c(G, \mathcal{E})$ by

$$(\tilde{T}\xi)(s) = T(\xi(s)) \quad \forall \xi \in C_c(G, \mathcal{E}),\ s \in G.$$

A similar descent also exists if we replace the reduced crossed products by full crossed products.

Now, if A is a G-C^*-algebra, we can consider the following chain of maps

$$\begin{aligned}\mu_X : KK_*^G(C_0(X), A) &\xrightarrow{J_G} KK_*(C_0(X) \rtimes_r G, A \rtimes_r G)\\ &\xrightarrow{[p_c]\otimes\cdot} KK_*(\mathbb{C}, A \rtimes_r G) \cong K_*(A \rtimes_r G),\end{aligned}$$

where J_G denotes Kasparov's descent homomorphism. If X and Y are two G-compact proper G-spaces and if $\varphi : X \to Y$ is a continuous G-equivariant map, then one can check that $\varphi : X \to Y$ is automatically proper, i.e., inverse images of compact sets are compact, and therefore it induces a G-equivariant $*$-homomorphism

$$\varphi^* : C_0(Y) \to C_0(X); f \mapsto f \circ \varphi.$$

Moreover, if $c : Y \to [0, \infty)$ is a cut-off function for (G, Y), then $\varphi^*(c) : X \to [0, \infty)$ is a cut-off function for (G, X) such that $p_{\varphi^*(c)} = (\varphi^* \rtimes_r G)(p_c)$. Using this fact, it is easy to check that the diagram

$$\begin{tikzcd}
KK_*^G(C_0(X), A) \arrow[r, "J_G"'] \arrow[d, "{[\varphi^*]\otimes\cdot}"'] & KK_*(C_0(X) \rtimes_r G, A \rtimes_r G) \arrow[r, "{[p_{\varphi^* c}]\otimes\cdot}"'] \arrow[d, "{[\varphi^* \rtimes_r G]\otimes\cdot}"'] & KK_*(\mathbb{C}, A \rtimes_r G) \arrow[d, "="] \\
KK_*^G(C_0(Y), A) \arrow[r, "J_G"'] & KK_*(C_0(Y) \rtimes_r G, A \rtimes_r G) \arrow[r, "{[p_c]\otimes\cdot}"'] & KK_*(\mathbb{C}, A \rtimes_r G)
\end{tikzcd}$$

commutes. Hence if we define the *topological K-theory of G with coefficients in A* as

$$K_*^G(\underline{EG}; A) := \lim_{\substack{X \subseteq \underline{EG} \\ X \text{ is } G\text{-compact}}} KK_*^G(C_0(X), A)$$

where the G-compact subsets of $\underline{EG}$ are ordered by inclusion, we get a well-defined homomorphism

$$\mu_{(G,A)} : K_*^G(\underline{EG}; A) = \lim_X KK_*^G(C_0(X), A) \xrightarrow{\lim_X \mu_X} K_*(A \rtimes_r G).$$

This is the Baum–Connes assembly map for the system (A, G, α). We say that G *satisfies BC for A*, if this map is bijective.

Remark 3.4.7. We should remark that almost the same construction yields an assembly map

$$\mu^{\text{full}}_{(G,A)} : K^G_*(\underline{EG}; A) \to K_*(A \rtimes_\alpha G)$$

for the full crossed product $A \rtimes_\alpha G$ such that

$$\Lambda_* \circ \mu^{\text{full}}_{(G,A)} = \mu_{(G,A)},$$

where $\Lambda : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ denotes the regular representation. The only difference is that we then use Kasparov's descent $J^{\text{full}}_G : KK^G(C_0(X), A) \to KK(C_0(X) \rtimes G, A \rtimes G)$ for the full crossed products. Note that, since G acts properly (and hence amenably) on X, we have $C_0(X) \rtimes G \cong C_0(X) \rtimes_r G$ (see Remark 3.4.16 below), so that we can use the same product of the fundamental class $[p_c] \in K_*(C_0(X) \rtimes G)$ as for the reduced assembly map. But it is well known that the full analogue of the conjecture must fail for all lattices Γ in any almost connected Lie group G that has Kazhdan's property (T). But for a large class of groups (including the K-amenable groups of Cuntz and Julg–Valette [Cun83, JV84]), the regular representation induces an isomorphism in K-theory for the full and the reduced crossed products, and then the full assembly map coincides up to this isomorphism with the reduced one.

Example 3.4.8 (The Green–Julg Theorem). If G is a compact group with normed Haar measure we have $\underline{EG} = \{\text{pt}\}$ and hence $K^G_*(\underline{EG}; A) = KK^G_*(\mathbb{C}, A)$ is the G-equivariant K-theory $K^G_*(A)$ of A. The isomorphism $K^G_*(A) \cong K_*(A \rtimes G)$ is the content of the Green–Julg theorem (see [Jul81]). Let us briefly look at the special form of the assembly map in this situation: First of all, if G is compact, we may realise $KK^G(\mathbb{C}, A)$ as the set of homotopy classes of triples $(\mathcal{E}, \gamma, T)$ in which $\mathcal{E}$ is a graded Hilbert A-module, $\gamma : G \to \operatorname{Aut}(\mathcal{E})$ is a compatible action and $T \in \mathcal{L}(\mathcal{E})$ is a G-invariant operator such that

$$T^* - T, T^2 - 1 \in \mathcal{K}(\mathcal{E}).$$

(If T is not G-invariant, it may be replaced by the G-invariant operator $\tilde{T} = \int_G \operatorname{Ad}\gamma_s(T)\, ds$.) The cut-off function of the one-point space is simply the function which sends this point to 1, and the projection $p \in C(G) \subseteq C^*(G)$ is the constant function 1_G. It acts on $\xi \in C(G, \mathcal{E}) \subseteq \mathcal{E} \rtimes G$ via $(p\xi)(s) = \int_G \gamma_t(\xi(t^{-1}s))\, dt$.

Now, given $(\mathcal{E}, \gamma, T) \in KK^G(\mathbb{C}, A)$ the assembly map sends this class to the class of the Kasparov cycle $(\mathcal{E} \rtimes G, p, \tilde{T})$, with $(\tilde{T}\xi)(s) = T(\xi(s))$ for all $s \in G$. Since T is G-invariant, a short computation shows that $\tilde{T}$ commutes with p, and hence we can decompose $(\mathcal{E} \rtimes_r G, p, \tilde{T}) = (p(\mathcal{E} \rtimes G), 1, \tilde{T}) \oplus ((1-p)(\mathcal{E} \rtimes G), 0, \tilde{T})$ in which the second summand is degenerate. Thus we get

$$\mu([\mathcal{E}, \gamma, T]) = [p(\mathcal{E} \rtimes G), \tilde{T}] \in KK(\mathbb{C}, A \rtimes G).$$

Note that a function $\xi \in C(G, \mathcal{E})$ lies in $p(\mathcal{E} \rtimes G)$ if and only if $\xi(s) = \gamma_s(\xi(e))$ for all $s \in G$, and it is clear that such functions are dense in $p(\mathcal{E} \rtimes G)$. Using this, the module $p(\mathcal{E} \rtimes G)$ can be described alternatively as follows: We equip $\mathcal{E}$ with the $A \rtimes G$ valued inner product and left action of $A \rtimes G$ given by

$$\langle e_1, e_2\rangle_{A\rtimes G}(s) = \langle e_1, \gamma_s(e_e)\rangle_A \quad \text{and} \quad e \cdot f = \int_G \gamma_s(e \cdot f(s^{-1}))\, ds$$

for $f \in C(G, A) \subseteq A \rtimes G$. We denote by $\mathcal{E}_{A\rtimes G}$ the completion of $\mathcal{E}$ as a Hilbert $A \rtimes G$-module with this action and inner product. It is then easy to check that every G-invariant operator $S \in \mathcal{L}(\mathcal{E})$ extends to an operator $S^G \in \mathcal{L}(\mathcal{E}_{A\rtimes G})$. A short computation shows that the map $\Phi : p(\mathcal{E} \rtimes G) \to \mathcal{E}_{A\rtimes G}$ given by $\Phi(\xi) = \xi(e)$ for $\xi \in C(G, \mathcal{E}) \cap p(\mathcal{E} \rtimes G)$ is an isomorphism of Hilbert $A \rtimes G$-modules that intertwines $\tilde{T}$ with $T^G \in \mathcal{L}(\mathcal{E}_{A\rtimes G})$. Using this, we get the following description of the assembly map

$$\mu : KK^G(\mathbb{C}, A) \to KK(\mathbb{C}, A \rtimes G); \quad \mu([\mathcal{E}, \gamma, T]) = [\mathcal{E}_{A\rtimes G}, T^G].$$

This map has a direct inverse given as follows: Let $L^2(G, A)$ be the Hilbert A-module with A-valued inner product given by $\langle f, g\rangle_A = \int_G \alpha_{t^{-1}}(f(t)^* g(t))\, dt$ and right A-action given by $(f \cdot a)(s) = f(s)\alpha_s(a)$. Then $A \rtimes G$ acts on $L^2(G, A)$ via the regular representation given by convolution. There is a canonical α-compatible action $\sigma : G \to \mathrm{Aut}(L^2(G, A))$ given by right translation $\sigma_s(f)(t) = f(ts)$. It is then not difficult to check that

$$\nu : KK(\mathbb{C}, A \rtimes G) \to KK^G(\mathbb{C}, A); \quad [\mathcal{F}, S] \mapsto [\mathcal{F} \otimes_{A\rtimes G} L^2(G, A), \mathrm{id} \otimes \sigma, S \otimes 1]$$

is an inverse of μ. For a few more details on these computations see [Ech08].

Example 3.4.9. If G is a discrete torsion free group, then $\underline{EG} = EG$, the universal principal G-bundle of G. Since G acts freely and properly on EG it follows from a theorem of Green [Gre77] that $C_0(EG) \rtimes G$ is Morita equivalent to $C_0(G\backslash EG) = C_0(BG)$, where $BG = G\backslash EG$ is the classifying space of G. Now, for any discrete group and any G-C^*-algebra A we have a canonical isomorphism

$$KK^G(A, \mathbb{C}) \cong KK(A \rtimes G, \mathbb{C})$$

which sends the class of an equivariant $A - \mathbb{C}$ KK-cycle $(\mathcal{H}, \Phi, \gamma, T)$ to the class of the $A \rtimes G - \mathbb{C}$ KK-cycle $(\mathcal{H}, \Phi \rtimes \gamma, T)$. Note that in this situation (Φ, γ) is a covariant representation of (A, G, α) on the Hilbert space $\mathcal{H}$, and hence sums up to a representation of $A \rtimes G$ by the universal property of the full crossed product. Since G is discrete, one checks that condition (ii) in Definition 3.3.2 for $(\mathcal{H}, \Phi \rtimes \gamma, T)$ is equivalent to the corresponding condition for $(\mathcal{H}, \Phi, \gamma, T)$. Thus, if in addition EG is G-compact, we get

$$\begin{aligned} K_*^G(\underline{EG}, \mathbb{C}) \cong KK_*^G(C_0(EG), \mathbb{C}) &\cong KK_*(C_0(EG) \rtimes G, \mathbb{C}) \\ &\overset{\text{Morita-eq.}}{\cong} KK_*(C(BG), \mathbb{C}) = K_*(BG). \end{aligned}$$

Hence in this situation the left-hand side of the Baum–Connes conjecture is the topological K-homology of the classifying space of G. If $\underline{EG}$ is not G-compact, a similar argument gives

$$K_*^G(\underline{EG}, \mathbb{C}) = \lim_{C \subseteq BG} K_*(C),$$

where C runs through the compact subsets of BG, which is the K-homology of BG with compact supports. Hence the Baum–Connes conjecture relates the K-theory of the (often quite complicated) C^*-algebra $C_r^*(G)$ to the K-homology of the classifying space BG of G, which can be handled by methods of classical algebraic topology (but can still be difficult to compute).

We close this section with an exercise:

Exercise 3.4.10. Suppose that A and B are G-C^*-algebras and let $x \in KK^G(A, B)$. Then x induces a map

$$\cdot \otimes_A x : K_*^G(\underline{EG}; A) \to K_*^G(\underline{EG}; A)$$

given on the level of $KK^G(C_0(X), A)$ for some G-compact subset $X \subseteq \underline{EG}$ via the map

$$KK^G(C_0(X), A) \to KK^G(C_0(X), B); y \mapsto y \otimes_A x.$$

On the other hand, we have a map

$$\cdot \otimes_{A \rtimes_r G} j_G(x) : KK(\mathbb{C}, A \rtimes_r G) \to KK(\mathbb{C}, B \rtimes_r G)$$

between the K-theory groups of the crossed products.
Show that the map $\cdot \otimes_A x : K_*^G(\underline{EG}; A) \to K_*^G(\underline{EG}; A)$ is well-defined and that the diagram

$$\begin{array}{ccc} K_*^G(\underline{EG}; A) & \xrightarrow{\mu_{(G,A)}} & K_*(A \rtimes_r G) \\ {\scriptstyle \cdot \otimes_A x}\downarrow & & \downarrow{\scriptstyle \cdot \otimes_{A \rtimes_r G} j_G(x)} \\ K_*^G(\underline{EG}; B) & \xrightarrow{\mu_{(G,B)}} & K_*(B \rtimes_r G) \end{array}$$

commutes. Show that it follows from this that if A and B are KK^G-equivalent, then $\mu_{(G,A)}$ is an isomorphism if and only if $\mu_{(G,B)}$ is an isomorphism. Check that a similar result holds for the full assembly maps $\mu_{(G,A)}^{full}$ and $\mu_{(G,B)}^{full}$ of Remark 3.4.7.

3.4.3 Proper G-algebras and the Dirac dual-Dirac method

As an extension of the Green–Julg theorem one can prove that the Baum–Connes assembly map is always an isomorphism if the coefficient algebra A is a proper G-C^*-algebra in the sense of Kasparov, which we are now going to explain. Recall that if X is a locally compact space, then a C^*-algebra A is called a $C_0(X)$-algebra, if there exists a nondegenerate $*$-homomorphism

$$\Phi : C_0(X) \to \mathcal{ZM}(A),$$

the center of the multiplier algebra of A. If A is a $C_0(X)$-algebra, then A can be realized as an algebra of C_0-sections of a (upper semicontinuous) bundle of C^*-algebras $\{A_x : x \in X\}$, where each fibre A_x is given by $A_x = A/I_x$ with $I_x = (C_0(X \smallsetminus \{x\}) \cdot A)$, where we write $f \cdot a := \Phi(f)a$ for $f \in C_0(X)$, $a \in A$. We refer to [Wil07, Appendix C] for a detailed discussion of $C_0(X)$-algebras.

Definition 3.4.11. Suppose that G is a locally compact group and A is a G-C^*-algebra. Suppose further that A is a $C_0(X)$-algebra such that the structure map $\Phi : C_0(X) \to \mathcal{ZM}(A)$ is G-equivariant. We then say that A is an $X \rtimes G$-C^*-algebra. If A is an $X \rtimes G$-C^*-algebra for some proper G-space X, then A is called a *proper G-C^*-algebra.*

Note that in the definition of a proper G-C^*-algebra we may always assume X to be a realisation of $\underline{EG}$: Since if $\varphi : X \to \underline{EG}$ is a G-equivariant continuous map, we get a nondegenerate G-equivariant $*$-homomorphism

$$\varphi^* : C_0(\underline{EG}) \to C_b(X) = \mathcal{M}(C_0(X)); \varphi^*(f) = f \circ \varphi$$

and then the composition $\Phi \circ \varphi^* : C_0(\underline{EG}) \to \mathcal{ZM}(A)$ makes A into an $\underline{EG} \rtimes G$-algebra. Recall from our discussion of proper G-spaces that proper actions behave very much like actions of compact groups since they are *locally induced* from actions of compact subgroups. It is therefore not very surprising that an analogue of the Green–Julg theorem should hold also for proper G-C^*-algebras.

Theorem 3.4.12. *Suppose that A is a proper G-C^*-algebra. Then the Baum–Connes assembly map*

$$\mu : K_*^G(\underline{EG}, A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

However, the proof of this result is much harder than the proof of the Green–Julg theorem shown in the previous section. In what follows we want to indicate at least some ideas towards this result. On the way we discuss some useful results about induced dynamical systems and their applications to the Baum–Connes conjecture.

In what follows suppose that H is a closed subgroup of the locally compact group G and that $\beta : H \to \mathrm{Aut}(B)$ is an action of H on the C^*-algebra B. Recall from Chapter 2.6 that the *induced C^*-algebra* $\mathrm{Ind}_H^G B$ is defined as the algebra

$$\mathrm{Ind}_H^G B := \left\{ f \in C_b(G, B) : \begin{array}{c} f(sh) = \beta_{h^{-1}}(f(s)) \text{ for all } s \in G \text{ and } h \in H \\ \text{and } (sH \mapsto \|f(s)\|) \in C_0(G/H) \end{array} \right\}.$$

This is a C^*-subalgebra of $C_b(G, B)$ which carries an action $\operatorname{Ind}\beta : G \to \operatorname{Aut}(\operatorname{Ind}_H^G B)$ given by

$$\big(\operatorname{Ind}\beta_s(f)\big)(t) = f(s^{-1}t).$$

If Y is a locally compact H-space, then $\operatorname{Ind}_H^G C_0(Y) \cong C_0(G \times_H Y)$ as G-algebras. Hence the above procedure extends the procedure of inducing G-spaces as discussed in Example 3.4.2 above. The following result is quite useful when working with induced algebras. For the formulation recall that for any G-C^*-algebra A we have a continuous action of G on the primitive ideal space $\operatorname{Prim}(A)$ given by $(s, P) \mapsto \alpha_s(P)$.

Theorem 3.4.13. *Suppose that A is a G-C^*-algebra and let H be a closed subgroup of G. Then the following are equivalent:*

(i) *There exists an H-algebra B such that $A \cong \operatorname{Ind}_H^G B$ as G-algebras.*

(ii) *A carries the structure of a $G/H \rtimes G$-C^*-algebra.*

(iii) *There exists a continuous G-equivariant map $\phi : \operatorname{Prim}(A) \to G/H$.*

Proof. (i) $\Leftrightarrow$ (iii) is Theorem 2.6.2. The proof of (iii) $\Leftrightarrow$ (ii) follows from the general correspondence between continuous maps $\phi : \operatorname{Prim}(A) \to X$ and nondegenerate $*$-homomorphisms $\Phi : C_0(X) \to \mathcal{ZM}(A)$ given by the Dauns–Hofmann theorem. We refer to [Wil07, Appendix C] for a discussion of this correspondence. $\square$

Let us briefly indicate how the objects in (i) and (ii) of the above theorem are related to each other. If $A = \operatorname{Ind}_H^G B$, then the G-equivariant $*$-homomorphism $\Phi : C_0(G/H) \to \mathcal{ZM}(\operatorname{Ind}_H^G B)$ is simply given by

$$\big(\Phi(g)f\big)(s) = g(sH)f(s), \quad g \in C_0(G/H), f \in \operatorname{Ind}_H^G B.$$

Conversely, if $\Phi : C_0(G/H) \to \mathcal{ZM}(A)$ is given, let

$$B := A_{eH} = A/(C_0(G/H \smallsetminus \{eH\}) \cdot A)$$

be the fibre of A over the coset eH. Since $\Phi : C_0(G/H) \to \mathcal{ZM}(A)$ is G-equivariant, it follows that the ideal $C_0(G/H \smallsetminus \{eH\}) \cdot A$ is H-invariant for the restriction of α to H. Thus $\alpha|_H$ induces an action $\beta : H \to \operatorname{Aut}(A_{eH}) = \operatorname{Aut}(B)$. The G-isomorphism $\Psi : A \to \operatorname{Ind}_H^G B$ is then given by

$$\Psi(a)(s) = q(\alpha_{s^{-1}}(a)),$$

where $q : A \to A_{eH}$ denotes the quotient map.

The induction of H-algebras to G-algebras extends to an induction map

$$\operatorname{Ind}_H^G : KK^H(B, C) \to KK^G(\operatorname{Ind}_H^G B, \operatorname{Ind}_H^G C)$$

given as follows: If $[\mathcal{E}, \Phi, \gamma, T] \in KK^H(B, C)$, then we define the induced Hilbert $\operatorname{Ind}_H^G C$-module $\operatorname{Ind}_H^G \mathcal{E}$ as

$$\operatorname{Ind}_H^G \mathcal{E} = \left\{ \xi \in C_b(G, \mathcal{E}) : \begin{array}{c} \xi(sh) = \gamma_{h^{-1}}(\xi(s)) \text{ for all } s \in G \text{ and } h \in H \\ \text{and } (sH \mapsto \|f(s)\|) \in C_0(G/H) \end{array} \right\}$$

with $\operatorname{Ind}_H^G C$-valued inner product and left $\operatorname{Ind}_H^G C$-action given as follows:

$$\langle \xi, \eta \rangle_{\operatorname{Ind}_H^G C}(s) = \langle \xi(s), \eta(s) \rangle_C \quad \text{and} \quad (\xi \cdot f)(s) = \xi(s) \cdot f(s).$$

Similarly, if $\Phi : B \to \mathcal{L}(\mathcal{E})$ is a left action of B on $\mathcal{E}$, then we get an action $\operatorname{Ind} \Phi : \operatorname{Ind}_H^G B \to \mathcal{L}(\operatorname{Ind}_H^G \mathcal{E})$ by

$$(\operatorname{Ind} \Phi(g)\xi)(s) = \Phi(g(s))\xi(s).$$

Finally, we define the operator $\tilde{T} \in \mathcal{L}(\operatorname{Ind}_H^G \mathcal{E})$ via $(\tilde{T}\xi)(s) = T(\xi(s))$. It is not difficult to check that $(\operatorname{ind}_H^G \mathcal{E}, \operatorname{Ind} \Phi, \operatorname{Ind} \beta, \tilde{T})$ is a G-equivariant $\operatorname{Ind}_H^G B - \operatorname{Ind}_H^G C$ Kasparov cycle and Kasparov's induction map in KK-theory is then defined as

$$\operatorname{Ind}_H^G([\mathcal{E}, \Phi, \gamma, T]) = [\operatorname{ind}_H^G \mathcal{E}, \operatorname{Ind} \Phi, \operatorname{Ind} \beta, \tilde{T}] \in KK^G(\operatorname{Ind}_H^G B, \operatorname{Ind}_H^G C).$$

We want to use this map to define an induction map

$$\operatorname{Ind}_H^G : K_*^H(\underline{EH}; B) \to K_*^G(\underline{EG}; \operatorname{Ind}_H^G B)$$

for every H-algebra B. For this suppose that $Y \subseteq \underline{EH}$ is an H-compact subset. Then the induced G-space $G \times_H Y$ is proper and G-compact and therefore maps equivariantly into $\underline{EG}$ via some continuous map $j : G \times_H Y \to \underline{EG}$ whose image is a G-compact subset $X(Y) \subseteq \underline{EG}$. One can check that the composition of maps

$$KK^G(C_0(Y), B) \xrightarrow{\operatorname{Ind}_H^G} KK^G(C_0(G \times_H Y), \operatorname{Ind}_H^G B) \xrightarrow{j_*} KK^G(C_0(X(Y)), \operatorname{Ind}_H^G B)$$

is compatible with taking limits and therefore induces a well-defined induction map

$$I_H^G : K_0^H(\underline{EH}; B) \to K_0^G(\underline{EG}; \operatorname{Ind}_H^G B).$$

Replacing B by $B \otimes Cl_1$ (or $B \otimes C_0(\mathbb{R})$) gives an analogous map from $K_1^H(\underline{EH}; B)$ to $K_1^G(\underline{EG}; \operatorname{Ind}_H^G B)$. We then have the following theorem, which has been shown in [CE01a, Theorem 2.2] (for G discrete and H finite the result has first been obtained earlier in [GHT00]):

Theorem 3.4.14. *Suppose that H is a closed subgroup of G. Then the induction map*

$$I_H^G : K_*^H(\underline{EH}; B) \to K_*^G(\underline{EG}; \operatorname{Ind}_H^G B)$$

is an isomorphism of abelian groups for every H-C^*-algebra B.

Now Green's imprimitivity theorem (see Theorem 2.6.4) says that the full (resp. reduced) crossed products $B \rtimes_{\beta,(r)} G$ and $\mathrm{Ind}_H^G B \rtimes_{\mathrm{Ind}\,\beta,(r)} G$ are Morita equivalent via a canonical $B \rtimes_{(r)} H - \mathrm{Ind}_H^G B \rtimes_{(r)} G$ equivalence bimodule $X_H^G(B)_{(r)}$. Since Morita equivalences provide KK-equivalences, we obtain the following diagram of maps

$$\begin{tikzcd} K_*^H(\underline{EH}; B) \arrow[r, "\mu_H"] \arrow[d, "I_H^G"'] & K_*(B \rtimes_r H) \arrow[d, "\otimes[X_H^G(B)_r]"] \\ K_*^G(\underline{EG}, \mathrm{Ind}_H^G B) \arrow[r, "\mu_G"'] & K_*(\mathrm{Ind}_H^G B \rtimes_r G) \end{tikzcd}$$

in which both vertical arrows are isomorphisms. It is shown in [CE01a, Proposition 2.3] that this diagram commutes. As a corollary we get

Corollary 3.4.15. *Suppose that H is a closed subgroup of G and B is an H-algebra. Then the assembly map*

$$\mu_H : K_*^H(\underline{EH}; B) \to K_*(B \rtimes_r H)$$

is an isomorphism if and only if

$$\mu_G : K_*^G(\underline{EG}, \mathrm{Ind}_H^G B) \to K_*(\mathrm{Ind}_H^G B \rtimes_r G)$$

is an isomorphism. In particular, if G satisfies BCC, then so does H. A similar result holds for the assembly maps into the K-theories of the full crossed products $B \rtimes H$ and $\mathrm{Ind}_H^G B \rtimes G$, respectively (see Remark 3.4.7).

We now come back to general proper G-algebras A. So suppose that X is a proper G-space and that A is an $X \rtimes G$-C^*-algebra. Since every proper G-space is locally induced from a compact subgroup, we find for every $x \in X$ an open G-invariant neighbourhood $U \subseteq X$ such that $U \cong G \times_K Y$ for some compact subgroup K of G and some K-space Y. Then $C_0(G \times_K Y) \cong \mathrm{Ind}_H^G C_0(Y)$ is a $G/K \rtimes G$-algebra by Theorem 3.4.13. Let $A(U) := \Phi(C_0(U))A \subseteq A$. Then $A(U)$ is a G-invariant ideal of A and carries the structure of a $U \rtimes G$-algebra in the canonical way. The composition

$$C_0(G/H) \to C_b(G \times_K Y) \cong C_b(U) \xrightarrow{\Phi} \mathcal{ZM}(A(U))$$

then gives $A(U)$ the structure of a $G/K \rtimes G$-algebra. Thus it follows from Theorem 3.4.13 that $A(U) \cong \mathrm{Ind}_K^G B$ for some K-algebra B. By the Green–Julg theorem we know that K satisfies BCC and hence it follows from Corollary 3.4.15 that the assembly map

$$\mu_U : K_*^G(\underline{EG}, A(U)) \to K_*(A(U) \rtimes_r G)$$

is an isomorphism. Thus we see that for proper G-algebras the Baum–Connes conjecture holds *locally*. Now every G-invariant open subset $W \subseteq X$ with G-compact closure $\overline{W}$ can be covered by a finite union of open sets $U_1, \ldots, U_l$ such that each U_i is isomorphic to some induced space $G \times_{K_i} Y_i$ for some compact subgroup $K_i \subseteq G$. Using six-term sequences and induction on the number l of open sets in this covering, we then conclude that

$$\mu_W : K_*^G(\underline{EG}, A(W)) \to K_*(A(W) \rtimes_r G)$$

for all such W. Now, taking inductive limits indexed by W, one can show that the assembly map

$$\mu : K_*^G(\underline{EG}, A) \to K_*(A \rtimes_r G)$$

is an isomorphism as well. This then finishes the proof of Theorem 3.4.12. (We refer to [CEM01] for the original proof and further details.)

Remark 3.4.16. An application of Green's imprimitivity theorem also implies that for any proper G-C^*-algebra A the full and reduced crossed products coincide. To see this, let A be an $X \rtimes G$-algebra for some proper G-space X. Let $\{U_i : i \in I\}$ be an open cover of X consisting of G-invariant open sets such that each of these sets is induced by some compact subgroup K of G. Suppose now that $\pi \rtimes U : A \rtimes G \to \mathcal{B}(\mathcal{H})$ is any irreducible representation of $A \rtimes G$. We claim that there exists at least one $i \in I$ such that π does not vanish on the ideal $A(U_i)$ of A, and hence $\pi \rtimes U$ does not vanish on the ideal $A(U_i) \rtimes G$ of the crossed product. Indeed, since $\{U_i : i \in I\}$ is a covering of X it is an easy exercise, using a partition of unity argument, to show that $\sum_{i \in I} A(U_i)$ is a dense ideal in A. The claim then follows since $\pi \neq 0$.

It now suffices to show that $A(U_i) \rtimes G = A(U_i) \rtimes_r G \subseteq A \rtimes_r G$, since this implies that every irreducible representation of the ideal $A(U_i) \rtimes G$ corresponds to an irreducible representation of $A \rtimes_r G$. To see this, recall that $A(U_i) \cong \mathrm{Ind}_K^G B$ for some compact subgroup K of G and some K-algebra B. Since K is compact, hence amenable, we have $B \rtimes K = B \rtimes_r K$. Since the $B \rtimes_r K - \mathrm{Ind}_K^G B \rtimes_r G$-equivalence bimodule $X_H^G(B)_r$ is the quotient of the $B \rtimes K - \mathrm{Ind}_K^G B \rtimes G$-equivalence bimodule $X_K^G(B)$ by the submodule $(\ker \Lambda_{(B,K)}) \cdot X_K^G(B) = \{0\}$, it follows from the Rieffel-correspondence (Proposition 2.5.4) that $A(U_i) \rtimes G \cong \mathrm{Ind}_K^G B \rtimes G = \mathrm{Ind}_K^G B \rtimes_r G \cong A(U_i) \rtimes_r G$, which finishes the proof.

We are now coming to Kasparov's Dirac-dual Dirac method for proving the Baum–Connes conjecture. As we shall discuss below, this has been the most successful method so far for proving that the conjecture holds for certain classes of groups. Since, as we saw above, the Baum–Connes conjecture always holds for proper G-C^*-algebras as coefficients, the basic idea is to show that for a given group G *every* G-C^*-algebra B is KK^G-equivalent to a proper G-C^*-algebra. Since by Exercise 3.4.10 the validity of the Baum–Connes conjecture is invariant under passing to KK^G-equivalent coefficient algebras, this would result in a proof that the group G satisfies BCC, i.e., Baum–Connes for all coefficients. Indeed, we need less:

Definition 3.4.17. Suppose that G is a second countable locally compact group and assume that there is a proper G-C^*-algebra $\mathcal{D}$ together with elements

$$\alpha \in KK_0^G(\mathcal{D}, \mathbb{C}) \quad \text{and} \quad \beta \in KK^G(\mathbb{C}, \mathcal{D})$$

such that

$$\gamma := \beta \otimes_{\mathcal{D}} \alpha = 1_{\mathbb{C}} \in KK^G(\mathbb{C}, \mathbb{C}).$$

We then say that G *has a γ-element equal to one*. If, in addition $\alpha \otimes_{\mathbb{C}} \beta = 1_{\mathcal{D}}$, then we say that G satisfies the *strong Baum–Connes conjecture.*

Note that in almost all cases where G has a γ-element equal to one there is also a proof of strong BC.

If G has a γ-element equal to one and B is any other G-C^*-algebra, it follows that

$$(\beta \hat{\otimes} 1_B) \otimes_{\mathcal{D}\hat{\otimes} B} (\alpha \hat{\otimes} 1_B) = \gamma \hat{\otimes} 1_B = 1_B \in KK^G(B, B)$$

and similarly, since the descent $KK^G(A, B) \to KK(A \rtimes_r G, B \rtimes_r G)$ is compatible with Kasparov products, we get

$$J_G(\beta \hat{\otimes} 1_B) \otimes_{(\mathcal{D}\hat{\otimes} B)\rtimes_r G} J_G(\alpha \hat{\otimes} 1_B) = J_G(1_B) = 1_{B\rtimes_r G} \in KK(B \rtimes_r G, B \rtimes_r G).$$

Moreover, it follows from Exercise 3.4.10 that the following diagram commutes

$$\begin{tikzcd}
K_*^G(\underline{EG}, B) \arrow[r, "\mu_{(G,B)}"] \arrow[d, "\cdot\otimes(\beta\otimes 1_B)"'] & K_*(B \rtimes_r G) \arrow[d, "\cdot\otimes J_G(\beta\otimes 1_B)"] \\
K_*^G(\underline{EG}, \mathcal{D}\hat{\otimes} B) \arrow[r, "\mu_{(G,\mathcal{D}\hat{\otimes}B)}", "\cong"'] \arrow[d, "\cdot\otimes(\alpha\otimes 1_B)"'] & K_*((\mathcal{D}\hat{\otimes} B) \rtimes_r G) \arrow[d, "\cdot\otimes J_G(\alpha\otimes 1_B)"] \\
K_*^G(\underline{EG}, B) \arrow[r, "\mu_{(G,B)}"] & K_*(B \rtimes_r G).
\end{tikzcd} \tag{3.4.1}$$

Since $\mathcal{D}\hat{\otimes}B$ is a proper G-algebra (via the composition of $\Phi : C_0(X) \to \mathcal{ZM}(\mathcal{D})$ with the canonical map of $\mathcal{M}(\mathcal{D})$ to $\mathcal{M}(\mathcal{D}\hat{\otimes}B)$), the middle horizontal map is an isomorphism of abelian groups, and by the above discussion it follows that the compositions of the vertical maps on either side are isomorphisms as well. It then follows by an easy diagram chase that the upper horizontal map is injective and the lower horizontal map is surjective, hence $\mu_{(G,B)}$ is an isomorphism as well. Thus we get

Corollary 3.4.18. *If G has a γ-element equal to one, then G satisfies the Baum–Connes conjecture with coefficients (BCC).*

Remark 3.4.19. In diagram (3.4.1) we can replace all reduced crossed products by the full ones and the (reduced) assembly map by the full assembly map to see that whenever G has a γ-element equal to one, then the full assembly map

$$\mu_{(G,B)}^{\text{full}} : K_*^G(\underline{EG}; B) \to K_*(B \rtimes_\beta G)$$

is an isomorphism as well. Moreover, if $\Lambda : B \rtimes_\beta G \to B \rtimes_{\beta,r} G$ is the regular representation, then the diagram

$$\begin{array}{ccc} K_*^G(\underline{EG}; B) & \xrightarrow{\mu^{\text{full}}_{(G,B)}} & K_*(B \rtimes G) \\ & \searrow^{\mu_{(G,B)}} & \downarrow \Lambda_* \\ & & K_*(B \rtimes_r G) \end{array}$$

commutes. Since both assembly maps are isomorphisms, it follows that the regular representation induces an isomorphism in K-theory between the maximal and the reduced crossed products by G. Indeed, it is shown by Tu in [Tu99b] that G is K-amenable in the sense of Cuntz [Cun83] and Julg–Valette [JV84] (which actually implies that Λ is a KK-equivalence) whenever G has a γ-element equal to one.

Remark 3.4.20. If G has a γ-element equal to one, then so does every closed subgroup of G. Indeed, if $\alpha \in KK^G(\mathcal{D}, \mathbb{C})$ and $\beta \in KK^G(\mathbb{C}, \mathcal{D})$ are as in the definition of strong BC, then the action of G on $\mathcal{D}$ restricts to a proper action of H on $\mathcal{D}$. Moreover, for every pair of G-C^*-algebras A, B we have a natural homomorphism

$$\mathrm{res}^G_H : KK^G(A, B) \to KK^H(A, B)$$

which is given by simply restricting all actions on algebras and Hilbert modules from G to H. It is easy to see that this restriction map is compatible with the Kasparov product, so that we get

$$\mathrm{res}^G_H(\beta) \otimes_{\mathcal{D}} \mathrm{res}^G_H(\alpha) = \mathrm{res}^G_H(\beta \otimes_{\mathcal{D}} \alpha) = \mathrm{res}^G_H(1_{\mathbb{C}}) = 1_{\mathbb{C}} \in KK^H(\mathbb{C}, \mathbb{C}).$$

Example 3.4.21. As a sample, we want to show that $\mathbb{R}$ and $\mathbb{Z}$ satisfy strong BC (and, in particular, have $\gamma = 1_{\mathbb{C}}$). For this recall the construction of the Dirac and dual Dirac elements in the proof of the Bott periodicity theorem in Section 3.3.4: Let $\mathcal{D} = C_0(\mathbb{R}) \hat{\otimes}\, Cl_1$. We constructed elements $\alpha \in KK(\mathcal{D}, \mathbb{C})$ and $\beta \in KK(\mathbb{C}, \mathcal{D})$ that are inverse to each other in KK. Let $\tau : \mathbb{R} \to \mathrm{Aut}(C_0(\mathbb{R}))$ denote the translation action $(\tau_s(f))(x) = f(x - s)$. Then $\mathcal{D}$ becomes a proper $\mathbb{R}$-algebra via the action $\tau \hat{\otimes} \mathrm{id}_{Cl_1}$. Now recall that the classes α and β have been given by

$$\alpha = \left[\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \Phi, T = \frac{D}{\sqrt{1 + D^2}}\right] \quad \text{and} \quad \beta = [\mathcal{D}, 1, S],$$

in which $D = \begin{pmatrix} 0 & -\frac{d}{dt} \\ \frac{d}{dt} & 0 \end{pmatrix}$, $\Phi : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ is given as in (3.3.2), and $S = S_\varphi$ is given by pointwise multiplication with a function $\varphi : \mathbb{R} \to [-1, 1]$, which can be any odd continuous function with $\varphi(x) = 0 \Leftrightarrow x = 0$ and $\lim_{t\to\infty} \varphi(t) = 1$. With the given $\mathbb{R}$-action on $\mathcal{D}$ we may view β as a class

$$\beta = [\mathcal{D}, \tau \hat{\otimes} \mathrm{id}_{Cl_1}, 1, S] \in KK^{\mathbb{R}}(\mathbb{C}, \mathcal{D}).$$

The only extra condition to check is the condition that $\mathrm{Ad}_{\tau\otimes\mathrm{id}(s)}(S)-S\in\mathcal{K}(\mathcal{D})=\mathcal{D}$, which follows from the fact that for any function φ as above, we have $\tau_s(\varphi)-\varphi\in C_0(\mathbb{R})$ for all $s\in\mathbb{R}$. Similarly, if we equip $\mathcal{H}=L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ with the representation $\lambda\oplus\lambda$, where $\big(\lambda_s(\xi)\big)(x)=\xi(x-s)$ denotes the regular representation of $\mathbb{R}$, we obtain a class

$$\alpha=[\mathcal{H},\lambda,\Phi,T]\in KK^{\mathbb{R}}(\mathcal{D},\mathbb{C}).$$

As above, the only extra condition to check is that $(\mathrm{Ad}\,\lambda_s(T)-T)\Phi(d)\in\mathcal{K}(\mathcal{H})$ for all $s\in\mathbb{R}$ and $d\in\mathcal{D}$, which we leave as an exercise for the reader. We claim that

$$\beta\otimes_{\mathcal{D}}\alpha=1_{\mathbb{C}}\in KK^{\mathbb{R}}(\mathbb{C},\mathbb{C}). \tag{3.4.2}$$

Note that this would be true if we equipped everything with the trivial $\mathbb{R}$-action instead of the translation action, since then it would be a direct consequence of the product $\beta\otimes_{\mathcal{D}}\alpha=1_{\mathbb{C}}\in KK(\mathbb{C},\mathbb{C})$, which we proved in Section 3.3.4. We show equation (3.4.2) by a simple trick, showing that the translation action of $\mathbb{R}$ is homotopic to the trivial action in the following sense: We consider the algebra $\mathcal{D}[0,1]=\mathcal{D}\hat{\otimes}C[0,1]$ equipped with the $\mathbb{R}$-action $\tilde{\tau}\hat{\otimes}\,\mathrm{id}_{Cl_1}$ where

$$\tilde{\tau}:\mathbb{R}\to\mathrm{Aut}(C_0(\mathbb{R}\times[0,1]));\,\big(\tilde{\tau}_s(f)\big)(x,t)=f(x-ts,t).$$

We then consider the class $\tilde{\alpha}\in KK^{\mathbb{R}}(\mathcal{D}[0,1],C[0,1])$, where $C[0,1]$ carries the trivial $\mathbb{R}$-action, given by

$$\tilde{\alpha}=\big[\mathcal{H}\hat{\otimes}C[0,1],\tilde{\lambda}\oplus\tilde{\lambda},\Phi\hat{\otimes}1,T\hat{\otimes}1\big],$$

where the $\mathbb{R}$-action $\tilde{\lambda}\oplus\tilde{\lambda}$ on $\mathcal{H}\hat{\otimes}C[0,1]$ is given by a formula similar to the one for $\tilde{\tau}$. On the other hand, we consider the class

$$\tilde{\beta}=\big[\mathcal{D}[0,1],\tilde{\tau}\hat{\otimes}1,1,S\hat{\otimes}1\big]\in KK^{\mathbb{R}}(\mathbb{C},\mathcal{D}[0,1]).$$

If we evaluate the class $\tilde{\beta}\otimes_{\mathcal{D}[0,1]}\tilde{\alpha}\in KK^{\mathbb{R}}(\mathbb{C},C[0,1])$ in 0, we obtain the product of β with α equipped with trivial $\mathbb{R}$-actions, which is $1_{\mathbb{C}}\in KK^{\mathbb{R}}(\mathbb{C},\mathbb{C})$ by the proof of Bott periodicity. If we evaluate at 1, we obtain the product $\beta\otimes\alpha$ with respect to the proper translation action on $\mathcal{D}$. Hence both classes are homotopic, which proves (3.4.2).

Hence we see that the Dirac dual-Dirac method applies to $\mathbb{R}$, and by Remark 3.4.20 it then also applies to $\mathbb{Z}\subseteq\mathbb{R}$ and both have γ-element equal to one. Using an easy product argument, this proof also implies that $\mathbb{R}^n$ and $\mathbb{Z}^n$ have γ-element equal to one. We leave it as an exercise to check that $\alpha\otimes_{\mathbb{C}}\beta=1_{\mathcal{D}}\in KK^{\mathbb{R}}(\mathcal{D},\mathcal{D})$, i.e., that $\mathbb{R}^n$ and $\mathbb{Z}^n$ do satisfy the strong Baum-Connes conjecture.

Extending Example 3.4.21 to higher dimensions, one can use Kasparov's equivariant Bott periodicity theorem as discussed in the last paragraph of Section 3.3.4

to show that the Dirac dual-Dirac method works for all groups which act properly and isometrically by affine transformations on a finite-dimensional euclidean space. This has already been pointed out by Kasparov in his conspectus [Kas95]. Later, in [Kas88], he extended this to show that the method works for all amenable Lie groups (and their closed subgroups) and, together with Pierre Julg in [JK95], they showed that the method works for the Lie groups $SO(n, 1)$ and $SU(n, 1)$ and their closed subgroups. But the most far-reaching positive result, which includes all cases mentioned above, has been obtained by Higson and Kasparov in [HK01]:

Theorem 3.4.22 (Higson–Kasparov)**.** *Suppose that the second countable locally compact group acts continuously and metrically properly by isometric affine transformations on a separable real Hilbert space $\mathcal{H}$. Then G satisfies the strong Baum–Connes conjecture.*

Note that the action of G on $\mathcal{H}$ is called *metrically proper* if for any $\xi \in \mathcal{H}$ and $R > 0$ there exists a compact subset $C \subseteq G$ such that $\|s \cdot \xi\| > R$ for all $s \in G \setminus C$. The basic idea of the proof is to construct the proper G-algebra as an inductive limit of algebras $C_0(V) \hat{\otimes} Cl(V)$, where $V \subseteq \mathcal{H}$ runs through the finite-dimensional subspaces of the Hilbert space $\mathcal{H}$. But the precise construction of the algebra $\mathcal{D}$ and the classes α and β is very complex and we refer to the original work [HK01] of Higson and Kasparov for more details. A detailed exposition of certain aspects of the proof can be found in the recent paper [Nis16]. We should also note that the original proof of Higson and Kasparov uses E-theory, a variant of KK-theory introduced by Connes and Higson in [CH90] (see also [Bla86, Chapter 25]). A groupoid version of the above theorem has been shown by Tu in [Tu99a].

The class of groups that satisfies the conditions of the Higon–Kasparov theorem was studied first by Gromov who called them *a-T-menable* groups. A second countable group G is a-T-menable if and only if it satisfies the Haagerup approximation property which says that the trivial representation 1_G can be approximated uniformly on compact sets by a net of positive definite functions (φ_i) on G such that each φ_i vanishes at ∞ on G. We refer to [CCJ+01] for a detailed exposition on the class of a-T-menable groups. As a consequence of the theorem we get

Corollary 3.4.23. *Every amenable second countable locally compact group satisfies strong BC. Also, the free groups F_n in n generators, $n \in \mathbb{N} \cup \{\infty\}$ and all closed subgroups of the Lie groups* $\mathrm{SU}(n, 1)$ *and* $\mathrm{SO}(n, 1)$ *satisfy strong BC.*

All groups in the above corollary satisfy the Haagerup property.

The Dirac dual-Dirac method can also be used in cases in which the element

$$\gamma = \beta \otimes_{\mathcal{D}} \alpha \in KK^G(\mathbb{C}, \mathbb{C})$$

is not necessarily equal to $1_{\mathbb{C}}$, but where it satisfies the following weaker condition:

Definition 3.4.24 (Kasparov's γ-element)**.** Suppose that $\mathcal{D}$ is a proper G-algebra, $\alpha \in KK^G(\mathcal{D}, \mathbb{C})$ and $\beta \in KK^G(\mathbb{C}, \mathcal{D})$. Then $\gamma = \beta \otimes_{\mathcal{D}} \alpha \in KK^G(\mathbb{C}, \mathbb{C})$ is called a *γ-element for G* iff

$$\gamma \otimes 1_{C_0(X)} = 1_{C_0(X)} \in KK^G(C_0(X), C_0(X))$$

for every proper G-space X.

The class of groups that admit a γ-element is huge. It has been shown by Kasparov in [Kas95, Kas88] that it contains almost all connected groups (i.e., groups with co-compact connected component of the identity) and it is clear that the existence of a γ-element passes to closed subgroups. In [KS91, KS03] Kasparov and Skandalis proved the existence of γ-elements for many other groups. Note that the above definition of a γ-element is slightly weaker than Kasparov's original definition, in which he required that $\gamma \otimes 1_{C_0(X)} = 1_{C_0(X)}$ in the $X \rtimes G$-equivariant group $KK^{X\rtimes G}(C_0(X), C_0(X))$, where $X \rtimes G$ denotes the transformation groupoid for the G-space X. Since the above definition suffices for our purposes and since we want to avoid talking about equivariant KK-theory for groupoids, we use it here. We have:

Theorem 3.4.25 (Kasparov)**.** *Suppose that G is a second countable group that admits a γ-element. Then for every G-C^*-algebra B the Baum–Connes assembly map*

$$\mu_{(G,B)} : K_*^G(\underline{EG}; B) \to K_*(B \rtimes_r G)$$

is split injective (the same holds true for the full assembly map $\mu^{\text{full}}_{(G,B)}$).

Proof. Going back to diagram (3.4.1), we see that for proving split injectivity it suffices to show that the composition of the left vertical arrows of the diagram is the identity map. So we need to check that the map $\cdot \otimes \gamma : K_*^G(\underline{EG}; B) \to K_*^G(\underline{EG}; B)$, which is given on the level of any G-compact subset $X \subseteq \underline{EG}$ by the map

$$KK_*^G(C_0(X), B) \to KK_*^G(C_0(X), B); x \mapsto x \otimes_B (1_B \otimes \gamma),$$

is the identity on $KK_*^G(C_0(X), B)$. But by commutativity of the Kasparov product over $\mathbb{C}$ we get

$$x \otimes (1_B \otimes \gamma) = x \otimes_{\mathbb{C}} \gamma = \gamma \otimes_{\mathbb{C}} x = (\gamma \otimes 1_{C_0(X)}) \otimes_{C_0(X)} x = 1_{C_0(X)} \otimes_{C_0(X)} x = x.$$

$\square$

The above proof relies heavily on Theorem 3.4.12, which in turn relies on Theorem 3.4.14. In the course of proving that theorem in [CE01a] the authors made heavy use of Kasparov's result that all almost connected groups have a γ-element and that this implies (without using BC for proper coefficients) that the Baum–Connes assembly map is injective whenever G has a γ-element in the stronger sense of Kasparov.

Remark 3.4.26. It is shown by Kasparov in [Kas95, Kas88] that for a discrete group G the rational injectivity (i.e., injectivity after tensoring both sides with $\mathbb{Q}$) of the assembly map

$$\mu_G : K_*^G(\underline{EG}; \mathbb{C}) \to K_*(C_r^*(G))$$

implies the famous Novikov conjecture from topology. We do not want to discuss this conjecture here (e.g., see [Kas88] for the formulation), but we want to mention that Theorem 3.4.25 shows that every group that admits a γ-element also satisfies the Novikov conjecture. This fact leads to the following notation: A group G is said to satisfy the *strong Novikov conjecture with coefficients*, if the assembly map $\mu_{(G,B)}$ is injective for every G-C^*-algebras B.

Remark 3.4.27. If G has a γ-element in the sense of Definition 3.4.24 and if B is any given G-C^*-algebra, then the assembly map $\mu_{(G,B)} : K_*^G(\underline{EG}; B) \to K_*(B \rtimes_r G)$ is surjective if and only if the map

$$F_\gamma : K_*(B \rtimes_r G) \to K_*(B \rtimes_r G); x \mapsto x \otimes_{B \rtimes_r G} J_G(1_B \otimes \gamma)$$

coincides with the identity map. This follows easily from the proof of Theorem 3.4.25 together with diagram (3.4.1). Indeed, more generally, we may conclude from the lower square of diagram (3.4.1) that every element in the image of F_γ lies in the image of the assembly map, and then the outer rectangle of (3.4.1) implies that we actually have

$$\mu_{(G,B)}(K_*^G(\underline{EG}; B)) = F_\gamma(K_*(B \rtimes_r G)).$$

Moreover, it follows also from (3.4.1) that F_γ is idempotent, so it is surjective if and only if it is the identity. Kasparov calls $F_\gamma(K_*(B \rtimes_r G))$ the *γ-part of* $K_*(B \rtimes_r G)$.

So one strategy for proving the Baum–Connes conjecture for given coefficients is to show that F_γ is the identity on $K_*(B \rtimes_r G)$. This method has been used quite effectively by Lafforgue in [Laf02] for proving the Baum–Connes conjecture with trivial coefficients for a large class of groups (including all real or p-adic reductive linear groups). For doing this he first introduced a Banach version of KK-theory in oder to show that the γ-element induces an isomorphism in K-theory of certain Banach algebras $\mathcal{S}(G)$, which can be viewed as algebras of Schwartz functions, which admit an embedding as dense subalgebras of $C_r^*(G)$ such that the inclusion $\mathcal{S}(G) \hookrightarrow C_r^*(G)$ induces an isomorphism in K-theory. As a result, the map F_γ is the identity on $K_*(C_r^*(G))$ which proves BC.

Extending his methods, Lafforgue later showed that all Gromov hyperbolic groups satisfy the Baum–Connes conjecture with coefficients (see [Laf12, Pus14]).

We close this section with a short argument of how Connes's Thom isomorphism for the K-theory of crossed products by $\mathbb{R}$ and the Pimsner–Voiculescu sequence for the K-theory of crossed products by $\mathbb{Z}$ can be deduced quite easily from the Dirac-dual Dirac method for actions of $\mathbb{R}$ as worked out in Example 3.4.21.

Corollary 3.4.28 (Connes's Thom isomorphism). *Let $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ be an action of $\mathbb{R}$ on the C^*-algebra A. Then the crossed product $A \rtimes_\alpha \mathbb{R}$ is KK-equivalent to $A \hat{\otimes} \mathit{Cl}_1$. In particular, there is a canonical isomorphism*

$$K_*(A \rtimes_\alpha \mathbb{R}) \cong K_*(A \hat{\otimes} \mathit{Cl}_1) = K_{*+1}(A).$$

Proof. To construct the KK-equivalence, let $\beta \in KK^{\mathbb{R}}(\mathbb{C}, C_0(\mathbb{R}) \otimes \mathit{Cl}_1)$ be as in Example 3.4.21. Then $1_A \otimes \beta \in KK^{\mathbb{R}}(A, A \hat{\otimes} C_0(\mathbb{R}) \hat{\otimes} \mathit{Cl}_1)$ is an $\mathbb{R}$-equivariant KK-equivalence between A and $A \hat{\otimes} C_0(\mathbb{R}) \hat{\otimes} \mathit{Cl}_1$ and its descent $J_{\mathbb{R}}(1_A \otimes \beta) \in KK(A \rtimes_\alpha \mathbb{R}, (A \hat{\otimes} C_0(\mathbb{R}) \hat{\otimes} \mathit{Cl}_1) \rtimes_{\alpha \otimes \tau \otimes \mathrm{id}_{\mathit{Cl}_1}} \mathbb{R})$ is a KK-equivalence as well. But $(A \hat{\otimes} C_0(\mathbb{R}) \hat{\otimes} \mathit{Cl}_1) \rtimes_{\alpha \otimes \tau \otimes \mathrm{id}_{\mathit{Cl}_1}} \mathbb{R}$ is isomorphic to $A \hat{\otimes} \mathcal{K}(L^2(\mathbb{R})) \hat{\otimes} \mathit{Cl}_1$ by an application of Lemma 2.4.1 and Example 2.6.6 (2). This finishes the proof. □

Theorem 3.4.29 (Pimsner–Voiculescu). *Let α be a fixed automorphism of the C^*-algebra A and let $n \mapsto \alpha^n$ be the corresponding action of $\mathbb{Z}$ on A. Then there is a six-term exact sequence*

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\mathrm{id} - \alpha_*} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\
{\scriptstyle\partial}\big\uparrow & & & & \big\downarrow{\scriptstyle\partial} \\
K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow[\iota_*]{} & K_1(A) & \xleftarrow[\mathrm{id} - \alpha_*]{} & K_1(A)
\end{array}$$

where $\iota : A \to A \rtimes_\alpha \mathbb{Z}$ denotes the canonical inclusion.

Scetch of proof. By Green's Theorem 2.6.4 the crossed product $A \rtimes_\alpha \mathbb{Z}$ is Morita equivalent to the crossed product $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}} A \rtimes_{\mathrm{Ind}\,\alpha} \mathbb{R}$ where the induced algebra $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}} A$ is isomorphic to the mapping cone $C_\alpha(A) = \{f : [0,1] \to A : f(0) = \alpha(f(1))\}$. Thus, by Connes's Thom isomophism, we get

$$K_*(A \rtimes_\alpha \mathbb{Z}) \cong K_*(C_\alpha(A) \rtimes \mathbb{R}) \cong K_{*+1}(C_\alpha(A)).$$

The mapping cone $C_\alpha(A)$ fits into a canonical short exact sequence

$$0 \to C_0(0,1) \otimes A \to C_\alpha(A) \to A \to 0,$$

where the quotient map is given by evaluation at 1, say. This gives the six-term exact sequence

$$\begin{array}{ccccc}
K_0(C_0(0,1) \otimes A) & \longrightarrow & K_0(C_\alpha(A)) & \longrightarrow & K_0(A) \\
\big\uparrow & & & & \big\downarrow \\
K_1(A) & \longleftarrow & K_1(C_\alpha(A)) & \longleftarrow & K_1(C_0(0,1) \otimes A).
\end{array}$$

Using $K_*(C_0(0,1) \otimes A) \cong K_{*+1}(A)$ and $K_{*+1}(C_\alpha(A)) \cong K_*(A \rtimes_\alpha \mathbb{Z})$ this turns into the six-term sequence of the theorem. (However, it is not completely trivial to check that the maps in the sequence coincide with the ones given in the theorem). □

The above method of proof of the Pimsner–Voiculescu theorem is taken from Blackadar's book [Bla86]. The original proof of Pimsner and Voiculescu in [PV80] was independent of Connes's Thom isomorphism and used a certain Toeplitz extension of $A \rtimes_\alpha \mathbb{Z}$.

3.4.4 The Baum–Connes conjecture for group extensions

Suppose that N is a closed normal subgroup of the second countable locally compact group G. Then, if A is a G-C^*-algebra, we would like to relate the Baum–Connes conjecture for G to the Baum–Connes conjecture for N and G/N. In order to do so, we first need to write the crossed product $A \rtimes_r G$ as an iterated crossed product $(A \rtimes_r N) \rtimes_r G/N$ for a suitable action of G/N on $A \rtimes_r N$. Unfortunately, this is not possible in general if we restrict ourselves to ordinary actions, but it can be done by using twisted actions as discussed in Section 2.8 above. In what follows we shall simply write $A \rtimes_r G/N$ for the reduced crossed product of a twisted action of the pair (G, N) in the sense of Green. We then get the desired isomorphism

$$A \rtimes_r G \cong (A \rtimes_r N) \rtimes_r G/N$$

(and similarly for the full crossed products). Recall that by Theorem 2.8.9 every Green-twisted (G, N)-action is equivariantly Morita equivalent to an ordinary action of G/N, which allows us to cheaply extend many results known for ordinary crossed products to the twisted case. In [CE01b] the authors extended the Baum–Connes assembly map to the category of twisted (G, N)-actions, and they constructed a partial assembly map

$$\mu_{(N,B)}^{(G,N)} : K_*^G(\underline{EG}; B) \to K_*^{G/N}(\underline{E(G/N)}, B \rtimes_r N) \tag{3.4.3}$$

such that the following diagram commutes

$$\begin{tikzcd}
K_*^G(\underline{EG}; B) \arrow[r, "\mu_{(G,B)}"] \arrow[d, "\mu_{(N,B)}^{(G,N)}"'] & K_*(B \rtimes_r G) \arrow[d, "\cong"] \\
K_*^{G/N}(\underline{E(G/N)}, B \rtimes_r N) \arrow[r, "\mu_{(G/N, B \rtimes_r N)}"'] & K_*((B \rtimes_r N) \rtimes_r G/N)
\end{tikzcd}$$

As a consequence, if the partial assembly map (3.4.3) is an isomorphism, then G satisfies BC for B if and only if G/N satisfies BC for $B \rtimes_r N$. Using these ideas, the following result has been shown in [CEOO04, Theorem 2.1] extending some earlier results of [CE01b, CE01a, Oyo01]:

Theorem 3.4.30. *Suppose that N is a closed normal subgroup of the second countable locally compact group G and let B be a G-C^*-algebra. Assume further, that the following condition* (A) *holds:*

(A) *Every closed subgroup $L \subseteq G$ such that $N \subseteq L$ and L/N is compact satisfies the Baum–Connes conjecture for B.*

Then G satisfies BC for G if and only if G/N satisfies BC for $B \rtimes_r N$.

Of course, the idea is that one should show that condition (A) implies that the partial assembly map (3.4.3) is an isomorphism. This has been the approach in [CE01b, CE01a], but in [CEOO04] a slightly different version of the partial assembly map has been used instead. Since every compact extension $N \subseteq L$ of an a-T-menable group N is a-T-menable (see [CCJ+01]), and since every a-T-menable group satisfies the Baum–Connes conjecture with coefficients, we get the following corollary:

Corollary 3.4.31. *Suppose that N is a closed normal subgroup of the second countable locally compact group G such that N is a-T-menable. Suppose further that G is any G-C^*-algebra. Then G satisfies BC for B if and only if G/N satisfies BC for $B \rtimes_r N$. In particular, if*

$$1 \to N \to G \to G/N \to 1$$

is a short exact sequence of second countable groups such that G and G/N are both a-T-menable, then G satisfies the Baum–Connes conjecture with coefficients.

Note that it is not true in general that G is a-T-menable if N and G/N are a-T-menable. For counterexamples see [CCJ+01].

Theorem 3.4.30 has been used extensively in [CEN03] to give the proof of the original Connes–Kasparov conjecture, which is equivalent to the Baum–Connes conjecture with trivial coefficients for the class of all (second countable) almost connected groups. The basic idea goes as follows: If G is any almost connected group, then one can use structure theory for such groups to see that there exists an amenable normal subgroup of N of G such that G/N is a reductive Lie-group. Since amenable groups are also a-T-menable, we can apply Corollary 3.4.31 to see that G satisfies BC with trivial coefficients if and only G/N satisfies BC with coefficient $C_r^*(N)$. Now by Lafforgue's results we know that the reductive group G/N satisfies BC with trivial coefficients and we somehow need to find good arguments that give us BC for the coefficient algebra $C_r^*(N)$ instead. It is this point where the arguments become quite complicated, and we refer to [CEN03] for the details of the proof.

3.5 The going-down (or restriction) principle and applications

3.5.1 The going-down principle

In this section we discuss an application of the Baum–Connes conjecture that helps, among other things, to give explicit K-theory computations in some interesting cases. Assume we have two G-C^*-algebras A and B and a G-equivariant

$*$-homomorphism $\phi : A \to B$. This map descends to a map

$$\phi \rtimes_r G : A \rtimes_r G \to B \rtimes_r G.$$

Suppose we want to prove that this map induces an isomorphism in K-theory. If G satisfies the Baum–Connes conjecture for A and B, then this problem is equivalent to the problem that the map

$$\phi_* : K_*^G(\underline{EG}; A) \to K_*^G(\underline{EG}; B)$$

is an isomorphism (use Exercise 3.4.10). The restriction (or Going-Down) principle allows us to deduce the isomorphism on the level of topological K-theory from the behaviour on compact subgroups of G. Let us state the theorem:

Theorem 3.5.1 (Going-down principle)**.** *Suppose that G is a second countable locally compact group, A and B are G-C^*-algebras, and $x \in KK^G(A, B)$ such that for all compact subgroups $K \subseteq G$ the class $\mathrm{res}_K^G(x) \in KK^K(A, B)$ induces an isomorphism*

$$\cdot \otimes_A \mathrm{res}_K^G(x) : KK_*^K(\mathbb{C}, A) \overset{\cong}{\to} KK_*^K(\mathbb{C}, B).$$

Then the map

$$\cdot \otimes_A x : K_*^G(\underline{EG}; A) \to K_*^G(\underline{EG}; B),$$

which is given on the level of $KK_*^G(C_0(X), A)$ for a G-compact $X \subseteq \underline{EG}$ by Kasparov product with x, is an isomorphism. As a consequence, if G satisfies the Baum–Connes conjecture for A and B, then the class x induces an isomorphism

$$\cdot \otimes_{A \rtimes_r G} J_G(x) : K_*(A \rtimes_r G) \xrightarrow{\cong} K_*(B \rtimes_r G).$$

Remark 3.5.2. There are many interesting groups G for which the trivial subgroup is the only compact subgroup (e.g., $G = \mathbb{R}^n, \mathbb{Z}^n$ or the free group F_n in n generators). For those groups, the condition on compact subgroups in the theorem reduces to the single condition that

$$\cdot \otimes_A x : K_*(A) \to K_*(B)$$

is an isomorphism. In many applications, this condition comes for free.

Remark 3.5.3. Instead of asking that $\cdot \otimes_A \mathrm{res}_K^G(x) : KK_*^K(\mathbb{C}, A) \overset{\cong}{\to} KK_*^K(\mathbb{C}, B)$ is an isomorphism for all compact subgroups K of G, we could alternatively require that

$$\cdot \otimes_{A \rtimes K} J_K(\mathrm{res}_K^G(x)) : K_*(A \rtimes K) \to K_*(B \rtimes K)$$

is an isomorphism for all such K. This follows from the commutativity of the diagram

$$\begin{tikzcd}
KK_*^K(\mathbb{C}, A) \arrow[r, "\cdot \otimes_A \mathrm{res}_K^G(x)"] \arrow[d, "\mu_{(K,A)}"'] & KK_*^K(\mathbb{C}, B) \arrow[d, "\mu_{(K,B)}"] \\
K_*(A \rtimes K) \arrow[r, "\cdot \otimes_{A \rtimes K} J_K(\mathrm{res}_K^G(x))"'] & K_*(B \rtimes K),
\end{tikzcd}$$

in which the vertical arrows are the isomorphisms of the Green–Julg theorem (see Example 3.4.8).

The proof of Theorem 3.5.1 in the above version is given in [ELPW10, Proposition 1.6.], but it relies very heavily on a more general going-down principle obtained by Chabert, Echterhoff and Oyono-Oyono in [CEOO04, Theorem 1.5]. In that paper we also show how Theorem 3.4.30 on the Baum–Connes conjecture for group extensions can be obtained as a consequence of the (more general) going-down principle. In what follows below we shall present the proof in the case where G is discrete. In this case the proof becomes much easier, but still reveals the basic ideas. Note that most of the relevant details for the discrete case first appeared (in the language of E-theory) in [GHT00, Chapter 12].

If G is discrete, then the first observation is that each G-compact proper G-space maps continuously and G-equivariantly into the geometric realisation of a G-finite G-simplicial complex. For this let $F \subseteq G$ be any finite subset of G that contains the identity of G. We then define

$$\mathcal{M}_F = \big\{ f \in C_c^+(G) : \sum_{g \in G} f(g) = 1 \text{ and } \forall g, h \in \operatorname{supp}(f) : g^{-1}h \in F \big\}.$$

Then $\mathcal{M}_F$ is the geometric realisation of a locally finite simplicial complex with vertices $\{g : g \in G\}$ and $\{g_1, \ldots, g_n\}$ is an n-simplex if and only if $g_i \neq g_j$ for $i \neq j$ and $g_i^{-1} g_j \in F$ for all $1 \leq i, j \leq n$. It follows directly from the definition that for any simplex $\{g_1, \ldots, g_n\}$ we have $g_1^{-1}\{g_1, \ldots, g_n\} \subseteq F$, hence $\mathcal{M}_F$ is G-finite in the sense that there exists a finite set $\mathcal{S}$ of simplices such that every other simplex is a translate of one in $\mathcal{S}$. Note that this implies that for all $f \in \mathcal{M}_F$ the formula

$$1 = \sum_{g \in G} f(g) = \sum_{g \in G} g \cdot f(e) \tag{3.5.1}$$

holds. Note also that if $F \subseteq F'$ for some finite set $F' \subseteq G$, then there is a canonical inclusion $\mathcal{M}_F \subseteq \mathcal{M}_{F'}$. With this we get:

Lemma 3.5.4. *Suppose that G is a discrete group and let X be a G-compact proper G-space. Suppose further that $c : X \to [0,1]$ is a cut-off function for X as in Section 3.4.2. Then there exists a finite subset $F \subseteq G$ such that $g(\operatorname{supp}(c)) \cap \operatorname{supp}(c) = \emptyset$ for all $g \notin F$ and a continuous G-map*

$$\varphi_c : X \to \mathcal{M}_F; \varphi_c(x) = [g \mapsto c^2(g^{-1}x)].$$

Moreover, for any other continuous G-map $\psi : X \to \mathcal{M}_F$ there is a finite set $F' \subseteq G$ containing F such that ψ is G-homotopic to φ in $\mathcal{M}_{F'}$.

Proof. Existence of the finite set F as in the lemma follows from compactness of the set $\{(g,x) : (gx, x) \in \operatorname{supp}(c) \times \operatorname{supp}(c)\} \subseteq G \times X$. It is compact since G acts properly on X. It is then straightforward to check that φ_c is a continuous

G-map. Suppose now that $\psi : X \to \mathcal{M}_F$ is any other continuous G-map. We define $\tilde{c} : X \to [0,1]$ as $\tilde{c}(x) := \sqrt{\psi(x)(e)}$. It follows then from (3.5.1) that $\tilde{c}$ is a cut-off function as well and that $\psi = \varphi_{\tilde{c}}$. Now let $d : X \times [0,1] \to [0,1]$ be given by $d(x,t) := \sqrt{(1-t)c^2(x) + t\tilde{c}^2(x)}$. Then there exists a finite set $F' \supseteq F$ such that $g(\mathrm{supp}(d)) \cap \mathrm{supp}(d) = \emptyset$ for all $g \notin F'$. The continuous G-map $\varphi_d : X \times [0,1] \to \mathcal{M}_{F'}$ then evaluates to φ_c at $t = 0$ (using $\mathcal{M}_F \subseteq \mathcal{M}_{F'}$) and to $\varphi_{\tilde{c}} = \psi$ at $t = 1$. $\square$

Lemma 3.5.5. *Let G be a discrete group. Then for every G-C^*-algebra A we have*

$$K_*^G(\underline{EG}; A) = \lim_F KK_*^G(C_0(\mathcal{M}_F), A),$$

where F runs through all finite subsets of G and the limit is taken with respect to the canonical inclusion $\mathcal{M}_F \subseteq \mathcal{M}_{F'}$ if $F \subseteq F'$.

Proof. This follows from the definition

$$K_*^G(\underline{EG}; A) = \lim_X KK_*^G(C_0(X), A),$$

where X runs through the G-compact subsets of $\underline{EG}$ and Lemma 3.5.4: By the universal property of $\underline{EG}$ there are G-compact subsets $X_F \subseteq \underline{EG}$ and G-continuous maps $\mathcal{M}_F \to X_F \subseteq \underline{EG}$, which, up to a possible enlargement of X_F, are unique up to G-homotopy. On the other hand, Lemma 3.5.4 provides maps $X_F \to \mathcal{M}_{F'}$ for some $F' \supseteq F$ which, up to passing to a bigger set F'' if necessary, is also unique up to G-homotopy. Thus we get a zigzag diagram

$$\begin{tikzcd}
 & KK_*^G(C_0(\mathcal{M}_F), A) \arrow[r] \arrow[d] & KK_*^G(C_0(\mathcal{M}_{F'}), A) \arrow[r] \arrow[d] & KK_*^G(C_0(\mathcal{M}_{F''}), A)\ldots.. \\
KK_*^G(C_0(X), A) \arrow[r] \arrow[ur] & KK_*^G(C_0(X_F), A) \arrow[r] \arrow[ur] & KK_*^G(C_0(X_{F'}), A) \arrow[r] \arrow[ur] & \ldots..
\end{tikzcd}$$

which commutes sufficiently well to induce an isomorphism of the inductive limits. $\square$

The next lemma gives the crucial point in the proof Theorem 3.5.1 in the case of discrete G. It has first been shown in the setting of E-theory in [GHT00, Lemma 12.11]. A more general version for arbitrary open subgroups H of a second countable locally compact group G has been shown in [CE01a, Proposition 5.14].

Lemma 3.5.6. *Suppose that $K \subseteq G$ is a finite subgroup of the discrete group G. Then, for every G-C^*-algebra B, there is a well defined* compression isomorphism

$$\mathrm{comp}_K : KK_*^G(C_0(G/K), B) \to KK_*^K(\mathbb{C}, B)$$

given as the composition of the maps

$$KK_*^G(C_0(G/K), B) \overset{\mathrm{res}_K^G}{\longrightarrow} KK_*^K(C_0(G/K), B) \overset{\iota^*}{\longrightarrow} KK_*^K(\mathbb{C}, B),$$

where $\iota : \mathbb{C} \hookrightarrow C_0(G/K)$ denotes the inclusion $\lambda \mapsto \lambda\delta_{eK}$ with δ_{eK} the characteristic function of the open one-point set $\{eK\} \subseteq G/K$.

Proof. We construct an inverse

$$\mathrm{ind}_K^G : KK_*^K(\mathbb{C}, B) \to KK_*^G(C_0(G/K), B)$$

for the compression map. We may restrict ourselves to the case of the K_0-groups, the K_1-case then follows from passing from B to $B \otimes C_0(\mathbb{R})$. Let $(\mathcal{E}, 1, \gamma, T)$ be a representative for a class $x \in KK^K(\mathbb{C}, B)$ where $T \in \mathcal{L}(\mathcal{E})$ is a K-invariant operator such that $T^* - T, T^2 - 1 \in \mathcal{K}(\mathcal{E})$. We then define a Hilbert B-module $\mathrm{ind}_K^G \mathcal{E}$ as

$$\mathrm{Ind}_K^G \mathcal{E} = \left\{ \xi : G \to \mathcal{E} : \begin{array}{r} \text{s.t. } \xi(gk) = \gamma_{k^{-1}}(\xi(g)) \text{ for all } g \in G, k \in K \\ \text{and } \sum_{g \in G} \beta_g(\langle \xi(g), \xi(g) \rangle_B) < \infty \end{array} \right\}$$

where $\sum_{g \in G} \beta_g(\langle \xi(g), \xi(g) \rangle) < \infty$ just means that the sum converges in the norm-topology of B. The grading on $\mathrm{Ind}_K^G \mathcal{E}$ is given by the grading of $\mathcal{E}$ applied pointwise to the elements of $\mathrm{Ind}_K^G \mathcal{E}$. We define the B-valued inner product and the right B-action on $\mathrm{Ind}_K^G \mathcal{E}$ by

$$\langle \xi, \eta \rangle_B = \frac{1}{|K|} \sum_{g \in G} \beta_g(\langle \xi(g), \eta(g) \rangle_B) \quad \text{and} \quad (\xi \cdot b)(g) = \xi(g) \beta_{g^{-1}}(b)$$

for all $\xi, \eta \in \mathrm{Ind}_K^G \mathcal{E}$, $b \in B$ and $g \in G$. Moreover, we define a $*$-homomorphism

$$M : C_0(G/K) \to \mathcal{L}(\mathrm{Ind}_K^G \mathcal{E}); (M(f)\xi)(g) := f(gK)\xi(g)$$

and an operator $\tilde{T} \in \mathcal{L}(\mathrm{Ind}_K^G \mathcal{E})$ by $(\tilde{T}\xi)(g) = T\xi(g)$. Finally, the G-action $\mathrm{Ind}\,\gamma : G \to \mathrm{Aut}(\mathrm{Ind}_K^G \mathcal{E})$ is given by $(\mathrm{Ind}\,\gamma_g \xi)(h) = \xi(g^{-1}h)$ for $g, h \in G$.

