Connections Between the Dirichlet and the Neumann Problem for Continuous and Integrable Boundary Data

Lucian Beznea, Mihai N. Pascu, and Nicolae R. Pascu

Dedicated to Rodrigo Banuelos on the occasion of his sixtieth birthday

Abstract We present results concerning the representation of the solution of the Neumann problem for the Laplace operator on the n-dimensional unit ball in terms of the solution of an associated Dirichlet problem. We show that the representation holds in the case of integrable boundary data, thus providing an explicit solution of the generalized solution of the Neumann problem.

Keywords Dirichlet problem • Dirichlet-to-Neumann operator • Infinite-dimensional Laplace operator • Laplace operator • Neumann problem

1991 Mathematics Subject Classification 31B05, 31B10, 42B37, 35J05, 35J25

L. Beznea (✉)
Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit No. 2, P.O. Box 1-764, RO-014700 Bucharest, Romania

Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania

Centre Francophone en Mathématique de Bucarest, Bucarest, Romania
e-mail: lucian.beznea@imar.ro

M.N. Pascu
Department of Mathematics and Computer Science, Transilvania University of Braşov, Str. Iuliu Maniu Nr. 50, 500091 Braşov, Romania
e-mail: mihai.pascu@unitbv.ro

N.R. Pascu
Department of Mathematics, Kennesaw State University, 1100 S. Marietta Parkway, Marietta, GA 30060-2896, USA
e-mail: npascu@kennesaw.edu

F. Baudoin, J. Peterson (eds.), *Stochastic Analysis and Related Topics*, Progress in Probability 72, DOI 10.1007/978-3-319-59671-6_4

1 Introduction

The classical Dirichlet and Neumann problems on a smooth bounded domain $D \subset \mathbb{R}^n$ $(n \geq 1)$ are the problem of finding $u \in C^2(D) \cap C(\overline{D})$ which solves

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = \varphi \text{ on } \partial D \end{cases}, \tag{1}$$

respectively $U \in C^2(D) \cap C^1(\overline{D})$ which solves

$$\begin{cases} \Delta U = 0 \text{ in } D \\ \frac{\partial U}{\partial \nu} = \phi \text{ on } \partial D \end{cases}, \tag{2}$$

where ν is the outward unit normal to the boundary of D.

As it is known, for continuous boundary data, the Dirichlet problem (1) has a unique solution and the Neumann problem (2) has a solution, unique up to additive constants, if we require in addition the condition $\int_{\partial D} \phi(z)\, \sigma(dz) = 0$. Note that this is a necessary condition for the existence of a solution, since by Green's first identity we have

$$\int_{\partial D} \phi(z)\, \sigma(dz) = \int_{\partial D} 1 \frac{\partial U}{\partial \nu}(z)\, \sigma(dz) = \int_D 1 \Delta U(z) + \nabla 1 \cdot \nabla U(z)\, dz = 0.$$

In this paper, we present explicit relations between the solutions of (1) and (2), which appeared recently in [4]. This shows that the Dirichlet and Neumann problems are "equally hard", in the sense that solving one of them leads to the solution of the other one. The central results for continuous boundary data (Theorem 1, and its extensions given in Theorems 2 and 5) provide an explicit relation between the solution(s) of (2) and (1), in the sense that the normalized solution of (2) can be found as a weighted average of the solution of (1).

The link between the solution of the Dirichlet problem and the Neumann problem is provided by the operator defined by (3). What is interesting here is that the same operator also provides a relationship between the solution of Dirichlet and Neumann problem in the infinite-dimensional setting of generalized Laplacian on an abstract Wiener space (see [4], Sect. 3). In Sect. 3 we show that the same operator can be used in order to construct a generalized solution of the Neumann boundary problem in the case of the unit ball in $\mathbb{R}^n$ $(n \geq 1)$ for integrable boundary data. While the existence of such a generalized solution for the Dirichlet boundary problem for integrable boundary data is known (the Perron-Wiener-Brelot theory [1, 9], or alternately the method of controlled convergence introduced by Cornea [5, 6]), in the case of the Neumann problem this is a new result, and it is the main result of the present paper, given in Theorem 12.

In Sect. 2, we consider the case of continuous boundary data for the Dirichlet and Neumann problems. This section is based on the recent results on the subject from

[4]. The main result giving the connection between the Dirichlet and the Neumann problem in the case of the unit ball is given in Theorem 1. The result can be extended to other operators besides the Laplacian, and in Theorem 2 we present such an extension.

