Decomposition and Limit Theorems for a Class of Self-Similar Gaussian Processes

Daniel Harnett and David Nualart

Abstract We introduce a new class of self-similar Gaussian stochastic processes, where the covariance is defined in terms of a fractional Brownian motion and another Gaussian process. A special case is the solution in time to the fractional-colored stochastic heat equation described in Tudor (Analysis of variations for self-similar processes: a stochastic calculus approach. Springer, Berlin, 2013). We prove that the process can be decomposed into a fractional Brownian motion (with a different parameter than the one that defines the covariance), and a Gaussian process first described in Lei and Nualart (Stat Probab Lett 79:619–624, 2009). The component processes can be expressed as stochastic integrals with respect to the Brownian sheet. We then prove a central limit theorem about the Hermite variations of the process.

Keywords Fractional Brownian motion • Hermite variations • Self-similar processes • Stochastic heat equation

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1 Introduction

The purpose of this paper is to introduce a new class of Gaussian self-similar stochastic processes related to stochastic partial differential equations, and to establish a decomposition in law and a central limit theorem for the Hermite variations of the increments of such processes.

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Consider the d-dimensional stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \dot{W},\ t \geq 0,\ x \in \mathbb{R}^d, \tag{1.1}$$

with zero initial condition, where $\dot{W}$ is a zero mean Gaussian field with a covariance of the form

$$\mathbb{E}\left[\dot{W}^H(t,x)\dot{W}^H(s,y)\right] = \gamma_0(t-s)\Lambda(x-y), \quad s,t \geq 0,\ x,y \in \mathbb{R}^d.$$

We are interested in the process $U = \{U_t, t \geq 0\}$, where $U_t = u(t,0)$.

Suppose that $\dot{W}$ is white in time, that is, $\gamma_0 = \delta_0$ and the spatial covariance is the Riesz kernel, that is, $\Lambda(x) = c_{d,\beta}|x|^{-\beta}$, with $\beta < \min(d,2)$ and $c_{d,\beta} = \pi^{-d/2}2^{\beta-d}\Gamma(\beta/2)/\Gamma((d-\beta)/2)$. Then U has the covariance (see [14])

$$\mathbb{E}[U_tU_s] = D\left((t+s)^{1-\frac{\beta}{2}} - |t-s|^{1-\frac{\beta}{2}}\right), \quad s,t \geq 0, \tag{1.2}$$

for some constant

$$D = (2\pi)^{-d}(1-\beta/2)^{-1}\int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{2}}\frac{d\xi}{|\xi|^{d-\beta}}. \tag{1.3}$$

Up to a constant, the covariance (1.2) is the covariance of the *bifractional Brownian motion* with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{\beta}{2}$. We recall that, given constants $H \in (0,1)$ and $K \in (0,1)$, the bifractional Brownian motion $B^{H,K} = \{B_t^{H,K}, t \geq 0\}$, introduced in [4], is a centered Gaussian process with covariance

$$R_{H,K}(s,t) = \frac{1}{2^K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right), \quad s,t \geq 0.$$

When $K = 1$, the process $B^H = B^{H,1}$ is simply the fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, with covariance $R_H(s,t) = R_{H,1}(s,t)$. In [5], Lei and Nualart obtained the following decomposition in law for the bifractional Brownian motion

$$B^{H,K} = C_1B^{HK} + C_2Y^K_{t^{2H}},$$

where B^{HK} is a fBm with Hurst parameter HK, the process Y^K is given by

$$Y_t^K = \int_0^\infty y^{-\frac{1+K}{2}}(1-e^{-yt})dW_y, \tag{1.4}$$

with $W = \{W_y, y \geq 0\}$ a standard Brownian motion independent of $B^{H,K}$, and C_1, C_2 are constants given by $C_1 = 2^{\frac{1-K}{2}}$ and $C_2 = \sqrt{\frac{2-K}{\Gamma(1-K)}}$. The process Y^K

has trajectories which are infinitely differentiable on $(0,\infty)$ and Hölder continuous of order $HK-\epsilon$ in any interval $[0,T]$ for any $\epsilon > 0$. In particular, this leads to a decomposition in law of the process U with covariance (1.2) as the sum of a fractional Brownian motion with Hurst parameter $\frac{1}{2}-\frac{\beta}{4}$ plus a regular process.

The classical one-dimensional space-time white noise can also be considered as an extension of the covariance (1.2) if we take $\beta = 1$. In this case the covariance corresponds, up to a constant, to that of a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$.

The case where the noise term $\dot{W}$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ in time and a spatial covariance given by the Riesz kernel, that is,

$$\mathbb{E}\left[\dot{W}^H(t,x)\dot{W}^H(s,y)\right] = \alpha_H c_{d,\beta}|s-t|^{2H-2}|x-y|^{-\beta},$$

where $0 < \beta < \min(d,2)$ and $\alpha_H = H(2H-1)$, has been considered by Tudor and Xiao in [14]. In this case the corresponding process U has the covariance

$$\mathbb{E}[U_t U_s] = D\alpha_H \int_0^t \int_0^s |u-v|^{2H-2}(t+s-u-v)^{-\gamma}\,du dv, \tag{1.5}$$

where D is given in (1.3) and $\gamma = \frac{d-\beta}{2}$. This process is self-similar with parameter $H-\frac{\gamma}{2}$ and it has been studied in a series of papers [1, 8, 12–14]. In particular, in [14] it is proved that the process U can be decomposed into the sum of a scaled fBm with parameter $H-\frac{\gamma}{2}$, and a Gaussian process V with continuously differentiable trajectories. This decomposition is based on the stochastic heat equation. As a consequence, one can derive the exact uniform and local moduli of continuity and Chung-type laws of the iterated logarithm for this process. In [12], assuming that $d = 1, 2$ or 3, a central limit theorem is obtained for the renormalized quadratic variation

$$V_n = n^{2H-\gamma-\frac{1}{2}} \sum_{j=0}^{n-1} \left\{(U_{(j+1)T/n} - U_{jT/n})^2 - \mathbb{E}\left[(U_{(j+1)t/n} - U_{jT/n})^2\right]\right\},$$

assuming $\frac{1}{2} < H < \frac{3}{4}$, extending well-known results for fBm (see for example [6, Theorem 7.4.1]).