It is then an easy exercise to check that $(\mathrm{Ind}_K^G \mathcal{E}, M, \mathrm{Ind}\,\gamma, \tilde{T})$ is a G-equivariant $C_0(G/K) - B$ Kasparov cycle such that

$$\mathrm{comp}_K\left([\mathrm{Ind}_K^G \mathcal{E}, M, \mathrm{Ind}\,\gamma, \tilde{T}]\right) = [\mathcal{E}, 1, \gamma, T].$$

For the converse, averaging over K, we may first assume that for a given class $x = [\mathcal{F}, \Phi, \nu, S] \in KK^G(C_0(G/K), B)$ the operator S is K-invariant and that $\Phi : C_0(G/K) \to \mathcal{L}(\mathcal{F})$ is non-degenerate. Let $\tilde{S} = \sum_{gK \in G/K} \Phi(\delta_{gK}) S \Phi(\delta_{gK})$. We claim that $\tilde{S}$ is a compact perturbation of S, i.e.,

$$(S - \tilde{S})\Phi(f) = \sum_{gK \in G/K} (S - \Phi(\delta_{gK})S)\Phi(\delta_{gK}) f(gK) \in \mathcal{K}(\mathcal{F})$$

for all $f \in C_0(G/K)$. For this we first observe that, since $[S, \Phi(\delta_{gK})] \in \mathcal{K}(\mathcal{F})$ for all $gK \in G/K$, each summand lies in $\mathcal{K}(\mathcal{F})$. Moreover, since $f \in C_0(G/K)$, the sum converges in norm, which proves the claim. Thus, replacing S by $\tilde{S}$ if necessary, we may assume that $[S, \Phi(f)] = 0$ for all $f \in C_0(G/K)$. In particular, if $p := \Phi(\delta_{eK})$, it follows that $S = pSp + (1 - p)S(1 - p)$.

The class $\mathrm{comp}_K(x)$ is represented by the KK-cycle $[\mathcal{F}, \Phi|_{\mathbb{C}\delta_{eK}}, \nu|_K, S]$. For $p = \Phi(\delta_{eK})$, let $\mathcal{E} := p\mathcal{F}$, $T = pSp$ and $\gamma : K \to \mathrm{Aut}(\mathcal{E})$ be the restriction of $\nu|_K$ to the summand $\mathcal{E}$ of $\mathcal{F}$. Since $S = pSp + (1-p)S(1-p)$ and since $[(1-p)\mathcal{F}, \Phi|_{\mathbb{C}\delta_{eK}}, \nu|_K, (1-p)S(1-p)]$ is degenerate, we see that $\mathrm{comp}_K(x) = [\mathcal{E}, 1, \gamma, T]$. It is then straightforward to check that

$$U : \mathrm{Ind}_K^G \mathcal{E} \to \mathcal{F}; U\xi = \frac{1}{|K|}\sum_{g\in G} \nu_g(\xi(g))$$

is a an isomorphism of Hilbert-B-modules that induces an isomorphism between the KK-cycles $(\mathrm{Ind}_K^G \mathcal{E}, M, \mathrm{Ind}\,\gamma, \tilde{T})$ and $(\mathcal{F}, \Phi, \nu, S)$. This finishes the proof. $\square$

Suppose now that X is a proper G-space, $U \subseteq X$ is an open G-invariant subset of X, and $Y := X \smallsetminus U$. Since $C_0(X)$ is nuclear, there exists a completely positive contractive section $\Phi : C_0(Y) \to C_0(X)$ for the restriction homomorphism $\mathrm{res}_Y : C_0(X) \mapsto C_0(Y) : f \mapsto f|_Y$. By properness of the action, we may average Φ with the help of a cut-off function $c : X \to [0, \infty)$ to get the G-equivariant completely positive and contractive section

$$\Phi^G(\varphi)(x) := \int_G c^2(g^{-1}x)\Phi(\varphi)(x)\, dg$$

for $\varphi \in C_0(Y)$. It follows then from Theorem 3.3.28 that for every G-algebra B there exists a six-term exact sequence

$$\begin{array}{ccccc}
KK_0^G(C_0(Y), B) & \xrightarrow{\mathrm{res}_Y^*} & KK_0^G(C_0(X), B) & \xrightarrow{\iota^*} & KK_0^G(C_0(U), B) \\
\partial\uparrow & & & & \downarrow\partial \\
KK_1^G(C_0(U), B) & \xleftarrow[\iota^*]{} & KK_1^G(C_0(X), B) & \xleftarrow[\mathrm{res}_K^*]{} & KK_0^G(C_0(Y), B).
\end{array}$$

We are now ready for

Proof of Theorem 3.5.1 *for G discrete.* By Lemma 3.5.5 it suffices to show that for every (geometric realisation) of a G-finite G-simplicial complex X, the map

$$\cdot \otimes_A x : KK_*^G(C_0(X), A) \to KK_*^G(C_0(X), B)$$

given by taking Kasparov product with the class $x \in KK^G(A, B)$ is an isomorphism. We do the proof by induction on the dimension of X. Suppose first that $\dim(X) = 0$. In that case X is discrete and therefore decomposes into a finite union of G-orbits

$$X = G(x_1) \,\dot\cup\, G(x_2) \,\dot\cup \cdots \dot\cup\, G(x_l)$$

for suitable elements $x_1, \ldots, x_l$ in X. Then we have $C_0(X) \cong \bigoplus_{i=1}^l C_0(G(x_i))$ and $KK_*^G(C_0(X), A) = \prod_{i=1}^l KK_*^G(C_0(G(x_i)), A)$ (and similarly for $KK_*^G(C_0(X), B)$).

Thus, it suffices to show that $\cdot \otimes_A x : KK_*^G(C_0(G(x_i)), A) \xrightarrow{\cong} KK_G^*(C_0(G(x_i)), B)$ for all $1 \leq i \leq l$. But $G(x_i) \cong G/G_{x_i}$ as a G-space, where $G_{x_i} = \{g \in G : gx_i = x_i\}$ denotes the stabiliser of x_i. By properness, we have G_{x_i} finite for all i. We then get a commutative diagram

$$\begin{tikzcd}
KK_*^G(C_0(G/G_{x_i}), A) \arrow[r, "\cdot\otimes_A x"] \arrow[d, "\mathrm{comp}_{G_{x_i}}"'] & KK_*^G(C_0(G/G_{x_i}), B) \arrow[d, "\mathrm{comp}_{G_{x_i}}"] \\
KK_*^{G_{x_i}}(\mathbb{C}, A) \arrow[r, "\cdot\otimes_A \mathrm{res}^G_{G_{x_i}}(x)"] & KK_*^{G_{x_i}}(\mathbb{C}, B)
\end{tikzcd}$$

in which all vertical arrows are isomorphisms by Lemma 3.5.6 and the lower horizontal arrow is an isomorphism by the assumption of the theorem. Hence the upper horzontal arrow is an isomorphism as well.
Suppose now that $\dim(X) = n$. After performing a baricentric subdivision of X, if necessary, we may assume that the action of G on X satisfies the following condition: If Δ is a simplex in X, then an element $g \in G$ either fixes all of Δ or $g{\cdot}\mathrm{int}(\Delta) \cap \mathrm{int}(\Delta) = \emptyset$, where $\mathrm{int}(\Delta)$ denotes the interior of Δ. Now let $\widetilde{X}$ denote the union of the interiors of all n-dimensional simplices in X. Then $X_{n-1} := X \smallsetminus \widetilde{X}$ is an $n-1$-dimensional G-simplicial complex and by the induction assumption we have $KK_*^G(C_0(X_{n-1}), A) \cong KK_*^G(C_0(X_{n-1}), B)$ via taking Kasparov product with x. We now show that $KK_*^G(C_0(\widetilde{X}), A) \cong KK_*^G(C_0(\widetilde{X}), B)$ as well. If this is done, then the five-lemma applied to the diagram

$$\begin{tikzcd}
KK_{*-1}^G(C_0(\widetilde{X}), A) \arrow[r, "\partial"] \arrow[d, "\cdot\otimes_A x"'] & KK_*^G(C_0(X_{n-1}), A) \arrow[r, "\mathrm{res}^*"] \arrow[d, "\cdot\otimes_A x"'] & KK_*^G(C_0(X), A) \arrow[d, "\cdot\otimes_A x"'] \\
KK_{*-1}^G(C_0(\widetilde{X}), B) \arrow[r, "\partial"] & KK_*^G(C_0(X_{n-1}), B) \arrow[r, "\mathrm{res}^*"] & KK_*^G(C_0(X), B)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\iota^*"] & KK_*^G(C_0(\widetilde{X}), A) \arrow[r, "\partial"] \arrow[d, "\cdot\otimes_A x"] & KK_{*+1}^G(C_0(X_{n-1}), A) \arrow[d, "\cdot\otimes_A x"] \\
{} \arrow[r, "\iota^*"] & KK_*^G(C_0(\widetilde{X}), B) \arrow[r, "\partial"] & KK_{*+1}^G(C_0(X_{n-1}), B)
\end{tikzcd}$$

shows that $KK_*^G(C_0(X), A) \cong KK_G^*(C_0(X), B)$.

To see that $KK_*^G(C_0(\widetilde{X}), A) \cong KK_*^G(C_0(\widetilde{X}), B)$ we first observe that $\widetilde{X}$ is a finite union of orbits of open simplices $\mathrm{int}(\Delta_1), \ldots, \mathrm{int}(\Delta_k)$ for some $k \in \mathbb{N}$. Via the corresponding product decomposition of the KK-groups, we may then assume that $\widetilde{X} = G{\cdot}\mathrm{int}(\Delta)$ for a single open n-simplex Δ. By our assumption on the action of G on X, we have

$$G \cdot \mathrm{int}(\Delta) \cong G/G_\Delta \times \mathrm{int}(\Delta)$$

where $G_\Delta = \{g \in G : g \cdot \Delta = \Delta\}$ denotes the (finite!) stabiliser of Δ and where the G-action on $G/G_\Delta \times \mathrm{int}(\Delta)$ is given by left translation on the first factor. We then get a diagram

$$\begin{array}{ccccc} KK_*^G(C_0(G/G_\Delta \times \mathrm{int}(\Delta)), A) & \xrightarrow[\cong]{\mathrm{Bott}} & KK_{*+n}^G(C_0(G/G_\Delta), A) & \xrightarrow[\cong]{\mathrm{comp}_{G_\Delta}} & KK_{*+n}^{G_\Delta}(\mathbb{C}, A) \\ \downarrow{\scriptstyle \cdot\otimes_A x} & & \downarrow{\scriptstyle \cdot\otimes_A x} & & \downarrow{\scriptstyle \cdot\otimes_A \mathrm{res}^G_{G_\Delta}(x)} \\ KK_*^G(C_0(G/G_\Delta \times \mathrm{int}(\Delta)), B) & \xrightarrow[\cong]{\mathrm{Bott}} & KK_{*+n}^G(C_0(G/G_\Delta), B) & \xrightarrow[\cong]{\mathrm{comp}_{G_\Delta}} & KK_{*+n}^{G_\Delta}(\mathbb{C}, B). \end{array}$$

Since, by assumption, the last vertical arrow is an isomorphism, the result follows. □

We should note that for groups that satisfy the strong Baum–Connes conjecture in the sense of Definition 3.4.17, a stronger version of Theorem 3.5.1 has been shown by Meyer and Nest in [MN06, Theorem 9.3]:

Theorem 3.5.7 (Meyer-Nest)**.** *Suppose that the second countable group G satisfies the strong Baum–Connes conjecture (e.g., this is satisfied if G is a-T-menable or, in particular, if G is amenable) and assume that $x \in KK^G(A, B)$ such that for every compact subgroup K of G the class $J_K(\mathrm{res}^G_K(x)) \in KK(A \rtimes K, B \rtimes K)$ is a KK-equivalence. Then $J_G(x) \in KK(A \rtimes_r G, B \rtimes_r G)$ is KK-equivalence as well.*

3.5.2 Applications of the going-down principle

We now give a number of applications. The first one is the proof that every exact locally compact group satisfies the strong Novikov conjecture. Recall that a locally compact group is called exact (in the sense of Kirchberg and Wassermann) if for every short exact sequence of G-C^*-algebras

$$0 \to I \xrightarrow{\iota} A \xrightarrow{q} A/I \to 0$$

the corresponding sequence of reduced crossed products

$$0 \to I \rtimes_r G \xrightarrow{\iota \rtimes_r G} A \rtimes_r G \xrightarrow{q \rtimes_r G} A/I \rtimes_r G \to 0$$

is also exact. It has been known for a long time by work of Ozawa and others (see [Oza00]) that a discrete group is exact if and only if it admits an amenable action on a compact space X. This means that the transformation groupoid $X \rtimes G$ is topologically amenable in the sense of [ADR00]. Very recently the result of Ozawa has been generalised by Brodzki, Cave and Li ([BCL16]) to second countable locally compact groups. The following result has been shown first for discrete G by Higson in [Hig00]. The result has been extended in [CEOO04] to the case of second countable locally compact groups acting amenably on a compact space.

Theorem 3.5.8. *Let G be an exact second countable locally compact group. Then G satisfies the strong Novikov conjecture with coefficients, i.e., for each G-C^*-algebra B the assembly map*

$$\mu_{(G,B)} : K_*^G(\underline{EG}; B) \to K_*(B \rtimes_r G)$$

is split injective. A similar statement holds true for the full assembly map $\mu^{\mathrm{full}}_{(G,B)}$.

Proof. By [BCL16], being exact is equivalent to the condition that there exists a compact amenable G-space X. Following the arguments given by Higson in [Hig00, Lemma 3.5 and Lemma 3.6] we may as well assume that X is a metrisable convex space and G acts by affine transformations. In particular, X is K-equivariantly contractible for every compact subgroup K of X. It then follows that the inclusion map $\iota : \mathbb{C} \to C(X)$ is a KK^K-equivalence for every compact subgroup K of G – it's inverse is given by the map $C(X) \to \mathbb{C}; f \mapsto f(x_0)$ for any K-fixed point $x_0 \in X$. It then follows, that for every G-C^*-algebra B, the $*$-homomorphism $B \to B \otimes C(X); b \mapsto b \otimes 1_X$ is a KK^K-equivalence as well. Thus it follows from Theorem 3.5.1 that

$$\Phi_* : K^G_*(\underline{EG}; B) \to K^G_*(\underline{EG}; B \otimes C(X))$$

is an isomorphism. Moreover, by Tu's extension of the Higson–Kasparov theorem to groupoids (see [Tu99a]), the assembly map

$$\mu_{(X\rtimes G, B\otimes C(X))} : K^{X\rtimes G}_*(\underline{E(X \rtimes G)}, B \otimes C(X)) \to K_*((B \otimes C(X)) \rtimes_r G)$$

is an isomorphism, since $X \rtimes G$ is an amenable, and hence a-T-menable groupoid. Moreover, it has been shown in [CEOO03] that the *forgetful map*

$$F : K^{X\rtimes G}_*(\underline{E(X \rtimes G)}, B \otimes C(X)) \to K^G_*(\underline{EG}, B \otimes C(X))$$

is an isomorphism and that the diagram

$$\begin{array}{ccc}
K^{X\rtimes G}_*(\underline{E(X \rtimes G)}, B \otimes C(X)) & \xrightarrow{\mu_{(X\rtimes G, B\otimes C(X))}} & K_*((B \otimes C(X)) \rtimes_r G) \\
{\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle =} \\
K^G_*(\underline{EG}, B \otimes C(X)) & \xrightarrow{\mu_{(G, B\otimes C(X))}} & K_*((B \otimes C(X)) \rtimes_r G)
\end{array}$$

commutes. The result then follows from the commutative diagram

$$\begin{array}{ccc}
K^G_*(\underline{EG}, B) & \xrightarrow{\mu_{(G, B\otimes C(X))}} & K_*(B \rtimes_r G) \\
{\scriptstyle \Phi_*}\big\downarrow & & \big\downarrow{\scriptstyle \Phi\rtimes_r G_*} \\
K^G_*(\underline{EG}, B \otimes C(X)) & \xrightarrow{\mu_{(G, B\otimes C(X))}} & K_*((B \otimes C(X)) \rtimes_r G)
\end{array}$$

in which the left vertical arrow and the bottom horizontal arrow are isomorphisms. □

The main application of the going-down principle in [CEOO04] was the proof of a version of the Künneth formula for $K^G_*(\underline{EG}, B)$ with applications for the Baum–Connes conjecture with trivial coefficients. We don't want to go into the details here. But we would like to mention some other useful applications. By a homotopy between two actions $\alpha^0, \alpha^1 : G \to \mathrm{Aut}(A)$ we understand a path of actions $\alpha^t : G \to \mathrm{Aut}(A)$, $t \in [0,1]$, such that

$$\big(\alpha_g(f)\big)(t) := \alpha_g^t(f(t)) \quad \forall f \in A[0,1], g \in G, t \in [0,1]$$

defines an action on $A[0,1] = C([0,1], A)$. The following is, of course, a direct consequence of Theorem 3.5.1:

Corollary 3.5.9. *Suppose that $\alpha : G \to \mathrm{Aut}(A[0,1])$ is a homotopy between the actions $\alpha^0, \alpha^1 : G \to \mathrm{Aut}(A)$ and assume that G satisfies BC for $(A[0,1], \alpha)$ and (A, α^t) for $t = 0, 1$. Suppose further that for $t = 0, 1$ and for every compact subgroup K of G the evaluation map $\epsilon_t : A[0,1] \to A; f \mapsto f(t)$ induces an isomorphism $\epsilon_t \rtimes K_* : K_*(A[0,1] \rtimes_\alpha K) \xrightarrow{\cong} K_*(A \rtimes_{\alpha^t} K)$. Then*

$$\epsilon_t \rtimes_r G_* : K_*(C([0,1], A) \rtimes_{\alpha,r} G) \to K_*(A \rtimes_{\alpha^t,r} G)$$

is an isomorphism as well. In particular, we have $K_*(A \rtimes_{\alpha^0,r} G) \cong K_*(A \rtimes_{\alpha^1,r} G)$.

Of course, the condition on the compact subgroups in the above corollary is quite annoying. However, for those groups that have no compact subgroups other than the trivial group, the corollary becomes very nice, since the evaluation maps $\epsilon_t : A[0,1] \to A; f \mapsto f(t)$ are always KK-equivalences.

Corollary 3.5.10. *Suppose that $\alpha : G \to \mathrm{Aut}(A[0,1])$ is a homotopy between the actions $\alpha^0, \alpha^1 : G \to \mathrm{Aut}(A)$ and assume that G satisfies BC for $(A[0,1], \alpha)$ and (A, α^0), (A, α^1). If $\{e\}$ is the only compact subgroup of G, then $K_*(A \rtimes_{\alpha^0,r} G) \cong K_*(A \rtimes_{\alpha^1,r} G)$.*

In [ELPW10] Corollary 3.5.9 has been used to show that for groups G that satisfy BC for suitable coefficients, the K-theory of reduced twisted group algebras $C_r^*(G, \omega)$, where $\omega : G \times G \to \mathbb{T}$ is a Borel 2-cocycle on G, only depends on the homotopy class of the 2-cocycle ω (with a suitable definition of homotopy). We don't want to go into the details here, but we do want to mention that if $(\omega_t)_{t \in [0,1]}$ is such a homotopy of 2-cocycles, it induces a homotopy $\alpha : G \to \mathrm{Aut}(\mathcal{K}[0,1])$ of actions of G on the compact operators $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ such that $\mathcal{K} \rtimes_{\alpha^t,r} G \cong \mathcal{K} \otimes C_r^*(G, \omega_t)$ for all $t \in [0,1]$. We refer to Section 2.8.6 for a discussion of twisted group algebras. It follows from the results in [EW01] that a homotopy of actions of a compact group K on $\mathcal{K}$ must be exterior equivalent to a constant path of actions, hence

$$\mathcal{K}[0,1] \rtimes K \cong (\mathcal{K} \rtimes K)[0,1]$$

for all compact subgroups of G, from which it follows that the evaluation maps

$$\epsilon_t \rtimes K : \mathcal{K}[0,1] \rtimes K \to \mathcal{K} \rtimes K$$

are KK-equivalences for all K. Thus, if G satisfies BC for $\mathcal{K}$ and $\mathcal{K}[0,1]$ (for the relevant actions), it follows from Corollary 3.5.9 that

$$K_*(C_r^*(G, \omega_0)) \cong K_*(\mathcal{K} \rtimes_{\alpha^0,r} G) \cong K_*(\mathcal{K} \rtimes_{\alpha^1,r} G) \cong K_*(C_r^*(G, \omega_1)).$$

Note that this result extends earlier results of Elliott ([Ell81]) for the case of finitely generated abelian groups G and of Packer and Raeburn [PR92] for a class of solvable groups G. The main application in [ELPW10] was given for the computation of the K-theory of the crossed products $A_\theta \rtimes F$ of the noncommutative 2-tori A_θ, $\theta \in [0,1]$ with finite subgroups $F \subseteq \mathrm{SL}(2,\mathbb{Z})$ acting canonically on A_θ. It turned out that $A_\theta \rtimes F \cong C_r^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$ for some cocycles ω_θ which depend continuously on the parameter θ. Since $\mathbb{Z}^2 \rtimes F$ is amenable it satisfies strong BC, and then it follows from the above results that

$$K_*(A_\theta \rtimes F) \cong K_*(C_r^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)) \cong K_*(C_r^*(\mathbb{Z}^2 \rtimes F, \omega_0)) = K_*(C(\mathbb{T}^2) \rtimes F).$$

The last group can be computed by methods from classical topology. We refer to [ELPW10] for further details on this. Note that in this situation we can also use Theorem 3.5.7 to deduce that all algebras $A_\theta \rtimes F$, $\theta \in [0,1]$, are pairwise KK-equivalent.

3.5.3 Crossed products by actions on totally disconnected spaces

We now want to apply our techniques to certain crossed products of groups G acting "nicely" on totally disconnected spaces Ω. The main application of this will be given for reduced semigroup algebras and crossed products by certain semigroups, which will be presented elsewhere in this book, in Chapters 5 and 6.

If Ω is a totally disconnected locally compact space we denote by $\mathcal{U}_c(\Omega)$ the collection of all compact open subsets of Ω. This set is countable if and only if Ω has a countable basis of its topology, i.e., Ω is second countable. For any set $\mathcal{V} \subseteq \mathcal{U}_c(\Omega)$ we say that $\mathcal{V}$ *generates* $\mathcal{U}_c(\Omega)$, if every subset $\mathcal{U} \subseteq \mathcal{U}_c(\Omega)$ that contains $\mathcal{V}$ and is closed under finite intersections, finite unions, and taking differences $U \smallsetminus W$ with $U, W \in \mathcal{U}$, must coincide with $\mathcal{U}_c(\Omega)$.

Let $C_c^\infty(\Omega)$ denotes the dense subalgebra of $C_0(\Omega)$ consisting of locally constant functions with compact supports on Ω. Then

$$C_c^\infty(\Omega) = \mathrm{span}\{1_U : U \in \mathcal{U}_c(\Omega)\},$$

where 1_U denotes the indicator function of $U \subseteq \Omega$. The straightforward proof of the following lemma is given in [CEL13, Lemma 2.2]:

Lemma 3.5.11. *Suppose that $\mathcal{V}$ is a family of compact open subsets of the totally disconnected locally compact space Ω. Then the following are equivalent:*

(i) *The set $\{1_V : V \in \mathcal{V}\}$ of characteristic functions of the elements in $\mathcal{V}$ generates $C_0(\Omega)$ as a C^*-algebra.*

(ii) *The set $\mathcal{V}$ generates $\mathcal{U}_c(\Omega)$ in the sense explained above.*

If, in addition, $\mathcal{V}$ is closed under finite intersections, then (i) *and* (ii) *are equivalent to*

(iii) $\mathrm{span}\{1_V : V \in \mathcal{V}\} = C_c^\infty(\Omega)$.

We see in particular that the commutative C^*-algebra $C_0(\Omega)$ is generated as a C^*-algebra by a (countable) set of projections. The converse is also true: If D is any commutative C^*-algebra generated by a set of projections $\{e_i : i \in I\} \subseteq D$ and if $\Omega = \mathrm{Spec}(D)$ is the Gelfand spectrum of D, then Ω is totally disconnected and the sets

$$\mathcal{V} = \{\mathrm{supp}(\hat{e}_i) : i \in I\},$$

where, for any $d \in D$, $\hat{d} \in C_0(\Omega)$ denotes the Gelfand transform of d, is a family of compact open subsets of Ω which generates $\mathcal{U}_c(\Omega)$. For a proof see [CEL13, Lemma 2.3]. Thus, there is an equivalence between studying sets of projections which generate D or sets of compact open subsets of Ω that generate $\mathcal{U}_c(\Omega)$.

Lemma 3.5.12. *Suppose that $\{e_i : i \in I\}$ is a set of projections in the commutative C^*-algebra D. Then for each finite subset $F \subseteq I$ there exists a smallest projection $e \in D$ such that $e_i \leq e$ for every $i \in F$. We then write $e := \vee_{i \in F} e_i$.*

Proof. By the above discussion we may assume that $D = C_0(\Omega)$ for some totally disconnected space Ω. For each $i \in F$ let $V_i := \mathrm{supp}(e_i)$. Then $e = 1_V$ with $V = \cup_{i=1}^{l} V_i$. □

The independence condition given in the following definition is central for the results of this section:

Definition 3.5.13. Suppose that $\{X_i : i \in I\}$ is a family of subsets of a set X. We then say that $\{X_i : i \in I\}$ is *independent* if for any finite subset $F \subseteq I$ and for any index $i_0 \in I$ we have

$$X_{i_0} = \cup_{i \in F} X_i \Rightarrow i_0 \in F.$$

Similarly, a family $\{e_i : i \in I\}$ of projections in the commutative C^*-algebra D is called *independent* if for any finite subset $F \subseteq I$ and every $i_0 \in I$ we have

$$e_{i_0} = \vee_{i \in F} e_i \Rightarrow i_0 \in F.$$

Of course, if $D = C_0(\Omega)$ then $\{e_i : i \in I\}$ is an independent family of projections in D if and only if $\{\mathrm{supp}(e_i) : i \in I\}$ is an independent family of compact open subsets of Ω. The following lemma is [CEL13, Lemma 2.8]. The proof follows from [Li13, Proposition 2.4]:

Lemma 3.5.14. *Suppose that $\{e_i : i \in I\}$ is an independent set of projections in the commutative C^*-algebra D which is closed under finite multiplication up to* 0*. Then $\{e_i : i \in I\}$ is independent if and only if it is linearly independent.*

Definition 3.5.15. Suppose that Ω is a totally disconnected locally compact Hausdorff space. An independent family $\mathcal{V}$ of nonempty compact open subsets of Ω is called a *regular basis* (for the compact open subsets of Ω) if it generates $\mathcal{U}_c(\Omega)$ and if $\mathcal{V} \cup \{\emptyset\}$ is closed under finite intersections.

A family of projections $\{e_i : i \in I\}$ in the commutative C^*-algebra D is called a *regular basis* for D if it is (linearly) independent, closed under finite multiplication up to 0 and generates D as a C^*-algebra.

The following lemma is a consequence of the above discussions. We leave the details to the reader.

Lemma 3.5.16. *A family of projections $\{e_i : i \in I\}$ in the commutative C^*-algebra D is a regular basis for D if and only if the set $\mathcal{V} = \{\operatorname{supp} \hat{e}_i : i \in I\}$ is a regular basis for the compact open subsets of $\Omega = \operatorname{Spec}(D)$. Conversely, $\mathcal{V}$ is a regular basis for the compact open subsets of the locally compact space Ω if and only if $\{1_V : V \in \mathcal{V}\}$ is a regular basis for $C_0(\Omega)$.*

It is not difficult to see that every totally disconnected locally compact space Ω has a regular basis for its compact open sets. A formal proof is given in [CEL13, Proposition 2.12]. The following example shows the existence for the Cantor set:

Example 3.5.17. Let $\Omega = \{0,1\}^{\mathbb{Z}}$ denote the direct product of copies of $\{0,1\}$ over $\mathbb{Z}$ equipped with the product topology. For each finite subset $F \subseteq \mathbb{Z}$ let

$$V_F = \{(\epsilon_n)_n \in \Omega : \epsilon_n = 0 \,\forall n \in F\}$$

be the corresponding *cylinder set* in Ω. It is then an easy exercise to check that the collection $\mathcal{V} = \{V_F : F \subseteq \mathbb{Z} \text{ finite}\}$ is a regular basis for the compact open subsets of Ω.

Assume now that G is a second countable locally compact group and Ω is a second countable totally disconnected G-space such that there exists a G**-invariant** regular basis $\mathcal{V} = \{V_i : i \in I\}$ for the compact open subsets of Ω. For all $i \in I$ let $e_i = 1_{V_i}$ be the characteristic function of V_i. Then, since $\mathcal{V}$ is G-invariant, the action of G on Ω induces an action of G on I. Consider the unitary representation $U : G \to \mathcal{U}(\ell^2(I)); (U_g\xi)(i) = \xi(g^{-1}i)$ and let $\operatorname{Ad} U : G \to \operatorname{Aut}(\mathcal{K}(\ell^2(I)))$ denote the corresponding adjoint action. For each $i \in I$ let δ_i denote the Dirac function at i and let $d_i : \ell^2(I) \to \mathbb{C}\delta_i$ denote the orthogonal projection. Then there exists a unique G-equivariant $*$-homomorphism

$$\Phi : C_0(I) \to C_0(\Omega) \otimes \mathcal{K}(\ell^2(I)) \quad \text{such that} \quad \Phi(\delta_i) = e_i \otimes d_i,$$

for all $i \in I$, where the action of G on $C_0(I)$ is induced by the action on I and the action on $C_0(\Omega) \otimes \mathcal{K}(\ell^2(I))$ is given by the diagonal action $\tau \otimes \operatorname{Ad} U$, where $\tau : G \to \operatorname{Aut}(C_0(\Omega))$ denotes the given action of G on $C_0(\Omega)$. More generally if $\alpha : G \to \operatorname{Aut}(A)$ is an action of G on a C^*-algebra A, then there exists a G-equivariant $*$-homomorphism

$$\Phi_A : C_0(I) \otimes A \to C_0(\Omega) \otimes A \otimes \mathcal{K}(\ell^2(I)) \quad \text{s.t.} \quad \Phi_A(\delta_i \otimes a) = e_i \otimes a \otimes d_i.$$

Note that the action $\tau \otimes \alpha \otimes \operatorname{Ad} U$ of G on $C_0(\Omega) \otimes A \otimes \mathcal{K}(\ell^2(I))$ is Morita equivalent to the action $\tau \otimes \alpha$ of G on $C_0(\Omega) \otimes A$ via the G-equivariant equivalence bimodule

$\mathcal{E} := (C_0(\Omega) \otimes A \otimes \ell^2(I), \tau \otimes \alpha \otimes U)$. Thus, we obtain a KK^G-class

$$x = [\Phi_A] \otimes_{C_0(\Omega)\otimes A \otimes \mathcal{K}} \mathcal{E} \in KK^G(C_0(I,A), C_0(\Omega, A)).$$

The following is the main result of this section:

Theorem 3.5.18 (cf. [CEL13]). *Suppose that $\{e_i : i \in I\}$ is a G-equivariant regular basis for $C_0(\Omega)$, A is any G-C^*-algebra, and G satisfies the Baum–Connes conjecture for $C_0(I,A)$ and $C_0(\Omega, A)$. Then the descent*

$$J_G(x) \in KK(C_0(I,A) \rtimes_r G, C_0(\Omega, A) \rtimes_r G)$$

of the class $x \in KK^G(C_0(I,A), C_0(\Omega, A))$ constructed above induces an isomorphism $K_*(C_0(I,A) \rtimes_r G) \cong K_*(C_0(\Omega, A) \rtimes_r G)$.

If, moreover, G satsfies the strong Baum–Connes conjecture and if A is type I, then $J_G(x) \in KK(C_0(I,A) \rtimes_r G, C_0(\Omega, A) \rtimes_r G)$ is a KK-equivalence.

The above theorem was originally shown in [CEL13, §3], extending an earlier result given in [CEL15]. Before we present some of the crucial ideas of the proof, we would like to discuss a bit why this result might be useful for explicit K-theory calculations. The main reason is due to the relatively easy structure of crossed products by groups acting on *discrete* spaces I. If such action is given (as in the situation of our theorem) and if A is any other G-C^*-algebra, we obtain a G-equivariant direct sum decomposition

$$C_0(I,A) \cong \bigoplus_{[i] \in G \backslash I} C_0(G\cdot i) \otimes A,$$

in which $G \cdot i = \{g \cdot i : g \in G\}$ denotes the G-orbit of the representative i of the class $[i] \in G\backslash I$. Let $G_i := \{g \in G : g \cdot i = i\}$ denote the stabiliser of i in G. Then G_i is open in G and we have a G-equivariant bijection

$$G/G_i \overset{\cong}{\to} G \cdot i; gG_i \mapsto g \cdot i.$$

Moreover, by Green's imprimitivity theorem (Theorem 2.6.4; see also Remark 2.6.9), there are natural Morita equivalences

$$C_0(G/G_i, A) \rtimes_r G \cong A \rtimes_r G_i.$$

Putting things together, we therefore get

$$C_0(I,A) \rtimes_r G \cong \bigoplus_{[i] \in G\backslash I} C_0(G/G_i, A) \rtimes_r G \sim_M \bigoplus_{[i]\in G\backslash I} A \rtimes_r G_i.$$

Since Morita equivalences are KK-equivalences, we get

$$K_*(C_0(I,A) \rtimes_r G) \cong \bigoplus_{[i]\in G\backslash I} K_*(A \rtimes_r G_i).$$

Thus,

Corollary 3.5.19. *Suppose that G, Ω, A and $\{e_i : i \in I\}$ are as in Theorem* 3.5.18. *Then there is an isomorphism*

$$K_*(C_0(\Omega, A) \rtimes_r G) \cong \bigoplus_{[i]\in G\backslash I} K_*(A \rtimes_r G_i).$$

In particular, if $A = \mathbb{C}$, there is an isomorphism

$$K_*(C_0(\Omega) \rtimes_r G) \cong \bigoplus_{[i]\in G\backslash I} K_*(C_r^*(G_i)).$$

If G satisfies strong BC and A is type I, the isomorphism is induced by a KK-equivalence between $C_0(\Omega, A) \rtimes_r G$ and $\bigoplus_{[i]\in G\backslash I} A \rtimes_r G_i$.

In many interesting examples coming from the theory of C^*-semigroup algebras and crossed products of semigroups by automorphic actions of semigroups $P \subseteq G$, the stabilisers for the action of G on I have very easy structure, so that the K-theory groups of the crossed products $A \rtimes_r G$ are computable. This is in particular true in the case $A = \mathbb{C}$. The applications of Theorem 3.5.18 to C^*-semigroup algebras will be discussed in more detail in Chapter 5 (Section 5.10) and Chapter 6 (Section 6.5).

Example 3.5.20. To illustrate the usefulness of our approach we consider the group algebra of the lamplighter group $\mathbb{Z}/2 \wr \mathbb{Z}$ which is the semi-direct product $\big(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\big) \rtimes \mathbb{Z}$, where the action is given via translation of the summation index. Since the dual group of $\bigoplus_{\mathbb{Z}} \mathbb{Z}/2$ is equal to the direct product $\Omega := \prod_{\mathbb{Z}}\{1, -1\} = \{1, -1\}^{\mathbb{Z}}$ the group algebra $C^*(\mathbb{Z}/2 \wr \mathbb{Z})$ is isomorphic to $C(\Omega) \rtimes \mathbb{Z}$. Moreover, by Example 3.5.17 there exists a regular basis $\mathcal{V} = \{V_F : F \subseteq \mathbb{Z} \text{ finite}\}$ for the compact open subsets of Ω consisting of the cylinder sets $V_F = \{(\epsilon_n)_n \in \Omega : \epsilon_n = 1\ \forall n \in F\}$ attached to the finite subsets $F \subseteq \mathbb{Z}$. This basis is clearly $\mathbb{Z}$-invariant, hence our theorem applies to the corresponding regular basis $\{e_F = 1_{V_F} : F \subseteq \mathbb{Z} \text{ finite}\}$ of $C(\Omega)$. Let $\mathcal{F} = \{F \subseteq \mathbb{Z} : F \text{ finite}\}$ denote the index set of this basis and let $\mathcal{F}^* = \mathcal{F} \smallsetminus \{\emptyset\}$. The action of $\mathbb{Z}$ on $\mathcal{F}$ fixes $\emptyset$ and acts freely on $\mathcal{F}^*$. Hence, our theorem gives

$$\begin{aligned} K_*(C^*(\mathbb{Z}/2 \wr \mathbb{Z})) = K_*(C(\Omega) \rtimes \mathbb{Z}) &\cong K_*(C_0(\mathcal{F}) \rtimes \mathbb{Z}) \\ &\cong K_*(C^*(\mathbb{Z})) \oplus \Big(\bigoplus_{[F]\in \mathbb{Z}\backslash \mathcal{F}^*} K_*(\mathbb{C})\Big). \end{aligned}$$

Since $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ and $K_0(C(\mathbb{T})) = \mathbb{Z} = K_1(C(\mathbb{T}))$ we get

$$K_0(C^*(\mathbb{Z}/2 \wr \mathbb{Z}) \cong \bigoplus_{[F]\in G\backslash \mathcal{F}} \mathbb{Z} \quad \text{and} \quad K_1(C^*(\mathbb{Z}/2 \wr \mathbb{Z})) = \mathbb{Z}.$$

Of course, the result can easily be extended to more general wreath products $\mathbb{Z}/2 \wr G = \bigoplus_G \mathbb{Z}/2 \rtimes G$, where G is a countable discrete group which satisfies appropriate versions of the Baum–Connes conjecture.

Proof of Theorem 3.5.18. For the sake of presentation, let us assume that G is countable discrete (which is the case in most applications). Since the (strong) Baum–Connes conjecture is invariant with respect to KK^G-equivalent actions, and since G-equivariant Morita equivalences are KK^G-equivalences, it suffices to prove that the descent

$$\Phi_A \rtimes_r G : C_0(I, A) \rtimes_r G \to \big(C_0(\Omega, A) \otimes \mathcal{K}(\ell^2(I))\big) \rtimes_r G$$

of the homomorphism Φ_A induces an isomorphism in K-theory (resp. a KK-equivalence in the case where G satisfies strong BC and A is type I). To see that this is the case we want to exploit the going-down principle of the previous section, i.e., we need to show that for every finite subgroup $F \subseteq G$, the map

$$\Phi_A \rtimes F : C_0(I, A) \rtimes F \to \big(C_0(\Omega, A) \otimes \mathcal{K}(\ell^2(I))\big) \rtimes F \tag{3.5.2}$$

induces an isomorphism in K-theory. Note that if A is type I, the same is true for $\big(C_0(\Omega, A) \otimes \mathcal{K}(\ell^2(I))\big) \rtimes F$ by Corollary 2.8.21 and therefore it follows from the universal coefficient theorem of KK-theory (e.g., see [Bla86, Chapter 23]) that $\Phi_A \rtimes F$ being an isomorphism already implies that it is a KK-equivalence. Thus, in this situation, the stronger result that $\Phi_A \rtimes_r G$ is a KK-equivalence will then follow from the Meyer–Nest theorem 3.5.7.

Using the Green–Julg theorem, the map $\Phi_A \rtimes F$ of (3.5.2) being an isomorphism is equivalent to

$$(\Phi_A)_* : K_*^F(C_0(I, A)) \to K_*^F\big(C_0(\Omega, A) \otimes \mathcal{K}(\ell^2(I))\big)$$

being an isomorphism.

So in what follows let us fix a finite subgroup F of G. Let $J \subseteq I$ be any finite F-invariant subset such that $\{e_i : i \in J\}$ is closed under multiplication (up to 0). Then $D_J := \operatorname{span}\{e_i : i \in J\}$ is a finite-dimensional commutative C^*-subalgeba of $C_0(\Omega)$ of dimension $\dim(D_J) = |J|$. Consider the map

$$\Phi_J : C_0(J) \to D_J \otimes \mathcal{K}(\ell^2(J)); \ \Phi_J(\delta_i) = e_i \otimes d_i. \tag{3.5.3}$$

We want to show that Φ_J is invertible in $KK^F\big(C_0(J), D_J \otimes \mathcal{K}(\ell^2(J))\big) \cong KK^F(C_0(J), D_J)$. If this happens to be true, then $\Phi_{A,J} := \Phi_J \otimes \mathrm{id}_A : C_0(J, A) \to D_J \otimes A \otimes \mathcal{K}(\ell^2(J))$ will be KK^F-invertible as well, and the desired result then follows from the following commutative diagram

$$\begin{tikzcd}
K_*^F(C_0(J, A)) \arrow[r, "(\Phi_{A,J})_*", "\cong"'] \arrow[d, "\iota_*"'] & K_*^F\big(D_J \otimes A \otimes \mathcal{K}(\ell^2(J))\big) \arrow[d, "\iota_*"] \\
\lim_J K_*^F(C_0(J, A)) \arrow[r, "\lim_J (\Phi_{A,J})_*", "\cong"'] \arrow[d, "\cong"'] & \lim_J K_*^F\big(D_J \otimes A \otimes \mathcal{K}(\ell^2(J))\big) \arrow[d, "\cong"] \\
K_*^F(C_0(I, A)) \arrow[r, "(\Phi_A)_*", "\cong"'] & K_*^F\big(C_0(\Omega) \otimes A \otimes \mathcal{K}(\ell^2(I))\big)
\end{tikzcd}$$

The KK^F-invertibility of Φ_J in (3.5.3) will be a consequence of a UCT-type result for finite-dimensional F-algebras, which we now explain: Suppose that C and D are commutative finite-dimensional F-algebras with $\dim(C) = n, \dim(D) = m$ (in our application, C will be $C_0(J)$, $D = D_J$, and $n = m = |J|$). Let $\{c_1, \ldots, c_n\}$ and $\{d_1, \ldots, d_m\}$ be choices of pairwise orthogonal projections, which then form a basis of C and D, respectively. Then we have isomorphisms $\mathbb{Z}^n \cong K_0(C)$ and $\mathbb{Z}^m \cong K_0(D)$ sending the jth unit vector e_j to $[c_j]$ (resp. $[d_j]$). If we ignore the F-action, the UCT-theorem for KK implies that

$$KK_0(C, D) \cong \operatorname{Hom}(K_0(C), K_0(D)) \cong M(m \times n, \mathbb{Z}), \tag{3.5.4}$$

where the first isomorphism is given by sending $x \in KK(C, D)$ to the map $\cdot \otimes_C x : K_0(C) \to K_0(D)$ and the second map is given via the above identifications of $K_0(C) \cong \mathbb{Z}^n$ and $K_0(D) \cong \mathbb{Z}^m$. Suppose now that C and D are F-algebras such that F acts via permutations of the basis elements in $\{c_1, \ldots, c_n\}$ and $\{d_1, \ldots, d_m\}$, respectively. Note that the actions of F on C and D are determined by two homomorphisms $\tau : F \to S_n$ and $\sigma : F \to S_m$ such that $g \cdot c_i = c_{\tau_g(i)}$ and $g \cdot d_j = d_{\sigma_g(j)}$ for all i, j. Then the equivariant version of (3.5.4) does not give an isomorphism in general, but we get a homomorphism

$$\Psi_{C,D} : KK_0^F(C, D) \to \operatorname{Hom}_F(K_0(C), K_0(D)) \cong M_F(m \times n, \mathbb{Z}) : x \mapsto \Gamma_x \tag{3.5.5}$$

where $\operatorname{Hom}_F(K_0(C), K_0(D))$ denotes the F-equivariant homomorphisms (with F acting on the basis elements $[c_i]$ and $[d_i]$ of $K_0(C)$ and $K_0(D)$, repectively) and $M_F(m \times n, \mathbb{Z})$ denotes the set of all $m \times n$-matrices $\Gamma = (\gamma_{ij})$ over $\mathbb{Z}$ that satisfy

$$\gamma_{ij} = \gamma_{\tau_g(i), \sigma_g(j)} \quad \forall g \in F. \tag{3.5.6}$$

We need to construct a section $M_F(m \times n, \mathbb{Z}) \to KK^F(C, D); \Gamma \mapsto x_\Gamma$ for $\Psi_{C,D}$ that is compatible with taking Kasparov products. For this let $\Gamma = (\gamma_{ij}) \in M_F(m \times n, \mathbb{Z})$ be given. Let $\mathcal{E}_{ij} = \mathbb{C}^{|\gamma_{ij}|} \otimes \mathbb{C} d_i$ viewed as a Hilbert D-module in the canonical way. Let $\varphi_{ij} : C \to \mathcal{K}(\mathcal{E}_{ij})$ be the $*$-homomorphism such that $\varphi_{ij}(c_j) = 1_{\mathcal{E}_{ij}}$ and $\varphi_{ij}(c_k) = 0$ for all $k \neq i$. Let

$$\mathcal{E}_\Gamma^+ = \bigoplus_{\gamma_{ij} > 0} \mathcal{E}_{ij} \quad \text{and} \quad \varphi^+ = \bigoplus_{\gamma_{ij} > 0} \varphi_{ij} : C \to \mathcal{K}(\mathcal{E}^+),$$

and, similarly,

$$\mathcal{E}_\Gamma^- = \bigoplus_{\gamma_{ij} < 0} \mathcal{E}_{ij} \quad \text{and} \quad \varphi^- = \bigoplus_{\gamma_{ij} < 0} \varphi_{ij} : C \to \mathcal{K}(\mathcal{E}^-).$$

Because of (3.5.6) there are canonical actions of F on $\mathcal{E}^+, \mathcal{E}^-$ such that $g \cdot \mathcal{E}_{ij} = \mathcal{E}_{\tau_g(i)\sigma_g(j)}$ for all i, j and such that $(\mathcal{E}_\Gamma^+, \varphi^+)$ and $(\mathcal{E}_\Gamma^-, \varphi^-)$ become F-equivariant $C - D$ correspondences. Finally, let $\mathcal{E}_\Gamma = \mathcal{E}_\Gamma^+ \oplus \mathcal{E}_\Gamma^-$ with $\mathbb{Z}/2$-grading given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $\varphi = \begin{pmatrix} \varphi^+ & 0 \\ 0 & \varphi^- \end{pmatrix}$. Since φ takes value in $\mathcal{K}(\mathcal{E})$, we get a class

$$x_\Gamma := [\mathcal{E}_\Gamma, \varphi, 0] \in KK^F(C, D).$$

The proof of the following lemma is left as an exercise for the reader (or see [CEL13, Lemma A.2]):

Lemma 3.5.21. *Suppose that B, C, D are finite-dimensional commutative F-algebras such that $\{b_1, \dots, b_k\}$, $\{c_1, \dots, c_n\}$, and $\{d_1, \dots, d_m\}$ are F-invariant bases consisting of orthogonal projections in B, C, D, respectively. Then, for all matrices $\Lambda \in M_F(n \times k, \mathbb{Z})$ and $\Gamma \in M_F(m \times n, \mathbb{Z})$ we get*

$$x_\Lambda \otimes_C x_\Gamma = x_{\Gamma \cdot \Lambda} \in KK^F(B, D).$$

In particular, if $n = \dim(C) = \dim(D)$ and $\Gamma \in M_F(n \times n, \mathbb{Z})$ is invertible over $\mathbb{Z}$, then $x_\Gamma \in KK^F(C, D)$ is invertible as well.

Let us come back to the class $\Phi_J : C_0(J) \to D_J \otimes \mathcal{K}(\ell^2(J))$ which sends δ_i to $e_i \otimes d_i$, where d_i denotes the orthogonal projection onto $\mathbb{C}\delta_i$. The following lemma gives the crucial point of how independence of the family $\{e_i : i \in I\}$ of the basis elements of $C_0(\Omega)$ enters the picture:

Lemma 3.5.22 (cf [CEL13, Lemma 3.8])**.** *Let D be a commutative C^*-algebra generated by a multplicatively closed (up to 0) and independent finite set of projections $\{e_i : i \in J\}$. For each $i \in J$ let $e_i' := e_i - \vee_{e_j < e_i} e_j$. Then $\{e_i' : i \in J\}$ is a family of nonzero pairwise orthogonal projections spanning D. Moreover, the transition matrix $\Gamma = (\gamma_{ij})$ determined by the equation*

$$e_j = \sum_{i \in J} \gamma_{ij} e_i'$$

is unipotent and therefore invertible over $\mathbb{Z}$. Its entries are either 0 or 1.

Proof. Independence implies that $e_i' \neq 0$ for all $i \in J$. If $i \neq j$, then we have either $e_i < e_j$, in which case $e_j' e_i = e_j e_i - \vee_{e_k < e_j} e_k e_i = 0$ or $e_i e_j < e_i$, in which case $e_i' e_j = e_i e_j - \vee_{e_k < e_i} e_k e_j = 0$. Either case implies $e_i' e_j' = 0$. Since $\dim(D) = |J|$, it follows that $D = \operatorname{span}\{e_i' : i \in I\}$.

If $e_i' \leq e_j$, then $e_i' \leq e_i e_j \leq e_i$ by definition of e_i'. This shows that $\gamma_{ij} = 1$ if $e_i \leq e_j$ and $\gamma_{ij} = 0$ otherwise. Thus, if we choose an ordering $\{i_1, \dots, i_n\}$ of J such that $e_i \leq e_j \Rightarrow i \leq j$, the matrix Γ is upper triangular with 1's on the diagonal, hence unipotent. Thus, $1 - \Gamma$ is nilpotent of order $n = |J|$ and Γ is invertible with inverse $\Gamma^{-1} = \sum_{k=0}^n (1 - \Gamma)^k$. □

To finish the proof of Theorem 3.5.18 we observe that the class $[\Phi_J] \in KK^F(C_0(J), D_J \otimes \mathcal{K}(\ell^2(J))) \cong KK^F(C_0(J), D_J)$ coincides with the class $x_\Gamma = [\mathcal{E}_\Gamma, \varphi, 0]$ as constructed in the above lemma with respect to the basis

$\{\delta_i : i \in J\}$ of $C_0(J)$ and the basis $\{e'_i : i \in J\}$ of D_J. A combination of Lemma 3.5.22 with Lemma 3.5.21 then implies that $[\Phi_J]$ is invertible. Indeed, since $\gamma_{ij} = 0$ or 1, it follows that $\mathcal{E}_\Gamma = \mathcal{E}_\Gamma^+ = \bigoplus_{i\in J} \left(\bigoplus_{j\in J,\gamma_{ij}=1} \mathbb{C}e'_i\right)$ embeds as a direct summand into $D_J \otimes \ell^2(J)$ such that $\Phi_J : C_0(J) \to D_J \otimes \mathcal{K}(\ell^2(J)) \cong \mathcal{K}(D_J \otimes \ell^2(J))$ decomposes as $\varphi \oplus 0$. $\square$

As remarked before, the main applications for Theorem 3.5.18 are given in case of computing the K-theory of reduced semigroup algebras $C^*_\lambda(P)$, where $e \in P \subseteq G$ is a sub-semigroup of the countable group G. In case where $P \subseteq G$ satisfies a certain Toeplitz condition (which is discussed in detail in Chapter 5), there exists a totally disconnected G-space $\Omega_{P\subseteq G}$ such that $C^*_\lambda(P)$ can be realised as a full corner in the crossed product $C_0(\Omega_{P\subseteq G}) \rtimes_r G$, hence $K_*(C^*_\lambda(P)) \cong K_*(C_0(\Omega_{P\subseteq G}) \rtimes_r G)$. Now, the existence of a G-invariant regular basis for $C_0(\Omega_{P\subseteq G})$ will follow from a certain independence condition for the inclusion $P \subseteq G$, which, somewhat surprisingly, is satisfied in a large number of interesting cases. Again, we refer to the Chapters 5 and 6 for more details on this.

Unfortunately, a G-invariant regular basis $\{e_i : i \in I\}$ for $C_0(\Omega)$, as required for the proof of Theorem 3.5.18, does not exist in general. In fact, we have the following result, which excludes a large number of interesting cases from our theory:

Proposition 3.5.23 ([CEL13, Proposition 3.18]). *Let G be a countable discrete group that acts minimally on the totally disconnected locally compact space Ω, i.e., for every nonempty open subset U of Ω, we have*

$$\Omega = \cup_{g\in G} gU.$$

Suppose further that there exists a nonzero G-invariant Borel measure μ on Ω (which holds if G is amenable and Ω is compact). Then Ω has a G-invariant regular basis for the compact open sets if and only if Ω is discrete.

Chapter 4

Quantitative K-theory for geometric operator algebras

Guoliang Yu

4.1 Introduction

The purpose of this chapter is to give a friendly introduction to *quantitative K-theory* of operator algebras and its applications. Quantitative operator K-theory was first introduced in my work on the Novikov conjecture for groups with finite asymptotic dimension [Yu98]. Hervé Oyono-Oyono and I developed a more general quantitative K-theory for C^*-algebras [OOY15]. Quantitative operator theory provides a constructive way to compute K-theory of C^*-algebras under certain finiteness conditions. The crucial idea is that quantitative operator K-theory is often computable by using a cutting-and-pasting technique in each scale under certain finite-dimensionality conditions and the usual K-theory is an inductive limit when the scale goes to infinity. The data necessary for defining quantitative K-theory of a C^*-algebra is a length function for the C^*-algebra. Such a length function gives a geometric structure on the C^*-algebra. For this reason, a C^*-algebra with a length function will be called a *geometric C^*-algebra*. A general quantitative K-theory was developed for geometric C^*-algebras in my joint work with Hervé Oyono-Oyono in [OOY15].

The concept of geometric C^*-algebras fits beautifully with geometric group theory. Any length function on a group G would give rise to a natural length function on

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the group C^*-algebra $C^*(G)$ and the crossed product C^*-algebra $A \rtimes G$ for any C^*-algebra A with a G action. In my joint work with Erik Guentner and Rufus Willett [GWY16b, GWY16a], we applied quantitative operator K-theory to give a constructive algorithm for computing the K-theory of crossed product C^*-algebras $C(X) \rtimes G$ when G acts on the compact space X with a finite dynamic asymptotic dimension.

Finiteness of the dynamic asymptotic dimension for the G action on the compact space X also implies finiteness of the nuclear dimension of the reduced crossed product C^*-algebra $C(X) \rtimes G$ [GWY16b]. The nuclear dimension is an important concept introduced by Winter and Zacharias [WZ10]. Finiteness of nuclear dimension plays an crucial role in recent spectacular work on classification by Gong–Lin–Niu, Elliott–Gong–Lin–Niu, and Tikuisis–White–Winter [GLN15, EGLN15, TWW15].

Hervé Oyono-Oyono and I generalized the concept of asymptotic dimension to geometric C^*-algebras and prove the Künneth formula for any geometric C^*-algebras with finite asymptotic dimension [OOY16]. It is our hope that the quantitative method will extend to the K-homology setting and will provide a way to attack the universal coefficient theorem for K-theory of operator algebras. The universal coefficient theorem is a key ingredient in the classification program for nuclear C^*-algebras.

More recently, Yeong Chyuan Chung developed a quantitative K-theory for Banach algebras and applied this theory to compute K-theory of Banach crossed product algebras [Chu16a, Chu16b]. Chung showed that the L^p-version of the Baum–Connes conjecture holds for a group G with coefficients in $C(X)$ if G acts on X with a finite dynamic asymptotic dimension. As a consequence, Chung was able to show that the K-theory of the L^p-crossed product algebra $B^p(X, G)$ is independent of p when the group G acts on the compact space X with a finite asymptotic dynamic dimension. Whether the K-theory of $B^p(X, G)$ depends on p remains an important open question in general. We emphasize that there is substantial difficulty to extend the standard Dirac-dual Dirac method to the L^p settings. The use of quantitative K-theory is crucial here.

I should mention that topologists have introduced similar ideas to compute algebraic K-theory and L-theory [Bar03, BLR08a, FJ86, FJ87, FJ89]. The analytic quantitative K-theory and algebraic quantitative K-theory were initially developed independently, but have had a productive exchange of ideas recently.

This chapter is composed essentially of the notes I prepared for the lectures I gave at the Oberwolfach seminar. I would like to thank my co-organizers Joachim Cuntz, Siegfried Echterhoff, and Xin Li for this wonderful opportunity.

4.2 Geometric C^*-algebras

In this section, we introduce the concept of geometric C^*-algebras. For simplicity, we assume that all C^*-algebras are complex in this chapter.

We need the following geometric structure on a C^*-algebra in order to define quantitative K-theory.

Definition 4.2.1. Let A be a C^*-algebra. A function $l : A \to [0, \infty]$, is called a length function if it satisfies the following conditions:

(0) $l(0) = 0$;

(1) $l(a + b) \leq \max\{l(a), l(b)\}$ and $l(ab) \leq l(a) + l(b)$ for any $a, b \in A$;

(2) $l(ca) \leq l(a)$ for any $a \in A$ and $c \in \mathbb{C}$, the set of all complex numbers;

(3) $l(a^*) = l(a)$ for all $a \in A$;

(4) the set $\{a \in A : l(a) < \infty\}$ is dense in A and, for each $r \geq 0$, $\{a : l(a) \leq r\}$ is a closed subset of A.

A C^*-algebra with a length function is called a *geometric C^*-algebra*. A geometric C^*-algebra has a natural filtration given by: $A_n = \{a \in A : l(a) \leq n\}$ for each nonnegative integer. Then $\{A_n\}$ satisfies the following filtration condition:

(1) A_n is a closed linear subspace of A and is closed under the $*$-operation for each nonnegative integer n;

(2) $A_n A_m \subseteq A_{n+m}$ for all pairs of nonnegative integers n and m;

(3) $\cup_{n=0}^{\infty} A_n$ is dense in A.

Next we give a few examples of geometric C^*-algebras.

The first example comes from geometric group theory. Let G be a countable group. In this chapter, all our groups are discrete. There exists a proper length function l on G, i.e., l satisfies the following conditions:

(1) $l(g) = 0$ iff $g = e$, the identity of G;

(2) $l(gh) \leq l(g) + l(h)$ for all g and h in G;

(3) $l(g^{-1}) = l(g)$ for all $g \in G$;

(4) l is proper, i.e., for each $r \geq 0$, $\{g \in G : l(g) \leq r\}$ is finite.

Proper length functions on G are all coarsely equivalent to one another. Properness is not essential or our purpose.

Let B be a C^*-algebra with a G action. Let $B \rtimes G$ be the reduced crossed product C^*-algebra defined as follows. The algebraic crossed product algebra $B \rtimes_{alg} G$ is defined to be the vector space of all formal finite sums $\sum_{g\in G} a_g g$ with the following product and $*$-operation:

$$\Big(\sum_{g\in G} a_g g\Big)(\sum_{h\in G} b_h h) = \sum_{g,h\in G} a_g \alpha_g(b_h) gh,$$

$$\Big(\sum_{g\in G} a_g g\Big)^* = \sum_{g\in G} \alpha_{g^{-1}}(a_g^*) g^{-1},$$

where $a_g, b_h \in B$, and α is the G action on B. We take a faithful $*$-representation of B on a Hilbert space H. There is a natural faithful $*$-representation of $B \rtimes_{alg} G$ on the Hilbert space $l^2(G, H)$ defined by:

$$(a\xi)(h) = \alpha_{g^{-1}}(a)\xi(h),$$

$$(g\xi)(h) = \xi(g^{-1}h)$$

for all $\xi \in l^2(G, H)$, $a \in B$, and $g, h \in G$. We define the reduced crossed product C^*-algebra $B \rtimes G$ to be the operator norm closure of $B \rtimes_{alg} G$ in $B(l^2(G, H))$, the algebra of all bounded linear operators acting on the Hilbert space $l^2(G, H)$.

We can extend the length function l on G to a length function on $B \rtimes G$ as follows:

$$l(a) = \sup_{g\in G: a_g \neq 0} l(g),$$

where $a = \sum_{g\in G} a_g g$ and $a_g \in B$. One can easily verify that l is a length function on the crossed product C^*-algebra $B \rtimes G$.

The second example comes from John Roe's index theory on noncompact manifolds [Roe93]. Let X be a discrete metric space with bounded geometry, i.e., for each $r \geq 0$, there exists a positive integer $N(r)$ such that each ball with radius r in X has at most $N(r)$ number of elements. The *propagation* of a bounded linear operator T acting on $l^2(X)$ is defined to be

$$\sup\{d(x, y) :< T\delta_x, \delta_y > \neq 0\},$$

where δ_x and δ_y are, respectively, Dirac functions at x and y in X. Let $C^*(X)$ be the operator norm closure of all finite propagation operators acting on $l^2(X)$. For every operator $T \in C^*(X)$, we define its length $l(T)$ to be its propagation. It is straightforward to verify that l is a length function on the C^*-algebra $C^*(X)$.

In the above example, if we choose $X = G$ with a G-invariant metric induced from a proper length function l on G, i.e., $d(g, h) = l(g^{-1}h)$, then there is a natural action of G on X by translations. Let $C^*(X)^G$ be the operator norm closure of all G-invariant finite propagation operators on $l^2(X)$. The C^*-algebra $C^*(X)^G$ is isomorphic to the reduced group C^*-algebra of G.

Finally, we give the third example of a geometric C^*-algebra. Let A be a finitely generated C^*-algebra. Choose a finite generating set S such that S is closed under the $*$-operation. For each $a \in A$, we define the length $l(a)$ to be the smallest nonnegative integer n such that a can be approximated by linear combinations of the products of n number of elements in S. It is not hard to check that l is a length function on A. This length function depends on the choice of the finite generating set. This example is generic in the sense that most C^*-algebras we are interested in are finitely generated.

4.3 Quantitative K-theory for C^*-algebras

In this section, we introduce the basic concepts of quantitative K-theory for geometric C^*-algebras. Let A be a geometric C^*-algebra with a length function l. Without loss of generality, we can assume that A is unital. Otherwise we can extend the length function l to A^+, the unital C^*-algebra obtained from A by adjoining a unit, as follows: $l(a + cI) = l(a)$ for any $a \in A$ and $c \in \mathbb{C}$.

Let $M_\infty(A) = \cup_{n=1}^{\infty} M_n(A)$, where $M_n(A)$ is embedded in $M_{n+1}(A)$ in a standard way. We can extend the length function l to $M_\infty(A)$ by taking the maximum of the length of the matrix entries.

Let $r > 0$ and $0 < \epsilon < 1/4$. An operator q in $M_\infty(A)$ is called an (ϵ, r)-*quasi-projection* if

$$\|q^2 - q\| < \epsilon, \quad q^* = q, \quad l(q) \leq r.$$

Let $P_\infty^{\epsilon,r}(A)$ be the set of all (ϵ, r)-quasi-projections in $M_\infty(A)$. Two (ϵ, r)-quasi-projections are said to be (ϵ, r)-equivalent if they are homotopic through a path of (ϵ, r)-quasi-projections.

Lemma 4.3.1. *$P_\infty^{\epsilon,r}(A)/\sim$ is an abelian semi-group with respect to the direct sum operation.*

Grothendick introduced a process of constructing an abelian group out of an abelian semi-group. Let S be an abelian semi-group. The Grothendick group $G(S)$ is an abelian group:

$$\{(s, t) : s, t \in S\}/\sim$$

with the following addition operation:

$$[(s_1, t_1)] + [(s_2, t_2)] = [(s_1 + s_2, t_1 + t_2)],$$

where the equivalence relation is defined as follows: $(s, t) \sim (s', t')$ if there exists $r \in S$ such that

$$s + t' + r = s' + t + r.$$

Definition 4.3.2. Let $r > 0$ and $0 < \epsilon < 1/4$. We define $K_0^{\epsilon,r}(A)$ to be the Grothendick group of $P_\infty^{\epsilon,r}(A)/\sim$.

Our definition is slightly different from but equivalent to the definition in [OOY15].

There exists a $\delta > 0$ dependent on ϵ such that the spectrum of every (ϵ, r)-quasi-projection is contained in $(-\infty, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, \infty)$. Let f be a continuous function on the real line such that $f(x) = 0$ on $(-\infty, \frac{1}{2} - \delta]$ and $f(x) = 1$ on $[\frac{1}{2} + \delta, \infty)$. We define a homomorphism ϕ from $K_0^{\epsilon,r}(A)$ to the K_0 group of A by mapping a (ϵ, r)-quasi-projection q in $M_\infty(A)$ to a projection $p = f(q)$ in $M_\infty(A)$.

Proposition 4.3.3. *The homomorphism ϕ induces an isomorphism from $\lim_{r\to\infty} K_0^{\epsilon,r}(A)$ to $K_0(A)$.*

Let $GL_\infty(A) = \cup_{n=1}^{\infty} GL_n(A)$. Let $r > 0$ and $0 < \epsilon < 1/4$. An operator v in $GL_\infty(A)$ is called an (ϵ, r)-*quasi-unitary* if

$$\|vv^* - I\| < \epsilon, \quad \|vv^* - I\| < \epsilon, \quad l(v) \leq r.$$

Let $U_\infty^{\epsilon,r}(A)$ be the set of all (ϵ, r)-unitaries in $GL_\infty(A)$. Two (ϵ, r)-quasi-unitaries are said to be (ϵ, r)-equivalent if they are homotopic through a path of (ϵ, r)-quasi-unitaries.

Lemma 4.3.4. *For any $0 < \epsilon < 1/4$, $U_\infty^{\epsilon,r}(A)/\sim$ is an abelian semi-group with respect to the direct sum operation.*

The following definition is slightly different from but equivalent to the definition in [OOY15].

Definition 4.3.5. Let $r > 0$ and $0 < \epsilon < 1/4$. We define $K_1^{\epsilon,r}(A)$ to be the Grothendick group of $U_\infty^{\epsilon,r}(A)/\sim$.

There is a natural homomorphism ψ from $K_1^{\epsilon,r}(A)$ to the K_1 group of A by mapping a (ϵ, r)-quasi-unitary v in $GL_\infty(A)$ to the invertible element v in $GL_\infty(A)$.

Proposition 4.3.6. *The homomorphism ψ induces an isomorphism from $\lim_{r\to\infty} K_1^{\epsilon,r}(A)$ to $K_1(A)$.*

There exists a six-term asymptotically exact sequence for quantitative K-theory of geometric C^*-algebras [OOY15].

4.4 A quantitative Mayer–Vietoris sequence

In this section, we formulate an asymptotically exact Mayer–Vietoris sequence for quantitative K-theory for geometric C^*-algebras. This quantitative Mayer–Vietoris sequence plays an essential role in the computation of quantitative K-theory for geometric C^*-algebras. The usual Mayer–Vietoris sequence for the K-theory of C^*-algebras requires a decomposition of the C^*-algebra into the sum of two ideals [HRY93]. The need to use ideals for the usual operator K-theory Mayer–Vietoris sequence makes it a limited tool in the computation of the K-theory of

C^*-algebras since generically C^*-algebras don't have nontrivial ideals. The advantage of the quantitative Mayer–Vietoris sequence is that it is unnecessary to use nontrivial ideals.

Let A be a geometric C^*-algebra with a length function l. For each $r \geq 0$, let $A_r = \{a \in A : l(a) \leq r\}$.

The following concept is the analogue of ideals in the quantitative sense [OOY15].

Definition 4.4.1. Let Δ be a closed linear subspace of A. Let $r \geq 0$.

(1) We define the r-neighborhood of Δ, $N_r(\Delta)$ to be

$$\Delta + A_r \cdot \Delta + \Delta \cdot A_r + A_r \cdot \Delta \cdot A_r.$$

(2) We define the r-neighborhood C^*-algebra, $C^*(N_r(\Delta))$, to be the C^*-algebra generated by $N_r(\Delta)$.

The following concept is essential in the formulation of the excisive condition for the quantitative Mayer–Vietoris sequence [OOY15].

Definition 4.4.2. Let S_1 and S_2 be two subsets of a C^*-algebra A. The pair (S_1, S_2) is said to have a complete intersection approximation property (CIA) if there exists $c > 0$ such that for any positive number ϵ, any $x \in M_n(S_1)$ and $y \in M_n(S_2)$ for some positive integer n and $\|x - y\| < \epsilon$, then there exists $z \in M_n(S_1 \cap S_2)$ satisfying

$$\|z - x\| < c\epsilon, \quad \|z - y\| < c\epsilon.$$

The above condition is often automatically satisfied in natural examples [OOY15, GWY16b, GWY16a].

Definition 4.4.3. Let Δ be a closed linear subspace of A. Let Δ_1 and Δ_2 be closed subspaces of Δ satisfying

$$\Delta = \Delta_1 + \Delta_2.$$

The decomposition $\Delta = \Delta_1 + \Delta_2$ is called *completely contractive* if for any $x \in M_n(\Delta)$ for some n, there exist $x_1 \in M_n(\Delta_1)$ and $x_2 \in M_n(\Delta_2)$ satisfying

$$x = x_1 + x_2, \quad \|x_i\| \leq \|x\|$$

for $i = 1, 2$.

Definition 4.4.4. Let $r \geq 0$. Let Δ_1 and Δ_2 be two closed linear subspaces of A_r. The pair (Δ_1, Δ_2) is said to be r-excisive if

(1) $A_s = \Delta_{1,s} + \Delta_{2,s}$ is a completely contractive decomposition for all $0 \leq s \leq r$, where $\Delta_{i,s} = \{a \in \Delta_i : l(a) \leq s\}$;

(2) The pair $(C^*(N_r(\Delta_1))_s, C^*(N_r(\Delta_2))_s)$ satisfies the CIA property for all $0 \leq s \leq r$, where $C^*(N_r(\Delta_i))_s = \{a \in C^*(N_r(\Delta_i)) : l(a) \leq s\}$.

The following asymptotically exact Mayer–Vietoris sequence is a main tool in the computation of quantitative K-theory [OOY15]. Note that the usual Mayer–Vietoris sequence for operator K-theory requires the crucial use of ideals [HRY93]. However, C^*-algebras don't always have nontrivial ideals. The great advantage of quantitative K-theory is that we only need to use the neighborhood algebras to establish a Mayer–Vietoris sequence. These neighborhood algebras play the role of ideals at a certain scale.

Theorem 4.4.5. *Let A be a geometric C^*-algebra. Let (Δ_1, Δ_2) be a r-excisive pair for A. There exists a universal constant $\lambda > 10$ (dependent only on the constant c in the definition of the CIA property) such that for each positive $\epsilon < \frac{1}{10}$ and $0 \leq s \leq \frac{r}{\lambda^2}$, the following sequence is asymptotically exact:*

$$\begin{aligned}
&K_1^{\epsilon,s}(C^*(N_s(\Delta_1)) \cap C^*(N_s(\Delta_2)))\\
&\xrightarrow{i} K_1^{\epsilon,s}(C^*(N_s(\Delta_1))) \oplus K_1^{\epsilon,s}(C^*(N_s(\Delta_2)))\\
&\xrightarrow{j} K_1^{\epsilon,s}(A)\\
&\xrightarrow{\partial} K_0^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1)) \cap C^*(N_{\lambda s}(\Delta_2)))\\
&\xrightarrow{i} K_0^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1))) \oplus K_0^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_2)))\\
&\xrightarrow{j} K_0^{\epsilon,\lambda s}(A)
\end{aligned}$$

in the following sense:

(1) $j \circ i = 0$;

(2) *the kernel of* $j : K_1^{\epsilon,s}(C^*(N_s(\Delta_1))) \oplus K_1^{\epsilon,s}(C^*(N_s(\Delta_2))) \to K_1^{\epsilon,s}(A)$ *in* $K_1^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1))) \oplus K_1^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_2)))$ *is contained in the image of*

$$\begin{aligned}
i: \quad & K_1^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1)) \cap C^*(N_{\lambda s}(\Delta_2)))\\
&\to K_1^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1))) \oplus K_1^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_2)));
\end{aligned}$$

(3) $\partial \circ j = 0$;

(4) *the kernel of* $\partial : K_1^{\epsilon,s}(A) \to K_0^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1)) \cap C^*(N_{\lambda s}(\Delta_2)))$ *is contained in*
the image of $j : K_1^{\epsilon,\lambda^2 s}(C^*(N_{\lambda^2 s}(\Delta_1))) \oplus K_1^{\epsilon,\lambda^2 s}(C^*(N_{\lambda^2 s}(\Delta_2))) \to K_1^{\epsilon,\lambda^2 s}(A)$;

(5) $i \circ \partial = 0$;

(6) *the kernel of*

$$\begin{aligned}
i: \quad & K_0^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1)) \cap C^*(N_{\lambda s}(\Delta_2)))\\
&\to K_0^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_1))) \oplus K_0^{\epsilon,\lambda s}(C^*(N_{\lambda s}(\Delta_2)))
\end{aligned}$$

in $K_0^{\epsilon,\lambda^2 s}(C^*(N_{\lambda^2 s}(\Delta_1)) \cap C^*(N_{\lambda^2 s}(\Delta_2)))$ *is contained in the image of* $\partial : K_1^{\epsilon,\lambda s}(A) \to K_0^{\epsilon,\lambda^2 s}(C^*(N_{\lambda^2 s}(\Delta_1)) \cap C^*(N_{\lambda^2 s}(\Delta_2)))$.

