Multiplicative Functional for the Heat Equation on Manifolds with Boundary

Cheng Ouyang

Abstract The multiplicative functional for the heat equation on k-forms with absolute boundary condition is constructed and a probabilistic representation of the solution is obtained. As an application, we prove a heat kernel domination that was previously discussed by Donnelly and Li, and Shigekawa.

Keywords Absolute boundary condition • Gradient inequality • Heat kernel domination • Hodge-de Rham Laplacian • Riemannian manifold with boundary

1 Introduction

Throughout this paper, we assume that M is an n-dimensional compact Riemannian manifold with boundary ∂M. Denote by $\Box$ the Hodge-de Rham Laplacian. Let θ_0 be a differential k-form on M and consider the following initial boundary valued problem on M:

$$\begin{cases} \frac{\partial\theta}{\partial t} = \frac{1}{2}\Box\theta, \\ \theta(\cdot, 0) = \theta_0, \\ \theta_{norm} = 0, (d\theta)_{norm} = 0. \end{cases} \tag{1.1}$$

The well known Weitzenböck formula shows that the difference between the Hodge-de Rham Laplacian and the covariant Laplacian for the differential forms on a Riemannian manifold M is a linear transformation at each $x \in M$. So the heat equation for differential forms is naturally associated with a matrix-valued Feynman-Kac multiplicative functional determined by the curvature tensor. The boundary condition

$$\theta_{norm} = 0, \text{ and } (d\theta)_{norm} = 0,$$

C. Ouyang (✉)
Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, MSCS (M/C 249), 851 South Morgan Street, Chicago, IL 60607, USA
e-mail: couyang@math.uic.edu

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is called the absolute boundary condition. The significance of the absolute boundary condition stems from the well-know work [7]. Since it is Dirichlet in the normal direction and Neumann in the tangential directions, the associated multiplicative functional is discontinuous and therefore difficult to handle. Ikeda and Watanabe [5, 6] have dealt with this situation by using an excursion theory. Later, Hsu [3] constructed the discontinuous multiplicative functional M_t for 1-forms by an approximating argument inspired by Ariault [1]. Following a similar idea, the same multiplicative functional M_t has been constructed for non-compact manifolds with boundary by Wang [9]. The solution to Eq. (1.1) for 1-forms thus can be represented in terms of M_t as

$$\theta(x,t) = u_0 \mathbb{E}_x\{M_t u_t^{-1}\theta_0(x_t)\}, \tag{1.2}$$

where $\{x_t\}$ is a reflecting Brownian motion on M, and $\{u_t\}$ its horizontal lift process to the orthonormal fame bundle $\mathscr{O}(M)$ starting from a frame $u_0 : \mathbb{R}^n \to T_xM$, which we will use to identify T_xM with $\mathbb{R}^n$. As a direct consequence, a gradient estimate

$$|\nabla P_t f(x)| \leq \mathbb{E}_x\left\{|\nabla f(x_t)| \exp\left[-\frac{1}{2}\int_0^t \kappa(x_s)ds - \int_0^t h(x_s)dl_s\right]\right\}$$

was obtained. Here l is the boundary local time for $\{x_t\}$, $\kappa(x)$ the lower bound of the Ricci curvature at $x \in M$, and $h(x)$ the lower bound of the second fundamental form at $x \in \partial M$.

The present paper extends Hsu's work [3] to multiplicative functional on the full exterior algebra $\wedge^* M$. We lift the absolute boundary condition onto the frame bundle $\mathscr{O}(M)$ and clarify the action of second fundamental form on k-forms in the absolute boundary condition. Then the multiplicative functional M_t for the heat equation (1.1) is constructed. With this M_t, the representation (1.2) still holds for k-forms, and we have the following estimate

$$|M_t|_{2,2} \leq \exp\left[\frac{1}{2}\int_0^t \lambda(x_s)ds - \int_0^t \sigma_k(x_s)dl_s\right]. \tag{1.3}$$

Here

$$\lambda(x) = \sup_{\theta\in\wedge_x^k M, \langle\theta,\theta\rangle=1} \langle D^* R(x)\theta, \theta\rangle, \tag{1.4}$$

with $D^*R\,\theta$ the curvature tensor acting on θ as the Lie algebra action, and $\sigma_k(x)$, $k = 1, 2, \ldots, n$ being combinations of eigenvalues of the second fundamental form at $x \in \partial M$, which we will specify later. It follows immediately with (1.2) and (1.3) our generalized gradient inequality

$$|dP_t\theta(x)| \leq \mathbb{E}_x\left\{\left|d\theta\right| \exp\left[\frac{1}{2}\int_0^t \lambda(x_s)ds - \int_0^t \sigma_{k+1}(x_s)dl_s\right]\right\}.$$

Let $\bar{\lambda} = \sup_{x\in\partial M} \lambda(x)$, we also prove the heat kernel domination

$$|p^k_M(t,x,y)|_{2,2} \le e^{\frac{1}{2}\bar{\lambda}t} p_M(t,x,y)\mathbb{E}_x\{e^{-\int_0^t \sigma_k(x_s)dl_s}|x_t = y\}.$$

Here $p^k(t,x,y)$ is the heat kernel on k-forms with absolute boundary condition and $p_M(t,x,y)$ is the heat kernel on functions with Neumann boundary condition. Note that when $\sigma_k \ge 0$, the above inequality reduces to

$$|p^k_M(t,x,y)|_{2,2} \le e^{\frac{1}{2}\bar{\lambda}t} p_M(t,x,y). \tag{1.5}$$

This special case was proved by Donnelly and Li [2]. We remark that the heat kernel domination was also discussed in Shigekawa [8] by an approach using theory of Dirichlet form. Inequality (1.5) was obtained as an example for 1-forms in [8].