As an application, in Theorem 4 we give an explicit representation of the inverse of the Dirichlet-to-Neumann operator (a particular case of the Poincaré-Steklov operator, which encapsulates the boundary response of a system modeled by a certain partial differential equation).

By using conformal mapping arguments (in the 2-dimensional case), the main result obtained in the case of the unit disk is extended (Theorem 5) to the general case of smooth bounded simply connected domains.

In what follows, we will identify as usual the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, that is we identify the vector $(x, y) \in \mathbb{R}^2$ with the complex number $z = x + iy \in \mathbb{C}$. In particular, the dot product of two vectors $a, b \in \mathbb{R}^2$ will be written in terms of multiplication of complex numbers as $a \cdot b = \operatorname{Re}\left(a\overline{b}\right)$, and for a complex number $z \in \mathbb{C}$ we denote the real part and the imaginary part of z by $\operatorname{Re}(z)$, respectively $\operatorname{Im}(z)$. Also, for a function u defined on a subset D of $\mathbb{R}^2$ (or $\mathbb{C}$), we will write equivalently $u(x, y)$ or $u(z)$, where $z = x + iy \in D$.

For a smooth bounded domain we will be denote by $\sigma(\cdot)$ and $\sigma_0(\cdot)$ the surface measure on its boundary, respectively the surface measure normalized to have total mass 1.

2 The Case of Continuous Boundary Data

We start by recalling some recent results [4] concerning the equivalence between the Dirichlet and the Neumann problem for the Laplace operator in the case of continuous boundary data.

Heuristic arguments from Complex analysis (in the 2-dimensional case) led us to consider the operator which associates to a continuous function $u : D\subset \mathbb{R}^n \to \mathbb{R}$ with $u(0) = 0$ the function $U : D\subset \mathbb{R}^n \to \mathbb{R}$ defined by

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in D, \tag{3}$$

where $D \subset \mathbb{R}^n$ is a smooth bounded subset, starlike with respect to the origin (i.e. $\rho z \in D$ for any $z \in D$ and $\rho \in [0, 1]$).

A first result concerning the operator defined above is that in the case of the n-dimensional unit ball $D = \mathbb{U} = \{z \in \mathbb{R}^n : |z| < 1\}$, the relation (3) provides an explicit solution of the Neumann problem (2) in terms of the Dirichlet problem (1) with the boundary condition $\varphi = \phi$. Conversely, since for a harmonic function the Laplacian and the partial derivatives commute, one can see that it is possible to solve the Dirichlet problem by solving an appropriate Neumann problem. The result is the following.

Theorem 1 **([4])** *The following assertions hold.*

(i) Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ *is continuous and satisfies* $\int_{\partial\mathbb{U}} \phi(z)\,\sigma_0(dz) = 0$. *If* u *is the solution of the Dirichlet problem (1) with boundary condition* $\varphi = \phi$ *on* $\partial\mathbb{U}$, *then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}}, \tag{4}$$

is the solution to the Neumann problem (2) with $U(0) = 0$.

(ii) Assume $\varphi : \partial\mathbb{U} \to \mathbb{R}$ *is continuous. If* U *is the solution of the Neumann problem (2) with boundary condition* $\phi = \varphi - \int_{\partial\mathbb{U}} \varphi(\xi)\,\sigma_0(d\xi)$, *then*

$$u(z) = z \cdot \nabla U(z) + \int_{\partial\mathbb{U}} \varphi(\xi)\,\sigma_0(d\xi), \qquad z \in \overline{\mathbb{U}}, \tag{5}$$

is the solution to the Dirichlet problem (1).

As shown in [4], the previous result can also be applied to other operators besides the Laplacian. For example, considering the operator $\mathscr{L}$ defined by

$$\mathscr{L}f(z) = \sum_{i,j=1}^{n} a_{ij}(z) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{i=1}^{n} a_i(z) \frac{\partial f}{\partial z_i}(z), \tag{6}$$

where the coefficients a_{ij} are smooth and homogeneous of degree $k \in [0, 1]$, i.e.

$$a_{ij}(\rho z) = \rho^k a_{ij}(z), \qquad 0 \le \rho \le 1, z \in \mathbb{U}, 1 \le i, j \le n, \tag{7}$$

and the coefficients a_i are also smooth and homogeneous of degree $k - 1$, i.e.

$$a_i(\rho z) = \rho^{k-1} a_{ij}(z), \qquad 0 \le \rho \le 1, z \in \mathbb{U}, 1 \le i \le n, \tag{8}$$

if u (with $u(0) = 0$) and U are related by (4), then

$$\mathscr{L}U(z) = \int_0^1 \rho^{1-k} \mathscr{L}u(\rho z)\, d\rho, \qquad z \in \mathbb{U},$$

and

$$\frac{\partial U}{\partial \nu}(z) = u(z), \qquad z \in \partial\mathbb{U}.$$

The previous observation leads to the following more general result.