We remark that there exists also a quantitative Bott periodicity [OOY15].

4.5 Dynamic asymptotic dimension and K-theory of crossed product C^*-algebras

In this section, we introduce the concept of dynamic asymptotic dimension and apply quantitative K-theory to compute K-theory of crossed product C^*-algebras when the dynamic asymptotic dimension is finite.

We first recall the definition of crossed product C^*-algebra. We focus on the following case of our special interest: group actions on compact spaces. Let G be a countable group and X be a compact space with a G action. Let $C(X)$ be the algebra of complex-valued continuous functions on X. The G action on X induces an action of G on $C(X)$ (denoted by α). The algebraic crossed product algebra $C(X) \rtimes_{\text{alg}} G$ is defined to be the vector space of all formal finite sums $\sum_{g \in G} a_g g$ with the following product and $*$-operation:

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g,h \in G} a_g \alpha_g(b_h) gh,$$

$$\left(\sum_{g \in G} a_g g\right)^* = \sum_{g \in G} \alpha_{g^{-1}}(\bar{a}_g) g^{-1},$$

where $a_g, b_h \in C(X)$. There is a natural faithful $*$-representation of $C(X) \rtimes_{\text{alg}} G$ on the Hilbert space $l^2(G, L^2(X))$. With the help of this $*$-representation, we can define the (reduced) crossed product C^*-algebra $C(X) \rtimes G$ to be the operator norm completion of the algebraic crossed product algebra $C(X) \rtimes_{\text{alg}} G$.

In order to describe the notion of dynamic asymptotic dimension, it is convenient to recall the transformation groupoid $X \rtimes G$ defined by

$$\{(x, g) : x \in X, g \in G\}$$

with partially defined product operation

$$(x, g)(y, h) = (x, gh)$$

when $x = gy$. The groupoid $X \rtimes G$ is given the natural product topology.

The following concept is introduced in my joint work with Erik Guentner and Rufus Willett [GWY16b, GWY16a].

Definition 4.5.1. A G action on X is said to have finite dynamic asymptotic dimension if there exists $d \in \mathbb{N}$ such that for all finite subsets $F \subseteq G$, there exist a finite subset $E \subseteq G$ and an open cover $\{U_0, \ldots, U_d\}$ of $\{(x, g) \in X \times G : \ g \in F\}$ satisfying:

(1) for each $x \in X$, the set $\{(x, g) \in X \times G : \ g \in F\}$ is contained in U_i for some i;

(2) the subgroupoid of $X \rtimes G$ generated by U_i is contained in the set

$$\{(x, gh) : (x, g) \in U_i, \quad h \in E\}.$$

The above concept of dynamic asymptotic dimension was inspired by Gromov's concept of asymptotic dimension [Gon15]. Let Z be a proper metric space, i.e., every closed ball is compact. The asymptotic dimension of Z is the smallest non-negative integer d for which, given any $r \geq 0$, there exists a uniformly bounded cover $\{Z_i\}_i$ for A such that each ball with radius r in Z intersects at most $d+1$ members of the cover $\{Z_i\}_i$. Any countable group has a proper length metric. Two choices of such metrics are coarsely equivalent. Asymptotic dimension is invariant under coarse equivalence and is therefore an intrinsic concept for any countable group. If G is a countable group with finite asymptotic dimension, then G acts on a compact space X with finite dynamic asymptotic dimension. In fact, we can take X to be the Stone–Čech compactification of G. The asymptotic dimension of G is equal to the smallest non-nonnegative integer d for which, given any $r \geq 0$, there exists a finite cover $\{G_k\}_{k=0}^{d}$ such that each $G_k = \cup_i G_{k,i}$, $\{G_{k,i}\}_i$ is uniformly bounded and $d(G_{k,i}, G_{k,i'}) > r$ for any pair $i \neq i'$. For each k, let U_k be the closure of G_k in X, the Stone–Čech compactification of G. Note that U_k is also an open subset of X. It is not difficult to verify that $\{U_k\}_k$ satisfies the conditions in the above definition of finite dynamic asymptotic dimension. One can prove that a countable group G acts on a compact space with finite dynamic asymptotic dimension if and only if G has finite asymptotic dimension. Examples of groups with finite asymptotic dimension include all hyperbolic groups, discrete subgroups of almost connected Lie groups. We also remark that the Bartels–Lück–Reich condition [BLR08a, BLR08b] in their work on the Farrell-Jones conjecture implies finite dynamic asymptotic dimension.

The importance of the concept of finite dynamic asymptotic dimension lies in the following result [GWY16a].

Theorem 4.5.2. *If a group G acts on a compact space X with a finite dynamic asymptotic dimension, then there exists an algorithm for computing the K-theory of the reduced crossed product C^*-algebra $C(X) \rtimes G$.*

As an application, one obtains a constructive proof of the Baum–Connes conjecture for G with coefficients in $C(X)$ when G acts on X with finite dynamic asymptotic dimension. The key tool in the proof of the above theorem is quantitative operator K-theory and the quantitative Mayer–Vietoris sequence. The main idea is that the condition of finite dynamic asymptotic dimension allows one to compute quantitative operator K-theory using an asymptotically exact Mayer–Vietoris sequence. The usual K-theory can then be computed by taking limits of the quantitative operator K-theory.

If a group G acts on a compact space X with a finite dynamic asymptotic dimension, then the nuclear dimension of the reduced crossed product C^*-algebra

$C(X) \rtimes G$ is finite [GWY16b]. The nuclear dimension is an important concept introduced by Winter and Zacharias [WZ10]. Finiteness of nuclear dimension plays an crucial role in recent spectacular work on classification by Gong–Lin–Niu, Elliott–Gong–Lin–Niu, and Tikuisis–White–Winter [GLN15, EGLN15, TWW15].

The concept of finite dynamic asymptotic dimension can be generalized to a more flexible notion of finite dynamic complexity just as the concept of a finite asymptotic dimension can be generalized to the notion of finite geometric complexity [GTY12, GTY13]. If a group G has finite geometric complexity, then G acts on a compact space X with finite dynamic complexity (again X can be taken as the Stone–Čech compactification of G). Examples of groups with finite geometric complexity include all linear groups [GTY13].

Erik Guentner, Rufus Willett and I are in the process of writing down the proof of the following result.

Theorem 4.5.3. *If a group G acts on a compact space X with finite dynamic complexity, then there exists an algorithm for computing K-theory of the crossed product C^*-algebra $C(X) \rtimes G$.*

It is an interesting project to introduce a notion of "nuclear complexity" for a nuclear C^*-algebra that is compatible with the concept of dynamic complexity in the case of crossed product C^*-algebras and then apply such a concept to classify nuclear C^*-algebras.

4.6 Asymptotic dimension for geometric C^*-algebras and the Künneth formula

In this section, we introduce a concept of asymptotic dimension for geometric C^*-algebras and then formulate a Künneth formula for geometric C^*-algebras with a finite asymptotic dimension. This is joint work with Hervé Oyono-Oyono [OOY16].

We first recall the bootstrap category introduced by Jonathan Rosenberg and Claude Schochet [RS87].

Definition 4.6.1. The bootstrap category N is the smallest class of nuclear separable C^*-algebras such that

(1) N contains $\mathbb{C}$;

(2) N is closed under countable inductive limits;

(3) N is stable under extension, i.e., for any extension of C^*-algebras

$$0 \to J \to A \to A/J \to 0,$$

if any two of the C^*-algebras are in N, then so is the third;

(4) N is closed under KK-equivalence.

Next we introduce the concept of locally bootstrap C^*-algebras.

Definition 4.6.2. A geometric C^*-algebra A is called locally bootstrap if for all positive number s there exists a positive number r with $r > s$ and a C^*-subalgebra $A(s)$ of A such that

(1) $A(s)$ belongs to the bootstrap class;

(2) $A_s \subseteq A(s) \subseteq A_r$, where $\{A_r\}_{r>0}$ is the filtration given by $A_r = \{a \in A : l(a) \leq r\}$.

The following is a uniform version of the locally bootstrap C^*-algebras.

Definition 4.6.3. A family of geometric C^*-algebras $\{A_i\}_{i\in I}$ is uniformly locally bootstrap if for all $k \in I$ and for all positive number s, there exists a positive number r with $r > s$ and a C^*-algebra $A_k(s)$ of A_k such that for all integer k,

(1) $A_k(s)$ belongs to the bootstrap class;

(2) $A_{k,s} \subseteq A_k(s) \subseteq A_{k,r}$, where $\{A_{k,r}\}_r$ is the filtration given by the geometric structure on A_k.

Definition 4.6.4. We first define C_0 to be the class of uniformly locally bootstrap families of C^*-algebras. Then we define by induction C_n to be the class of family $\mathcal{A}$ such that for every positive number r, the following is satisfied:

there exists a family $\mathcal{B}$ in C_{n-1} and, for any C^*-algebra A in $\mathcal{A}$, there is an r-excisive pair (Δ_1, Δ_2) of A with $C^*(N_r(\Delta_1)), C^*(N_r(\Delta_2))$ and $C^*(N_r(\Delta_1)) \cap C^*(N_r(\Delta_2))$ in $\mathcal{B}$.

Finally, we can define the concept of asymptotic dimension for geometric C^*-algebras.

Definition 4.6.5. Let A be a geometric C^*-algebra. The asymptotic dimension of A is defined to be the smallest integer d such that $\{A\} \in C_d$. If there is no such integer d, we say that the asymptotic dimension of A is infinite.

In [OOY16], we prove the following result.

Theorem 4.6.6. *If A is a geometric C^*-algebra with a finite asymptotic dimension, then A satisfies the Künneth formula in K-theory, i.e., there exists a natural short exact sequence*

$$0 \to K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \to \mathrm{Tor}(K_*(A), K_*(B)) \to 0$$

for any C^*-algebra B.

In my joint work with Rufus Willett, we are developing a quantitative K-homology theory with the goal of studying the universal coefficients theorem for geometric C^*-algebras with finite asymptotic dimension.

Definition 4.6.7. A geometric C^*-algebra A is said to have (polynomial) subexponential growth if the dimension of A_n has (polynomial) subexponential growth in n, where $A_n = \{a \in A : l(a) \leq n\}$.

In [KV92], Kirchberg and G. Vaillant proved that any geometric C^*-algebras with subexponential growth is nuclear.

We raise the following open question:

Problem 4.6.8. Does every geometric C^*-algebra with (polynomial) subexponential growth have finite asymptotic dimension?

A positive answer to this question would help us understand the structure of this class of nuclear C^*-algebras.

4.7 Quantitative K-theory for Banach algebras

In this section, we give a brief overview for quantitative K-theory for Banach algebras and its applications. Quantitative K-theory for Banach algebra was developed by Yeong Chyuan Chung in his thesis [Chu16a, Chu16b]. As an application, Chung obtained an algorithm for computing K-theory for Banach crossed product algebras when the action has finite dynamic asymptotic dimension. There is a great deal of technical difficulty in extending the standard Dirac-dual Dirac method to the Banach algebra context. However, Chung showed that the quantitative K-theory method works perfectly well in this Banach setting.

For simplicity, we shall assume that our Banach algebra A is a subalgebra of $B(L^p(Z, \mu))$ for some $1 \leq p < \infty$ and some space Z with a measure μ. In this case, for each positive integer n, we have a natural norm on $M_\infty(A)$ inherited from $B(L^p(\mathbb{N} \times Z, m \times \mu)$, where $\mathbb{N}$ is the set of all positive integers with the counting measure m. For general Banach algebra, it is necessary to choose a norm on $M_\infty(A)$ in order to define quantitative K-theory [Chu16a, Chu16b].

Definition 4.7.1. Let A be a Banach algebra. A function $l : A \to [0, \infty]$, is called a *length function* if it satisfies the following conditions:

(0) $l(0) = 0$;

(1) $l(a + b) \leq \max\{l(a), l(b)\}$ and $l(ab) \leq l(a) + l(b)$ for any $a, b \in A$;

(2) $l(ca) \leq l(a)$ for any $a \in A$ and $c \in \mathbb{C}$, the set of all complex numbers;

(3) the set $\{a \in A : l(a) < \infty\}$ is dense in A and, for each $r \geq 0$, $\{a : l(a) \leq r\}$ is a closed subset of A.

A Banach algebra with a length function is called a *geometric Banach algebra*. A geometric Banach algebra has a natural filtration given by: $A_n = \{a \in A : l(a) \leq n\}$ for every nonnegative integer. $\{A_n\}$ satisfies the usual filtration condition:

(1) A_n is a closed linear subspace of A for every nonnegative integer n;

(2) $A_n A_m \subseteq A_{n+m}$ for all pairs of nonnegative integers n and m;

(3) $\cup_{n=0}^{\infty} A_n$ is dense in A.

A crucial change in quantitative K-theory for geometric Banach algebra is to replace the almost projection condition with a norm control.

Let $r > 0$, $0 < \epsilon < 1/4$, and $N \geq 1$. An operator q in $M_\infty(A)$ is called an (ϵ, r, N)-*quasi-idempotent* if

$$\|q^2 - q\| < \epsilon, \quad \|q\| \leq N, \quad l(q) \leq r.$$

Let $\mathrm{Idem}_\infty^{\epsilon,r,N}(A)$ be the set of all (ϵ, r, N)-quasi-idempotents in $M_\infty(A)$. Two (ϵ, r, N)-quasi-idempotents are said to be (ϵ, r, N)-equivalent if they are homotopic through a path of (ϵ, r, N)-quasi-idempotents.

Lemma 4.7.2. $\mathrm{Idem}_\infty^{\epsilon,r,N}(A)/\sim$ *is an abelian semi-group with respect to the direct sum operation.*

Definition 4.7.3. Let $r > 0$, $0 < \epsilon < 1/4$, and $N \geq 1$. We define $K_0^{\epsilon,r,N}(A)$ to be the Grothendick group of $\mathrm{Idem}_\infty^{\epsilon,r,N}(A)/\sim$.

Similar to the C^*-algebra case, there is a natural homomorphism ϕ from $K_0^{\epsilon,r,N}(A)$ to the K_0 group of A.

Proposition 4.7.4. *The homomorphism ϕ induces an isomorphism from $\lim_{r,N\to\infty} K_0^{\epsilon,r,N}(A)$ to $K_0(A)$.*

Let $GL_\infty(A) = \cup_{n=1}^{\infty} GL_n(A)$. Let $r > 0$, $0 < \epsilon < 1/4$, and $N \geq 1$. An operator v in $GL_\infty(A)$ is called an (ϵ, r, N)-*invertible* if there exists another element $w \in GL_\infty(A)$ such that

$$\|vw - I\| < \epsilon, \quad \|wv - I\| < \epsilon,$$
$$\|v\| \leq N, \quad \|w\| \leq N,$$
$$l(v) \leq r, \quad l(w) \leq r.$$

Let $GL_\infty^{\epsilon,r,N}(A)$ be the set of all (ϵ, r, N)-invertibles in $GL_\infty(A)$. Two (ϵ, r, N)-invertibles are said to be (ϵ, r, N)-equivalent if they are homotopic through a path of (ϵ, r, N)-invertibles.

Lemma 4.7.5. *For any $0 < \epsilon < 1/4$, $GL_\infty^{\epsilon,r,N}(A)/\sim$ is an abelian semi-group with respect to the direct sum operation.*

The following definition is slightly different from but equivalent to the definition in [Chu16a, Chu16b].

Definition 4.7.6. Let $r > 0$ and $0 < \epsilon < 1/4$. We define $K_1^{\epsilon,r,N}(A)$ to be the Grothendick group of $GL_\infty^{\epsilon,r,N}(A)/\sim$.

There is a natural homomorphism ψ from $K_1^{\epsilon,r,N}(A)$ to the K_1 group of A by mapping a (ϵ, r, N)-invertible v in $GL_\infty(A)$ to the invertible element v in $GL_\infty(A)$.

Proposition 4.7.7. *The homomorphism ψ induces an isomorphism from $\lim_{r,N\to\infty} K_1^{\epsilon,r,N}(A)$ to $K_1(A)$.*

Let B be a Banach subalgebra of $B(L^p(Z,\mu))$, where Z is a space with measure μ. For simplicity, we will denote $L^p(Z,\mu)$ by $L^p(Z)$. Let G be a countable group acting on B by norm preserving automorphisms. Let $B \rtimes G$ be the reduced crossed product Banach algebra defined as follows. The algebraic crossed product algebra $B \rtimes_{\mathrm{alg}} G$ is defined to be the vector space of all formal finite sums $\sum_{g\in G} a_g g$ with the following product:

$$\Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{h\in G} b_h h\Big) = \sum_{g,h\in G} a_g \alpha_g(b_h) gh,$$

where $a_g, b_h \in B$, and α is the G action on B. There is a natural faithful representation of $B \rtimes_{\mathrm{alg}} G$ on the Banach space $l^p(G, L^p(Z))$ defined by:

$$(a\xi)(h) = \alpha_{h^{-1}}(a)\xi(h),$$

$$(g\xi)(h) = \xi(g^{-1}h)$$

for all $\xi \in l^p(G, L^p(Z))$, $a \in B$, and $g, h \in G$. We define the reduced Banach crossed product algebra $B \rtimes G$ to be the operator norm closure of $B \rtimes_{\mathrm{alg}} G$ in $B(l^p(G, L^p(Z)))$, the algebra of all bounded linear operators acting on the Banach space $l^p(G, L^p(Z))$.

Let l be a proper length function on G. We can extend the length function l on G to a length function on the Banach crossed product algebra $B \rtimes G$ just as in the C^*-algebra case.

Let X be a compact space and let G be a countable group acting on X. For each $1 \le p < \infty$, $C(X)$ is a Banach subalgebra of $\mathrm{B}(L^p(X))$, the algebra of all bounded linear operators acting on the Banach space $L^p(X)$. We denote the crossed product Banach algebra of $C(X)$ with G by $B^p(X,G)$. When X is a point with a trivial action of G, $B^p(X,G)$ is the Banach group algebra $B^p(G)$, where $B^p(G)$ is the operator norm completion of the group algebra $\mathbb{C}G$ in $B(l^p(G))$, the Banach algebra of all bounded linear operators on the Banach space $l^p(G)$.

The crossed product Banach algebra $B^p(X,G)$ can be viewed a deformation of the reduced crossed C^*-algebra $C(X) \rtimes G$.

An important open question is the following:

Problem 4.7.8. Is the K-theory of the crossed product Banach algebra $B^p(X,G)$ independent of p?

In [LY16], Liao and I proved a semi-continuity for the K-group of $B^p(X,G)$ in p.

There is also a duality between $B^p(X,G)$ and $B^q(X,G)$ for $1 < p < \infty$ and $\frac{1}{q}+\frac{1}{p} = 1$ [KY16, LY16]. To be more precise, $B^p(X,G)$ is anti-isomorphic to $B^q(X,G)$ in the sense that there exists a bijective linear map $\psi : B^p(X,G) \to B^q(X,G)$ such that ψ preserves the norm and $\psi(ab) = \psi(b)\psi(a)$ for all $a, b \in B^p(X,G)$. This implies that the K-groups of $B^p(X,G)$ and $B^q(X,G)$ are isomorphic.

In his thesis [Chu16a, Chu16b], Chung developed a quantitative Mayer–Vietoris sequence for geometric Banach algebras. As an application, he obtained the following result:

Theorem 4.7.9. *If a group G acts on a compact space X with finite dynamic asymptotic dimension, then there exists an algorithm for computing the K-theory of the crossed product Banach algebra $B^p(X,G)$.*

As a consequence, Chung answers the above question positively when the group acts on the compact space with finite dynamic asymptotic dimension. Chung also verifies an L^p-version of the Baum–Connes conjecture for G with coefficients in $C(X)$ when the group G acts on a compact space X with finite dynamic asymptotic dimension. Chung's result is currently not accessible to the Dirac-dual Dirac method.

Corollary 4.7.10. *If a group G acts on a compact space X with finite dynamic asymptotic dimension, then the K-theory of the crossed product Banach algebra $B^p(X,G)$ is independent of p.*

When the space X is a point with a trivial action of the group G, the crossed product Banach algebra $B^p(X,G)$ is the Banach group algebra $B^p(G)$. The question of independence of K-theory of $B^p(G)$ on p remains open in general. This question was answered positively when G is hyperbolic or amenable in my joint work with Gennadi Kasparov [KY16] and Benben Liao [LY16]. The importance of this question is that sometimes it is easier to compute the K-groups of $B^p(G)$ for large p [KY16]. If the K-group of $B^p(G)$ is independent of p, then such a computation would give a computation of the K-groups of $B^p(G)$ for all p including the special case $p = 2$. On the other hand, if the K-groups of $B^p(G)$ depends on p, this would be an even more exciting phenomenon—indicating a breakdown of the Baum–Connes conjecture at some level.

A very interesting Banach algebra associated to a group G is the Banach group $*$-algebra $B^{p,*}(G)$ defined to be the norm completion of the group algebra $\mathbb{C}G$ under the norm:

$$\left\|\sum c_g g\right\| = \max\left\{\left\|\sum c_g g\right\|_{B(l^p(G))}, \ \left\|\sum \bar{c}_g g^{-1}\right\|_{B(l^p(G))}\right\}$$

with the $*$-operation:

$$\left(\sum c_g g\right)^* = \bar{c}_g g^{-1}.$$

We can similarly define a Banach crossed product $*$-algebra $B^{p,*}(X,G)$ to be the norm completion of the algebrac crossed product $C(X) \rtimes_{\text{alg}} G$ under the norm:

$$\left\|\sum a_g g\right\| = \max\left\{\left\|\sum a_g g\right\|_{B(l^p(G))}, \left\|\sum \alpha_{g^{-1}}(\bar{a}_g)g^{-1}\right\|_{B(l^p(G))}\right\}$$

with the $*$-operation:

$$\left(\sum_{g\in G} a_g g\right)^* = \sum_{g\in G} \alpha_{g^{-1}}(\bar{a}_g)g^{-1},$$

where $a_g \in C(X)$ and α is the G-action on $C(X)$.

The following question is open.

Problem 4.7.11. Is the K-theory of the crossed product Banach algebra $B^{p,*}(X,G)$ independent of p?

By interpolation theory, for every $p > 1$, there is a natural bounded homomorphism:

$$\phi_p : B^{p,*}(X,G) \to C(X) \rtimes G,$$

where $C(X) \rtimes G$ is the reduced C^*-algebra crossed product algebra.

We have the following open question:

Problem 4.7.12. Is $(\phi_p)_* : K_*(B^{p,*}(X,G)) \to K_*(C(X) \rtimes G)$ an isomorphism.

Chung's work can be used to prove that the answer is yes when the group G acts on the compact space X with finite dynamic asymptotic dimension.

Finally we raise the following open question:

Problem 4.7.13. Is $j_* : K_*(B^{p,*}(X,G)) \to K_*(B^p(X,G))$ an isomorphism, where j is the natural homomorphism from $B^{p,*}(X,G)$ to $B^p(X,G)$.

When X is a point, the answer to the above question is yes if G satisfies the Banach property RD [LY16].

Chapter 5

Semigroup C^*-algebras

Xin Li

5.1 Introduction

A semigroup C^*-algebra is the C^*-algebra generated by the left regular representation of a left cancellative semigroup. In the case of groups, this is the classical construction of reduced group C^*-algebras, which received great interest and serves as a motivating class of examples in operator algebras.

For semigroups that are far from being groups, we encounter completely new phenomena which are not visible in the group case. It is therefore a natural and interesting task to try to understand and explain these new phenomena. This challenge has been taken up by several authors in many works, and our present goal is to give a unified treatment of this endeavour.

We point out that particular classes of semigroups have played a predominant role in the development of our work, as they have served as our motivation and have guided us towards important properties of semigroups that allow for a systematic study of their C^*-algebras. The examples include positive cones in totally ordered groups, semigroups given by particular presentations and semigroups coming from rings of number-theoretic origin. Important properties that isolate from the general and wild class of all left cancellative semigroups a manageable subclass were first given by Nica's quasi-lattice order [Nic92] and later on by the independence condition [Li12] and the Toeplitz condition [Li13].

Aspects of semigroup C^*-algebras that we discuss in the following include descriptions as crossed products and groupoid C^*-algebras, the connection between amenability and nuclearity, boundary quotients, and the classification problem for semigroup C^*-algebras. The first three topics are discussed in detail, and we give

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a more or less self-contained presentation. The last topic puts together many results. In particular, it builds on the K-theory computations that are explained in detail by S. Echterhoff in Chapter 3 of this book. Since a detailed account of classification results would take too much space, we briefly summarize the main results, and refer the interested reader to the relevant papers for more details and complete proofs.

Our discussion of semigroup C^*-algebras builds on previous work of J. Renault on groupoids and their C^*-algebras [Ren80], and the work of R. Exel on C^*-algebras of inverse semigroups, their quotients corresponding to tight representations of inverse semigroups, and on partial actions [Exe08, Exe09, Exe15].

Inevitably, certain interesting aspects of semigroup C^*-algebras are not covered in this book. This includes a discussion of C^*-algebras of semigroups which do not embed into groups such as general right LCM semigroups (see [Sta15b]) or Zappa–Szép products (see [BRRW14]), or C^*-algebras of certain topological semigroups (see [RS15, Sun14]). Moreover, we do not discuss KMS-states in detail, but we refer the reader to [LR10, BaHLR11, CDL13, CaHR16] for more information. We also mention that in §6.6 of this book, J. Cuntz describes KMS-states for particular examples. We apologize for these omissions and try to make up for them by pointing the interested reader to the relevant literature. To this end, we have included a long (but not complete) list of references.

5.2 C^*-algebras generated by left regular representations

Let P be a semigroup. We assume that P is left cancellative, i.e., for all $p, x, y \in P$, $px = py$ implies $x = y$. In other words, the map

$$P \to P, \quad x \mapsto px$$

given by left multiplication with $p \in P$ is injective for all $p \in P$.

The left regular representation of P is given as follows: The Hilbert space $\ell^2 P$ comes with a canonical orthonormal basis $\{\delta_x : x \in P\}$. Here δ_x is the delta-function in $x \in P$, defined by

$$\delta_x(y) = 1 \text{ if } y = x \quad \text{and} \quad \delta_x(y) = 0 \text{ if } y \neq x.$$

For every $p \in P$, the map

$$P \to P, \quad x \mapsto px$$

is injective by left cancellation, so that the mapping

$$\delta_x \mapsto \delta_{px} \quad (x \in P)$$

extends (uniquely) to an isometry

$$V_p : \ell^2 P \to \ell^2 P.$$

The assignment

$$p \mapsto V_p \quad (p \in P)$$

represents our semigroup P as isometries on $\ell^2 P$. This is called the left regular representation of P. It generates the following C^*-algebra:

Definition 5.2.1.

$$C^*_\lambda(P) := C^*(\{V_p : p \in P\}) \subseteq \mathcal{L}(\ell^2 P).$$

By definition, $C^*_\lambda(P)$ is the smallest subalgebra of $\mathcal{L}(\ell^2 P)$ containing $\{V_p : p \in P\}$ which is invariant under forming adjoints and closed in the operator norm topology. We call $C^*_\lambda(P)$ the semigroup C^*-algebra of P, or more precisely, the left reduced semigroup C^*-algebra of P.

Note that left cancellation is a crucial assumption for our construction. In general, without left cancellation, the mapping $\delta_x \mapsto \delta_{px}$ does not even extend to a bounded linear operator on $\ell^2 P$. Moreover, we point out that we view our semigroups as discrete objects. Our construction, and some of the analysis, carries over to certain topological semigroups (see [RS15, Sun14]). Finally, $C^*_\lambda(P)$ will be separable if P is countable. This helps to exclude pathological cases. Therefore, for convenience, we assume from now on that all our semigroups are countable, although this is not always necessary in our discussion.

5.3 Examples

We have already pointed out the importance of examples. Therefore, it is appropriate to start with a list of examples of semigroups where we can apply our construction. All our examples are actually semigroups with an identity, so that they are all monoids.

5.3.1 The natural numbers

Our first example is given by $P = \mathbb{N} = \{0, 1, 2, \ldots\}$, the set of natural numbers including zero, viewed as an additive monoid. By construction, V_1 is the unilateral shift. Since $\mathbb{N}$ is generated by 1 as a monoid, it is clear that $C^*_\lambda(\mathbb{N})$ is generated as a C^*-algebra by the unilateral shift. This C^*-algebra has been studied by Coburn (see [Cob67, Cob69]). It turns out that it is the universal C^*-algebra generated by one isometry, i.e.,

$$C^*_\lambda(\mathbb{N}) \cong C^*(v \mid v^* v = 1), \quad V_1 \mapsto v.$$

$C^*_\lambda(\mathbb{N})$ is also called the Toeplitz algebra. This name comes from the observation that $C^*_\lambda(\mathbb{N})$ can also be described as the C^*-algebra of Toeplitz operators on the Hardy space, defined on the circle. This interpretation connects our semigroup C^*-algebra $C^*_\lambda(\mathbb{N})$ with index theory and K-theory.

5.3.2 Positive cones in totally ordered groups

Motivated by connections to index theory and K-theory, several authors including Coburn and Douglas studied the following examples in [CD71, CDSS71, Dou72, DH71]:

Let G be a subgroup of $(\mathbb{R}, +)$, and consider the additive monoid $P = [0, \infty) \cap G$. The case $G = \mathbb{Z}$ gives our previous example $P = \mathbb{N}$. The case where $G = \mathbb{Z}[\lambda, \lambda^{-1}]$ for some positive real number λ is discussed in [CPPR11, Li15].

These examples belong to the bigger class of positive cones in totally ordered groups. A left invariant total order on a group G is a relation $\leq$ on G such that

- For all $x, y \in G$, we have $x = y$ if and only if $x \leq y$ and $y \leq x$.
- For all $x, y \in G$, we always have $x \leq y$ or $y \leq x$.
- For all $x, y, z \in G$, $x \leq y$ and $y \leq z$ imply $x \leq z$.
- For all $x, y, z \in G$, $x \leq y$ implies $zx \leq zy$.

Given a left invariant total order $\leq$ on G, define $P := \{x \in G : e \leq x\}$. Here e is the identity in G. P is called the positive cone in G. It is a monoid satisfying

$$G = P \cup P^{-1} \quad \text{and} \quad P \cap P^{-1} = \{e\}. \tag{5.1}$$

Conversely, every submonoid $P \subseteq G$ of a group G satisfying (5.1) gives rise to a left invariant total order $\leq$ by setting, for $x, y \in G$, $x \leq y$ if $y \in xP$. Here $xP = \{xp : p \in P\} \subseteq G$.

In the examples mentioned above of subgroups of $(\mathbb{R}, +)$, we have canonical left invariant total orders given by restricting the canonical order on $(\mathbb{R}, +)$.

The study of left invariant total orders on group is of great interest in group theory. For instance, the existence of a left invariant total order on a group G implies the Kaplansky conjecture for G. This conjecture says that for a torsion-free group G and a ring R, the group ring RG does not have zero-divisors if R does not have zero-divisors. We refer to [MR77, DNR14] for more details.

While it is known that every torsion-free nilpotent group admits a left invariant total order, it is an open conjecture that lattices in simple Lie groups of rank at least two have no left invariant total order. It is also an open question whether an infinite property (T) group can admit a left invariant total order (see [DNR14] for more details).

5.3.3 Monoids given by presentations

Another source for examples of monoids comes from group presentations. One way to define a group is to give a presentation, i.e., generators and relations. For instance, the additive group of integers is the group generated by one element with no relation, $\mathbb{Z} = \langle a \rangle$. The nonabelian free group on two generators is the group generated by two elements with no relations, $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$. And $\mathbb{Z} \times \mathbb{Z}$ is the group generated by two elements that commute, $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$. If we look at the semigroups (or rather monoids) defined by the same presentations, we get $\mathbb{N} = \langle a \rangle^+$, $\mathbb{N}^{*2} = \mathbb{N} * \mathbb{N} = \langle a, b \rangle^+$, $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N} = \langle a, b \mid ab = ba \rangle^+$. Here, we write $\langle \cdot \mid \cdot \rangle^+$ for the universal monid given by a particular presentation, while we write $\langle \cdot \mid \cdot \rangle$ for the universal group given by a particular representation. This is to distinguish between group presentations and monoid presentations.

Of course, in general, it is not clear whether this procedure of taking generators and relations from group presentations to define monoids leads to interesting semigroups, or whether we can apply our C^*-algebraic construction to the resulting semigroups. For instance, it could be that the monoid given by a presentation actually coincides with the group given by the same presentation. Another problem that might arise is that the canonical homomorphism from the monoid to the group given by the same presentation, sending generator to generator, is not injective. In that case, our monoid might not even be left cancellative. However, there are conditions on our presentations that ensure that these problems do not appear. There is, for instance, the notion of completeness (see [Deh03]), explained in §5.6.5. Now let us give a list of examples.

The presentations for $\mathbb{Z}$, $\mathbb{F}_2$ and $\mathbb{Z}^2$ all have in common that two generators either commute or satisfy no relation (i.e., they are free), and these are the only relations we impose. This can be generalized. Let $\Gamma = (V, E)$ be an undirected graph, where we connect two vertices by at most one edge and no vertex to itself. This means that we can think of E as a subset of $V \times V$.

We then define

$$\begin{aligned} A_\Gamma &:= \langle \{\sigma_v : v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \text{ for all } (v, w) \in E \rangle , \\ A_\Gamma^+ &:= \langle \{\sigma_v : v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \text{ for all } (v, w) \in E \rangle^+ . \end{aligned}$$

For instance, the graph for $\mathbb{Z}$ only consists of one vertex and no edge, the graph for $\mathbb{F}_2$ consists of two vertices and no edges, and the graph for $\mathbb{Z}^2$ consists of two vertices and one edge joining them.

The groups A_Γ are called right-angled Artin groups and the monoids A_Γ^+ are called right-angled Artin monoids. Their C^*-algebras are discussed in [CL02, CL07, Iva10, ELR16].

Right-angled Artin monoids and the corresponding groups are special cases of graph products. Let $\Gamma = (V, E)$ be a graph as above, with $E \subseteq V \times V$. Assume that for every $v \in V$, G_v is a group containing a submonoid P_v. Then let $\Gamma_{v \in V} G_v$ be the group obtained from the free product $*_{v \in V} G_v$ by introducing the relations $xy = yx$ for all $x \in G_v$ and $y \in G_w$ with $(v, w) \in E$. Similarly, define $\Gamma_{v \in V} P_v$ as the monoid obtained from the free product $*_{v \in V} P_v$ by introducing the relations

$xy = yx$ for all $x \in P_v$ and $y \in P_w$ with $(v,w) \in E$. It is explained in [CL02] (see also [Gre90, HM95]) that the embeddings $P_v \hookrightarrow G_v$ induce an embedding

$$\Gamma_{v \in V} P_v \hookrightarrow \Gamma_{v \in V} G_v.$$

In the case that $P_v \subseteq G_v$ is given by $\mathbb{N} \subseteq \mathbb{Z}$ for all $v \in V$, we obtain right-angled Artin monoids and the corresponding groups.

We will have more to say about general graph products in §5.4.2 and §5.9.

As the name suggests, there is a more general class of Artin groups that contains right-angled Artin groups. Let I be a countable index set,

$$\{m_{ij} \in \{2,3,4,\dots\} \cup \{\infty\} : i,j \in I,\, i \neq j\}$$

be such that $m_{ij} = m_{ji}$ for all i and j. Then define

$$G := \left\langle \{\sigma_i : i \in I\} \mid \underbrace{\sigma_i\sigma_j\sigma_i\sigma_j\cdots}_{m_{ij}} = \underbrace{\sigma_j\sigma_i\sigma_j\sigma_i\cdots}_{m_{ji}} \text{ for all } i,j \in I,\, i \neq j \right\rangle.$$

For $m_{ij} = \infty$, there is no relation involving σ_i and σ_j, i.e., σ_i and σ_j are free. And define

$$P := \left\langle \{\sigma_i : i \in I\} \mid \underbrace{\sigma_i\sigma_j\sigma_i\sigma_j\cdots}_{m_{ij}} = \underbrace{\sigma_j\sigma_i\sigma_j\sigma_i\cdots}_{m_{ji}} \text{ for all } i,j \in I,\, i \neq j \right\rangle^+.$$

If $m_{ij} \in \{2,\infty\}$ for all i and j, then we get right-angled Artin groups and monoids. To see some other groups, take, for instance, $I = \{1,2\}$ and $m_{1,2} = m_{2,1} = 3$. We get the (third) Braid group and the corresponding Braid monoid

$$B_3 := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$
$$B_3^+ := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle^+.$$

In general, for $n \geq 1$, the braid group B_n and the corresponding braid monoid B_n^+ are given by

$$B_n := \left\langle \sigma_1,\dots,\sigma_{n-1} \,\middle|\, \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle,$$
$$B_n^+ := \left\langle \sigma_1,\dots,\sigma_{n-1} \,\middle|\, \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle^+.$$

This corresponds to the case where $I = \{1,\dots,n-1\}$ and $m_{i,i+1} = m_{i+1,i} = 3$ for all $1 \leq i \leq n-2$ and $m_{i,j} = m_{j,i} = 2$ for all $|i-j| \geq 2$.

These Artin groups form an interesting class of examples that is of interest for group theorists.

Another family of examples is given by Baumslag–Solitar groups and their presentations: For $k, l \geq 1$, define the group

$$B_{k,l} := \langle a, b \mid ab^k = b^l a \rangle$$

and the monoid

$$B_{k,l}^+ := \langle a, b \mid ab^k = b^l a \rangle^+ .$$

Also, again for $k, l \geq 1$, define the group

$$B_{-k,l} := \langle a, b \mid a = b^l a b^k \rangle$$

and the monoid

$$B_{-k,l}^+ := \langle a, b \mid a = b^l a b^k \rangle^+ .$$

These are the Baumslag–Solitar groups and the Baumslag–Solitar monoids. The reader may find more about the semigroup C^*-algebras attached to Baumslag–Solitar monoids in [Spi12, Spi14].

Finally, let us mention the Thompson group and the Thompson monoid. The Thompson group is given by

$$F := \langle x_0, x_1, \ldots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle .$$

This is just one possible presentation defining the Thompson group. There are others, for instance,

$$F = \langle A, B \mid [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = e \rangle .$$

The first presentation, however, has the advantage that it leads naturally to the definition of the Thompson monoid as

$$F^+ := \langle x_0, x_1, \ldots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle^+ .$$

The Thompson group is of great interest in group theory; in particular the question whether it is amenable or not is currently attracting a lot of attention. Therefore, it would be very interesting to study the Thompson monoid and its semigroup C^*-algebra.

5.3.4 Examples from rings in general, and number theory in particular

Let us present another source for examples. This time, our semigroups come from rings. Let R be a ring without zero-divisors ($x \neq 0$ is a zero-divisor if there exists $0 \neq y \in R$ with $xy = 0$). Then $R^\times = R \setminus \{0\}$ is a cancellative semigroup with respect to multiplication.

We can also construct the $ax+b$-semigroup $R \rtimes R^\times$. The underlying set is $R \times R^\times$, and multiplication is given by $(d,c)(b,a) = (d+cb, ca)$. It is a semidirect product for the canonical multiplicative action of $R^\times$ on R.

Another possibility would be to take an integral domain R, i.e., a commutative ring with unit not containing zero-divisors, and form the semigroup $M_n(R)^\times$ of $n \times n$-matrices over R with nonvanishing determinant. We could also form the semidirect product $M_n(R) \rtimes M_n(R)^\times$ for the canonical multiplicative action of $M_n(R)^\times$ on $M_n(R)$.

In particular, rings from number theory are interesting. Let K be a number field, i.e., a finite extension of $\mathbb{Q}$. Then the ring of algebraic integers R in K is given by

$$\left\{ x \in K : \text{There are } n \geq 1,\ a_{n-1}, \dots, a_0 \in \mathbb{Z} \text{ with } x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0 \right\}.$$

For instance, for the classical case $K = \mathbb{Q}$, the ring of algebraic integers is given by the usual integers, $R = \mathbb{Z}$. For the number field of Gaussian numbers, $K = \mathbb{Q}[i]$, the ring of algebraic integers are given by the Gaussian integers, $R = \mathbb{Z}[i]$. More generally, for the number field $K = \mathbb{Q}[\zeta]$ generated by a root of unity ζ, the ring of algebraic integers is given by $R = \mathbb{Z}[\zeta]$. For the real quadratic number field $K = \mathbb{Q}[\sqrt{2}]$, the ring of algebraic integers is given by $\mathbb{Z}[\sqrt{2}]$, while for the real quadratic number field $K = \mathbb{Q}[\sqrt{5}]$, the ring of algebraic integers is given by $R = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

Let us briefly mention an interesting invariant of number fields. Let K be a number field with ring of algebraic integers R. We introduce an equivalence relation for nonzero ideals $\mathcal{I}$ of R by saying that $\mathfrak{a} \sim \mathfrak{b}$ if there exist $a, b \in R^\times$ with $b\mathfrak{a} = a\mathfrak{b}$. It turns out that with respect to multiplication of ideals, $\mathcal{I}/_\sim$ becomes a finite abelian group. This is the class group Cl_K of K. An outstanding open question in number theory is how to compute Cl_K, or even just the class number $h_K = \#Cl_K$, in a systematic and efficient way. It is not even known whether there are infinitely many (nonisomorphic) number fields with trivial class group (i.e., class number one). We refer the interested reader to [Neu99] for more details.

It is possible to consider more general semidirect products, in the more flexible setting of semigroups acting by endomorphisms on a group. Particular cases are discussed in Chapter 6. We also refer to [CV13, BLS17, BS16, Sta15a] and the references therein for more examples and for results on the corresponding semigroup C^*-algebras.

5.3.5 Finitely generated abelian cancellative semigroups

Finally, one more class of examples that illustrates quite well that the world of semigroups can be much more complicated than the world of groups: Consider finitely generated abelian cancellative semigroups, or monoids. For groups, we have a well-understood structure theorem for finitely generated abelian groups. But for semigroups, this class of examples is interesting and challenging to understand. For instance, particular examples are given by numerical semigroups, i.e., semigroups of the type $P = \mathbb{N} \setminus F$, where F is a finite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus F$ is additively closed. For instance, we could take $F = \{1\}$ or $F = \{1, 3\}$. We refer the interested reader to [RGS09] and the references therein for more about numerical semigroups, and also to Chapter 7.

5.4 Preliminaries

5.4.1 Embedding semigroups into groups

As we mentioned earlier, we need left cancellation for semigroups in our construction of semigroup C^*-algebras. One way to ensure cancellation is to embed our semigroups into groups, i.e., to find an injective semigroup homomorphism from our semigroup into a group. In general, the question of which semigroups embed into groups is quite complicated. Cancellation is necessary but not sufficient. Malcev gave the complete answer. He found an infinite list of conditions that are necessary and sufficient for group embeddability, and showed that any finite subset of his list is no longer sufficient. His list includes cancellation, which means both left cancellation and right cancellation. The latter means that for every $p, x, y \in P$, $xp = yp$ implies $x = y$. But Malcev's list also consists of conditions such as the following:

$$\text{For every } a, b, c, d, u, v, x, y \in P,$$
$$xa = yb,\ xc = yd,\ ua = vb \text{ implies } uc = vd.$$

We refer to [CP67, §12] for more details.

As explained in [CP67, §12], if a semigroup P embeds into a group, then there is a universal group embedding $P \hookrightarrow G_{\text{univ}}$, meaning that for every homomorphism $P \to G$ of the semigroup P to a group G, there is a unique homomorphism $G_{\text{univ}} \to G$ that makes the diagram

$$\begin{tikzcd} P \arrow[r, hook] \arrow[dr] & G_{\text{univ}} \arrow[d] \\ & G \end{tikzcd}$$

commutative.

Group embeddability is in general a complicated issue. Therefore, whenever it is convenient, we will simply assume that our semigroups can be embedded into groups. Verifying this assumption might be a challenge, for instance, in the case of Artin monoids (compare [Par02]).

However, we would like to mention one sufficient condition for group embeddability. Let P be a cancellative semigroup, i.e., P is left and right cancellative. Furthermore, assume that P is right reversible, i.e., for every $p, q \in P$, we have $Pp \cap Pq \neq \emptyset$. Here $Pp = \{xp : x \in P\}$. Then P embeds into a group. Actually, the universal group in the universal group embedding of P is given by an explicit construction as the group G of left quotients. This means that G consists of formal quotients of the form $q^{-1}p$, for all $q \in P$ and $p \in P$. We say that two such formal expressions $\tilde{q}^{-1}\tilde{p}$ and $q^{-1}p$ represent the same element in G if there is $r \in P$ with $\tilde{q} = rq$ and $\tilde{p} = rp$. To multiply elements in G, we make use of right reversibility: Given $p, q, r, s \in P$, suppose we want to multiply $s^{-1}r$ with $q^{-1}p$. As $Pq \cap Pr \neq \emptyset$, there exist x and y in P with $q = yr$. Thus, $q^{-1}p = (xq)^{-1}(xp) = (yr)^{-1}(xp)$. Let us now make the following formal computation:

$$(s^{-1}r)(q^{-1}p) = (s^{-1}r)(yr)^{-1}(xp) = s^{-1}rr^{-1}y^{-1}(xp) = s^{-1}y^{-1}(xp) = (ys)^{-1}(xp).$$

Motivated by this computation, we set

$$(s^{-1}r)(q^{-1}p) := (ys)^{-1}(xp).$$

It is now straightforward to check that this indeed defines a group $G = P^{-1}P$, and that $P \to G$, $p \mapsto e^{-1}p$ is an embedding of our semigroup P into our group G. Here e is the identity of P. We can always arrange that P has an identity by simply adjoining one if necessary. It is easy to see that this group embedding that we just constructed is actually the universal group embedding for P. We refer the reader to [CP61, §1.10] for more details.

Obviously, by symmetry, we also obtain that a cancellative semigroup P embeds into a group, if P is left reversible, i.e., if for every $p, q \in P$, we have $pP \cap qP \neq \emptyset$. In that case, P embeds into its group G of right quotients, $G = PP^{-1}$, and this is the universal group embedding for P.

For instance, both of these necessary conditions for group embeddability are satisfied for cancellative abelian semigroups. They are also satisfied for the Braid monoids B_n^+ introduced above.

The $ax+b$-semigroup $R \rtimes R^\times$ over an integral domain R is right reversible, but if R is not a field, then $R \rtimes R^\times$ is not left reversible.

The Thompson monoid is left reversible but not right reversible.

Finally, the nonabelian free monoid $\mathbb{N}^{*n}$ is neither left nor right reversible.

5.4.2 Graph products

We collect some basic facts about graph products that we will use later on in §5.9. Basically, we follow [CL02, §2]. Let $\Gamma = (V, E)$ be a graph with vertices V and edges E. Two vertices in V are connected by at most one edge, and no vertex is connected to itself. Hence we view E as a subset of $V \times V$. For every $v \in V$, assume that we are given a submonoid P_v of a group G_v. We can then form the graph products $P := \Gamma_{v \in V} P_v$ and $G := \Gamma_{v \in V} G_v$. As we explained, the group G is obtained from the free product $*_{v \in V} G_v$ by introducing the relations $xy = yx$ for all $x \in G_v$ and $y \in G_w$ with $(v, w) \in E$. Similarly, P is defined as the monoid obtained from the free product $*_{v \in V} P_v$ by introducing the relations $xy = yx$ for all $x \in P_v$ and $y \in P_w$ with $(v, w) \in E$. As explained in [CL02], it turns out that for every v, the monoid P_v sits in a canonical way as a submonoid inside the monoid $\Gamma_{v \in V} P_v$. Similarly, for each $v \in V$, the group G_v sits in a canonical way as a subgroup inside the group $\Gamma_{v \in V} G_v$. Moreover, the monoid $P = \Gamma_{v \in V} P_v$ can be canonically embedded as a submonoid of the group $G = \Gamma_{v \in V} G_v$.

A typical element g of $G = \Gamma_{v \in V} G_v$ is a product of the form $x_1 x_2 \cdots x_l$, where $x_i \in G_{v_i}$ are all nontrivial. (To obtain the identity, we would have to allow the empty word, i.e., the case $l = 0$.) We distinguish between words like $x_1 x_2 \cdots x_l$ and the element g they represent in the graph product G by saying that $x_1 x_2 \cdots x_l$ is an expression for g. Let us now explain when two words are expressions for the same group element.

First of all, for a word like $x_1 x_2 \cdots x_l$, we call the x_is the syllables and l the length of the word. We write $v(x_i)$ for the vertex $v_i \in V$ with the property that x_i lies in G_{v_i}. Given a word

$$x_1 \cdots x_i x_{i+1} \cdots x_l$$

with the property that $(v(x_i), v(x_{i+1})) \in E$, we can replace the subword $x_i x_{i+1}$ by $x_{i+1} x_i$. In this way, we transform the original word

$$x_1 \cdots x_i x_{i+1} \cdots x_l$$

to the new word

$$x_1 \cdots x_{i+1} x_i \cdots x_l.$$

This procedure is called a shuffle. Two words are called shuffle equivalent if one can be obtained from the other by performing finitely many shuffles.

Moreover, given a word

$$x_1 \cdots x_i x_{i+1} \cdots x_l$$

with the property that $v(x_i) = v(x_{i+1})$, then we say that our word admits an amalgamation. In that case, we can replace the subword $x_i x_{i+1}$ by the product $x_i \cdot x_{i+1} \in G_{v_i}$, where $v_i = v(x_i) = v(x_{i+1})$. Furthermore, if $x_i \cdot x_{i+1} = e$ in G_{v_i}, then we delete this part of our word. In this way, we transform the original word

$$x_1 \cdots x_i x_{i+1} \cdots x_l$$

to the new word

$$x_1 \cdots (x_i \cdot x_{i+1}) \cdots x_l$$

if $x_i \cdot x_{i+1} \neq e$ in G_{v_i} and

$$x_1 \cdots x_{i-1} x_{i+2} \cdots x_l$$

if $x_i \cdot x_{i+1} = e$ in G_{v_i}. This procedure is called an amalgamation.

Finally, we say that a word is reduced if it is not shuffle equivalent to a word that admits an amalgamation.

We have the following:

Lemma 5.4.1 (Lemma 1 in [CL02])**.** *A word*

$$x_1 \cdots x_l$$

is reduced if and only if for all $1 \leq i < j \leq l$ *with* $v(x_i) = v(x_j)$*, there exists* $1 \leq k \leq l$ *with* $i < k < j$ *such that* $(v(x_i), v(x_k)) \notin E$*.*

Suppose that we are given two words, and we can transform one word into the other by finitely many shuffles and amalgamations. Then it is clear that these two words are expressions for the same element in our group G. The converse is also true, this is the following result due to Green (see [Gre90]):

Theorem 5.4.2 (Theorem 2 in [CL02])**.** *Any two reduced words that are expressions for the same group element in* G *are shuffle equivalent.*

In other words, two words that are expressions for the same group element in G can be transformed into one another by finitely many shuffles and amalgamations. This is because, with the help of Lemma 5.4.1, it is easy to see that every word can be transformed into a reduced one by finitely many shuffles and amalgamations.

Because of Theorem 5.4.2, we may introduce the notion of length:

Definition 5.4.3. The *length* of an element g in our graph product G is the length of a reduced word that is an expression for g.

We also introduce the following:

Definition 5.4.4. Suppose we are given a reduced word

$$x = x_1 \cdots x_l.$$

Then we call x_i an *initial syllable* and $v(x_i)$ an *initial vertex* of our word, if for every $1 \leq h < i$, $(v(x_h), v(x_i)) \in E$. The set of all initial vertices of x is denoted by $V^i(x)$ (in [CL02], the notation $\Delta(x)$ is used).

Similarly, we call x_j a *final syllable* and $v(x_j)$ a *final vertex* of our word, if for every $j < k \leq l$, $(v(x_j), v(x_k)) \in E$. The set of all final vertices of x is denoted by $V^f(x)$ (it is denoted by $\Delta^r(x)$ in [CL02]).

The following is an easy observation:

Lemma 5.4.5 (Lemma 3 in [CL02]). *Let*

$$x = x_1 \cdots x_l$$

be a reduced word. Then:

(1) *If x_i is an initial syllable of x, then x is shuffle equivalent to $x_i x_1 \cdots x_{i-1} x_{i+1} \cdots x_l$.*

(2) *For all $v, w \in V^i(x)$, we have $(v, w) \in E$.*

(3) *For every $v \in V^i(x)$, there is a unique initial syllable x_i of x with $v(x_i) = v$. Let us denote this syllable by $S_v^i(x)$.*

(4) *If x' is shuffle equivalent to x, then $V^i(x) = V^i(x')$ and for every $v \in V^i(x) = V^i(x')$,*

$$S_v^i(x) = S_v^i(x').$$

The last three statements are also true for final vertices and final syllables. So we denote for a reduced word x with final syllable v the unique final syllable x_j of x with $v(x_j) = v$ by $S_v^f(x)$.

Definition 5.4.6. Let g be an element in our graph product G, and let x be a reduced word that is an expression for g. Then we set

$$V^i(g) := V^i(x),$$

and for $v \in V$,

$$S_v^i(g) := \begin{cases} S_v^i(x) & \text{if } v \in V^i(g) \\ e & \text{if } v \notin V^i(g). \end{cases}$$

Similarly, we define

$$V^f(g) := V^f(x),$$

and for $v \in V$,

$$S_v^f(g) := \begin{cases} S_v^f(x) & \text{if } v \in V^f(g) \\ e & \text{if } v \notin V^f(g). \end{cases}$$

We need the following:

Lemma 5.4.7 ([CL02], Lemma 5). *Given g and h in our graph product G, let*

$$W := V^f(g) \cap V^i(h),$$

and suppose that

$$z_w := S_w^f(g) S_w^i(h) \neq e$$

for all $w \in W$. Define

$$z := \prod_{w \in W} z_w,$$

in any order. Then, if $x \cdot \prod_{w \in W} S_w^f(g)$ is a reduced expression for g and $\prod_{w \in W} S_w^i(h) \cdot y$ is a reduced expression for h, then $x \cdot z \cdot y$ is a reduced expression for $g \cdot h$.

5.4.3 Krull rings

Since we want to study $ax+b$-semigroups over integral domains and their semigroup C^*-algebras later on, we collect a few basic facts in this context.

Let R be an integral domain.

Definition 5.4.8. The *constructible* (ring-theoretic) *ideals* of R are given by

$$\mathcal{I}(R) := \left\{ c^{-1}\left(\bigcap_{i=1}^{n} a_i R \right) : a_1, \ldots, a_n, c \in R^{\times} \right\}.$$

Here, for $c \in R^{\times}$ and an ideal I of R, we set

$$c^{-1}I := \{ r \in R : cr \in I \}.$$

Now let Q be the quotient field of R.

$$\mathcal{I}(R \subseteq Q) := \left\{ (x_1 \cdot R) \cap \ldots \cap (x_n \cdot R) : x_i \in Q^{\times} \right\}. \tag{5.2}$$

Note that for $c \in R^{\times}$ and $X \subseteq R$, we set

$$c^{-1}X = \{ r \in R : cr \in X \}, \text{ but } c^{-1} \cdot X = \left\{ c^{-1}x : x \in X \right\}.$$

Moreover, note that $\mathcal{I}(R) = \{ J \cap R : J \in \mathcal{I}(R \subseteq Q) \}$.

By construction, the family $\mathcal{I}(R)$ consists of integral divisorial ideals of R, and $\mathcal{I}(R \subseteq Q)$ consists of divisorial ideals of R. By definition, a divisorial ideal of an integral domain R is a fractional ideal I that satisfies $I = (R : (R : I))$, where $(R : J) = \{ q \in Q : qJ \subseteq R \}$. Equivalently, divisorial ideals are nonzero intersections of some nonempty family of principal fractional ideals (ideals of the form qR, $q \in Q$). Let $D(R)$ be the set of divisorial ideals of R. In our situation, we only consider finite intersections of principal fractional ideals (see (5.2)). So in general, our family $\mathcal{I}(R \subseteq Q)$ will only be a proper subset of $D(R)$.

However, for certain rings, the set $\mathcal{I}(R \subseteq Q)$ coincides with $D(R)$. For instance, this happens for noetherian rings. It also happens for Krull rings. The latter have a number of additional favourable properties which are very helpful for our purposes. Let us start with the following

Definition 5.4.9. An integral domain R is called a *Krull ring* if there exists a family of discrete valuations $(v_i)_{i \in I}$ of the quotient field Q of R such that

$(K1)$ $R = \{ x \in Q : v_i(x) \geq 0 \text{ for all } i \in I \}$,

$(K2)$ for every $0 \neq x \in Q$, there are only finitely many valuations in $(v_i)_i$ such that $v_i(x) \neq 0$.

The following result gives us many examples of Krull rings.

Theorem 5.4.10 ([Bou06b, Chapitre VII, §1.3, Corollaire]). *A noetherian integral domain is a Krull ring if and only if it is integrally closed.*

Let us collect some basic properties of Krull rings:
[Bou06b, Chapitre VII, §1.5, Corollaire 2] yields

Lemma 5.4.11. *For a Krull ring R, $\mathcal{I}(R \subseteq Q) = D(R)$ and $\mathcal{I}(R)$ is the set of integral divisorial ideals.*

Moreover, the prime ideals of height 1 play a distinguished role in a Krull ring.

Theorem 5.4.12. *[Bou06b, Chapitre VII, §1.6, Théorème 3 and Chapitre VII, §1.7, Théorème 4] Let R be a Krull ring. Every prime ideal of height 1 of R is a divisorial ideal. Let*

$$\mathcal{P}(R) = \{\mathfrak{p} \lhd R \text{ prime}: \operatorname{ht}(\mathfrak{p}) = 1\}.$$

For every $\mathfrak{p} \in \mathcal{P}(R)$, the localization $R_\mathfrak{p} = (R\backslash\mathfrak{p})^{-1}R$ is a principal valuation ring. Let $v_\mathfrak{p}$ be the corresponding (discrete) valuations of the quotient field Q of R. Then the family $(v_\mathfrak{p})_{\mathfrak{p}\in\mathcal{P}(R)}$ satisfies the conditions $(K1)$ and $(K2)$ from Definition 5.4.9.

Proposition 5.4.13. *[Bou06b, Chapitre VII, §1.5, Proposition 9] Let R be a Krull ring and $(v_\mathfrak{p})_{\mathfrak{p}\in\mathcal{P}(R)}$ be the valuations from the previous theorem. Given finitely many integers $n_1, \ldots, n_r$ and finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in $\mathcal{P}(R)$, there exists x in the quotient field Q of R with*

$$v_{\mathfrak{p}_i}(x) = n_i \text{ for all } 1 \leq i \leq r \text{ and } v_\mathfrak{p}(x) \geq 0 \text{ for all } \mathfrak{p} \in \mathcal{P}(R) \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}.$$

Moreover, given a fractional ideal I of R, we let $I^\sim := (R : (R : I))$ be the divisorial closure of I. $I^\sim$ is the smallest divisorial ideal of R that contains I. We can now define the product of two divisorial ideals I_1 and I_2 to be the divisorial closure of the (usual ideal-theoretic) product of I_1 and I_2, i.e., $I_1 \bullet I_2 := (I_1 \cdot I_2)^\sim$. $D(R)$ becomes a commutative monoid with this multiplication.

Theorem 5.4.14 ([Bou06b, Chapitre VII, §1.2, Théorème 1; Chapitre VII, §1.3, Théorème 2 and Chapitre VII, §1.6, Théorème 3]). *For a Krull ring R, $(D(R), \bullet)$ is a group. It is the free abelian group with free generators given by $\mathcal{P}(R)$, the set of prime ideals of R that have height 1.*

This means that every $I \in \mathcal{I}(R \subseteq Q)$ (Q is the quotient field of the Krull ring R) is of the form $I = \mathfrak{p}_1^{(n_1)} \bullet \cdots \bullet \mathfrak{p}_r^{(n_r)}$, with $n_i \in \mathbb{Z}$. Here for $\mathfrak{p} \in \mathcal{P}(R)$ and $n \in \mathbb{N}$, we write

$$\mathfrak{p}^{(n)} \text{ for } \underbrace{\mathfrak{p} \bullet \cdots \bullet \mathfrak{p}}_{n \text{ times}}, \text{ and } \mathfrak{p}^{(-n)} \text{ for } \underbrace{\mathfrak{p}^{-1} \bullet \cdots \bullet \mathfrak{p}^{-1}}_{n \text{ times}},$$

where $\mathfrak{p}^{-1} = (R : \mathfrak{p})$. We set for $\mathfrak{p} \in \mathcal{P}(R)$:

$$v_\mathfrak{p}(I) := \begin{cases} n_i \text{ if } \mathfrak{p} = \mathfrak{p}_i, \\ 0 \text{ if } \mathfrak{p} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}. \end{cases}$$

With this notation, we have $I = \prod_{\mathfrak{p} \in \mathcal{P}(R)} \mathfrak{p}^{(v_{\mathfrak{p}}(I))}$, where the product is taken in $D(R)$. In addition, we have for $I \in \mathcal{I}(R \subseteq Q)$ that $I \in \mathcal{I}(R)$ if and only if $v_{\mathfrak{p}}(I) \geq 0$ for all $\mathfrak{p} \in \mathcal{P}(R)$. And combining the last statement in [Bou06b, Chapitre VII, §1.3, Théorème 2] with [Bou06b, Chapitre VII, §1.4, Proposition 5], we obtain for every $I \in \mathcal{I}(R \subseteq Q)$:

$$I = \{x \in Q : v_{\mathfrak{p}}(x) \geq v_{\mathfrak{p}}(I) \text{ for all } \mathfrak{p} \in \mathcal{P}(R)\}. \tag{5.3}$$

Finally, the principal fractional ideals $F(R)$ form a subgroup of $(D(R), \bullet)$ which is isomorphic to $Q^{\times}$. Suppose that R is a Krull ring. Then the quotient group $C(R) := D(R)/F(R)$ is called the divisor class group of R.

These are basic properties of Krull rings. We refer the interested reader to [Bou06b, Chapitre VII] or [Fos73] for more information.

5.5 C^*-algebras attached to inverse semigroups, partial dynamical systems and groupoids

We refer the interested reader to [Ren80, Exe08, Exe15, Pat99] for more references for this section.

5.5.1 Inverse semigroups

Inverse semigroups play an important role in the study of semigroup C^*-algebras.

Definition 5.5.1. An *inverse semigroup* is a semigroup S with the property that for every $x \in S$, there is a unique $y \in S$ with $x = xyx$ and $y = yxy$.

We write $y = x^{-1}$ and call y the inverse of x.

Definition 5.5.2. An inverse semigroup S is called an *inverse semigroup with zero* if there is a distinguished element $0 \in S$ satisfying $0 \cdot s = 0 = s \cdot 0$ for all $s \in S$.

Usually, if we write "inverse semigroup", we mean an inverse semigroup with or without zero. Sometimes we write "inverse semigroups without zero" for ordinary inverse semigroups that do not have a distinguished zero element.

Every inverse semigroup can be realized as partial bijections on a fixed set. Multiplication is given by composition. However, a partial bijection is only defined on its domain. Therefore, if we want to compose the partial bijection $s : \mathrm{dom}(s) \to \mathrm{im}\,(s)$ with another partial bijection $t : \mathrm{dom}(t) \to \mathrm{im}\,(t)$, we have to restrict t to

$$\mathrm{dom}(t) \cap t^{-1}(\mathrm{dom}(s))$$

to make sure that the image of the restriction of t lies in the domain of s. Only then we can form $s \circ t$. The inverse of a partial bijection is the usual inverse, in the category of sets.

Inverse semigroups can also be realized as partial isometries on a Hilbert space. To make sure that the product of two partial isometries is again a partial isometry, we have to require that the source and range projections of our partial isometries commute. Then multiplication in the inverse semigroup is just the usual multiplication of operators on a fixed Hilbert space, i.e., composition of operators. The inverse in our inverse semigroup is given by the adjoint operation for operators in general or partial isometries in our particular situation.

Let us explain how to attach an inverse semigroup to a left cancellative semigroup. Assume that P is a left cancellative semigroup. Its left inverse hull $I_l(P)$ is the inverse semigroup generated by the partial bijections

$$P \to pP, \quad x \mapsto px,$$

whose domain is P and whose image is $pP = \{px : x \in P\}$. Its inverse is given by

$$pP \to P, \quad px \mapsto x.$$

So $I_l(P)$ is the smallest semigroup of partial bijections on P that is closed under inverses and contains

$$\{P \to P, \ x \mapsto px : p \in P\}.$$

Given $p \in P$, we denote the partial bijection

$$P \to pP, \quad x \mapsto px$$

by p. In this way, we obtain an embedding of P into $I_l(P)$ by sending $p \in P$ to the partial bijection $p \in I_l(P)$. This allows us to view P as a subsemigroup of $I_l(P)$. We say that $I_l(P)$ is an inverse semigroup with zero if the partial bijection that is nowhere defined, $\emptyset \to \emptyset$, is in $I_l(P)$. In that case, $\emptyset \to \emptyset$ is the distinguished zero element 0.

Alternatively, we can also describe $I_l(P)$ as the smallest inverse semigroup of partial isometries on $\ell^2 P$ generated by the isometries $\{V_p : p \in P\}$. This means that $I_l(P)$ can be identified with the smallest semigroup of partial isometries on $\ell^2 P$ containing the isometries $\{V_p : p \in P\}$ and their adjoints $\left\{V_p^* : p \in P\right\}$ and that is closed under multiplication. In this picture, $I_l(P)$ is an inverse semigroup with zero if and only if the zero operator is in $I_l(P)$.

An important subsemigroup of an inverse semigroup S is its semilattice of idempotents.

Definition 5.5.3. The *semilattice* E of idempotents in an inverse semigroup S is given by

$$E := \left\{x^{-1}x : x \in S\right\} = \left\{xx^{-1} : x \in S\right\} = \left\{e \in S : e = e^2\right\}.$$

Define an order on E by setting, for $e, f \in E$, $e \leq f$ if $e = ef$.

If S is an inverse semigroup with zero, E becomes a *semilattice with zero*, and the distinguished zero element of S becomes the *distinguished zero element of* E.

In the case of partial bijections, the semilattice of idempotents is given by all domains and images. Multiplication in this semilattice is intersection of sets, and $\leq$ is $\subseteq$ for sets, i.e., containment.

Definition 5.5.4. For the left inverse hull $I_l(P)$ attached to a left cancellative semigroup P, the *semilattice of idempotents* is denoted by $\mathcal{J}_P$.

It is easy to see that $\mathcal{J}_P$ is given by

$$\mathcal{J}_P = \left\{p_n \cdots q_1^{-1} p_1(P) : q_i, p_i \in P\right\} \cup \left\{q_n^{-1} p_n \cdots q_1^{-1} p_1(P) : q_i, p_i \in P\right\}.$$

Here, for $X \subseteq P$ and $p, q \in P$, we write

$$p(X) = \{px : x \in X\}$$

and

$$q^{-1}(X) = \{y \in P : qy \in X\}.$$

Subsets of the form $p_n \cdots q_1^{-1} p_1(P)$ or $q_n^{-1} p_n \cdots q_1^{-1} p_1(P)$ are right ideals of P. Here, we call $X \subseteq P$ a right ideal if for every $x \in X$ and $r \in P$, we always have $xr \in X$.

Definition 5.5.5. The elements in $\mathcal{J}_P$ are called *constructible right ideals* of P.

We will work out the set of constructible right ideals explicitly for classes of examples in §5.6.5.

There is a duality between semilattices, i.e., abelian semigroups of idempotents, and totally disconnected locally compact Hausdorff spaces. Given a semilattice E, we construct its space of characters $\widehat{E}$ as follows:

$$\widehat{E} = \{\chi : E \to \{0,1\} \text{ nonzero semigroup homomorphism}\}.$$

In other words, elements in $\widehat{E}$ are multiplicative maps from E to $\{0,1\}$, where the latter set is equipped with the usual multiplication when we view it as a subspace of $\mathbb{R}$ (or $\mathbb{C}$). In addition, we require that these multiplicative maps must take the value 1 for some element $e \in E$. If our semilattice E is a semilattice with zero, and 0 is its distinguished zero element, then we require that $\chi(0) = 0$ for all $\chi \in \widehat{E}$.

The topology on $\widehat{E}$ is given by pointwise convergence. Every $\chi \in \widehat{E}$ is uniquely determined by

$$\chi^{-1}(1) = \{e \in E : \chi(e) = 1\}.$$

$\chi^{-1}(1)$ is an E-valued filter (which we simply call a filter from now on), i.e., a subset of E satisfying:

- $\chi^{-1}(1) \neq \emptyset$.
- For all $e, f \in E$ with $e \leq f$, $e \in \chi^{-1}(1)$ implies $f \in \chi^{-1}(1)$.
- For all $e, f \in E$ with $e, f \in \chi^{-1}(1)$, ef lies in $\chi^{-1}(1)$.

Conversely, every filter, i.e., every subset $\mathcal{F} \in E$ satisfying these three conditions determines a unique $\chi \in \widehat{E}$ with $\chi^{-1}(1) = \mathcal{F}$. Therefore, we have a one-to-one correspondence between characters $\chi \in \widehat{E}$ and filters.

If E is a semilattice with zero, and 0 is the distinguished zero element, then we have $\chi(0) = 0$ for all $\chi \in \widehat{E}$. In terms of filters, this amounts to saying that 0 is never an element of a filter.

As an illustrative example, the reader is encouraged to work out the set of constructible right ideals $\mathcal{J}_P$ and the space of characters $\widehat{\mathcal{J}_P}$ for the nonabelian free semigroup on two generators $P = \mathbb{N} * \mathbb{N}$, or in other words, the semilattice of idempotents E and the space of its characters $\widehat{E}$ for the inverse semigroup $S = I_l(\mathbb{N} * \mathbb{N})$.

Now assume that we are given a subsemigroup P of a group G. We define

$$I_l(P)^\times := I_l(P) \setminus \{0\}$$

if $I_l(P)$ is an inverse semigroup with zero, and 0 is its distinguished zero element, and

$$I_l(P)^\times := I_l(P)$$

otherwise.

Now it is easy to see that for every partial bijection s in $I_l(P)^\times$, there exists a unique $\sigma(s) \in G$ such that s is of the form

$$s(x) = \sigma(s) \cdot x \text{ for } x \in \mathrm{dom}(s).$$

Here we view P as a subset of the group G and make use of multiplication in G.

In the alternative picture of $I_l(P)$ as the inverse semigroup of partial isometries on $\ell^2 P$ generated by the isometries $\{V_p : p \in P\}$, $I_l(P)^\times$ is given by all nonzero partial isometries in $I_l(P)$. Every element in $I_l(P)^\times$ is of the form

$$V_{q_1} \cdots V_{p_n}^*,\ V_{p_1}^* V_{q_1} \cdots V_{p_n}^*,\ V_{q_1} \cdots V_{p_n}^* V_{q_n}, \text{ or } V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n}.$$

The map σ which we introduced above is then given by

$$\begin{aligned} \sigma(V_{q_1} \cdots V_{p_n}^*) &= q_1 \cdots p_n^{-1} \in G, \\ \sigma(V_{p_1}^* V_{q_1} \cdots V_{p_n}^*) &= p_1^{-1} q_1 \cdots p_n^{-1} \in G, \\ \sigma(V_{q_1} \cdots V_{p_n}^* V_{q_n}) &= q_1 \cdots p_n^{-1} q_n \in G, \\ \text{or } \sigma(V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n}) &= p_1^{-1} q_1 \cdots p_n^{-1} q_n \in G. \end{aligned}$$

To see that σ is well-defined, note that, similarly as above, every partial isometry $V \in I_l(P)^\times$ has the property that there exists a unique $g \in G$ such that for every $x \in P$, either $V\delta_x = 0$ or $V\delta_x = \delta_{g \cdot x}$. And σ is defined in such a way that $\sigma(V) = g$.

It is easy to see that the map $\sigma : I_l(P)^\times \to G$ satisfies

$$\sigma(st) = \sigma(s)\sigma(t)$$

for all $s, t \in I_l(P)^\times$, as long as the product st lies in $I_l(P)^\times$, i.e., is nonzero. Moreover, setting

$$\mathcal{J}_P^\times := \mathcal{J}_P \setminus \{0\}$$

if $\mathcal{J}_P$ is a semilattice with zero, and 0 is the distinguished zero element, and

$$\mathcal{J}_P^\times := \mathcal{J}_P$$

otherwise, it is also easy to see that

$$\sigma^{-1}(e) = \mathcal{J}_P^\times.$$

Here e is the identity in our group G.

We formalize this in the next definition: Let S be an inverse semigroup and E the semilattice of idempotents of S. We set $S^\times := S \setminus \{0\}$ if S is an inverse semigroup with zero, and 0 is the distinguished zero element, and $S^\times := S$ otherwise. Similarly, let $E^\times := E \setminus \{0\}$ if E is a semilattice with zero, and 0 is the distinguished zero element, and $E^\times := E$ otherwise. Moreover, let G be a group.