The purpose of this paper is to establish a decomposition in law, similar to that obtained by Lei and Nualart in [5] for the bifractional Brownian motion, and a central limit theorem for the Hermite variations of the increments, for a class of self-similar processes that includes the covariance (1.5). Consider a centered Gaussian process $\{X_t, t \geq 0\}$ with covariance

$$R(s,t) = \mathbb{E}[X_s X_t] = \mathbb{E}\left[\left(\int_0^t Z_{t-r}\,dB_r^H\right)\left(\int_0^s Z_{s-r}\,dB_r^H\right)\right], \tag{1.6}$$

where

(i) $B^H = \{B_t^H, t \geq 0\}$ is a fBm with Hurst parameter $H \in (0,1)$.
(ii) $Z = \{Z_t, t > 0\}$ is a zero-mean Gaussian process, independent of B^H, with covariance

$$\mathbb{E}[Z_s Z_t] = (s+t)^{-\gamma}, \tag{1.7}$$

where $0 < \gamma < 2H$.

In other words, X is a Gaussian process with the same covariance as the process $\{\int_0^t Z_{t-r} dB_r^H, t \geq 0\}$, which is not Gaussian.

When $H \in (\frac{1}{2}, 1)$, the covariance (1.6) coincides with (1.5) with $D = 1$. However, we allow the range of parameters $0 < H < 1$ and $0 < \gamma < 2H$. In other words, up to a constant, X has the law of the solution in time of the stochastic heat equation (1.1), when $H \in (0,1)$, $d \geq 1$ and $\beta = d - 2\gamma$. Also of interest is that X can be constructed as a sum of stochastic integrals with respect to the Brownian sheet (see the proof of Theorem 1).

1.1 Decomposition of the Process X

Our first result is the following decomposition in law of the process X as the sum of a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2} = H - \frac{\gamma}{2}$ plus a process with regular trajectories.

Theorem 1 *The process X has the same law as $\{\sqrt{\kappa} B_t^{\frac{\alpha}{2}} + Y_t, t \geq 0\}$, where here and in what follows, $\alpha = 2H - \gamma$,*

$$\kappa = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{z^{\gamma-1}}{1+z^2}\, dz, \tag{1.8}$$

$B^{\frac{\alpha}{2}}$ is a fBm with Hurst parameter $\frac{\alpha}{2}$, and Y (up to a constant) has the same law as the process Y^K defined in (1.4), with $K = 2\alpha + 1$, that is, Y is a centered Gaussian process with covariance given by

$$\mathbb{E}\,[Y_t Y_s] = \lambda_1 \int_0^\infty y^{-\alpha-1}(1-e^{-yt})(1-e^{-ys})\, dy,$$

where

$$\lambda_1 = \frac{4\pi}{\Gamma(\gamma)\Gamma(2H+1)\sin(\pi H)} \int_0^\infty \frac{\eta^{1-2H}}{1+\eta^2} d\eta.$$

The proof of this theorem is given in Sect. 3.

1.2 Hermite Variations of the Process

For each integer $q \geq 0$, the qth Hermite polynomial is given by

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}.$$

See [6, Sect. 1.4] for a discussion of properties of these polynomials. In particular, it is well known that the family $\{\frac{1}{\sqrt{q!}} H_q, q \geq 0\}$ constitutes an orthonormal basis of the space $L^2(\mathbb{R}, \gamma)$, where γ is the $N(0,1)$ measure.

Suppose $\{Z_n, n \geq 1\}$ is a stationary, Gaussian sequence, where each Z_n follows the $N(0,1)$ distribution with covariance function $\rho(k) = \mathbb{E}[Z_n Z_{n+k}]$. If $\sum_{k=1}^{\infty} |\rho(k)|^q < \infty$, it is well known that as n tends to infinity, the Hermite variation

$$V_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} H_q(Z_j) \tag{1.9}$$

converges in distribution to a Gaussian random variable with mean zero and variance given by $\sigma^2 = q! \sum_{k=1}^{\infty} \rho(k)^q$. This result was proved by Breuer and Major in [3]. In particular, if B^H is a fBm, then the sequence $\{Z_{j,n}, 0 \leq j \leq n-1\}$ defined by

$$Z_{j,n} = n^H \left(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}} \right)$$

is a stationary sequence with unit variance. As a consequence, if $H < 1 - \frac{1}{q}$, we have that

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} H_q \left(n^H \left(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}} \right) \right)$$

converges to a normal law with variance given by

$$\sigma_q^2 = \frac{q!}{2^q} \sum_{m \in \mathbb{Z}} \left(|m+1|^{2H} - 2|m|^{2H} + |m-1|^{2H} \right)^q. \tag{1.10}$$

See [3] and Theorem 7.4.1 of [6].

The above Breuer-Major theorem can not be applied to our process because X is not necessarily stationary. However, we have a comparable result.

Theorem 2 *Let $q \geq 2$ be an integer and fix a real $T > 0$. Suppose that $\alpha < 2 - \frac{1}{q}$, where α is defined in Theorem 1. For $t \in [0, T]$, define,*

$$F_n(t) = n^{-\frac{1}{2}} \sum_{j=0}^{\lfloor nt \rfloor - 1} H_q \left(\frac{\Delta X_{\frac{j}{n}}}{\left\| \Delta X_{\frac{j}{n}} \right\|_{L^2(\Omega)}} \right),$$

where $H_q(x)$ denotes the qth Hermite polynomial. Then as $n \to \infty$, the stochastic process $\{F_n(t), t \in [0,T]\}$ converges in law in the Skorohod space $D([0,T])$, to a scaled Brownian motion $\{\sigma B_t, t \in [0,T]\}$, where $\{B_t, t \in [0,T]\}$ is a standard Brownian motion and $\sigma = \sqrt{\sigma^2}$ is given by

$$\sigma^2 = \frac{q!}{2^q} \sum_{m \in \mathbb{Z}} \left(|m+1|^\alpha - 2|m|^\alpha + |m-1|^\alpha\right)^q . \tag{1.11}$$

The proof of this theorem is given in Sect. 4.