Theorem 2 ([4]) *Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous. If u is the solution of the Dirichlet problem*

$$\begin{cases} \mathscr{L}u = 0 \text{ in } \mathbb{U} \\ u = \phi \text{ on } \partial\mathbb{U} \end{cases} \tag{9}$$

where $\mathscr{L}$ is the operator given by (6) which satisfies (7) and (8), and if $u(0) = 0$, then

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}}, \tag{10}$$

is the solution to the Neumann problem

$$\begin{cases} \mathscr{L}U = 0 \ \text{ in } \mathbb{U} \\ \frac{\partial U}{\partial \nu} = \phi \ \text{ on } \partial\mathbb{U} \end{cases}, \tag{11}$$

with $U(0) = 0$.

Remark 3 The above result was stated in [4], Theorem 2, under the condition $\int_{\partial\mathbb{U}} \phi(z)\sigma_0(dz) = 0$ instead of $u(0) = 0$. If $\mathscr{L} = \Delta$, then these two conditions are equivalent, due to the Poisson formula.

As an application of the correspondence between the solutions of the Dirichlet and Neumann problems given above, we obtained an explicit representation of the inverse of the Dirichlet-to-Neumann operator Λ_n in the case of the unit ball $\mathbb{U} \subset \mathbb{R}^n$, $n \geq 2$. See for example [10, Sect. 5.0], or [4] for details on the Dirichlet-to-Neumann operator Λ_n and its inverse.

Theorem 4 *Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and satisfies $\int_{\partial\mathbb{U}} \phi(\xi)\sigma(d\xi) = 0$. We have*

$$\Lambda_n^{-1}(\phi)(z) = \int_{\partial\mathbb{U}} \phi(\xi) k_n(z,\xi)\sigma_0(d\xi), \qquad z \in \partial\mathbb{U}, \tag{12}$$

where $k_n(z,\zeta) = \int_0^1 \frac{1}{\rho}\left(\frac{1-\rho^2}{|\rho z - \xi|^n} - 1\right) d\rho$, $z, \xi \in \partial\mathbb{U}$.

Explicitly, $k_2(z,\xi) = -2\ln|z-\xi|$, $k_3(z,\xi) = \frac{2}{|z-\xi|} - 2 + \ln 2 - \ln\left(\frac{|z-\xi|^2}{2} + |z-\xi|\right)$, and for $n > 4$ the kernel $k_n(z,\xi)$ can be computed using the recurrence formulae

$$k_n(z,\xi) = k_{n-2}(z,\xi) + \frac{2\left(1 - |z-\xi|^{n-2}\right)}{(n-2)|z-\xi|^{n-2}} - \frac{1-|z-\xi|^{n-4}}{(n-4)|z-\xi|^{n-4}} + \left(1 - \frac{|z-\xi|^2}{2}\right) J_{n-2}(z,\xi), \tag{13}$$

where $J_n(z,\xi) = \int_0^1 \frac{1}{|\rho z - \xi|^n} d\rho$ *satisfies*

$$J_n(z,\xi) = \frac{4(n-3) J_{n-2}(z,\xi)}{(n-2)\left(4-|z-\xi|^2\right)|z-\xi|^2} + \frac{2\left(1+4|z-\xi|^{n-4}-|z-\xi|^{n-2}\right)}{(n-2)\left(4-|z-\xi|^2\right)|z-\xi|^{n-2}}. \tag{14}$$

Using conformal mapping arguments (in the 2-dimensional case), the result in Theorem 1 can be extended to the general case of a smooth bounded simply connected domain $D \subset \mathbb{C}$ ($C^{1,\alpha}$ boundary with $0 < \alpha < 1$ will suffice). The result is the following.