Definition 5.5.6. A map $\sigma : S^\times \to G$ is called a *partial homomorphism* if $\sigma(st) = \sigma(s)\sigma(t)$ for all $s, t \in S^\times$ with $st \in S^\times$.

A map $\sigma : S^\times \to G$ is called *idempotent pure* if $\sigma^{-1}(e) = E^\times$.

The existence of an idempotent pure partial homomorphism will allow us to describe C^*-algebras attached to inverse semigroups as crossed products of partial dynamical systems later on.

The following is a useful observation which we need later on.

Lemma 5.5.7. *Assume that S is an inverse semigroup and $\sigma : S^\times \to G$ is an idempotent pure partial homomorphism to a group G. Whenever two elements s and t in $S^\times$ satisfy $s^{-1}s = t^{-1}t$ and $\sigma(s) = \sigma(t)$, then we must have $s = t$.*

Proof. It is clear that st^{-1} lies in $S^\times$. Since $\sigma(st^{-1}) = e$, we must have $st^{-1} \in E$. Hence

$$st^{-1} = ts^{-1}st^{-1} = tt^{-1},$$

and therefore

$$s = ss^{-1}s = st^{-1}t = tt^{-1}t = t. \qquad \square$$

Let us now explain the construction of reduced and full C^*-algebras for inverse semigroups.

Let S be an inverse semigroup, and define $S^\times$ as above. For $s \in S$, define

$$\lambda_s : \ell^2 S^\times \to \ell^2 S^\times$$

by setting

$$\lambda_s(\delta_x) := \delta_{sx} \text{ if } s^{-1}s \geq xx^{-1}, \text{ and } \lambda_s(\delta_x) := 0 \text{ otherwise.}$$

Note that we require $s^{-1}s \geq xx^{-1}$ because on

$$\left\{x \in S : \, s^{-1}s \geq xx^{-1}\right\},$$

the map $x \mapsto sx$ given by left multiplication with s is injective. This is because we can reconstruct x from sx due to the computation

$$x = xx^{-1}x = s^{-1}sxx^{-1}x = s^{-1}(sx).$$

Therefore, for each s, we obtain a partial isometry λ_s by our construction. The assignment $s \mapsto \lambda_s$ is a $*$-representation of S by partial isometries on $\ell^2 S^\times$. It is called the left regular representation of S. The star in $*$-representation indicates that we have $\lambda_{s^{-1}} = \lambda_s^*$.

Definition 5.5.8. We define

$$C^*_\lambda(S) := C^*(\{\lambda_s : \, s \in S\}) \subseteq \mathcal{L}(\ell^2 S^\times).$$

$C^*_\lambda(S)$ is called the *reduced inverse semigroup C^*-algebra* of S.

The full C^*-algebra of an inverse semigroup S is given by a universal property.

Definition 5.5.9. We define

$$C^*(S) := C^*\left(\{v_s\}_{s \in S} \;\middle|\; v_s v_t = v_{st}, \; v_s^* = v_{s^{-1}}, \; v_0 = 0 \text{ if } 0 \in S\right).$$

$C^*(S)$ is the *full inverse semigroup C^*-algebra* of S.

Here, $0 \in S$ is short for

"S is an inverse semigroup with zero, and 0 is the distinguished zero element."

This means that $C^*(S)$ is uniquely determined by the property that given any C^*-algebra B with elements $\{w_s : \, s \in S\}$ satisfying the above relations, i.e.,

$$w_s w_t = w_{st}, \; w_s^* = w_{s^{-1}}, \; w_0 = 0 \text{ if } 0 \in S,$$

then there exists a unique $*$-homomorphism from $C^*(S)$ to B sending v_s to w_s.

In other words, $C^*(S)$ is the C^*-algebra universal for $*$-representation of S by partial isometries (in a C^*-algebra, or on a Hilbert space). Note that we require that if $0 \in S$, then the zero element of S should be represented by the partial isometry 0. That is why $v_0 = 0$ in case $0 \in S$. This is different from the definition in [Pat99, §2.1], where the partial isometry representing 0 in the full C^*-algebra of S is a nonzero, minimal and central projection. We will come back to this difference in the definitions later on.

By construction, there is a canonical $*$-homomorphism $\lambda : \, C^*(S) \to C^*_\lambda(S)$, $v_s \mapsto \lambda_s$. It is called the left regular representation (of $C^*(S)$).

We refer the reader to [Pat99] for more about inverse semigroups and their C^*-algebras.

5.5.2 Partial dynamical systems

Whenever we have a semigroup embedded into a group, or an inverse semigroup with an idempotent pure partial homomorphism to a group, we can construct a partial dynamical system. Let us first present the general framework.

In the following, our convention will be that all our groups are discrete and countable, and all our topological spaces are locally compact, Hausdorff and second countable.

Definition 5.5.10. Let G be a group with identity e, and let X be a topological space. A *partial action* α of G on X consists of

- a collection $\{U_g\}_{g\in G}$ of open subspaces $U_g \subseteq X$,
- a collection $\{\alpha_g\}_{g\in G}$ of homeomorphisms $\alpha_g : U_{g^{-1}} \to U_g$, $x \mapsto g.x$ such that
  - $U_e = X$, $\alpha_e = \mathrm{id}_X$;
  - for all $g_1, g_2 \in G$, we have

$$g_2.(U_{(g_1g_2)^{-1}} \cap U_{g_2^{-1}}) = U_{g_2} \cap U_{g_1^{-1}},$$
$$\text{and } (g_1g_2).x = g_1.(g_2.x) \text{ for all } x \in U_{(g_1g_2)^{-1}} \cap U_{g_2^{-1}}.$$

We call such a triple (X, G, α) a *partial dynamical system*, and denote it by $\alpha : G \curvearrowright X$ or simply $G \curvearrowright X$.

Let $\alpha : G \curvearrowright X$ be a partial dynamical system. The dual action α^* of α is the partial action (in the sense of [McC95]) of G on $C_0(X)$ given by

$$\alpha_g^* : C_0(U_{g^{-1}}) \to C_0(U_g),\ f \mapsto f(g^{-1}.\sqcup).$$

We set out to describe a canonical partial action attached to a semigroup P embedded into a group G. Let $C^*_\lambda(P)$ be the reduced semigroup C^*-algebra of P. It contains a canonical commutative subalgebra $D_\lambda(P)$, which is given by

$$D_\lambda(P) := C^*(\{1_X : X \in \mathcal{J}_P\}) \subseteq C^*_\lambda(P).$$

It is clear that $D_\lambda(P)$ coincides with $\overline{\mathrm{span}}(\sigma^{-1}(e))$. Recall that the map $\sigma : I_l(P)^\times \to G$ is given as follows: Every partial isometry $V \in I_l(P)^\times$ has the property that there exists a unique $g \in G$ such that for every $x \in P$, either $V\delta_x = 0$ or $V\delta_x = \delta_{g\cdot x}$. And σ is defined in such a way that $\sigma(V) = g$.

Let us now describe the canonical partial action $G \curvearrowright D_\lambda(P)$. We will think of it as a dual action α^*. For $g \in G$, let

$$D_{g^{-1}} := \overline{\mathrm{span}}(\{V^*V : V \in I_l(P)^\times,\ \sigma(V) = g\}).$$

By construction, we have that $D_e = D_\lambda(P)$. Moreover, it is easy to see that $D_{g^{-1}}$ is an ideal of $D_\lambda(P)$. Here is the argument: Suppose we are given $V \in I_l(P)^\times$ with $\sigma(V) = g$, and $W \in I_l(P)^\times$ with $\sigma(W) = e$. Then W must be a projection since for every $x \in P$, either $W\delta_x = 0$ or $W\delta_x = \delta_{e\cdot x} = \delta_x$. Moreover, W and V^*V commute as both of these are elements in the commutative C^*-algebra $\ell^\infty(P)$. Hence WV^*V is nonzero if and only if V^*VW is nonzero, and if that is the case, we obtain

$$WV^*V = V^*VW = WV^*VW = (VW)^*(VW).$$

As $\sigma(VW) = g$, this implies that both WV^*V and V^*VW lie in $D_{g^{-1}}$. Therefore, as we claim, $D_{g^{-1}}$ is an ideal of $D_\lambda(P)$.

We then define α_g^* as $\alpha_g^* : D_{g^{-1}} \to D_g$, $V^*V \to VV^*$ for $V \in I_V^\times$ with $\sigma(V) = g$. This is well-defined: If we view $\ell^2 P$ as a subspace $\ell^2 G$ and let λ be the left regular representation of G, then every $V \in I_V^\times$ with $\sigma(V) = g$ satisfies $V = \lambda_g V^*V$. Therefore, $VV^* = \lambda_g V^*V\lambda_g^*$. This shows that α_g^* is just conjugation with the unitary λ_g. This also explains why α_g^* is an isomorphism.

Of course, we can also describe the dual action α. Set

$$\Omega_P := \mathrm{Spec}\,(D_\lambda(P))$$

and for every $g \in G$, let

$$U_{g^{-1}} := \widehat{D_{g^{-1}}}.$$

It is easy to see that

$$U_{g^{-1}} = \left\{\chi \in \Omega_P : \chi(V^*V) = 1 \text{ for some } V \in I_V^\times \text{ with } \sigma(V) = g\right\}.$$

We then define α_g by setting $\alpha_g(\chi) := \chi \circ \alpha_{g^{-1}}^*$. These α_g, $g \in G$, give rise to the canonical partial dynamical system $G \curvearrowright \Omega_P$ attached to a semigroup P embedded into a group G.

Our next goal is to describe a canonical partial dynamical system attached to inverse semigroups equipped with a idempotent pure partial homomorphism to a group. Let S be an inverse semigroup and E be the semilattice of idempotents of S. Let G be a group. Assume that σ is a partial homomorphism $S^\times \to G$ that is idempotent pure.

In this situation, we describe a partial dynamical system $G \curvearrowright \widehat{E}$, and we will show later (see Corollary 5.5.23) that the reduced C^*-algebra $C_\lambda^*(S)$ of S is canonically isomorphic to $C_0(\widehat{E}) \rtimes_r G$.

Consider the sub-C^*-algebra

$$C^*(E) := C^*(\{\lambda_e : e \in E\}) \subseteq C_\lambda^*(S).$$

As we will see, we have a canonical isomorphism $\mathrm{Spec}\,(C^*(E)) \cong \widehat{E}$, so that $C_0(\widehat{E}) \cong C^*(E)$.

Now let us describe the partial action $G \curvearrowright C^*(E)$. For $g \in G$, define a sub-C^*-algebra of $C^*(E)$ by

$$C^*(E)_{g^{-1}} := \overline{\mathrm{span}}(\{\lambda_{s^{-1}s} : s \in S^\times, \sigma(s) = g\}).$$

As σ is idempotent pure, we have $C^*(E)_e = C^*(E)$. For every $g \in G$, we have a C^*-isomorphism

$$\alpha_g^* : C^*(E)_{g^{-1}} \to C^*(E)_g, \, \lambda_{s^{-1}s} \mapsto \lambda_{ss^{-1}}.$$

The corresponding dual action is given as follows: We identify $\mathrm{Spec}\,(C^*(E))$ with $\widehat{E}$. Then, for every $g \in G$, we set

$$U_g = \mathrm{Spec}\,(C^*(E)_g) \subseteq \widehat{E}.$$

It is easy to see that

$$U_{g^{-1}} = \left\{ \chi \in \widehat{E} : \chi(s^{-1}s) = 1 \text{ for some } s \in S^\times \text{ with } \sigma(s) = g \right\}.$$

For every $g \in G$, the homeomorphism $\alpha_g : U_{g^{-1}} \to U_g$ defining the partial dynamical system $G \curvearrowright \widehat{E}$ is given by $\alpha_g(\chi) = \chi \circ \alpha^*_{g^{-1}}$. More concretely, given $\chi \in U_{g^{-1}}$ and $s \in S^\times$ with $\sigma(s) = g$ and $\chi(s^{-1}s) = 1$, we have $\alpha_g(\chi)(e) = \chi(s^{-1}es)$. These α_g, $g \in G$, give rise to the canonical partial dynamical system $G \curvearrowright \widehat{E}$ attached to an inverse semigroup S equipped with an idempotent pure partial homomorphism to a group G.

At this point, a natural question arises. Assume we are given a semigroup P embedded into a group G. We have seen above that this leads to an idempotent pure partial homomorphism on the left inverse hull $I_l(P)$ to our group G. How is the partial dynamical system $G \curvearrowright \Omega_P$ related to the partial dynamical system $G \curvearrowright \widehat{\mathcal{J}_P}$? We will see the answer in §5.6.7.

Let us now recall the construction, originally defined in [McC95], of the reduced and full crossed products $C_0(X) \rtimes_{\alpha^*,r} G$ and $C_0(X) \rtimes_{\alpha^*} G$ attached to our partial dynamical system $\alpha : G \curvearrowright X$. We usually omit α^* in our notation for the crossed products for the sake of brevity.

First of all,

$$C_0(X) \rtimes^{\ell^1} G := \left\{ \sum_g f_g \delta_g \in \ell^1(G, C_0(X)) : f_g \in C_0(U_g) \right\}$$

becomes a $*$-algebra under component-wise addition, multiplication given by

$$\left(\sum_g f_g \delta_g \right) \cdot \left(\sum_h \tilde{f}_h \delta_h \right) := \sum_{g,h} \alpha_g^*(\alpha^*_{g^{-1}}(f_g) \tilde{f}_h) \delta_{gh}$$

and involution

$$\left(\sum_g f_g\delta_g\right)^* := \sum_g \alpha_g^*(f_{g^{-1}}^*)\delta_g.$$

As in [McC95], we construct a representation of $C_0(X) \rtimes^{\ell^1} G$. Viewing X as a discrete set, we define $\ell^2 X$ and the representation

$$M:\ C_0(X) \to \mathcal{L}(\ell^2 X),\ f \mapsto M(f),$$

where $M(f)$ is the multiplication operator $M(f)(\xi) := f \cdot \xi$ for $\xi \in \ell^2 X$. M is obviously a faithful representation of $C_0(X)$. Every $g \in G$ leads to a twist of M, namely,

$$M_g:\ C_0(X) \to \mathcal{L}(\ell^2 X) \text{ given by } M_g(f)\xi := f|_{U_g}(g.\sqcup) \cdot \xi|_{U_{g^{-1}}}.$$

Here we view $f|_{U_g}(g.\sqcup)$ as an element in $C_b(U_{g^{-1}})$, and $C_b(U_{g^{-1}})$ acts on $\ell^2 U_{g^{-1}}$ just by multiplication operators. Given $\xi \in \ell^2 X$, we set

$$\xi|_{U_{g^{-1}}}(x) := \xi(x) \text{ if } x \in U_{g^{-1}} \text{ and } \xi|_{U_{g^{-1}}}(x) := 0 \text{ if } x \notin U_{g^{-1}}.$$

In other words, $\xi|_{U_{g^{-1}}}$ is the component of ξ in $\ell^2 U_{g^{-1}}$ with respect to the decomposition

$$\ell^2 X = \ell^2 U_{g^{-1}} \oplus \ell^2 U_{g^{-1}}^c.$$

So we have

$$M_g(f)\xi(x) = f(g.x)\xi(x) \text{ if } x \in U_{g^{-1}} \text{ and } M_g(f)\xi(x) = 0 \text{ if } x \notin U_{g^{-1}}.$$

Consider now the Hilbert space

$$H := \ell^2(G, \ell^2 X) \cong \ell^2 G \otimes \ell^2 X,$$

and define the representation

$$\mu:\ C_0(X) \to \mathcal{L}(H) \text{ given by } \mu(f)(\delta_g \otimes \xi) := \delta_g \otimes M_g(f)\xi.$$

For $g \in G$, let E_g be the orthogonal projection onto $\overline{\mu(C_0(U_{g^{-1}}))H}$. Moreover, let λ denote the left regular representation of G on $\ell^2 G$, and set $V_g := (\lambda_g \otimes I) \cdot E_g$. Here I is the identity operator on H.

We can now define the representation

$$\mu \times \lambda:\ C_0(X) \rtimes^{\ell^1} G \to \mathcal{L}(H),\ \sum_g f_g\delta_g \mapsto \sum_g \mu(f_g)V_g.$$

Following the original definition in [McC95], we set

Definition 5.5.11.
$$C_0(X) \rtimes_r G := \overline{C_0(X) \rtimes^{\ell^1} G}^{\|\cdot\|_{\mu\times\lambda}}.$$

To define the full crossed product $C_0(X) \rtimes G$ attached to our partial dynamical system $G \curvearrowright X$, recall that we have already introduced the $*$-algebra $C_0(X) \rtimes^{\ell^1} G$.

Definition 5.5.12. Let $C_0(X) \rtimes G$ be the *universal enveloping C^*-algebra* of the $*$-algebra $C_0(X) \rtimes^{\ell^1} G$.

This means that $C_0(X) \rtimes G$ is universal for $*$-representations of $C_0(X) \rtimes^{\ell^1} G$ as bounded operators on Hilbert spaces or to C^*-algebras. To construct this universal C^*-algebra, we follow the usual procedure of completing $C_0(X) \rtimes^{\ell^1} G$ with respect to the maximal C^*-norm on $C_0(X) \rtimes^{\ell^1} G$. Usually, we only obtain a C^*-seminorm and have to divide out vectors with trivial seminorm, but because the $*$-representation $\mu \times \lambda$ constructed above is faithful, we get a C^*-norm. So there is an embedding $C_0(X) \rtimes^{\ell^1} G \hookrightarrow C_0(X) \rtimes G$, and the universal property of $C_0(X) \rtimes G$ means that whenever we have a $*$-homomorphism $C_0(X) \rtimes^{\ell^1} G \to B$ to some C^*-algebra B, there is a unique $*$-homomorphism $C_0(X) \rtimes G \to B$ that makes the diagram

$$\begin{tikzcd} C_0(X) \rtimes^{\ell^1} G \arrow[r, hook] \arrow[dr] & C_0(X) \rtimes G \arrow[d] \\ & B \end{tikzcd}$$

commutative.

By construction, there is a canonical $*$-homomorphism $C_0(X) \rtimes G \to C_0(X) \rtimes_r G$ extending the identity on $C_0(X) \rtimes^{\ell^1} G$.

The reader may consult [McC95, Exe15] for more information about partial dynamical systems and their C^*-algebras.

5.5.3 Étale groupoids

Groupoids play an important role in operator algebras in general and for our topic of semigroup C^*-algebras in particular. This is because many C^*-algebras can be written as groupoid C^*-algebras. This also applies to many semigroup C^*-algebras.

Let us first introduce groupoids. In the language of categories, a groupoid is simply a small category with inverses. Very roughly speaking, this means that a groupoid is a group where multiplication is not globally defined. Roughly speaking, a groupoid $\mathcal{G}$ is a set, whose elements γ are arrows $r(\gamma) \longleftarrow s(\gamma)$. Here $r(\gamma)$ and $s(\gamma)$ are elements in $\mathcal{G}^{(0)}$, the set of units. r stands for range and s stands for source. For every $u \in \mathcal{G}^{(0)}$, there is a distinguished arrow $u \overset{\mathrm{id}_u}{\longleftarrow} u$ in our groupoid $\mathcal{G}$. This allows us to define an embedding

$$\mathcal{G}^{(0)} \hookrightarrow \mathcal{G},\ u \mapsto \mathrm{id}_u,$$

which in turn allows us to view $\mathcal{G}^{(0)}$ as a subset of $\mathcal{G}$.

$\mathcal{G}$ comes with a multiplication

$$\{(\gamma,\eta) \in \mathcal{G} \times \mathcal{G} : \ s(\gamma) = r(\eta)\} \longrightarrow \mathcal{G}, \ (\gamma,\eta) \mapsto \gamma\eta.$$

We think of this multiplication as a concatenation of arrows. With this picture in mind, the condition $s(\gamma) = r(\eta)$ makes sense. Also, $\mathcal{G}$ comes with an inversion

$$\mathcal{G} \to \mathcal{G}, \ \gamma \to \gamma^{-1}.$$

We think of this inversion as reversing arrows. The picture of arrows, with concatenation as multiplication and reversing as inversion, leads to obvious axioms, which, once imposed, give rise to the formal definition of a groupoid. Let us present the details.

Definition 5.5.13. A *groupoid* is a set $\mathcal{G}$, together with a bijective map $\mathcal{G} \to \mathcal{G}$, $\gamma \mapsto \gamma^{-1}$, a subset $\mathcal{G} * \mathcal{G} \subseteq \mathcal{G} \times \mathcal{G}$, and a map $\mathcal{G} * \mathcal{G} \to \mathcal{G}$, $(\gamma,\eta) \mapsto \gamma\eta$, such that

$$\begin{aligned}(\gamma^{-1})^{-1} &= \gamma \text{ for all } \gamma \in \mathcal{G},\\ (\gamma\eta)\zeta &= \gamma(\eta\zeta) \text{ for all } (\gamma,\eta), (\eta,\zeta) \in \mathcal{G} * \mathcal{G},\\ \gamma^{-1}\gamma\eta &= \eta, \ \gamma\eta\eta^{-1} = \gamma \text{ for all } (\gamma,\eta) \in \mathcal{G} * \mathcal{G}.\end{aligned}$$

Note that we implicitly impose conditions on $\mathcal{G} * \mathcal{G}$ so that these equations make sense. For instance, the second equation implicitly requires that for all (γ,η) and (η,ζ) in $\mathcal{G} * \mathcal{G}$, $((\gamma\eta),\zeta)$ and $(\gamma,(\eta\zeta))$ must lie in $\mathcal{G} * \mathcal{G}$ as well.

Elements in $\mathcal{G} * \mathcal{G}$ are called *composable pairs*.

The set of units is now defined by

$$\mathcal{G}^{(0)} := \left\{\gamma^{-1}\gamma : \ \gamma \in \mathcal{G}\right\},$$

it is also given by

$$\mathcal{G}^{(0)} = \left\{\gamma\gamma^{-1} : \ \gamma \in \mathcal{G}\right\}.$$

Moreover, we define the source map by setting

$$s : \ \mathcal{G} \to \mathcal{G}^{(0)}, \ \gamma \mapsto \gamma^{-1}\gamma$$

and the range map by setting

$$r : \ \mathcal{G} \to \mathcal{G}^{(0)}, \ \gamma \mapsto \gamma\gamma^{-1}.$$

It is now an immediate consequence of the axioms that

$$\mathcal{G} * \mathcal{G} = \{(\gamma,\eta) \in \mathcal{G} \times \mathcal{G} : \ s(\gamma) = r(\eta)\}.$$

A groupoid $\mathcal{G}$ is called a *topological* groupoid if the set $\mathcal{G}$ comes with a topology such that multiplication and inversion become continuous maps. A topological groupoid is called *étale* if r and s are local homeomorphisms. A topological groupoid is called *locally compact* if it is locally compact (and Hausdorff) as a topological space.

As an example, let us describe the partial transformation groupoid attached to the partial dynamical system $\alpha : G \curvearrowright X$. It is denoted by $G \,_\alpha\!\ltimes X$ and is given by

$$G \,_\alpha\!\ltimes X := \left\{(g,x) \in G \times X : \, g \in G, \; x \in U_{g^{-1}}\right\},$$

with source map $s(g,x) = x$, range map $r(g,x) = g.x$, composition

$$(g_1, g_2.x)(g_2, x) = (g_1 g_2, x)$$

and inverse

$$(g,x)^{-1} = (g^{-1}, g.x).$$

We equip $G \,_\alpha\!\ltimes X$ with the subspace topology from $G \times X$. Usually, we write $G \ltimes X$ for $G \,_\alpha\!\ltimes X$ if the action α is understood. The unit space of $G \ltimes X$ coincides with X. Since G is discrete, $G \ltimes X$ is an étale groupoid. Actually, if we set

$$G_x := \left\{g \in G : \, x \in U_{g^{-1}}\right\} \text{ and } G^x := \{g \in G : \, x \in U_g\}$$

for $x \in X$, then we have canonical identifications

$$s^{-1}(x) \cong G_x, \, (g,x) \mapsto g \text{ and } r^{-1}(x) \cong G^x, \, (g, g^{-1}.x) \mapsto g.$$

Let $\mathcal{G}$ be an étale locally compact groupoid. For $x \in \mathcal{G}^{(0)}$, let $\mathcal{G}_x = s^{-1}(x)$ and $\mathcal{G}^x = r^{-1}(x)$. $C_c(\mathcal{G})$ is a $*$-algebra with respect to the multiplication

$$(f * g)(\gamma) = \sum_{\beta \in \mathcal{G}_{s(\gamma)}} f(\gamma\beta^{-1}) g(\beta)$$

and the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

For every $x \in \mathcal{G}^{(0)}$, define a $*$-representation π_x of $C_c(\mathcal{G})$ on $\ell^2\mathcal{G}_x$ by setting

$$\pi_x(f)(\xi)(\gamma) = (f * \xi)(\gamma) = \sum_{\beta \in \mathcal{G}_x} f(\gamma\beta^{-1}) \xi(\beta).$$

Alternatively, if we want to highlight why these representations play the role of the left regular representation, attached to left multiplication, we could define π_x by setting

$$\pi_x(f)\delta_\gamma = \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\alpha)\delta_{\alpha\gamma}.$$

Here $\{\delta_\gamma : \, \gamma \in \mathcal{G}_x\}$ is the canonical orthonormal basis of $\ell^2\mathcal{G}_x$.

With these definitions, we are ready to define groupoid C^*-algebras.

Definition 5.5.14. Let
$$\|f\|_{C_r^*(\mathcal{G})} := \sup_{x\in\mathcal{G}^{(0)}} \|\pi_x(f)\|$$
for $f \in C_c(\mathcal{G})$. We define $C_r^*(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{C_r^*(\mathcal{G})}}$.

$C_r^*(\mathcal{G})$ is called the *reduced groupoid C^*-algebra* of $\mathcal{G}$.

Alternatively, we could set
$$\pi = \bigoplus_{x\in\mathcal{G}^{(0)}} \pi_x$$
and
$$C_r^*(\mathcal{G}) = \overline{\pi(C_c(\mathcal{G}))} \subseteq \mathcal{L}(\bigoplus_x \ell^2\mathcal{G}_x).$$

Let us now define the full groupoid C^*-algebra. Let $\mathcal{G}$ be an étale locally compact groupoid. Then $\mathcal{G}^{(0)}$ is a clopen subspace of $\mathcal{G}$. Therefore, we can think of $C_c(\mathcal{G}^{(0)})$ as a subspace of $C_c(\mathcal{G})$ simply by extending functions on $\mathcal{G}^{(0)}$ by 0 to functions on $\mathcal{G}$. This allows us to define the full groupoid C^*-algebra.

Definition 5.5.15. For $f \in C_c(\mathcal{G})$, let
$$\|f\|_{C^*(\mathcal{G})} = \sup_{\pi} \|\pi(f)\|,$$
where the supremum is taken over all $*$-representations of $C_c(\mathcal{G})$ that are bounded on $C_c(\mathcal{G}^{(0)})$ (with respect to the supremum norm $\|\cdot\|_\infty$). We then set
$$C^*(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{C^*(\mathcal{G})}}.$$

$C^*(\mathcal{G})$ is called the *full groupoid C^*-algebra* of $\mathcal{G}$.

Remark 5.5.16. *We will only deal with second countable locally compact étale groupoids. In that case, [Ren80, Chapter II, Theorem 1.21] tells us that every $*$-representation of $C_c(\mathcal{G})$ on a separable Hilbert space is automatically bounded. In other words, the full groupoid C^*-algebra of $\mathcal{G}$ is the universal enveloping C^*-algebra of $C_c(\mathcal{G})$. This notion has been explained after Definition 5.5.12.*

By construction, there is a canonical $*$-homomorphism $C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$ extending the identity on $C_c(\mathcal{G})$. It is called the left regular representation.

5.5.4 The universal groupoid of an inverse semigroup

We attach groupoids to inverse semigroups so that full and reduced C^*-algebras coincide. The groupoids we construct are basically Paterson's universal groupoid, as in [Pat99, §4.3] or [MS14]. There is, however, a small difference. In the case of inverse semigroups with zero, our construction differs from Paterson's because we want the distinguished zero element to be represented by zero in the reduced and full C^*-algebras.

Let us first explain our construction. We start with an inverse semigroup S with a semilattice of idempotents denoted by E. Set

$$\Sigma := \left\{(s,\chi) \in S \times \widehat{E} : \chi(s^{-1}s) = 1\right\}.$$

Note that in the case $0 \in S$, we must have $s \neq 0$ since $\chi(0) = 0$ by our convention.

We introduce an equivalence relation on Σ. Given (s,χ) and (t,ψ) in Σ, we define

$$(s,\chi) \sim (t,\psi) \text{ if there exists } e \in E \text{ with } se = te \text{ and } \chi(e) = 1.$$

The equivalence class of $(s,\chi) \in \Sigma$ with respect to $\sim$ is denoted by $[s,\chi]$. We set

$$\mathcal{G}(S) := \Sigma/_\sim, \text{ i.e., } \mathcal{G}(S) = \{[s,\chi] : (s,\chi) \in \Sigma\}.$$

To define a multiplication on $\mathcal{G}(S)$, we need to introduce the following notation: Let $s \in S$ and $\chi \in \widehat{E}$ be such that $\chi(s^{-1}s) = 1$. Then we define a new element $s.\chi$ of $\widehat{E}$ by setting

$$(s.\chi)(e) := \chi(s^{-1}es).$$

Then we say that $[t,\psi]$ and $[s,\chi]$ are composable if $\psi = s.\chi$. In that case, we define their product as

$$[t,\psi][s,\chi] := [ts,\chi].$$

The inverse map is given by

$$[s,\chi]^{-1} := [s^{-1}, s.\chi].$$

It is easy to see that multiplication and inverse are well-defined, and they give rise to a groupoid structure on $\mathcal{G}(S)$.

Moreover, we introduce a topology on $\mathcal{G}(S)$ by choosing a basis of open subsets. Given $s \in S$ and an open subspace

$$U \subseteq \left\{\chi \in \widehat{E} : \chi(s^{-1}s) = 1\right\},$$

we define

$$D(s,U) := \{[s,\chi] : \chi \in U\}.$$

We equip $\mathcal{G}(S)$ with the topology that has as a basis of open subsets

$$D(s,U), \text{ for } s \in S \text{ and } U \subseteq \left\{\chi \in \widehat{E} : \chi(s^{-1}s) = 1\right\} \text{ open.}$$

It is easy to check that with this topology, $\mathcal{G}(S)$ becomes a locally compact étale groupoid. In all our examples, S will be countable, in which case $\mathcal{G}(S)$ will be second countable.

Let us explain the difference between our groupoid $\mathcal{G}(S)$ and the universal groupoid attached to S in [Pat99, §4.3]. Assume that S is an inverse semigroup with zero, and 0 is the distinguished zero element. The starting point is that our space $\widehat{E}$ and the space of semi-characters X introduced in [Pat99, §2.1] and [Pat99, §4.3] do not coincide. They are related by

$$X = \widehat{E} \sqcup \{\chi_0\}\,.$$

Here χ_0 is the semi-character on E that sends every element of E to 1, even 0. The disjoint union above is not only a disjoint union of sets, but also of topological spaces, i.e., χ_0 is an isolated point in X (it is open and closed).

Now it is easy to see that our $\mathcal{G}(S)$ is the restriction of the universal groupoid $G_{\mathbf{u}}$ attached to S in [Pat99, §4.3] to $\widehat{E}$. This means that

$$\mathcal{G}(S) = \left\{\gamma \in \mathcal{G}_{\mathbf{u}} : \, r(\gamma) \in \widehat{E},\, s(\gamma) \in \widehat{E}\right\}.$$

Actually, the only element in $\mathcal{G}_{\mathbf{u}}$ that does not have range and source in $\widehat{E}$ is χ_0 itself. It follows that

$$G_{\mathbf{u}} = \mathcal{G}(S) \sqcup \{\chi_0\}\,. \tag{5.4}$$

5.5.5 Inverse semigroup C^*-algebras as groupoid C^*-algebras

We begin by identifying the full C^*-algebras. Given an inverse semigroup S with a semilattice of idempotents E, let us introduce the notation that for $e \in E$, we write

$$U_e := \left\{\chi \in \widehat{E} : \, \chi(e) = 1\right\}.$$

Theorem 5.5.17. *For every inverse semigroup S, there is a canonical isomorphism*

$$C^*(S) \xrightarrow{\cong} C^*(\mathcal{G}(S))$$

sending the generator $v_s \in C^*(S)$ to the characteristic function on $D(s, U_{s^{-1}s})$, viewed as an element in $C_c(\mathcal{G}) \subseteq C^*(\mathcal{G})$.

Recall that

$$D(s, U_{s^{-1}s}) = \{[s, \chi] : \, \chi \in U_{s^{-1}s}\}\,.$$

Proof. In the case of inverse semigroups without zero, our theorem is just [Pat99, Chapter 4, Theorem 4.4.1].

Now let us assume that $0 \in S$. Then the full C^*-algebra attached to S in [Pat99, §2.1] is canonically isomorphic to

$$C^*(S) \oplus \mathbb{C}v_0,$$

where $C^*(S)$ is our full inverse semigroup C^*-algebra in the sense of Definition 5.5.9, and v_0 is a (nonzero) projection.

For the full groupoid C^*-algebra of the universal groupoid $G_{\mathbf{u}}$ attached to S in [Pat99, §4.3], we get because of (5.4):

$$C^*(G_{\mathbf{u}}) \cong C^*(\mathcal{G}(S)) \oplus \mathbb{C}1_{\chi_0}.$$

Here 1_{χ_0} is the characteristic function of the one-point set $\{\chi_0\}$, and it is easy to see that 1_{χ_0} is a (nonzero) projection.

With these observations in mind, it is easy to see that the identification in [Pat99, Chapter 4, Theorem 4.4.1] of the full C^*-algebra attached to S in [Pat99, §2.1] with the full groupoid C^*-algebra $C^*(G_{\mathbf{u}})$ respects these direct sum decompositions, i.e., it sends $C^*(S)$ in the sense of Definition 5.5.9 to $C^*(\mathcal{G}(S))$. Finally, it is also easy to see that the identification we get in this way really sends $v_s \in C^*(S)$ to the characteristic function on $D(s, U_{s^{-1}s})$. □

Next, we identify the reduced C^*-algebras.

Theorem 5.5.18. *For every inverse semigroup S, there is a canonical isomorphism*

$$C^*_\lambda(S) \xrightarrow{\cong} C^*_r(\mathcal{G}(S))$$

sending the generator $\lambda_s \in C^*_\lambda(S)$ to the characteristic function on $D(s, U_{s^{-1}s})$, viewed as an element in $C_c(\mathcal{G}) \subseteq C^*_r(\mathcal{G})$.

We could give a proof of this result in complete analogy to the case of the full C^*-algebras, using [Pat99, Chapter 4, Theorem 4.4.2] instead of [Pat99, Chapter 4, Theorem 4.4.1]. Instead, since all these C^*-algebras are defined using concrete representations, we give a concrete proof identifying certain representations.

Proof. For $e \in E^\times$, define

$$S^\times_e := \left\{x \in S^\times : x^{-1}x = e\right\}.$$

It is then easy to see that

$$S^\times = \bigsqcup_{e \in E^\times} S^\times_e.$$

This yields the direct sum decomposition

$$\ell^2 S^\times = \bigoplus_{e \in E^\times} \ell^2 S^\times_e.$$

The left regular representation of S respects this direct sum decomposition. This is because given $s \in S$ and $x \in S^\times_e$ with $s^{-1}s \geq xx^{-1}$, we have that $sx \in S^\times_e$ since

$$(sx)^{-1}(sx) = x^{-1}(s^{-1}s)x = x^{-1}(s^{-1}sxx^{-1})x = x^{-1}(xx^{-1})x = x^{-1}x = e.$$

Therefore, for every $s \in S$, we have

$$\lambda_s = \bigoplus_{e \in E^\times} \lambda_s\big|_{\ell^2 S^\times_e}.$$

Now define for every $e \in E^\times$ the character $\chi_e \in \widehat{E}$ by setting

$$\chi_e(f) = 1 \text{ if } e \leq f,$$
$$\chi_e(f) = 0 \text{ if } e \not\leq f.$$

The map

$$S_e^\times \longrightarrow \mathcal{G}(S)_{\chi_e},\ x \mapsto [x, \chi_e]$$

is surjective as every $(x, \chi_e) \in \Sigma$ is equivalent to (xe, χ_e), and xe lies in $S_e^\times$ as $\chi_e(x^{-1}x) = 1$ implies $e \leq x^{-1}x$. It is also injective as $[x, \chi_e] = [y, \chi_e]$ for $x, y \in S_e^\times$ implies that $xf = yf$ for some $f \in E^\times$ with $e \leq f$, and thus $x = y$. Therefore, the map above is a bijection. It induces a unitary

$$U : \ell^2 S_e^\times \xrightarrow{\cong} \ell^2 \mathcal{G}(S)_{\chi_e},\ \delta_x \mapsto \delta_{[x,\chi_e]}.$$

Now let $1_{D(s,U_{s^{-1}s})}$ be the characteristic function on $D(s, U_{s^{-1}s})$, viewed as an element in $C_c(\mathcal{G})$. Then we have

$$U \circ \lambda_s\big|_{\ell^2 S_e^\times} = \pi_{\chi_e}(1_{D(s,U_{s^{-1}s})}) \circ U. \tag{5.5}$$

This is because

$$(U \circ \lambda_s\big|_{\ell^2 S_e^\times})(\delta_x) = U(\delta_{sx}) = [sx, \chi_e]$$

and

$$(\pi_{\chi_e} \circ 1_{D(s,U_{s^{-1}s})} \circ U)(\delta_x) = \pi_{\chi_e}(1_{D(s,U_{s^{-1}s})})([x, \chi_e]) = [sx, \chi_e]$$

if $s^{-1}s \geq xx^{-1}$, and both sides of (5.5) are zero if $s^{-1}s \not\geq xx^{-1}$.

Hence it follows that the left regular representation of $C^*(S)$ is unitarily equivalent to

$$\bigoplus_{e \in E^\times} \pi_{\chi_e}$$

under the isomorphism from Theorem 5.5.17.

Thus, all we have to show in order to conclude our proof is that

$$\sup_{\chi \in \widehat{E}} \|\pi_\chi(f)\| = \sup_{e \in E^\times} \|\pi_{\chi_e}(f)\|, \tag{5.6}$$

for all $f \in C_c(\mathcal{G}(S))$. To show this, we first need to observe that

$$\{\chi_e : e \in E^\times\}$$

is dense in $\widehat{E}$. This is because a basis of open subsets for the topology of $\widehat{E}$ are given by

$$U(e; e_1, \ldots, e_n) := \left\{\chi \in \widehat{E} : \chi(e) = 1;\ \chi(e_1) = \ldots = \chi(e_n) = 0\right\},$$

for $e, e_1, \ldots, e_n \in E^\times$ with $e_i \not\leq e$. It is then clear that χ_e lies in $U(e; e_1, \ldots, e_n)$. Because of density, (5.6) follows from [Pat99, Chapter 3, Proposition 3.1.2]. $\square$

Remark 5.5.19. *It is clear that the explicit isomorphisms provided by Theorem 5.5.17 and Theorem 5.5.18 give rise to a commutative diagram*

$$\begin{tikzcd} C^*(S) \arrow[r] \arrow[d, "\cong"] & C^*_\lambda(S) \arrow[d, "\cong"] \\ C^*(\mathcal{G}(S)) \arrow[r] & C^*_r(\mathcal{G}(S)) \end{tikzcd}$$

where the horizontal arrows are the left regular representations and the vertical arrows are the identifications provided by Theorem 5.5.17 and Theorem 5.5.18.

5.5.6 C^*-algebras of partial dynamical systems as C^*-algebras of partial transformation groupoids

Our goal is to identify the full and reduced crossed products attached to partial dynamical systems with full and reduced groupoid C^*-algebras for the corresponding partial transformation groupoids.

Given a partial dynamical system $G \curvearrowright X$, we have constructed its partial transformation groupoid $G \ltimes X$ in §5.5.3.

The following result is [Aba04, Theorem 3.3]:

Theorem 5.5.20. *The canonical homomorphism*

$$C_c(G \ltimes X) \to C_0(X) \rtimes^{\ell^1} G, \, \theta \mapsto \sum_g \theta(g, g^{-1}.\sqcup)\delta_g,$$

where $\theta(g, g^{-1}.\sqcup)$ is the function $U_{g^{-1}} \to \mathbb{C}$, $x \mapsto \theta(g, g^{-1}.x)$, extends to an isomorphism

$$C^*(G \ltimes X) \xrightarrow{\cong} C_0(X) \rtimes G.$$

Here we use the same notation for partial dynamical systems and their crossed products as in §5.5.2.

Let us now identify reduced crossed products.

Theorem 5.5.21. *The canonical homomorphism*

$$C_c(G \ltimes X) \to C_0(X) \rtimes^{\ell^1} G, \, \theta \mapsto \sum_g \theta(g, g^{-1}.\sqcup)\delta_g, \tag{5.7}$$

where $\theta(g, g^{-1}.\sqcup)$ is the function

$$U_{g^{-1}} \to \mathbb{C}, \, x \mapsto \theta(g, g^{-1}.x),$$

extends to an isomorphism

$$C^*_r(G \ltimes X) \xrightarrow{\cong} C_0(X) \rtimes_r G.$$

We include a proof of this result. It is taken from [Li16b].

Proof. We use the same notation as in the construction of the reduced crossed product in §5.5.2. As above, let $\mu \times \lambda$ be the representation $C_0(X) \rtimes^{\ell^1} G \to \mathcal{L}(H)$ which we used to define $C_0(X) \rtimes_r G$. Our first observation is

$$\overline{\mathrm{im}\,(\mu \times \lambda)(H)} = \bigoplus_{h \in G} \delta_h \otimes \ell^2 U_{h^{-1}}. \tag{5.8}$$

To see this, observe that for all $g \in G$,

$$\mathrm{im}\,(E_g) \subseteq \bigoplus_h \delta_h \otimes \ell^2(U_{h^{-1}} \cap U_{(gh)^{-1}}).$$

This holds since for

$$x \notin h^{-1}.(U_h \cap U_{g^{-1}}) = U_{(gh)^{-1}} \cap U_{h^{-1}},$$

$f|_{U_h}(h.x) = 0$ for $f \in C_0(U_{g^{-1}})$. Therefore,

$$\pi(C_0(U_{g^{-1}}))(\delta_h \otimes \ell^2 X) \subseteq \delta_h \otimes \ell^2(U_{h^{-1}} \cap U_{(gh)^{-1}}).$$

Hence

$$\mathrm{im}\,(E_g) \subseteq \bigoplus_h \delta_h \otimes \ell^2(U_{h^{-1}} \cap U_{(gh)^{-1}}),$$

and thus,

$$\mathrm{im}\,(V_g) \subseteq \bigoplus_h \delta_{gh} \otimes \ell^2(U_{h^{-1}} \cap U_{(gh)^{-1}}) \subseteq \bigoplus_h \delta_h \otimes \ell^2 U_{h^{-1}}.$$

This shows "$\subseteq$" in (5.8). For "$\supseteq$", note that for $f \in C_0(X)$,

$$(\mu \times \lambda)(f\delta_e) = \mu(f)E_e,$$

and for $\xi \in \ell^2 U_{h^{-1}}$,

$$\mu(f)E_e(\delta_h \otimes \xi) = \delta_h \otimes f|_{U_h}(h.\sqcup)\xi.$$

So $(\mu \times \lambda)(f\delta_e)(H)$ contains $\delta_h \otimes f \cdot \xi$ for all $f \in C_0(U_{h^{-1}})$ and $\xi \in \ell^2 U_{h^{-1}}$, hence also $\delta_h \otimes \ell^2 U_{h^{-1}}$. This proves "$\supseteq$".

For $x \in X$, let $G_x = \{g \in G : x \in U_{g^{-1}}\}$ as before. Our second observation is that for every $x \in X$, the subspace $H_x := \ell^2 G_x \otimes \delta_x$ is $(\mu \times \lambda)$-invariant. It is clear that $\mu(f)$ leaves H_x invariant for all $f \in C_0(X)$. For $g, h \in G$,

$$E_g(\delta_h \otimes \delta_x) = \delta_h \otimes \delta_x$$

if $x \in U_{h^{-1}} \cap U_{(gh)^{-1}}$, and if that is the case, then

$$V_g(\delta_h \otimes \delta_x) = \delta_{gh} \otimes \delta_x \in H_x.$$

Therefore,

$$H = \left(\bigoplus_{x \in X} H_x \right) \oplus (\mu \times \lambda)(C_0(X) \rtimes^{\ell^1} G)(H)^\perp$$

is a decomposition of H into $\mu \times \lambda$-invariant subspaces. For $x \in X$, set

$$\rho_x := (\mu \times \lambda)|_{H_x}.$$

Then

$$C_0(X) \rtimes_r G = \overline{C_0(X) \rtimes^{\ell^1} G}^{\|\cdot\|_{\oplus_x \rho_x}}.$$

Moreover, we have for $x \in U_{h^{-1}}$,

$$\begin{aligned} \rho_x \left(\sum_g f_g \delta_g \right) (\delta_h \otimes \delta_x) &= \sum_g \mu(f_g) V_g (\delta_h \otimes \delta_x) \\ = \sum_{g \colon x \in U_{(gh)^{-1}}} \mu(f_g)(\delta_{gh} \otimes \delta_x) &= \sum_{g \colon x \in U_{(gh)^{-1}}} \delta_{gh} \otimes f_g(gh.x) \delta_x \\ = \sum_{k \in G_x} \delta_k \otimes f_{kh^{-1}}(k.x) \delta_x. & \end{aligned} \tag{5.9}$$

Let us compare this construction with the construction of the reduced groupoid C^*-algebra of $G \ltimes X$. Obviously, (5.7) is an embedding of $C_c(G \ltimes X)$ as a subalgebra which is $\|\cdot\|_{\ell^1}$-dense in $C_0(X) \rtimes^{\ell^1} G$. Therefore,

$$C_0(X) \rtimes_r G = \overline{C_c(G \ltimes X)}^{\|\cdot\|_{\oplus_x \rho_x}}.$$

Now, to construct the reduced groupoid C^*-algebra $C_r^*(G \ltimes X)$, we follow our explanations in §5.5.3 and construct for every $x \in X$ the representation

$$\pi_x : C_c(G \ltimes X) \to \mathcal{L}(\ell^2(s^{-1}(x)))$$

by setting

$$\pi_x(\theta)(\xi)(\zeta) := \sum_{\eta \in s^{-1}(x)} \theta(\zeta \eta^{-1}) \xi(\eta).$$

In our case, using $s^{-1}(x) = G_x \times \{x\}$, we obtain for $\xi = \delta_h \otimes \delta_x$ with $h \in G_x$:

$$\pi_x(\theta)(\delta_h \otimes \delta_x)(k, x) = \theta((k.x)(h, x)^{-1}) = \theta(kh^{-1}, h.x).$$

Thus,

$$\pi_x(\theta)(\delta_h \otimes \delta_x)(k, x) = \sum_{k \in G_x} \theta(kh^{-1}, h.x) \delta_k \otimes \delta_x. \tag{5.10}$$

By definition,

$$C_r^*(G \ltimes X) = \overline{C_c(G \ltimes X)}^{\|\cdot\|_{\oplus_x \pi_x}}.$$

Therefore, in order to show that $\|\cdot\|_{\bigoplus_x \rho_x}$ and $\|\cdot\|_{\bigoplus_x \rho_x}$ coincide on $C_c(G \ltimes X)$, it suffices to show that for every $x \in X$, π_x and the restriction of ρ_x to $C_c(G \ltimes X)$ are unitarily equivalent. Given $x \in X$, using $s^{-1}(x) = G_x \times \{x\}$, we obtain the canonical unitary

$$\ell^2(s^{-1}(x)) \cong H_x = \ell^2(G_x) \otimes \delta_x,$$

so that we may think of both ρ_x and π_x as representations on $\ell^2(G_x) \otimes \delta_x$. We then have for $x \in X$, $\theta \in C_c(G \ltimes X)$ and $h \in G_x$:

$$\begin{aligned}\rho_x(\theta)(\delta_h \otimes \delta_x) &\overset{(5.7)}{=} \rho_x\left(\sum_g \theta(g, g^{-1}.\sqcup)\delta_g\right)(\delta_h \otimes \delta_x)\\ &\overset{(5.9)}{=} \sum_{k \in G_x} \delta_k \otimes \theta(kh^{-1}, h.x)\delta_x \overset{(5.10)}{=} \pi_x(\theta)(\delta_h \otimes \delta_x).\end{aligned}$$

This yields the canonical identification

$$C_0(X) \rtimes_r G \cong C_r^*(G \ltimes X),$$

as desired. □

5.5.7 The case of inverse semigroups admitting an idempotent pure partial homomorphism to a group

We now show that in the case of inverse semigroups that admit an idempotent pure partial homomorphism to a group, all our constructions above coincide.

Let S be an inverse semigroup and E the semilattice of idempotents of S. Let G be a group. Assume that σ is a partial homomorphism $S^\times \to G$ that is idempotent pure.

In this situation, we constructed a partial dynamical system $G \curvearrowright \widehat{E}$ in §5.5.2. Our first observation is that the partial transformation groupoid of $G \curvearrowright \widehat{E}$ can be canonically identified with the groupoid $\mathcal{G}(S)$ we attached to S in §5.5.4.

Lemma 5.5.22. *In the situation described above, we have a canonical identification*

$$\mathcal{G}(S) \xrightarrow{\cong} G \ltimes \widehat{E},\ [s, \chi] \mapsto (\sigma(s), \chi).$$

of topological groupoids.

Proof. We use the notations from §5.5.2 and §5.5.4.

To see that the mapping $[s, \chi] \mapsto (\sigma(s), \chi)$ is well defined, suppose that (s, χ) and (t, χ) in Σ are equivalent. Then there exists $e \in E^\times$ such that $se = te$, and se (or te) cannot be zero in the case $0 \in S$. Therefore,

$$\sigma(s) = \sigma(se) = \sigma(te) = \sigma(t).$$

To see that $[s, \chi] \mapsto (\sigma(s), \chi)$ is a morphism of groupoids, note that $[s, \chi]^{-1} = [s^{-1}, s.\chi]$ is sent to $(\sigma(s^{-1}), s.\chi) = (\sigma(s), \chi)^{-1}$. Hence our mapping respects inverses. For multiplication, observe that

$$s([s, \chi]) = \chi = s(\sigma(s), \chi)$$

and

$$r([s, \chi]) = s.\chi = \sigma(s).\chi = r(\sigma(s), \chi).$$

Moreover, $[t, s.\chi] \cdot [s, \chi] = [ts, \chi]$ is mapped to $[\sigma(ts), \chi] = [\sigma(t), s.\chi] \cdot [\sigma(s), \chi]$. Hence it follows that our mapping is a groupoid morphism.

We now set out to construct an inverse. Define the map

$$G \ltimes \widehat{E} \longrightarrow \mathcal{G}(S),\ (g, \chi) \mapsto [s, \chi]$$

where for every g in G, we choose $s \in S$ with $\sigma(s) = g$ and $\chi(s^{-1}s) = 1$. This is well-defined: Given $t \in S$ with $\sigma(t) = g$ and $\chi(t^{-1}t) = 1$, set $e := s^{-1}st^{-1}t$. Then $\chi(e) = 1$. Moreover, $se = st^{-1}t$ and $te = ts^{-1}s$. As $\sigma(se) = \sigma(s) = g = \sigma(t) = \sigma(te)$ and $(se)^{-1}(se) = e = (te)^{-1}(te)$, we deduce by Lemma 5.5.7 that $se = te$. Hence $(s, \chi) \sim (t, \chi)$.

It is easy to see that we have just constructed the inverse of

$$\mathcal{G}(S) \longrightarrow G \ltimes \widehat{E},\ [s, \chi] \mapsto (\sigma(s), \chi).$$

Moreover, it is also easy to see that both our mappings are open, so that they give rise to the desired identification of topological groupoids. □

Combining Theorem 5.5.17 with Theorem 5.5.20 and Theorem 5.5.18 with Theorem 5.5.21, we obtain the following

Corollary 5.5.23. *Let S be an inverse semigroup and E be the semilattice of idempotents of S. Let G be a group. Assume that σ is a partial homomorphism $S^\times \to G$ that is idempotent pure. In this situation, we have canonical isomorphisms*

$$C^*(S) \to C^*(E) \rtimes G,\ v_s \mapsto \lambda_{ss^{-1}} \delta_{\sigma(s)}$$

and

$$C^*_\lambda(S) \to C^*(E) \rtimes_r G,\ \lambda_s \mapsto \lambda_{ss^{-1}} V_{\sigma(s)}.$$

5.6 Amenability and nuclearity

Amenability is an important structural property for groups and groupoids, while nuclearity plays a crucial role in the structure theory for C^*-algebras, in particular in the classification program. In the case of groups and groupoids, it is known that amenability and nuclearity of C^*-algebras are closely related. Moreover, there are further alternative ways to characterize amenability in terms of C^*-algebras. Our goal now is to explain to what extent analogous results hold true in the semigroup context.

5.6.1 Groups and groupoids

Let us start by reviewing the case of groups and groupoids.

Let G be a discrete group. We recall three conditions.

Definition 5.6.1. Our group G is said to be *amenable* if there exists a left invariant state on $\ell^\infty(G)$.

This means that we require the existence of a state $\mu : \ell^\infty(G) \to \mathbb{C}$ with the property that $\mu(f(s\sqcup)) = \mu(f)$ for every $f \in \ell^\infty(G)$ and $s \in G$. Here $f(s\sqcup)$ is the function $G \to \mathbb{C}$, $x \mapsto f(sx)$.

Definition 5.6.2. Our group G is said to satisfy *Reiter's condition* if there exists a net $(\theta_i)_i$ of probability measures on G such that

$$\lim_{i\to\infty} \|\theta_i - g\theta_i\| = 0$$

for all $g \in G$.

Here $g\theta$ is the pushforward of θ under

$$G \cong G,\, x \mapsto gx.$$

Definition 5.6.3. Our group G is said to satisfy *Følner's condition* if for every finite subset $E \subseteq G$ and every $\varepsilon > 0$, there exists a nonempty finite subset $F \subseteq G$ with

$$|(sF)\triangle F| \,/\, |F| < \varepsilon$$

for all $s \in E$.

Here $sF = \{sx : x \in F\}$, and $\triangle$ stands for symmetric difference.

It turns out that a group is amenable if and only if it satisfies Reiter's condition if and only if it satisfies Følner's condition. We refer the reader to [BO08, Chapter 2, §6] for more details.

All abelian, nilpotent and solvable groups are amenable, to mention some examples. Nonabelian free groups are not amenable.

We now turn to groupoids.

Definition 5.6.4. An étale locally compact groupoid $\mathcal{G}$ is *amenable* if there is a net $(\theta_i)_i$ of continuous systems of probability measures $\theta_i = (\theta_i^x)_{x\in\mathcal{G}^{(0)}}$ with

$$\lim_{i\to\infty} \left\|\theta_i^{r(\gamma)} - \gamma\theta_i^{s(\gamma)}\right\| = 0 \text{ for all } \gamma \in \mathcal{G}.$$

Here θ^x is a probability measure on $\mathcal{G}$ with support contained in $\mathcal{G}^x$. "Continuous" means that for every $f \in C_c(\mathcal{G})$, the function

$$\mathcal{G}^{(0)} \to \mathbb{C},\, x \mapsto \int f \mathrm{d}\theta^x$$

is continuous. As above, $\gamma\theta$ is the pushforward of θ under

$$\mathcal{G}^{s(\gamma)} \to \mathcal{G}^{r(\gamma)}, \; \eta \mapsto \gamma\eta.$$

Note that what we call amenability of groupoids is really Reiter's condition for groupoids. Moreover, we may require that the convergence in our definition happens uniform on compact subsets of $\mathcal{G}$. This is because of [Ren15].

For instance, if G is an amenable group, and $G \curvearrowright \Omega$ is a partial dynamical system on a locally compact Hausdorff space Ω, then the partial transformation groupoid $G \ltimes \Omega$ is amenable by [Exe15, Theorem 20.7 and Theorem 25.10]. But we can get amenable partial transformation groupoids even if G is not amenable.

Let us now introduce nuclearity for C^*-algebras.

Definition 5.6.5. A C^*-algebra A is *nuclear* if there exists a net of contractive completely positive maps $\varphi_i : A \to F_i$ and $\psi_i : F_i \to A$, where F_i are finite-dimensional C^*-algebras, such that

$$\lim_{i \to \infty} \|\psi_i \circ \varphi_i(a) - a\| = 0$$

for all $a \in A$.

For instance, all commutative C^*-algebras are nuclear, and all finite dimensional C^*-algebras are nuclear.

The reader may find more about nuclearity for C^*-algebras, for example, in [BO08, Chapter 2].

Let us now relate amenability and nuclearity. Let us start with the case of groups.

Recall that the full group C^*-algebra $C^*(G)$ of a discrete group G is the C^*-algebra universal for unitary representations of G. This means that $C^*(G)$ is generated by unitaries u_g, $g \in G$, satisfying

$$u_{gh} = u_g u_h \text{ for all } g, h \in G,$$

and whenever we find unitaries v_g, $g \in G$, in another C^*-algebra B satisfying

$$v_{gh} = v_g v_h \text{ for all } g, h \in G,$$

then there exists a (unique) $*$-homomorphism $C^*(G) \to B$ sending u_g to v_g.

The reduced group C^*-algebra $C^*_\lambda(G)$ of a discrete group G is the C^*-algebra generated by the left regular representations of G. The left regular representation is exactly what we get when we apply the construction at the beginning of §5.2 to G. Therefore, $C^*_\lambda(G)$ is the C^*-algebra we get when we apply Definition 5.2.1 to G in place of P.

By construction, we have a canonical $*$-homomorphism

$$\lambda : \ C^*(G) \to C^*_\lambda(G), \ u_g \to \lambda_g.$$

It is called the left regular representation (of $C^*(G)$).

Here are a couple of C^*-algebraic characterizations of amenability for groups. We refer the reader to [BO08, Chapter 2, §6] for details and proofs.

Theorem 5.6.6. *Let G be a discrete group. The following are equivalent:*

- *G is amenable.*
- *$C^*(G)$ is nuclear.*
- *$C^*_\lambda(G)$ is nuclear.*
- *The left regular representation $\lambda : \ C^*(G) \to C^*_\lambda(G)$ is an isomorphism.*
- *There exists a character on $C^*_\lambda(G)$.*

Here, by a character on a unital C^*-algebra A, we simply mean a unital $*$-homomorphism from A to $\mathbb{C}$.

We now turn to groupoids and C^*-algebraic characterizations of amenability for them. We already introduced full and reduced groupoid C^*-algebras in §5.5.3. We also introduced the left regular representation (of the full groupoid C^*-algebra)

$$\lambda : \ C^*(\mathcal{G}) \to C^*_r(\mathcal{G}).$$

Theorem 5.6.7. *Let $\mathcal{G}$ be an étale locally compact groupoid. Consider the statements*

(i) *$\mathcal{G}$ is amenable.*

(ii) *$C^*(\mathcal{G})$ is nuclear.*

(iii) *$C^*_\lambda(\mathcal{G})$ is nuclear.*

(iv) *$\lambda : \ C^*(\mathcal{G}) \to C^*_\lambda(\mathcal{G})$ is an isomorphism.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

We refer to [BO08, Chapter 5, §6] and [ADR00] for more details.

It was an open question whether statement (iv) implies the other statements. But Rufus Willett gave a counterexample in [Wil15]. There are, however, results saying that statement (iv) does imply the other statements for particular classes of groupoids. For instance, we mention [Mat14].

5.6.2 Amenability for semigroups

Let us now turn to amenability for semigroups. As in the group case, we have the following definitions:

Definition 5.6.8. A discrete semigroup P is called *left amenable* if there exists a left invariant mean on $\ell^\infty(P)$, i.e., a state μ on $\ell^\infty(P)$ such that for every $p \in P$ and $f \in \ell^\infty(P)$, $\mu(f(p\sqcup)) = \mu(f)$.

Here $f(p\sqcup)$ is the function $P \to \mathbb{C}$, $x \mapsto f(px)$.

For instance, every abelian semigroup is left amenable.

Definition 5.6.9. A discrete semigroup P is said to satisfy *Reiter's condition* if there is a net $(\theta_i)_i$ of probability measures on P with the property that

$$\lim_i \|\theta_i - p\theta_i\| = 0 \text{ for all } p \in P.$$

Here $p\theta$ is the pushforward of θ under $P \to P$, $x \mapsto px$.

Definition 5.6.10. A discrete semigroup P satisfies the *strong Følner condition* if for every finite subset $E \subseteq P$ and every $\varepsilon > 0$, there exists a nonempty finite subset $F \subseteq P$ such that

$$|(pF) \triangle F| \,/\, |F| < \varepsilon$$

for all $p \in C$.

Here $pF = \{px : x \in F\}$ and $\triangle$ stands for symmetric difference.

As in the group case, a discrete left cancellative semigroup is left amenable if and only if it satisfies Reiter's condition if and only if it satisfies the strong Følner condition. The reader may consult [Li12] for a proof, and we also refer to [Pat88] for more details.

Our goal now is to find the analogues of Theorem 5.6.6 and Theorem 5.6.7 in the context of semigroups and their C^*-algebras. The motivation is to understand and explain – in a conceptual way – the following two observations:

Let $P = \mathbb{N} \times \mathbb{N}$, the universal monoid generated by two commuting elements. This is an abelian semigroup, so it is left amenable. So far, we have not discussed the question of how to construct full semigroup C^*-algebras. But a natural candidate for the full semigroup C^*-algebra of $\mathbb{N} \times \mathbb{N}$ would be

$$C^* \left(v_a, v_b \mid v_a^* v_a = 1,\ v_b^* v_b = 1,\ v_a v_b = v_b v_a \right).$$

In other words, this is the universal C^*-algebra generated by two commmuting isometries. It is the C^*-algebra universal for isometric representations of our semigroup. This is a very natural candidate for the full semigroup C^*-algebra. But Murphy showed that this C^*-algebra is not nuclear in [Mur96, Theorem 6.2].

Next, consider $P = \mathbb{N} * \mathbb{N}$, the nonabelian free monoid on two generators. As in the group case, nonabelian free semigroups are examples of semigroups that are not left amenable. But it is easy to see that $C^*_\lambda(\mathbb{N} * \mathbb{N})$ is generated as a C^*-algebra by two isometries V_a and V_b with orthogonal range projections, i.e.,

$$(V_a V_a^*) \cdot (V_b V_b^*) = 0.$$

Therefore, $C^*_\lambda(\mathbb{N} * \mathbb{N})$ is isomorphic to the canonical extension of the Cuntz algebra $\mathcal{O}_2$, as introduced in [Cun77, §3]. It fits into an exact sequence

$$0 \to \mathcal{K} \to C^*_\lambda(\mathbb{N} * \mathbb{N}) \to \mathcal{O}_2 \to 0,$$

where $\mathcal{K}$ is the C^*-algebra of compact operators on a infinite-dimensional and separable Hilbert space. Hence it follows that $C^*_\lambda(\mathbb{N} * \mathbb{N})$ is nuclear. Moreover, $C^*_\lambda(\mathbb{N} * \mathbb{N})$ can be described as a universal C^*-algebra, because

$$C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong C^* \left(v_a, v_b \mid v_a^* v_a = 1,\ v_b^* v_b = 1,\ v_a v_a^* v_b v_b^* = 0\right).$$

So this is a hint that for the semigroup $\mathbb{N} * \mathbb{N}$, the full and reduced semigroup C^*-algebras are isomorphic. But, as we remarked above, $\mathbb{N} * \mathbb{N}$ is not left amenable.

Our goal now is to explain these phenomena, to clarify the relation between amenability and nuclearity, and to obtain analogues of Theorem 5.6.6 and Theorem 5.6.7 in the context of semigroups. The first step for us will be to find a systematic and reasonable way to define full semigroup C^*-algebras. It turns out that left inverse hulls attached to left cancellative semigroups, as introduced in §5.5.1, give rise to an approach to this problem. However, before we come to the construction of full semigroup C^*-algebras, we first need to compare the reduced C^*-algebras of left cancellative semigroups and their left inverse hulls.

5.6.3 Comparing reduced C^*-algebras for left cancellative semigroups and their left inverse hulls

Let P be a left cancellative semigroup and $I_l(P)$ the left inverse hull attached to P, as in §5.5.1. As we explained in §5.5.1, we have a canonical embedding of P into $I_l(P)$, denoted by

$$P \hookrightarrow I_l(P),\ p \mapsto p.$$

It gives rise to the isometry

$$\mathbf{I}:\ \ell^2 P \to \ell^2 S^\times,\ \delta_p \mapsto \delta_p.$$

Thus, we may think of $\ell^2 P$ as a subspace of $\ell^2 S^\times$.

The following observation appears in [Nor14, §3.2].

Lemma 5.6.11. *Assume that P is a left cancellative semigroup with left inverse hull $I_l(P)$. Then the subspace $\ell^2 P$ of $\ell^2 I_l(P)^\times$ is invariant under $C^*_\lambda(I_l(P))$. Moreover, we obtain a well-defined surjective $*$-homomorphism*

$$C^*_\lambda(I_l(P)) \to C^*_\lambda(P),\, T \mapsto \mathbf{I}^* T \mathbf{I}$$

sending λ_p to V_p for every $p \in P$.

Proof. We first claim that every $s \in I_l(P)$ has the following property:

$$\text{For every } x \in \mathrm{dom}(s) \text{ and every } r \in P,\ xr \text{ lies in } \mathrm{dom}(s), \text{ and } s(xr) = s(x)r. \tag{5.11}$$

To prove our claim, first observe that for every $p \in P$, the partial bijection $p \in I_l(P)$ certainly has this property, as it is just given by left multiplication with p. Moreover, p^{-1} is the partial bijection

$$pP \to P,\, px \mapsto x.$$

Certainly, for every $px \in pP$ and every $r \in P$, pxr lies in pP, and

$$p^{-1}(pxr) = xr = p^{-1}(px)r.$$

Hence p^{-1} has the desired property as well. To conclude the proof of our claim, suppose that $s, t \in I_l(P)$ both have the desired property. Choose $x \in \mathrm{dom}(st)$. Then for every $r \in P$, xr lies in $\mathrm{dom}(t)$, and $t(xr) = t(x)r$. Since $t(x)$ lies in $\mathrm{dom}(s)$, $t(x)r$ lies in $\mathrm{dom}(s)$ as well. The conclusion is that xr lies in $\mathrm{dom}(st)$, and we have

$$(st)(xr) = s(t(x)r) = s(t(x))r = (st)(x)r.$$

As every element in $I_l(P)$ is a finite product of partial bijections in

$$\{p : p \in P\} \cup \left\{p^{-1} : p \in P\right\},$$

this proves our claim.

The second step is to show that for every $s \in I_l(P)$ and $x \in P$ with $s^{-1}s \geq pp^{-1}$, we must have $sx = s(x) \in P$. This is because we have, for every $y \in P$:

$$(sx)(y) = s(x(y)) = s(xy) = s(x)y = (s(x))(y).$$

Here we used our first claim from above.

Now let $s \in I_l(P)$ be arbitrary. We want to show that $\lambda_s(\ell^2 P) \subseteq \ell^2 P$. Given $x \in P$, we have $\lambda_s(\delta_x) = 0$ if $s^{-1}s \not\geq pp^{-1}$. If $s^{-1}s \geq pp^{-1}$, then what we showed in the second step implies that $\lambda_s(\delta_x) = \delta_{s(x)}$ lies in $\ell^2 P$. As s was arbitrary, this shows that

$$C^*_\lambda(I_l(P))(\ell^2 P) \subseteq \ell^2 P.$$

Therefore, every $T \in C^*_\lambda(I_l(P))$ satisfies $T\mathbf{I}\mathbf{I}^* = \mathbf{I}\mathbf{I}^* T\mathbf{I}\mathbf{I}^*$, and since $C^*_\lambda(I_l(P))$ is $*$-invariant, we even obtain that every $T \in C^*_\lambda(I_l(P))$ satisfies $T\mathbf{I}\mathbf{I}^* = \mathbf{I}\mathbf{I}^* T$. This shows that the map

$$C^*_\lambda(I_l(P)) \to \mathcal{L}(\ell^2 P),\, T \mapsto \mathbf{I}^* T \mathbf{I}$$

is a $*$-homomorphism. Its image is $C^*_\lambda(P)$ because we have, for $p \in P$ and $x \in P$:

$$\lambda_p(\delta_x) = \delta_{px} = V_p(\delta_x),$$

so that $\mathbf{I}^* \lambda_p \mathbf{I} = V_p$ for all $p \in P$. □

Recall that we denote the semilattice of idempotents in $I_l(P)$ by $\mathcal{J}_P$, and we identified this semilattice with the constructible right ideals of P (see §5.5.1). Moreover, we also introduced in §5.5.2 the sub-C^*-algebra of $C^*_\lambda(I_l(P))$ generated by $\mathcal{J}_P$:

$$C^*(\mathcal{J}_P) = C^*(\{\lambda_X : X \in \mathcal{J}_P\}).$$

It is easy to see that for every $X \in \mathcal{J}_P$, we get

$$\mathbf{I}^* \lambda_X \mathbf{I} = 1_X,$$

where 1_X is the characteristic function of X, viewed as an element in $\ell^\infty(P)$.

Hence, restricting the $*$-homomorphism

$$C^*_\lambda(I_l(P)) \to C^*_\lambda(P)$$

from Lemma 5.6.11 to $C^*(\mathcal{J}_P)$, we obtain a $*$-homomorphism from $C^*(\mathcal{J}_P)$ onto the sub-C^*-algebra $D_\lambda(P) = C^*(\{1_X : X \in \mathcal{J}_P\})$ of $C^*_\lambda(P)$, which is generated by $\{1_X : X \in \mathcal{J}_P\}$,

$$C^*(\mathcal{J}_P) \twoheadrightarrow D_\lambda(P),\, T \mapsto \mathbf{I}^* T \mathbf{I}.$$

Obviously, if the $*$-homomorphism from Lemma 5.6.11 is an isomorphism, then its restriction to $C^*(\mathcal{J}_P)$ must be an isomorphism (onto its image) as well. Let us now discuss a situation when the converse holds.

We need the following:

Lemma 5.6.12. *Let X be a set. There exists a faithful conditional expectation*

$$\Theta_X : \mathcal{L}(\ell^2 X) \twoheadrightarrow \ell^\infty(X)$$

such that, for every $T \in \mathcal{L}(\ell^2 X)$, we have

$$\langle \Theta_X(T)\delta_x, \delta_y \rangle = \delta_{x,y} \langle T\delta_x, \delta_y \rangle \tag{5.12}$$

for all $x, y \in X$.

Proof. Let $e_{x,x}$ be the rank one projection onto $\mathbb{C}\delta_x \subseteq \ell^2 X$, given by

$$e_{x,x}(\xi) = \langle \xi, \delta_x \rangle \, \delta_x \text{ for all } \xi \in \ell^2 X.$$

Consider the linear map

$$\operatorname{span}(\{\delta_x : x \in X\}) \to \operatorname{span}(\{\delta_x : x \in X\}), \sum_x \alpha_x \delta_x \mapsto \sum_x \alpha_x (e_{x,x} \circ T)(\delta_x). \tag{5.13}$$

We have

$$\begin{aligned} \left\| \sum_x \alpha_x (e_{x,x} \circ T)(\delta_x) \right\|^2 &= \left\langle \sum_x \alpha_x (e_{x,x} \circ T)(\delta_x), \sum_x \alpha_x (e_{x,x} \circ T)(\delta_x) \right\rangle \\ &= \sum_x |\alpha_x|^2 \left\langle (e_{x,x} \circ T)(\delta_x), (e_{x,x} \circ T)(\delta_x) \right\rangle \\ &\leq \|T\|^2 \sum_x |\alpha_x|^2 = \|T\|^2 \left\| \sum_x \alpha_x \delta_x \right\|^2 \end{aligned}$$

So the linear map in (5.13) extends to a bounded linear operator $\ell^2 X \to \ell^2 X$, which we denote by $\Theta_X(T)$. Our computation shows that

$$\|\Theta_X(T)\| \leq \|T\| \, .$$

By definition,

$$\Theta_X(T)(\delta_x) = \langle T\delta_x, \delta_x \rangle \, \delta_x.$$

This shows that $\Theta_X(T)$ lies in $\ell^\infty(X)$. It also shows that $\Theta_X(T)$ satisfies (5.12).

Moreover, by construction, $\Theta_X(T) = T$ for all $T \in \ell^\infty(X)$. Therefore, the map

$$\Theta_X : \mathcal{L}(\ell^2 X) \to \ell^\infty(X), \, T \mapsto \Theta_X(T)$$

is a projection of norm 1. Hence it follows by [Bla06, Theorem II.6.10.2] that Θ_X is a conditional expectation.