2 Preliminaries

2.1 Analysis on the Wiener Space

The reader may refer to [6, 7] for a detailed coverage of this topic. Let $Z = \{Z(h), h \in \mathcal{H}\}$ be an *isonormal Gaussian process* on a probability space $(\Omega, \mathcal{F}, P)$, indexed by a real separable Hilbert space $\mathcal{H}$. This means that Z is a family of Gaussian random variables such that $\mathbb{E}[Z(h)] = 0$ and $\mathbb{E}[Z(h)Z(g)] = \langle h, g\rangle_{\mathcal{H}}$ for all $h, g \in \mathcal{H}$.

For integers $q \geq 1$, let $\mathcal{H}^{\otimes q}$ denote the qth tensor product of $\mathcal{H}$, and $\mathcal{H}^{\odot q}$ denote the subspace of symmetric elements of $\mathcal{H}^{\otimes q}$.

Let $\{e_n, n \geq 1\}$ be a complete orthonormal system in $\mathcal{H}$. For elements $f, g \in \mathcal{H}^{\odot q}$ and $p \in \{0, \ldots, q\}$, we define the pth-order contraction of f and g as that element of $\mathcal{H}^{\otimes 2(q-p)}$ given by

$$f \otimes_p g = \sum_{i_1,\ldots,i_p=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p}\rangle_{\mathcal{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p}\rangle_{\mathcal{H}^{\otimes p}} , \tag{2.1}$$

where $f \otimes_0 g = f \otimes g$. Note that, if $f, g \in \mathcal{H}^{\odot q}$, then $f \otimes_q g = \langle f, g\rangle_{\mathcal{H}^{\odot q}}$. In particular, if f, g are real-valued functions in $\mathcal{H}^{\otimes 2} = L^2(\mathbb{R}^2, \mathcal{B}^2, \mu^2)$ for a non-atomic measure μ, then we have

$$f \otimes_1 g = \int_{\mathbb{R}} f(s, t_1) g(s, t_2)\, \mu(ds). \tag{2.2}$$

Let $\mathcal{H}_q$ be the qth Wiener chaos of Z, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(Z(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_q(x)$ is the qth Hermite polynomial. It can be shown (see [6, Proposition 2.2.1]) that if $Z, Y \sim N(0,1)$ are jointly Gaussian, then

$$\mathbb{E}\left[H_p(Z) H_q(Y)\right] = \begin{cases} p!\,(\mathbb{E}[ZY])^p & \textit{if } p = q \\ 0 & \text{otherwise} \end{cases} . \tag{2.3}$$

For $q \geq 1$, it is known that the map

$$I_q(h^{\otimes q}) = H_q(Z(h)) \tag{2.4}$$

provides a linear isometry between $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\|\cdot\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$, where $I_q(\cdot)$ is the generalized Wiener-Itô stochastic integral (see [6, Theorem 2.7.7]). By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$.

We use the following integral multiplication theorem from [7, Proposition 1.1.3]. Suppose $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$. Then

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\widetilde{\otimes}_r g), \tag{2.5}$$

where $f\widetilde{\otimes}_r g$ denotes the symmetrization of $f \otimes_r g$. For a product of more than two integrals, see Peccati and Taqqu [9].

2.2 Stochastic Integration and fBm

We refer to the 'time domain' and 'spectral domain' representations of fBm. The reader may refer to [10, 11] for details. Let $\mathcal{E}$ denote the set of real-valued step functions on $\mathbb{R}$. Let B^H denote fBm with Hurst parameter H. For this case, we view B^H as an isonormal Gaussian process on the Hilbert space $\mathfrak{H}$, which is the closure of $\mathcal{E}$ with respect to the inner product $\langle f, g\rangle_{\mathfrak{H}} = \mathbb{E}\left[I(f)I(g)\right]$. Consider also the inner product space

$$\tilde{\Lambda}_H = \left\{ f : f \in L^2(\mathbb{R}), \int_{\mathbb{R}} |\mathcal{F}f(\xi)|^2 |\xi|^{1-2H} d\xi < \infty \right\},$$

where $\mathcal{F}f = \int_{\mathbb{R}} f(x)e^{i\xi x}dx$ is the Fourier transform, and the inner product of $\tilde{\Lambda}_H$ is given by

$$\langle f, g\rangle_{\tilde{\Lambda}_H} = \frac{1}{C_H^2}\int_{\mathbb{R}} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}|\xi|^{1-2H}d\xi, \tag{2.6}$$

where $C_H = \left(\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}\right)^{\frac{1}{2}}$. It is known (see [10, Theorem 3.1]) that the space $\tilde{\Lambda}_H$ is isometric to a subspace of $\mathfrak{H}$, and $\tilde{\Lambda}_H$ contains $\mathcal{E}$ as a dense subset. This inner product (2.6) is known as the 'spectral measure' of fBm. In the case $H \in (\frac{1}{2}, 1)$, there is another isometry from the space

$$|\Lambda_H| = \left\{ f : \int_0^\infty \int_0^\infty |f(u)||f(v)||u-v|^{2H-2}du\, dv < \infty \right\}$$

to a subspace of $\mathfrak{H}$, where the inner product is defined as

$$\langle f, g\rangle_{|\Lambda_H|} = H(2H-1)\int_0^\infty \int_0^\infty f(u)g(v)|u-v|^{2H-2}du\, dv,$$

see [10] or [7, Sect. 5.1].

3 Proof of Theorem 1

For any $\gamma > 0$ and $\lambda > 0$, we can write

$$\lambda^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty y^{\gamma-1}e^{-\lambda y}dy,$$

where Γ is the Gamma function defined by $\Gamma(\gamma) = \int_0^\infty y^{\gamma-1}e^{-y}dy$. As a consequence, the covariance (1.7) can be written as

$$\mathbb{E}[Z_sZ_t] = \frac{1}{\Gamma(\gamma)}\int_0^\infty y^{\gamma-1}e^{-(t+s)y}dy. \tag{3.1}$$

Notice that this representation implies the covariance (1.7) is positive definite. Taking first the expectation with respect to the process Z, and using formula (3.1), we obtain

$$\begin{aligned} R(s,t) &= \frac{1}{\Gamma(\gamma)}\int_0^\infty \mathbb{E}\left[\left(\int_0^t e^{yu}dB_u^H\right)\left(\int_0^s e^{yu}dB_u^H\right)\right]y^{\gamma-1}e^{-(t+s)y}dy \\ &= \frac{1}{\Gamma(\gamma)}\int_0^\infty \left\langle e^{yu}\mathbf{1}_{[0,t]}(u), e^{yv}\mathbf{1}_{[0,s]}(v)\right\rangle_{\mathfrak{H}} y^{\gamma-1}e^{-(t+s)y}dy. \end{aligned}$$