Finally, we would like to remark that although the present work focuses on compact manifolds, we believe similar results can be obtained for non-compact manifolds (under suitable conditions on curvatures and second fundamental forms) by using the treatment discussed in Wang [9, Chap. 3].

The rest of the paper is organized as follows. In Sect. 2, we briefly recall the Weitzenböck formula and corresponding actions on differential forms. In Sect. 3, we give an explicit expression for the absolute boundary condition. The reflecting Brownian motion with Neumann boundary condition is briefly introduced in Sect. 4. Then, we focus on the construction of the multiplicative functional on k-forms for heat equation (1.1) in Sect. 5. Finally we provide some applications in Sect. 6.

2 Weitzenböck Formula on Orthonormal Frame Bundle

For our purpose, it is more convenient to lift equation (1.1) onto the orthonormal frame bundle $\mathscr{O}(M)$. In this section, we give a brief review of Weitzenböck formula and it's lift on the frame bundle $\mathscr{O}(M)$. More detailed discussion can be found in [4].

Let $\triangle = \mathrm{trace}\nabla^2$ be the Laplace-Beltrami operator and $\Box = -(dd^* + d^*d)$ the Hodge-de Rham Laplacian. They are related by the Weitzenböck formula

$$\Box = \triangle + D^*R.$$

We first explain the action of the curvature tensor R on differential forms in the above formula. Suppose that $T : T_xM \to T_xM$ is a linear transformation and T^* : $\wedge^1_x M \to \wedge^1_x M$ its dual. The linear map T^* on $\wedge^1_x M$ can be extended to the full exterior algebra $\wedge^*_x M = \sum_{k=0}^n \bigoplus \wedge^k_x M$ as a Lie algebra action (derivation) D^*T by

$$D^*T(\theta_1 \wedge \theta_2) = D^*T\theta_1 \wedge \theta_2 + \theta_1 \wedge D^*T\theta_2.$$

Let $\mathrm{End}(T_xM)$ be the space of linear maps from T_xM to itself. We define a bilinear map

$$D^* : \mathrm{End}(T_xM) \otimes \mathrm{End}(T_xM) \to \mathrm{End}(\wedge_x^* M)$$

by

$$D^*(T_1 \otimes T_2) = D^* T_1 \circ D^* T_2.$$

From elementary algebra we know that $\mathrm{End}(T_xM)$=$(T_xM)^* \otimes T_xM$. By the definition of the curvature tensor R and using the isometry $(T_xM)^* \to T_xM$ induced by the inner product, we can identify R as an element in $\mathrm{End}(T_xM) \otimes \mathrm{End}(T_xM)$. Thus by the above definition, we obtain a linear map

$$D^* R : \wedge_x^* M \to \wedge_x^* M,$$

which, by the Weiztenböck formula, is the difference between the covariance Laplacian and the Hodge-de Rham Laplacian.

A frame $u \in \mathscr{O}(M)$ is an isometry $u : \mathbb{R}^n \to T_xM$, where $x = \pi u$ and $\pi : \mathscr{O}(M) \to M$ is the canonical projection. A curve $\{u_t\}$ in $\mathscr{O}(M)$ is horizontal if, for any $e \in \mathbb{R}^n$, the vector field $\{u_t e\}$ is parallel along the curve $\{\pi u_t\}$. A vector on $\mathscr{O}(M)$ is horizontal if it is the tangent vector of a horizontal curve. For each $v \in T_xM$ and a frame $u \in \mathscr{O}(M)$ such that $\pi u = x$, there is a unique horizontal vector V, called the horizontal lift of v, such that $\pi_* V = v$. For each $i = 1, \ldots, n$, let $H_i(u)$ be the horizontal lift of $ue_i \in T_xM$. Each H_i is a horizontal vector field on $\mathscr{O}(M)$, and $H_1, \ldots, H_n$ are called the fundamental horizontal vector fields on $\mathscr{O}(M)$.

On the orthonormal frame bundle $\mathscr{O}(M)$, a k-form θ is lifted to its scalarization $\tilde{\theta}$ defined by

$$\tilde{\theta}(u) = u^{-1}\theta(\pi u).$$

Here a frame $u : \mathbb{R}^n \to T_xM$ is assumed to be extended canonically to an isometry $u : \wedge^* \mathbb{R}^n \to \wedge_x^* M$. By definition, $\tilde{\theta}$ is a function on $\mathscr{O}(M)$ taking values in the vector space $\wedge^k \mathbb{R}^n$ and is $O(n)$-invariant in the sense that $\tilde{\theta}(gu) = g\tilde{\theta}(u)$ for $g \in O(n)$. We remark that through the isometry $u : \wedge^* \mathbb{R}^n \to \wedge_x^* M$, a linear transformation $T(x) : \wedge_x^* M \to \wedge_x^* M$ can also be lifted onto $\mathscr{O}(M)$ as a linear map

$$\tilde{T}(u) = u^{-1} H(\pi u) u : \wedge^* \mathbb{R}^n \to \wedge^* \mathbb{R}^n.$$

To simplify the notation, whenever feasible, we still use T for the more precise $\tilde{T}$ throughout our discussion.

Bochner's horizontal Laplacian on the frame bundle $\mathscr{O}(M)$ is defined to be $\triangle_{\mathscr{O}(M)} = \sum_{i=1}^{n} H_i^2$. It is the lift of the Laplace-Beltrami operator $\triangle$ in the sense that

$$\triangle_{\mathscr{O}(M)}\tilde{\theta}(u) = \widetilde{\triangle\theta(x)}, \quad \pi u = x.$$

To write the Weitzenbök formula on the frame bundle, we lift $D^*R : \wedge_x^* M \to \wedge_x^* M$ to the frame bundle $\mathscr{O}(M)$, which will be denoted by $D^*\Omega$, and let

$$\Box_{\mathscr{O}(M)} = \triangle_{\mathscr{O}(M)} + D^*\Omega. \tag{2.1}$$

Then $\Box_{\mathscr{O}(M)}$ is the lift of the Hodged-de Rham Laplacian in the sense that $\Box_{\mathscr{O}(M)}\tilde{\theta}(u) = \widetilde{\Box\theta(x)}$, where $\pi u = x$. The identity (2.1) is the lifted Weiztenböck formula on the orthonormal frame bundle $\mathscr{O}(M)$.