Finally, Θ_X is faithful because given $T \in \mathcal{L}(\ell^2 X)$, $\Theta_X(T^*T) = 0$ implies that

$$0 = \langle T^*T\delta_x, \delta_x \rangle = \|T\delta_x\|^2 \, ,$$

so that $T\delta_x = 0$ for all $x \in X$, and hence $T = 0$. □

Applying Lemma 5.6.12 to $X = I_l(P)^\times$ and $X = P$, we obtain faithful conditional expectations

$$\Theta_{I_l(P)} : \mathcal{L}(\ell^2 I_l(P)^\times) \twoheadrightarrow \ell^\infty(I_l(P)^\times)$$

and

$$\Theta_P : \mathcal{L}(\ell^2 P) \twoheadrightarrow \ell^\infty(P).$$

They fit into the following commutative diagram:

$$\begin{tikzcd} \mathcal{L}(\ell^2 I_l(P)^\times) \arrow[r, "\mathbf{I}^* \sqcup \mathbf{I}"] \arrow[d, "\Theta_{I_l(P)}"'] & \mathcal{L}(\ell^2 P) \arrow[d, "\Theta_P"] \\ \mathcal{L}(\ell^2 I_l(P)^\times) \arrow[r, "\mathbf{I}^* \sqcup \mathbf{I}"] & \ell^\infty(P) \end{tikzcd} \tag{5.14}$$

Here $\mathbf{I}^* \sqcup \mathbf{I}$ is our notation for the map sending T to $\mathbf{I}^* T \mathbf{I}$. Commutativity of the diagram above follows from the following computation:

$$\Theta_P(\mathbf{I}^* T \mathbf{I})\, \delta_x = \langle \mathbf{I}^* T \mathbf{I} \delta_x, \delta_x \rangle\, \delta_x = \langle T \delta_x, \delta_x \rangle\, \delta_x = (\mathbf{I}^* \Theta_{I_l(P)}(T) \mathbf{I})\, \delta_x.$$

This leads us to:

Corollary 5.6.13. *Assume that*

$$\Theta_{I_l(P)}(C^*_\lambda(I_l(P))) = C^*(\mathcal{J}_P). \tag{5.15}$$

Then the $*$-homomorphism

$$C^*_\lambda(I_l(P)) \to C^*_\lambda(P),\, T \mapsto \mathbf{I}^* T \mathbf{I}$$

from Lemma 5.6.11 *is an isomorphism if and only if its restriction to $C^*(\mathcal{J}_P)$,*

$$C^*(\mathcal{J}_P) \twoheadrightarrow D_\lambda(P),\, T \mapsto \mathbf{I}^* T \mathbf{I},$$

is an isomorphism.

Proof. Take the commutative diagram (5.14) and restrict the upper left corner to

$$C^*_\lambda(I_l(P)) \subseteq \mathcal{L}(\ell^2 I_l(P)^\times).$$

As $\mathbf{I}^* C^*_\lambda(I_l(P)) \mathbf{I} = C^*_\lambda(P)$ by Lemma 5.6.11, and because of (5.15), we obtain the commutative diagram

$$\begin{tikzcd} C^*_\lambda(I_l(P)) \arrow[r, "\mathbf{I}^* \sqcup \mathbf{I}"] \arrow[d, "\Theta_{I_l(P)}"'] & C^*_\lambda(P) \arrow[d, "\Theta_P"] \\ C^*(\mathcal{J}_P) \arrow[r, "\mathbf{I}^* \sqcup \mathbf{I}"] & D_\lambda(P) \end{tikzcd} \tag{5.16}$$

As the vertical arrows are faithful, it is now easy to see that if the lower horizontal arrow is faithful, the upper horizontal arrow has to be faithful as well. This proves our corollary. □

Remark 5.6.14. *The condition* (5.15)*, i.e.,*

$$\Theta_{I_l(P)}(C^*_\lambda(I_l(P))) = C^*(\mathcal{J}_P),$$

implies that

$$C^*(\mathcal{J}_P) = C^*_\lambda(I_l(P)) \cap \ell^\infty(I_l(P)^\times),$$

and

$$D_\lambda(P) = C^*_\lambda(P) \cap \ell^\infty(P).$$

This is because we always have

$$C^*(\mathcal{J}_P) \subseteq C^*_\lambda(I_l(P)) \cap \ell^\infty(I_l(P)^\times) \subseteq \Theta_{I_l(P)}(C^*_\lambda(I_l(P))), \tag{5.17}$$

and

$$D_\lambda(P) \subseteq C^*_\lambda(P) \cap \ell^\infty(P) \subseteq \Theta_P(C^*_\lambda(P)), \tag{5.18}$$

and (5.15) *implies that all these inclusions are equalities in* (5.17)*, and also in* (5.18) *because*

$$\begin{aligned} &D_\lambda(P) \\ &\quad = \mathbf{I}^* \, C^*(\mathcal{J}_P) \, \mathbf{I} \overset{(5.15)}{=} \mathbf{I}^* \, \Theta_{I_l(P)}(C^*_\lambda(I_l(P))) \, \mathbf{I} = \Theta_P(\mathbf{I}^* \, C^*_\lambda(I_l(P)) \, \mathbf{I}) \\ &\quad = \Theta_P(C^*_\lambda(P)). \end{aligned}$$

Here we used commutativity of the diagram in (5.16) *and Lemma 5.6.11.*

It remains to find out when condition (5.15) holds. We follow [Nor14, §3.2]. Let us introduce the following

Definition 5.6.15. An inverse semigroup S is called E*-*unitary* if for every $s \in S$, we must have $s \in E$ if there exists $x \in S^\times$ with $sx = x$.

Remark 5.6.16. *If there exists an idempotent pure partial homomorphism $\sigma : S^\times \to G$ to some group G, then S is E*-unitary. This is because if we are given $s \in S$, and there exists $x \in S^\times$ with $sx = x$, then $\sigma(x) = \sigma(s)\sigma(x)$, so that $\sigma(s) = e$, where e is the identity element in G. Since σ is idempotent pure, s must lie in E.*

Now we apply Lemma 5.6.12 to $X = S^\times$. Then we get a faithful conditional expectation

$$\Theta_{S^\times} : \mathcal{L}(\ell^2 S^\times) \twoheadrightarrow \ell^\infty(S^\times),$$

and we may apply it to elements in $C^*_\lambda(S)$.

Lemma 5.6.17. *In the situation above, our inverse semigroup S is E*-unitary if and only if for every $s \in S$, we always have*

$$\Theta_{S^\times}(\lambda_s) = 0$$

or

$$s \in E \text{ and } \Theta_{S^\times}(\lambda_s) = \lambda_s.$$

Proof. For "$\Rightarrow$", assume that $\Theta_{S^\times}(\lambda_s) \neq 0$. This is equivalent to saying that there exists $x \in S^\times$ with $sx = x$. But since S is E*-unitary, this implies $s \in E$. And since λ_s lies in $\ell^\infty(S)$ for all $s \in E$, we must have $\Theta_{S^\times}(\lambda_s) = \lambda_s$.

Conversely, for "$\Leftarrow$", take $s \in S$ and suppose that there is $x \in S^\times$ with $sx = x$. Then $sxx^{-1} = xx^{-1}$, so that sxx^{-1} is idempotent, and we conclude that

$$s^{-1}sxx^{-1} = (xx^{-1}s^{-1})(sxx^{-1}) = sxx^{-1} = xx^{-1},$$

i.e., $s^{-1}s \geq xx^{-1}$. Hence

$$\lambda_s(\delta_x) = \delta_{sx} = \delta_x.$$

Hence it follows that $\Theta_{S^\times}(\lambda_s) \neq 0$, and this implies, by assumption, that s lies in E. □

In particular, we can draw the following conclusion.

Corollary 5.6.18. *If S is an E*-unitary inverse semigroup, then $\Theta_{S^\times}(C^*_\lambda(S)) = C^*(E)$.*

Combining Corollary 5.6.13, Remark 5.6.14, Corollary 5.6.18, Remark 5.6.16 and the observation that $I_l(P)$ admits an idempotent pure partial homomorphism to a group if P embeds into a group (see §5.5.1), we obtain

Corollary 5.6.19. *Assume that P is a semigroup that embeds into a group G. Then condition* (5.15) *holds, i.e.,*

$$\Theta_{I_l(P)}(C^*_\lambda(I_l(P))) = C^*(\mathcal{J}_P),$$

and the $*$-homomorphism

$$C^*_\lambda(I_l(P)) \to C^*_\lambda(P),\ T \mapsto \mathbf{I}^*T\mathbf{I}$$

from Lemma 5.6.11 *is an isomorphism if and only if its restriction to $C^*(\mathcal{J}_P)$,*

$$C^*(\mathcal{J}_P) \twoheadrightarrow D_\lambda(P),\ T \mapsto \mathbf{I}^*T\mathbf{I},$$

is an isomorphism.
Moreover,

$$C^*(\mathcal{J}_P) = C^*_\lambda(I_l(P)) \cap \ell^\infty(I_l(P)^\times),$$

and

$$D_\lambda(P) = C^*_\lambda(P) \cap \ell^\infty(P). \tag{5.19}$$

Corollary 5.6.19 prompts the question when the $*$-homomorphism

$$C^*(\mathcal{J}_P) \twoheadrightarrow D_\lambda(P),\ T \mapsto \mathbf{I}^*T\mathbf{I},$$

is an isomorphism. Note that both $C^*(\mathcal{J}_P)$ and $D_\lambda(P)$ are generated by a family of commuting projections, closed under multiplication, and our $*$-homomorphism sends generator to generator, i.e., λ_X to 1_X for all $X \in \mathcal{J}_P$. Let us now investigate when such a $*$-homomorphism is an isomorphism.

5.6.4 C^*-algebras generated by semigroups of projections

We basically follow [Li12, §2.6] in this subsection.

If we think of elements of an inverse semigroup as partial isometries on a Hilbert space, then the semilattice of idempotents is a family of commuting projections, closed under multiplication, or in other words, a semigroup of projections.

Let us consider the general setting of a semilattice E of idempotents, i.e., E is an abelian semigroup consisting of idempotents. Suppose that D is a C^*-algebra generated by a multiplicatively closed family $\{d_e : e \in E\}$ of projections such that

$$E \to D,\, e \mapsto d_e$$

is a semigroup homomorphism.

We make the following easy observation:

Lemma 5.6.20. *For every finite subset F of E, there exists a projection in D, denoted by $\bigvee_{f \in F} d_f$, which is the smallest projection dominating all the projections d_f, $f \in F$.*

Moreover, with $E(F)$ denoting the subsemigroup of E generated by F, $\bigvee_{f \in F} d_f$ lies in

$$\mathrm{span}(\{d_e : e \in E(F)\}).$$

Just to be clear, the projection $\bigvee_{f \in F} d_f$ is uniquely characterized by

$$d_f \leq \bigvee_{f \in F} d_f \text{ for all } f \in F,$$

and whenever a projection $d \in D$ satisfies

$$d_f \leq d \text{ for all } f \in F,$$

then we must have

$$\bigvee_{f \in F} d_f \leq d.$$

Proof. We proceed inductively on the cardinality of F. The case $|F| = 1$ is trivial. Now assume that our claim holds for a finite subset F, and take an arbitrary element $\tilde{f} \in E$. We want to check our claim for $F \cup \left\{\tilde{f}\right\}$. Consider the element

$$\bigvee_{f \in F} d_f + d_{\tilde{f}} - \left(\bigvee_{f \in F} d_f\right) \cdot d_{\tilde{f}}. \tag{5.20}$$

It is easy to see that this is a projection in D, which dominates all the d_f, $f \in F$, as well as $d_{\tilde{f}}$. Moreover, if d is a projection in D that dominates all the d_f, $f \in F$, and also $d_{\tilde{f}}$, then d obviously also dominates the projection in (5.20). Furthermore, since $\bigvee_{f \in F} d_f$ lies in

$$\mathrm{span}(\{d_e : e \in E(F)\})$$

by induction hypothesis, the projection in (5.20) lies in

$$\mathrm{span}(\left\{d_e : e \in E(F \cup \left\{\tilde{f}\right\})\right\}). \qquad \square$$

As above, let E be a semilattice of idempotents. Suppose that D is a C^*-algebra generated by projections $\{d_e : e \in E\}$ such that $d_0 = 0$ if $0 \in E$ and $d_{ef} = d_e d_f$ for all $e, f \in E$. We prove the following result about $*$-homomorphisms out of D.

Proposition 5.6.21. *Let B be a C^*-algebra containing a semigroup of projections $\{b_e : e \in E\}$ such that $b_0 = 0$ if $0 \in E$ and $b_{ef} = b_e b_f$ for all $e, f \in E$.*

There exists a $*$-homomorphism $D \to B$ sending d_e to b_e for all $e \in E$ if and only of for every $e \in E$ and every finite subset $F \subseteq E$ such that $f \lneq e$ for all $f \in F$, the equation

$$d_e = \bigvee_{f \in F} d_f \text{ in } D$$

implies that

$$b_e = \bigvee_{f \in F} b_f \text{ in } B.$$

In that case, the kernel of the $*$-homomorphism

$$D \to B,\, d_e \to b_e$$

is generated by

$$\left\{ d_e - \bigvee_{f \in F} d_f \in D : \, e \in E,\, F \subseteq \{f \in E : \, f \lneq e\} \text{ finite, } b_e = \bigvee_{f \in F} b_f \text{ in } B \right\}.$$

Proof. Let us start with the first part. Our condition is certainly a necessary condition for the existence of a $*$-homomorphism $D \to B$, $d_e \to b_e$. To prove that it is also sufficient, write E as an increasing union of finite subsemigroups E_i, i.e.,

$$E = \bigcup_i E_i.$$

Let $D_i := C^*(\{d_e : e \in E_i\})$. Obviously,

$$D = \overline{\bigcup_i D_i}.$$

For every $e \in E_i$, let $F_e := \{f \in E_i : f \precneqq e\}$. Then, by Lemma 5.6.20,

$$d_e - \bigvee_{f \in F_e} d_f$$

is a projection in D_i. It is easy to see that

$$\left\{ d_e - \bigvee_{f \in F_e} d_f : e \in E_i \right\}$$

is a family of pairwise orthogonal projections which generates D_i. Moreover, it is also easy to see that

$$\left\{ b_e - \bigvee_{f \in F_e} b_f : e \in E_i \right\}$$

is a family of pairwise orthogonal projections in B. Hence it follows that there exists a $*$-homomorphism $D_i \to B$ sending

$$d_e - \bigvee_{f \in F_e} d_f$$

to

$$b_e - \bigvee_{f \in F_e} b_f$$

for all $e \in E_i$ if and only if

$$d_e - \bigvee_{f \in F_e} d_f = 0 \text{ in } D_i$$

implies

$$b_e - \bigvee_{f \in F_e} b_f = 0 \text{ in } B,$$

for all $e \in E_i$. But this is precisely the condition in the first part of our proposition. Moreover, it is easy to see that the $*$-homomorphism $D_i \to B$ we just constructed sends d_e to b_e for all $e \in E_i$. Hence these $*$-homomorphisms, taken together for all i, are compatible and give rise to the desired $*$-homomorphism from $D = \overline{\bigcup_i D_i}$ to B.

For the second part of the proposition, let I be the ideal of D generated by

$$\left\{ d_e - \bigvee_{f \in F} d_f \in D : F \subseteq E \text{ finite}, b_e = \bigvee_{f \in F} b_f \text{ in } B \right\}.$$

Obviously, I is contained in the kernel of $D \to B$, $d_e \mapsto b_e$. It remains to show that the induced $*$-homomorphism $D/I \to B$ is injective. With the D_is as above, set $I_i := I \cap D_i$. Obviously, we have

$$I = \overline{\bigcup_i I_i} \quad \text{and} \quad D/I = \overline{\bigcup_i D_i/I_i}.$$

Hence it suffices to prove that the restriction $D_i/I_i \to B$ is injective, or in other words, that the $*$-homomorphism $D_i \to B$ we constructed above has kernel equal to I_i. But we have seen that

$$\left\{ d_e - \bigvee_{f \in F_e} d_f \, : \, e \in E_i \right\}$$

is a family of pairwise orthogonal projections that generates D_i. So the kernel is generated by those projections

$$d_e - \bigvee_{f \in F_e} d_f$$

for which we have

$$b_e - \bigvee_{f \in F_e} b_f = 0 \text{ in } B.$$

Therefore, the kernel is I_i, as required. $\square$

As before, let D be a C^*-algebra generated by a semigroup $\{d_e : e \in E\}$ of projections such that $d_0 = 0$ if $0 \in E$ and $d_{ef} = d_e d_f$ for all $e, f \in E$. We set $E^\times := E$ if E is a semilattice without zero, and $E^\times := E \setminus \{0\}$ if $0 \in E$.

Proposition 5.6.22. *The following are equivalent:*

(i) *Our C^*-algebra D is universal for representations of E by projections, i.e., we have an isomorphism*

$$D \xrightarrow{\cong} C^*(\{v_e : e \in E\} \mid v_e^* = v_e = v_e^2,\, v_0 = 0 \text{ if } 0 \in E,\, v_{ef} = v_e v_f)$$

sending d_e to v_e.

(ii) *For every $e \in E$ and every finite subset $F \subseteq E$ with $f \lneqq e$ for all $f \in F$, we have*

$$\bigvee_{f \in F} d_f \lneqq d_e.$$

(iii) *The projections $\{d_e : e \in E^\times\}$ are linearly independent in D.*

Proof. Obviously, (iii) implies (ii).

Moreover, (ii) implies (i) by Proposition 5.6.21, because if (ii) holds, we can never have

$$d_e = \bigvee_{f \in F} d_f \text{ in } D$$

for any finite subset $F \subseteq E$ with $f \lneq e$ for all $f \in F$.

It remains to prove that (i) implies (iii). First of all, consider the left regular representation λ on $\ell^2 E^\times$ as in §5.5.1. It is given by $\lambda_e \delta_x = \delta_x$ if $e \geq x$ and $\lambda_e \delta_x = 0$ if $e \not\geq x$. By the universal property of D, there is a $*$-homomorphism $D \to \mathcal{L}(\ell^2 E^\times)$ sending d_e to λ_e. But it is easy to see that $\lambda_e = \lambda_f$ if and only if $e = f$. Hence it follows that $d_e = d_f$ if and only if $e = f$.

Furthermore, again by the universal property of D, there exists a $*$-homomorphism

$$D \to D \otimes D,\, d_e \mapsto d_e \otimes d_e.$$

Let

$$\mathcal{D} = \mathrm{span}(\{d_e : e \in E\}) \subseteq D.$$

Restricting the $*$-homomorphism $D \to D \otimes D$ from above to $\mathcal{D}$, we obtain a homomorphism $\Delta : \mathcal{D} \to \mathcal{D} \odot \mathcal{D}$ that is determined by $d_e \mapsto d_e \otimes d_e$ for every $e \in E$.

We now deduce from the existence of such a homomorphism Δ that $\{d_e : e \in E^\times\}$ is a $\mathbb{C}$-basis of D. As $\{d_e : e \in E^\times\}$ generates $\mathcal{D}$ as a $\mathbb{C}$-vector space, we can always find a subset $\mathcal{S}$ of $E^\times$ such that $\{d_e : e \in \mathcal{S}\}$ is a $\mathbb{C}$-basis for $\mathcal{D}$. It then follows that $\{d_e \otimes d_f : e, f \in \mathcal{S}\}$ is a $\mathbb{C}$-basis of $\mathcal{D} \odot \mathcal{D}$.

Now take $e \in E^\times$. We can find finitely many $e_i \in \mathcal{S}$ and $\alpha_i \in \mathbb{C}$ with $d_e = \sum_i \alpha_i d_{e_i}$. Applying Δ yields

$$\sum_{i,j} \alpha_i \alpha_j d_{e_i} \otimes d_{e_j} = d_e \otimes d_e = \Delta(d_e) = \sum_i \alpha_i \Delta(d_{e_i}) = \sum_i \alpha_i d_{e_i} \otimes d_{e_i}.$$

Hence it follows that among the α_i, there can only be one nonzero coefficient which must be 1. The corresponding vector d_{e_i} must then coincide with d_e. This implies $e = e_i \in \mathcal{S}$, i.e., $\{d_e : e \in E^\times\}$ is a $\mathbb{C}$-basis of $\mathcal{D}$. This proves (iii). □

Now let S be an inverse semigroup with a semilattice of idempotents E, and let $C^*_\lambda(S)$ be its reduced C^*-algebra. Recall that we defined

$$C^*(E) := \{\lambda_e : e \in E\}.$$

Lemma 5.6.23. *The C^*-algebra $C^*(E)$ is universal for representations of E by projections.*

Proof. By Proposition 5.6.22, all we have to show is that for every $e \in E$ and every finite subset $F \subseteq E$ with $f \subsetneq e$ for all $f \in F$, we have

$$\bigvee_{f \in F} \lambda_f \subsetneq \lambda_e.$$

But this follows from $\lambda_f(\delta_e) = 0$ for all $f \in E$ with $f \subsetneq e$, while $\lambda_e(\delta_e) = \delta_e$ for all $e \in E^{\times}$. □

It turns out that $C^*(E)$ can be identified with the corresponding sub-C^*-algebra of the full C^*-algebra of S.

Corollary 5.6.24. *We have an isomorphism*

$$C^*(E) \xrightarrow{\cong} C^*(\{v_e : e \in E\}) \subseteq C^*(S)$$

sending λ_e to v_e for all $e \in E$.

Proof. By Lemma 5.6.23, there is a $*$-homomorphism

$$C^*(E) \xrightarrow{\cong} C^*(\{v_e : e \in E\}) \subseteq C^*(S)$$

sending λ_e to v_e for all $e \in E$. It is an isomorphism because the inverse is given by restricting the left regular representation $C^*(S) \to C^*_\lambda(S)$ to $C^*(\{v_e : e \in E\}) \subseteq C^*(S)$. □

This justifies why we denote the sub-C^*-algebra $C^*(\{\lambda_e : e \in E\})$ of $C^*_\lambda(S)$ by $C^*(E)$.

Corollary 5.6.25. *We have a canonical identification $\widehat{E} \cong \mathrm{Spec}\,(C^*(E))$.*

Proof. This is because by the universal property of $C^*(E)$ (see Lemma 5.6.23), there is a one-to-one correspondence between nonzero $*$-homomorphisms $C^*(E) \to \mathbb{C}$ and nonzero semigroup homomorphisms $E \to \{0, 1\}$ (sending 0 to 0 if $0 \in E$). □

Now suppose that we have an inverse semigroup S with a semilattice of idempotents E, and that we have a surjective $*$-homomorphism $C^*(E) \to D$ sending $\lambda_e \to d_e$. Then D is a commutative C^*-algebra, and we can describe its spectrum as follows:

Corollary 5.6.26. *Viewing* $\mathrm{Spec}\,(D)$ *as a closed subspace of $\widehat{E}$,* $\mathrm{Spec}\,(D)$ *is given by the subspace of all $\chi \in \widehat{E}$ with the property that whenever we have $e \in E$ with $\chi(e) = 1$ and a finite subset $F \subseteq E$ with $f \subsetneq e$ for every $f \in F$ satisfying $d_e = \bigvee_{f \in F} d_f$ in D, then we must have $\chi(f) = 1$ for some $f \in F$.*

Proof. This is an immediate consequence of Proposition 5.6.21. □

Now let us suppose that we have a left cancellative semigroup P. We now apply Corollary 5.6.26 and Proposition 5.6.22 to the situation where $S = I_l(P)$, $E = \mathcal{J}_P$ and $D = D_\lambda(P) \subseteq C^*_\lambda(P)$. First, we make the following easy observation:

Lemma 5.6.27. *Suppose that we are given finitely many $X_i \in \mathcal{J}_P$. Then we have*

$$\bigvee_i 1_{X_i} = 1_{\bigcup_i X_i} \text{ in } D_\lambda(P) \subseteq \ell^\infty(P).$$

The following follows immediately from Corollary 5.6.26:

Corollary 5.6.28. *The spectrum $\Omega_P = \mathrm{Spec}\,(D_\lambda(P))$ is given by the closed subspace of $\widehat{\mathcal{J}_P}$ consisting of all $\chi \in \widehat{\mathcal{J}_P}$ with the property that for all $X \in \mathcal{J}_P$ with $\chi(X) = 1$ and all $X_1, \dots, X_n \in \mathcal{J}_P$ with $X = \bigcup_{i=1}^n X_i$ in P, we must have $\chi(X_i) = 1$ for some $1 \leq i \leq n$.*

Proposition 5.6.22 yields in our situation:

Corollary 5.6.29. *The following are equivalent:*

- *We have an isomorphism*

$$D_\lambda(P) \overset{\cong}{\longrightarrow} C^* \left(\{v_X : X \in \mathcal{J}_P\} \,\middle|\, \begin{array}{c} v_X^* = v_X = v_X^2, \\ v_0 = 0 \text{ if } 0 \in \mathcal{J}_P, \\ v_{X \cap Y} = v_X v_Y) \end{array} \right), \; 1_X \mapsto v_X.$$

- *We have an isomorphism*

$$C^*(E) \overset{\cong}{\longrightarrow} D_\lambda(P),\, \lambda_X \mapsto 1_X.$$

- *For every $X \in \mathcal{J}_P$ and all $X_1, \dots, X_n \in \mathcal{J}_P$,*

$$X = \bigcup_{i=1}^n X_i$$

  implies that $X = X_i$ for some $1 \leq i \leq n$.

- *The projections $\left\{1_X : X \in \mathcal{J}_P^\times\right\}$ are linearly independent in $D_\lambda(P)$.*

5.6.5 The independence condition

Corollary 5.6.29 justifies the following:

Definition 5.6.30. We say that our left cancellative semigroup P satisfies the *independence condition* (or simply independence) if for every $X \in \mathcal{J}_P$ and all $X_1, \dots, X_n \in \mathcal{J}_P$,

$$X = \bigcup_{i=1}^n X_i$$

implies that $X = X_i$ for some $1 \leq i \leq n$.

Let us now discuss examples of left cancellative semigroups that satisfy independence, and also some examples that do not. We start with the following

Lemma 5.6.31. *Suppose that P is a left cancellative semigroup with identity e. If every nonempty constructible right ideal of P is principal, i.e.,*

$$\mathcal{J}_P^{\times} = \{pP : p \in P\},$$

then P satisfies independence.

Proof. Suppose that

$$pP = \bigcup_{i=1}^{n} p_i P$$

for some $p, p_1, \ldots, p_n \in P$. Then, since P has an identity, the element p lies in pP, hence we must have $p \in p_i P$ for some $1 \leq i \leq n$. But then, since $p_i P$ is a right ideal, we conclude that $pP \subseteq p_i P$. Hence it follows that $pP = p_i P$, since we always have $pP \supseteq p_i P$. □

When are all nonempty constructible right ideals principal? Here is a necessary and sufficient condition:

Lemma 5.6.32. *For a left cancellative semigroup P (with or without identity), we have*

$$\mathcal{J}_P^{\times} = \{pP : p \in P\}$$

if and only if the following criterion holds:

For all $p, q \in P$ with $pP \cap qP \neq \emptyset$, there exists $r \in P$ with $pP \cap qP = rP$.

Proof. Our criterion is certainly necessary, since $\mathcal{J}_P$ is a semilattice, hence closed under intersections. To show that our condition is also sufficient, we first observe that $\mathcal{J}_P$ can be characterized as the smallest family of subsets of P containing P itself and closed under left multiplication, i.e.,

$$X \in \mathcal{J}_P,\ p \in P \ \Rightarrow\ p(X) \in cJ_P,$$

as well as pre-images under left multiplication, i.e.,

$$X \in \mathcal{J}_P,\ q \in P \ \Rightarrow\ q^{-1}(X) \in cJ_P.$$

Now $\{pP : p \in P\}$ is obviously closed under left multiplication. Hence it suffices to prove that principal right ideals are also closed under pre-images under left multiplication, up to $\emptyset$. Take $p, q \in P$. We always have

$$q^{-1}(pP) = q^{-1}(pP \cap qP).$$

Therefore, if $pP \cap qP = \emptyset$, then $q^{-1}(pP) = \emptyset$. If $pP \cap qP \neq \emptyset$, then by our criterion, there exists $r \in P$ with $pP \cap qP = rP$. As $rP \subseteq qP$, we must have $r \in qP$, so that we can write $r = qx$ for some $x \in P$. Therefore, we conclude that

$$q^{-1}(pP) = q^{-1}(pP \cap qP) = q^{-1}(rP) = q^{-1}(qxP) = xP. \qquad \square$$

For instance, positive cones in totally ordered groups (as in §5.3.2) always satisfy independence. This is because if P is such a positive cone, then for $p, q \in P$, we have $pP \cap qP = pP$ if $p \geq q$ and $pP \cap qP = qP$ if $p \leq q$. Hence, all constructible right ideals are principal by Lemma 5.6.32.

Moreover, right-angled Artin monoids (see §5.3.3) satisfy independence. Actually, all nonempty constructible right ideals are principal, because the criterion of Lemma 5.6.32 is true. This will come out of our general discussion of graph products in §5.9.

To discuss more examples, let us explain a general method for verifying the criterion in Lemma 5.6.32. This is based on [Deh03].

Suppose that we are given a monoid P defined by a presentation, i.e., generators Σ and relations R, so that $P = \langle \Sigma \,|\, R \rangle^+$. Assume that all the relations in R are of the form $w_1 = w_2$, where w_1 and w_2 are formal words in Σ. Now we introduce formal symbols

$$\left\{ \sigma^{-1} : \sigma \in \Sigma \right\} =: \Sigma^{-1},$$

and look at formal words in Σ and Σ^{-1}. For two such words w and w', we write $w \curvearrowright_R w'$ if w can be transformed into w' be finitely many of the following two possible steps:

- Delete $\sigma^{-1}\sigma$.
- Replace $\sigma_i^{-1}\sigma_j$ by uv^{-1} if $\sigma_i u = \sigma_j v$ is a relation in R.

We then say that our presentation (Σ, R) is complete for $\curvearrowright_R$ if for two formal words u and v in Σ, we have

$$u^{-1}v \curvearrowright_R \varepsilon \quad \text{(where } \varepsilon \text{ is the empty word)}$$

if and only if u and v define the same element in our monoid $P = \langle \Sigma \,|\, R \rangle^+$.

There are criteria on (Σ, R) which ensure completeness for $\curvearrowright_R$ (see [Deh03]).

If completeness for $\curvearrowright_R$ is given, then we can read of properties of our monoid $P = \langle \Sigma \,|\, R \rangle^+$ from the presentation (Σ, R). We refer the reader to [Deh03] for a general and more complete discussion. For our purposes, the following observation is important: If (Σ, R) is complete for $\curvearrowright_R$, then $P = \langle \Sigma \,|\, R \rangle^+$ has the property that

$$\begin{gathered} \text{for all } p, q \in P \text{ with } pP \cap qP \neq \emptyset, \text{ there exists } r \in P \text{ with } pP \cap qP = rP \\ \text{if and only if} \\ \text{for all } \sigma_i, \sigma_j \in \Sigma, \text{ there is at most one relation of the form } \sigma_i u = \sigma_j v \text{ in } R. \end{gathered}$$

Coming back to examples, it turns out that the presentations for Artin monoids, discussed in §5.3.3, are complete for $\curvearrowright_R$. Also, the presentations for Baumslag–Solitar monoids $B_{k,l}^+$, for $k, l \geq 1$, are complete for $\curvearrowright_R$. Furthermore, the presentation for the Thompson monoid F^+ is complete for $\curvearrowright_R$.

Following our discussion above, it is now easy to see that for Artin monoids, the Baumslag–Solitar monoids $B_{k,l}^{+}$, for $k, l \geq 1$, and the Thompson monoid F^{+}, all nonempty constructible right ideals are principal. In particular, all these examples satisfy independence.

For semigroups coming from rings, we have the following result:

Lemma 5.6.33. *Let R be a principal ideal domain. For both semigroups $M_n^{\times}(R)$ and $M_n(R) \rtimes M_n^{\times}(R)$, every nonempty constructible right ideal is principal.*

For the proof, we need the following

Lemma 5.6.34. *For every a, c in $M_n^{\times}(R)$, there exists $x \in M_n^{\times}(R)$ such that*

$$aM_n(R) \cap cM_n(R) = xM_n(R) \text{ and } aM_n^{\times}(R) \cap cM_n^{\times}(R) = xM_n^{\times}(R).$$

Proof. For brevity, we write M for $M_n(R)$ and $M^{\times}$ for $M_n^{\times}(R)$.

We will use the observation that for every $z \in M^{\times}$, there exist u and v in $GL_n(R)$ such that uzv is a diagonal matrix (see, for instance, [Kap49]).

To prove our lemma, let us first of all define x. Let $\tilde{c} \in M^{\times}$ satisfy $c\tilde{c} = \tilde{c}c = \det(c) \cdot 1_n$ (1_n is the identity matrix). Choose u and v in $GL_n(R)$ with

$$\tilde{c}a = u \cdot \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \cdot v,$$

where $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ is the diagonal matrix with $\alpha_1, \ldots, \alpha_n$ on the diagonal. For all $1 \leq i \leq n$, set $\beta_i := \mathrm{lcm}(\alpha_i, \det(c))$ and $\gamma_i := \det(c)^{-1}\beta_i$. Then our claim is that we can choose x as $x = c \cdot u \cdot \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$. In the following, we verify our claim:

$$\begin{aligned} aM \cap cM &= \tilde{c}^{-1}(\tilde{c}aM \cap (\det(c) \cdot 1_n)M) \\ &= \tilde{c}^{-1}((u \cdot \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \cdot v)M \cap (\det(c) \cdot 1_n)M) \\ &= \tilde{c}^{-1}u(\mathrm{diag}(\alpha_1, \ldots, \alpha_n)M \cap (\det(c) \cdot 1_n)M) \\ &= \tilde{c}^{-1} \cdot u \cdot \mathrm{diag}(\beta_1, \ldots, \beta_n)M \\ &= \tilde{c}^{-1}(\det(c) \cdot 1_n) \cdot u \cdot \mathrm{diag}(\gamma_1, \ldots, \gamma_n)M \\ &= c \cdot u \cdot \mathrm{diag}(\gamma_1, \ldots, \gamma_n)M. \end{aligned}$$

Thus, we have shown $aM \cap cM = xM$. Exactly the same computation shows that $aM^{\times} \cap cM^{\times} = xM^{\times}$. □

Proof of Lemma 5.6.33. For $M_n^{\times}(R)$, our claim is certainly a consequence of the Lemma 5.6.34. For $M_n(R) \rtimes M_n^{\times}(R)$, first note that given (b, a) and (d, c) in $M_n(R) \rtimes M_n^{\times}(R)$, we have

$$\begin{aligned} (b, a)(M_n(R) \rtimes M_n^{\times}(R)) &= (b + aM_n(R)) \times (aM_n^{\times}(R)), \\ (d, c)(M_n(R) \rtimes M_n^{\times}(R)) &= (d + cM_n(R)) \times (cM_n^{\times}(R)). \end{aligned}$$

Moreover, the intersection

$$(b + aM_n(R)) \cap (d + cM_n(R))$$

is either empty or of the form

$$y + (aM_n(R) \cap cM_n(R))$$

for some $y \in M_n(R)$. Now Lemma 5.6.34 provides an element $x \in M_n^\times(R)$ with

$$aM_n(R) \cap cM_n(R) = xM_n(R) \text{ and } aM_n^\times(R) \cap cM_n^\times(R) = xM_n^\times(R).$$

Thus either

$$(b, a)(M_n(R) \rtimes M_n^\times(R)) \cap (d, c)(M_n(R) \rtimes M_n^\times(R))$$

is empty or we obtain

$$(b, a)(M_n(R) \rtimes M_n^\times(R)) \cap (d, c)(M_n(R) \rtimes M_n^\times(R)) = (y, x)(M_n(R) \rtimes M_n^\times(R)). \qquad \square$$

In general, however, given an integral domain R, the semigroups $R^\times$ and $R \rtimes R^\times$ do not have the property that all nonempty constructible right ideals are principal. For example, just take a number field with nontrivial class number, and let R be its ring of algebraic integers. The property that all nonempty constructible right ideals are principal, for $R^\times$ or $R \rtimes R^\times$, translates to the property of the ring R of being a principal ideal domain. But this is not the case if the class number is bigger than 1. However, for all rings of algebraic integers, and more generally, for all Krull rings R, the semigroups $R^\times$ and $R \rtimes R^\times$ do satisfy independence.

Let R be an integral domain. Recall that we introduced the set $\mathcal{I}(R)$ of constructible ideals in §5.4.3. It is now easy to see that

$$\mathcal{J}_{R^\times} = \left\{ I^\times : I \in \mathcal{I}(R) \right\}$$

and

$$\mathcal{J}_{R \rtimes R^\times} = \left\{ (r + I) \times I^\times : r \in R, a, I \in \mathcal{I}(R) \right\},$$

where $I^\times = I \setminus \{0\}$.

Let us make the following observation about the relationship between the independence condition for multiplicative semigroups and $ax + b$-semigroups:

Lemma 5.6.35. *Let R be an integral domain. Then $R^\times$ satisfies independence if and only if $R \rtimes R^\times$ satisfies independence.*

Proof. If $\mathcal{J}_{R \rtimes R^\times}$ is not independent, then we have a nontrivial equation of the form

$$(r + I) \rtimes I^\times = \bigcup_{i=1}^{n} (r_i + I_i) \times I_i^\times \text{ with } (r_i + I_i) \times I_i^\times \subsetneq (r + I) \rtimes I^\times.$$

It is clear that

$$(r_i + I_i) \times I_i^\times \subsetneq (r + I) \rtimes I^\times$$

implies that $I_i \subsetneq I$, for all $1 \leq i \leq n$. Projecting onto the second coordinate of $R \times R^\times$, we obtain

$$I^\times = \bigcup_{i=1}^{n} I_i^\times.$$

This means that $R^\times$ does not satisfy independence.

Conversely, assume that $R^\times$ does not satisfy independence, so that we have a nontrivial equation of the form

$$I^\times = \bigcup_{i=1}^{n} I_i^\times$$

with $I_i^\times \subsetneq I^\times$. Hence it follows that

$$I = \bigcup_{i=1}^{n} I_i,$$

and $I_i \subsetneq I$ for all $1 \leq i \leq n$. By [Got94, Theorem 18], we may assume without loss of generality that

$$[I : I_i] < \infty \text{ for all } 1 \leq i \leq n.$$

But then we have

$$I \times I^\times = \bigcup_{i=1}^{n} \bigcup_{r+I_i \in I/I_i} (r + I_i) \times I_i^\times.$$

This shows that $\mathcal{J}_{R \rtimes R^\times}$ does not satisfy independence. □

Lemma 5.6.36. *For a Krull ring R, both semigroups $R^\times$ and $R \rtimes R^\times$ satisfy independence.*

Proof. We use the same notations as in §5.4.3.

Let Q be the quotient field of R, and let $I, I_1, \dots, I_n$ be ideals in $\mathcal{I}(R)$ with $I_i \subsetneq I$ for all $1 \leq i \leq n$. Then for every $1 \leq i \leq n$, there exists $\mathfrak{p}_i \in \mathcal{P}(R)$ with

$$v_{\mathfrak{p}_i}(I_i) > v_{\mathfrak{p}_i}(I).$$

By Proposition 5.4.13, there exists $x \in Q^\times$ with

$$v_{\mathfrak{p}_i}(x) = v_{\mathfrak{p}_i}(I) \text{ for all } 1 \leq i \leq n$$

and

$$v_{\mathfrak{p}}(x) \geq v_{\mathfrak{p}}(I) \text{ for all } \mathfrak{p} \in \mathcal{P}(R) \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}.$$

Thus x lies in I, but does not lie in I_i for any $1 \leq i \leq n$. Therefore,

$$\bigcup_{i=1}^{n} I_i \subsetneq I,$$

and thus

$$\bigcup_{i=1}^{n} I_i^{\times} \subsetneq I^{\times}.$$

This shows that $R^{\times}$ satisfies independence. By Lemma 5.6.35, $R \rtimes R^{\times}$ must satisfy independence as well. $\square$

Let us present an example of a semigroup coming from a ring that does not satisfy independence. Consider the ring $R := \mathbb{Z}[i\sqrt{3}]$. Its quotient field is given by $Q = \mathbb{Q}[i\sqrt{3}]$. R is not integrally closed in Q. Let $\alpha := \frac{1}{2}(1 + i\sqrt{3})$. α is a primitive sixth root of unity. It is clear that $\alpha \notin R$. But $2\alpha = 1 + i\sqrt{3}$ lies in R.

The integral closure of R is given by $\bar{R} := \mathbb{Z}[\alpha]$. We claim that

$$2\bar{R} = 2^{-1}(2\alpha R) = 2^{-1}(1 + i\sqrt{3})R.$$

To prove "$\subseteq$", observe that $\bar{R} = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \alpha$. Now

$$2 \cdot (2 \cdot 1) = 4 = (1 + i\sqrt{3}) \cdot (1 - i\sqrt{3}) \in (1 + i\sqrt{3})R,$$

and

$$2 \cdot (2\alpha) = 2 \cdot (1 + i\sqrt{3}) \in (1 + i\sqrt{3})R.$$

For "$\supseteq$", let $x = m + n \cdot i\sqrt{3}$ be in R such that $2x \in 2\alpha R$. As

$$2\alpha R = (1+i\sqrt{3})R = \mathbb{Z}\cdot(1+i\sqrt{3})+\mathbb{Z}\cdot((1+i\sqrt{3})i\sqrt{3}) = \mathbb{Z}\cdot(1+i\sqrt{3})+\mathbb{Z}\cdot(-3+i\sqrt{3}),$$

there exist $k, l \in \mathbb{Z}$ with

$$2x = 2m + 2n \cdot i\sqrt{3} = k(1 + i\sqrt{3}) + l(-3 + i\sqrt{3}) = (k - 3l) + (k + l)(i\sqrt{3}),$$

so that $2m = k - 3l$ and $2n = k + l$. It follows that $2n = 2m + 4l$, and thus $n = m + 2l$ or $m = n - 2l$. We conclude that

$$x = -2l + n \cdot (1 + i\sqrt{3}) \in 2\bar{R}.$$

This shows that $2\bar{R} = 2^{-1}(1 + i\sqrt{3})R$. Hence it follows that $2\bar{R}$ is a constructible (ring-theoretic) ideal of R.

We have $\bar{R} = R \cup \alpha R \cup \alpha^2 R$ in Q. This is because

$$R = \mathbb{Z} + \mathbb{Z}(2\alpha),\ \alpha R = \mathbb{Z}\alpha + \mathbb{Z}(2\alpha^2) = \mathbb{Z}\alpha + \mathbb{Z}(2\alpha - 2) \text{ and } \alpha^2 R = \mathbb{Z}(\alpha - 1) + \mathbb{Z}2.$$

Now take $x = m + n\alpha \in \bar{R}$ with $m, n \in \mathbb{Z}$. If n is even, then x is contained in R. If n is odd and m is even, then write $l = \frac{m}{2}$. We have

$$x = (n + m) \cdot \alpha + (-l) \cdot (2\alpha - 2) \in \alpha R.$$

Finally, if n is odd and m is odd, we write $k = \frac{m+n}{2}$. Then

$$x = n \cdot (\alpha - 1) + k \cdot 2 \in \alpha^2 R.$$

This shows $\bar{R} = R \cup \alpha R \cup \alpha^2 R$. Therefore,

$$2\bar{R} = 2R \cup 2\alpha R \cup 2\alpha^2 R = 2R \cup (1 + i\sqrt{3})R \cup (-1 + i\sqrt{3})R.$$

But $2R \subsetneq 2\bar{R}$, $(1+i\sqrt{3})R \subsetneq 2\bar{R}$ and $(-1+i\sqrt{3})R \subsetneq 2\bar{R}$. This means that $R^\times$ does not satisfy independence. By Lemma 5.6.35, $R \rtimes R^\times$ does not satisfy independence, either.

Let us present another example of a left cancellative semigroup not satisfying independence. Consider $P = \mathbb{N} \setminus \{1\}$. Clearly, P is a semigroup under addition. We have the following constructible right ideals

$$2 + P = \{2, 4, 5, 6, \dots\} \text{ and } 3 + P = \{3, 5, 6, 7, \dots\}.$$

Hence

$$5 + \mathbb{N} = \{5, 6, 7, \dots\} = (2 + P) \cap (3 + P)$$

is also a constructible right ideal of P. Moreover, it is clear that

$$5 + \mathbb{N} = (5 + P) \cup (6 + P).$$

But since $5 + P \subsetneq 5 + \mathbb{N}$ and $6 + P \subsetneq 5 + \mathbb{N}$, it follows that P does not satisfy independence.

A similar argument shows that for every numerical semigroup of the form $\mathbb{N} \setminus F$, where F is a nonempty finite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus F$ is still closed under addition, the independence condition does not hold. The reader may also compare Chapter 7 for more examples of a similar kind (which are two-dimensional versions), where the independence condition typically fails.

Now let us come back to the comparison of reduced C^*-algebras for left cancellative semigroups and their left inverse hulls. Combining Corollary 5.6.19 and Proposition 5.6.29, we get:

Proposition 5.6.37. *Let P be a subsemigroup of a group. The $*$-homomorphism*

$$C^*_\lambda(I_l(P)) \to C^*_\lambda(P),\ \lambda_p \mapsto V_p$$

is an isomorphism if and only if P satisfies independence.

5.6.6 Construction of full semigroup C^*-algebras

Proposition 5.6.37 explains when we can identify $C^*_\lambda(I_l(P))$ and $C^*_\lambda(P)$ in a canonical way, in case P embeds into a group. Motivated by this result, we construct full semigroup C^*-algebras.

Definition 5.6.38. Let P be a left cancellative semigroup, and $I_l(P)$ its left inverse hull. We define the *full semigroup C^*-algebra* of P as the full inverse semigroup C^*-algebra of $I_l(P)$, i.e.,

$$C^*(P) := C^*(I_l(P)).$$

Recall that $C^*(I_l(P))$ is the C^*-algebra universal for $*$-representations of the inverse semigroup $I_l(P)$ by partial isometries (see §5.5.1).

As we saw in §5.5.1, there is a canonical $*$-homomorphism

$$C^*(I_l(P)) \to C^*_\lambda(I_l(P)),\ v_p \mapsto \lambda_p.$$

Composing with the $*$-homomorphism

$$C^*_\lambda(I_l(P)) \to C^*_\lambda(P),\ \lambda_p \mapsto V_p,$$

we obtain a canonical $*$-homomorphism

$$C^*(P) \to C^*_\lambda(P),\ v_p \mapsto V_p.$$

We call it the left regular representation of $C^*(P)$.

Remark 5.6.39. *It is clear that if the left regular representation of $C^*(P)$ is an isomorphism, then P must satisfy independence. This is because the restriction of $C^*(P) \to C^*_\lambda(P)$ to $C^*(\{v_X : X \in \mathcal{J}_P\})$ is the composition*

$$C^*(\{v_X : X \in \mathcal{J}_P\}) \to C^*(E) \to D_\lambda(P),$$

and we know that the first $*$-homomorphism is always an isomorphism (see Corollary 5.6.24), while the second one is an isomorphism if and only if P satisfies independence (see Corollary 5.6.29).

Given a concrete left cancellative semigroup P, it is usually possible to find a natural and simple presentation for $C^*(P)$ as a universal C^*-algebra generated by isometries and projections, subject to relations. Let us discuss some examples.

For the example $P = \mathbb{N}$, the full semigroup C^*-algebra $C^*(\mathbb{N})$ is the universal unital C^*-algebra generated by one isometry,

$$C^*(\mathbb{N}) \cong C^*(v \mid v^*v = 1).$$

For $P = \mathbb{N} \times \mathbb{N}$, $C^*(\mathbb{N} \times \mathbb{N})$ is the universal unital C^*-algebra generated by two isometries that $*$-commute, i.e.,

$$C^*(\mathbb{N} \times \mathbb{N}) \cong C^*(v_a, v_b \mid v_a^*v_a = 1 = v_b^*v_b,\ v_av_b = v_bv_a,\ v_a^*v_b = v_bv_a^*).$$

Note that this C^*-algebra is a quotient of

$$C^*(v_a, v_b \mid v_a^* v_a = 1 = v_b^* v_b,\ v_a v_b = v_b v_a).$$

As we remarked in §5.6.2, the latter C^*-algebra is not nuclear by [Mur96, Theorem 6.2]. However, as we will see in §5.6.8, this quotient, and hence $C^*(\mathbb{N} \times \mathbb{N})$, is nuclear.

For the nonabelian free monoid on two generators $P = \mathbb{N} * \mathbb{N}$, $C^*(\mathbb{N} * \mathbb{N})$ is the universal unital C^*-algebra generated by two isometries with orthogonal range projections, i.e.,

$$C^*(\mathbb{N} * \mathbb{N}) \cong C^*(v_a, v_b \mid v_a^* v_a = 1 = v_b^* v_b,\ v_a v_a^* v_b v_b^* = 0).$$

More generally, for a right-angled Artin monoid P, a natural and simple presentation for $C^*(P)$ has been established in [CL02] (see also [ELR16]).

Let us also mention that for a class of left cancellative semigroups, full semigroup C^*-algebras can be identified in a canonical way with semigroup crossed products by endomorphisms. Let P be a left cancellative semigroup with constructible right ideals $\mathcal{J}_P$. We then have a natural action α of P by endomorphisms on

$$D(P) := C^*(\{v_X : X \in \mathcal{J}_P\}) \subseteq C^*(P),$$

where $p \in P$ acts by the endomorphism

$$\alpha_p : D(P) \to D(P),\ v_X \mapsto v_{pX}.$$

If P is right reversible, i.e., $Pp \cap Pq \neq \emptyset$ for all $p, q \in P$, or if every nonempty constructible right ideal of P is principal, i.e., $\mathcal{J}_P^\times = \{pP : p \in P\}$, then we have a canonical isomorphism

$$C^*(P) \cong D(P) \rtimes_\alpha P.$$

We refer to [Li12, §3] for more details. Writing out the definition of the crossed product, we get the following presentation:

$$C^*(P) \cong C^*\left(\{e_X : X \in \mathcal{J}_P\} \cup \{v_p : p \in P\} \;\middle|\; \begin{array}{c} e_X^* = e_X = e_X^2;\ v_p^* v_p = 1; \\ e_\emptyset = 0 \text{ if } \emptyset \in \mathcal{J}_P,\ e_P = 1, \\ e_{X \cap Y} = e_X \cdot e_Y; \\ v_{pq} = v_p v_q; \\ v_p e_X v_p^* = e_{pX} \end{array} \right)$$

In particular, for an integral domain R, we obtain the following presentation for the full semigroup C^*-algebra of $R \rtimes R^\times$:

$$C^*(R \rtimes R^\times)$$
$$\cong C^*\left(\begin{array}{c|c} \{e_I : I \in \mathcal{I}(R)\} & e_I^* = e_I = e_I^2; \\ \cup \{u^b : b \in R\} & u^b(u^b)^* = 1 = (u^b)^* u^b;\ v_a^* v_a = 1 \\ \cup \{s_a : a \in R^\times\} & e_R = 1,\ e_{I \cap J} = e_I \cdot e_J; \\ & s_{ac} = s_a s_c,\ u^{b+d} = u^b u^d,\ s_a u^b = u^{ab} s_a; \\ & s_a e_I s_a^* = e_{aI}; \\ & u^b e_I = e_I u^b \text{ if } b \in I,\ e_I u^b e_I = 0 \text{ if } b \notin I \end{array} \right)$$

We refer to [CDL13, §2] as well as [Li12, §2.4].

In order to explain how this definition of full semigroup C^*-algebras is related to previous constructions in the literature, we mention first of all that our definition generalizes Nica's construction in the quasi-lattice ordered case [Nic92]. Moreover, in the case of $ax+b$-semigroups over rings of algebraic integers (or more generally Dedekind domains), our definition includes the construction in [CDL13]. In the case of subsemigroups of groups, our definition coincides with the construction, denoted by $C_s^*(P)$, in [Li12, Definition 3.2]. Last but not least, we point out that in comparison with another construction in [Li12, Definition 2.2], our definition is always a quotient of the construction in [Li12, Definition 2.2], and in certain cases (see [Li12, §3.1] for details), our definition is actually isomorphic to the construction in [Li12, Definition 2.2].

5.6.7 Crossed product and groupoid C^*-algebra descriptions of reduced semigroup C^*-algebras

We now specialize to the case where our semigroup P embeds into a group G. To explain the connection between amenability and nuclearity, we would like to write the reduced C^*-algebra $C_\lambda^*(P)$ of P as a reduced crossed product attached to a partial dynamical system, and hence as a reduced groupoid C^*-algebra. Let us start with the underlying partial dynamical system.

We already saw that $\Omega_P = \mathrm{Spec}\,(D_\lambda(P))$ may be identified with the subspace of $\widehat{\mathcal{J}_P}$ given by the characters χ with the property that for all $X, X_1, \dots, X_n$ in $\mathcal{J}_P$ with $X = \bigcup_{i=1}^n X_i$, $\chi(X) = 1$ implies that $\chi(X_i) = 1$ for some $1 \leq i \leq n$ (see Corollary 5.6.28).

Moreover, we introduced the partial dynamical system $G \curvearrowright \widehat{\mathcal{J}_P}$ in §5.5.1. It is given as follows: Every $g \in G$ acts on

$$U_{g^{-1}} = \left\{ \chi \in \widehat{\mathcal{J}_P} : \chi(s^{-1}s) = 1 \text{ for some } s \in I_l(P)^\times \text{ with } \sigma(s) = g \right\},$$

and for $\chi \in U_{g^{-1}}$, $g.\chi = \chi(s^{-1} \sqcup s)$ where $s \in I_l(P)^\times$ is an element satisfying $\chi(s^{-1}s) = 1$ and $\sigma(s) = g$.

We now claim:

Lemma 5.6.40. *Ω_P is an G-invariant subspace of $\widehat{\mathcal{J}_P}$.*

Proof. Take $g \in G$ and $\chi \in U_{g^{-1}} \cap \Omega_P$, and suppose that $s \in I_l(P)^\times$ satisfies $\chi(s^{-1}s) = 1$ and $\sigma(s) = g$. We have to show that $g.\chi = \chi(s^{-1} \sqcup s)$ lies in Ω_P.

Suppose that $X, X_1, \ldots, X_n$ in $\mathcal{J}_P$ satisfy $X = \bigcup_{i=1}^n X_i$. Then, identifying $s^{-1}s$ with $\mathrm{dom}(s)$, we have

$$s^{-1}Xs = (g^{-1}X) \cap \mathrm{dom}(s) = \bigcup_{i=1}^{n} (g^{-1}X_i) \cap \mathrm{dom}(s) = \bigcup_{i=1}^{n} s^{-1}X_i s.$$

Hence, if $g.\chi(X) = 1$, then $\chi(s^{-1}Xs) = 1$, and hence $g.\chi(X_i) = \chi(s^{-1}X_i s) = 1$ for some $1 \leq i \leq n$. This shows that $g.\chi$ lies in Ω_P. □

Hence we obtain a partial dynamical system $G \curvearrowright \Omega_P$ by restricting $G \curvearrowright \widehat{\mathcal{J}_P}$ to Ω_P. A moment's thought shows that this partial dynamical system coincides with the one introduced in §5.5.2.

If our group G were exact, then this observation, together with Corollary 5.5.23, would immediately imply that $C^*_\lambda(P) \cong C(\Omega_P) \rtimes_r G$ with respect to the G-action $G \curvearrowright \Omega_P$. However, it turns out that we do not need exactness here.

Theorem 5.6.41. *There is a canonical isomorphism $C^*_\lambda(P) \cong C(\Omega_P) \rtimes_r G$ determined by $V_p \mapsto W_p$. Here W_g denote the canonical partial isometries in $C(\Omega_P) \rtimes_r G$.*

Proof. We work with the dual action $G \curvearrowright D_\lambda(P)$ as described in §5.5.2. Our strategy is to describe both $C^*_\lambda(P)$ and $D_\lambda(P) \rtimes_r G$ as reduced (cross-sectional) algebras of Fell bundles, and then to identify the underlying Fell bundles.

Let us start with $C^*_\lambda(P)$. As in §5.5.1, we think of $I_l(P)$ as partial isometries. Recall that we defined the partial homomorphism $\sigma : I_l(P)^\times \to G$ in §5.5.1. Now we set

$$B_g := \overline{\mathrm{span}}(\sigma^{-1}(g))$$

for every $g \in G$. We want to see that $(B_g)_{g \in G}$ is a grading for $C^*_\lambda(P)$, in the sense of [Exe97, Definition 3.1]. Conditions (i) and (ii) are obviously satisfied. For (iii), we use the faithful conditional expectation $\Theta_P : C^*_\lambda(P) \twoheadrightarrow D_\lambda(P) = B_e$ from §5.6.3. Given a finite sum

$$x = \sum_g x_g \in C^*_\lambda(P)$$

of elements $x_g \in B_g$ such that $x = 0$, we conclude that

$$0 = x^*x = \sum_{g,h} x_g^* x_h,$$

and hence

$$0 = \Theta_P(x^*x) = \sum_g x_g^* x_g.$$

Here we used that $\Theta_P|_{B_g} = 0$ if $g \neq e$. This implies that $x_g = 0$ for all g. Therefore, the subspaces B_g are independent. It is clear that the linear span of all the B_g is dense in $C^*_\lambda(P)$. This proves (iii). If we let $\mathcal{B}$ be the Fell bundle given by $(B_g)_{g\in G}$, then [Exe97, Proposition 3.7] implies $C^*_\lambda(P) \cong C^*_r(\mathcal{B})$ because $\Theta_P : C^*_\lambda(P) \twoheadrightarrow D_\lambda(P) = B_e$ is a faithful conditional expectation satisfying $\Theta_P|_{B_e} = \mathrm{id}_{B_e}$ and $\Theta_P|_{B_g} = 0$ if $g \neq e$.

Let us also describe $D_\lambda(P) \rtimes_r G$ as a reduced algebra of a Fell bundle. We denote by W_g the partial isometry in $D_\lambda(P) \rtimes_r G$ corresponding to $g \in G$, and we set $B'_g := D_g W_g$. Recall that we defined

$$D_{g^{-1}} = \overline{\mathrm{span}}(\{V^*V : V \in I_l(P)^\times, \sigma(V) = g\})$$

in §5.5.2. It is easy to check that $(B'_g)_{g\in G}$ satisfy (i), (ii) and (iii) in [Exe97, Definition 3.1]. Moreover, $B'_e = D_e = D_\lambda(P)$, and it follows immediately from the construction of the reduced partial crossed product that there is a faithful conditional expectation $D_\lambda(P) \rtimes_r G \twoheadrightarrow D_\lambda(P) = B'_e$ which is identity on B'_e and 0 on B'_g for $g \neq e$. Hence if we let $\mathcal{B}'$ be the Fell bundle given by $(B'_g)_{g\in G}$, then [Exe97, Proposition 3.7] implies $D_\lambda(P) \rtimes_r G \cong C^*_r(\mathcal{B}')$.

To identify $C^*_\lambda(P)$ and $D_\lambda(P) \rtimes_r G$, it now remains to identify $\mathcal{B}$ with $\mathcal{B}'$. We claim that the map

$$\mathrm{span}(\{V : \sigma(V) = g\}) \to \mathrm{span}(\{VV^*W_g : \sigma(V) = g\}), \sum_i \alpha_i V_i \mapsto \sum_i \alpha_i V_i V_i^* W_g$$

is well defined and extends to an isometric isomorphism $B_g \to B'_g$, for all $g \in G$.

All we have to show is that our map is isometric. We have

$$\left\| \sum_i \alpha_i V_i \right\|^2 = \left\| \sum_{i,j} \alpha_i \overline{\alpha_j} V_i V_j^* \right\|_{D_\lambda(P)}$$

and

$$\left\| \sum_i \alpha_i V_i V_i^* W_g \right\|^2 = \left\| \sum_{i,j} \alpha_i \overline{\alpha_j} V_i V_i^* V_j V_j^* \right\|_{D_\lambda(P)}.$$

Since $V_i = V_i V_i^* \lambda_g$ and $V_j^* = \lambda_{g^{-1}} V_j V_j^*$, we have

$$V_i V_j^* = V_i V_i^* \lambda_g \lambda_{g^{-1}} V_j V_j^* = V_i V_i^* V_j V_j^*.$$

Hence, indeed,

$$\left\| \sum_i \alpha_i V_i \right\|^2 = \left\| \sum_i \alpha_i V_i V_i^* W_g \right\|^2,$$

and we are done.

All in all, we have proven that

$$C^*_\lambda(P) \cong C^*_r(\mathcal{B}) \cong C^*_r(\mathcal{B}') \cong D_\lambda(P) \rtimes_r G.$$

Our isomorphism sends V_p to $V_p V_p^* W_p$, but a straightforward computation shows that actually, $V_p V_p^* W_p = W_p$ for all $p \in P$. Thus the isomorphism we constructed is given by $V_p \mapsto W_p$ for all $p \in P$. □

In particular, in combination with Theorem 5.5.21, we get an isomorphism

$$C^*_\lambda(P) \xrightarrow{\cong} C^*_r(G \ltimes \Omega_P),\ V_p \mapsto 1_{\{p\}\times\Omega_P}. \tag{5.21}$$

Together with Remark 5.5.19 and Lemma 5.5.22, we see that we obtain a commutative diagram

$$\begin{tikzcd} C^*(P) = C^*(I_l(P)) \arrow[r, "\cong"] \arrow[d] & C^*(G \ltimes \widehat{\mathcal{J}_P}) \arrow[d] \\ C^*_\lambda(I_l(P)) \arrow[r, "\cong"] \arrow[d] & C^*_r(G \ltimes \widehat{\mathcal{J}_P}) \arrow[d] \\ C^*_\lambda(P) \arrow[r, "\cong"] & C^*_r(G \ltimes \Omega_P) \end{tikzcd} \tag{5.22}$$

Here the upper left vertical arrow is the left regular representation of $C^*(I_l(P))$. The lower left vertical arrow is the $*$-homomorphism provided by Lemma 5.6.11. The upper right vertical arrow is the left regular representation of $C^*(G \ltimes \widehat{\mathcal{J}_P})$. The lower right vertical arrow is the canonical projection map; it corresponds to the canonical map $C(\widehat{\mathcal{J}_P}) \rtimes_r G \twoheadrightarrow C(\Omega) \rtimes_r G$ under the identification from Theorem 5.5.21. The first horizontal arrow is the identifications from Theorem 5.5.20. The second horizontal arrow is the isomorphism from Theorem 5.5.21. For both of these horizontal arrows, we also need Lemma 5.5.22. The third horizontal arrow is provided by the isomorphism (5.21).

Now we are ready to discuss the relationship between amenability and nuclearity and thereby explain the strange phenomena mentioned at the beginning of §5.6.2.

5.6.8 Amenability of semigroups in terms of C^*-algebras

Let us start by explaining how to characterize amenability of semigroups in terms of their C^*-algebras.

Theorem 5.6.42. *Let P be a cancellative semigroup, i.e., P is both left and right cancellative. Assume that P satisfies the independence condition. Then the following are equivalent:*

1) *P is left amenable.*

2) *$C^*(P)$ is nuclear and there is a character on $C^*(P)$.*

3) *$C^*_\lambda(P)$ is nuclear and there is a character on $C^*(P)$.*

4) *The left regular representation $C^*(P) \to C^*_\lambda(P)$ is an isomorphism and there is a character on $C^*(P)$.*

5) *There is a character on $C^*_\lambda(P)$.*

By a character, we mean a unital $*$-homomorphism to $\mathbb{C}$.
For the proof, we need the following

Lemma 5.6.43. *Let P be a left cancellative semigroup. The following are equivalent:*

1. *There is a character on $C^*(P)$.*
2. *P is left reversible, i.e., $pP \cap qP \neq \emptyset$ for all $p, q \in P$.*
3. *$I_l(P)$ does not contain $\emptyset \to \emptyset$, the partial bijection that is nowhere defined.*

Recall that in the convention we introduced in §5.5.1, if $\emptyset \to \emptyset$ lies in $I_l(P)$, then we say that $I_l(P)$ is an inverse semigroup with zero, and let $\emptyset \to \emptyset$ be its distinguished zero element, which we denote by 0.

Proof. 1. $\Rightarrow$ 2.: If χ is a character on $C^*(P)$, then for every $p, q \in P$, we have

$$\chi(1_{pP \cap qP}) = \chi(1_{pP})\chi(1_{qP}) = \chi(V_p V_p^*)\chi(V_q V_q^*) = |\chi(V_p)|^2 \, |\chi(V_q)|^2 = 1.$$

Hence $pP \cap qP \neq \emptyset$.

2. $\Rightarrow$ 3.: Every partial bijection in $I_l(P)$ is a finite product of elements in

$$\{p : p \in P\} \cup \left\{q^{-1} : q \in P\right\}.$$

Hence, by an inductive argument, it suffices to show that if $s \in I_l(P)$ is not $\emptyset \to \emptyset$, then for all $p, q \in P$, ps and $q^{-1}s$ are not $\emptyset \to \emptyset$. For ps, this is clear. For $q^{-1}s$, choose $x \in \mathrm{dom}(s)$. Then $xP \subseteq \mathrm{dom}(s)$ and $s(xr) = s(x)r$ for all $r \in P$ by property (5.11). As P is left reversible, there exists $y \in P$ with $y \in qP \cap s(x)P$. Hence $y = s(x)r = qz$ for some $r, z \in P$. Therefore,

$$(q^{-1}s)(xz) = q^{-1}(s(xr)) = q^{-1}(s(x)r) = q^{-1}(qz) = z.$$

Hence $q^{-1}s$ is not $\emptyset \to \emptyset$, as desired.

3. $\Rightarrow$ 1.: Since $I_l(P)$ does not contain $\emptyset \to \emptyset$, we have by definition that

$$C^*(P) = C^*(I_l(P)) = C^*(\{v_s : s \in I_l(P)\} \, | v_{st} = v_s v_t, \, v_{s^{-1}} = v_s^*).$$

Obviously, by universal property, we obtain a character $C^*(P) \to \mathbb{C}$, $v_s \to 1$. $\square$

Proof of Theorem 5.6.42. 1) $\Rightarrow$ 2): If P is left amenable, then there exists a left invariant state μ on $\ell^\infty(P)$ by definition. Hence, for every $p \in P$, we have

$$\mu(1_{pP}) = \mu(1_{pP}(p\sqcup)) = \mu(1_P) = 1.$$

Now, if there were $p, q \in P$ with $pP \cap qP = \emptyset$, then $1_{pP} + 1_{qP}$ would be a projection in $\ell^\infty(P)$ with $1_{pP} + 1_{qP} \leq 1_P$, so that

$$1 = \mu(1_P) \geq \mu(1_{pP} + 1_{qP}) = \mu(1_{pP}) + \mu(1_{qP}) = 1 + 1 = 2.$$

This is a contradiction. Therefore, P must be left reversible. By Lemma 5.6.43, it follows that $C^*(P)$ has a character.

In addition, by our discussion of group embeddability in §5.4.1, we see that P embeds into its group G of right quotients. Moreover, as P is left amenable, G must be amenable by [Pat88, Proposition (1.27)]. Hence, statement 2) follows from Theorem 5.6.44 (see also Corollary 5.6.45).

2) $\Rightarrow$ 3) is obvious.

3) $\Rightarrow$ 4) follows again from Theorem 5.6.44.

4) $\Rightarrow$ 5) is obvious.

5) $\Rightarrow$ 1): We follow [Li12, §4.2]. Let $\chi : C^*_\lambda(P) \to \mathbb{C}$ be a nonzero character. Viewing χ as a state, we can extend it by the theorem of Hahn–Banach to a state on $\mathcal{L}(\ell^2(P))$. We then restrict the extension to $\ell^\infty(P) \subseteq \mathcal{L}(\ell^2(P))$ and call this restriction μ. The point is that by construction, $\mu|_{C^*_\lambda(P)} = \chi$ is multiplicative, hence $C^*_\lambda(P)$ is in the multiplicative domain of μ. Thus we obtain for every $f \in \ell^\infty(P)$ and $p \in P$

$$\mu(f(p\sqcup)) = \mu(V_p^* f V_p) = \mu(V_p^*)\mu(f)\mu(V_p) = \mu(V_p)^*\mu(V_p)\mu(f) = \mu(f).$$

Thus μ is a left invariant mean on $\ell^\infty(P)$. This shows “5) $\Rightarrow$ 1)”. $\square$

Theorem 5.6.42 tells us that for the example $P = \mathbb{N} \times \mathbb{N}$ discussed in §5.6.2, our definition of full semigroup C^*-algebras leads to a full C^*-algebra $C^*(\mathbb{N} \times \mathbb{N})$ which is nuclear and whose left regular representation is an isomorphism. This explains and resolves the strange phenomenon described in §5.6.2.

At the same time, we see why it is not a contradiction that $\mathbb{N} * \mathbb{N}$ is not amenable while its C^*-algebra behaves like those of amenable semigroups. The point is that there is no character on $C^*(\mathbb{N} * \mathbb{N})$ because $\mathbb{N} * \mathbb{N}$ is not left reversible.

However, we still need an explanation why the semigroup C^*-algebra of $\mathbb{N} * \mathbb{N}$ behaves like those of amenable semigroups. This leads us to our next result.

5.6.9 Nuclearity of semigroup C^*-algebras and the connection to amenability

Theorem 5.6.44. *Let P be a semigroup that embeds into a group G. Consider*

(i) $C^*(P)$ *is nuclear.*

(ii) $C^*_\lambda(P)$ *is nuclear.*

(iii) $G \ltimes \Omega_P$ *is amenable.*

(iv) *The left regular representation* $C^*(P) \to C^*_\lambda(P)$ *is an isomorphism.*

We always have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii), *and* (iv) *implies that P satisfies independence.*

If P satisfies independence, then we also have (iii) $\Rightarrow$ (i) *and* (iii) $\Rightarrow$ (iv).

Note that the étale locally compact groupoid $G \ltimes \Omega_P$ really only depends on P, not on the embedding $P \hookrightarrow G$. This follows from Lemma 5.5.22.

Proof. The first claim follows from the description of $C^*(P) = C^*(I_l(P))$ as a full groupoid C^*-algebra (see Theorem 5.5.17), the description of $C^*_\lambda(P)$ as a reduced groupoid C^*-algebra (see §5.6.7 and the isomorphism (5.21)), the commutative diagram (5.22), and Theorem 5.6.7. That (iv) implies that P satisfies independence was explained in Remark 5.6.39.

The second claim follows from the observation that if P satisfies independence, then $\Omega_P = \widehat{\mathcal{J}_P}$ (see Corollary 5.6.28 and equation (5.19)), so that the partial dynamical systems $G \curvearrowright \Omega_P$ and $G \curvearrowright \widehat{\mathcal{J}_P}$, and hence their partial transformation groupoids coincide, and Theorem 5.6.7. □

Corollary 5.6.45. *If P is a subsemigroup of an amenable group G, then statements* (i), (ii) *and* (iii) *from Theorem* 5.6.44 *hold, and* (iv) *holds if and only if P satisfies independence.*

Proof. This is because if G is amenable, then the partial transformation groupoid $G \ltimes \Omega_P$ is amenable by [Exe15, Theorem 20.7 and Theorem 25.10]. □

This explains the second strange phenomenon mentioned at the beginning of §5.6.2, that the semigroup C^*-algebra of $\mathbb{N} * \mathbb{N}$ behaves like those of amenable semigroups. The underlying reason is that $\mathbb{N} * \mathbb{N}$ embeds into an amenable group: Let $\mathbb{F}_2$ be the free group on two generators. By [Hoc69], we have an embedding $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2 / \mathbb{F}_2''$, where $\mathbb{F}_2''$ is the second commutator subgroup of $\mathbb{F}_2$. But $\mathbb{F}_2 / \mathbb{F}_2''$ is solvable, in particular amenable. Moreover, $\mathbb{N} * \mathbb{N}$ satisfies independence (see §5.6.5). This is why statements (i) to (iv) from Theorem 5.6.44 are all true for the semigroup $P = \mathbb{N} * \mathbb{N}$.

Remark 5.6.46. *If we modify the definition of full semigroup C^*-algebras, then we can get the same results as in Theorem* 5.6.42, *Theorem* 5.6.44 *and Corollary* 5.6.45

without having to mention the independence condition. Simply define $C^*(P)$ as the full groupoid C^*-algebra of the restriction

$$\mathcal{G}(I_l(P))|\Omega_P = \{\gamma \in \mathcal{G}(I_l(P)) : r(\gamma), s(\gamma) \in \Omega_P\}$$

of the universal groupoid $\mathcal{G}(I_l(P))$ of $I_l(P)$ to Ω_P. This means that we would set

$$C^*(P) := C^*(\mathcal{G}(I_l(P))|\Omega_P).$$

Then, in Theorem 5.6.44*, we would have* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)*, and all these statements imply* (iv)*. Corollary* 5.6.45 *would say that statements* (i) *to* (iv) *from Theorem* 5.6.44 *hold whenever G is amenable. Moreover, Theorem* 5.6.42 *would be true without the assumption that P satisfies independence.*

We have chosen not to follow this route and keep the definition of full semigroup C^*-algebras as full C^*-algebras of left inverse hulls because the C^*-algebras $C^*(I_l(P))$ usually have a nicer presentation, i.e., a nicer and simpler description as universal C^*-algebras given by generators and relations. Moreover, in the case of semigroups embeddable into groups, we know that these two definitions of full semigroup C^*-algebras differ precisely by the (failure of the) independence condition.