Using the isometry between $\tilde{\Lambda}_H$ and a subspace of $\mathfrak{H}$ (see Sect. 2.2), we can write

$$\begin{aligned} \left\langle e^{yu}\mathbf{1}_{[0,t]}(u), e^{yv}\mathbf{1}_{[0,s]}(v)\right\rangle_{\mathfrak{H}} &= C_H^{-2}\int_{\mathbb{R}} |\xi|^{1-2H}(\mathcal{F}\mathbf{1}_{[0,t]}e^{y\cdot})(\overline{\mathcal{F}\mathbf{1}_{[0,s]}e^{y\cdot}})\, d\xi \\ &= C_H^{-2}\int_{\mathbb{R}} \frac{|\xi|^{1-2H}}{y^2+\xi^2}\left(e^{yt+i\xi t}-1\right)\left(e^{ys-i\xi s}-1\right)\, d\xi, \end{aligned}$$

where $(\mathcal{F}\mathbf{1}_{[0,t]}e^{x\cdot})$ denotes the Fourier transform and $C_H = \left(\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}\right)^{\frac{1}{2}}$. This allows us to write, making the change of variable $\xi = \eta y$,

$$R(s,t) = \frac{1}{\Gamma(\gamma)C_H^2}\int_0^\infty \int_{\mathbb{R}} y^{\gamma-1}\frac{|\xi|^{1-2H}}{y^2+\xi^2}\left(e^{i\xi t}-e^{-yt}\right)\left(e^{-i\xi s}-e^{-ys}\right)\, d\xi\, dy$$

$$= \frac{1}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_{\mathbb{R}} y^{-\alpha-1} \frac{|\eta|^{1-2H}}{1+\eta^2} \left(e^{i\eta yt} - e^{-yt}\right)\left(e^{-i\eta ys} - e^{-ys}\right)\, d\eta\, dy, \tag{3.2}$$

where $\alpha = 2H - \gamma$. By Euler's identity, adding and subtracting 1 to compensate the singularity of $y^{-\alpha-1}$ at the origin, we can write

$$e^{i\eta yt} - e^{-yt} = (\cos(\eta yt) - 1 + i\sin(\eta yt)) + (1 - e^{-yt}). \tag{3.3}$$

Substituting (3.3) into (3.2) and taking into account that the integral of the imaginary part vanishes because it is an odd function, we obtain

$$\begin{aligned} R(s,t) = {} & \frac{2}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_0^\infty y^{-\alpha-1} \frac{\eta^{1-2H}}{1+\eta^2} \Big((1-\cos(\eta yt))(1-\cos(\eta ys)) \\ & + \sin(\eta yt)\sin(\eta ys) + (\cos(\eta ys) - 1)(1-e^{-yt}) + (\cos(\eta yt)-1)(1-e^{-ys}) \\ & + (1-e^{-yt})(1-e^{-ys})\Big)\, d\eta\, dy. \end{aligned}$$

Let $B^{(j)} = \{B^{(j)}(\eta,t), \eta \geq 0, t \geq 0\}, j = 1,2$ denote two independent Brownian sheets. That is, for $j = 1,2$, $B^{(j)}$ is a continuous Gaussian field with mean zero and covariance given by

$$\mathbb{E}\left[B^{(j)}(\eta,t)B^{(j)}(\xi,s)\right] = \min(\eta,\xi) \times \min(t,s).$$

We define the following stochastic processes:

$$U_t = \frac{\sqrt{2}}{\sqrt{\Gamma(\gamma)}C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2}-\frac{1}{2}} \sqrt{\frac{\eta^{1-2H}}{1+\eta^2}}\, (\cos(\eta yt) - 1)\, B^{(1)}(d\eta, dy), \tag{3.4}$$

$$V_t = \frac{\sqrt{2}}{\sqrt{\Gamma(\gamma)}C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2}-\frac{1}{2}} \sqrt{\frac{\eta^{1-2H}}{1+\eta^2}}\, (\sin(\eta yt))\, B^{(2)}(d\eta, dy), \tag{3.5}$$

$$Y_t = \frac{\sqrt{2}}{\sqrt{\Gamma(\gamma)}C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2}-\frac{1}{2}} \sqrt{\frac{\eta^{1-2H}}{1+\eta^2}}\, \left(1 - e^{-yt}\right) B^{(1)}(d\eta, dy), \tag{3.6}$$

where the integrals are Wiener-Itô integrals with respect to the Brownian sheet. We then define the stochastic process $X = \{X_t, t \geq 0\}$ by $X_t = U_t + V_t + Y_t$, and we have $\mathbb{E}[X_s X_t] = R(s,t)$ as given in (3.2). These processes have the following properties:

(I) The process $W_t = U_t + V_t$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2}$ scaled with the constant $\sqrt{\kappa}$. In fact, the covariance of this process is

$$\mathbb{E}[W_t W_s] = \frac{2}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_0^\infty y^{-\alpha-1} \frac{\eta^{1-2H}}{1+\eta^2}\Big((\cos(\eta y t) - 1)(\cos(\eta y s) - 1) + \sin(\eta y t)\sin(\eta y s)\Big) d\eta dy$$

$$= \frac{1}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_{\mathbb{R}} y^{\gamma-1} \frac{|\xi|^{1-2H}}{y^2+\xi^2}(e^{i\xi t} - 1)(e^{-i\xi s} - 1) d\xi dy.$$

Integrating in the variable y we finally obtain

$$\mathbb{E}[W_t W_s] = \frac{c_1}{\Gamma(\gamma)C_H^2} \int_{\mathbb{R}} \frac{(e^{i\xi t} - 1)(e^{-i\xi s} - 1)}{|\xi|^{\alpha+1}} d\xi,$$

where $c_1 = \int_0^\infty \frac{z^{\gamma-1}}{1+z^2} dz = \kappa\Gamma(\gamma)$. Taking into account the Fourier transform representation of fBm (see [11, p. 328]), this implies $\kappa^{-\frac{1}{2}} W$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2}$.