3 Absolute Boundary Condition

The purpose of this section is to give an explicit expression for the absolute boundary condition on forms. Once the boundary condition is identified, the multiplicative functional M_t could be constructed accordingly.

Fix an $x \in \partial M$, we let $n(x)$ be the inward unit normal vector at x. For a k-form θ, we may decompose θ into its tangential and normal component, $\theta = \theta_{tan} + n(x) \wedge \beta$, with $\theta_{tan} \in \wedge_x^k \partial M$ and $\beta \in \wedge_x^{k-1}\partial M$. We denote $\theta_{norm} = \theta - \theta_{tan}$. The form θ is said to satisfy the absolute boundary condition if

$$\theta_{norm} = 0 \ \text{ and } \ (d\theta)_{norm} = 0.$$

Let $Q(x) : \wedge_x^* M \to \wedge_x^* M$ be the orthogonal projection to the tangential component, i.e., $Q(x)\theta = \theta_{tan}$. We extend Q (indeed $\tilde{Q}$) to a smooth, projection linear map on the whole bundle $\mathscr{O}(M)$ and let $P(x) = I - Q(x)$. $P(x)$ is the orthogonal projection to the normal component.

Recall that the second fundamental form $H : T_x\partial M \otimes_{\mathbb{R}} T_x\partial M \to \mathbb{R}$ is defined by

$$H(x)(X, Y) = \langle \nabla_X Y, n(x) \rangle, \quad X, Y \in T_x\partial M.$$

By duality, $H(x)$ can also be regarded as a linear map $H(x) : T_x\partial M \to T_x\partial M$ via the relation

$$\langle HX, Y \rangle = H\langle X, Y \rangle.$$

It is clear that $H(x)$ is symmetric on $T_x\partial M$. We extend H to the whole tangent space T_xM by letting $H(x)n(x) = 0$, and denote the dual of H still by $H : \wedge_x^1 M \to \wedge_x^1 M$.

The following lemma gives an explicit expression for the absolute boundary condition on differential forms. Let

$$\partial\mathscr{O}(M) = \{u \in \mathscr{O}(M) : \pi u \in \partial M\}.$$

Lemma 3.1 *For any k-form θ on M, it satisfies the absolute boundary condition if and only if*

$$Q[N-H]\tilde{\theta} - P\tilde{\theta} = 0 \ \ on \ \ \partial\mathscr{O}(M).$$

Note that $\tilde{\theta}$ is the scalarization of θ, and N is the horizontal lift of n along the boundary ∂M.

Before we proceed to the proof of the above lemma, let us explain the various actions that appear in the above expression. Recall that N is a vector field on $\partial\mathscr{O}(M)$ and $\tilde{\theta}$ is a $\wedge^k\mathbb{R}^n$-valued function on $\mathscr{O}(M)$, thus $N\tilde{\theta}$ is naturally understood as the vector field acting on functions. The action $H\tilde{\theta}$ is more important. We know that H is a linear transformation on $\wedge^1_x M$ for $x \in \partial M$. For $\theta \in \wedge^k_x M$, the action $H\theta$ is the extension of H to $\wedge^* M$ as the Lie-algebra action(derivation) specified in Sect. 2. More specifically,

$$H(\theta_1 \wedge \ldots \wedge \theta_k) = \sum_{i=1}^{k} \theta_1 \wedge \ldots \wedge H\theta_i \wedge \ldots \wedge \theta_k,$$

where θ_i are 1-forms. Now $H\tilde{\theta}$ is simply $\tilde{H}\tilde{\theta}$.

Proof It is enough to show that

$$\theta_{norm} = 0 \Leftrightarrow P\tilde{\theta} = 0$$

and that, if $\theta_{norm} = 0$, then

$$(d\theta)_{norm} = 0 \Leftrightarrow Q[N-H]\tilde{\theta} = 0.$$

Fix any $x \in \partial M$. Let $\{E_i\}$ be a frame in a neighborhood of x with $E_1 = n$, the inward pointing unit normal vector field along the boundary and all other E_i's being tangent to the boundary. Furthermore we can choose the frame such that$\{E_i\}$ are orthonormal at x and $\nabla_{E_1} E_i = 0$ for all $i = 2, \ldots, n$ in a small neighborhood of x in M. To illustrate, we only prove the case when θ is a 2-form. The proof for k-forms will be clear, and actually identical when we understand what happens to 2-forms.

Let $\theta = \theta_{ij} E^i \wedge E^j$ be any 2-form, where $\{E^i\}$ is the dual of $\{E_i\}$. It's easy to see that $\theta_{norm} = 0$ is equivalent to $\theta_{1j} = \theta_{i1} = 0$ for all i, j, i.e., $P\tilde{\theta} = 0$.