Theorem 5 ([4]) *Let* $D \subset \mathbb{C}$ *be a smooth bounded simply connected domain (*$C^{1,\alpha}$ *boundary with* $0 < \alpha < 1$ *will suffice), and for an arbitrarily fixed* $w_0 \in D$ *let* $f : \mathbb{U} \to D$ *be the conformal map of the unit disk* $\mathbb{U}$ *onto* D *with* $f(0) = w_0$, $\arg f'(0) = 0$, *and let* $g = f^{-1} : D \to \mathbb{U}$ *be its inverse.*

Assume $\phi : \partial D \to \mathbb{R}$ *is continuous and satisfies* $\int_{\partial D} \phi(w)\,\sigma(dw) = 0$. *If* u *is the solution of the Dirichlet problem (1) with boundary condition*

$$\varphi(w) = \frac{1}{|g'(w)|}\phi(w), \qquad w \in \partial D, \tag{15}$$

then

$$U(w) = \int_0^1 \frac{u(f(\rho g(w)))}{\rho} d\rho, \qquad w \in D, \tag{16}$$

is the solution to the Neumann problem (2) with $U(w_0) = 0$.

The result in Theorem 1 can also be extended to the case of Dirichlet and Neumann problems for the infinite-dimensional ball on an abstract Wiener space, in the setup stated in [7, 8], and [3]; for details see Sect. 3 from [4].

3 The Case of Integrable Boundary Data

In order to extend the result in Theorem 1 to a correspondence between the solutions of the Dirichlet problem and the Neumann problem for the unit ball in the general case of integrable boundary data, we will use Cornea's notion of *controlled convergence* [5, 6]. Even in the case of the unit ball $\mathbb{U} \subset \mathbb{R}^n$ ($n \geq 2$) which we consider here this is a new result, and it provides an explicit solution to the general Neumann problem for the Laplace operator.

It can be shown that in the case of the unit ball Cornea's approach is equivalent to the Perron-Wiener-Brelot approach for the generalized solution of the Dirichlet problem. More precisely, it can be shown that for integrable boundary data, both methods indicate that the generalized solution of the Dirichlet problem is given by the stochastic solution $H_{\mathbb{U}}^f$ defined by (18) below (see [5], Corollary 2, [6], Corollary 2.13, [1], Theorem 6.4.6, and [3], Theorem 4.5).

We will first recall the notion of *controlled convergence* introduced in [5, 6].

Definition 6 (Controlled convergence (A. Cornea, [5, 6])) Let $D \subset \mathbb{R}^d$ be a bounded open set, $\partial D \subset \Delta \subset \overline{D}$, $f : \partial D \to \mathbb{R}$ and $h, k : D \to \overline{\mathbb{R}}$, $k \geq 0$. The function h converges to f controlled by k (we write $h \xrightarrow{k} f$) if the following conditions hold:

For any set $A \subset D$ and any point $z_0 \in \overline{A} \cap \Delta$ we have

(*) If $\limsup_{A \ni z \to z_0} k(z) < +\infty$, then $f(z_0) \in \mathbb{R}$ and $\lim_{A \ni z \to z_0} h(z) = f(z)$.
(**) If $\lim_{A \ni z \to z_0} k(z) = +\infty$, then $\lim_{A \ni z \to z_0} \frac{h(z)}{1+k(z)} = 0$.

The function k will be called a *control function* for f.

Remark 7 It can be shown (see [6], Theorem 1.5, or [5], Theorem 1) that h converges to f controlled by k, in the sense of the above definition if and only if for any $z_0 \in \partial D$ the following equivalent conditions are satisfied:

(a) If $\liminf_{D \ni z \to z_0} k(z) < +\infty$, then $f(z_0) \in \mathbb{R}$ and $\lim_{D \ni z \to z_0} \frac{h(z)-f(z)}{1+k(z)} = 0$.
(b) If $\lim_{D \ni z \to z_0} k(z) = +\infty$, then $\lim_{D \ni z \to z_0} \frac{h(z)}{1+k(z)} = 0$.

Using the above definition, Cornea [5, 6] introduced the notion of generalized solution of the Dirichlet problem (1) as follows.

Definition 8 ([5, 6]) A *generalized solution* of the Dirichlet problem (1) is a harmonic function $u : D \to \mathbb{R}$ which satisfies

$$\lim_{z \to z_0} u(z) = \varphi(z), \qquad z_0 \in \partial D, \tag{17}$$

controlled by a continuous, non-negative (super)harmonic function $k : D \to \mathbb{R}_+$.

A function $\varphi : \partial D \to \overline{\mathbb{R}}$ for which the Dirichlet problem has a generalized solution is called *resolutive*. We denote by $\mathscr{R}(D)$ the set of resolutive functions $\varphi : \partial D \to \overline{\mathbb{R}}$.

In the same spirit, we propose the following definition for the generalized solution of the Neumann problem (2).