(II) The process Y coincides, up to a constant, with the process Y^K introduced in (1.4) with $K = 2\alpha + 1$. In fact, the covariance of this process is given by

$$\mathbb{E}[Y_t Y_s] = \frac{2c_2}{\Gamma(\gamma)C_H^2} \int_0^\infty y^{-\alpha-1}(1 - e^{-yt})(1 - e^{-ys}) dy, \tag{3.7}$$

where

$$c_2 = \int_0^\infty \frac{\eta^{1-2H}}{1+\eta^2} d\eta.$$

Notice that the process X is self-similar with exponent $\frac{\alpha}{2}$. This concludes the proof of Theorem 1.

4 Proof of Theorem 2

Along the proof, the symbol C denotes a generic, positive constant, which may change from line to line. The value of C will depend on parameters of the process and on T, but not on the increment width n^{-1}.

For integers $n \geq 1$, define a partition of $[0, \infty)$ composed of the intervals $\{[\frac{j}{n}, \frac{j+1}{n}), j \geq 0\}$. For the process X and related processes U, V, W, Y defined in Sect. 3, we introduce the notation

$$\Delta X_{\frac{j}{n}} = X_{\frac{j+1}{n}} - X_{\frac{j}{n}} \text{ and } \Delta X_0 = X_{\frac{1}{n}},$$

with corresponding notation for U, V, W, Y. We start the proof of Theorem 2 with two technical results about the components of the increments.

4.1 Preliminary Lemmas

Lemma 3 *Using above notation with integers $n \geq 2$ and $j, k \geq 0$, we have*

(a) $\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right] = \frac{\kappa}{2} n^{-\alpha} \left(|j-k-1|^{\alpha} - 2|j-k|^{\alpha} + |j-k-1|^{\alpha}\right)$, *where κ is defined in (1.8).*

(b) For $j+k \geq 1$,

$$\left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right]\right| \leq C n^{-\alpha} (j+k)^{\alpha-2}$$

for a constant $C > 0$ that is independent of j, k and n.

Proof Property (a) is well-known for fractional Brownian motion. For (b), we have from (3.7):

$$\begin{aligned}
\mathbb{E}\left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right] &= \frac{2c_2}{\Gamma(\gamma) C_H^2 n^{\alpha}} \int_0^{\infty} y^{-\alpha-1} \left(e^{-yj} - e^{-y(j+1)}\right) \left(e^{-yk} - e^{-y(k+1)}\right) dy \\
&= \frac{2c_2}{\Gamma(\gamma) C_H^2 n^{\alpha}} \int_0^{\infty} y^{-\alpha+1} \int_0^1 \int_0^1 e^{-y(j+k+u+v)} du\, dv\, dy.
\end{aligned}$$

Note that the above integral is nonnegative, and we can bound this with

$$\begin{aligned}
\left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right]\right| &\leq C n^{-\alpha} \int_0^{\infty} y^{-\alpha+1} e^{-y(j+k)}\, dy \\
&= C n^{-\alpha} (j+k)^{\alpha-2} \int_0^{\infty} u^{-\alpha+1} e^{-u} du \\
&\leq C n^{-\alpha} (j+k)^{\alpha-2}.
\end{aligned}$$

□

Lemma 4 *For $n \geq 2$ fixed and integers $j, k \geq 1$,*

$$\left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right]\right| \leq C n^{-\alpha} j^{2H-2} k^{-\gamma}$$

for a constant $C > 0$ that is independent of j, k and n.

Proof From (3.4)–(3.6) in the proof of Theorem 1, observe that

$$\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right] = \mathbb{E}\left[(\Delta U_{\frac{j}{n}} + \Delta V_{\frac{j}{n}}) \Delta Y_{\frac{k}{n}}\right] = \mathbb{E}\left[\Delta U_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right].$$

Assume $s, t > 0$. By self-similarity we can define the covariance function ψ by $\mathbb{E}[U_t Y_s] = s^\alpha \mathbb{E}\left[U_{t/s} Y_1\right] = s^\alpha \psi(t/s)$, where, using the change-of-variable $\theta = \eta x$,

$$\psi(x) = \int_0^\infty \int_0^\infty y^{-\alpha-1} \frac{\eta^{1-2H}}{1+\eta^2} (\cos(y\eta x) - 1)(1 - e^{-y})\, d\eta\, dy$$
$$= \int_0^\infty y^{-\alpha-1}(1 - e^{-y}) \int_0^\infty \frac{\theta^{1-2H} x^{2H}}{x^2 + \theta^2} (\cos(y\theta) - 1)\; d\theta\; dy.$$

Then using the fact that

$$\left| \frac{\theta^{1-2H} x^{2H}}{x^2 + \theta^2} \right| \le |\theta^{-2H}|\, |x|^{2H-1}, \tag{4.1}$$

we see that $|\psi(x)| \le C x^{2H-1}$, and

$$\psi'(x) = 2H \int_0^\infty y^{-\alpha-1}(1 - e^{-y}) \int_0^\infty \frac{\theta^{1-2H} x^{2H-1}}{x^2 + \theta^2} (\cos(y\theta) - 1)\; d\theta\; dy$$
$$- 2 \int_0^\infty y^{-\alpha-1}(1 - e^{-y}) \int_0^\infty \frac{\theta^{1-2H} x^{2H+1}}{(x^2 + \theta^2)^2} (\cos(y\theta) - 1)\; d\theta\; dy.$$

Using (4.1) and similarly

$$\left| \frac{\theta^{1-2H} x^{2H+1}}{(x^2 + \theta^2)^2} \right| \le |\theta^{-2H}|\, |x|^{2H-2}, \tag{4.2}$$

we can write

$$\left|\psi'(x)\right| \le x^{2H-2} |2H - 2| \int_0^\infty y^{-\alpha-1}(1 - e^{-y}) \int_0^\infty \theta^{-2H} (\cos(y\theta) - 1)\; d\theta\; dy \le C x^{2H-2}.$$