Now we assume $P\tilde{\theta} = 0$ (i.e., $\theta_{1j} = \theta_{i1} = 0$ for all i, j). To see what $(d\theta)_{norm}$ means, we compute

$$\begin{aligned} d\theta &= E^k \wedge \nabla_{E_k}(\theta_{ij} E^i \wedge E^j) \\ &= E_k \theta_{ij} E^k \wedge E^i \wedge E^j + \theta_{ij} E^k \wedge \nabla_{E_k}(E^i \wedge E^j) \\ &= I_1 + I_2. \end{aligned}$$

Apparently

$$(I_1)_{norm} = E_1 \theta_{ij} E^1 \wedge E^i \wedge E^j, \tag{3.1}$$

since $\theta_{1j} = \theta_{i1} = 0$. On the other hand, we have

$$\begin{aligned} I_2 &= \theta_{ij} E^k \wedge (\nabla_{E_k} E^i \wedge E^j) + \theta_{ij} E^k \wedge (E^i \wedge \nabla_{E_k} E^j) \\ &= J_1 + J_2. \end{aligned}$$

Note that at x,

$$(\nabla_{E_k} E^i)(E_l) = -E^i(\nabla_{E_k} E_l) = -\langle \nabla_{E_k} E_l, E_i \rangle,$$

we therefore have

$$\nabla_{E_k} E^i = -\langle \nabla_{E_k} E_l, E_i \rangle E^l.$$

Hence at x,

$$J_1 = -\langle \nabla_{E_k} E_l, E_i \rangle \theta_{ij} E^k \wedge E^l \wedge E^j.$$

Keeping in mind that $\theta_{1j} = \theta_{i1} = 0$ and $\nabla_{E_1} E_i = 0$ for $i \neq 1$, we obtain

$$(J_1)_{norm} = -\langle \nabla_{E_k} E_1, E_i \rangle \theta_{ij} E^k \wedge E^1 \wedge E^j.$$

Re-indexing it we have

$$(J_1)_{norm} = \langle \nabla_{E_i} E_1, E_k \rangle \theta_{kj} E^1 \wedge E^i \wedge E^j. \tag{3.2}$$

Similarly

$$(J_2)_{norm} = \langle \nabla_{E_j} E_1, E_k \rangle \theta_{ik} E^1 \wedge E^i \wedge E^j. \tag{3.3}$$

Note that here $-\langle \nabla_{E_i} E_1, E_j \rangle$ is the matrix of second fundamental form on 1-forms. So we conclude that, by (3.1)–(3.3), when $\theta_{norm} = 0$, $(d\theta)_{norm} = 0$ is equivalent to

$$(E_1\theta_{ij} + \langle \nabla_{E_i} E_1, E_k \rangle \theta_{kj} + \langle \nabla_{E_j} E_1, E_k \rangle \theta_{ik}) E^1 \wedge E^i \wedge E^j = 0,$$

i.e., $Q(N - H)\tilde{\theta} = 0$. The proof is completed. □

Remark 3.2 Lemma 3.1 gives us a clear picture of the role the second fundamental form plays in the absolute boundary condition. Together with the discussion in Sect. 2, the initial boundary valued problem (1.1) can be lifted onto $\mathscr{O}(M)$ as

$$\begin{cases} \frac{\partial \tilde{\theta}}{\partial t} = \frac{1}{2}[\triangle_{\mathscr{O}(M)} + D^*\Omega]\tilde{\theta}, \\ \tilde{\theta}(\cdot, 0) = \tilde{\theta}_0, \\ Q[N - H]\tilde{\theta} - P\tilde{\theta} = 0. \end{cases} \tag{3.4}$$

Finally, we state an easy corollary of Lemma 3.1, which will be needed later. For each $x \in \partial M$, by the way we extended H to a linear map on $T_x M$, $\gamma_1 = 0$ is an eigenvalue of H associated to the eigenvector $n(x)$. Suppose that $\gamma_2(x), \ldots, \gamma_n(x)$ are other eigenvalues of H on $T_x \partial M$. We may define a real-valued function σ_k on ∂M by (see Donnelly-Li [2]),

$$\sigma_k(x) = \min_I \big(\gamma_{i_1}(x) + \gamma_{i_2}(x) + \ldots + \gamma_{i_k}(x)\big), \tag{3.5}$$

where $I = \{(i_1, \ldots, i_k)\}$ is the collection of multi-indices $(i_1, \ldots, i_k)$ such that $i_s \neq i_l$ if $s \neq l$; $s, l = 2, 3, \ldots, k$. Apparently, $\sigma_k(x)$ is a combination of eigenvalues of the second fundamental form H on $T_x \partial M$.

Corollary 3.3 *For any $x \in \partial M$ we have*

$$\sigma_k(x) = \inf_{\theta \in \wedge^k \partial M, |\theta| = 1} \langle H(x)\theta, \theta \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product on forms and $|\theta|^2 := \langle \theta, \theta \rangle$.

Proof Fix $x \in \partial M$, let $\{E_2, \ldots, E_n\}$ be a the set of orthonormal eigenvectors corresponding to the eigenvalues $\{\gamma_2, \ldots, \gamma_n\}$, and $\{E^i\}$ its dual. We first prove for any k-form θ with $|\theta| = 1$ we have

$$\sigma_k(x) \leq \langle H(x)\theta, \theta \rangle. \tag{3.6}$$

Let $\theta = \theta_{i_1,\ldots,i_k} E^{i_1} \wedge \ldots \wedge E^{i_k}$ with $|\theta|^2 = \sum \theta^2_{i_1,\ldots,i_k} = 1$. By the previous lemma we have

$$H(x)\theta = (\gamma_{i_1} + \ldots + \gamma_{i_k})\theta_{i_1,\ldots,i_k} E^{i_1} \wedge \ldots \wedge E^{i_k}.$$

Hence

$$\langle H(x)\theta, \theta\rangle = \sum(\gamma_{i_1} + \ldots + \gamma_{i_k})\theta^2_{i_1,\ldots,i_k} \geq \sigma_k(x)\sum \theta^2_{i_1,\ldots,i_k} = \sigma_k(x),$$

which proves (3.6). On the other hand, it's not hard to see that the equality can be achieved. The proof is completed. □

4 Reflecting Brownian Motion

Let $\omega = \{\omega_t\}$ be an n-dimensional Euclidean Brownian motion. Recall the definition of N in the previous section, and consider the following stochastic differential equation on the fame bundle $\mathscr{O}(M)$

$$du_t = \sum_{i=1}^{n} H_i(u_t) \circ d\omega_t^i + N(u_t)dl_t. \tag{4.1}$$

The solution $\{u_t\}$ is a horizontal reflecting Brownian motion starting at an initial frame u_0. Let $x_t = \pi u_t$. Then $\{x_t\}$ is a reflecting Brownian motion on M, with its transition density the Neumann heat kernel $p_M(t, x, y)$. The nondecreasing process l_t is the boundary local time, which increases only when $x_t \in \partial M$.