Definition 9 Let $D \subset \mathbb{R}^d$ be a bounded open set, $\partial D \subset \Delta \subset \overline{D}$, $h, k : D \to \overline{\mathbb{R}}$, $k \geq 0$. We say that the function h has a continuous extension to $\overline{D}$ controlled by k if the following conditions hold:

For any set $A \subset D$ and any point $z_0 \in \overline{A} \cap \Delta$ we have:

(i) If $\limsup_{A \ni z \to z_0} k(z) < +\infty$, we have $h(z_0) := \lim_{A \ni z \to z_0} h(z) \in \mathbb{R}$.
(ii) If $\lim_{A \ni z \to z_0} k(z) = +\infty$, then $\lim_{A \ni z \to z_0} \frac{h(z)}{1+k(z)} = 0$.

Remark 10 The previous remark shows that h has a continuous extension to $\overline{D}$ iff the equivalent conditions (a)–(b) above are satisfied.

If h has a continuous extension to $\overline{D}$ controlled by k, then the function h can be extended by continuity at the set of points z_0 belonging to the set $\left\{z_0 \in \partial D : \limsup_{D \ni z \to z_0} k(z) < +\infty\right\}$. On the set, $E = \left\{z_0 \in \partial D : \limsup_{D \ni z \to z_0} k(z) = +\infty\right\}$, the limit $\lim_{D \ni z \to z_0} h(z)$ may not exist, and the function h may fail to be continuous (this set of points is "controlled" by the function k).

Definition 11 A *generalized solution* of the Neumann problem (2) is a harmonic function $U : D \to \mathbb{R}$ which has a continuous extension to ∂D, controlled by a non-negative harmonic function $k : D \to \mathbb{R}_+$, and for any $z_0 \in \partial D$ for which $\limsup_{[0,z_0] \ni z \to z_0} k(z) < +\infty$ we have

$$\lim_{\varepsilon \nearrow 0} \frac{U(z_0 + \varepsilon \nu(z_0)) - U(z_0)}{\varepsilon} = \phi(z_0),$$

where $\nu(z)$ denotes the outward unit normal to the boundary of D at $z \in \partial D$.

In [5], the author showed that in the case of the unit ball $D = \mathbb{U} \subset \mathbb{R}^n$, every function $f \in L^1(\partial\mathbb{U}, \sigma_0)$ is resolutive for the Dirichlet problem. Moreover, by Beznea [2], the generalized solution coincides in fact with the stochastic solution, that is

$$u(z) = H_{\mathbb{U}}^{f}(z) = E^z f(B_\tau), \tag{18}$$

where $(B_t)_{t \geq 0}$ is a n-dimensional Brownian motion starting at $z \in \mathbb{U}$ and $\tau = \tau_{\partial\mathbb{U}} = \inf\{t \geq 0 : B_t \in \partial\mathbb{U}\}$ is the hitting time of the boundary of $\mathbb{U}$, and the controlled convergence to the boundary data f holds outside an exceptional (polar) set. It is also known (see [2], Corollary 4.3) that the generalized solution of the Dirichlet problem is unique.

With this preparation, we can now prove the main result, as follows.

Theorem 12 *Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is integrable and satisfies $\int_{\partial\mathbb{U}} \phi(z)\, \sigma_0(dz) = 0$. If u is the generalized solution of the Dirichlet problem (1) with boundary condition $\varphi = \phi$ on $\partial\mathbb{U}$, then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}}, \tag{19}$$

is a generalized solution to the Neumann problem (2) with $U(0) = 0$.

Proof Before proceeding with the proof, note that by symmetry, the exit distribution from $\mathbb{U}$ of the Brownian motion starting at the origin is the (normalized) surface measure σ_0 on $\partial\mathbb{U}$, and using the hypothesis we obtain

$$u(0) = E^0 \phi(B_{\tau_{\mathbb{U}}}) = \int_{\partial\mathbb{U}} \phi(z)\, \sigma_0(dz) = 0. \tag{20}$$

Using this, we have

$$\lim_{\rho\searrow 0}\frac{u(\rho z)}{\rho}=\lim_{\rho\searrow 0} z\frac{u(\rho z)-u(0)}{\rho z-0}=z\cdot\nabla u(0)\,,\qquad z\in\mathbb{U}, \tag{21}$$

which shows that the integrand in (19) can be extended by continuity at $\rho=0$, so U is well defined for all $z\in\mathbb{U}$. Note that the relation (20) also shows that $U(0)=0$.