By continuing the computation, we can find that $|\psi''(x)| \le C x^{2H-3}$. We have for $j, k \ge 1$,

$$\mathbb{E}\left[\Delta U_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right] = n^{-\alpha}(k+1)^\alpha \left(\psi\left(\frac{j+1}{k+1}\right) - \psi\left(\frac{j}{k+1}\right)\right)$$
$$- n^{-\alpha} k^\alpha \left(\psi\left(\frac{j+1}{k}\right) - \psi\left(\frac{j}{k}\right)\right)$$
$$= n^{-\alpha}\left((k+1)^\alpha - k^\alpha\right)\left(\psi\left(\frac{j+1}{k+1}\right) - \psi\left(\frac{j}{k+1}\right)\right)$$
$$+ n^{-\alpha} k^\alpha \left(\psi\left(\frac{j+1}{k+1}\right) - \psi\left(\frac{j}{k+1}\right) - \psi\left(\frac{j+1}{k}\right) + \psi\left(\frac{j}{k}\right)\right).$$

With the above bounds on ψ and its derivatives, the first term is bounded by

$$
\begin{aligned}
n^{-\alpha} \left|(k+1)^{\alpha} - k^{\alpha}\right| & \left|\psi\left(\frac{j+1}{k+1}\right) - \psi\left(\frac{j}{k+1}\right)\right| \\
&\leq \alpha n^{-\alpha} \int_0^1 (k+u)^{\alpha-1} du \int_0^{\frac{1}{k+1}} \left|\psi'\left(\frac{j}{k+1} + v\right)\right| dv \\
&\leq C n^{-\alpha} k^{\alpha-2} \left(\frac{j}{k}\right)^{2H-2} \leq C n^{-\alpha} k^{-\gamma} j^{2H-2},
\end{aligned}
$$

and

$$
\begin{aligned}
n^{-\alpha} k^{\alpha} & \left|\psi\left(\frac{j+1}{k+1}\right) - \psi\left(\frac{j}{k+1}\right) - \psi\left(\frac{j+1}{k}\right) + \psi\left(\frac{j}{k}\right)\right| \\
&= n^{-\alpha} k^{\alpha} \left|\int_0^{\frac{1}{k+1}} \psi'\left(\frac{j}{k+1} + u\right) du - \int_0^{\frac{1}{k}} \psi'\left(\frac{j}{k} + u\right) du\right| \\
&\leq n^{-\alpha} k^{\alpha} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left|\psi'\left(\frac{j}{k} + u\right)\right| du + \int_0^{\frac{1}{k+1}} \int_{\frac{j}{k+1}}^{\frac{j}{k}} \left|\psi''(u+v)\right| dv\, du \\
&\leq C n^{-\alpha} k^{\alpha-2} \left(\frac{j}{k}\right)^{2H-2} + C n^{-\alpha} k^{\alpha-3} j \left(\frac{j}{k}\right)^{2H-3} \leq C n^{-\alpha} k^{-\gamma} j^{2H-2}.
\end{aligned}
$$

This concludes the proof of the lemma. □

4.2 *Proof of Theorem 2*

We will make use of the notation $\beta_{j,n} = \left\|\Delta X_{\frac{j}{n}}\right\|_{L^2(\Omega)}$. We have for integer $j \geq 1$,

$$
\beta_{j,n}^2 = \mathbb{E}\left[\Delta W_{\frac{j}{n}}^2\right] + \mathbb{E}\left[\Delta Y_{\frac{j}{n}}^2\right] + 2\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{j}{n}}\right] = \kappa n^{-\alpha}(1 + \theta_{j,n}),
$$

where

$$
\kappa n^{-\alpha} \theta_{j,n} = \mathbb{E}\left[\Delta Y_{\frac{j}{n}}^2\right] + 2\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{j}{n}}\right].
$$

It follows from Lemmas 3 and 4 that $|\theta_{j,n}| \leq C j^{\alpha-2}$ for some constant $C > 0$. Notice that, in the definition of $F_n(t)$, it suffices to consider the sum for $j \geq n_0$ for a fixed n_0. Then, we can choose n_0 in such a way that $C n_0^{\alpha-2} \leq \frac{1}{2}$, which implies

$$
\beta_{j,n}^2 \geq \kappa n^{-\alpha}(1 - C j^{\alpha-2}) \tag{4.3}
$$

for any $j \geq n_0$.

By (2.4),

$$\beta_{j,n}^q H_q\left(\beta_{j,n}^{-1}\Delta X_{\frac{j}{n}}\right) = I_q^X\left(\left(\mathbf{1}_{[\frac{j}{n},\frac{j+1}{n})}\right)^{\otimes q}\right),$$

where I_q^X denotes the multiple stochastic integral of order q with respect to the process X. Thus, we can write

$$F_n(t) = n^{-\frac{1}{2}}\sum_{j=n_0}^{\lfloor nt\rfloor-1}\beta_{j,n}^{-q} I_q^X\left(\mathbf{1}_{\left[\frac{j}{n},\frac{j+1}{n}\right)}^{\otimes q}\right).$$

The decomposition $X = W + Y$ leads to

$$I_q^X\left(\mathbf{1}_{\left[\frac{j}{n},\frac{j+1}{n}\right)}^{\otimes q}\right) = \sum_{r=0}^{q}\binom{q}{r} I_r^W\left(\mathbf{1}_{\left[\frac{j}{n},\frac{j+1}{n}\right)}^{\otimes r}\right) I_{q-r}^Y\left(\mathbf{1}_{\left[\frac{j}{n},\frac{j+1}{n}\right)}^{\otimes q-r}\right).$$

We are going to show that the terms with $r = 0, \ldots, q-1$ do not contribute to the limit. Define

$$G_n(t) = n^{-\frac{1}{2}}\sum_{j=n_0}^{\lfloor nt\rfloor-1}\beta_{j,n}^{-q} I_q^W\left(\mathbf{1}_{\left[\frac{j}{n},\frac{j+1}{n}\right)}^{\otimes q}\right)$$

and

$$\widetilde{G}_n(t) = n^{-\frac{1}{2}}\sum_{j=n_0}^{\lfloor nt\rfloor-1}\left\|\Delta W_{j/n}\right\|_{L^2(\Omega)}^{-q} I_q^W\left(\mathbf{1}_{\left[\frac{j}{n},\frac{j+1}{n}\right)}^{\otimes q}\right).$$

Consider the decomposition

$$F_n(t) = (F_n(t) - G_n(t)) + (G_n(t) - \widetilde{G}_n(t)) + \widetilde{G}_n(t).$$