5.7 Topological freeness, boundary quotients, and C^*-simplicity

Given a semigroup P that embeds into a group G, we have constructed a partial dynamical system $G \curvearrowright \Omega_P$ and identified the reduced semigroup C^*-algebra $C^*_\lambda(P)$ with the reduced crossed product $C(\Omega_P) \rtimes_r G$. Let us now present a criterion for topological freeness of $G \curvearrowright \Omega_P$. First recall (compare [ELQ02] and [Li16b]) that a partial dynamical system $G \curvearrowright X$ is called topologically free if for every $e \neq g \in G$,

$$\left\{x \in U_{g^{-1}} : g.x \neq x\right\}$$

is dense in $U_{g^{-1}}$. Here, we use the same notation as in §5.5.2.

We first need the following observation: Let P be a monoid. For $p \in P$, let $\chi_{pP} \in \widehat{\mathcal{J}_P}$ be defined by $\chi_{pP}(X) = 1$ if and only if $pP \subseteq X$, for $X \in \mathcal{J}_P$. Since P is a monoid, χ_{pP} lies in Ω_P for all $p \in P$.

Lemma 5.7.1. *The subset $\{\chi_{pP} : p \in P\}$ is dense in Ω_P.*

Proof. Basic open sets in Ω_P are of the form

$$U(X; X_1, \dots, X_n) = \{\chi \in \Omega_P : \chi(X) = 1, \, \chi(X_i) = 0 \text{ for all } 1 \leq i \leq n\}.$$

Here $X, X_1, \dots, X_n$ are constructible ideals of P. Clearly, $U(X; X_1, \dots, X_n)$ is empty if $X = \bigcup_{i=1}^n X_i$. Thus, for a nonempty basic open set $U(X; X_1, \dots, X_n)$, we may choose $p \in X$ such that $p \notin \bigcup_{i=1}^n X_i$, and then $\chi_{pP} \in U(X; X_1, \dots, X_n)$. $\square$

Theorem 5.7.2. *Let P be a monoid with identity e which embeds into a group G. If P has trivial units $P^* = \{e\}$, then $G \curvearrowright \Omega_P$ is topologically free.*

Proof. For $p \in P$, let $\chi_{pP} \in \widehat{\mathcal{J}_P}$ be defined as in Lemma 5.7.1, i.e., $\chi_{pP}(X) = 1$ if and only if $pP \subseteq X$, for $X \in \mathcal{J}_P$. Assume that $g \in G$ satisfies $g.\chi_{pP} = \chi_{pP}$ for some $p \in P$. This equality only makes sense if $\chi_p \in U_{g^{-1}}$, i.e., there exists $s \in I_l(P)$ with $\sigma(s) = g$ and $\chi_p(s^{-1}s) = 1$. The latter condition is equivalent to $pP \subseteq \mathrm{dom}(s)$. Then

$$g.\chi_{pP}(X) = \chi_{pP}(s^{-1}Xs) = \chi_{pP}(s^{-1}(X \cap \mathrm{im}\,(s))) = \chi_{pP}(g^{-1}(X \cap \mathrm{im}\,(s))).$$

So for $X \in \mathcal{J}_P$,

$$g.\chi_{pP}(X) = 1$$

if and only if

$$pP \subseteq g^{-1}(X \cap \mathrm{im}\,(s)) = g^{-1}X \cap \mathrm{dom}(s).$$

But since $pP \subseteq \mathrm{dom}(s)$ holds, we have that $g.\chi_{pP}(X) = 1$ if and only if $pP \subseteq g^{-1}X$ if and only if $gpP \subseteq X$. Therefore, $\chi_{pP} = g.\chi_{pP}$ means that for $X \in \mathcal{J}_P$, we have $pP \subseteq X$ if and only if $gpP \subseteq X$. Note that $gpP = s(pP)$ lies in $\mathcal{J}_P$. Hence, for $X = pP$, we obtain $gpP \subseteq pP$, and for $X = gpP$, we get $pP \subseteq gpP$. Hence there exist $x, y \in P$ with

$$gp = px \text{ and } p = gpy.$$

So $p = gpy = pxy$ and $gp = px = gpyx$. Thus, $xy = yx = e$. Hence $x, y \in P^*$. Since $P^* = \{e\}$ by assumption, we must have $x = y = e$, and hence $gp = p$. This implies $g = e$. In other words, for every $e \neq g \in G$, we have $g.\chi_{pP} \neq \chi_{pP}$ for all $p \in P$ such that $\chi_{pP} \in U_{g^{-1}}$. Hence it follows that

$$\left\{\chi \in U_{g^{-1}} : g.\chi \neq \chi\right\} \text{ contains } \left\{\chi_{pP} \in U_{g^{-1}} : p \in P\right\},$$

and the latter set is dense in $U_{g^{-1}}$ as $\{\chi_{pP} : p \in P\}$ is dense in Ω_P. □

Note that $G \curvearrowright \Omega_P$ can be topologically free if $P^* \neq \{e\}$. For instance, partial dynamical systems attached to $ax + b$-semigroups over rings of algebraic integers in number fields are shown to be topologically free in [EL13]. A generalization of this result is obtained in [Li16c, Proposition 5.8].

By [ELQ02, Theorem 2.6] and because of Theorem 5.6.41, we obtain the following:

Corollary 5.7.3. *Suppose that P is a monoid with trivial units which embeds into a group. Let I be an ideal of $C^*_\lambda(P)$. If $I \cap D_\lambda(P) = (0)$, then $I = (0)$.*

In other words, a representation of $C^*_\lambda(P)$ is faithful if and only if it is faithful on $D_\lambda(P)$.

Let us now discuss boundary quotients. We start with general inverse semigroups (with or without zero). In many situations, we are not only interested in the reduced C^*-algebra of an inverse semigroup, but also in its boundary quotient.

This is a notion going back to Exel (see [Exe08, Exe09, Exe15, EGS12]). Let us recall the construction. Given a semilattice E, let $\widehat{E}_{\max}$ be the subset of $\widehat{E}$ consisting of those $\chi \in \widehat{E}$ such that $\{e \in E : \chi(e) = 1\}$ is maximal among all characters $\chi \in \widehat{E}$. Note that if E is a semilattice without zero, then $\widehat{E}_{\max}$ consists of only one element, namely, the character χ satisfying $\chi(e) = 1$ for all $e \in E$. For later purposes, we make the following observation:

Lemma 5.7.4. *Let E be a semilattice with zero, and let 0 be its distinguished zero element. Suppose that $\chi \in \widehat{E}_{\max}$ satisfies $\chi(e) = 0$ for some $e \in E^{\times}$. Then there exists $f \in E^{\times}$ with $\chi(f) = 1$ and $ef = 0$.*

Proof. If every $f \in E^{\times}$ with $\chi(f) = 1$ satisfies $ef \neq 0$, then we can define a filter $\mathcal{F}$ by defining, for every $\tilde{f} \in E^{\times}$,

$$\tilde{f} \in \mathcal{F} \text{ if there exists } f \in E^{\times} \ \text{ with } \chi(f) = 1 \text{ and } ef \leq \tilde{f}.$$

It is obvious that $\mathcal{F}$ is a filter, so that there exists a character $\chi_F \in \widehat{E}$ with $\chi_F^{-1} = \mathcal{F}$. By construction,

$$\left\{f \in E^{\times} : \chi(f) = 1\right\} \subseteq \left\{f \in E^{\times} : \chi_F(f) = 1\right\},$$

but $\chi_F(e) = 1$ while $\chi(e) = 0$. This contradicts maximality of

$$\{f \in E : \chi(f) = 1\}. \qquad \square$$

We define

$$\partial\widehat{E} := \overline{\widehat{E}_{\max}} \subseteq \widehat{E}.$$

Now let E be the semilattice of idempotents in an inverse semigroup S. As $\partial\widehat{E} \subseteq \widehat{E}$ is closed, we obtain a short exact sequence

$$0 \to I \to C_0(\widehat{E}) \to C_0(\partial\widehat{E}) \to 0.$$

Now there are two options. We could view I as a subset of $C^*_\lambda(S)$ and form the ideal $\langle I \rangle$ of $C^*_\lambda(S)$ generated by I. The boundary quotient in Exel's sense (see [Exe08, Exe09, Exe15, EGS12]) is given by

$$\partial C^*_\lambda(S) := C^*_\lambda(S) / \langle I \rangle.$$

Alternatively, we could take the universal groupoid $\mathcal{G}(S)$ of our inverse semigroup, form its restriction to $\partial\widehat{E}$,

$$\mathcal{G}(S) \,|\, \partial\widehat{E} := \left\{\gamma \in \mathcal{G}(S) : r(\gamma), s(\gamma) \in \partial\widehat{E}\right\},$$

and form the reduced groupoid C^*-algebra

$$C^*_r(\mathcal{G}(S) \,|\, \partial\widehat{E}).$$

As the canonical homomorphism

$$C^*_\lambda(S) \cong C^*_r(\mathcal{G}(S)) \twoheadrightarrow C^*_r(\mathcal{G}(S) \,|\, \partial\widehat{E})$$

contains $\langle I \rangle$ in its kernel, we obtain canonical projections

$$C^*_\lambda(S) \twoheadrightarrow C^*_\lambda(S)/\langle I \rangle \twoheadrightarrow C^*_r(\mathcal{G}(S) \,|\, \partial\widehat{E}).$$

Under an exactness assumption, the second $*$-homomorphism actually becomes an isomorphism, so that our two alternatives for the boundary quotient coincide. For our purposes, it is more convenient to work with $C^*_r(\mathcal{G}(S) \,|\, \partial\widehat{E})$ because it is, by its very definition, a reduced groupoid C^*-algebra, so that groupoid techniques apply.

Now let us assume that our inverse semigroup S admits an idempotent pure partial homomorphism $\sigma : S^\times \to G$ to a group G. In that situation, we can define the partial dynamical system $G \curvearrowright \widehat{E}$ (see §5.5.2) and identify $\mathcal{G}(S)$ with the partial transformation groupoid $G \ltimes \widehat{E}$ (see Lemma 5.5.22). We have the following:

Lemma 5.7.5. *Let S be an inverse semigroup with an idempotent pure partial homomorphism to a group G. Let $G \curvearrowright \widehat{E}$ be its partial dynamical system. Then $\partial\widehat{E}$ is G-invariant.*

Proof. Let us first show that for every $g \in G$,

$$g.(U_{g^{-1}} \cap \widehat{E}_{\max}) \subseteq U_g \cap \widehat{E}_{\max}.$$

Take $\chi \in \widehat{E}_{\max}$ with $\chi(s^{-1}s) = 1$ for some $s \in S$ with $\sigma(s) = g$. Then $g.\chi(e) = \chi(s^{-1}es)$. Assume that $g.\chi \notin \widehat{E}_{\max}$. This means that there is $\psi \in \widehat{E}_{\max}$ such that $\psi(e) = 1$ for all $e \in E$ with $g.\chi(e) = 1$, and there exists $f \in E$ with $\psi(f) = 1$ but $\chi(s^{-1}fs) = 0$. Then $\psi \in U_g$ since $g.\chi(ss^{-1}) = 1$, which implies $\psi(ss^{-1}) = 1$. Consider $g^{-1}.\psi$ given by $g^{-1}.\psi(e) = \psi(ses^{-1})$. Then for every $e \in E$, $\chi(e) = 1$ implies $\chi(s^{-1}ses^{-1}s) = 1$, hence $\chi(s^{-1}(ses^{-1})s) = 1$, so that $g^{-1}.\psi(e) = \psi(ses^{-1}) = 1$. But $\chi(s^{-1}fs) = 0$ and $g^{-1}.\psi(s^{-1}fs) = \psi(ss^{-1}fss^{-1}) = \psi(f) = 1$. This contradicts $\chi \in \widehat{E}_{\max}$. Hence $g.(U_{g^{-1}} \cap \widehat{E}_{\max}) \subseteq U_g \cap \widehat{E}_{\max}$.

To see that

$$g.(U_{g^{-1}} \cap \partial\widehat{E}) \subseteq U_g \cap \partial\widehat{E},$$

let $\chi \in U_{g^{-1}} \cap \partial\widehat{E}$ and choose a net $(\chi_i)_i$ in $\widehat{E}_{\max}$ with $\lim_i \chi_i = \chi$. As $U_{g^{-1}}$ is open, we may assume that all the χ_i lie in $U_{g^{-1}}$. Then $g.\chi_i \in \widehat{E}_{\max}$, and $\lim_i g.\chi_i = g.\chi$. This implies $g.\chi \in \partial\widehat{E}$. □

Corollary 5.7.6. *In the situation of Lemma 5.7.5, we have canonical isomorphisms*

$$\mathcal{G}(S) \,|\, \partial\widehat{E} \cong G \ltimes \partial\widehat{E}$$

and

$$C^*_r(\mathcal{G}(S) \,|\, \partial\widehat{E}) \cong C_0(\partial\widehat{E}) \rtimes_r G.$$

Proof. The first identification follows immediately from Lemma 5.7.5, while the second one is a consequence of the first one and Theorem 5.5.21. □

Let us now specialize to the case where S is the left inverse hull of a left cancellative semigroup P. First, we observe the following:

Lemma 5.7.7. *We have* $\partial \widehat{\mathcal{J}_P} \subseteq \Omega_P$.

Proof. Let $X, X_1, \dots, X_n \in \mathcal{J}_P$ satisfy $X = \bigcup_{i=1}^n X_i$. Then for $\chi \in (\widehat{\mathcal{J}_P})_{\max}$, $\chi(X_i) = 0$ implies that there exists $X_i' \in \mathcal{J}$ with $\chi(X_i') = 1$ and $X_i \cap X_i' = \emptyset$ (see Lemma 5.7.4). Thus, if $\chi(X_i) = 0$ for all $1 \leq i \leq n$, then let X_i', $1 \leq i \leq n$ be as above. Then for $X' = \bigcap_{i=1}^n X_i'$, $\chi(X') = 1$ and $X \cap X' = \emptyset$. Thus, $\chi(X) = 0$. This shows $(\widehat{\mathcal{J}_P})_{\max} \subseteq \Omega_P$. As Ω_P is closed, we conclude that $\partial \widehat{\mathcal{J}_P} \subseteq \Omega_P$. □

Definition 5.7.8. We write $\partial \Omega_P := \partial \widehat{\mathcal{J}_P}$.

For simplicity, let us now restrict to semigroups that embed into groups.

Definition 5.7.9. We call $C_r^*(\mathcal{G}(I_l(P)) \,|\, \partial \Omega_P)$ the *boundary quotient* of $C_\lambda^*(P)$, and denote it by $\partial C_\lambda^*(P)$.

Note that by Corollary 5.7.6, given a semigroup P embedded into a group G, we have a canonical isomorphism

$$\partial C_\lambda^*(P) \cong C(\partial \Omega_P) \rtimes_r G.$$

Let us discuss some examples. Assume that our semigroup P is cancellative, and that it is left reversible, i.e., $pP \cap qP \neq \emptyset$ for all $p, q \in P$. This is, for instance, the case for positive cones in totally ordered groups. Given such a semigroup, we know because of Lemma 5.6.43 that $\mathcal{J}_P$ is a semilattice without zero, so that $(\widehat{\mathcal{J}_P})_{\max}$ degenerates to a point. Therefore, $\partial \Omega_P$ degenerates to a point. Hence it follows that the boundary quotient $\partial C_\lambda^*(P)$ coincides with the reduced group C^*-algebra of the group of right quotients of P.

For the nonabelian free monoid $\mathbb{N} * \mathbb{N}$ on two generators, the boundary quotient $\partial C_\lambda^*(\mathbb{N} * \mathbb{N})$ is canonically isomorphic to the Cuntz algebra $\mathcal{O}_2$. More generally, boundary quotients for right-angled Artin monoids are worked out and studied in [CL07].

Given an integral domain R, the boundary quotient $\partial C_\lambda^*(R \rtimes R^\times)$ of the $ax+b$-semigroup over R is canonically isomorphic to the ring C^*-algebra $\mathfrak{A}_r[R]$ of R (see [CL10, CL11a, Li10]). It is given as follows:

Consider the Hilbert space $\ell^2 R$ with canonical orthonormal basis $\{\delta_x : x \in R\}$. For every $a \in R^\times$, define $S_a(\delta_x) := \delta_{ax}$, and for every $b \in R$, define $U^b(\delta_x) := \delta_{b+x}$. Then the ring C^*-algebra of R is the C^*-algebra generated by these two families of operators, i.e,

$$\mathfrak{A}_r[R] := C^*(\{S_a : a \in R^\times\} \cup \{U^b : b \in R\}) \subseteq \mathcal{L}(\ell^2 R).$$

We refer to [CL10, CL11a, Li10] and also [Li13, §8.3] for details.

Let us now establish structural properties for boundary quotients. From now on, let us suppose that our semigroup P embeds into a group G.

Lemma 5.7.10. *$\partial\Omega_P$ is the minimal nonempty closed G-invariant subspace of $\widehat{\mathcal{J}_P}$.*

Proof. Let $C \subseteq \widehat{\mathcal{J}_P}$ be nonempty, closed and G-invariant. Let $\chi \in (\widehat{\mathcal{J}_P})_{\max}$ be arbitrary, and choose $X \in \mathcal{J}_P$ with $\chi(X) = 1$. Choose $p \in X$ and $\chi \in C$. As $U_{p^{-1}} = \widehat{\mathcal{J}_P}$, we can form $p.\chi$, and we know that $p.\chi \in C$. We have $p.\chi(pP) = \chi(P) = 1$, so that $p.\chi(X) = 1$ as $p \in X$ implies $pP \subseteq X$ (X is a right ideal). Set $\chi_X := p.\chi$. Consider the net $(\chi_X)_X$ indexed by $X \in \mathcal{J}$ with $\chi(X) = 1$, ordered by inclusion. Passing to a convergent subnet if necessary, we may assume that $\lim_X \chi_X$ exists. But it is clear because of $\chi \in (\widehat{\mathcal{J}_P})_{\max}$ that $\lim_X \chi_X = \chi$. As $\chi_X \in C$ for all X, we deduce that $\chi \in C$. Thus, $(\widehat{\mathcal{J}_P})_{\max} \subseteq C$, and hence $\partial\Omega_P \subseteq C$. □

In particular, $\partial\Omega_P$ is the minimal nonempty closed G-invariant subspace of Ω_P. Another immediate consequence is:

Corollary 5.7.11. *The transformation groupoid $G \ltimes \partial\Omega_P$ is minimal.*

To discuss topological freeness of $G \curvearrowright \partial\Omega_P$, let

$$G_0 = \left\{g \in G : \, X \cap gP \neq \emptyset \neq X \cap g^{-1}P \text{ for all } \emptyset \neq X \in \mathcal{J}_P\right\},$$

as in [Li13, §7.3]. Clearly,

$$G_0 = \left\{g \in G : \, pP \cap gP \neq \emptyset \neq pP \cap g^{-1}P \text{ for all } p \in P\right\}.$$

Furthermore, we have the following:

Lemma 5.7.12. *G_0 is a subgroup of G.*

Proof. Take g_1, g_2 in G_0. Then for all $\emptyset \neq X \in \mathcal{J}$, we have

$$\begin{aligned}((g_1g_2)P) \cap X &= g_1((g_2P) \cap (g_1^{-1}X)) \\ &\supseteq g_1((g_2P) \cap (g_1^{-1}X)) \cap (g_1P) = g_1((g_2P) \cap ((g_1^{-1}X) \cap P)).\end{aligned}$$

Now

$$(g_1^{-1}X) \cap P = g_1^{-1}(X \cap (g_1P)) \neq \emptyset.$$

Thus there exists $x \in P$ such that $x \in (g_1^{-1}X) \cap P$. Hence $xP \subseteq (g_1^{-1}X) \cap P$. Thus

$$\emptyset \neq g_1((g_2P) \cap (xP)) \subseteq ((g_1g_2)P) \cap X. \qquad \square$$

Proposition 5.7.13. *$G \curvearrowright \partial\Omega_P$ is topologically free if and only if $G_0 \curvearrowright \partial\Omega_P$ is topologically free.*

Proof. "$\Rightarrow$" is clear. For "$\Leftarrow$", assume that $G_0 \curvearrowright \partial\Omega_P$ is topologically free, and suppose that $G \curvearrowright \partial\Omega_P$ is not topologically free, i.e., there exists $g \in G$ and $U \subseteq U_{g^{-1}} \cap \partial\Omega_P$ such that $g.\chi = \chi$ for all $\chi \in U$. As $(\widehat{\mathcal{J}_P})_{\max} = \partial\Omega_P$, we can find $\chi \in U_{g^{-1}} \cap (\widehat{\mathcal{J}_P})_{\max}$ with $g.\chi = \chi$.

For every $X \in \mathcal{J}_P$ with $\chi(X) = 1$, choose $x \in X$ and $\psi_X \in (\widehat{\mathcal{J}_P})_{\max}$ with $\psi_X(xP) = 1$, so that $\psi_X(X) = 1$. Consider the net $(\psi_X)_X$ indexed by $X \in \mathcal{J}_P$ with $\chi(X) = 1$, ordered by inclusion. Passing to a convergent subnet if necessary, we may assume that $\lim_X \psi_X = \chi$. As U is open, we may assume that $\psi_X \in U$ for all X. Then $\psi_X(xP) = 1$ implies that $\psi_X \in U_x \cap U$.

Hence for sufficiently small $X \in \mathcal{J}_P$ with $\chi(X) = 1$, there exists $x \in X$ such that $x^{-1}.(U_x \cap U)$ is a nonempty open subset of $\partial\Omega_P$. We conclude that $(x^{-1}gx).\psi = \psi$ for all $\psi \in x^{-1}.(U_x \cap U)$. This implies that $x^{-1}gx \notin G_0$ as $G_0 \curvearrowright \partial\Omega_P$ is topologically free. So there exists $p \in P$ with

$$pP \cap x^{-1}gxP = \emptyset \text{ or } pP \cap x^{-1}g^{-1}xP = \emptyset.$$

Let $\chi_X \in (\widehat{\mathcal{J}_P})_{\max}$ satisfy $\chi_X(xpP) = 1$. If $pP \cap x^{-1}gxP = \emptyset$, then

$$xpP \cap gxP = \emptyset, \text{ so that } xpP \cap g^{-1}xpP = \emptyset.$$

Hence $g.\chi_X \neq \chi_X$ if $\chi_X \in U_{g^{-1}}$. If $pP \cap x^{-1}g^{-1}xP = \emptyset$, then

$$xpP \cap g^{-1}xP = \emptyset, \text{ so that } xpP \cap g^{-1}xpP = \emptyset.$$

Again, $g.\chi_X \neq \chi_X$ if $\chi_X \in U_{g^{-1}}$.

For every sufficiently small $X \in \mathcal{J}_P$ with $\chi(X) = 1$, we can find $x \in X$ and χ_X as above. Hence we can consider the net $(\chi_X)_X$ as above, and assume after passing to a convergent subnet that $\lim_X \chi_X = \chi$. As $\chi \in U \subseteq U_{g^{-1}} \cap \partial\Omega_P$, it follows that $\chi_X \in U \subseteq U_{g^{-1}} \cap \partial\Omega_P$ for sufficiently small X. So we obtain $g.\chi_X \neq \chi_X$, although g acts trivially on U. This is a contradiction. □

Corollary 5.7.14. *If $G_0 \curvearrowright \partial\Omega_P$ is topologically free, then $\partial C^*_\lambda(P)$ is simple.*

Proof. This follows from Lemma 5.7.10, Proposition 5.7.13 and [Ren80, Chapter II, Proposition 4.6]. □

We present a situation where Corollary 5.7.14 applies. Recall that we introduced the notion of "completeness for $\curvearrowright_R$" for presentations after Lemma 5.6.32. Moreover, a pair $P \subseteq G$ consisting of a monoid P embedded into a group G is called quasi-lattice ordered (see [Nic92]) if P has trivial units $P^* = \{e\}$ and for every $g \in G$ with $gP \cap P \neq \emptyset$, we can find an element $p \in P$ such that $gP \cap P = pP$.

Theorem 5.7.15. *Let $P = \langle \Sigma, R \rangle^+$ be a monoid given by a presentation (Σ, R) which is complete for $\curvearrowright_R$, in the sense of* [Deh03]. *Assume that for all $u \in \Sigma$, there is $v \in \Sigma$ such that there is no relation of the form $u \cdots = v \cdots$ in R. Also, suppose that P embeds into a group G such that $P \subseteq G$ is quasi-lattice ordered in the sense of* [Nic92]. *Then $G_0 = \{e\}$ and $\partial C^*_\lambda(P)$ is simple.*

Proof. In view of Corollary 5.7.14, it suffices to prove $G_0 = \{e\}$. Let $g \in G_0$. Assume that $gP \cap P \neq P$. Then $g \in G_0$ implies that this intersection is not empty. Hence, we must have $gP \cap P = pP$ for some $p \in P$ because $P \subseteq G$ is quasi-lattice ordered. If $p \neq e$, then there exists $u \in \Sigma$ with $pP \subseteq uP$. By assumption, there exists $v \in \Sigma$ such that no relation in R is of the form $u \cdots = v \cdots$. Because (Σ, R) is complete for $\curvearrowright_R$, we know that $uP \cap vP = \emptyset$ (see [Deh03, Proposition 3.3]), so that $gP \cap vP = \emptyset$. This contradicts $g \in G_0$. Hence, we must have $gP \cap P = P$, and similarly, $g^{-1}P \cap P = P$. These two equalities imply $g \in P^*$. But $P^* = \{e\}$ because $P \subseteq G$ is quasi-lattice ordered. Thus, $g = e$. $\square$

Theorem 5.7.15 implies that for every right-angled Artin monoid A_Γ^+ (see §5.3.3) with the property that (A_Γ, A_Γ^+) is graph-irreducible in the sense of [CL07], the boundary quotient $\partial C_\lambda^*(A_\Gamma^+)$ is simple.

Moreover, assume that we have a cancellative semigroup. By going over to the opposite semigroup, the left regular representation becomes the right regular representation. In this way, our discussion about C^*-algebra generated by left regular representations applies to C^*-algebras of right regular representations. In particular, we can define boundary quotients for C^*-algebras generated by right regular representations of semigroups. For instance, for the Thompson monoid

$$F^+ = \langle x_0, x_1, \dots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle^+ ,$$

it is easy to see that Theorem 5.7.15 applies to the opposite monoid, so that the boundary quotient of the C^*-algebra generated by the right regular representation of F^+ is simple.

We now turn to the property of pure infiniteness. As we mentioned, the boundary quotient $\partial C_\lambda^*(\mathbb{N} * \mathbb{N})$ is isomorphic to $\mathcal{O}_2$, a purely infinite C^*-algebra. We will now see that this is not a coincidence.

First of all, it is easy to see that for a partial dynamical system $G \curvearrowright X$, the transformation groupoid $G \ltimes X$ is purely infinite in the sense of [Mat15] if and only if every compact open subset of X is $(G, \mathcal{CO})$-paradoxical in the sense of [GS14, Definition 4.3], where $\mathcal{CO}$ is the set of compact open subsets of X. We recall that a nonempty subset $V \subseteq X$ is called $(G, \mathcal{CO})$-paradoxical in [GS14, Definition 4.3] if there exist

$$V_1, \dots, V_{n+m} \in \mathcal{O} \text{ and } t_1, \dots, t_{n+m} \in G$$

such that

$$\bigcup_{i=1}^{n} V_i = V = \bigcup_{i=n+1}^{m} V_i,$$

and

$$V_i \in U_{t_i^{-1}}, \ t_i.V_i \subseteq V, \text{ and } t_i.V_i \cap t_j.V_j = \emptyset \text{ for all } i \neq j.$$

Theorem 5.7.16. *The groupoid $G \ltimes \partial\Omega_P$ is purely infinite if and only if there exist $p, q \in P$ with $pP \cap qP = \emptyset$.*

Proof. Obviously, if $pP \cap qP \neq \emptyset$ for all $p, q \in P$, then $\partial\Omega_P$ degenerates to a point.

Let us prove the converse. Every compact open subset of $\widehat{\mathcal{J}_P}$ can be written as a disjoint union of basic open sets

$$U = \{\psi \in \partial\Omega_P : \psi(X) = 1, \psi(X_1) = \cdots = \psi(X_n) = 0\},$$

for some $X, X_1, \ldots, X_n \in \mathcal{J}_P$. Hence it suffices to show that U is $(G, \mathcal{CO})$-paradoxical. Since $(\widehat{\mathcal{J}_P})_{\max}$ is dense in $\partial\Omega_P$, there exists $\chi \in (\widehat{\mathcal{J}_P})_{\max}$ with $\chi \in U$. As χ lies in $(\widehat{\mathcal{J}_P})_{\max}$, $\chi(X_i) = 0$ implies that there exists $Y_i \in \mathcal{J}$ with $X_i \cap Y_i = \emptyset$ and $\chi(Y_i) = 1$ (see Lemma 5.7.4). Let

$$Y := X \cap \bigcap_{i=1}^{n} Y_i.$$

Certainly, $Y \neq \emptyset$ as $\chi(Y) = 1$. Moreover, for every $\psi \in \partial\Omega_P$, $\psi(Y) = 1$ implies $\psi \in U$. Now choose $x \in Y$. By assumption, we can find $p, q \in P$ with $pP \cap qP = \emptyset$. For $\psi \in \partial\Omega_P$, $xp.\psi(xpP) = \psi(P) = 1$. Similarly, for all $\psi \in \partial\Omega_P$, we have $xq.\psi(xqP) = 1$. Thus

$$\begin{aligned} &xp.U \subseteq xp.\partial\Omega_P \subseteq U, \ xq.U \subseteq xq.\partial\Omega_P \subseteq U \\ &\text{and } (xp.U) \cap (xq.U) \subseteq (xp.\partial\Omega_P) \cap (xq.\partial\Omega_P) = \emptyset \end{aligned}$$

since $xpP \cap xqP = \emptyset$. □

Corollary 5.7.17. *If P is not the trivial monoid, $P \neq \{e\}$, and if $G_0 \curvearrowright \partial\Omega_P$ is topologically free, then the boundary quotient $\partial C^*_\lambda(P)$ is a purely infinite simple C^*-algebra.*

Proof. First of all, by Corollary 5.7.14, the boundary quotient is simple.

Furthermore, we observe that our assumptions that $P \neq \{e\}$ and that G_0 acts topologically freely on $\partial\Omega_P$ imply that P is not left reversible: If P were left reversible, then $\partial\Omega_P$ would consist of only one point. Also, if P were left reversible, then we would have $P \subseteq G_0$. Since every element in P obviously leaves $\partial\Omega_P$ fixed, and by our assumption that $P \neq \{e\}$, we conclude that G_0 cannot act topologically freely on $\partial\Omega_P$ if P were left reversible. Hence Theorem 5.7.16 implies that the groupoid $G \ltimes \partial\Omega_P$ is purely infinite.

This, together with [GS14, Theorem 4.4], implies that the boundary quotient $\partial C^*_\lambda(P)$ is purely infinite. This completes our proof. □

Corollary 5.7.18. *If P is not the trivial monoid, $P \neq \{e\}$, if $G \ltimes \partial\Omega_P$ is amenable, and if $G_0 \curvearrowright \partial\Omega_P$ is topologically free, then the boundary quotient $\partial C^*_\lambda(P)$ is a unital UCT Kirchberg algebra.*

Proof. By assumption, our semigroups are countable, so that all the C^*-algebras we construct are separable. Clearly, the boundary quotient $\partial C_\lambda^*(P)$ is unital.

Since $G \ltimes \partial\Omega_P$ is amenable, the boundary quotient $\partial C_\lambda^*(P)$ is nuclear and satisfies the UCT.

Now our claim follows from Corollary 5.7.17 $\square$

Note that this shows that [Li13, Corollary 7.23] holds without the independence and the Toeplitz condition.

Let us now study simplicity of reduced semigroup C^*-algebras. Let P be a semigroup that embeds into a group. If $C_\lambda^*(P)$ is simple, then the groupoid $G \ltimes \Omega_P$ must be minimal, as $C_\lambda^*(P) \cong C_r^*(G \ltimes \Omega_P)$ (see the isomorphism (5.21)). In particular, we must have $\Omega_P = \partial\Omega_P$. This equality can be characterized in terms of the semigroup as follows:

Lemma 5.7.19. *Let P be a monoid. We have $\Omega_P = \partial\Omega_P$ if and only if for every $X_1, \dots, X_n \in \mathcal{J}_P$ with $X_i \subsetneq P$ for all $1 \le i \le n$, there exists $p \in P$ with $pP \cap X_i = \emptyset$ for all $1 \le i \le n$.*

Proof. Let χ_P be the character in Ω_P determined by $\chi_P(X) = 1$ if and only if $X = P$, for all $X \in \mathcal{J}_P$. Such a character exists in $\widehat{\mathcal{J}_P}$, and our assumption that P has an identity element ensures that χ_P lies in Ω_P. This is because an equation of the form

$$P = \bigcup_{i=1}^{n} X_i$$

for some $X_i \in \mathcal{J}_P$ implies that $X_i = P$ for some $1 \le i \le n$, as one of the X_i must contain the identity element.

First, we claim that $\Omega_P = \partial\Omega_P$ holds if and only if χ_P lies in $\partial\Omega_P$. This is certainly necessary. It is also sufficient as $\partial\Omega_P$ is G-invariant, and

$$\{p.\chi_P = \chi_{pP} : p \in P\}$$

is dense in Ω_P (see Lemma 5.7.1).

Now basic open subsets containing χ_P are of the form

$$U(P; X_1, \dots, X_n) = \{\chi \in \Omega_P : \chi(X_1) = \ldots = \chi(X_n) = 0\},$$

for $X_1, \dots, X_n \in \mathcal{J}_P$ with $X_i \subsetneq P$ for all $1 \le i \le n$.
Then $\chi_P \in \partial\Omega_P$ if and only if $\chi_P \in \overline{(\widehat{\mathcal{J}_P})_{\max}}$ if and only if for all $X_1, \dots, X_n \in \mathcal{J}_P$ with $X_i \subsetneq P$ for all $1 \le i \le n$, there is $\chi \in (\widehat{\mathcal{J}_P})_{\max}$ with $\chi \in U(P; X_1, \dots, X_n)$. Hence it follows that our proof is complete once we show that there exists $\chi \in (\widehat{\mathcal{J}_P})_{\max}$ with $\chi \in U(P; X_1, \dots, X_n)$ if and only if there exists $p \in P$ with $pP \cap X_i = \emptyset$ for all $1 \le i \le n$.

For "$\Rightarrow$", assume that $\chi \in (\widehat{\mathcal{J}_P})_{\max}$ lies in $\chi \in U(P; X_1, \dots, X_n)$. Then $\chi(X_i) = 0$ for all $1 \leq i \leq n$. But this means that there must exist $Y_i \in \mathcal{J}_P$, for $1 \leq i \leq n$, such that $\chi(Y_i) = 1$ and $X_i \cap Y_i = \emptyset$ (see Lemma 5.7.4). Take the intersection

$$Y := \bigcap_{i=1}^{n} Y_i.$$

As $\chi(Y) = 1$, Y is not empty. Therefore, we may choose some $p \in Y$. Obviously, $pP \subseteq Y$ as Y is a right ideal. Moreover, for every $1 \leq i \leq n$, we have

$$X_i \cap pP \subseteq X_i \cap Y \subseteq X_i \cap Y_i = \emptyset.$$

For "$\Leftarrow$", suppose that there exists $p \in P$ with $pP \cap X_i = \emptyset$ for all $1 \leq i \leq n$. An easy application of Zorn's Lemma yields a character $\chi \in (\widehat{\mathcal{J}_P})_{\max}$ with $\chi(pP) = 1$. Hence $\chi(X_i) = \emptyset$ for all $1 \leq i \leq n$, and it follows that χ lies in $U(P; X_1, \dots, X_n)$. □

Let us derive some immediate consequences.

Corollary 5.7.20. *If $G \curvearrowright \Omega_P$ is topologically free, then $C^*_\lambda(P)$ is simple if and only if for every $X_1, \dots, X_n \in \mathcal{J}_P$ with $X_i \subsetneq P$ for all $1 \leq i \leq n$, there exists $p \in P$ with $pP \cap X_i = \emptyset$ for all $1 \leq i \leq n$.*

Proof. This follows immediately from Corollary 5.7.11 and Lemma 5.7.19 (see also Corollary 5.7.14). □

Corollary 5.7.21. *Let P be a monoid with identity e, and suppose that P embeds into a group. Suppose that P has trivial units $P^* = \{e\}$. Then $C^*_\lambda(P)$ is simple if and only if for every $X_1, \dots, X_n \in \mathcal{J}_P$ with $X_i \subsetneq P$ for all $1 \leq i \leq n$, there exists $p \in P$ with $pP \cap X_i = \emptyset$ for all $1 \leq i \leq n$.*

Proof. This follows from Corollary 5.7.20 and Lemma 5.7.2. □

As an example, the countable free product $P = *_{i=1}^{\infty} \mathbb{N}$ satisfies the criterion in Lemma 5.7.19. Moreover, we obviously have $P^* = \{e\}$. Hence Corollary 5.7.21 applies, and we deduce that $C^*_\lambda(*_{i=1}^{\infty} \mathbb{N})$ is simple. Actually, $C^*_\lambda(*_{i=1}^{\infty} \mathbb{N})$ is canonically isomorphic to the Cuntz algebra $\mathcal{O}_\infty$.

5.8 The Toeplitz condition

So far, we were able to derive all our results about semigroup C^*-algebras just using descriptions as partial crossed products. However, it turns out that when we want to compute K-theory or the primitive ideal space, we need descriptions (at least up to Morita equivalence) as ordinary crossed products, attached to globally defined dynamical systems. Let us now introduce a criterion that guarantees such descriptions as ordinary crossed products.

Definition 5.8.1. Let $P \subseteq G$ be a semigroup embedded into a group G. We say that $P \subseteq G$ satisfies the *Toeplitz condition* (or simply that $P \subseteq G$ is Toeplitz) if for every $g \in G$ with $g^{-1}P \cap P \neq \emptyset$, the partial bijection

$$g^{-1}P \cap P \to P \cap gP, \; x \mapsto gx$$

lies in the inverse semigroup $I_l(P)$.

We can also think of $I_l(P)$ as partial isometries on $\ell^2 P$. In this picture, we can give an equivalent characterization of the Toeplitz condition. First, using the embedding $P \subseteq G$, we pass to the bigger Hilbert space $\ell^2 G$. Let 1_P be the characteristic function of P, viewed as an element in $\ell^\infty(G)$. Moreover, let λ be the left regular representation of G on $\ell^2 G$. Then $P \subseteq G$ is Toeplitz if and only if for every $g \in G$ with $1_P \lambda_g 1_P \neq 0$, we can write $1_P \lambda_g 1_P$ as a finite product of isometries and their adjoints from the set

$$\{V_p : p \in P\} \cup \left\{V_q^* : q \in P\right\}.$$

Let us now explain why the reduced semigroup C^*-algebra $C^*_\lambda(P)$ is a full corner in an ordinary crossed product if $P \subseteq G$ is Toeplitz. In terms of the partial dynamical system $G \curvearrowright \Omega_P$, this amounts to showing that if $P \subseteq G$ is Toeplitz, then $G \curvearrowright \Omega_P$ has an enveloping action, in the sense of [Aba03], on a locally compact Hausdorff space. This is because if $P \subseteq G$ is Toeplitz, then $g^{-1}P \cap P$ lies in the semilattice $\mathcal{J}_P$. Hence, for every $g \in G$,

$$U_{g^{-1}} = \left\{\chi \in \Omega_P : \chi(g^{-1}P \cap P) = 1\right\},$$

since among all $s \in I_l(P)^\times$ with $\sigma(s) = g$, $g^{-1}P \cap P$ is the maximal domain. This means that for every $g \in G$, the subspace $U_{g^{-1}}$ is clopen. Whenever this is the case, our partial dynamical system will have an enveloping action on a locally compact Hausdorff space. This follows easily from [Aba03].

In the following, we give a direct argument describing $C^*_\lambda(P)$ as a full corner in an ordinary crossed product in a very explicit way. First, we introduce some notation.

Fix an embedding $P \subseteq G$ of a semigroup P into a group G.

Definition 5.8.2. We let $\mathcal{J}_{P \subseteq G}$ be the smallest G-invariant semilattice of subsets of G containing $\mathcal{J}_P$.

Lemma 5.8.3. *We have*

$$\mathcal{J}_{P \subseteq G} = \left\{\bigcap_{i=1}^n g_i P : g_i \in G\right\}. \tag{5.23}$$

If $P \subseteq G$ is Toeplitz, then

$$\mathcal{J}_P^\times = \left\{\emptyset \neq Y \cap P : Y \in \mathcal{J}_{P \subseteq G}^\times\right\}.$$

Proof. Clearly,

$$\left\{ \bigcap_{i=1}^{n} g_i P : g_i \in G \right\}$$

is a G-invariant semilattice of subsets of G. It remains to show that it contains $\mathcal{J}_P$. It certainly includes the subset P of G. Moreover, for every subset $X \in P$ and all $p, q \in P$, we have $p(X) = pX$ and $q^{-1}(X) = q^{-1}X \cap P$. Here, pX and $q^{-1}X$ are products taken in G. Therefore, we see that $\mathcal{J}_{P \subseteq G}$ is closed under left multiplication and pre-images under left multiplication. But $\mathcal{J}_P$ may be characterized as the smallest semilattice of subsets of P containing P and closed under left multiplication and pre-images under left multiplication. Therefore, $\mathcal{J}_P$ is contained in $\mathcal{J}_{P \subseteq G}$.

Our argument above also shows that we always have

$$\mathcal{J}_P^{\times} \subseteq \left\{ \emptyset \neq Y \cap P : Y \in \mathcal{J}_{P \subseteq G}^{\times} \right\}.$$

Now let us assume that $P \subseteq G$ is Toeplitz, and let us prove "$\supseteq$". By assumption, the partial bijection

$$g^{-1}P \cap P \to P \cap gP, \, x \mapsto gx$$

lies in $I_l(P)$ as long as $g^{-1}P \cap P \neq \emptyset$. Therefore, as long as $g^{-1}P \cap P \neq \emptyset$, the image of this partial bijection, $P \cap gP$, lies in $\mathcal{J}_P$. Hence it follows, because of (5.23), that

$$\left\{ \emptyset \neq Y \cap P : Y \in \mathcal{J}_{P \subseteq G}^{\times} \right\}$$

is contained in $\mathcal{J}_P^{\times}$. □

Definition 5.8.4. We define

$$D_{P \subseteq G} := C^*(\{1_Y : Y \in \mathcal{J}_{P \subseteq G}\}) \subseteq \ell^\infty(G).$$

Obviously, $D_{P \subseteq G}$ is G-invariant with respect to the canonical action of G on $\ell^\infty(G)$ by left multiplication. Therefore, we can form the crossed product $D_{P \subseteq G} \rtimes_r G$. It is easy to see, and explained in [CEL15, §2.5], that we can identify this crossed product $D_{P \subseteq G} \rtimes_r G$ with the C^*-algebra

$$C^*(\{1_Y \lambda_g : Y \in \mathcal{J}_{P \subseteq G}, \, g \in G\}) \subseteq \mathcal{L}(\ell^2 G)$$

concretely represented on $\ell^2 G$.

Proposition 5.8.5. *In the situation above, 1_P is a full projection in $D_{P \subseteq G} \rtimes_r G$. If $P \subseteq G$ is Toeplitz, then*

$$C_\lambda^*(P) = 1_P(D_{P \subseteq G} \rtimes_r G)1_P. \tag{5.24}$$

In particular, $C_\lambda^*(P)$ is a full corner in $D_{P \subseteq G} \rtimes_r G$.

Equation (5.24) is meant as an identity of sub-C^*-algebras of $\mathcal{L}(\ell^2 G)$.

Proof. As the linear span of elements of the form

$$1_Y \lambda_g, \; Y \in \mathcal{J}_{P \subseteq G}, \, g \in G$$

is dense in $D_{P \subseteq G} \rtimes_r G$, it suffices to show that, for all $Y \in \mathcal{J}_{P \subseteq G}$ and $g \in G$,

$$1_Y \lambda_g \in (D_{P \subseteq G} \rtimes_r G) \, 1_P \, (D_{P \subseteq G} \rtimes_r G)$$

in order to show that 1_P is a full projection. Let

$$Y = \bigcap_{i=1}^{n} g_i P.$$

Then

$$1_Y \lambda_g = (\lambda_{g_1} 1_P) \, 1_P \left(\lambda_{g_1}^* 1_{\bigcap_{i=2}^n g_i P} \lambda_g \right)$$

lies in

$$(D_{P \subseteq G} \rtimes_r G) \, 1_P \, (D_{P \subseteq G} \rtimes_r G) \, .$$

Let us prove that

$$C_\lambda^*(P) = 1_P (D_{P \subseteq G} \rtimes_r G) 1_P$$

if $P \subseteq G$ is Toeplitz. First, observe that "$\subseteq$" always holds as for all $p \in P$, we have $V_p = 1_P \lambda_p 1_P$. Conversely, it suffices to show that for every $Y \in \mathcal{J}_{P \subseteq G}$ and $g \in G$, $1_P 1_Y \lambda_g 1_P$ lies in $C_\lambda^*(P)$. But

$$1_P 1_Y \lambda_g 1_P = (1_P 1_Y 1_P) \, (1_P \lambda_g 1_P) \, ,$$

and $1_P 1_Y 1_P$ lies in $C_\lambda^*(P)$ as $P \cap Y$ lies in $\mathcal{J}_P$ as long as it is not empty by Lemma 5.8.3, and $1_P \lambda_g 1_P$ lies in $C_\lambda^*(P)$ because $P \subseteq G$ is Toeplitz. $\square$

Let us discuss some examples. First, assume that P is cancellative, and right reversible, i.e., $Pp \cap Pq \neq \emptyset$ for all $p, q \in P$. Then P embeds into its group G of left quotients. We have $G = P^{-1} P$. We claim that $P \subseteq G$ is Toeplitz in this case: Take $g \in G$, and write $g = q^{-1} p$ for some $p, q \in P$. Then the partial bijection

$$g^{-1} P \cap P \to P \cap gP, \, x \mapsto gx$$

is the composition of

$$q^{-1} : \, qP \to P, \, qx \mapsto x \text{ and } p : \, P \to pP, \, x \mapsto px.$$

This is because

$$g^{-1} P \cap P = p^{-1} q P \cap P = p^{-1}(qP) \cap P = p^{-1}(\mathrm{dom}(q^{-1})) = \mathrm{dom}(q^{-1} p),$$

and for $x \in g^{-1} P \cap P = \mathrm{dom}(q^{-1} p)$, we have $gx = q^{-1} p x = (q^{-1} p)(x)$.

In particular, if P is the positive cone in a totally ordered group G, then $P \subseteq G$ is Toeplitz. Also, the inclusion $B_n^+ \subseteq B_n$ of the Braid monoid into the corresponding Braid group is Toeplitz. Furthermore, if R is an integral domain with quotient field Q, then for the $ax+b$-semigroup $R \rtimes R^\times$, we have that $R \rtimes R^\times \subseteq Q \rtimes Q^\times$ is Toeplitz.

Let us discuss a second class of examples. Suppose that we have a monoid P with identity e, and that $P \subseteq G$ is an embedding of P into a group G. Furthermore, we assume that

$$\mathcal{J}_{P\subseteq G}^\times = \{gP : g \in G\}.$$

In this situation, we claim that $P \subseteq G$ is Toeplitz.
To see this, take $g \in G$. If $g^{-1}P \cap P \neq \emptyset$, then we can find $p \in P$ such that $g^{-1}P \cap P = pP$. This is because we have $\mathcal{J}_{P\subseteq G}^\times = \{gP : g \in G\}$ by assumption. Here, we used the hypothesis that P has an identity element. Therefore, we can find $q \in P$ with $g^{-1}q = p$. We now claim that the partial bijection

$$g^{-1}P \cap P \to P \cap gP,\ x \mapsto gx$$

is the composition of

$$p : P \to pP,\ x \mapsto px \text{ and } q^{-1} : qP \to P,\ qx \mapsto x.$$

This is because

$$g^{-1}P \cap P = qP = \mathrm{dom}(pq^{-1}),$$

and for $x \in g^{-1}P \cap P = \mathrm{dom}(pq^{-1})$, we have $gx = pq^{-1}x = (pq^{-1})(x)$.

In particular, for every graph Γ as in §5.3.3, the inclusion $A_\Gamma^+ \subseteq A_\Gamma$ of the right-angled Artin monoid in the corresponding right-angled Artin group is Toeplitz. For instance, the canonical embedding $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2$ is Toeplitz. Also, the inclusion $B_{k,l}^+ \subseteq B_{k,l}$ of the Baumslag–Solitar monoid into the corresponding Baumslag–Solitar group is Toeplitz, for $k, l \geq 1$. Moreover, the inclusion $F^+ \subseteq F$ of the Thompson monoid into the Thompson group is Toeplitz.

We make the following observation, which is an immediate consequence of our preceding discussion and Lemma 5.8.3:

Remark 5.8.6. *Suppose that P is a monoid that is embedded into a group G. If*

$$\mathcal{J}_P^\times = \{pP : p \in P\},$$

then $P \subseteq G$ is Toeplitz if and only if

$$\mathcal{J}_{P\subseteq G}^\times = \{gP : g \in G\}.$$

Let us present two examples of semigroup embeddings into groups that are not Toeplitz. In both of our examples, the semigroup will be given by the nonabelian free monoid $\mathbb{N} * \mathbb{N}$ on two generators.

First, consider the canonical homomorphism $\mathbb{N} * \mathbb{N} \to \mathbb{F}_2/\mathbb{F}_2''$. Here, $\mathbb{F}_2''$ is the second commutator subgroup of the nonabelian free group $\mathbb{F}_2$ on two generators. By [Hoc69], this canonical homomorphism $\mathbb{N} * \mathbb{N} \to \mathbb{F}_2/\mathbb{F}_2''$ is injective. We want to see that $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2/\mathbb{F}_2''$ is not Toeplitz.

Let us denote both the canonical generators of $\mathbb{N} * \mathbb{N}$ and $\mathbb{F}_2$ by a and b. We use the notation $[g,h] = ghg^{-1}h^{-1}$ for commutators. Obviously,

$$[(ab)^{-1},(ba)^{-1}][ba,bab][(ab)^{-1},(ba)^{-1}]^{-1}[ba,bab]^{-1}$$

lies in $\mathbb{F}_2''$. Thus,

$$(ba)(ab)[(ab)^{-1},(ba)^{-1}][ba,bab][(ab)^{-1},(ba)^{-1}]^{-1}[ba,bab]^{-1}(ab)^{-1}(ba)^{-1}$$

lies in $\mathbb{F}_2''$. Now set

$$\begin{aligned} p &= (ab)(ba)(ba)(bab) \\ q &= (ab)(ba)(bab)(ba) \\ x &= (ba)(ab)(bab)(ba) \\ y &= (ba)(ab)(ba)(bab). \end{aligned}$$

Then

$$\begin{aligned} &pq^{-1}yx^{-1} \\ &= (ba)(ab)[(ab)^{-1},(ba)^{-1}][ba,bab][(ab)^{-1},(ba)^{-1}]^{-1}[ba,bab]^{-1}(ab)^{-1}(ba)^{-1} \end{aligned}$$

lies in $\mathbb{F}_2''$. Therefore, we have $pq^{-1} = xy^{-1}$ in $\mathbb{F}_2/\mathbb{F}_2''$. Now we consider $g = pq^{-1}$. Obviously, $P \cap gP \neq \emptyset$ as $p \in gP$. Moreover, we know that for $P = \mathbb{N} * \mathbb{N}$, the nonempty constructible right ideals are given by $\mathcal{J}_P^{\times} = \{pP : p \in P\}$. Hence by Remark 5.8.6, if $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2/\mathbb{F}_2''$ were Toeplitz, we would have $P \cap gP = zP$ for some $z \in P$, as $P \cap gP$ must lie in $\mathcal{J}_P^{\times}$.

We already know that p lies in $P \cap gP$. Moreover, x lies in $P \cap gP$ as $x = gy$ in $\mathbb{F}_2/\mathbb{F}_2''$. But the only element $z \in P$ with $p \in zP$ and $x \in zP$ is the identity element $z = e$. This is because p starts with a while x starts with b.

Hence, if $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2/\mathbb{F}_2''$ were Toeplitz, we would have $P \cap gP = P$, or in other words, $P \subseteq gP$. In particular, the identity element $e \in P$ must be of the form $e = pq^{-1}r$ for some $r \in P$. Hence it would follow that $q = rp$ in $\mathbb{F}_2/\mathbb{F}_2''$, and therefore in $\mathbb{N} * \mathbb{N}$. But this is absurd as $p \neq q$ while p and q have the same word length with respect to the generators a and b.

All in all, this shows that $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2/\mathbb{F}_2''$ is not Toeplitz.

Our second example is given as follows: Again, we take $P = \mathbb{N} * \mathbb{N}$. But this time, we let our group be the Thompson group

$$F := \langle x_0, x_1, \dots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle .$$

Let a and b be the canonical free generators of $\mathbb{N}*\mathbb{N}$. Consider the homomorphism

$$\mathbb{N}*\mathbb{N} \to F,\ a \mapsto x_0,\ b \mapsto x_1.$$

This is an embedding. For instance, this follows from uniqueness of the normal form in [BG84, (1.3) in §1]. We claim that this embedding $\mathbb{N}*\mathbb{N} \hookrightarrow F$ is not Toeplitz.

To simplify notations, let us identify $\mathbb{N}*\mathbb{N}$ with the monoid $\langle x_0, x_1\rangle^+$ generated by x_0 and x_1 in F. Consider

$$q = x_0^4 x_1 \text{ and } p = x_0^3.$$

Set $g := pq^{-1}$. Then we have $p \in P \cap gP$. But we also have that $x_0 x_1 x_0^2$ lies in $P \cap gP$ because

$$x_0^3 x_1 x_0 x_1 = x_0^4 x_2 x_1 = x_0^4 x_1 x_3 \text{ in } F,$$

so that

$$pq^{-1} x_0^3 x_1 x_0 x_1 = pq^{-1} x_0^4 x_1 x_3 = p x_3 = x_0^3 x_3 = x_0 x_1 x_0^2 \text{ in } F.$$

If $\mathbb{N}*\mathbb{N} \hookrightarrow F$ were Toeplitz, we would have that $P \cap gP$ is of the form zP for some $z \in P$. The argument is the same as in the previous example. But as we saw that x_0^3 and $x_0 x_1 x_0^2$ both lie in $P \cap gP$, our element z can only be either the identity element e or the generator x_0.

If $z = e$, then we would have $P \cap gP = P$, hence the identity e must lie in gP. This means that there exists $r \in P$ with $e = gr = pq^{-1}r$ and therefore $q = rp$. But this is absurd.

If $z = x_0$, then we would have $P \cap gP = x_0 P$, hence $x_0 \in gP$. Thus there must exist an element $r \in P$ with $x_0 = gr = pq^{-1}r$, and thus $qp^{-1}x_0 = r$. We conclude that

$$r = qp^{-1}x_0 = x_0^4 x_1 x_0^{-3} x_0 = x_0^4 x_1 x_0^{-2}$$

so that

$$x_0^4 x_1 = r x_0^2.$$

But this is again absurd.
All in all, this shows that $\mathbb{N}*\mathbb{N} \hookrightarrow F$ is not Toeplitz.

Looking at the preceding two examples, and comparing with our observation above that the canonical embedding $\mathbb{N}*\mathbb{N} \hookrightarrow \mathbb{F}_2$ is Toeplitz, we get the feeling that it is easier for the universal group embedding of a semigroup to satisfy the Toeplitz condition than for any other group embedding. Indeed, this is true. Let us explain the reason. We need the following equivalent formulation of the Toeplitz condition:

Lemma 5.8.7. *Let P be a semigroup, and suppose that $P \subseteq G$ is an embedding of P into a group G. The inclusion $P \subseteq G$ satisfies the Toeplitz condition if and only if for all $p, q \in P$, there exists a partial bijection $s \in I_l(P)$ with $s(q) = p$ and the intersection $P \cap qp^{-1}P$, taken in G, is contained in the domain* $\mathrm{dom}(s)$*.*

Proof. If $g \in G$ satisfies $g^{-1}P \cap P \neq \emptyset$, then there exists p, q in P with $g^{-1}p = q$, i.e., $g = pq^{-1}$. This shows that $P \subseteq G$ is Toeplitz if and only if for all $p, q \in P$, the partial bijection

$$qp^{-1}P \cap P \to P \cap pq^{-1}P,\ x \mapsto pq^{-1}x$$

lies in $I_l(P)$. But this is precisely what our condition says. □

Corollary 5.8.8. *Suppose that we have a semigroup P with two group embeddings $P \hookrightarrow G$ and $P \hookrightarrow \tilde{G}$. Furthermore, assume that there is a group homomorphism $\tilde{G} \to G$ such that the diagram*

$$\begin{tikzcd} P \arrow[r, hook] \arrow[dr] & \tilde{G} \arrow[d] \\ & G \end{tikzcd} \tag{5.25}$$

commutes. Then if $P \hookrightarrow G$ is Toeplitz, then the inclusion $P \hookrightarrow \tilde{G}$ must be Toeplitz as well.

Proof. In our equivalent formulation of the Toeplitz condition (see Lemma 5.8.7), the only part that depends on the group embedding of our semigroup is the intersection $P \cap qp^{-1}P$. In our particular situation, the intersection $P \cap qp^{-1}P$ taken in $\tilde{G}$ is given by

$$\left\{x \in P : pq^{-1}x \in P \text{ in } \tilde{G}\right\},$$

while the intersection $P \cap qp^{-1}P$ taken in G is given by

$$\left\{x \in P : pq^{-1}x \in P \text{ in } G\right\}.$$

Because of the commutative diagram (5.25), the condition $pq^{-1}x \in P$ in $\tilde{G}$ implies the condition $pq^{-1}x \in P$ in G. Hence the intersection $P \cap qp^{-1}P$, taken in $\tilde{G}$, is contained in the intersection $P \cap qp^{-1}P$, taken in G, where we view both intersections as subsets of P. Our claim follows. □

As an immediate consequence, we obtain:

Corollary 5.8.9. *Let P be a semigroup that embeds into a group, and assume that $P \hookrightarrow G_{\text{univ}}$ is its universal group embedding. If $P \hookrightarrow G_{\text{univ}}$ does not satisfy the Toeplitz condition, then for any other embedding $P \hookrightarrow G$ of our semigroup into a group G, we must have that $P \hookrightarrow G$ does not satisfy the Toeplitz condition either.*

5.9 Graph products

We discuss the independence condition and the Toeplitz condition for graph products.

Let $\Gamma = (V, E)$ be a graph with vertices V and edges E. Assume that two vertices in V are connected by at most one edge, and no vertex is connected to itself. Hence we view E as a subset of $V \times V$. For every $v \in V$, let P_v be a submonoid of a group G_v. We then form the graph products

$$P := \Gamma_{v \in V} P_v$$

and

$$G := \Gamma_{v \in V} G_v,$$

as in §5.4.2. As explained in §5.4.2, we can think of P as a submonoid of G in a canonical way.

Our goal is to prove that if each of the individual semigroups P_v, for all $v \in V$, satisfy the independence condition, then the graph product P also satisfies the independence condition. Similarly, if each of the pairs $P_v \subseteq G_v$, for all $v \in V$, are Toeplitz, then the pair $P \subseteq G$ satisfies the Toeplitz condition as well. Along the way, we give an explicit description for the constructible right ideals of P.

We use the same notation as in §5.4.2.

5.9.1 Constructible right ideals

Let us start with some easy observations.

Lemma 5.9.1. *Let $x_1 \cdots x_s$ be a reduced expression for $x \in G$, with $x_i \in G_{v_i}$. Assume that $v_1, \dots, v_j \in V^i(x)$. Then for all $1 \leq i \leq j$, $x_1 \cdots x_{i-1} x_{i+1} \cdots x_s$ is a reduced expression (for $x_i^{-1} x$). Similarly, if $v_{s-j}, \dots, v_s \in V^f(x)$, then for all $1 \leq i \leq j$, $x_1 \cdots x_{s-i-1} x_{s-i+1} \cdots x_s$ is a reduced expression (for $x x_{s-i}^{-1}$).*

Proof. By assumption, the expressions $x_1 \cdots x_s$ and $x_i x_1 \cdots x_{i-1} x_{i+1} \cdots x_s$ are shuffle equivalent. In particular, the latter expression is reduced. Our first claim follows. The second assertion is proven analogously. □

Lemma 5.9.2. *For $w \in V$, let g be an element in G_w. Then for every $x \in G$, we have $g S_w^i(x) = S_w^i(gx)$.*

Proof. Let $x_1 \cdots x_s$ be a reduced expression for x. If $w \notin V^i(x)$, then Lemma 5.4.7 implies that $g x_1 \cdots x_s$ is a reduced expression for gx, and our claim follows. If $w \in V^i(x)$, we may assume that $x_1 = S_w^i(x)$. If $g x_1 \neq e$, then obviously $(g x_1) x_2 \cdots x_s$ is a reduced expression for gx, and we are done. If $g x_1 = e$, then $x_2 \cdots x_s$ is a reduced expression for gx by Lemma 5.9.1. Clearly, $w \notin V^i(gx)$, and our claim follows. □

Definition 5.9.3. Let $W \subseteq V$ be a subset with $W \times W \subseteq E$, i.e., for every w_1, w_2 in W, we have $(w_1, w_2) \in E$. Given constructible right ideals $X_w \in \mathcal{J}_{P_w}$ for every $w \in W$, we set

$$\left(\prod_{w \in W} X_w \right) \cdot P := \left\{ x \in P : \, S_w^i(x) \in X_w \text{ for all } w \in W \right\}.$$

If for some $w \in W$, we have $X_w = \emptyset$, then we set $\left(\prod_{w \in W} X_w\right) \cdot P = \emptyset$. If $W = \emptyset$, we set $\left(\prod_{w \in W} X_w\right) \cdot P = P$.

By construction, we clearly have

$$\left(\prod_{w \in W} X_w\right) \cdot P = \bigcap_{w \in W} (X_w \cdot P).$$

Lemma 5.9.4. *Assume that $X_w = p_1^{-1} q_1 \cdots p_n^{-1} q_n(P_w)$ for some $p_i, q_i \in P_w$. Then we have $X_w \cdot P = p_1^{-1} q_1 \cdots p_n^{-1} q_n(P)$. Here we view p_i, q_i as elements of P (via the canonical embedding $P_w \subseteq P$).*

Proof. We proceed inductively on n. The case $n = 0$ is trivial. Let p_i, q_i be elements of P_w, for $1 \leq i \leq n+1$. Set $Y_w := p_2^{-1} q_2 \cdots p_{n+1}^{-1} q_{n+1}(P_w)$. We compute

$$\begin{aligned}(q_1(Y_w)) \cdot P &= \left\{x \in P : S_w^i(x) \in q_1(Y_w)\right\} \\ &= \left\{x \in q_1 P : q_1^{-1}(x) \in Y_w \cdot P\right\} \\ &= \{x \in q_1 P : x \in q_1(Y_w \cdot P)\} \\ &= \left\{x \in P : x \in q_1 p_2^{-1} q_2 \cdots p_{n+1}^{-1} q_{n+1}(P)\right\}.\end{aligned}$$

Finally,

$$\begin{aligned}(p_1^{-1} q_1(Y_w)) \cdot P &= \left\{x \in P : S_w^i(x) \in p_1^{-1} q_1(Y_w)\right\} \\ &= \left\{x \in P : p_1 S_w^i(x) \in q_1(Y_w)\right\} \\ &= \left\{x \in P : S_w^i(p_1 x) \in q_1(Y_w)\right\} \qquad \text{by Lemma 5.9.2} \\ &= \{x \in P : p_1 x \in (q_1(Y_w)) \cdot P\} \\ &= p_1^{-1}(q_1(Y_w)) \cdot P = p_1^{-1} q_1 \cdots p_{n+1}^{-1} q_{n+1}(P). \qquad \square\end{aligned}$$

Lemma 5.9.5. *Assume that we are given $p \in P$ and W, $\{X_w : w \in W\}$ as in Definition 5.9.3. Assume that $\emptyset \neq p\left(\prod_{w \in W} X_w\right) \cdot P \neq P$. Then there exist $\tilde{p}$ in P, $\tilde{W} \subseteq V$ with $\tilde{W} \times \tilde{W} \subseteq E$, $\tilde{X}_w \in \mathcal{J}_{P_w}$ for $w \in \tilde{W}$ with*

- *$\tilde{W} \neq \emptyset$ and $\emptyset \neq \tilde{X}_w \neq P_w$ for every $w \in \tilde{W}$,*
- *either $\tilde{p} = e$ or for all $v \in V^f(\tilde{p})$, there exists $w \in \tilde{W}$ with $(v, w) \notin E$,*

such that

$$p\left(\prod_{w \in W} X_w\right) \cdot P = \tilde{p}\left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P.$$

Proof. We proceed inductively on the length $l(p)$ of p. If $l(p) = 0$, i.e., $p = e$, then for all $w \in W$, we must have $X_w \neq \emptyset$, and there must exist $w \in W$ with $X_w \neq P_w$. Thus, we can set

$$\tilde{W} := \{w \in W : X_w \neq P_w\} \text{ and } \tilde{X}_w := X_w \text{ for } w \in \tilde{W}.$$

Now assume that $l(p) > 0$. Without changing $p\left(\prod_{w \in W} X_w\right) \cdot P$, we can replace W by $\{w \in W : X_w \neq P_w\}$. So we may just as well assume that for every $w \in W$, we have $\emptyset \neq X_w \neq P_w$. If for every $v \in V^f(p)$, there exists $w \in W$ with $(v, w) \notin E$, then we can just set $\tilde{W} = W$ and $\tilde{X}_w = X_w$ for all $w \in \tilde{W}$. If not, then we choose $v \in V^f(p)$ with $(v, w) \in E$ for every $w \in W$. Let $p_1 \cdots p_r$ be a reduced expression for p, with $p_r \in P_v$. Set $X_v := P_v$ if $v \notin W$. Using Lemma 5.9.1 and Lemma 5.9.2, we deduce

$$\begin{aligned} p_r \left(\prod_{w \in W} X_w\right) \cdot P &= \left\{ y \in P : y = p_r x \text{ for some } x \in \left(\prod_{w \in W} X_w\right) \cdot P \right\} \\ &= \left\{ y \in P : S_w^i(y) \in X_w \text{ for all } v \neq w \in W \text{ and } S_v^i(y) \in p_r X_v \right\} \\ &= (p_r X_v) \cdot \left(\prod_{v \neq w \in W} X_w\right) \cdot P. \end{aligned}$$

Thus

$$p\left(\prod_{w \in W} X_w\right) \cdot P = (p_1 \cdots p_{r-1}) \left[(p_r X_v) \cdot \left(\prod_{v \neq w \in W} X_w\right) \right] \cdot P.$$

Now our claim follows once we apply the induction hypothesis with $p_1 \cdots p_{r-1}$ in place of p. $\square$

Definition 5.9.6. Assume that we are in the situation of Lemma 5.9.5, i.e., we are given $p \in P$ and W, $\{X_w : w \in W\}$ as in Definition 5.9.3. Assume that

$$\emptyset \neq p\left(\prod_{w \in W} X_w\right) \cdot P \neq P.$$

Then

$$p\left(\prod_{w \in W} X_w\right) \cdot P$$

is said to be in *standard form* if both conditions from the Lemma 5.9.5 are satisfied, i.e.,

- $W \neq \emptyset$ and $\emptyset \neq X_w \neq P_w$ for every $w \in W$,
- either $p = e$ or for all $v \in V^f(p)$, there exists $w \in W$ with $(v, w) \notin E$.

Lemma 5.9.7. *Assume that $p\left(\prod_{w \in W} X_w\right) \cdot P$ is in standard form. Given reduced expressions $p_1 \cdots p_r$ for p and $x_1 \cdots x_s$ for $x \in \left(\prod_{w \in W} X_w\right) \cdot P$, $p_1 \cdots p_r x_1 \cdots x_s$ is a reduced expression for px. In particular, if in addition $p \neq e$, then for every $v \in V^i(p)$, we have $S_v^i(px) = S_v^i(p)$.*

Proof. For our first claim, the case $p = e$ is trivial. So let us assume $p \neq e$. As $X_w \neq P_w$ for all $w \in W$, we know that $W \subseteq V^i(x)$, so that $V^f(p) \cap V^i(x) = \emptyset$ because $p \left(\prod_{w \in W} X_w\right) \cdot P$ is in standard form. Then our assertion that $p_1 \cdots p_r x_1 \cdots x_s$ is reduced follows from Lemma 5.4.7. $\square$

Proposition 5.9.8. *The nonempty constructible right ideals of P are precisely given by all the nonempty subsets of P of the form $p \left(\prod_{w \in W} X_w\right) \cdot P$, with $p \in P$ and W, $\{X_w : w \in W\}$ as in Definition 5.9.3.*

Proof. First, we prove that $p \left(\prod_{w \in W} X_w\right) \cdot P$ is constructible. It certainly suffices to check that $\left(\prod_{w \in W} X_w\right) \cdot P$ is constructible. But Lemma 5.9.4 tells us that $X_w \cdot P$ is constructible for every $w \in W$. Therefore, $\left(\prod_{w \in W} X_w\right) \cdot P = \bigcap_{w \in W} (X_w \cdot P)$ is constructible itself.
Secondly, we show that every nonempty constructible right ideal is of the form $p \left(\prod_{w \in W} X_w\right) \cdot P$. For this purpose, let $\mathcal{J}'$ be the set of all nonempty constructible right ideals which are of the form $p \left(\prod_{w \in W} X_w\right) \cdot P$. Clearly, P lies in $\mathcal{J}'$. Also, if $\emptyset \neq X \in \mathcal{J}'$ and $p \in P$, then obviously pX lies in $\mathcal{J}'$. It remains to prove that for $\emptyset \neq X \in \mathcal{J}'$ and $q \in P$, we have $q^{-1}(X) \in \mathcal{J}'$ if $q^{-1}(X) \neq \emptyset$. Since the set $\mathcal{J}_P^{\times}$ of nonempty constructible right ideals of P is minimal with respect to these properties, this would then show that $\mathcal{J}_P^{\times} \subseteq \mathcal{J}'$, as desired. By induction on $l(q)$, we may assume that $q \in P_v$, and it even suffices to consider the case $q \in P_v \setminus P_v^*$. For $X = p \left(\prod_{w \in W} X_w\right) \cdot P$, we want to show that $q^{-1}(X) = \emptyset$ or $q^{-1}(X) \in \mathcal{J}'$. We distinguish between the following cases:

1. $p = e$:

1.a) There exists $w \in W$ with $(v, w) \notin E$. Without loss of generality we may assume that $X_w \neq P_w$ for all $w \in W$. Then for every $x \in P$, $w \notin V^i(qx)$ since $v \in V^i(qx)$. Thus, $S_w^i(qx) = e \notin X_w$. Therefore,

$$q^{-1} \left(\prod_{w \in W} X_w \right) \cdot P = \emptyset.$$

1.b) We have $(v, w) \in E$ for all $w \in W$ and $v \notin W$. Then

$$\begin{aligned} q^{-1} \left(\prod_{w \in W} X_w \right) \cdot P &= \left\{ x \in P : S_w^i(qx) \in X_w \text{ for all } w \in W \right\} \\ &= \left\{ x \in P : S_w^i(x) \in X_w \text{ for all } w \in W \right\} \\ &= \left(\prod_{w \in W} X_w \right) \cdot P \in \mathcal{J}'. \end{aligned}$$

1.c) We have $(v,w) \in E$ for all $w \in W$ and $v \in W$. Then

$$q^{-1}\left(\prod_{w\in W} X_w\right)\cdot P = \left\{x \in P :\ S^i_w(qx) \in X_w \text{ for all } w \in W\right\}$$
$$= \left[(q^{-1}(X_v))\cdot\left(\prod_{v\neq w\in W} X_w\right)\right]\cdot P \in \mathcal{J}'.$$

2. $p \neq e$: We can clearly assume that

$$\emptyset \neq p\left(\prod_{w\in W} X_w\right)\cdot P \neq P.$$

By Lemma 5.9.5, we may assume that $p\left(\prod_{w\in W} X_w\right)\cdot P$ is in standard form. And because we have already finished the case $p = e$, we can in addition assume that $p \neq e$. Without loss of generality, we may assume $v \in V^i(p)$, as we would otherwise have $q^{-1}[p(\prod_{w\in W} X_w)\cdot P] = \emptyset$ or $q^{-1}[p(\prod_{w\in W} X_w]\cdot P] = p(\prod_{w\in W} X_w)\cdot P$. Lemma 5.9.7 gives $S^i_v(px) = S^i_v(p)$ for every $x \in \left(\prod_{w\in W} X_w\right)\cdot P$. Now y lies in $q^{-1}\left[p\left(\prod_{w\in W} X_w\right)\cdot P\right]$ if and only if there exists $x \in \left(\prod_{w\in W} X_w\right)\cdot P$ such that $qy = px$. Hence if there exists $y \in q^{-1}\left[p\left(\prod_{w\in W} X_w\right)\cdot P\right]$, we must have

$$qS^i_v(y) = S^i_v(qy) = S^i_v(px) = S^i_v(p)$$

by Lemma 5.9.2. Thus $p \in S^i_v(p)P \subseteq qP$. This implies that

$$q^{-1}\left[p\left(\prod_{w\in W} X_w\right)\cdot P\right] = (q^{-1}p)\left(\prod_{w\in W} X_w\right)\cdot P \in \mathcal{J}'. \qquad \square$$

5.9.2 The independence condition

Lemma 5.9.9. *Assume that*

$$\emptyset \neq p\left(\prod_{w\in W} X_w\right)\cdot P \neq P \text{ and } \emptyset \neq \tilde{p}\left(\prod_{w\in \tilde{W}} \tilde{X}_w\right)\cdot P \neq P$$

are in standard form, with $p \neq e$. If

$$\tilde{p}\left(\prod_{w\in \tilde{W}} \tilde{X}_w\right)\cdot P \subseteq p\left(\prod_{w\in W} X_w\right)\cdot P,$$

then $\tilde{p} \in pP$.