Now suppose that we have two smooth functions

$$R : \mathscr{O}(M) \to \mathrm{End}(\wedge^*\mathbb{R}^n), \quad A : \partial\mathscr{O}(M) \to \mathrm{End}(\wedge^*\mathbb{R}^n).$$

Define the $\mathrm{End}(\wedge^*\mathbb{R}^n)$-valued, continuous multiplicative functional $\{M_t\}$ by

$$dM_t + M_t\{-\frac{1}{2}R(u_t)dt + A(u_t)dl_t\} = 0, \ \ M_0 = I.$$

Since M_t takes values in $\mathrm{End}(\wedge^k\mathbb{R}^n)$, it is also helpful to think $\{M_t\}$ as a matrix-valued process.

Lemma 4.1 *Let $\mathcal{L} = \frac{\partial}{\partial s} - \frac{1}{2}[\triangle_{\mathscr{O}(M)} + R]$ and $F : \mathscr{O}(M) \times \mathbb{R}_+ \to \wedge^*\mathbb{R}^n$ be a solution to*

$$\begin{cases} \mathcal{L}F = 0 & u \in \mathscr{O}(M)/\partial\mathscr{O}(M) \\ (N - A)F = 0 & u \in \partial\mathscr{O}(M), \end{cases} \tag{4.2}$$

we have

$$M_t F(u_t, T-t) = F(u_0, T) + \int_0^t \langle M_s \nabla^H F(u_s, T-s), d\omega\rangle,$$

where $\nabla^H F = \{H_1F, H_2F, \ldots, H_nF\}$ *is the horizontal gradient of a function* F *on* $\mathscr{O}(M)$*. In this case, we say that* $\{M_t\}$ *is the multiplicative functional associated with the operator* $\mathcal{L}$ *with the boundary condition* $(N - A)F = 0$.

Proof Apply Itô's formula to $M_tF(u_t, T - t)$. □

5 Discontinuous Multiplicative Functional

We have shown that the heat equation on k-forms with absolute boundary condition is equivalent to the following heat equation on $O(n)$-invariant functions F: $\mathscr{O}(M) \times \mathbb{R}_+ \to \wedge^k\mathbb{R}^n$:

$$\begin{cases} \frac{\partial F}{\partial t} = \frac{1}{2}[\triangle_{\mathscr{O}(M)} + D^*\Omega]F, \\ F(\cdot, 0) = f, \\ QNF - (H + P)F = 0. \end{cases} \tag{5.1}$$

Compared with the boundary condition in (4.2), $QN-(H+P)$ is degenerate, because Q is a projection (hence is not of full rank as a linear map). Thus Lemma 4.1 cannot be applied directly. In this section we follow closely the idea of Hsu [3] to construct the End($\wedge^k\mathbb{R}^n$)-valued multiplicative functional associated to (5.1).

Observe that the boundary condition in (5.1) consists of two orthogonal components:

$$Q[N - H]F = 0, \quad PF = 0. \tag{5.2}$$

We replace PF above by $(-\varepsilon PN + P)F$ and rewrite the boundary condition as

$$\left[N - H - \frac{P}{\varepsilon}\right]F = 0.$$

According to Lemma 4.1, the multiplicative functional for this approximate boundary condition is given by

$$dM_t^\varepsilon + M_t^\varepsilon\left\{-\frac{1}{2}D^*\Omega(u_t)dt + \left[\frac{1}{\varepsilon}P(u_t) + H(u_t)\right]dl_t\right\} = 0. \tag{5.3}$$

In the rest of this section, we show that $\{M_t^\varepsilon\}$ converges to a discontinuous multiplicative functional $\{M_t\}$ which turns out to be the right one for the boundary condition (5.2).

Recall the definition of σ_k in (3.5) and let

$$\lambda(x) = \sup_{\theta\in\wedge_x^k M, \langle\theta,\theta\rangle=1} \langle D^*R(x)\theta, \theta\rangle. \tag{5.4}$$

When $k = 1$, it is well known that $D^*R(x) = -Ric(x)$, where $Ric(x)$ is the Ricci transformation at x (see Hsu [4], for example), hence $\lambda(x)$ is the negative lower bound of the Ricci transform at x.

Proposition 5.1 *Let $|\cdot|_{2,2}$ be the norm of a linear transform on $\wedge^k\mathbb{R}^n$ with the standard Euclidean norm. Then for all positive ε such that $\varepsilon^{-1} \geq \min_{x\in\partial M}\sigma_k(x)$, we have*

$$|M_t^\varepsilon|_{2,2} \leq \exp\left[\frac{1}{2}\int_0^t \lambda(x_s)ds - \int_0^t \sigma_k(x_s)dl_s\right].$$

Proof We only outline the proof here, the technical details being mostly the same as that in [3]. Instead of considering M_t^ε, we prove for the adjoint (transpose, if we think M_t^ε as a matrix-valued process) of M_t^ε , namely $(M_t^\varepsilon)^T$. Let $f(t) = |(M_t^\varepsilon)^T\tilde{\theta}|^2 = \langle (M_t^\varepsilon)^T\tilde{\theta}, (M_t^\varepsilon)^T\tilde{\theta}\rangle$. Differentiate f with respect to t. By (5.3), our assumption on ε and standard estimate we have

$$df(t) \leq f(t)\{\lambda(x_t)dt - 2\sigma_k(x_t)dl_t\},$$

which gives us the desired result. □

The integrability of M_t^ε is given by the following lemma.