Next, we show that under the given hypotheses the function U has a continuous extension (controlled by k) to the boundary $\partial\mathbb{U}$, and it has the appropriate normal derivative. To be precise, for an arbitrary $z_0\in\partial\mathbb{U}$ we'll show the following:

a) if $\liminf_{\mathbb{U}\ni z\to z_0} k(z)<\infty$ then there exists $U(z_0)\in\mathbb{R}$ such that $\lim_{\mathbb{U}\ni z\to z_0}\frac{U(z)-U(z_0)}{1+k(z)}=0$. Moreover, if $\limsup_{[0,z_0]\ni z\to z_0} k(z)<\infty$, then $\lim_{\varepsilon\nearrow 0}\frac{U(z_0+\varepsilon\nu(z_0))-U(z_0)}{\varepsilon}=\phi(z_0)$.

b) if $\lim_{\mathbb{U}\ni z\to z_0} k(z)=\infty$, then $\lim_{\mathbb{U}\ni z\to z_0}\frac{U(z)}{1+k(z)}=0$

Consider $z_0\in\partial\mathbb{U}$ and assume that $\liminf_{z\to z_0} k(z)<\infty$. Since $u\to\phi$ controlled by k, we have $\lim_{z\to z_0}\frac{u(z)-u(z_0)}{1+k(z)}=0$. Since u is continuous in $\mathbb{U}$, it follows that the function $(\rho,z)\mapsto\frac{u(\rho z)-u(\rho z_0)}{1+k(z)}$ is bounded on the set

$$\{(\rho,z)\in[0,1]\times\mathbb{U}:|\rho z-z_0|<\delta\}\,,$$

for some $\delta>0$. Since u and k are also bounded (being continuous) on the compact cone

$$C_\delta=\left\{\rho z:\rho\in[0,1]\,,z\in\overline{\mathbb{U}}\text{ s.t. }|z-z_0|=\frac{\delta}{2}\right\}\cap\{z\in\mathbb{U}:|z-z_0|\geq\delta\}\subset\mathbb{U},$$

(see Fig. 1), it follows that $\frac{u(\rho z)-u(\rho z_0)}{1+k(z)}$ is bounded on $[0,1]\times\{z\in\mathbb{U}:|z-z_0|<\delta/2\}$.

Using the bounded convergence theorem and the above, we obtain

$$\lim_{z\to z_0}\frac{U(z)-U(z_0)}{1+k(z)}=\lim_{z\to z_0}\int_0^1\frac{u(\rho z)-u(\rho z_0)}{1+k(z)}d\rho=\int_0^1\lim_{z\to z_0}\frac{u(\rho z)-u(\rho z_0)}{1+k(z)}d\rho=0.$$

Suppose now that $z_0\in\partial\mathbb{U}$ is such that $\lim_{z\to z_0}k(z)=\infty$. In order to show that $\lim_{z\to z_0}\frac{U(z)}{1+k(z)}=0$, we will first show that for $\rho_0\in(0,1)$ arbitrarily fixed we have

$$\lim_{z\to z_0}\frac{1}{1+k(z)}\int_0^{\rho_0}\frac{u(\rho z)}{\rho}d\rho=0.$$

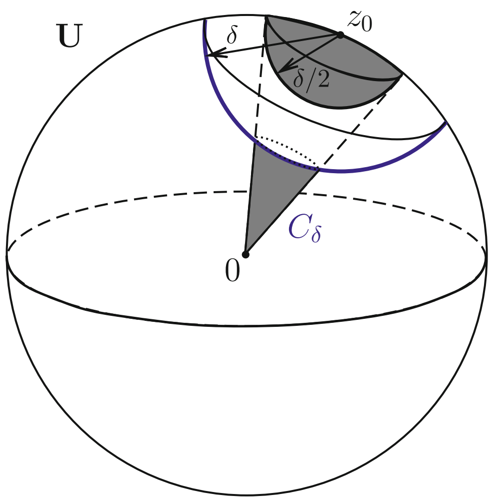

Fig. 1 The cone C_δ in the proof of Theorem 12

Since $u \in C^1(\mathbb{U})$, and using the substitution $w = \rho z$ we obtain

$$\lim_{\rho \searrow 0} \frac{u(\rho z)}{\rho z} = \lim_{w \to 0} \frac{u(w)}{w} = \nabla u(0),$$

uniformly with respect to $z \in \mathbb{U}$. It follows that

$$\left| \frac{u(\rho z)}{(1 + k(z))\rho} \right| \leq 1 + |z \cdot \nabla u(0)| \leq 1 + |\nabla u(0)|$$

is bounded for $\rho < \rho_1$ sufficiently small, uniformly with respect to $z \in \mathbb{U}$. For $\rho \in [\rho_1, \rho_0]$, we have