Proof. First of all, let us show that $\tilde{p} \neq e$. Namely, assume the contrary, i.e., $\tilde{p} = e$. Take $\tilde{x} \in \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P$. By assumption, we can find $x \in \left(\prod_{w \in W} X_w\right) \cdot P$ so that $\tilde{x} = px$. Moreover, choose $v \in V^i(p)$. By Lemma 5.9.7, it follows that $S_v^i(\tilde{x}) = S_v^i(px) = S_v^i(p)$. Thus, we have proven that every $\tilde{x} \in \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P$ must satisfy $S_v^i(\tilde{x}) = S_v^i(p)$. But this is obviously a wrong statement. Thus, we must have $\tilde{p} \neq e$.

Now we proceed inductively on $l(p)$. We start with the case $l(p) = 1$, i.e., $p \in P_v$. For $\tilde{x} \in \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P$ with $S_v^i(\tilde{p}\tilde{x}) = S_v^i(\tilde{p})$, we can always find $x \in \left(\prod_{w \in W} X_w\right) \cdot P$ so that $\tilde{p}\tilde{x} = px$. By Lemma 5.9.7, we deduce that $p = S_v^i(px) = S_v^i(\tilde{p}\tilde{x}) = S_v^i(\tilde{p})$. Therefore, $\tilde{p} \in S_v^i(\tilde{p})P = pP$.

For the induction step, take $v \in V^i(p)$. For $\tilde{x} \in \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P$ with $S_v^i(\tilde{p}\tilde{x}) = S_v^i(\tilde{p})$, again choose $x \in \left(\prod_{w \in W} X_w\right) \cdot P$ so that $\tilde{p}\tilde{x} = px$. Then $S_v^i(p) = S_v^i(px) = S_v^i(\tilde{p}\tilde{x}) = S_v^i(\tilde{p})$. This shows that both $\tilde{p}$ and p lie in $S_v^i(p)P$. We deduce that

$$(S_v^i(p)^{-1}\tilde{p}) \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P \subseteq (S_v^i(p)^{-1}p) \left(\prod_{w \in W} X_w\right) \cdot P.$$

Since $l(S_v^i(p)^{-1}p) < l(p)$, we can now apply the induction hypothesis, and we are done. □

Lemma 5.9.10. *As above, let*

$$\emptyset \neq \left(\prod_{w \in W} X_w\right) \cdot P \neq P \text{ and } \emptyset \neq \tilde{p} \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P \neq P$$

be in standard form, this time with $\tilde{p} \neq e$ (and $p = e$). If

$$\tilde{p} \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P \subseteq \left(\prod_{w \in W} X_w\right) \cdot P,$$

then

$$\tilde{p} \in \left(\prod_{w \in W} X_w\right) \cdot P.$$

Proof. For $\tilde{x} \in \left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P$ with $S_v^i(\tilde{p}\tilde{x}) = S_v^i(\tilde{p})$, $\tilde{p}\tilde{x}$ lies in $\left(\prod_{w \in W} X_w\right) \cdot P$ by assumption. Hence, Lemma 5.9.7 tells us that for all $w \in W$, $S_w^i(\tilde{p}) = S_w^i(\tilde{p}\tilde{x})$ lies in X_w. Thus, $\tilde{p}$ lies in $\left(\prod_{w \in W} X_w\right) \cdot P$. □

Lemma 5.9.11. *Let*

$$\emptyset \neq \left(\prod_{w \in W} X_w \right) \cdot P \neq P \text{ and } \emptyset \neq \left(\prod_{w \in \tilde{W}} \tilde{X}_w \right) \cdot P \neq P$$

be in standard form. Then

$$\left(\prod_{w \in \tilde{W}} \tilde{X}_w \right) \cdot P \subseteq \left(\prod_{w \in W} X_w \right) \cdot P$$

if and only if $W \subseteq \tilde{W}$ and $\tilde{X}_w \subseteq X_w$ for every $w \in W$.

Proof. The direction "$\Leftarrow$" is obvious. To prove the reverse direction, first assume that $W \nsubseteq \tilde{W}$. Choose for every $\tilde{w} \in \tilde{W}$ an element $x_{\tilde{w}} \in \tilde{X}_{\tilde{w}}$. Then the product $\prod_{\tilde{w} \in \tilde{W}} x_{\tilde{w}}$ obviously lies in $\left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P$. But for $w \in W \setminus \tilde{W}$, we have $S_w^i(\prod_{\tilde{w} \in \tilde{W}} x_{\tilde{w}}) = e \notin X_w$ as $X_w \neq P_w$. This contradicts $\left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P \subseteq \left(\prod_{w \in W} X_w\right) \cdot P$. So we must have $W \subseteq \tilde{W}$. If for some $w \in W$, we have $\tilde{X}_w \nsubseteq X_w$, then choose $x_w \in \tilde{X}_w \setminus X_w$. For all remaining $\tilde{w} \in \tilde{W} \setminus \{w\}$, choose $x_{\tilde{w}} \in \tilde{X}_{\tilde{w}}$. Then the product $\prod_{\tilde{w} \in \tilde{W}} x_{\tilde{w}}$ lies in $\left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P$. But $S_w^i(\prod_{\tilde{w} \in \tilde{W}} x_{\tilde{w}}) = x_w \notin X_w$. This again contradicts $\left(\prod_{w \in \tilde{W}} \tilde{X}_w\right) \cdot P \subseteq \left(\prod_{w \in W} X_w\right) \cdot P$. $\square$

Proposition 5.9.12. *If for every $v \in V$, the semigroup P_v satisfies independence, then the graph product P satisfies independence.*

Proof. Let

$$\emptyset \neq p \left(\prod_{w \in W} X_w \right) \cdot P \neq P$$

be in standard form, and let

$$\emptyset \neq p_i \left(\prod_{w \in W_i} X_w^{(i)} \right) \cdot P \neq P$$

be finitely many constructible right ideals of P in standard form. If

$$p \left(\prod_{w \in W} X_w \right) \cdot P = \bigcup_i p_i \left(\prod_{w \in W_i} X_w^{(i)} \right) \cdot P,$$

then either $p = e$ or $p_i \in pP$ for all i by Lemma 5.9.9. Hence

$$\left(\prod_{w \in W} X_w \right) \cdot P = \bigcup_i (p^{-1} p_i) \left(\prod_{w \in W_i} X_w^{(i)} \right) \cdot P$$

Therefore, we may without loss of generality assume that $p = e$, i.e.,

$$\left(\prod_{w \in W} X_w\right) \cdot P = \bigcup_i p_i \left(\prod_{w \in W_i} X_w^{(i)}\right) \cdot P. \tag{5.26}$$

Let $I = \{i : p_i \neq e\}$ and $J = \{i : p_i = e\}$. By Lemma 5.9.10, we have for all $i \in I$ and $w \in W$ that $S_w^i(p_i) \in X_w$. We define for every $i \in I$:

$$p_i' = \prod_{w \in W} S_w^i(p_i).$$

For each $i \in I$, we obviously have

$$p_i \left(\prod_{w \in W_i} X_w^{(i)}\right) \cdot P \subseteq p_i P \subseteq p_i' P \subseteq \left(\prod_{w \in W} X_w\right) \cdot P.$$

Therefore,

$$\left(\prod_{w \in W} X_w\right) \cdot P = \bigcup_{i \in I} (p_i' P) \cup \bigcup_{i \in J} \left(\prod_{w \in W_i} X_w^{(i)}\right) \cdot P.$$

Set $\tilde{W}_i := W$ if $i \in I$, $\tilde{W}_i := W_i$ for $i \in J$ and

$$\tilde{X}_w^{(i)} := \begin{cases} S_w^i(p_i) P_w & \text{if } i \in I,\ w \in \tilde{W}_i, \\ X_w^{(i)} & \text{if } i \in J,\ w \in \tilde{W}_i. \end{cases}$$

Since $\left(\prod_{w \in \tilde{W}_i} \tilde{X}_w^{(i)}\right) \cdot P = p_i' P$ for all $i \in I$, we obviously again have

$$\left(\prod_{w \in W} X_w\right) \cdot P = \bigcup_i \left(\prod_{w \in \tilde{W}_i} \tilde{X}_w^{(i)}\right) \cdot P. \tag{5.27}$$

Moreover, $\tilde{X}_w^{(i)} \neq P_w$ for all i and $w \in \tilde{W}_i$.

By Lemma 5.9.11, we must have $\tilde{X}_w^{(i)} \subseteq X_w$ for all i and $w \in \tilde{W}_i$. Assume that for all i with $\tilde{W}_i = W$, there exists $w(i) \in W$ with $\tilde{X}_{w(i)}^{(i)} \subsetneq X_{w(i)}$. Choose for every $w \in \{w(i)\}_i$ an element

$$x_w \in X_w \setminus \bigcup_{\{i:\, w(i)=w\}} \tilde{X}_{w(i)}^{(i)}.$$

This is possible since $\mathcal{J}_{P_v}$ is independent for every $v \in V$, so that

$$X_w \setminus \bigcup_{\{i:\, w(i)=w\}} \tilde{X}_{w(i)}^{(i)} \neq \emptyset.$$

For all remaining $w \in W$, just choose some $x_w \in X_w$. Then $x := \prod_{w \in W} x_w$ lies in $\left(\prod_{w \in W} X_w\right) \cdot P$, but for all i with $\tilde{W}_i = W$, $S^i_{w(i)}(x)$ does not lie in $\tilde{X}^{(i)}_{w(i)}$. Therefore, x does not lie in $\left(\prod_{w \in \tilde{W}_i} \tilde{X}^{(i)}_w\right) \cdot P$ whenever i satisfies $\tilde{W}_i = W$. For i with $\tilde{W}_i \neq W$, take $\tilde{w} \in \tilde{W}_i \setminus W$. Then $S^i_{\tilde{w}}(x) = e \notin \tilde{X}^{(i)}_{\tilde{w}}$. Thus also for i with $\tilde{W}_i \neq W$, we have $x \notin \left(\prod_{w \in \tilde{W}_i} \tilde{X}^{(i)}_w\right) \cdot P$. Since this contradicts (5.27), there must exist an index i with $W_i = W$ and $\tilde{X}^{(i)}_w = X_w$ for all $w \in W$. In particular, for that index i, we must have

$$\left(\prod_{w \in W} X_w\right) \cdot P = \left(\prod_{w \in \tilde{W}_i} \tilde{X}^{(i)}_w\right) \cdot P.$$

If this index i lies in I, then we have shown that $\left(\prod_{w \in W} X_w\right) \cdot P$ is a principal right ideal, and we are done. If this index i lies in J, then we have proven that $\left(\prod_{w \in W} X_w\right) \cdot P$ coincides with one of the (constructible right) ideals on the right-hand side of (5.26) (since $p_i = e$ for $i \in J$), and we are also done. □

5.9.3 The Toeplitz condition

Definition 5.9.13. Let $x \in G$, and assume that $x_1 \cdots x_s$ is a reduced expression for x. We set $S(x) := \{x_1, \dots, x_s\}$.

Note that this is well defined by Theorem 5.4.2.

Lemma 5.9.14. *Let $g, x \in G$, $v \in V^f(g)$, and assume $S^f_v(g) S^i_v(x) \neq e$. Then $S^f_v(g) S^i_v(x)$ lies in $S(gx)$.*

Proof. Let $g_1 \cdots g_r$ be a reduced expression for g, with $g_r = S^f_v(g)$.

First of all, if $V^f(g) \cap V^i(x) = \emptyset$, then Lemma 5.4.7 tells us that for every reduced expression $x_1 \cdots x_s$ for x, $g_1 \cdots g_r x_1 \cdots x_s$ is a reduced expression for gx. Hence $g_r = S^f_v(g) S^i_v(x)$ lies in $S(gx)$.

Secondly, assume that $V^f(g) \cap V^i(x) = \{v\}$. If $x_1 \cdots x_s$ is a reduced expression for x with $x_1 = S^i_v(x)$, then since $g_r x_1 \neq e$, Lemma 5.4.7 tells us that $g_1 \cdots g_{r-1}(g_r x_1) x_2 \cdots x_s$ is a reduced expression for gx. Again, our claim follows.

Finally, it remains to treat the case

$$\emptyset \neq V^f(g) \cap V^i(x) \neq \{v\}.$$

We proceed inductively on $l(g)$. The cases $l(g) = 0$ and $l(g) = 1$ are taken care of by the previous cases. As

$$\emptyset \neq V^f(g) \cap V^i(x) \neq \{v\},$$

we can choose $w \in V^f(g) \cap V^i(x)$ with $w \neq v$. If v lies in $V^f(g) \cap V^i(x)$, then choose a reduced expression $g_1 \cdots g_r$ for g with $g_{r-1} \in G_w$ and $g_r \in G_v$, and let $x_1 \cdots x_s$ be a reduced expression for x with $x_1 \in G_v$ and $x_2 \in G_w$. Then

$$gx = g_1 \cdots g_{r-2}(g_r x_1)(g_{r-1} x_2) x_3 \cdots x_s.$$

Set

$$g' := g_1 \cdots g_{r-2} g_r \text{ and } x' := x_1 (g_{r-1} x_2) x_3 \cdots x_s.$$

By Lemma 5.9.1, we know that $g_1 \cdots g_{r-2} g_r$ is a reduced expression, so that $g_r = S_v^f(g')$. Also, $x_1(g_{r-1}x_2)x_3 \cdots x_s$ is a reduced expression. This is clear if $g_{r-2}x_2 \neq e$, and it follows from Lemma 5.9.1 in case $g_{r-2}x_2 = e$. Thus, $x_1 = S_v^i(x')$. So we again have

$$S_v^f(g') S_v^i(x') = S_v^f(g) S_v^i(x) \neq e.$$

Since $l(g') < l(g)$, induction hypothesis tells us that $S_v^f(g) S_v^i(x) = S_v^f(g') S_v^i(x')$ lies in $S(g'x') = S(gx)$. The case $v \notin V^f(g) \cap V^i(x)$ is treated similarly. Just set $x_1 = e$. □

For $g \in G$, let us denote the partial bijection

$$g^{-1}P \cap P \to P \cap gP, \; x \mapsto gx$$

by g_P.

Lemma 5.9.15. *Let $g_1 \cdots g_r$ be a reduced expression for $g \in G$. Then*

$$g_P = (g_1)_P \cdots (g_r)_P.$$

Proof. We proceed inductively on $l(g)$. The case $l(g) = 1$ is trivial. First, we show that for $x \in P$, $gx \in P$ implies $g_r x \in P$. Let $g_r \in G_v$. Then by Lemma 5.9.14, $g_r S_v^i(x)$ lies in $S(gx)$ or $g_r S_v^i(x) = e$. Since $gx \in P$, we conclude that in any case, we have $g_r S_v^i(x) \in P_v$. Obviously, $S(g_r x) \subseteq \{g_r S_v^i(x)\} \cup S(x)$. So we obtain $g_r x \in P$. Therefore, we compute

$$\begin{aligned} \mathrm{dom}(g_P) &= \{x \in P : \, gx \in P\} = \{x \in P : \, gx \in P \text{ and } g_r x \in P\} \\ &= \{x \in P : \, g_r x \in P \text{ and } (g_1 \ldots g_{r-1})(g_r x) \in P\} \\ &= \mathrm{dom}((g_1 \cdots g_{r-1})_P \; g_P). \end{aligned}$$

Hence it follows that $g_P = (g_1 \cdots g_{r-1})_P \; g_P$.

By induction hypothesis, $(g_1 \cdots g_{r-1})_P = (g_1)_P \ldots (g_{r-1})_P$, and we are done. □

Lemma 5.9.16. *For $g \in G_v$, we have $g^{-1}P_v \cap P_v \neq \emptyset$ if and only if $g^{-1}P \cap P \neq \emptyset$. Assume that this is the case, and that there are p_i, q_i in G_v with*

$$g_{P_v} = p_1^{-1} q_1 \cdots p_n^{-1} q_n$$

in $I_l(P_v)$. Then

$$g_P = p_1^{-1} q_1 \cdots p_n^{-1} q_n$$

in $I_l(P)$.

Proof. Let us start proving the first claim. Since $P_v \subseteq P$, the implication "$\Rightarrow$" is obvious. For the reverse direction, assume that $g^{-1}P \cap P \neq \emptyset$, i.e., there exists $x \in P$ with $gx \in P$. Then obviously, $S_v^i(x) \in P_v$, and $gS_v^i(x) = S_v^i(gx)$ lies in P_v (here we used Lemma 5.9.2), so $g^{-1}P_v \cap P_v \neq \emptyset$.

Secondly, we show $g^{-1}P \cap P = q_n^{-1}p_n \cdots q_1^{-1}p_1(P)$:

$$\begin{aligned}
g^{-1}P \cap P &= \{x \in P : \, gx \in P\} = \left\{x \in P : \, S_v^i(gx) \in P_v\right\} \\
&= \left\{x \in P : \, gS_v^i(x) \in P_v\right\} \quad \text{by Lemma 5.9.2} \\
&= \left\{x \in P : \, S_v^i(x) \in g^{-1}P_v \cap P_v\right\} \\
&= \left\{x \in P : \, S_v^i(x) \in q_n^{-1}p_n \cdots q_1^{-1}p_1(P_v)\right\} \\
&= \left(q_n^{-1}p_n \cdots q_1^{-1}p_1(P_v)\right) \cdot P \\
&= q_n^{-1}p_n \cdots q_1^{-1}p_1(P) \quad \text{by Lemma 5.9.4.}
\end{aligned}$$

Therefore, we have

$$\mathrm{dom}(g_P) = \mathrm{dom}(p_1^{-1}q_1 \cdots p_n^{-1}q_n)$$

as subsets of P. Hence it follows that

$$g_P = p_1^{-1}q_1 \cdots p_n^{-1}q_n$$

in $I_l(P)$ because we have $p_1^{-1}q_1 \cdots p_n^{-1}q_n = g$ in $G_v \subseteq G$. Here we are taking products of p_i^{-1} and q_i as group elements in G_v and G. □

Proposition 5.9.17. *If for all $v \in V$, $P_v \subseteq G_v$ is Toeplitz, then $P \subseteq G$ is Toeplitz.*

Proof. Let $g_1 \cdots g_r$ be a reduced expression for $g \in G$, with $g_i \in G_{v_i}$. Assume that $g^{-1}P \cap P \neq \emptyset$. By Lemma 5.9.15, we know that

$$g_P = (g_1)_P \cdots (g_r)_P.$$

In particular, $g_i^{-1}P \cap P \neq \emptyset$ for all $1 \leq i \leq r$. By Lemma 5.9.16, we conclude that $g_i^{-1}P_{v_i} \cap P_{v_i} \neq \emptyset$ for all $1 \leq i \leq r$. Since for all $1 \leq i \leq r$, the embedding $P_{v_i} \subseteq G_{v_i}$ is Toeplitz, we can find $p_{i,j}$, $q_{i,j}$ in P_{v_i} (for $1 \leq j \leq n_i$) with

$$(g_i)_{P_{v_i}} = p_{i,1}^{-1}q_{i,1} \cdots p_{i,n_i}^{-1}q_{i,n_i} \text{ in } I_l(P_{v_i}).$$

Lemma 5.9.16 implies that

$$(g_i)_P = p_{i,1}^{-1}q_{i,1} \cdots p_{i,n_i}^{-1}q_{i,n_i} \text{ in } I_l(P)$$

for all $1 \leq i \leq r$. Thus we have, in $I_l(P)$:

$$g_P = (g_1)_P \cdots (g_r)_P = \left(p_{1,1}^{-1}q_{1,1} \cdots p_{1,n_1}^{-1}q_{1,n_1}\right) \cdots \left(p_{r,1}^{-1}q_{r,1} \cdots p_{r,n_r}^{-1}q_{r,n_r}\right). \qquad \square$$

5.10 K-theory

Let us apply the K-theory results from Chapter 3 to semigroups and their reduced semigroup C^*-algebras.

Let P be a semigroup that embeds into a group. Assume that P satisfies independence, and that we have an embedding $P \subseteq G$ into a group G such that $P \subseteq G$ is Toeplitz. Furthermore, suppose that G satisfies the Baum–Connes conjecture with coefficients.

As $\mathcal{J}_{P\subseteq G}^{\times} = G.\mathcal{J}_P^{\times}$ by Lemma 5.8.3, we can choose a set of representatives $\mathfrak{X} \subseteq \mathcal{J}_P$ for the G-orbits $G \backslash \mathcal{J}_{P\subseteq G}^{\times}$. For every $X \in \mathfrak{X}$, let

$$G_X := \{g \in G : \, gX = X\},$$

and let

$$\iota_X : \, C_\lambda^*(G_X) \to C_\lambda^*(P), \, \lambda_g \mapsto \lambda_g 1_X.$$

Here we identify $C_\lambda^*(P)$ with the crossed product $D_{P\subseteq G} \rtimes_r G$ as in Proposition 5.8.5. This is possible because of our assumption that $P \subseteq G$ is Toeplitz.

Theorem 5.10.1. *In the situation above, we have that*

$$\bigoplus_{X\in\mathfrak{X}} (\iota_X)_* : \, \bigoplus_{X\in\mathfrak{X}} K_*(C_\lambda^*(G_X)) \xrightarrow{\cong} K_*(C_\lambda^*(P))$$

is an isomorphism.

To see how Theorem 5.10.1 follows from Corollary 3.5.19, we explain how to choose Ω, I and e_i, $i \in I$ (in the notation of Corollary 3.5.19). Let Ω be the spectrum of $D_{P\subseteq G}$ ($D_{P\subseteq G}$ was introduced in Definition 5.8.4), so that our semigroup C^*-algebra is a full corner in $C_0(\Omega) \rtimes_r G$ by Proposition 5.8.5. Moreover, let I be $\mathcal{J}_{P\subseteq G}^{\times}$, and let e_X be given by 1_X for all $X \in \mathcal{J}_{P\subseteq G}^{\times}$. Applying Corollary 3.5.19, with coefficient algebra $A = \mathbb{C}$, to this situation yields Theorem 5.10.1.

If, in addition, P is a monoid and we have $\mathcal{J}_P^{\times} = \{pP : \, p \in P\}$, then we must have $\mathcal{J}_{P\subseteq G}^{\times} = \{gP : \, g \in P\}$, so that we may choose $\mathfrak{X} = \{P\}$. Then the stabilizer group $G_P = P^*$ becomes the group of units in P. The theorem above then says that the $*$-homomorphism

$$\iota : \, C_\lambda^*(P^*) \xrightarrow{\cong} C_\lambda^*(P), \, \lambda_g \mapsto V_g$$

induces an isomorphism

$$\iota_* : \, K_*(C_\lambda^*(P^*)) \xrightarrow{\cong} K_*(C_\lambda^*(P)).$$

In particular, if we further have that P has trivial unit group, then we obtain that the unique unital $*$-homomorphism $\mathbb{C} \to C_\lambda^*(P)$ induces an isomorphism

$$K_*(\mathbb{C}) \xrightarrow{\cong} K_*(C_\lambda^*(P)).$$

This applies to positive cones in total ordered groups, as long as the group satisfies the Baum–Connes conjecture with coefficients. It also applies to right-angled Artin monoids, to Braid monoids, to Baumslag–Solitar monoids of the type $B_{k,l}^+$ for $k, l \geq 1$, and to the Thompson monoid.

Let us also discuss the case of $ax + b$-semigroups over rings of algebraic integers in number fields. This case is also discussed in detail in Section 6.5. Let K be a number field with ring of algebraic integers R. We apply our K-theory result to the semigroup $P = R \rtimes R^{\times}$. This semigroup embeds into the $ax + b$-group $K \rtimes K^{\times}$. All our conditions are satisfied, so that we only need to compute orbits and stabilizers. We have a canonical identification

$$G \backslash \mathcal{J}_{R \rtimes R^{\times} \subseteq K \rtimes K^{\times}} \xrightarrow{\cong} Cl_K, \; [\mathfrak{a} \times \mathfrak{a}^{\times}] \mapsto [\mathfrak{a}].$$

Moreover, for the stabilizer group $G_{\mathfrak{a} \times \mathfrak{a}^{\times}}$, we obtain

$$G_{\mathfrak{a} \times \mathfrak{a}^{\times}} = \mathfrak{a} \rtimes R^*.$$

Here, R^* is the group of multiplicative units in R.

Hence, our K-theory formula reads in this case

$$\bigoplus_{[\mathfrak{a}] \in Cl_K} K_*(C_\lambda^*(\mathfrak{a} \rtimes R^*)) \xrightarrow{\cong} K_*(C_\lambda^*(R \rtimes R^{\times})).$$

There is a generalization of this formula to $ax+b$-semigroups over Krull rings (see [Li16c]). Let us explain this, using the notation from §5.4.3.

Let R be a countable Krull ring with group of multiplicative units R^* and divisor class group $C(R)$. Then our K-theory formula gives

$$\bigoplus_{[\mathfrak{a}] \in C(R)} K_*(C_\lambda^*(\mathfrak{a} \rtimes R^*)) \xrightarrow{\cong} K_*(C_\lambda^*(R \rtimes R^{\times})).$$

The reader may also consult Corollary 6.5.4 in Chapter 6.

Building on our discussion of graph products in §5.4.2 and §5.9, we can also present a K-theory formula for graph products.

As in §5.4.2 and §5.9, let $\Gamma = (V, E)$ be a graph with vertices V and edges E, such that two vertices in V are connected by at most one edge, and no vertex is connected to itself. So we view E as a subset of $V \times V$. For every $v \in V$, let P_v be a submonoid of a group G_v. We then form the graph products

$$P := \Gamma_{v \in V} P_v$$

and

$$G := \Gamma_{v \in V} G_v.$$

We have a canonical embedding $P \subseteq G$.

For every $v \in V$, choose a system $\mathfrak{X}_v$ of representatives for the orbits $G_v \backslash \mathcal{J}^{\times}_{P_v \subseteq G_v}$ which do not contain P_v. Moreover, for every nonempty subset $W \subseteq V$, define $\mathfrak{X}_W := \prod_{w \in W} \mathfrak{X}_w$. Combining Proposition 5.9.8, Proposition 5.9.12, Proposition 5.9.17 and Theorem 5.10.1, we obtain:

Theorem 5.10.2. *Assume that for every vertex v in V, our semigroup P_v satisfies independence, and that $P_v \subseteq G_v$ is Toeplitz. Moreover, assume that G satisfies the Baum–Connes conjecture with coefficients. Then the K-theory of the reduced C^*-algebra of P is given by*

$$K_*(C^*_\lambda(P^*)) \oplus \bigoplus_{\substack{\emptyset \neq W \subseteq V \\ W \times W \in E}} \bigoplus_{(X_w)_w \in \mathfrak{X}_W} K_*\Bigg(C^*_\lambda\Bigg(\prod_{w \in W} G_{X_w}\Bigg)\Bigg) \xrightarrow{\cong} K_*(C^*_\lambda(P)).$$

Proof. We know that P satisfies independence by Proposition 5.9.12, and we know that $P \subseteq G$ is Toeplitz by Proposition 5.9.17. Moreover, it is an immediate consequence of Proposition 5.9.8 that

$$\begin{aligned} &G \setminus \mathcal{J}^{\times}_{P \subseteq G} \\ &= \{P\} \sqcup \left\{ \left[\left(\prod_{w \in W} X_w \right) \cdot P \right] : \emptyset \neq W \subseteq V,\, W \times W \subseteq E,\, (X_w)_w \in \mathfrak{X}_W \right\}. \end{aligned}$$

As we get for the stabilizer groups

$$G_{(\prod_{w \in W} X_w) \cdot P} = \prod_{w \in W} G_{X_w},$$

our theorem follows from Theorem 5.10.1. □

Note that the graph product G satisfies the Baum–Connes conjecture with coefficients if for every vertex $v \in V$, the group G_v has the Haagerup property. This is because, by [AD13], the graph product G has the Haagerup property in this case.

5.11 Further developments, outlook, and open questions

Based on the result we presented, in particular descriptions as partial or ordinary crossed products as well as our K-theory formula, we obtain classification results for semigroup C^*-algebras.

For instance, the case of positive cones in countable subgroups of the real line, where these groups are equipped with the canonical total order coming from $\mathbb{R}$, have been studied in [Dou72, JX88, CPPR11, Li15]. It turns out that the semigroup C^*-algebra of such positive cones remembers the semigroup completely.

Actually, we can replace the semigroup C^*-algebra by the ideal corresponding to the boundary quotient. It turns out that also these ideals determine the positive cones completely.

For right-angled Artin monoids, a complete classification result was obtained in [ELR16], building on previous work in [CL02, CL07, Iva10, LR96]. The final classification result allows us to decide which right-angled Artin monoids have isomorphic semigroup C^*-algebras by looking at the underlying graphs defining our right-angled Artin monoids. The invariants of the graphs deciding the isomorphism class of the semigroup C^*-algebras are explicitly given, and easy to compute in concrete examples.

For Baumslag–Solitar monoids, important structural results about their semigroup C^*-algebras were obtained in [Spi12, Spi14].

In the case of $ax + b$-semigroups over rings of algebraic integers in number fields, partial classification results have been obtained in [Li14], building on previous work in [CDL13, EL13]. It turns out that for two number fields with the same number of roots of unity, if the $ax + b$-semigroups over their rings of algebraic integers have isomorphic semigroup C^*-algebras, then our number fields must have the same zeta function. In other words, they must be arithmetically equivalent (see [Per77, SP95]).

In addition to these classification results, another observation is that the canonical commutative sub-C^*-algebra (denoted by $D_\lambda(P)$) of our semigroup C^*-algebra often provides interesting extra information. In many situations, the partial dynamical system attached to our semigroup (embedded into a group) is topologically free, and then this canonical commutative sub-C^*-algebra is a Cartan subalgebra in the sense of [Ren08]. For instance, for rings of algebraic integers in number fields, it is shown in [Li16a] that Cartan-isomorphism for two semigroup C^*-algebras of the $ax+b$-semigroups implies that the number fields are arithmetically equivalent and have isomorphic class groups. This is a strictly stronger statement then just being arithmetically equivalent, as there are examples of number fields that are arithmetically equivalent but have difference class numbers (see [dSP94]).

It would be interesting to obtain structural results for semigroup C^*-algebras of the remaining examples mentioned in §5.3.

For instance, for more general totally ordered groups, the semigroup C^*-algebras of their positive cones have not been studied and would be interesting to investigate. Their boundary quotients are given by the reduced group C^*-algebras of our totally ordered groups. It would be interesting to study the structure of the ideals corresponding to these boundary quotients.

For Artin monoids that are not right-angled, it would be interesting to find out more about their semigroup C^*-algebras. For example, the case of Braid monoids would already be interesting. Here the boundary quotients are given by the reduced group C^*-algebras of Braid groups. Therefore, the semigroup C^*-algebras of Braid monoids cannot be nuclear. But what about the ideals corresponding to the boundary quotients?

It would also be very interesting to study the semigroup C^*-algebra of the Thompson monoid. While the boundary quotient of the semigroup C^*-algebra attached to the left regular representation is isomorphic to the reduced group C^*-algebra of the Thompson group, the boundary quotient of the semigroup C^*-algebra generated by the right regular representation is a purely infinite simple C^*-algebra (see our discussion after Theorem 5.7.15, and also Corollary 5.7.17). Is it nuclear?

In the case of $ax+b$-semigroups over rings of algebraic integers in number fields, is it possible to find a complete classification result for their semigroup C^*-algebras? This means that we want to know when precisely two such $ax+b$-semigroups have isomorphic semigroup C^*-algebras. It would be interesting to find a characterization in terms of the underlying number fields and their invariants.

Finally, it seems that not much is known about semigroup C^*-algebras of finitely generated abelian cancellative semigroups. However, we remark that it is not difficult to see that all numerical semigroups have isomorphic semigroup C^*-algebras. Moreover, subsemigroups of $\mathbb{Z}^2$ are discussed in Chapter 7.

Moreover, apart from the issue of classification, we would like to mention a couple of interesting further questions.

Given a semigroup P that is cancellative, i.e., both left and right cancellative, we can form the semigroup C^*-algebra $C^*_\lambda(P)$ generated by the left regular representation, and also the semigroup C^*-algebra $C^*_\rho(P)$ generated by the right regular representation. It was observed in [CEL13, Li16c] that these two types of semigroup C^*-algebras are completely different. However, strangely enough, they seem to share some properties. For instance, in all the examples we know, our semigroup C^*-algebras $C^*_\lambda(P)$ and $C^*_\rho(P)$ have isomorphic K-theory (see [CEL13, Li16c]). There is even an example when this is the case, where our semigroup does not satisfy independence (see [LN16]). Is this a general phenomenon? Do $C^*_\lambda(P)$ and $C^*_\rho(P)$ always have isomorphic K-theory? What other properties do $C^*_\lambda(P)$ and $C^*_\rho(P)$ have in common? For instance, what about nuclearity?

Looking at Theorem 5.6.44, and in particular Corollary 5.6.45, the following task seems interesting: Find a semigroup P that embeds into a group, whose semigroup C^*-algebra is nuclear, such that P does not embed into an amenable group.

With our discussion of the Toeplitz condition in mind (see §5.8), it would be interesting to find a semigroup that embeds into a group, for which the universal group embedding is not Toeplitz.

Finally, we remark that it would be an interesting project to try to generalize our K-theory computations to subsemigroups of groups without using the Toeplitz condition.

Chapter 6

Algebraic actions and their C^*-algebras

Joachim Cuntz

6.1 Introduction

In this chapter we study examples of semigroup C^*-algebras and semigroup actions that have been instrumental for an important part of the recent development on semigroup C^*-algebras described in Chapter 5, as well as for the design of new methods to compute K-theory such as the method described in Section 3.5.3. Some of these are standard examples in ergodic theory, while others arise from semigroups and semigroup actions of number-theoretic origin. All this can be subsumed under the heading "algebraic actions".

By an algebraic action we mean here an action of a semigroup by algebraic endomorphisms on a compact abelian group or, dually, by endomorphisms on a discrete abelian group. Such actions are much studied in ergodic theory, but they also give rise to interesting C^*-algebras. In fact, quite a few of the standard examples of simple C^*-algebras such as $\mathcal{O}_n$-algebras, Bunce–Deddens algebras, UHF-algebras etc. arise from canonical representations of such endomorphisms. But the class of C^*-algebras obtained from general algebraic actions is much vaster and exhibits new interesting phenomena.

We start our survey with the discussion, following [CV13], of the C^*-algebra $\mathfrak{A}[\alpha]$ generated by the so-called Koopman representation on L^2H of a single endomorphism α of a compact abelian group H, together with the natural representation of the algebra $C(H)$ of continuous functions on H. Under natural conditions on α, this C^*-algebra is always simple purely infinite and can be described by a natural set of generators and relations. It contains a canonical maximal commutative

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C^*-algebra $\mathcal{D}$ with spectrum a Cantor space. This subalgebra is generated by the range projections $s^n s^{n*}$, where s is the isometry implementing the given endomorphism, and by their conjugates $u_\gamma s^n s^{n*} u_\gamma^*$, under the unitaries u_γ given by the characters γ of H. Then, the subalgebra $\mathcal{B}$ generated by $\mathcal{D}$ together with the u_γ is of Bunce–Deddens type and simple with unique trace. Moreover, $\mathfrak{A}[\alpha]$ can be considered as a crossed product of $\mathcal{B}$ by a single endomorphism. This leads to a natural exact sequence determining the K-theory of $\mathfrak{A}[\alpha]$. We briefly discuss a number of important examples.

The next case we consider is the C^*-algebra generated analogously by the Koopman representation of a family of commuting endomorphisms. We consider the important special case of endomorphisms arising from the ring of integers R in a number field K. The multiplicative semigroup $R^\times$ acts by commuting endomorphisms on the additive group $R \cong \mathbb{Z}^n$ or equivalently on the dual group $\widehat{R} \cong \mathbb{T}^n$ (n being the degree of the field extension K over $\mathbb{Q}$). The commutative semigroup $R^\times$ has a nontrivial structure and acts by interesting endomorphisms on $\mathbb{T}^n$. The study of the C^*-algebra $\mathfrak{A}[R]$ generated by the Koopman representation in this situation goes back to [Cun08] and was originally motivated by connections to Bost–Connes systems [BC95].

Again, $\mathfrak{A}[R]$ is simple purely infinite and is described by natural generators and relations. It has analogous subalgebras $\mathcal{D}$ and $\mathcal{B}$, and $\mathfrak{A}[R]$ can be viewed as a semigroup crossed product $\mathcal{B} \rtimes R^\times$. The new and challenging problem is the computation of the K-theory of $\mathfrak{A}[R]$. The key to this computation is a duality result for adele-groups and corresponding crossed products, [CL11a].

Since $\mathfrak{A}[R]$ is generated by the Koopman representation, on $\ell^2 R$, of the semidirect product semigroup $R \rtimes R^\times$, the next very natural step in our program is the consideration of the C^*-algebra generated by the natural representation of this semigroup on $\ell^2(R \rtimes R^\times)$ rather than on $\ell^2 R$, i.e., of the left regular C^*-algebra $C^*_\lambda(R \rtimes R^\times)$. This algebra is still purely infinite but no longer simple. It can be described by natural generators and relations. The algebra $\mathfrak{A}[R]$ is a quotient of $C^*_\lambda(R \rtimes R^\times)$ and the latter algebra is defined by relaxing the relations defining $\mathfrak{A}[R]$ in a systematic way. The best way to do so is to add a family of projections, indexed by the ideals of the ring R, as additional generators and to incorporate those into the relations. This way of defining the relations also guided Xin Li in his description of the left regular C^*-algebras for more general semigroups [Li12]; see also Chapter 5 in this book.

The (nontrivial) problem of computing the K-theory of $C^*_\lambda(R \rtimes R^\times)$ turned out to be particularly fruitful [CEL15], [CEL13]. It led to a powerful new method for computing the K-groups, for regular C^*-algebras of more general semigroups and of crossed products by automorphic actions of such more general semigroups, as well as for crossed products of certain actions of groups on totally disconnected spaces, [CEL15], [CEL13]. In the special case of $C^*_\lambda(R \rtimes R^\times)$ we get the interesting result that the K-theory is described by a formula that involves the basic number-theoretic structure of the number field K, namely, the ideal class group and the action of the group of units (invertible elements in R) on the additive group of an ideal.

Finally, we include a brief discussion of the rich KMS-structure on $C^*_\lambda(R \rtimes R^\times)$ for the natural one-parameter action on this C^*-algebra. Just as the K-theory for $C^*_\lambda(R \rtimes R^\times)$, this structure is related to the number-theoretic invariants of R, resp. K and there is a strong resemblance of the formula for KMS-states for large inverse temperature and the one for the K-theory of $C^*_\lambda(R \rtimes R^\times)$. We sketch a proof for this formula that is somewhat different from the one in [CDL13] and which explains this similarity.

Our goal in this survey, which is an extended version of [Cun15], is limited. We try to describe a leitmotif in this line of research and to explain the connections and similarities between the various results. The original articles contain more information and many additional finer, more sophisticated and more general results, which we omit. We also do not describe the results in the order they were obtained originally, but rather in the order that seems more systematic in hindsight. Some of the results described below are part of more general structures discussed in Chapter 3 and Chapter 5 of this book.

6.2 Single algebraic endomorphisms

Let H be a compact abelian group and $G = \widehat{H}$ its dual discrete group. We assume that G is countable. Let α be a surjective endomorphism of H with finite kernel. We denote by φ the dual endomorphism $\chi \mapsto \chi \circ \alpha$ of G (i.e., $\varphi = \widehat{\alpha}$). By duality, φ is injective and has finite cokernel, i.e., the quotient $G/\varphi G$ will be finite. Both α and φ induce isometric endomorphisms s_α and s_φ of the Hilbert spaces L^2H and $\ell^2 G$, respectively. This isometric representation of α on L^2H is called the Koopman representation in ergodic theory.

We will also assume that

$$\bigcap_{n\in\mathbb{N}} \varphi^n G = \{0\}$$

which, by duality, means that

$$\bigcup_{n\in\mathbb{N}} \operatorname{Ker} \alpha^n$$

is dense in H (this implies in particular that H and G cannot be finite). These conditions on α are quite natural and apply, for instance, to the usual examples considered in ergodic theory. We list a few important examples of compact groups and endomorphisms satisfying our conditions at the end of this section.

We want to describe the C^*-algebra $C^*(s_\alpha, C(H))$ generated in $\mathcal{L}(L^2H)$ by $C(H)$, acting by multiplication operators, and by the isometry s_α. Via Fourier transform it is isomorphic to the C^*-algebra $C^*(s_\varphi, C^*G)$ generated in $\mathcal{L}(\ell^2 G)$ by C^*G, acting via the left regular representation, and by the isometry s_φ. These two unitarily equivalent representations are useful for different purposes.

Now, $C^*(s_\varphi, C^*G)$ is generated by the isometry $s = s_\varphi$ together with the unitary operators u_g, $g \in G$ and these operators satisfy the relations

$$u_g u_h = u_{g+h}, \quad s u_g = u_{\varphi(g)} s, \quad \sum_{g(\varphi G) \in G/\varphi G} u_g s s^* u_g^* = 1. \tag{6.1}$$

Definition 6.2.1. Let H, G and α, φ be as above. We denote by $\mathfrak{A}[\varphi]$ the universal C^*-algebra generated by an isometry s and unitary operators u_g, $g \in G$ satisfying the relations (6.1).

It is shown in [CV13] that $\mathfrak{A}[\varphi] \cong C^*(s_\alpha, C(H)) \cong C^*(s_\varphi, C^*G)$, i.e., that the natural map from the universal C^*-algebra to the C^*-algebra generated by the concrete Koopman representation is an isomorphism. Particular situations of interest arise when $H = (\mathbb{Z}/n)^\infty$ with α the left shift (this gives rise to $\mathfrak{A}[\varphi] \cong \mathcal{O}_n$) or when $H = \mathbb{T}^n$.

Lemma 6.2.2. *The C^*-subalgebra $\mathcal{D}$ of $\mathfrak{A}[\varphi]$ generated by all projections of the form $u_g s^n s^{*n} u_g^*$, $g \in G$, $n \in \mathbb{N}$ is commutative. Its spectrum is the "φ-adic completion"*

$$G_\varphi = \varprojlim_n G/\varphi^n G.$$

It is an inverse limit of the finite spaces $G/\varphi^n G$ and becomes a Cantor space with the natural topology.

G acts on $\mathcal{D}$ via $d \mapsto u_g d u_g^*$, $g \in G$, $d \in \mathcal{D}$. This action corresponds to the natural action of the dense subgroup G on its completion G_φ via translation. The map $\mathcal{D} \to \mathcal{D}$ given by $x \mapsto s x s^*$ corresponds to the map induced by φ on G_φ.

From now on we will denote the compact abelian group G_φ by M. By construction, G is a dense subgroup of M. The dual group of M is the discrete abelian group

$$L = \varinjlim_n \operatorname{Ker}(\alpha^n : H \to H).$$

Because of the condition that we impose on α, L can be considered as a dense subgroup of H.

The groups M and L play an important role in the analysis of $\mathfrak{A}[\varphi]$. They are in a sense complementary to H and G. By Lemma 6.2.2, the C^*-algebra $\mathcal{D}$ is isomorphic to $C(M)$ and to $C^*(L)$.

Theorem 6.2.3. *The C^*-subalgebra B_φ of $\mathfrak{A}[\varphi]$ generated by $C(H)$ together with $C(M)$ (or equivalently by C^*G together with C^*L) is isomorphic to the crossed product $C(M) \rtimes G$. It is simple and has a unique trace.*

Proof. The action of the dense subgroup G by translation on M is obviously minimal (every orbit is dense). Therefore the crossed product $C(M) \rtimes G$ is simple. It also has a unique trace, the Haar measure on M being the only invariant measure. The fact that an invariant measure on M extends uniquely to a trace on the crossed product, if all the stabilizer groups are trivial, is well known, but not easy to pin down in the literature. Here is a very simple argument in the present case: Let $E : C(M) \rtimes G \to C(M)$ be the canonical conditional expectation and let $e_1^{(n)}, e_2^{(n)}, \ldots, e_{N(\varphi^n)}^{(n)}$, with $N(\varphi^n) = |G/\varphi^n(G)|$, be the minimal projections in $C(G/\varphi^n(G)) \subset C(M)$. Then, for any x in the crossed product, $\sum_{i=1}^{N(\varphi^n)} e_i^{(n)} x e_i^{(n)}$ converges to $E(x)$ for $n \to \infty$. For any trace τ on the crossed product, we have

$$\tau(x) = \tau\left(\sum_{i=1}^{N(\varphi^n)} e_i^{(n)} x\right) = \tau\left(\sum_{i=1}^{N(\varphi^n)} e_i^{(n)} x e_i^{(n)}\right)$$

and therefore $\tau(x) = \tau(E(x))$.

Finally, by Lemma 6.2.2, B_φ is generated by a covariant representation of the system $(C(M), G)$. The induced surjective map $C(M) \rtimes G \to B_\varphi$ has to be injective, thus an isomorphism. □

The map $x \mapsto sxs^*$ defines a natural endomorphism γ_φ of B_φ.

Theorem 6.2.4. *The algebra $\mathfrak{A}[\varphi]$ is simple, nuclear and purely infinite. Moreover, it is isomorphic to the semigroup crossed product $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$ (i.e., to the universal unital C^*-algebra generated by B_φ together with an isometry t such that $txt^* = \gamma_\varphi(x)$, $x \in B_\varphi$).*

Proof. $\mathfrak{A}[\varphi]$ contains B_φ as a unital subalgebra. The condition $\alpha_\lambda(s) = \lambda s$, $\alpha_\lambda(b) = b$, $b \in B_\varphi$ defines for each $\lambda \in \mathbb{T}$ an automorphism of $\mathfrak{A}[\varphi]$ and integration of $\alpha_\lambda(x)$ over $\mathbb{T}$ determines a faithful conditional expectation $\mathfrak{A}[\varphi] \to B_\varphi$. The proof now is very similar to the corresponding proof in [CL10]. The representation as a crossed product $C(M) \rtimes G$ of B_φ gives a natural faithful conditional expectation $B_\varphi \to \mathcal{D} \cong C(M)$. The composition of these expectations gives a faithful conditional expectation $E : \mathfrak{A}[\varphi] \to \mathcal{D}$.

Now, this expectation can be represented in a different way using only the internal structure of $\mathfrak{A}[\varphi]$. The relations (6.1) immediately show that the linear combinations of elements of the form $z = s^{*n} d u_g s^m$, $n, m \in \mathbb{N}$, $g \in G$, $d \in \mathcal{D}$ are dense in $\mathfrak{A}[\varphi]$. For such an element z we have $E(z) = s^{*n} d s^n$ if $n = m$, $g = 0$, and $E(z) = 0$ otherwise.

The subalgebra $\mathcal{D}$ of $\mathfrak{A}[\varphi]$ is the inductive limit of the finite-dimensional subalgebras $\mathcal{D}_n \cong C(G/\varphi^n G)$. Note that the minimal projections in $\mathcal{D}_n$ are all of the form $u_g s^n s^{*n} u_g^*$. Let

$$z = d + \sum_{i=1}^{m} s^{*k_i} d_i u_{g_i} s^{l_i}$$

be an element of $\mathfrak{A}[\varphi]$ such that for each i, $k_i \neq l_i$ or $g_i \neq e$ and such that $d, d_i \in \mathcal{D}_n$ for some large n (such elements are dense in $\mathfrak{A}[\varphi]$).

Let also n be large enough so that the projections $u_{g_i} e u_{g_i}^*$, $i = 1, \dots, m$, are pairwise orthogonal for each minimal projection e in $\mathcal{D}_n$ (this means that the g_i are pairwise distinct mod $\varphi^n G$).

We have $E(z) = d$ and there is a minimal projection e in $\mathcal{D}_n$ such that $E(z)e = \lambda e$ with $|\lambda| = \|E(z)\|$. Since E is faithful, $\lambda > 0$ if z is positive $\neq 0$.

Let $h \in G$ such that $e = u_h s^n s^{*n} u_h^*$. Then the product $es^{*k_i} d_i u_{g_i} s^{l_i} e$ is nonzero only if $g_i \varphi^{l_i}(h) = \varphi^{k_i}(h)$ or $g_i = \varphi^{k_i}(h)\varphi^{l_i}(h)^{-1} \mod \varphi^n(G)$. Let $f \in \varphi^n G$ such that $\varphi^{k_i}(f) \neq \varphi^{l_i}(f)$ for all i for which $k_i \neq l_i$ (such an f obviously exists) and let $k \geq 0$ such that $\varphi^{k_i}(f) \neq \varphi^{l_i}(f) \mod \varphi^{n+k} G$ for those i.

Then, setting $h' = hf$, we obtain

$$g_i \varphi^{l_i}(h') \neq \varphi^{k_i}(h') \mod \varphi^{n+k}(G),\ i = 1, \dots, m.$$

If we now set $e' = u_{h'} s^{n+k} s^{*(n+k)} u_{h'}^*$, then e' is a minimal projection in $\mathcal{D}_{n+k}$, $e' \leq e$ and $e' s^{*k_i} d_i u_{g_i} s^{l_i} e' = 0$ for $i = 1, \dots, m$.

Every positive element $x \neq 0$ of $\mathfrak{A}[\varphi]$ can be approximated up to an arbitrary ε by a positive element z as above. Thus, if ε is small enough, $e'xe'$ is close to $\lambda e'$ and therefore invertible in $e'\mathfrak{A}[\varphi]e'$. Thus, the product $s^{*n+k} u_{h'}^* x u_{h'} s^{n+k}$ is invertible in $\mathfrak{A}[\varphi]$. This shows, at the same time, that $\mathfrak{A}[\varphi]$ is purely infinite and simple. Moreover, it follows that the natural map from $\mathfrak{A}[\varphi]$ to the semigroup crossed product $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$, is an isomorphism. The fact that this crossed product is nuclear (B_φ is nuclear and, using a standard dilation, $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$ is Morita equivalent to a crossed product $B_\varphi^\infty \rtimes_{\gamma_\varphi^\infty} \mathbb{Z}$, where B_φ^∞ is nuclear) then shows that $\mathfrak{A}[\varphi]$ is nuclear. □

6.2.1 The K-theory of $\mathfrak{A}[\varphi]$

The fact that $\mathfrak{A}[\varphi]$ is a crossed product $B_\varphi \rtimes \mathbb{N}$ can be used to compute the K-theory of $\mathfrak{A}[\varphi]$.

By Theorem 6.2.3, $B_\varphi = C(M) \rtimes G$. Since, by definition,

$$M = \varprojlim_n G/\varphi^n G$$

is the φ-adic completion of G, we can represent B_φ as an inductive limit $B = \varinjlim_n B_n$ with $B_n = C(G/\varphi^n G) \rtimes G$.

It is well known (as an easy case of the "imprimitivity" theorem, see Example 2.6.6) that, for this crossed product,

$$C(G/\varphi^n G) \rtimes G \cong M_{N(\varphi)}(C^*(\varphi^n G))$$

where $N(\varphi) = |G/\varphi^n G|$ and $M_{N(\varphi)}$ denotes $N(\varphi) \times N(\varphi)$ matrices. Consider the natural inclusion

$$C^*G \cong C^*(\varphi^n G) \longrightarrow M_{N(\varphi)}(C^*(\varphi^n G)) \cong B_n$$

into the upper left corner of $M_{N(\varphi)}$, considered as a map $C^*G \to B_n$. This map induces an isomorphism $\kappa_n : K_*(C^*G) \to K_*(B_n)$ in K-theory.

Moreover, we let ι_n denote the map $K_*(B_n) \to K_*(B_{n+1})$ induced by the inclusion $B_n \hookrightarrow B_{n+1}$ and define

$$b(\varphi)_n : K_*(C^*(G)) \longrightarrow K_*(C^*(G))$$

by $b(\varphi)_n = \kappa_{n+1}^{-1}\iota_n\kappa_n$. Now, the commutative diagram

$$\begin{tikzcd}
C^*(G) \arrow[r, "\kappa_0"] \arrow[ddr, bend right, "\kappa_n"'] & B_0 \arrow[r, hook] \arrow[d, "\cong"] & B_1 \arrow[d, "\cong"'] & C^*(G) \arrow[l, "\kappa_1"'] \arrow[ddl, bend left, "\kappa_{n+1}"] \\
 & C^*(\varphi^n G) \arrow[r, hook] \arrow[d, hook] & C(X_n) \rtimes \varphi^n G \arrow[d, hook] & \\
 & B_n \arrow[r, hook] & B_{n+1} &
\end{tikzcd}$$

with $X_n = \varphi^n G/\varphi^{n+1}G$, shows that $b(\varphi)_n = b(\varphi)_0$ for all n. We write $b(\varphi)$ for this common map.

We obtain the following commutative diagram:

$$\begin{tikzcd}
K_*(C^*(G)) \arrow[r, "b(\varphi)"] \arrow[d, equal] & K_*(C^*(G)) \arrow[r, "b(\varphi)"] \arrow[d, "\kappa_1"] & K_*(C^*(G)) \arrow[r] \arrow[d, "\kappa_2"] & {} \\
K_*(B_0) \arrow[r, "\iota_0"] & K_*(B_1) \arrow[r, "\iota_1"] & K_*(B_2) \arrow[r] & {}
\end{tikzcd}$$

One immediate consequence is the following formula for the K-theory of B_φ:

$$K_*(B_\varphi) = \varinjlim_{b(\varphi)} K_*(C^*(G)). \tag{6.2}$$

We note, however, that the problem remains to determine a suitable formula for the map $b(\varphi)$, given a specific endomorphism φ.

Since, by Theorem 6.2.4, $\mathfrak{A}[\varphi]$ can be represented as a crossed product $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$, we are now in a position to derive a formula for the K-theory of $\mathfrak{A}[\varphi]$. Recall that $C^*(G) = C(H)$.

Theorem 6.2.5. (cf. [CV13]) *The K-groups of $\mathfrak{A}[\varphi]$ fit into an exact sequence as follows:*

$$\begin{tikzcd} K_*C(H) \arrow[r, "1-b(\varphi)"] & K_*C(H) \arrow[r] & K_*\mathfrak{A}[\varphi] \arrow[ll, bend left] \end{tikzcd} \tag{6.3}$$

where the map $b(\varphi) : K_*C(H) \to K_*C(H)$ satisfies $b(\varphi)\alpha_* = N(\alpha)\mathrm{id}$ with $N(\alpha) := |\mathrm{Ker}\,\alpha| = |G/\varphi G|$.

Remark 6.2.6. *In examples it is usually easy to determine $b(\varphi)$ using the formula $b(\varphi)\alpha_* = N(\alpha)\mathrm{id}$.*

Proof. From Theorem 6.2.4 we know that $\mathfrak{A}[\varphi]$ is isomorphic to the semigroup crossed product $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$. Using the Pimsner–Voiculescu sequence [PV80] in combination with a simple dilation argument as in [Cun81] (or directly appealing to the results in [Pim97] or in [KS97]) we see that there is an exact sequence, with γ_φ defined as before 6.2.4,

$$\begin{tikzcd} K_*B_\varphi \arrow[r, "1-\gamma_{\varphi*}"] & K_*B_\varphi \arrow[r] & K_*\mathfrak{A}[\varphi] \arrow[ll, bend left] \end{tikzcd} \tag{6.4}$$

In order to determine the kernel and cokernel of the map $K_*B_\varphi \overset{1-\gamma_{\varphi*}}{\longrightarrow} K_*B_\varphi$, consider the commutative diagram

$$\begin{tikzcd} K_*C^*(G) \arrow[r, "b(\varphi)"] \arrow[d, "\kappa_0", "="'] & K_*C^*(G) \arrow[r, "b(\varphi)"] \arrow[d, "\kappa_1"] & K_*C^*(G) \arrow[r, "b(\varphi)"] \arrow[d, "\kappa_2"] & {} \\ K_*B_0 \arrow[r, "\iota_0"] \arrow[d] & K_*B_1 \arrow[r, "\iota_1"] \arrow[d] & K_*B_2 \arrow[r, "\iota_2"] \arrow[d] & {} \\ K_*B_\varphi \arrow[r, "="] & K_*B_\varphi \arrow[r, "="] & K_*B_\varphi \arrow[r, "="] & {} \end{tikzcd}$$

By construction, it is clear that $\gamma_{\varphi*}\kappa_n = \kappa_{n+1}$ (where we still denote the composition $K_*C^*(G) \overset{\kappa_n}{\longrightarrow} K_*B_n \to K_*B_\varphi$ by κ_n). Let κ denote the map (isomorphism)

$$\kappa : \varinjlim_{b(\varphi)} K_*C^*(G) \longrightarrow K_*(B_\varphi)$$

induced by the commutative diagram. For an element of the form $[x_0, x_1, \ldots]$ in the inductive limit we then obtain

$$\gamma_{\varphi*} \circ \kappa([x_0, x_1, x_2, \ldots]) = \kappa([a, x_0, x_1, \ldots])$$

(where a is arbitrary). Therefore, the exact sequence (6.4) becomes isomorphic to

$$\begin{tikzcd} \varinjlim_{b(\varphi)} K_*C^*(G) \arrow[r, "1-\sigma"] & \varinjlim_{b(\varphi)} K_*C^*(G) \arrow[r] & K_*\mathfrak{A}[\varphi] \arrow[ll, bend left] \end{tikzcd} \tag{6.5}$$

where σ is the shift defined by

$$\sigma([x_0, x_1, x_2, \ldots]) = [a, x_0, x_1, \ldots]$$

Consider the natural map $j : K_*C^*(G) \to \varinjlim_{b(\varphi)} K_*C^*(G)$ defined by

$$j(x) = [x, b(\varphi)(x), b(\varphi)^2(x), \ldots]$$

If $(1-\sigma)[x_0, x_1, \ldots] = 0$, then there is n such that $x_n = x_{n+1} = b(\varphi)(x_n)$ and thus $[x_0, x_1, \ldots] = [x_n, x_n, \ldots]$. This shows that $\mathrm{Ker}\,(1-\sigma) = j(\mathrm{Ker}\,(1-b(\varphi))) \cong \mathrm{Ker}\,(1-b(\varphi))$.

If we divide $\varinjlim_{b(\varphi)} K_*C^*(G)$ by $\mathrm{Im}\,(1-\sigma)$, then $[x_0, x_1, \ldots]$ becomes identified with $[x_1, x_2, \ldots]$ and thus to an element of the form $[x, b(\varphi)(x), \ldots]$ which is in the image of j. Also j maps $\mathrm{Ker}\,(1-b(\varphi))$ to $\mathrm{Ker}\,(1-\sigma)$ and thus induces an isomorphism from the cokernel of $1-b(\varphi)$ to the cokernel of $1-\sigma$. This shows that j induces a transformation from the sequence

$$\begin{tikzcd} K_*C^*(G) \arrow[r, "1-b(\varphi)"] & K_*C^*(G) \arrow[r] & K_*\mathfrak{A}[\varphi] \arrow[ll, bend left] \end{tikzcd}$$

into the exact sequence (6.5), which is an isomorphism on kernels and cokernels (in fact, j transforms $1-b(\varphi)$ not into $1-\sigma$ but into $1-\sigma^{-1}$; this, however, does not affect exactness).

Let us finally prove the formula $b(\varphi)\alpha_* = N(\alpha)\mathrm{id}$. Under the identification $B_1 \cong M_{N(\varphi)}(C^*(G))$, the map $\iota_0\kappa_0\varphi_*$ is induced by the embedding of $C^*(G) \cong C^*(\varphi G)$ along the diagonal of $M_{N(\varphi)}(C^*(G))$. Therefore $\iota_0\kappa_0\varphi_* = N(\varphi)\kappa_1$ (κ_1 is induced by the embedding in the upper left corner). The assertion now follows from the definition of $b(\varphi)$ as $\kappa_1^{-1}\iota_0\kappa_0$. $\square$

6.2.2 Examples

Here are some examples of endomorphisms in the class we consider.

1. Let $H = \prod_{k\in\mathbb{N}} \mathbb{Z}/n$, $G = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}/n$ and α the one-sided shift on H defined by $\alpha((a_k)) = (a_{k+1})$.

   We obtain $M = \prod_{k\in\mathbb{N}} \mathbb{Z}/n \cong H$ and $L = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}/n \cong G$. The algebra B_φ is a UHF-algebra of type n^∞ and $\mathfrak{A}[\varphi]$ is isomorphic to $\mathcal{O}_n$. It is interesting to note that the UHF-algebra B_φ is generated by two maximal abelian subalgebras both isomorphic to $C(M)$.

2. Let $H = \mathbb{T}$, $G = \mathbb{Z}$ and α the endomorphism of H defined by $\alpha(z) = z^n$. The algebra B_φ is a Bunce–Deddens algebra of type n^∞ and $\mathfrak{A}[\varphi]$ is isomorphic to a natural subalgebra of the algebra $\mathcal{Q}_\mathbb{N}$ considered in [Cun08]. In this case,

we also get for B_φ the interesting isomorphism $C(\mathbb{Z}_n) \rtimes \mathbb{Z} \cong C(\mathbb{T}) \rtimes L$ where $\mathbb{Z}$ acts on the n-adic completion $\mathbb{Z}_n$ by the odometer action (addition of 1) and L denotes the subgroup of $\mathbb{T}$ given by all n^k-th roots of unity, acting on $\mathbb{T}$ by translation.

3. Let $H = \mathbb{T}^n$, $G = \mathbb{Z}^n$ and α an endomorphism of H determined by an integral matrix T with nonzero determinant. We assume that the condition

$$\bigcap_{n\in\mathbb{N}} \varphi^n G = \{0\}$$

is satisfied (this is in fact not very restrictive).

The algebra B_φ is a higher-dimensional analogue of a Bunce–Deddens algebra. In the case where H is the additive group of the ring R of algebraic integers in a number field of degree n and the matrix T corresponds to an element of R, the algebra $\mathfrak{A}[\varphi]$ is isomorphic to a natural subalgebra of the algebra $\mathfrak{A}[R]$ considered in the following section. It is also isomorphic to the algebra studied in [EaHR11].

4. As another natural example related to number theory consider the additive group of the polynomial ring $\mathbb{F}_p[t]$ over a finite field. An endomorphism satisfying our conditions is given by multiplication by a nonzero element in $\mathbb{F}_p[t]$. In this case $\mathfrak{A}[\varphi]$ is related to certain graph C^*-algebras, see [CL11b].

5. Let p and q be natural numbers that are relatively prime and γ the endomorphism of $\mathbb{T}$ defined by $z \mapsto z^p$. We take

$$H = \varprojlim_{\gamma} \mathbb{T}, \quad G = \mathbb{Z}\left[\frac{1}{p}\right],$$

α_q the endomorphism of H induced by $z \mapsto z^q$ and φ_q the endomorphism of G defined by $\varphi_q(x) = qx$. These endomorphisms satisfy our hypotheses. We find that $M = \mathbb{Z}_q$ (the q-adic completion of $\mathbb{Z}$).

In all these examples one can work out the K-theory of $\mathfrak{A}[\varphi]$ using formula (6.3); see [CV13].

In [CV13], the analysis of $\mathfrak{A}[\varphi]$ and the formula (6.3) for its K-theory was also extended to the case where α is replaced by a so-called rational polymorphism.

There are quite a few papers in the literature containing special cases or parts of the results described in this section. We mention only [Hir02] where it was shown that $\mathfrak{A}[\varphi]$ is simple and characterized by generators and relations and [EaHR11] where in particular a formula similar to (6.3) was derived for an expansive endomorphism of $\mathbb{T}^n$ – both papers using methods different from [CV13].

6.3 Actions by a family of endomorphisms, ring C^*-algebras

It is a natural problem to extend the results of Section 6.2 to actions of a family (semigroup) of several commuting endomorphisms of a compact abelian group, satisfying the conditions of Section 6.2. It turns out that the structural results such as simplicity, pure infiniteness, canonical subalgebras carry over without problem. However, the computation of the K-groups needs completely new ideas.

The most prominent example for us arises as follows. Let K be a number field, i.e., a finite algebraic extension of $\mathbb{Q}$. The ring of algebraic integers $R \subset K$ is defined as the integral closure of $\mathbb{Z}$ in K, i.e., as the set of elements $a \in K$ that annihilate some monic polynomial with coefficients in $\mathbb{Z}$. This ring is always a Dedekind domain (a Dedekind domain is by definition an integral domain in which every nonzero proper ideal factors into a product of prime ideals). It has many properties similar to the ordinary ring of integers $\mathbb{Z} \subset \mathbb{Q}$, but it is not a principal ideal domain in general. Its additive group is always isomorphic to $\mathbb{Z}^n$ where n is the degree of the field extension.

Consider the multiplicative semigroup $R^\times = R \backslash \{0\}$ of R. It acts as endomorphisms on the additive group R and thus also on the compact abelian dual group $\widehat{R} \cong \mathbb{T}^n$. Such endomorphisms of $\mathbb{T}^n$ are a frequent object of study in ergodic theory. If R is not a principal ideal domain, the semigroup $R^\times$ has an interesting structure.

As in Section 6.2 we consider the Koopman representation of $R^\times$ on $L^2\widehat{R} \cong \ell^2 R$.

Definition 6.3.1. We define the *ring C^*-algebra* $\mathfrak{A}[R]$ as the C^*-algebra generated by $C(\widehat{R})$ and $R^\times$ on $L^2(\widehat{R})$ (or equivalently as the C^*-algebra generated by the action of $C^*(R)$ and of $R^\times$ on $\ell^2 R$).

$\mathfrak{A}[R]$ is generated by the isometries s_n, $n \in R^\times$ and the unitaries u_j, $j \in R$. The s_n define a representation of the abelian semigroup $R^\times$ by isometries, the u_j define a representation of the abelian group R by unitaries and together they satisfy the relations

$$s_n u_k = u_{kn} s_n,\ k \in R,\ n, m \in R^\times, \qquad \sum_{j \in R/nR} u_j s_n s_n^* u_{-j} = 1. \tag{6.6}$$

The basic analysis of the structure of $\mathfrak{A}[R]$ is completely parallel to the discussion in Section 6.2 (in fact, historically the article [CL10] preceded [CV13]). One obtains:

Theorem 6.3.2. (cf. [CL10]) *The C^*-algebra $\mathfrak{A}[R]$ is simple, purely infinite and nuclear. It is the universal C^*-algebra generated by a unitary representation u of R together with an isometric representation s of $R^\times$ satisfying the relations* (6.6).

As for $\mathfrak{A}[\varphi]$ in Section 6.2 there are canonical subalgebras $\mathcal{D}$ and $\mathcal{B}$ of $\mathfrak{A}[R]$. The spectrum of the commutative C^*-algebra $\mathcal{D}$ is a Cantor space canonically homeomorphic to the maximal compact subring of the space of finite adeles for the number field K. The subalgebra $\mathcal{B}$ is generated by $\mathcal{D}$ together with the u_j, $j \in R$. It is simple and has a unique trace (a higher-dimensional Bunce–Deddens-type algebra). The general structure of C^*-algebras associated like this with a ring has been developed further by Xin Li in [Li10].

In order to compute the K-groups for $\mathfrak{A}[R]$ the natural strategy would appear to be an iteration of the formula (6.3) of Theorem 6.2.5. Since the proof of formula (6.3) is based on the usual Pimsner–Voiculescu sequence this would amount to iterating this sequence in order to compute the K-groups for the crossed product by $\mathbb{Z}^n$ by a commuting family of n automorphisms. However, this strategy immediately runs into problems since, assuming the K-groups for the crossed product by the first automorphism are determined, it is not at all clear how the second automorphism will act on these groups. In other words, there is a spectral sequence abutting to the K-theory for the crossed product by $\mathbb{Z}^n$, but it is useless for actual computations without further knowledge of the higher boundary maps in the spectral sequence. An analysis of relevant properties of the spectral sequence for actions as here is contained in [Bar15].

The key to the computation of the K-groups for $\mathfrak{A}[R]$ in [CL11a] is the following duality result.

Theorem 6.3.3. *Let $\mathbb{A}_f$ and $\mathbb{A}_\infty$ denote the locally compact spaces of finite, resp. infinite adeles of K both with the natural action of the additive group K. Then the crossed product C^*-algebras $C_0(\mathbb{A}_f) \rtimes K$ and $C_0(\mathbb{A}_\infty) \rtimes K$ are Morita equivalent, equivariantly for the action of $K^\times$ on both algebras (with the inverted natural action on the second algebra, i.e., $K^\times$ acts on $\mathbb{A}_\infty$ not by multiplication but by division).*

Note that the space $\mathbb{A}_\infty$ is simply $\mathbb{R}^n$ where n is the degree of the field extension. From this theorem the K-groups of $\mathfrak{A}[R]$ can be computed, at least in the case where the only roots of unit in K are ± 1.

We explain this here only for the case where $K = \mathbb{Q}$, $R = \mathbb{Z}$. In this case everything becomes rather concrete. The spectrum of the canonical commutative subalgebra $\mathcal{D}$ is the profinite completion $\overline{\mathbb{Z}}$ of $\mathbb{Z}$ (we use here $\overline{\mathbb{Z}}$ rather than the more standard notation $\widehat{\mathbb{Z}}$ in order not to create confusion with the dual group of $\mathbb{Z}$). It is homeomorphic to the infinite product of the p-adic completions $\mathbb{Z}_p$ for all primes p in $\mathbb{Z}$. Moreover $\mathbb{A}_f$ is the restricted infinite product of the $\mathbb{Q}_p$ and $\mathbb{A}_\infty$ simply is $\mathbb{R}$.

Thus, Theorem 6.3.3 gives a Morita equivalence between $C_0(\mathbb{A}_f) \rtimes \mathbb{Q}$ and $C_0(\mathbb{R}) \rtimes \mathbb{Q}$. Moreover, the first crossed product is Morita equivalent to the full corner $\mathcal{B} \cong C(\overline{\mathbb{Z}}) \rtimes \mathbb{Z}$.

Denote by $\mathcal{B}'$ the C^*-algebra generated by $\mathcal{B}$ together with the symmetry s_{-1}, i.e., $\mathcal{B}' \cong \mathcal{B} \rtimes \mathbb{Z}/2$ for the action of s_{-1}. Since $\mathcal{B}' \cong (C(\overline{\mathbb{Z}}) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}/2$ is an inductive limit of $C((\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}/2$ and this latter algebra is isomorphic to $M_n(C^*(\mathbb{Z} \rtimes \mathbb{Z}/2))$ it is not difficult to compute the K-theory of $\mathcal{B}$ as $K_0(\mathcal{B}') = \mathbb{Z} \oplus \mathbb{Q}$ and $K_1(\mathcal{B}') = 0$.

Now, we can use the Pimsner–Voiculescu sequence to compute the K-theory of the crossed product $\mathfrak{A}_1 = \mathcal{B}' \rtimes \mathbb{N} = C^*(\mathcal{B}', s_2)$ as

$$K_0(\mathfrak{A}_1) = \mathbb{Z}, \quad K_1(\mathfrak{A}_1) = \mathbb{Z}.$$

By a slight refinement of the statement in Theorem 6.3.3, $\mathfrak{A}_1$ is Morita equivalent to $(C_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z})$ where $\mathbb{Z}/2 \times \mathbb{Z}$ acts by multiplication by -1 and by 2.

Denote now by $\mathfrak{A}_n$ the C^*-algebra generated by $\mathcal{B}'$ together with $s_{p_1}, \dots, s_{p_n}$, where $p_1, \dots, p_n$ denote the first n prime numbers (with $p_1 = 2$). Then again, $\mathfrak{A}_n$ is Morita equivalent to $(C_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}^n)$, where $\mathbb{Z}/2$ acts by multiplication by -1 and $\mathbb{Z}^n$ by multiplication by $p_1, \dots, p_n$. Moreover $\mathfrak{A}[R]$ is the inductive limit of the $\mathfrak{A}_n$.

We can now consider the canonical inclusions

$$\iota_n : C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}^n) \to (C_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}^n) \sim_{\text{Morita}} \mathfrak{A}_n \tag{6.7}$$

into the crossed product where we leave out the action of the additive $\mathbb{Q}$ by translation on the left-hand side.

By the discussion above, ι_1 induces an isomorphism in K-theory. Now we obtain ι_{n+1} from ι_n by taking the crossed product by $\mathbb{Z}$ (acting by multiplication by p_{n+1}) on both sides in (6.7). Therefore, applying the Pimsner–Voiculescu sequence on both sides, we deduce, using the five-lemma, from the fact that ι_n induces an isomorphism on K-theory that the same holds for ι_{n+1}. The important point is that the action of $\mathbb{Z}^n$ on the left-hand side is homotopic to the trivial action, simply because multiplication by $p_1, \dots, p_n$ is homotopic to multiplication by 1 on $\mathbb{R}$. Therefore, $K_*(\mathfrak{A}_n) \cong K_*((C_0(\mathbb{R}) \rtimes \mathbb{Z}/2) \otimes C^*\mathbb{Z}^n)$.