Lemma 5.2 *For any positive constant C, there is a constant C_1 depending on C but independent of x such that*

$$\mathbb{E}_x e^{Cl_t} \leq C_1 e^{C_1 t}.$$

Proof This can be obtained by a heat kernel upper bound and the strong Markov property of reflecting Brownian motion. See [3, Lemma 3.2] for a detailed proof. □

If we formally let $\varepsilon \downarrow 0$ in (5.3), one should expect $M_t^\varepsilon P(u_t) \to 0$ for all t such that $u_t \in \partial\mathscr{O}(M)$. The next lemma shows it is indeed the case. Define

$$T_{\partial M} = \inf\{s \geq 0 : x_s \in \partial M\} = \text{the first hitting time of } \partial M.$$

A point $t \geq T_{\partial M}$ such that $l_t - l_{t-\delta} > 0$ for all positive $\delta \leq t$ is called a *left support point* of the boundary local time l.

Proposition 5.3 *When $\varepsilon \downarrow 0$, $M_t^\varepsilon P(u_t) \to 0$ for all left support points $t \geq T_{\partial M}$.*

Proof The proof is almost identical to the one for 1-forms in [3]. For the convenience of the reader, we still provide some details here. We drop the superscript ε for simplicity. Let $\theta \in \wedge^k M$ be a k-form and define

$$f(s) = \langle M_s^T\tilde{\theta}, P(u_t)M_s^T\tilde{\theta}\rangle = \langle\tilde{\theta}, M_s P(u_t)M_s^T\tilde{\theta}\rangle.$$

Differentiating f with respect to s, by (5.3) we have $df(s) = -\frac{2}{\varepsilon}f(s) + dN_s$, which gives us

$$f(t) = e^{-2(l_t - l_{t-\delta})/\varepsilon} f(t-\delta) + \int_{t-\delta}^{t} e^{-2(l_t - l_s)/\varepsilon} dN_s. \tag{5.5}$$

Here dN_s is equal to

$$\begin{aligned}
&\frac{1}{\varepsilon}\langle \tilde{\theta}, M_s(2P(u_t) - P(u_s)P(u_t) - P(u_t)P(u_s))M_s^T \tilde{\theta}\rangle dl_s \\
&\quad + \langle \tilde{\theta}, \frac{1}{2} M_s(D^*\Omega(u_s)P(u_t) + P(u_t)(D^*\Omega(u_s))^T)M_s^T \tilde{\theta}\rangle ds \\
&\quad - \langle \tilde{\theta}, M_s(H(u_s)P(u_t) + P(u_t)H(u_s))M_s^T \tilde{\theta}\rangle dl_s.
\end{aligned}$$

In the above we used the fact that $H^T = H$ and $P^T = P$. By continuity of P and Proposition 5.1, for any $\eta > 0$ there exists a $\delta > 0$ such that, for all $s \in [t-\delta, t]$ with $x_s \in \partial M$,

$$\langle \tilde{\theta}, M_s(2P(u_t) - P(u_s)P(u_t) - P(u_t)P(u_s))M_s^T \tilde{\theta}\rangle \le \eta |\tilde{\theta}|^2.$$

Also by Proposition 5.1, there is a constant C such that, for all $s \in [t-\delta, t]$ with $x_s \in \partial M$,

$$\langle \tilde{\theta}, \frac{1}{2} M_s(D^*\Omega(u_s)P(u_t) + P(u_t)(D^*\Omega(u_s))^T)M_s^T \tilde{\theta}\rangle \le C|\tilde{\theta}|^2$$

and

$$\langle \tilde{\theta}, M_s(H(u_s)P(u_t) + P(u_t)H(u_s))M_s^T \tilde{\theta}\rangle \le C|\tilde{\theta}|^2.$$

It follows that

$$|dN_s| \le |\tilde{\theta}|^2 \left[\left(\frac{\eta}{\varepsilon} + C \right) dl_s + C ds \right].$$

Substituting in (5.5), we obtain

$$\begin{aligned}
|M_t P(u_t)|_{2,2}^2 \le & e^{-2(l_t - l_{t-\delta})/\varepsilon} |M_{t-\delta}|_{2,2}^2 + \frac{\eta + C\varepsilon}{2} \{1 - e^{-2(l_t - l_{t-\delta})/\varepsilon}\} \\
& + C \int_{t-\delta}^{t} e^{-2(l_t - l_{t-\delta})/\varepsilon} ds.
\end{aligned} \tag{5.6}$$

Because t is a left support point, $l_t - l_s > 0$ for all $s < t$. We first let $\varepsilon \downarrow 0$ and then $\eta \to 0$ in (5.6), we have $M_t P(u_t) \to 0$. $\square$

We now come to the main result of this section, namely, the limit $\lim_{\varepsilon\to 0} M_t^\varepsilon = M_t$ exists. From the definition of M_t^ε, if t is such that $x_t \notin \partial M$ we have

$$dM_t^\varepsilon - \frac{1}{2} M_t^\varepsilon D^* \Omega(u_t) dt = 0.$$

Let $\{e(s,t), t \geq s\}$ be the solution of

$$\frac{d}{dt} e(s,t) - \frac{1}{2} e(s,t) D^* \Omega(u_t) = 0, \quad e(s,s) = I.$$

Then, for $t \geq T_{\partial M}$ we have $M_t^\varepsilon = M_{t*}^\varepsilon e(t_*, t)$. Here for each $t \geq T_{\partial M}$, t_* is defined to be the last exit time from ∂M, more precisely, $t_* = \sup\{s \leq t : x_s \in \partial M\}$.