$$\left| \frac{u(\rho z)}{(1 + k(z))\rho} \right| \leq \frac{1}{\rho_1} \max_{|w| \leq \rho_0} |u(w)|,$$

and combining with the above we conclude that $\frac{u(\rho z)}{(1+k(z))\rho}$ is bounded for $\rho \in [0, \rho_0]$, uniformly with respect to $z \in \mathbb{U}$. Using the bounded convergence theorem and $\lim_{z \to z_0} k(z) = \infty$, we conclude

$$\lim_{z \to z_0} \frac{1}{1 + k(z)} \int_0^{\rho_0} \frac{u(\rho z)}{\rho} d\rho = \int_0^{\rho_0} \lim_{z \to z_0} \frac{u(\rho z)}{(1 + k(z))\rho} d\rho = 0, \qquad (22)$$

thus proving the claim. In order to prove that $\lim_{z\to z_0}\frac{U(z)}{1+k(z)}=0$, it remains to show that

$$\lim_{z\to z_0}\frac{1}{1+k(z)}\int_{\rho_0}^{1}\frac{u(\rho z)}{\rho}d\rho=0,$$

for an arbitrarily fixed $\rho_0\in(0,1)$.

For $\varepsilon>0$ arbitrarily fixed, consider $n_0\in\mathbb{N}$ such that $n_0\geq\frac{1}{\varepsilon}$, and let $\phi_0=\frac{1}{n_0}\sum_{i=1}^{n_0}f_i$. We have

$$\begin{aligned}
u(z)+\varepsilon k(z) &\geq u(z)+\frac{1}{n_0}k(z)=H_{\phi}^{\mathbb{U}}(z)+\frac{1}{n_0}H^{\mathbb{U}}_{\sum_{n\geq 1}(f_n-g_n)}(z)\\
&\geq H_{\phi}^{\mathbb{U}}(z)+\frac{1}{n_0}H^{\mathbb{U}}_{\sum_{n=1}^{n_0}(f_n-g_n)}(z)\\
&=H_{\phi_0}^{\mathbb{U}}(z)+\frac{1}{n_0}H^{\mathbb{U}}_{\sum_{n=1}^{n_0}(f-g_n)}(z)\\
&\geq H_{\phi_0}^{\mathbb{U}}(z),
\end{aligned}$$

for any $z\in\mathbb{U}$.

Since by construction the functions f_n are lower bounded, there exists $M>0$ such $\phi_0\geq M$, and therefore $H_{\phi_0}^{\mathbb{U}}(z)\geq M$ for any $z\in\mathbb{U}$. We obtain

$$\begin{aligned}
\frac{1}{1+k(z)}\int_{\rho_0}^{1}\frac{u(\rho z)}{\rho}d\rho &= \frac{1}{1+k(z)}\int_{\rho_0}^{1}\left(\frac{u(\rho z)+\varepsilon k(\rho z)}{\rho}-\varepsilon\frac{k(\rho z)}{\rho}\right)d\rho\\
&\geq \frac{1}{1+k(z)}\int_{\rho_0}^{1}\left(\frac{M}{\rho}-\varepsilon\frac{k(\rho z)}{\rho}\right)d\rho\\
&\geq \frac{-M\ln\rho_0}{1+k(z)}-\frac{\varepsilon}{\rho_0(1+k(z))}\int_{\rho_0}^{1}k(\rho z)\,d\rho
\end{aligned}$$

An argument similar to the one in the beginning of the proof shows that $\frac{k(\rho z)}{1+k(z)}$ is bounded for $\rho\in[\rho_0,1]$ and z in a neighborhood of z_0. Passing to the limit in the above inequality, and using $\lim_{z\to z_0}k(z)=\infty$ (which in particular implies $\lim_{z\to z_0}\frac{k(\rho z)}{1+k(z)}=0$ for any $\rho\in[\rho_0,1)$, and $\lim_{z\to z_0}\frac{k(z)}{1+k(z)}=1$), we obtain

$$\begin{aligned}
\liminf_{z\to z_0}\frac{1}{1+k(z)}\int_{\rho_0}^{1}\frac{u(\rho z)}{\rho}d\rho &\geq \liminf_{z\to z_0}\frac{-M\ln\rho_0}{1+k(z)}-\frac{\varepsilon}{\rho_0}\int_{\rho_0}^{1}\limsup_{z\to z_0}\frac{k(\rho z)}{1+k(z)}\\
&\quad\times d\rho\geq-\frac{\varepsilon}{\rho_0}(1-\rho_0).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary chosen and $\rho_0 \in (0, 1)$, the above shows that

$$\liminf_{z \to z_0} \frac{1}{1 + k(z)} \int_{\rho_0}^{1} \frac{u(\rho z)}{\rho} d\rho \geq 0.$$