As a consequence we obtain:

Theorem 6.3.4. ([CL11a]) *The map ι_n induces an isomorphism on K-theory for all n. The K-theory of $\mathfrak{A}[R]$ is isomorphic to the K-theory of $(C_0(\mathbb{R}) \rtimes \mathbb{Z}/2) \rtimes \mathbb{Q}^\times$.*

Note that the K-theory of $C_0(\mathbb{R}) \rtimes \mathbb{Z}/2$ is the same as the one of $\mathbb{C}$ and that therefore the K-theory of $\mathfrak{A}[R]$ is the same as the one of an infinite-dimensional torus.

The argument that we sketched for $K = \mathbb{Q}$ works in a very similar, though somewhat more involved way for a number field with ± 1 as only roots of unit. In this case one has to determine the K-theory of $C_0(\mathbb{R}^n) \rtimes \mathbb{Z}/2$ rather than that of $C_0(\mathbb{R}) \rtimes \mathbb{Z}/2$; see [CL12]. The case of an arbitrary number field K can be treated in the same fashion. The important difference comes from the more general group $\mu(K)$ of roots of unit. For the computation one needs nontrivial information on the K-theory of the crossed product $C_0(\mathbb{R}^n) \rtimes \mu(K)$ and thus on the equivariant K-theory of $\mathbb{R}^n$ with respect to the action of $\mu(K)$. This nontrivial computation has been carried through by Li and Lück in [LL12b] using previous work by Langer and Lück [LL12a].

The analysis of the structure and of the K-theory of $\mathfrak{A}[R]$ can also be carried out in the case where R is a polynomial ring over a finite field (i.e., ring of integers in a certain function field). The structure of the C^*-algebra in this case is more closely related to the example of the shift endomorphism of $(\mathbb{Z}/p\mathbb{Z})^\infty$ mentioned above and to certain Cuntz–Krieger algebras. Nevertheless for the computation of the K-theory one can again use the duality result in Theorem 6.3.3 and the result for the K-theory is again similar, [CL11b].

6.4 Regular C^*-algebras for $ax + b$ -semigroups

By definition, the ring C^*-algebra $\mathfrak{A}[R]$ discussed in Section 6.3 is obtained from the natural representations of $C^*(R) \cong C(\widehat{R})$ and of the semigroup $R^\times$ on the Hilbert space $\ell^2 R \cong L^2(\widehat{R})$. Another way to view this is to say that it is defined by the natural representation of the semidirect product semigroup $R \rtimes R^\times$ on $\ell^2 R$.

Now, this semidirect product semigroup has another natural representation, given by the left regular representation on the Hilbert space $\ell^2(R \rtimes R^\times)$. The study of the left regular C^*-algebra $C^*_\lambda(R \rtimes R^\times)$ was begun in [CDL13]. This C^*-algebra is no longer simple but still purely infinite and has an intriguing structure. In particular, it has a very interesting KMS-structure and the determination of its K-theory leads to new challenging problems.

The first obvious observation concerning $C^*_\lambda(R \rtimes R^\times)$ is that, just as $\mathfrak{A}[R]$, it is generated by a unitary representation u_x, $x \in R$ of the additive group R and a representation by isometries s_a, $a \in R^\times$ of the multiplicative semigroup $R^\times$ satisfying the additional relation $s_a u_x = u_{ax} s_a$. However, the last relation $\sum_{x \in R/aR} u_x s_a s_a^* u_{-x} = 1$ in (6.6) becomes

$$\sum_{x \in R/aR} u_x s_a s_a^* u_{-x} \leq 1. \tag{6.8}$$

In fact, it turns out that this weakened relation (6.8) (of course, together with the relations on the u_x, s_a in the previous paragraph) determines $C^*_\lambda(R \rtimes R^\times)$ in the case where R is a principal ideal domain. The general case, however, is more intricate. In general, it is still possible to describe $C^*_\lambda(R \rtimes R^\times)$ by natural defining relations. However, the most natural way to do so uses an incorporation of the natural idempotents obtained as range projections of the partial isometries given by products of the u_x, s_a and their adjoints. It turns out that these range projections correspond exactly to translates $I + x$, $x \in R$ of the ideals I in R. Denote by $\mathcal{I}$ the set of such translates $I + x$. Note that $I + x = I + y$ if $x \equiv y$ mod I.

The C^*-algebra $C^*_\lambda(R \rtimes R^\times)$ contains the elements $u_x, x \in R$, $s_a, a \in R^\times$ coming from the left regular representation of R and $R^\times$, respectively. It contains, moreover, for $J = I + x$ in $\mathcal{I}$ the orthogonal projection e_J from $\ell^2(R \rtimes R^\times)$ onto the subspace $\ell^2((I + x) \times I^\times)$ (note that $I^\times$ is uniquely determined by $I + x$). The e_J are elements of $C^*_\lambda(R \rtimes R^\times)$, because for every ideal I in R there are a, b in R such that $bI = aI \cap bR$ and therefore $s_b^* s_a s_a^* s_b = e_I$ and $u_x s_b^* s_a s_a^* s_b u_x^* = e_{I+x}$ (see [CDL13] Lemma 4.15). These elements satisfy the following relations.

1. The u_x are unitary and satisfy $u_x u_y = u_{x+y}$, the s_a are isometries and satisfy $s_a s_b = s_{ab}$. Moreover $s_a u_x = u_{ax} s_a$ for all $x \in R$, $a \in R^\times$.
2. The e_J, $J \in \mathcal{I}$ are projections and satisfy
$$e_R = 1, \quad e_\emptyset = 0, \quad e_{J \cap J'} = e_J e_{J'}.$$
3. We have $s_a e_{I+x} s_a^* = e_{a(I+x)}$ and $u_y e_{I+x} u_y^* = e_{I+x+y}$.

This choice of generators and relations is slightly different from the one in [CDL13], but easily seen to be equivalent.

The universal C^*-algebra with generators $u_x, s_a, e_{(I+x)}$ and relations as above is no longer simple, but to some extent its structure remains similar to the one of the ring C^*-algebra $\mathfrak{A}[R]$. There are canonical subalgebras D and B that are analogous to the subalgebras $\mathcal{D}$ and $\mathcal{B}$ of $\mathfrak{A}[R]$ considered in Sections 6.2 and 6.3. The subalgebra D generated by the projections e_{I+x} is maximal commutative and has totally disconnected spectrum, and there is a is a Bunce–Deddens-type subalgebra B generated by D together with the u_x, $x \in R$. Using this structure one shows:

Theorem 6.4.1. (cf. [CDL13]) *The universal C^*-algebra with generators u_x, s_a, e_I satisfying the relations* 1, 2, 3 *above is canonically isomorphic to $C^*_\lambda(R \rtimes R^\times)$. As a consequence $C^*_\lambda(R \rtimes R^\times)$ is also isomorphic to the semigroup crossed product $D \rtimes (R \rtimes R^\times)$ (i.e., to the universal C^*-algebra generated by D together with a representation of the semigroup $R \rtimes R^\times$ by isometries implementing the given endomorphisms of D).*

The relations 1, 2, 3 above turned out to also give the right framework for describing the left regular C^*-algebra of more general semigroups. The theory of these regular C^*-algebras and also of the correct notion of a full C^*-algebra has been developed by Xin Li in great generality [Li12], [Li13]; see his contribution to this book.

The proof of Theorem 6.4.1 in [CDL13], giving at the same time insight into the structure of $C^*_\lambda(R \rtimes R^\times)$, proceeds as follows: Consider the universal C^*-algebra A with generators e_J, u_x, s_a as above satisfying the relations 1, 2, 3 and its C^*-subalgebra D generated by all projections e_J, $J \in \mathcal{I}$. First, it is shown that D is the universal C^*-algebra generated by projections e_J, $J \in \mathcal{I}$, satisfying relation 2 above. At this point one already has to use the fact that the projections e_J form a regular basis for D in the sense of Definition 6.5.1 below. This fact follows from elementary properties of the Dedekind domain R.

Consider next the C^*-subalgebra B generated in A by D together with all the u_x, $x \in R$. In the second step, it is shown that B is isomorphic to the crossed product $D \rtimes R$ for the action of the additive group R on D, where $y \in R$ acts by the automorphism $e_{(I+x)} \mapsto e_{(I+x+y)}$. This is achieved by showing that there is a natural *-homomorphism from $D \rtimes R$ onto B and that the crossed product is simple – using the fact that the action of the additive group R on the spectrum of D is minimal and topologically free.

Thirdly, it is shown that A is the crossed product $B \rtimes R^\times$ of B by the action of the multiplicative semigroup $R^\times$ where an element $a \in R^\times$ acts by the endomorphism $\beta_a : e_{(I+x)} \mapsto e_{a(I+x)}$, $u_x \mapsto u_{ax}$. Here, by the crossed product of the unital C^*-algebra B by the semigroup $R^\times$ we mean by definition the universal C^*-algebra generated by B together with isometries s_a, $a \in R^\times$ satisfying $s_a b s_a^* = \beta_a(b)$.

Finally then, it is argued that the action of $R \rtimes R^\times$ on the spectrum of D is minimal and topologically free. Using the faithful conditional expectation from the crossed product $D \rtimes (R \rtimes R^\times)$ to D, obtained by composing the natural expectations $A \to B \to D$, it is deduced from this that the crossed product is simple. Since the crossed product maps surjectively onto A, and A maps surjectively onto $C^*_\lambda(R \rtimes R^\times)$, it follows that these maps are both isomorphisms.

Since the ring C^*-algebra $\mathfrak{A}[R]$ is also generated by elements u_x, s_a, e_{I+x} satisfying relations 1, 2, 3, it follows that $\mathfrak{A}[R]$ is a quotient of $C^*_\lambda(R \rtimes R^\times)$. As mentioned above, in the simple case of a principal ideal domain, $\mathfrak{A}[R]$ is obtained from the left regular algebra $C^*_\lambda(R \rtimes R^\times)$ by "tightening" the relation $\sum_{j \in R/nR} u_j s_n s_n^* u_{-j} \leq 1$, which is a consequence of relation 2 together with relation, to $\sum_{j \in R/nR} u_j s_n s_n^* u_{-j} = 1$ which was used in (6.6) in Section 6.3. This kind of tightening was introduced by Exel [Exe08] and has occurred in many places in the literature under the name tight representation or boundary quotient etc.; see also Section 5.7in this book.

Remark 6.4.2. *The Exel boundary of Spec D has a very natural description in number-theoretic terms. The spectrum of D can be described as a completion of the disjoint union $\bigsqcup R/I$ over all ideals I in R, [CDL13]. This completion contains the profinite completion*

$$\overline{R} = \varprojlim_I R/I$$

where the limit is taken over the directed set of all ideals I in R. This is the minimal closed invariant subset of Spec D and thus the boundary in the sense of Exel. By the discussion in Section 6.3, the profinite completion $\overline{R}$ is the spectrum of the canonical commutative subalgebra $\mathcal{D}$ of $\mathfrak{A}[R]$. Recall that $\overline{R}$ is the maximal compact subring of the ring of finite adeles in K. The restriction map $C(\overline{\bigsqcup R/I}) \rtimes (R \rtimes R^\times) \to C(\overline{R}) \rtimes (R \rtimes R^\times)$ is exactly the quotient map $C^*_\lambda(R \rtimes R^\times) \to \mathfrak{A}[R]$.

6.5 The K-theory for $C^*_\lambda(R \rtimes R^\times)$

As in Section 6.3 the key to the computation of $K_*(C^*_\lambda(R \rtimes R^\times))$, for the ring R of integers in a number field K, lies in a KK-equivalence between the given action by endomorphisms of our semigroup with a much simpler situation.

The semigroup $S = R \rtimes R^\times$ admits $G = K \rtimes K^\times$ as a canonical enveloping group. The action of S on the commutative subalgebra D of $C^*_\lambda(R \rtimes R^\times)$ has a natural dilation to an action of G. This means that D can be embedded into a larger commutative C^*-algebra $\bar{D} \supset D$ with an action of G that extends the action of S on D (this uses the fact that S is a directed set ordered by right divisibility). The crossed product $\bar{D} \rtimes G$ is then Morita equivalent to $D \rtimes S \cong C^*_\lambda(R \rtimes R^\times)$ (the last isomorphism follows from Theorem 6.4.1).

A fractional ideal in K is a subset of K of the form aI where I is an ideal in R and $a \in K^\times$. We will also consider translates of fractional ideals and denote by $\mathcal{F}$ the set of all "translated fractional ideals", i.e., the set of all subsets of K of the form $a(I + x)$ with I an ideal in R, $x \in R$ and $a \in K^\times$.

It is easy to see that, in the dilated system, there is a bijection $J \mapsto f_J$ between $\mathcal{F}$ and the translates under G of the projections e_J, $J \in \mathcal{I}$. Moreover the f_J, $J \in \mathcal{F}$ generate $\bar{D} \supset D$. Using the fact that R is a Dedekind domain it is not difficult to show that the family $\{f_J\}$ forms a regular basis of $\bar{D}$ in the sense of the following definition. The importance of the regularity condition (or, in another guise, of the "independence" of the family of constructible left ideals of the semigroup) has been noted by Xin Li.

Definition 6.5.1. If $\{f_J : J \in \mathcal{F}\}$ is a countable set of nonzero projections in a commutative C^*-algebra C, we say that $\{f_J\}$ is a *regular basis* for C if it is linearly independent, closed under multiplication (up to 0) and generates C as a C^*-algebra (this means that $\operatorname{span}\{f_J : J \in \mathcal{F}\}$ is a dense subalgebra of C).

Now the group G acts on $\bar{D}$, on $\mathcal{F}$ and on the algebra $\mathcal{K} = \mathcal{K}(\ell^2(\mathcal{F}))$ of compact operators. We can trivially define an equivariant *-homomorphism $\kappa : C_0(\mathcal{F}) \to \mathcal{K} \otimes \bar{D}$ by mapping δ_J to $\varepsilon_J \otimes f_J$. Here, δ_J denotes the indicator function of the one-point set $\{J\}$ and ε_J denotes the matrix in $\mathcal{K}$ that is 1 in the diagonal place (J, J) and 0 otherwise (matrix unit).

As we will explain in a moment, one can then show the following Theorem (see also Theorem 3.5.18 and Corollary 3.5.19).

Theorem 6.5.2. ([CEL15, CEL13]) *The equivariant map κ induces an isomorphism*

$$K_*(C_0(\mathcal{F}) \rtimes G) \longrightarrow K_*(\bar{D} \rtimes G) \cong K_*(C^*_\lambda(R \rtimes R^\times)).$$

But now, by Green's imprimitivity theorem, the crossed product $C_0(\mathcal{F}) \rtimes G$ figuring on the left-hand side is simply Morita equivalent to the direct sum, over the G-orbits in $\mathcal{F}$, of the C^*-algebras of the stabilizer groups of each orbit, cf. Remark

2.6.9 (2) in Chapter 2. Thus, Theorem 6.5.2 reduces the problem of computing $K_*(C^*_\lambda(R \rtimes R^\times))$ to the much easier problem of computing the K-theory of this direct sum.

In the case at hand, the orbit space, as well as the stabilizer groups, correspond to well-known objects in number theory.

Definition 6.5.3. The *ideal class group* Cl_K is the quotient of the group of fractional ideals in K under the equivalence relation where J is equivalent to J' iff there is $a \in K^\times$ such that $J' = aJ$.

If R is the ring of algebraic integers in the number field K, then the class group is a finite abelian group; see, e.g., [Neu99].

Two translated fractional ideals $I+x$ and $I'+y$ are certainly in the same orbit for $K \subset G = K \rtimes K^\times$ if $I = I'$. Thus, they are in the same orbit under $G = K \rtimes K^\times$ iff there is $a \in K^\times$ such that $I' = aI$. Therefore, by Definition 6.5.3 the orbits are labeled exactly by the elements of the class group Cl_K. Moreover, for $g = (z, a)$ in G and $x+I \in \mathcal{F}$, we have $g(x+I) = (z+ax)+aI$; and therefore $g(x+I) = x+I$ iff $a \in R^*$ and $z \in I$, where R^* denotes the group of units (i.e., of invertible elements in $R^\times$). Thus, the stabilizer group of the class of a translated fractional ideal $x+I$ is given by the semidirect product $I \rtimes R^*$ of the additive group I by R^*. Obviously, for $a \in K^\times$, the semidirect product group $I \rtimes R^*$ is isomorphic to $aI \rtimes R^*$. As a corollary to Theorem 6.5.2 we thus obtain:

Corollary 6.5.4. *For each element γ of the class group Cl_K choose any ideal I_γ representing the class γ. Then, up to isomorphism, the group $I_\gamma \rtimes R^*$ does not depend on the choice of I_γ and*

$$K_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{\gamma \in Cl_K} K_*\big(C^*(I_\gamma) \rtimes R^*\big).$$

In the situation at hand, Theorem 6.5.2 can be proven directly, essentially in a similar way as at the end of Section 6.3. In fact, it follows easily from the fact that the projections f_J, $J \in \mathcal{F}$ satisfy the regularity condition of Definition 6.5.1, that the equivariant map κ induces a K-theory isomorphism $K_*(C_0(\mathcal{F})) \to K_*(\bar{D})$. This can be used as a starting point for an iteration of the Pimsner–Voiculescu sequence to compute first the K-theory for the crossed product by K and then by $K^\times$. The reader may also consult Section 5.10.

There is, however, a much more powerful approach based on techniques from work on the Baum–Connes conjecture as explained in Chapter 3 (Section 3.5) based on the following principle:

> Assume that the group G satisfies the Baum–Connes conjecture with coefficients in the G-algebras A and B. Let $\kappa : A \to B$ be an equivariant homomorphism that induces, via descent, isomorphisms $K_*(A \rtimes H) \cong K_*(B \rtimes H)$ for all compact subgroups H of G. Then κ also induces an isomorphism $K_*(A \rtimes_r G) \cong K_*(B \rtimes_r G)$.

The reader may consult Theorem 3.5.1 for more details about this principle. Theorem 6.5.2 then follows from checking that the equivariant map $C_0(\mathcal{F}) \to \mathcal{K} \otimes \bar{D}$ used there satisfies this condition for all finite subgroups of G.

This approach to Theorem 6.5.2 has a much broader scope of applications. It allows us to extend the argument to general actions of a group G, which satisfies the Baum–Connes conjecture with coefficients, on a commutative C^*-algebra C admitting a G-invariant regular basis of projections in the sense of Definition 6.5.1. In particular, it can then be used to compute the K-theory of the left regular C^*-algebra for a large class of semigroups as well as for crossed products by automorphic actions by such semigroups. Moreover, this more general method also allows us to compute the K-theory for crossed products for an action of a group on a totally disconnected space that admits an invariant regular basis as in Definition 6.5.1, [CEL15], [CEL13]. For instance, the semigroups $R^\times$ and $R^\times/R^*$ are simpler than the $ax+b$-semigroup $R \rtimes R^\times$, but still have a very interesting structure. The general method applies to compute the K-theory for their left regular C^*-algebras and for crossed products by these semigroups. One obtains:

Theorem 6.5.5. ([CEL13]) *Let R be a Dedekind domain with quotient field $Q(R)$ and A a C^*-algebra. Then the following are true:*

1. *For every action $\alpha : R^\times \to \mathrm{Aut}\,(A)$ there is a canonical isomorphism*
$$K_*(A \rtimes_{\alpha,r} R^\times) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(A \rtimes_{\alpha,r} R^*).$$
2. *For every action $\alpha : R^\times/R^* \to \mathrm{Aut}\,(A)$ there is a canonical isomorphism*
$$K_*(A \rtimes_{\alpha,r} (R^\times/R^*)) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(A).$$
3. *For every action $\alpha : R \rtimes R^\times \to \mathrm{Aut}\,(A)$ there is a canonical isomorphism*
$$K_*\big(A \rtimes_{\alpha,r} (R \rtimes R^\times)\big) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*\big(A \rtimes_{\alpha,r} (I_\gamma \rtimes R^*)\big).$$

The above method of computing K-theory for semigroup C^*-algebras and for certain crossed products for actions on totally disconnected spaces has been developed further by Li–Norling in [LN15], [LN16].

To end this section, let us briefly discuss a phenomenon concerning the regular C^*-algebras of semigroups that does not occur for groups. Let S be a cancellative semigroup. Then besides the left regular C^*-algebra $C^*_\lambda(S)$, one may also consider the right regular C^*-algebra $C^*_\rho(S)$ generated by the right regular (anti)representation ρ of S on $\ell^2(S)$. Alternatively, we may think of $C^*_\rho(S)$ as of the left regular C^*-algebra of the opposite semigroup. In the case, where S is a group, it is an elementary fact, that the left and right regular C^*-algebras are isomorphic, but for semigroups this is far from being true. The C^*-algebra $C^*_\lambda(R \rtimes R^\times)$ is an intriguing example for this situation. In fact, in this case, the opposite semigroup does not satisfy the left Ore condition and $C^*_\lambda(R \rtimes R^\times)$ and $C^*_\rho(R \rtimes R^\times)$ are wildly different. For instance, the second algebra admits nontrivial abelian quotients, while the first one does not.

Nevertheless it is shown in [CEL13] that their K-theory is the same, in fact, that they are KK-equivalent. This is done by showing that the method sketched for the proof of Theorem 6.5.2 above can also be applied to compute $K_*(C^*_\rho(R \rtimes R^\times))$, after replacing the Ore condition by the "Toeplitz condition" (see Section 5.8) in order to construct a suitable dilation. It is further shown in [CEL13] that the phenomenon of nonisomorphism, but KK-equivalence of right and left regular C^*-algebras occurs for other natural examples of semigroups, but it is unclear how far this can be pushed.

We refer the reader to Section 5.10 for more K-theory computations for semigroup C^*-algebras.

6.6 KMS-states

To end this survey we briefly discuss the KMS-structure for the natural one-parameter automorphism group of $C^*_\lambda(R \rtimes R^\times)$ where, again, R is the ring of algebraic integers in a number field K. After all, part of the motivation for the study of ring C^*-algebras came from Bost–Connes systems and a main feature of such systems is the rich KMS-structure. Also, one of the reasons in [CDL13] for passing from the ring C^*-algebra $\mathfrak{A}[R]$ to $C^*_\lambda(R \rtimes R^\times)$ was the existence of many KMS-states on the latter algebra.

Recall that, for a nonzero ideal I in R, we denote by $N(I)$ the norm of I, i.e., the number $N(I) = |R/I|$ of elements in R/I. For $a \in R^\times$ we also write $N(a) = N(aR)$. The norm is multiplicative, [Neu99]. Using the norm one defines a natural one-parameter automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ on $C^*_\lambda(R \rtimes R^\times)$, given on the generators by

$$\sigma_t(u_x) = u_x, \quad \sigma_t(e_J) = e_J, \quad \sigma_t(s_a) = N(a)^{it} s_a$$

(this assignment manifestly respects the relations between the generators and thus induces an automorphism). Let β be a real number ≥ 0. Recall that a β-KMS state with respect to a one-parameter automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ is a state φ that satisfies $\varphi(yx) = \varphi(x\sigma_{i\beta}(y))$ for a dense set of analytic vectors x, y and for the natural extension of (σ_t) to complex parameters on analytic vectors, [BR97]. For the one-parameter automorphism group σ defined above, the β-KMS condition for a state φ translates to

$$\varphi(e_J z) = \varphi(z\, e_J), \quad \varphi(u_x z) = \varphi(z\, u_x), \quad \varphi(s_a z) = N(a)^{-\beta}\varphi(z\, s_a) \tag{6.9}$$

for the generators u_x, e_J, s_a of $C^*_\lambda(R \rtimes R^\times)$ and for all polynomials z in these generators.

Theorem 6.6.1. ([CDL13]) *The KMS-states on $C^*_\lambda(R \rtimes R^\times)$ at inverse temperature β can be described. One has*

1. *no KMS-states for $\beta < 1$;*
2. *for each $\beta \in [1, 2]$ a unique β-KMS state;*
3. *for $\beta \in (2, \infty)$ a bijection between β-KMS states and tracial states on*

$$\bigoplus_{\gamma \in Cl_K} C^*(I_\gamma) \rtimes R^*$$

where Cl_K is the ideal class group, I_γ is any ideal representing γ and R^* denotes the multiplicative group of invertible elements in R (units).

Loosely speaking, the uniqueness or nonuniqueness of a β-KMS-state φ is due to the fact that, depending on β, certain projections in the weak closure of $C^*_\lambda(R \rtimes R^\times)$ in the GNS-representation for φ take more space and therefore are nonzero or not. The value of φ on these projections is determined by the third condition in (6.9), and the question whether they can be nonzero, depends on the question if the series representing the partial Dedekind ζ-functions for K converge at $\beta - 1$ or not.

There is a striking parallel between the formula for the KMS-states for $\beta > 2$ in the theorem above and the formula for the K-theory of $C^*_\lambda(R \rtimes R^\times)$ in Corollary 6.5.4. The K-theory is isomorphic to the K-theory of the C^*-algebra $\bigoplus_{\gamma \in \Gamma} C^*(I_\gamma) \rtimes R^*$ while the simplex of KMS-states is in bijection with the trace simplex of this direct sum C^*-algebra. Note that both results are nontrivial, as $\bigoplus_{\gamma \in \Gamma} C^*(I_\gamma) \rtimes R^*$ is not a subalgebra of $C^*_\lambda(R \rtimes R^\times)$ in a natural way.

The proof of Theorem 6.6.1 given in [CDL13] does not really explain the similarity with the formula for the K-theory of $C^*_\lambda(R \rtimes R^\times)$ in Theorem 6.5.5. We will now sketch a proof of point (3) in 6.6.1 that is closer to the proof of the formula for $K_*(C^*_\lambda(R \rtimes R^\times))$ in 6.5.5 sketched in Section 6.5. As in Section 6.5 we consider the dilation $D \subset \bar{D}$ of the action of $R \rtimes R^\times$ to an action of the enveloping group $G = K \rtimes K^\times$ so that $C^*_\lambda(R \rtimes R^\times) \cong D \rtimes (R \rtimes R^\times)$ becomes Morita equivalent to $\bar{D} \rtimes G$. As before, we also denote by $\mathcal{F}$ the set of translated fractional ideals and by e_J, $J \in \mathcal{F}$ the projections generating $\bar{D}$.

In Section 6.5 the computation of the K-theory of $C^*_\lambda(R \rtimes R^\times)$ was based on comparing this K-theory to the K-theory of the much simpler crossed product $C_0(\mathcal{F}) \rtimes G$ and noting that, by Green imprimitivity, this latter algebra is isomorphic to $\bigoplus_{\gamma \in Cl_K} C^*(I_\gamma) \rtimes R^*$. We will now employ a similar strategy to prove 6.6.1 (3). We can define a phantom version (α_t) of the one-parameter group (σ_t) on $C_0(\mathcal{F}) \rtimes G$ by letting (α_t) act trivially on $C_0(\mathcal{F})$ and defining $\alpha_t(\bar{u}_x) = \bar{u}_x$, $x \in K$,

$\alpha_t(\bar{s}_a) = N(a)^{it}\bar{s}_a$, $a \in K^\times$ for the unitary multipliers $\bar{u}_x$ and $\bar{s}_a$ implementing the action of $G = K \rtimes K^\times$ on $C_0(\mathcal{F})$ (note that $\bar{s}_a$ is here no longer a proper isometry, but a unitary, and note also that the norm N extends to a multiplicative map $N : K^\times \to \mathbb{R}$).

Now, it is easy to analyze the KMS-structure for (α_t). Given $\beta > 0$, any KMS weight μ on $C_0(\mathcal{F}) \rtimes G$ must, just as in equation (6.9) above, satisfy the conditions

$$\mu(\delta_J z) = \mu(z\,\delta_J), \quad \mu(\bar{u}_x z) = \mu(z\,\bar{u}_x), \quad \mu(\bar{s}_a z) = N(a)^{-\beta}\mu(z\,\bar{s}_a), \tag{6.10}$$

where, as in Section 6.5, the δ_J, $J \in \mathcal{F}$ are the characteristic functions of the one-point sets $\{J\}$ and z is a linear combination of elements of the form $\bar{s}_a\bar{u}_x\delta_J$.

The first condition in (6.10) shows that μ factors through the conditional expectation $C_0(\mathcal{F}) \rtimes G \to C_0(\mathcal{F})$. The third condition implies that μ is uniquely determined by its restrictions to $\delta_{I_\gamma}(C_0(\mathcal{F}) \rtimes G)\delta_{I_\gamma}$, $\gamma \in CL_K$ and the second and third conditions together imply that these restrictions have to be bounded traces. Since we know that $\delta_{I_\gamma}(C_0(\mathcal{F}) \rtimes G)\delta_{I_\gamma}$ is isomorphic to the C^*-algebra of the stabilizer group $I_\gamma \rtimes R^*$ of γ we trivially obtain:

Proposition 6.6.2. *There is a bijection between KMS-weights on $C_0(\mathcal{F}) \rtimes G$ and bounded traces on $\bigoplus_{\gamma\in Cl_K} C^*(I_\gamma) \rtimes R^*$.*

We now define a faithful representation π of $C^*_\lambda(R \rtimes R^\times)$ in the multiplier algebra $\mathcal{M}(C_0(\mathcal{F}) \rtimes G)$ as follows. Let E be the projection in $\mathcal{M}(C_0(\mathcal{F}) \rtimes G)$ defined by $E = \sum \delta_{I+x}$ where the sum is over all (nonfractional!) ideals I of R and $x \in R$.

Then we can set

$$\pi(u_x) = E\bar{u}_xE, \quad \pi(s_a) = E\bar{s}_aE, \quad \pi(e_{x+I}) = \sum_{(I'+x')\subset(I+x)} \delta_{(I'+x')}.$$

It is clear that π is equivariant for the action (σ_t) on $C^*_\lambda(R \rtimes R^\times)$ and the action induced by (α_t) on the multiplier algebra.

Lemma 6.6.3. *Let μ be a β-KMS weight on $C_0(\mathcal{F}) \rtimes G \cong \bigoplus_{\gamma\in Cl_K} C^*(I_\gamma) \rtimes R^*$ with components μ_γ, $\gamma \in Cl_K$, and let $\bar{\mu}$ be its extension to a weight on $\mathcal{M}(C_0(\mathcal{F}) \rtimes G)$ via the GNS-representation. Then $\bar{\mu}$ is bounded on the image $\pi(C^*_\lambda(R \rtimes R^\times))$ if and only if the series $\sum_{I\in\gamma} N(I)^{1-\beta}$ representing $\zeta_\gamma(\beta - 1)$ converges. Here ζ_γ denotes the partial Dedekind ζ-function, for the number field K and $\gamma \in Cl_K$.*

Proof. Obviously $\bar{\mu}$ is bounded iff $\bar{\mu}(E) < \infty$ and this is the case iff $\bar{\mu_\gamma}(E) < \infty$ for each γ. But now

$$\bar{\mu_\gamma}(E) = \sum_{I\in\gamma,\, x\in R/I} N(I) = \sum_{I\in\gamma} |R/I| N(I)^{-\beta} = \sum_{I\in\gamma} N(I)^{1-\beta}. \qquad \square$$

Now, it is well known that the series representing $\zeta_\gamma(s)$ converges iff $\mathrm{Re}\, s > 1$. Therefore, Lemma 6.6.3 shows that a β-KMS weight μ on $C_0(\mathcal{F}) \rtimes G$ induces a

β-KMS state on $C^*_\lambda(R \rtimes R^\times)$ iff $\beta > 2$. In particular, for each $\beta > 2$, we get an injective map from trace states on $C^*(I_\gamma \rtimes R^*)$ to β-KMS states on $C^*_\lambda(R \rtimes R^\times)$.

To finish the proof of Theorem 6.6.1 (3), one has to show that any β-KMS state for $\beta > 2$ arises that way. This is done in the following way. Given a prime ideal P in R and $k = 0, 2, \ldots$ we consider the projection

$$\delta_{P^k} = e_{P^k} - \sum_{x \in P^k/P^{k+1}} u_x e_{P^{k+1}} u_x^*$$

with the convention that $P^0 = R$. Let now $P_1, P_2, \ldots$ be an enumeration of the prime ideals in R and denote by $\mathcal{I}_n$ the set of ideals in R that are products of powers only of the first n prime ideals $P_1, \ldots, P_n$.

Given an ideal $I \in \mathcal{I}_n$ of the form $I = P_1^{k_1} \cdots P_n^{k_n}$ and $x \in R/I$ we set

$$\delta_{I+x,n} = u_x \delta_{P^{k_1}} \cdots \delta_{P^{k_n}} u_x^*.$$

Then the $\delta_{I+x,n}$ are pairwise orthogonal and in bijection with the set $\mathcal{F}_n$ of translates $I + x$ of ideals I in $\mathcal{I}_n$.

Consider now a β-KMS-state φ for $\beta > 2$ and let (π_φ, H_φ) be the corresponding GNS-representation. It is easily checked that the KMS-condition implies that

$$\varphi(\delta_{I+x,n}) = \frac{N(I)^{-\beta}}{N(I_\gamma)^{-\beta}} \varphi(\delta_{I_\gamma,n})$$

for each I in $\mathcal{I}_n \cap \gamma$. Moreover, the closed subspace E_n generated by the $\varphi(\delta_{I+x,n})$ in H_φ is invariant under $\pi_\varphi(C^*_\lambda(R \rtimes R^\times))$. Using the fact that φ has to induce a β-KMS-state on the restriction of π_φ to the complement $E_n^\perp$ and that on that complement one has the algebraic relation $1 = \sum_{x \in R/P} \pi_\varphi(\delta_P)$ for each prime ideal $P \in \mathcal{I}_n$, which can only hold for $\beta = 1$, one concludes that $E_n = H_\varphi$.

This shows that

$$\varphi(1) = \sum_{I \in \mathcal{I}_n,\, x \in R/I} \varphi(\delta_{I+x,n}) = \sum_{\gamma \in CL_K} \sum_{I \in \mathcal{I}_n \cap \gamma} \frac{N(I)N(I)^{-\beta}}{N(I_\gamma)^{-\beta}} \varphi(\delta_{I_\gamma,n})$$

and, rewriting the last sum,

$$\varphi(1) = \sum_{\gamma \in CL_K} \sum_{I \in \mathcal{I}_n \cap \gamma} N(I_\gamma)^\beta N(I)^{1-\beta} \varphi(\delta_{I_\gamma,n}). \tag{6.11}$$

The sequence of projections $\pi_\varphi(\delta_{I,n})$ is eventually defined (since I is eventually in $\mathcal{I}_n$) and decreasing for each ideal I in R, and therefore has a strong limit d_I as $n \to \infty$. Since for $(1 - \beta) > 1$, the series $\sum_{I \in \gamma} N(I)^{1-\beta}$ converges and represents the partial Dedekind ζ-function $\zeta_\gamma(\beta - 1)$, equation (6.11) becomes in the limit $n \to \infty$

$$\varphi(d_{I_\gamma}) = \big(N(I_\gamma)^\beta \zeta_\gamma(\beta - 1)\big)^{-1}.$$

This identity shows that for $\beta > 2$ we get a representation of $E(C_0(\mathcal{F}) \rtimes G)E$ in $H_\varphi \cong C^*_\lambda(R \rtimes R^\times)$ by mapping the δ-function for $(I+x) \in \mathcal{F}$ to $\pi_\varphi(u_x)d_I\pi_\varphi(u_x)^*$ and $E\bar{u_x}E$, $E\bar{s_a}E$ to $\pi_\varphi(u_x)$, $\pi_\varphi(s_a)$. Thus, π_φ is unitarily equivalent to the representation of $C^*_\lambda(R \rtimes R^\times)$ that we constructed above from a KMS-weight on $C_0(\mathcal{F}) \rtimes G$. This finishes the proof of point (3) in Theorem 6.6.1. We do not discuss the (nontrivial) proof of point (2) here and refer for this to [CDL13] and [Nes13]

Remark 6.6.4. *Given a KMS-state φ, the relation $\varphi(e_J z) = \varphi(z e_J)$ in* (6.10) *above shows that φ factors through the conditional expectation $C^*_\lambda(R \rtimes R^\times) \to D$. For $\beta > 1$, the induced measure on Spec D has to be concentrated on the complement of the boundary $\overline{R} \subset$ Spec D in Remark 6.4.2.*

Remark 6.6.5. *Just as for the K-theory computation in Section 6.5, the method for describing the KMS-states for large β sketched here also works to determine the KMS-states for the natural one-parameter automorphism groups on the left regular C^*-algebras $C^*_\lambda(R^\times)$ and $C^*_\lambda(R^\times/R^*)$. One obtains:*

1. *For $\beta > 1$, there is a bijection between β-KMS-states on $C^*_\lambda(R^\times)$ and trace states on $\bigoplus_{\gamma \in Cl_K} C^*(R^*)$.*
2. *For $\beta > 1$, there is a bijection between β-KMS-states on $C^*_\lambda(R^\times/R^*)$ and trace states on $\bigoplus_{\gamma \in Cl_K} \mathbb{C}$.*

The simpler case of Theorem 6.6.1, where $R = \mathbb{Z}$, $K = \mathbb{Q}$, had essentially been treated already by Laca–Raeburn in [LR10]. The first assertion in Theorem 6.6.1 is basically obvious. The original proof of point (3), in [CDL13], implicitly uses the representations of $C^*_\lambda(R \rtimes R^\times)$ arising from the homomorphism into the multiplier algebra of $C_0(\mathcal{F}) \rtimes G$ that we described above – but in a different (more complicated) guise. The proof of (2) in [CDL13] uses a result, also of some independent interest, on asymptotics of partial Dedekind ζ-functions. An alternative subsequent proof of Theorem 6.6.1, due to Neshveyev, is obtained by relating the problem to a general result on KMS-states for C^*-algebras of nonprincipal groupoids, and using previous results on KMS-states for Bost–Connes systems by Neshveyev and others, [Nes13].

Chapter 7

Semigroup C^*-algebras and toric varieties

Joachim Cuntz

7.1 Introduction

Let S be a finitely generated subsemigroup of $\mathbb{Z}^n$. Then its monoid algebra $\mathbb{C}S$ is a finitely generated $\mathbb{C}$-algebra with no nonzero nilpotent elements. It is therefore the coordinate ring of an affine variety over $\mathbb{C}$. Such varieties are called affine toric varieties (they carry an action of an n-dimensional torus). Of course, here we may replace $\mathbb{C}$ by an arbitrary field. General references for toric varieties and the corresponding semigroups are, for instance, [CLS11] or [Nee92].

In this chapter we study the left regular semigroup C^*-algebra $C^*_\lambda S$ which, in contrast to $\mathbb{C}S$, is generated not only by the elements of S but also by their adjoints (it is, in fact, generated by the enveloping inverse semigroup to S). It also carries a natural action of $\mathbb{T}^n$.

As we shall see (Lemma 7.3.5), the case $n = 1$ is without interest: for a nontrivial subsemigroup S of $\mathbb{Z}$ the C^*-algebra $C^*_\lambda S$ is, in fact, always isomorphic (noncanonically) to the ordinary Toeplitz algebra $C^*_\lambda \mathbb{N}$. But our main objective in this chapter is the computation of the K-theory of $C^*_\lambda S$ for a finitely generated subsemigroup S of $\mathbb{Z}^2$. In [CEL15], [CEL13] we had determined the K-theory of a large class of semigroup C^*-algebras using Xin Li's independence condition, cf. Definition 5.6.30 (this condition plays an important role in large parts of Chapter 5). The computation of K-theory on the basis of independence is described and used in this book in different chapters: Section 3.5.3, Section 5.10 and Section 6.5. The interesting feature of the semigroups that we meet in the present chapter, however, is that they do not satisfy independence except in trivial cases. One con-

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sequence is that the K-theory contains a torsion part and, as another consequence, we cannot rely here on an elegant general method for its determination.

But, by a more detailed study of the structure of arbitrary finitely generated subsemigroups of $\mathbb{Z}^2$, we are able to show that the K-theory of the C^*-algebra is always described by a simple formula involving only the "faces" of the semigroup.

In [PS91], on the basis of previous results in [JK88], [JX88] and [Par90], a formula for the K-theory of $C^*_\lambda S$ (which looks different from ours, but gives the same result) had been established in the important special case of a "saturated" finitely generated subsemigroup of $\mathbb{Z}^2$. These authors consider "Toeplitz algebras" associated with cones, but, upon inspection, their Toeplitz algebra is exactly the left regular C^*-algebra of the (automatically saturated) semigroup defined by a cone in $\mathbb{Z}^2$. In the saturated case our computation is somewhat more direct than the one in [PS91]. But much of our analysis here is really concerned with the nonsaturated case. The result in [PS91] actually also covers saturated subsemigroups of $\mathbb{Z}^2$ which are defined by a cone where one face has irrational slope. Such semigroups are not finitely generated. But the semigroup defined by a half-plane with irrational slope in $\mathbb{Z}^2$ does satisfy the independence condition. Thus, one easily derives the main result in [JK88], on the K-theory of the corresponding Toeplitz algebra, by applying the general method based on independence from Chapters 3 and 5. Using this, the case of a cone with one irrational face considered in [PS91] also lies within the scope of our methods.

Even though the exact sequences in K-theory that we use become significantly more complicated in higher dimensions, it may well be possible that our argument can be extended to subsemigroups of $\mathbb{Z}^n$ for $n > 2$.

7.2 Toric varieties

Much of the material in this section is well known. We consider a finitely generated subsemigroup S of $\mathbb{Z}^n$. We write the semigroup operation as addition and always assume that a semigroup contains 0. It is then easily seen that the subgroup $S - S$ generated by S in $\mathbb{Z}^n$ is the enveloping group of S and is, of course, isomorphic to $\mathbb{Z}^k$ for some k. Thus, without restriction of generality, we may assume that $\mathbb{Z}^n$ already is the enveloping group and therefore that S generates $\mathbb{Z}^n$ and that this embedding is natural. We will also assume that $S \cap (-S) = \{0\}$, i.e that S contains no invertible elements besides 0.

Summarizing, we assume from now on that S is a finitely generated subsemigroup of $\mathbb{Z}^n$, for some n, generating $\mathbb{Z}^n$ as a group. We also assume that $S \cap (-S) = \{0\}$. The following easy lemma will be used in two places.

Lemma 7.2.1. *Let $Y = \{y_1, \ldots, y_m\}$ be a finite set in $\mathbb{Z}^n$. Then there is $z \in S$ such that $z + Y \subset S$.*

Proof. Let $x_1, \dots, x_l$ denote the generators of S. Since S generates $\mathbb{Z}^n$ as a group, there are $k^i_j \in \mathbb{Z}$ such that $y_i = \sum_j k^i_j x_j$. We may assume that the k^i_j are ordered in such a way that $k^i_j < 0$ for $i \leq r_i$ and $k^i_j \geq 0$ for $i > r_i$. Denote by $\bar{y}_i = \sum_{j=1}^{r_i} k^i_j x_j$ the "negative part" of y_i. Then $z = -\sum_i \bar{y}_i$ has the required property. □

A subsemigroup $F \subset S$ is said to be a face of S if $x + y \in F$ with $x, y \in S$ implies that $x, y \in F$. A semigroup $S \subset \mathbb{Z}^n$ is said to be saturated if $kx \in S$, $x \in \mathbb{Z}^n$, $k \in \mathbb{N} \setminus \{0\}$ implies that $x \in S$. For an arbitrary subsemigroup S of $\mathbb{Z}^n$ let $\bar{S}$ denote the saturation of S, i.e., the semigroup consisting of all $s \in \mathbb{Z}^n$ for which an integral multiple ks, $k \in \mathbb{N} \setminus \{0\}$ lies in S. There is a bijection between faces in S and faces in $\bar{S}$ as follows:

Lemma 7.2.2. *(see also [Nee92] Lemma II.7) Let $F \subset S$ be a face of S. Then $\bar{F}$ is a face in $\bar{S}$ and $F = \bar{F} \cap S$. Conversely, if G is a face in $\bar{S}$, then $G \cap S$ is a face in S.*

Proof. Assume that F is a face and let x', y' be in $\bar{S}$ and f' in $\bar{F}$ such that $x' + y' = f'$. We have that kx', jy', nf' are in S and F, respectively, for suitable k, j, n in $\mathbb{N}$. Let m be the least common multiple of k, j, n. It follows that $mx' + my' = mf'$ with $mx', my' \in S$ and $mf' \in F$. Since F is a face, mx' and my' are in F and thus x', y' are in $\bar{F}$. By the defining property of a face we have that $F = \bar{F} \cap S$.

Conversely, assume that $\bar{F}$ is a face. If $x, y \in S$ and $f \in \bar{F} \cap S$ are such that $x + y = f$, then x, y are in $\bar{F}$ and thus also in $F = \bar{F} \cap S$. □

A subsemigroup T of S is said to be one-dimensional, if the subgroup of $\mathbb{Z}^2$ it generates, is isomorphic to $\mathbb{Z}$. Since a saturated generating semigroup of $\mathbb{Z}^2$ is determined by a convex cone (see e.g. [Nee92] Lemma II.7), Lemma 7.2.2 implies that any generating subsemigroup S of $\mathbb{Z}^2$ has exactly two one-dimensional faces. Note, that a subsemigroup of $\mathbb{Z}$ that contains no nonzero invertibles has to lie entirely in $\mathbb{N}$ or in $-\mathbb{N}$. Therefore, the structure of one-dimensional subsemigroups of S is determined by the following Lemma.

Lemma 7.2.3. *Let F be a finitely generated subsemigroup of $\mathbb{N}$. Then there is $d \in \mathbb{N}$ such that $F \subset d\mathbb{N}$ and such that $d\mathbb{N} \setminus F$ is finite.*

Proof. Let $F' = F - F$ be the subgroup of $\mathbb{Z}$ generated by F. Then there is $d \in \mathbb{N}$ such that $F' = d\mathbb{Z}$. It follows that $F \subset d\mathbb{N}$. Let $m \in \mathbb{N}$ such that $md \in F$. By Lemma 7.2.1 there is $z \in F$ such that $z + jd$ is in F for $j = 1, \dots, m-1$. Since $md \in F$, it follows that $z + \mathbb{N}d$ is contained in F. □

Let $F \subset S$ be a one-dimensional subsemigroup. By Lemma 7.2.3 there exists a unique $a \in \mathbb{Z}^n$ such that F is contained in $\mathbb{N}a$ with finite complement. Moreover, F then generates $\mathbb{Z}a$ as a group.

Definition 7.2.4. Given $F \subset S$ as above, we say that the element a is the *asymptotic generator* of F.

Recall that the quotient of a commutative semigroup S by a subsemigroup F is the semigroup consisting of equivalence classes of elements s in S for the equivalence relation $s_1 \sim s_2 \iff \exists f_1, f_2 \in F$ such that $s_1 + f_1 = s_2 + f_2$.

Lemma 7.2.5. *Let $F \subset S$ be as above and a the asymptotic generator of F. Denote by $x \mapsto \dot{x}$ the quotient map $S \to S/F$. Then $\dot{x} = \dot{y}$ for $x, y \in S$ if and only if $(x + \mathbb{Z}a) \cap S = (y + \mathbb{Z}a) \cap S$. If $\dot{x} \neq \dot{y}$, then $(x + \mathbb{Z}a) \cap (y + \mathbb{Z}a) = \emptyset$.*

Proof. If $\dot{x} = \dot{y}$, then there are f_1, f_2 in F such that $x + f_1 = y + f_2$ and thus that $x - y = f_2 - f_1 \in \mathbb{Z}a$. This implies that $x \in y + \mathbb{Z}a$ and $y \in x + \mathbb{Z}a$.

Conversely, assume that $x = y + ka$ with $k \in \mathbb{Z}$. By Lemma 7.2.3 there is $n \in \mathbb{N}$ such that $ka + na$ and na are in F. It follows that $x + na = y + ka + na$ and thus that $\dot{x} = \dot{y}$. The same argument shows that, if $x + k_1 a = y + k_2 a$ for $k_1, k_2 \in \mathbb{Z}$, then $\dot{x} = \dot{y}$. □

Corollary 7.2.6. *Let S and F be as in Lemma 7.2.5. Then S is a disjoint union*

$$S = \bigsqcup_{\dot{x} \in S/F} (x + \mathbb{Z}a) \cap S.$$

Proof. Since, for $\dot{x} \in S/F$, the set $x + \mathbb{Z}a$ does not depend on the representative x, this is an immediate consequence of Lemma 7.2.5. □

Lemma 7.2.7. *Let F be a nontrivial face in S. Then for each $x \in S$ with $x \notin F$, we have that $F \subset S \setminus (S + x)$.*

Proof. If F is a face, then $x \notin F$ implies that $F \cap (S + x) = \emptyset$. □

Lemma 7.2.8. *Let $x \in S$ and $\langle x \rangle$ the subsemigroup generated by x. Then the quotient map $y \mapsto \dot{y} \in S/\langle x \rangle$ induces a bijection between $S \setminus (S + x)$ and $S/\langle x \rangle$.*

Proof. Assume that $\dot{y}_1 = \dot{y}_2$ for $y_1, y_2 \in S \setminus (S + x)$. Then there is $n \in \mathbb{N}$ such that $y_1 + nx = y_2$ or $y_2 + nx = y_1$. Since $y_1, y_2 \in S \setminus (S + x)$, n has to be zero, so that $y_1 = y_2$. This shows injectivity.

To show surjectivity, take $\dot{y} \in S/\langle x \rangle$ represented by $y \in S$. There is a minimal $n \in \mathbb{N}$ such $y \in S + nx$, i.e., $y = z + nx$ and $z \notin S + x$. Then $\dot{z} = \dot{y}$. □

Lemma 7.2.9. *Let $S \subset \mathbb{Z}^2$ and let F_1, F_2 denote the two one-dimensional faces of S. Let a_1, a_2 be the asymptotic generators of F_1, F_2 and C the cone in $\mathbb{Z}^2$ spanned by a_1 and a_2 (i.e., $C = \bar{S}$ using the notation above). Then there is $z \in S$ such that $z + C \subset S$.*

Proof. Let $x_1, \ldots, x_n$ denote the generators of S and let b_1, b_2 be multiples of a_1, a_2 such that $b_i \in F_i$, $i = 1, 2$. Let $P = \{y_1, \ldots, y_m\}$ denote the set of all elements in $\mathbb{Z}^2$ that lie inside the parallelogram spanned by b_1 and b_2. Then $P + F' = C$ for the subsemigroup F' of S spanned by b_1, b_2. By Lemma 7.2.1 there is $z \in S$ such that $z + P \subset S$. Then, since $C = F' + P$, also $z + C \subset S$. □

Lemma 7.2.10. *Let $F \subset S$ be a one-dimensional face and a the asymptotic generator of F. Then $F = S \cap \mathbb{Z}a$.*

For each $x \in S$, $S \setminus (S + x)$ is a finite union of finitely many translates of F_1 and F_2, and of a finite set.

Proof. It is clear that $F \subset S \cap \mathbb{Z}a$. Conversely, let $ka \in S$ for $k \in \mathbb{Z}$. By Lemma 7.2.3 there is n in $\mathbb{N}$ such that na and $na + ka$ are in F. Since F is a face, this implies that $ka \in F$.

Let C be as in 7.2.9. If $x \in S$, then by Lemma 7.2.9, there is $z \in S$ such that $C+z \subset S+x$. Now, $S \setminus (C+z)$ is a finite union of subsets of the form $(y+\mathbb{Z}a_1) \cap S$ or $(y + \mathbb{Z}a_2) \cap S$ (each diagonal in $\mathbb{Z}^2$ parallel to a_i is a finite union of subsets of the form $y + \mathbb{Z}a_i$), and thus, up to a finite set, a finite union of translates $y + F_i$, $i = 1, 2$. Therefore also $S \setminus (S+x)$ is a finite union of subsets $(y+F_i) \cap (S \setminus (S+x))$, $i = 1, 2$, $y \in S$.

By Corollary 7.2.6, for each translate $y + F_i$, the intersection with $S + x$ is empty or has finite complement in $y + F_i$. □

7.3 The regular C^*-algebra for a toric semigroup

We consider a finitely generated generating subsemigroup S of $\mathbb{Z}^2$ and denote by F_1, F_2 the two one-dimensional faces of S. We denote by λ the left regular representation of S on $\ell^2 S$ and by $C^*_\lambda S$ the C^*-algebra generated by $\lambda(S)$. As usual, there is the commutative sub-C^*-algebra D of $C^*_\lambda S$ which is generated by all range projections of the partial isometries obtained as all possible products of the $\lambda(s), s \in S$ and their adjoints.

Lemma 7.3.1. *D contains all orthogonal projections onto $\ell^2(X)$ where X is a finite subset of S. Consequently, $C^*_\lambda S$ contains the algebra $\mathcal{K}$ of all compact operators on $\ell^2 S$.*

Proof. D contains the orthogonal projection onto $\ell^2(X)$ where $X = (S \setminus (S + f_1)) \cap (S \setminus (S + f_2))$, $f_1 \in F_1, f_2 \in F_2$. Lemma 7.2.10 implies that X is finite. Consider now all subsets of S obtained as the intersection of X with finitely many translates $s + X$ with $s \in S - S$. Let Y denote a minimal set in this family. Then $(Y + s_1) \cap (Y + s_2) = \emptyset$ whenever $s_1 \neq s_2$. Let now $y_1, y_2 \in Y$. Then, since the enveloping group $S - S$ is $\mathbb{Z}^2$, there are $s_1, s_2 \in S$ such that $y_1 + s_1 = y_2 + s_2$. This implies $s_1 = s_2$ and thus also $y_1 = y_2$. We see that Y consists of only one point. The one-dimensional projection onto $\ell^2(Y)$ is in D and therefore also all of its translates. □

Given a subset X of S, denote by e_X the orthogonal projection onto the subspace $\ell^2 X \subset \ell^2 S$.

Lemma 7.3.2. *D is generated by the projections of the form $\lambda(s)e_F\lambda(s)^*$, for $s \in S$ and F a face of S.*

Proof. We show first that e_F is in D for each face. This is clear for the trivial faces $\{0\}$ and S. Thus, let F be one of the two one-dimensional faces, and C and z as in Lemma 7.2.9. There is $d \in \mathbb{Z}^2$ such that $C \setminus ((C+d) \cap C)$ is the face of C which contains F. Replacing z by a translate $z + x$, for a suitable x if necessary, we may clearly assume that $z+d+C \subset S$. It follows that $F = (((z+S) \setminus (z+d+S)) - z) \cap S$ and thus that

$$e_F = \lambda(z)^* \left(\lambda(z)\lambda(z)^* - \lambda(z+d)\lambda(z+d)^*\right) \lambda(z).$$

Denote by D_0 the subalgebra generated by all projections $\lambda(s)e_F\lambda(s)^*$. Since e_F is in D, we have $D_0 \subset D$. Moreover, D_0 then contains all diagonal projections of finite rank. Lemma 7.2.9 also implies that the complement of any range projection of $\lambda(s)$ for $s \in S$ is a linear combination of finitely many translates of projections of the form e_F. This shows that $D_0 = D$. □

Remark 7.3.3. *Toric semigroups typically do not satisfy the independence condition which says that the projections in D, obtained as range projections of products of elements $\lambda(s)$, $s \in S$ and their adjoints, should be linearly independent; see Definition 5.6.30. As a simple example, consider the semigroup $S \subset \mathbb{Z}^2$ defined by the cone spanned by the vectors $(2,1)$ and $(2,-1)$. Then the intersection of $(2,1)+S$ and $(2,0)+S$ equals the union of $(4,1)+S$ and $(4,0)+S$. This kind of phenomenon occurs for all toric semigroups except for the trivial ones.*

Lemma 7.3.4. *Let F be a two-dimensional subsemigroup of S. Then the quotient S/F is a finite abelian group and equal to the quotient $(S-S)/(F-F)$ of the enveloping groups. If $a = (k,l)$ and $b = (m,n)$ are generators of $F - F$, then the number of elements in S/F is given by the absolute value of the determinant*

$$\det \begin{pmatrix} k & m \\ l & n \end{pmatrix}.$$

Proof. Elements x, y in F become equal in F/S if and only if there are f, g in F such that $x + f = y + g$ and thus if and only if $x - y = g - f$, i.e., iff x, y become equal in $(S-S)/(F-F)$. This means that the map $S/F \to (S-S)/(F-F)$ is injective. Now, $F - F$ is a two-dimensional subgroup of $S - S = \mathbb{Z}^2$ and therefore $(S-S)/(F-F)$ is finite. Thus, the image of S/F in $(S-S)/(F-F)$ is a subsemigroup of a finite group and therefore already a group. The formula for the number of elements in S/F is well known and follows from the elementary divisor theorem. □

Lemma 7.3.5. *Let F be a finitely generated subsemigroup of $\mathbb{N}$ generating $\mathbb{Z}$ as a group. Then $C^*_\lambda F \cong C^*_\lambda \mathbb{N}$. Moreover, viewed as subalgebras of $\mathcal{L}(\ell^2\mathbb{N})$, the algebra $C^*_\lambda F$ is a subalgebra of $C^*_\lambda \mathbb{N}$ such that $C^*_\lambda F/\mathcal{K} = C^*_\lambda \mathbb{N}/\mathcal{K}$.*

Proof. Lemma 7.2.3 shows that $M = \mathbb{N} \setminus F$ is a finite set. Let $n \in \mathbb{N}$ be large enough so that $M \subset \{0, \ldots, n\}$ and e_n be the projection onto $\ell^2\{0, \ldots, n\}$. Then we can find $f, g \in F$ such that $\lambda_{\mathbb{N}}(1)(1 - e_n) = \lambda_F(g)^*\lambda_F(f)(1 - e_n)$ (we denote here by $\lambda_{\mathbb{N}}$, λ_F the left regular representations on $\ell^2\mathbb{N}$ and $\ell^2 F$, respectively). It follows that $C^*_\lambda F = (1 - e_M)\, C^*_\lambda(\mathbb{N})\, (1 - e_M)$. Moreover, using the fact that $C^*_\lambda\mathbb{N}$ is the universal C^*-algebra generated by a single isometry, it is trivially seen that $C^*_\lambda(\mathbb{N}) \cong (1 - e_M)\, C^*_\lambda(\mathbb{N})\, (1 - e_M)$. □

Let F_1, F_2 be the two one-dimensional faces of S and denote by I_1, I_2 the closed ideals generated in $C^*_\lambda S$ by e_{F_1}, and by e_{F_2}, respectively.

Lemma 7.3.6. *The intersection $I_1 \cap I_2$ is equal to $\mathcal{K}(\ell^2 S)$. Each quotient $I_j/\mathcal{K}(\ell^2 S)$ is isomorphic to $\mathcal{K}(\ell^2(S/F_j)) \otimes C(\mathbb{T})$.*

Moreover, the quotient $C^*_\lambda S/(I_1 + I_2)$ is isomorphic to $C(\mathbb{T}^2)$.

Proof. The first assertion follows from the fact that each intersection of a translate of F_1 and a translate of F_2 contains at most one point. The second assertion is a consequence of Corollary 7.2.6 in combination with Lemma 7.2.10 and Lemma 7.3.5. Finally, Lemma 7.2.10 also shows that any element $\lambda(s)$, $s \in S$, becomes unitary in the quotient $C^*_\lambda S/(I_1 + I_2)$ so that the quotient is isomorphic to the C^*-algebra of the enveloping group $\mathbb{Z}^2$ of S. □

As customary, we will, from now on, not distinguish between the algebras of compact operators on different separable infinite-dimensional Hilbert spaces and just write $\mathcal{K}$. For the K-theory of the C^*-algebra $C^*_\lambda S/\mathcal{K}$ we obtain the following six-term exact sequence

$$\begin{tikzcd} K_*(\dot{I}_1) \oplus K_*(\dot{I}_2) \arrow[r] & K_*(C^*_\lambda S/\mathcal{K}) \arrow[r] & K_*(C\mathbb{T}^2) \arrow[ll, bend left] \end{tikzcd} \tag{7.1}$$

where $\dot{I}_j$ denotes the quotient $I_j/\mathcal{K}$.

Lemma 7.3.7. *Let a_1, a_2 be the asymptotic generators of the faces F_1, F_2 ordered in such a way that $\det(a_1, a_2)$ is positive (this implies that $\det(a_1, s)$ and $\det(s, a_2)$ are positive for all $s \in S$). Denote by π the quotient map $C^*_\lambda S \to C(\mathbb{T}^2)$. Let $a_j = (x_j, y_j)$ and let $s = (m, n) \in S \backslash F_j$. The index map $K_1(C\mathbb{T}^2) = K_1(C^*_\lambda S/(I_1 + I_2)) \to K_0(\dot{I}_j) \cong \mathbb{Z}$, for the extension (7.1), maps the class of $\pi(\lambda(s))$ to*

$$(-1)^{j+1} \det(a_j, s) = (-1)^{j+1} \det \begin{pmatrix} x_j & m \\ y_j & n \end{pmatrix}.$$

Proof. By Lemma 7.2.10, the set $S \setminus (S + s)$ that represents the index of $\pi(\lambda(s))$ is, up to finite subsets, a union of finitely many translates of F_1 and F_2. When we project to $\dot{I}_j$, the number of translates of F_j that we obtain is, according to Lemma 7.2.8, given by the number of elements in $S/(F_j + \langle s \rangle)$, which in turn by Lemma 7.3.4 is determined by the absolute value of the determinant above. □

Lemma 7.3.8. *Let C and z be as in Lemma 7.2.9 and let a_1, a_2 denote the asymptotic generators of the faces F_1, F_2. Then the projection E onto $\ell^2(z + C)$ is in*

$C^*_\lambda S$. The formulas $v_i = \lambda(a_i)E$, $i = 1, 2$ define elements in $C^*_\lambda S$. The isometries v_1, v_2 are relatively prime in the sense that

$$v_1 v_2^* = v_2^* v_1.$$

In particular, the C^*-subalgebra of $C^*_\lambda S$ generated by v_1, v_2 is isomorphic to the Toeplitz algebra $C^*_\lambda \mathbb{N}^2$.

Proof. Each diagonal in S parallel to a_i is invariant under addition of F_i and therefore, up to finite sets, a finite union of translates of F_i. It follows that the complement of $z + C$ in S is, up to finite sets, a finite union of translates of F_1 and of F_2. Since the projection onto ℓ^2 of such a translate is in $C^*_\lambda S$, we see that $E = e_{z+C}$ is in $C^*_\lambda S$.

Consider now v_1 and v_2. Using Lemmas 7.2.3 and 7.3.5 in combination with the fact that $z+C$ is invariant under addition of a_i, we may assume that the formulas for v_1, v_2 actually define elements of $C^*_\lambda S$ (if $x \in F_i$ such that $x + a_i \in F_i$, then $\lambda(a_i)E = \lambda(x)^*\lambda(x + a_i)E$). The primeness condition is equivalent to the fact that the range projection of the product $v_1 v_2$ is equal to the product of the range projections of the v_1, v_2. Therefore, we have to show that $(z+C+a_1) \cap (z+C+a_2) = (z+C+a_1+a_2)$. But, since a_1, a_2 span the boundary of the cone, one clearly has that $(C + a_1) \cap (C + a_2) = (C + a_1 + a_2)$. □

Let S be a finitely generated subsemigroup of $\mathbb{Z}^2$ generating $\mathbb{Z}^2$ as a group. Let $a_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2)$ denote the asymptotic generators of the two one-dimensional faces F_1 and F_2 of S. In the following we use the integral 2×2 matrices:

$$M = \begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}, \quad M^\perp = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

Here again we order a_1, a_2 so that $\det M^\perp$ is positive. Note that M is the adjugate matrix to $M^\perp$ in the sense of Cramer's rule so that $\det M = \det M^\perp$ and $MM^\perp = \det M\,\mathbf{1}$.

Lemma 7.3.9. *Consider the extensions $0 \to I \to C^*_\lambda S \to C(\mathbb{T}^2) \to 0$ and $0 \to I' \to C^*_\lambda \mathbb{N}^2 \to C(\mathbb{T}^2) \to 0$, where I, I' denote the kernels of the quotient maps. By Lemma 7.3.8 there is a natural map $\kappa : C^*_\lambda \mathbb{N}^2 \to C^*_\lambda S$ which maps the generators of $C^*_\lambda \mathbb{N}^2$ to v_1, v_2, where v_1, v_2 are as in Lemma* 7.3.8*. Then we have the following:*

1. *$K_0(I) = K_0(I') = \mathbb{Z}^2$ and $K_1(I) = K_1(I') = \mathbb{Z}$. The generator of $K_1(I)$ is represented by $w = \lambda(a_1)e_{F_1} + \lambda(a_2)^* e_{F_2}$ (this is unitary mod $\mathcal{K}$).*
2. *The map $K_0(C\mathbb{T}^2) \to K_0(C\mathbb{T}^2)$ induced by κ maps the Bott element b to* $(\det M)\, b$.
3. *The boundary map $K_1(C\mathbb{T}^2) = \mathbb{Z}^2 \to K_0(I) = \mathbb{Z}^2$ is given by multiplication by M.*
4. *The map $K_1 I' \cong \mathbb{Z} \to K_1(I) \cong \mathbb{Z}$ induced by κ is multiplication by* $\det M$.

Proof. (1) It follows from Corollary 7.2.6 and Lemma 7.3.4 that the ideals I and I' are stably isomorphic (and, thus, since both are stable, even isomorphic). In the long exact K-theory sequence for the extension $0 \to I' \to C^*_\lambda\mathbb{N}^2 \to C(\mathbb{T}^2) \to 0$ we know that $K_0(C^*_\lambda\mathbb{N}^2) = \mathbb{Z}$ with generator $[1]$ and that $K_1(C^*_\lambda\mathbb{N}^2) = 0$. This shows that $K_1(I) \cong \mathbb{Z}$. The fact that the generator is represented by w follows from the K-theory sequence for the extension $0 \to \mathcal{K} \to I \to I/\mathcal{K} \to 0$ and the fact that $I/\mathcal{K} \cong (\mathcal{K} \otimes C(\mathbb{T})) \oplus (\mathcal{K} \otimes C(\mathbb{T}))$.

(2) It is obvious by definition that $\kappa_* : K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2 \longrightarrow K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ is given by multiplication by the matrix $M^\perp$. The Bott element is represented by the exterior product of the generators of $K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$. The map induced by multiplication by M on the exterior product is $\det M$ by definition of the determinant.

(3) In the isomorphism $K_0(\dot{I}) \cong \mathbb{Z}^2$ we identify $K_0(\dot{I}_2)$ with the first component and $K_0(\dot{I}_1)$ with the second component of $\mathbb{Z}^2$ (this convention is used for the identification of the maps in diagram (7.3) below). By Lemma 7.3.7 (and keeping in mind the reverse identification of the components in $\mathbb{Z}^2$) we know that the boundary map for the extension (7.1) maps an element $s = (m, n) \in K_1(C\mathbb{T}^2) \cong \mathbb{Z}^2$ to the element (k_2, k_1) in $\mathbb{Z}^2 \cong K_0(\dot{I})$ with components

$$k_i = (-1)^{i+1} \det \begin{pmatrix} x_i & m \\ y_i & n \end{pmatrix}$$

where (x_i, y_i) are the components of the asymptotic generators a_i of F_i, $i = 1, 2$. In other words,

$$\begin{pmatrix} k_2 \\ k_1 \end{pmatrix} = \begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

This describes the boundary map to $K_0(\dot{I})$ (with $\dot{I} = I/\mathcal{K}$). But the long exact sequence for the extension $0 \to \mathcal{K} \to I \to \dot{I} \to 0$ shows that the map $K_0(I) \to K_0(\dot{I})$ is an isomorphism (using the fact that the induced map $K_0(\mathcal{K}) \to K_0(I)$ is 0).

(4) As above, we write $\dot{I}, \dot{I}'$ for the quotients of I, I' by $\mathcal{K}$. We have

$$\dot{I} = \dot{I}_1 \oplus \dot{I}_2 \cong \mathcal{K} \otimes C(\mathbb{T}) \oplus \mathcal{K} \otimes C(\mathbb{T}) \tag{7.2}$$

and similarly for $\dot{I}'$. In particular, $K_1(\dot{I}) = K_0(\dot{I}) = \mathbb{Z}^2$ and the isomorphism (7.2) shows that the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ induced by κ acting on K_1 is the same as the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ induced by κ on K_0. This latter map μ fits into the following commutative diagram of boundary maps

$$\begin{tikzcd} K_1(C\mathbb{T}^2) = \mathbb{Z}^2 \arrow[r, "\mathrm{id}"] \arrow[d, "M^\perp\cdot"] & K_0(\dot{I}') = \mathbb{Z}^2 \arrow[d, "\mu"] \\ K_1(C\mathbb{T}^2) = \mathbb{Z}^2 \arrow[r, "M\cdot"] & K_0(\dot{I}) = \mathbb{Z}^2 \end{tikzcd} \tag{7.3}$$

Here, as in point (3) we identify $K_0(\dot{I}_2'), K_0(\dot{I}_2)$ with the first component and $K_0(\dot{I}_1'), K_0(\dot{I}_1)$ with the second component of $\mathbb{Z}^2$. By commutativity, μ is the same as multiplication by $MM^\perp = \det M\mathbf{1}$. Finally, the maps $K_1(I) \to K_1(\dot{I})$ and $K_1(I') \to K_1(\dot{I}')$ are injections so that the map $K_1I' \cong \mathbb{Z} \to K_1(I) \cong \mathbb{Z}$ induced by κ is the restriction of μ to the images of these maps. $\square$

Remark 7.3.10. *Since $S - S = \mathbb{Z}^2$, Lemma 7.3.4 shows that $\mathbb{Z}^2/M^\perp\mathbb{Z}^2$ is isomorphic to the quotient S/F where $F = F_1 + F_2$ is the subsemigroup generated by the faces F_1 and F_2. Note also, that M and $M^\perp$ have the same elementary divisors so that $\mathbb{Z}^2/M\mathbb{Z}^2 \cong \mathbb{Z}^2/M^\perp\mathbb{Z}^2$.*

Theorem 7.3.11. *Let S be a finitely generated subsemigroup of $\mathbb{Z}^2$ as above. The K-theory of $C^*_\lambda S$ is determined by the formula*

$$K_0(C^*_\lambda S) = S/F \oplus \mathbb{Z}, \quad K_1(C^*_\lambda S) = 0,$$

where F is the sum of the two one-dimensional faces in S.

Proof. We use the natural map $\kappa : C^*_\lambda \mathbb{N}^2 \to C^*_\lambda S$ mapping the generators of $C^*_\lambda \mathbb{N}^2$ to $v_1, v_2 \in C^*_\lambda S$ (see Lemma 7.3.8). We then compare the long exact sequence for the extension $0 \to I \to C^*_\lambda S \to C(\mathbb{T}^2) \to 0$ with the corresponding long exact sequence for the extension $0 \to I' \to C^*_\lambda \mathbb{N}^2 \to C(\mathbb{T}^2) \to 0$ and use the fact that the long exact K-theory sequence for the second extension is explicitly known. Using then that $K_0(I) = K_0(I') = \mathbb{Z}^2$, $K_1(I') = \mathbb{Z}$ and that $K_0(C\mathbb{T}^2) = \mathbb{Z}^2$, $K_1(C\mathbb{T}^2) = \mathbb{Z}^2$ we obtain the following morphism of exact sequences

$$\begin{tikzcd}
{} \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r, "\beta'"] \arrow[d, "\varphi"] & \mathbb{Z} \arrow[r] \arrow[d, "\psi"] & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r] \arrow[d] & {} \\
{} \arrow[r, "M\cdot"] & \mathbb{Z}^2 \arrow[r, "\gamma"] & K_0(C^*_\lambda S) \arrow[r] & \mathbb{Z}^2 \arrow[r, "\beta"] & \mathbb{Z} \arrow[r] & K_1(C^*_\lambda S) \arrow[r, "\alpha"] & \mathbb{Z}^2 \arrow[r, "M\cdot"] & {}
\end{tikzcd}$$

According to Lemma 7.3.9(2) and (4), the map φ maps the class [1] in $K_0(C\mathbb{T}^2)$ to [1] and multiplies the class b of the Bott element in $K_0(C\mathbb{T}^2)$ by $\det M$, while the map ψ is multiplication by $\det M$. Since $\beta'(b) = 1$, it follows that also $\beta(b) = 1$ and of course we have $\beta([1]) = 0$. Moreover, α is 0 since the subsequent map $M\cdot$ is injective. Thus, we see that $K_1(C^*_\lambda S) = 0$.

It also follows that $K_0(C^*_\lambda S)$ is an extension of $\mathrm{Im}\gamma \cong \mathbb{Z}^2/M\mathbb{Z}^2$ by $\mathrm{Ker}\beta = \mathbb{Z}$ and thus that $K_0(C^*_\lambda S) = \mathbb{Z}^2/M\mathbb{Z}^2 \oplus \mathbb{Z}$. Finally, the remark above shows that $\mathbb{Z}^2/M\mathbb{Z}^2 \cong S/F$. $\square$

Remark 7.3.12. *The proof of the theorem shows that the torsion part of $K_0(C^*_\lambda S)$ is generated by the classes of the projections e_{F_1}, e_{F_2}.*

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