Define

$$Y_t^\varepsilon = M_t^\varepsilon P(u_t), \quad Z_t^\varepsilon = M_t^\varepsilon Q(u_t).$$

Since when $t \leq T_{\partial M}$ we have $M_t^\varepsilon = e(0,t)$; and when $t \geq T_{\partial M}$ we have

$$M_t^\varepsilon = M_{t*}^\varepsilon e(t_*, t) = \{Z_{t*}^\varepsilon + Y_{t*}^\varepsilon\} e(t_*, t),$$

we can write

$$\begin{aligned} Y_t^\varepsilon &= I_{\{t \leq T_{\partial M}\}} M_t^\varepsilon P(u_t) + I_{\{t > T_{\partial M}\}} M_t^\varepsilon P(u_t) \\ &= I_{\{t \leq T_{\partial M}\}} e(0,t) P(u_t) + I_{\{t > T_{\partial M}\}} Z_{t*}^\varepsilon e(t_*, t) P(u_t) + \alpha_t^\varepsilon, \end{aligned} \tag{5.7}$$

where

$$\alpha_t^\varepsilon = I_{\{t > T_{\partial M}\}} Y_{t*}^\varepsilon e(t_*, t) P(u_t). \tag{5.8}$$

If $t > T_{\partial M}$, then t_* is a left support point of l. By Proposition 5.3, $Y_{t*}^\varepsilon \to 0$ as $\varepsilon \downarrow 0$; hence $\alpha_t^\varepsilon \to 0$. On the other hand, by Eq. (5.3) for M_t^ε we have

$$\begin{aligned} Z_t^\varepsilon &= Q(u_0) + \int_0^t dM_s^\varepsilon Q(u_s) + \int_0^t M_s^\varepsilon dQ(u_s) \\ &= Q(u_0) + \int_0^t [Y_s^\varepsilon + Z_s^\varepsilon] d\chi_s, \end{aligned} \tag{5.9}$$

where

$$d\chi_s = -H(u_s) dl_s + \frac{1}{2} D^* \Omega(u_s) Q(u_s) ds + dQ(u_s).$$

Formally letting $\varepsilon \downarrow 0$ in (5.7) and (5.9) above, we expect that the limit (Y_t, Z_t) satisfies following equations:

$$\begin{cases} Y_t = I_{\{t \le T_{\partial M}\}} e(0,t) P(u_t) + I_{\{t > T_{\partial M}\}} Z_{t*} e(t_*, t) P(u_t), \\ Z_t = Q(u_0) + \int_0^t (Y_s + Z_s) d\chi_s. \end{cases} \tag{5.10}$$

Substituting the first equation into the second, we obtain an equation for Z itself in the form

$$Z_t = Q(u_0) + \int_0^t \Phi(Z)_s d\chi_s, \tag{5.11}$$

where

$$\Phi(Z)_s = Z_s + I_{\{s \le T_{\partial M}\}} e(0,s) P(u_s) + I_{\{s > T_{\partial M}\}} Z_{s*} e(s_*, s) P(u_s).$$

Now we can state the main result in this section. For an End$(\wedge^k \mathbb{R}^n)$-valued stochastic process $M = \{M_t\}$, we define

$$|M|_t = \sup_{0 \le s \le t} |M_s|_{2,2}.$$

Theorem 5.4 *We have*

(1) *Equation (5.11) has a unique solution Z. Define Y by the first equation in (5.10) and let $M_t = Y_t + Z_t$. Then $\{M_t\}$ is right continuous with left limits and $M_t P(u_t) = 0$ whenever $x_t \in \partial M$.*
(2) *For each fixed t,*

$$\mathbb{E}|Z^\varepsilon - Z|_t \to 0, \quad \mathbb{E}|Y_t^\varepsilon - Y_t|_{2,2}^2 \to 0, \ as\ \varepsilon \downarrow 0.$$

Hence $\mathbb{E}|M_t^\varepsilon - M_t|_{2,2}^2 \to 0$ as $\varepsilon \downarrow 0$.

Proof The proof of the stated results follow the lines of proofs of Theorem 3.4 and Theorem 3.5 of [3]. □

Corollary 5.5 *For the limit process $\{M_t\}$ we have*

$$|M_t|_{2,2} \le \exp\left[\frac{1}{2}\int_0^t \lambda(x_s) ds - \int_0^t \sigma_k(x_s) dl_s\right].$$

Proof Letting $\varepsilon \downarrow 0$ in Lemma 5.1, the result follows immediately. □

Corollary 5.6 *The End($\wedge^k \mathbb{R}^n$)-valued process M_t is the multiplicative functional associated to Eq. (5.1).*

Proof Since F is a solution to (5.1), from Lemma 4.1 with $\mathcal{L} = \frac{\partial}{\partial s} - \frac{1}{2}[\triangle_{\mathscr{O}(M)} + D^*\Omega]$, we have

$$M_t^\varepsilon F(u_t, T-t) = F(u_0, T) + \int_0^t \langle M_s^\varepsilon \nabla^H F(u_s, T-s), d\omega_s \rangle + \int_0^t M_s^\varepsilon \left[N - \frac{1}{\varepsilon} P - H \right] F(u_s, T-s) dl_s.$$

The terms involving $1/\varepsilon$ vanish because, by the assumption on F, $P(u_s)F(u_s, T-s) = 0$ for $u_s \in \partial\mathscr{O}(M)$. Using the previous theorem, we let $\varepsilon \downarrow 0$ and note that $Q[N-H]F = [N-H]F$ and $M_s = MQ(u_s)$ when $u_s \in \partial\mathscr{O}(M)$ (by Theorem 5.4), we obtain the desired equality. $\square$

6 Heart Kernel Representation and Applications

With the multiplicative functional M_t constructed in the previous section, we have the following probabilistic representation of the solution to (1.1).