Repeating the proof above with $\tilde{\phi} = -\phi$ in place of ϕ (for which the corresponding functions are $\tilde{u} = -u$, $\tilde{k} = k$, and $\tilde{U} = -U$), we also have

$$\limsup_{z \to z_0} \frac{1}{1 + k(z)} \int_{\rho_0}^{1} \frac{u(\rho z)}{\rho} d\rho \leq 0,$$

and therefore

$$\lim_{z \to z_0} \frac{1}{1 + k(z)} \int_{\rho_0}^{1} \frac{u(\rho z)}{\rho} d\rho = 0.$$

This, combined with (22) shows that

$$\lim_{z \to z_0} \frac{1}{1 + k(z)} \int_{0}^{1} \frac{u(\rho z)}{\rho} d\rho = 0,$$

concluding the proof of part b) of claim.

To see that U has the prescribed normal derivative on $\partial\mathbb{U}$ (recall that we are using the outward normal $\nu(z_0) = z_0$ to the boundary of $\partial\mathbb{U}$), fix $z_0 \in \partial\mathbb{U}$ such that $\limsup_{[0,z_0] \ni z \to z_0} k(z) < \infty$. Since $u \to \phi$ controlled by k, choosing the particular set $A = [0, z_0]$ in the Definition 6 of controlled convergence, we have that $\lim_{\rho \nearrow 1} u(\rho z_0) = \phi(z_0) \in \mathbb{R}$.

Using a change of variables and the mean value theorem, we obtain

$$\begin{aligned} \lim_{\varepsilon \nearrow 0} \frac{U(z_0 + \varepsilon\nu(z_0)) - U(z_0)}{\varepsilon} &= \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon}\Big(\int_{0}^{1+\varepsilon} \frac{u(\rho z_0)}{\rho} d\rho - \int_{0}^{1} \frac{u(\rho z_0)}{\rho} d\rho\Big) \\ &= \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon} \int_{1}^{1+\varepsilon} \frac{u(\rho z_0)}{\rho} d\rho = \lim_{\varepsilon \nearrow 0} \frac{u(\rho^* z_0)}{\rho^*} = \phi(z_0), \end{aligned}$$

where we denoted by $\rho^* \in (1 + \varepsilon, 1)$ the intermediate point given by the mean value theorem. This shows that the directional derivative of the function U in the direction of the normal to the boundary of $\mathbb{U}$ has the appropriate value $\frac{\partial U}{\partial \nu}(z_0) = \phi(z_0)$ at z_0, thus concluding the proof. □

Acknowledgements The first author acknowledges support from the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2016-0372. The second author kindly acknowledges the support by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PNII-ID-PCCE-2011-2-0015.

References

1. D.H. Armitage, S.J. Gardiner, *Classical Potential Theory*. Springer Monographs in Mathematics (Springer, London, 2001)
2. L. Beznea, The stochastic solution of the Dirichlet problem and controlled convergence. Lect. Notes Semin. Interdisciplinare Mat. **10**, 115–136 (2011)
3. L. Beznea, A. Cornea, M. Röckner, Potential theory of infinite dimensional Lévy processes. J. Funct. Anal. **261**(10), 2845–2876 (2011)
4. L. Beznea, M.N. Pascu, N.R. Pascu, An equivalence between the Dirichlet and the Neumann problem for the Laplace operator. Potential Anal. **44**(4), 655–672 (2016)
5. A. Cornea, Résolution du probléme de Dirichlet et comportement des solutions à la frontière à l'aide des fonctions de contrôle. C. R. Acad. Sci. Paris Ser. I Math. **320**, 159–164 (1995)
6. A. Cornea, Applications of controlled convergence in analysis, in *Analysis and Topology* (World Science, Singapore, 1998)
7. V. Goodman, Harmonic functions on Hilbert space. J. Funct. Anal. **10**, 451–470 (1972)
8. L. Gross, Potential theory on Hilbert space. J. Funct. Anal. **1**, 123–181 (1967)
9. L.L. Helms, *Introduction to Potential Theory, Pure and Applied Mathematics*, vol. XXII. (Robert E. Krieger Publishing Co., Huntington, 1975)
10. V. Isakov, *Inverse Problems for Partial Differential Equations*. Applied Mathematical Sciences, vol. 127, 2nd edn. (Springer, New York, 2006)