Notice that all these processes vanish at $t = 0$. We claim that for any $0 \leq s < t \leq T$, we have

$$\mathbb{E}[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^2] \leq \frac{(\lfloor nt\rfloor - \lfloor ns\rfloor)^\delta}{n} \tag{4.4}$$

and

$$\mathbb{E}[|G_n(t) - \widetilde{G}_n(t) - (G_n(s) - \widetilde{G}_n(s))|^2] \leq \frac{(\lfloor nt\rfloor - \lfloor ns\rfloor)^\delta}{n}, \tag{4.5}$$

where $0 \le \delta < 1$. By Lemma 3, $\|\Delta W_{j/n}\|^2_{L^2(\Omega)} = \kappa n^{-\alpha}$ for every j. As a consequence, using (2.4) we can also write

$$\widetilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} H_q\left(\kappa^{-\frac{1}{2}} n^{\frac{\alpha}{2}} \Delta W_{\frac{j}{n}}\right).$$

Since $\kappa^{-\frac{1}{2}} W$ is a fractional Brownian motion, the Breuer-Major theorem implies that the process $\widetilde{G}$ converges in $D([0, T])$ to a scaled Brownian motion $\{\sigma B_t, t \in [0, T]\}$, where σ^2 is given in (1.11). By the fact that all the p-norms are equivalent on a fixed Wiener chaos, the estimates (4.4) and (4.5) lead to

$$\mathbb{E}[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^{2p}] \le \frac{(\lfloor nt \rfloor - \lfloor ns \rfloor)^{\delta p}}{n^p} \tag{4.6}$$

and

$$\mathbb{E}[|G_n(t) - \widetilde{G}_n(t) - (G_n(s) - \widetilde{G}_n(s))|^{2p}] \le \frac{(\lfloor nt \rfloor - \lfloor ns \rfloor)^{\delta p}}{n^p}, \tag{4.7}$$

for all $p \ge 1$. Letting n tend to infinity, we deduce from (4.6) and (4.7) that for any $t \in [0, T]$ the sequences $F_n(t) - G_n(t)$ and $G_n(t) - \widetilde{G}_n(t)$ converge to zero in $L^{2p}(\Omega)$ for any $p \ge 1$. This implies that the finite dimensional distributions of the processes $F_n - G_n$ and $G_n - \widetilde{G}_n$ converge to zero in law. Moreover, by Billingsley [2, Theorem 13.5], (4.6) and (4.7) also imply that the sequences $F_n - G_n$ and $G_n - \widetilde{G}_n$ are tight in $D([0, T])$. Therefore, these sequences converge to zero in the topology of $D([0, T])$.

Proof of (4.4) We can write

$$\mathbb{E}\left[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^2\right] \le C \sum_{r=0}^{q-1} \mathbb{E}[\Phi_{r,n}^2],$$

where

$$\Phi_{r,n} = n^{-\frac{1}{2}} \sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \beta_{j,n}^{-q} I_r^W\left(\mathbf{1}^{\otimes r}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}\right) I_{q-r}^Y\left(\mathbf{1}^{\otimes q-r}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}\right).$$

We have, using (4.3),

$$\mathbb{E}[\Phi_{r,n}^2] \le n^{-1+q\alpha} \times \sum_{j,k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left|\mathbb{E}\left[I_r^W\left(\mathbf{1}^{\otimes r}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}\right) I_{q-r}^Y\left(\mathbf{1}^{\otimes q-r}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}\right) I_r^W\left(\mathbf{1}^{\otimes r}_{\left[\frac{k}{n}, \frac{k+1}{n}\right)}\right) I_{q-r}^Y\left(\mathbf{1}^{\otimes q-r}_{\left[\frac{k}{n}, \frac{k+1}{n}\right)}\right)\right]\right|.$$

Using a diagram method for the expectation of four stochastic integrals (see [9]), we find that, for any j,k, the above expectation consists of a sum of terms of the form

$$\left(\mathbb{E}\left[\Delta W_{\frac{j}{n}}\Delta W_{\frac{k}{n}}\right]\right)^{a_1}\left(\mathbb{E}\left[\Delta Y_{\frac{j}{n}}\Delta Y_{\frac{k}{n}}\right]\right)^{a_2}\left(\mathbb{E}\left[\Delta W_{\frac{j}{n}}\Delta Y_{\frac{k}{n}}\right]\right)^{a_3}\left(\mathbb{E}\left[\Delta Y_{\frac{j}{n}}\Delta W_{\frac{k}{n}}\right]\right)^{a_4},$$

where the a_i are nonnegative integers such that $a_1+a_2+a_3+a_4=q$, $a_1\le r\le q-1$, and $a_2\le q-r$. First, consider the case with $a_3=a_4=0$, so that we have the sum

$$n^{-1+q\alpha}\sum_{j,k=\lfloor ns\rfloor\vee n_0}^{\lfloor nt\rfloor-1}\left(\mathbb{E}\left[\Delta W_{\frac{j}{n}}\Delta W_{\frac{k}{n}}\right]\right)^{a_1}\left(\mathbb{E}\left[\Delta Y_{\frac{j}{n}}\Delta Y_{\frac{k}{n}}\right]\right)^{q-a_1},$$

where $0\le a_1\le q-1$. Applying Lemma 3, we can control each of the terms in the above sum by

$$n^{-q\alpha}(|j-k+1|^{\alpha}-2|j-k|^{\alpha}+|j-k-1|^{\alpha})^{a_1}(j+k)^{(q-a_1)(\alpha-2)},$$

which gives

$$\begin{aligned}
&n^{-1+q\alpha}\sum_{j,k=\lfloor ns\rfloor\vee n_0}^{\lfloor nt\rfloor-1}\left|\mathbb{E}\left[\Delta W_{\frac{j}{n}}\Delta W_{\frac{k}{n}}\right]\right|^{a_1}\left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}}\Delta Y_{\frac{k}{n}}\right]\right|^{q-a_1}\\
&\le Cn^{-1}\left(\sum_{j=\lfloor ns\rfloor\vee n_0}^{\lfloor nt\rfloor-1}j^{\alpha-2}+\sum_{j,k=\lfloor ns\rfloor\vee n_0,j\neq k}^{\lfloor nt\rfloor-1}|j-k|^{(q-1)(\alpha-2)}(j+k)^{\alpha-2}\right)\\
&\le Cn^{-1}\sum_{j=\lfloor ns\rfloor\vee n_0}^{\lfloor nt\rfloor-1}\left(j^{\alpha-2}+j^{q(\alpha-2)+1}\right)\\
&\le Cn^{-1}\left((\lfloor nt\rfloor-\lfloor ns\rfloor)^{(\alpha-1)\vee 0}+(\lfloor nt\rfloor-\lfloor ns\rfloor)^{[q(\alpha-2)+2]\vee 0}\right).
\end{aligned}\tag{4.8}$$