Theorem 6.1 *Let $\theta \in \wedge^k M$ be the solution of the initial boundary value problem (1.1). Then*

$$\tilde{\theta}(u, t) = \mathbb{E}_u\{M_t \tilde{\theta}_0(u_t)\}. \tag{6.1}$$

Hence θ is given by

$$\theta(x, t) = u\mathbb{E}_x\{M_t u_t^{-1} \theta_0(x_t)\} \tag{6.2}$$

for any $u \in \mathscr{O}(M)$ such that $\pi u = x$.

Proof By Corollary 5.6, $\{M_s \tilde{\theta}(u_s, t-s), 0 \le s \le t\}$ is a martingale. Equating the expected values at $s = 0$ and $s = t$ gives us (6.1). The second equality is a restatement of the first one on the manifold M. $\square$

There are several application with the above representation. We will examine two of them below. Let

$$p_M^*(t, x, y) : \wedge_y^* M \to \wedge_x^* M$$

be the heat kernel on differential forms with absolute boundary condition. By the above theorem we have

$$u\mathbb{E}_x\{M_t u_t^{-1} \theta(x_t)\} = \int_M p_M^*(t, x, y)\theta(y) dy, \quad \pi u = x. \tag{6.3}$$

On the other hand, we can also write

$$\begin{aligned} u\mathbb{E}_x\{M_t u_t^{-1}\theta(x_t)\} &= u\mathbb{E}_x\mathbb{E}_x\{M_t u_t^{-1}\theta(x_t)|x_t = y\} \\ &= \int_M p_M(t,x,y)u\mathbb{E}_x\{M_t u_t^{-1}\theta(x_t)|x_t = y\}dy. \end{aligned} \tag{6.4}$$

Here $p_M(t,x,y)$ is the heat kernel on functions with Neumann boundary condition, i.e., the transition probability of $\{x_t\}$. From (6.3) and (6.4) the heat kernel on differential forms can be written as

$$p_M^*(t,x,y) = p_M(t,x,y)u\mathbb{E}_x\{M_t u_t^{-1}|x_t = y\}. \tag{6.5}$$

Recall that

$$\sigma_k = \min_I \gamma_{i_1} + \gamma_{i_2} + \ldots + \gamma_{i_k},$$

where $\gamma_2, \ldots, \gamma_n$ are eigenvalues of the second fundamental form of ∂M, and $I = \{(i_1, \ldots, i_k)\}$ is the collection of multi-indices $(i_1, \ldots, i_k)$ with $i_s = 2, 3, \ldots, k$ and $i_s = i_l$ if $s \neq l$; and that

$$\lambda(x) = \sup_{\theta \in \wedge_x^k M, \langle\theta,\theta\rangle = 1} \langle D^* R(x)\theta, \theta\rangle. \tag{6.6}$$

We have the following heat kernel domination.

Theorem 6.2 *Let $p_M^k(t,x,y)$ be the heat kernel on k-forms. Define*

$$\bar{\sigma}_k = \inf_{x\in\partial M} \sigma_k \quad \text{and} \quad \bar{\lambda} = \sup_{x\in\partial M} \lambda(x).$$

We have

$$|p_M^k(t,x,y)|_{2,2} \leq e^{\frac{1}{2}\bar{\lambda}t - \bar{\sigma}_k l_t} p_M(t,x,y),$$

where l_t is the Brownian motion boundary local time.

Proof This is a direct application of representation (6.5) and Proposition 5.1. □

Remark 6.3 When $\bar{\sigma}_k \geq 0$ then we have

$$|p_M^k(t,x,y)|_{2,2} \leq e^{\frac{1}{2}\bar{\lambda}t} p_M(t,x,y).$$

This special case was proved by Donnelly and Li [2], and Shigekawa [8].

For $\theta \in \wedge^k M$, let $P_t\theta(x) = \int_M p^*(t,x,y)\theta(y)dy$. Then we have the following generalized gradient inequality.

Theorem 6.4 *Keep all the notation above, we have*

$$|dP_t\theta(x)| \le \mathbb{E}_x\left\{|d\theta|\exp\left[\frac{1}{2}\int_0^t \lambda(x_s)ds - \int_0^t \sigma_{k+1}(x_s)dl_s\right]\right\}.$$

Proof Let $\eta(x,t) = dP_t\theta(x)$. Then η is a $k+1$-form satisfying the absolute boundary condition, since $d\eta = d(dP_t\theta) = 0$ and $(\eta)_{norm} = (dP_t\theta)_{norm} = 0$. On the other hand, because d commute with the Hodge-de Rham Laplacian, we have

$$\frac{\partial\eta}{\partial t} = d\left(\frac{\partial P_t\theta}{\partial t}\right) = \frac{1}{2}d\,\Box P_t\theta = \frac{1}{2}\Box\, dP_t\theta = \frac{1}{2}\Box\eta.$$

So η is a solution to the heat equation (1.1). The rest of the proof is thus again an easy application of (6.2) and Proposition 5.1. ☐

Remark 6.5 When θ is a 0-form, i.e., a function on M, denoted as f. Then the above inequality reduces to

$$|\nabla P_t f(x)| \le \mathbb{E}_x\left\{|\nabla f(x_t)|\exp\left[\frac{1}{2}\int_0^t \lambda(x_t)ds - \int_0^t \sigma_1(x_s)dl_s\right]\right\},$$

where σ_1 is just the smallest eigenvalue of the second fundamental form at x and $-\lambda$ is the low bound of Ricci curvature(since in one dimension $D^*R = -Ricci$). This special case was proved by Hsu [3]. In the case when M is a non-compact manifold, similar bound was obtained in [9, Chap. 3].

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