Next, we consider the case where $a_3+a_4\ge 1$. By Lemma 3, we have that, up to a constant C,

$$\left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}}\Delta Y_{\frac{k}{n}}\right]\right|\le C\left|\mathbb{E}\left[\Delta W_{\frac{j}{n}}\Delta W_{\frac{k}{n}}\right]\right|,$$

so we may assume $a_2=0$, and have to handle the term

$$n^{-1+q\alpha}\sum_{j,k=\lfloor ns\rfloor\vee n_0}^{\lfloor nt\rfloor-1}\left|\mathbb{E}\left[\Delta W_{\frac{j}{n}}\Delta W_{\frac{k}{n}}\right]\right|^{q-a_3-a_4}\left|\mathbb{E}\left[\Delta W_{\frac{j}{n}}\Delta Y_{\frac{k}{n}}\right]\right|^{a_3}\left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}}\Delta W_{\frac{k}{n}}\right]\right|^{a_4}\tag{4.9}$$

for all allowable values of a_3, a_4 with $a_3 + a_4 \geq 1$. Consider the decomposition

$$n^{-1+q\alpha} \sum_{j,k=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{q-a_3-a_4} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right]\right|^{a_3} \left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{a_4}$$
$$= n^{q\alpha-1} \sum_{j=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} \left|\mathbb{E}\left[\Delta W^2_{\frac{j}{n}}\right]\right|^{q-a_3-a_4} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{j}{n}}\right]\right|^{a_3+a_4}$$
$$+ n^{q\alpha-1} \sum_{j=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} \sum_{k=\lfloor ns\rfloor \vee n_0}^{j-1} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{q-a_3-a_4} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right]\right|^{a_3} \left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{a_4}$$
$$+ n^{q\alpha-1} \sum_{k=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} \sum_{j=\lfloor ns\rfloor \vee n_0}^{k-1} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{q-a_3-a_4} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right]\right|^{a_3} \left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{a_4}.$$

We have,by Lemmas 3 and 4,

$$n^{-1+q\alpha} \sum_{j,k=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{q-a_3-a_4} \left|\mathbb{E}\left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}}\right]\right|^{a_3} \left|\mathbb{E}\left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}}\right]\right|^{a_4}$$
$$\leq Cn^{-1} \sum_{j=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} j^{(a_3+a_4)(\alpha-2)}$$
$$+ Cn^{-1} \sum_{j=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} j^{a_3(2H-2)-a_4\gamma} \sum_{k=\lfloor ns\rfloor \vee n_0}^{j-1} k^{-a_3\gamma+a_4(2H-2)} |j-k|^{(q-a_3-a_4)(\alpha-2)}$$
$$+ Cn^{-1} \sum_{k=\lfloor ns\rfloor \vee n_0}^{\lfloor nt\rfloor -1} k^{-a_3\gamma+a_4(2H-2)} \sum_{j=\lfloor ns\rfloor \vee n_0}^{k-1} j^{a_3(2H-2)-a_4\gamma} |k-j|^{(q-a_3-a_4)(\alpha-2)}$$
$$\leq Cn^{-1}\left((\lfloor nt\rfloor - \lfloor ns\rfloor)^{[(a_3+a_4)(\alpha-2)+1]\vee 0} + (\lfloor nt\rfloor - \lfloor ns\rfloor)^{[q(\alpha-2)+2]\vee 0}\right.$$
$$\left.+(\lfloor nt\rfloor - \lfloor ns\rfloor)^{[a_3(2H-2)-a_4\gamma+1]\vee 0} + (\lfloor nt\rfloor - \lfloor ns\rfloor)^{[a_4(2H-2)-a_3\gamma+1]\vee 0}\right). \tag{4.10}$$

Then (4.8) and (4.10) imply (4.4) because $\alpha < 2 - \frac{1}{q}$.

Proof of (4.5) We have

$$G_n(t) - \widetilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt\rfloor -1} \left(\beta_{j,n}^{-q} - \left\|\Delta W_{\frac{j}{n}}\right\|_{L^2(\Omega)}^{-q}\right) I_q^W\left(\mathbf{1}^{\otimes q}_{[\frac{j}{n},\frac{j+1}{n})}\right)$$

and we can write, using (4.3) for any $j \geq n_0$,

$$\left|\beta_{j,n}^{-q} - \left\|\Delta W_{\frac{j}{n}}\right\|_{L^2(\Omega)}^{-q}\right| = (\kappa^{-1}n^{\alpha})^{\frac{q}{2}} \left|(1+\theta_{j,n})^{-\frac{q}{2}} - 1\right| \leq C\left(\kappa^{-1}n^{\alpha}j^{\alpha-2}\right)^{\frac{q}{2}}.$$

This leads to the estimate

$$\mathbb{E}\left[\left|G_n(t)-\widetilde{G}_n(t)-(G_n(s)-\widetilde{G}_n(s))\right|^2\right]\leq Cn^{-1}$$
$$\times\left(\sum_{j=\lfloor ns\rfloor\vee n_0}^{\lfloor nt\rfloor-1} j^{\alpha-2}+\sum_{j,k=\lfloor ns\rfloor\vee n_0, j\neq k}^{\lfloor nt\rfloor-1}|j-k|^{q(\alpha-2)}\right)$$
$$\leq Cn^{-1}\left(\left(\lfloor nt\rfloor-\lfloor ns\rfloor\right)^{(\alpha-1)\vee 0}+\left(\lfloor nt\rfloor-\lfloor ns\rfloor\right)^{[q(\alpha-2)+2]\vee 0}\right),$$

which implies (4.5).

This concludes the proof of Theorem 2.

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