

Chapter 3

Ordered Fuzzy Numbers: Sources and Intuitions

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Abstract Most emerging methodologies, before they become well settled, stem from careful analysis of previous solutions. In that respect, this chapter refers to the roots of the Ordered Fuzzy Number (OFN) model. First, we outline some drawbacks of the most popular fuzzy number representations, which inspired us to search for a new approach. Then we discuss the idea of looking at fuzzy numbers from an alternative viewpoint. This leads towards formulation of the OFN model comprising three conceptual steps: (1) representing membership functions of fuzzy numbers as the pairs of increasing/decreasing components; (2) for each of two components treated as a locally defined function, inverting the meanings of its domain and its set of values; and finally (3) treating the obtained pairs of components as the ordered pairs. By introducing arithmetic operations on such ordered pairs, we obtain the framework, which is in many cases equivalent to the previous approaches but it also enables the representation of new information aspects.

3.1 Introduction

The Ordered Fuzzy Number (OFN) model was defined as a result of searching for simple and flexible algorithms performing calculations on fuzzy numbers [10]. A more formal description of the OFN model is provided in Chap. 1. Here we focus on initial inspirations, which can be helpful to understand the meaning behind mathematical formulas for OFN operations. Indeed, if someone with a thorough background in the theory of fuzzy sets jumps immediately to Sect. 3.5, the first impression might be confusing. Thus we encourage the readers to study the contents of this chapter

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step by step, in order to realize that the considered model developed primarily by Witold Kosiński is truly straightforward and easy to handle.

The chapter is organized as follows. In Sect. 3.2, some potential drawbacks of classically defined fuzzy numbers are explained. In Sect. 3.3, some alternative proposals regarding how to deal with those drawbacks are outlined. In Sect. 3.4, we discuss an idea of decomposing the shapes of fuzzy numbers onto ascending/constant/descending components, that is, expressing fuzzy membership functions in so-called *quasi-invertible* form that was utilized to redefine fuzzy numbers in [11]. In Sect. 3.5, we discuss how the idea of operating with quasi-invertible functions led us towards the OFN model. Section 3.6 concludes the chapter.

3.2 Problems with Calculations on Fuzzy Numbers

Basic operations on standard fuzzy numbers were discussed in Chap. 1. Two popular mechanisms of introducing them were mentioned: via the extension principle [3] and via interval calculations on α -cuts [1]. In both cases, for both addition and subtraction operations, their outcomes support increases compared to their inputs. Thus, their fuzziness increases. After performing several calculation steps, a resulting number's support usually becomes extremely broad, whereby information represented by that number is no longer practically useful.

In [12], we can find a more general summary of problems with original approaches to fuzzy arithmetics. Certainly, those problems can be handled using some more advanced models, thus propagation of fuzziness becomes more tractable during computations [21]. It should also be noted that standard fuzzy arithmetic methods turned out to be very useful in a number of practical applications [7]. Nevertheless, in some cases, one may require more straightforward tools for limiting or – sometimes – even reversing a degree of imprecision represented by fuzzy numbers. This observation inspired us to search for a new way of representing and computing fuzzy numbers. Two major goals in front of us were:

1. To introduce an intuitive model enabling us to decrease (not only increase) imprecision/inaccuracy as a result of arithmetic operations
2. In addition, to introduce such mechanisms of fuzzy number calculations that would be easy to understand and implement in practice

The general concept that allowed us to control imprecision during calculations can be interpreted as a kind of direction of fuzziness [18]. Chapter 4 contains a complete description of that idea from a mathematical perspective. It shows how the considered OFN model allows us to think about canceling fuzziness while adding/subtracting opposite or reversed fuzzy numbers. The remainder of this chapter can be treated as introductory background for that formalism.

3.3 Related Work

Let us present a short preview of other existing approaches to deal with the problem of increasing imprecision during fuzzy calculations. Generally, we can categorize such approaches as those refining standard operations [21], those introducing new operations [19], and those specifying a kind of context of operations [8].

In [19], two additional operations on fuzzy numbers were introduced: nonstandard subtraction and nonstandard division. Those operations are quite complicated but indeed it is true that, for fuzzy numbers A , B , C , equation $A + B = C$ is equivalent to equation $C \ominus B = A$, where \ominus denotes nonstandard subtraction. In our research, we kept looking for another solution, as it is not always a good idea to introduce new operations. It may be problematic from both conceptual and technical perspectives. In the space of real numbers, subtraction is equivalent to addition of an opposite number. By analogy, there should be no need to define a separate subtraction operation for fuzzy numbers. The same reasoning can be carried out for multiplication and division. This can be represented by requirements of a form $A - B = A + (-B)$ and $A/B = A \cdot B^{-1}$, which are not addressed in [19].

In [8], Klir presented another approach to operations on fuzzy numbers (by means of fuzzy intervals). His idea takes into account a context of relationship between two numbers – referred to as a *requisite constraint* – which may optionally allow a decrease of fuzziness in calculations. It is effective in solving equations of a type $A + X = B$, if we know that $X = B - A$. However, just as above, this method may make calculations complicated. The assumption that we are able to set up requisite constraints for all relevant pairs of quantities for a given calculation is difficult to track for more complex scenarios. Still, the ideas proposed in [8] seem to be closer to our way of understanding operations on fuzzy numbers than those in the case of [19]. However, in the OFN model, additional information, a kind of context, is assigned to particular numbers rather than relationships between them.

The above ideas became a source of our inspiration in 2000–2003, when the OFN model was formulated. However, the problem of expanding fuzziness is also present in more recent research. For instance, Dymova et al. proposed the operation called *interval extended zero* [6]. It is used to solve linear fuzzy equations. As another example, Piegat and Landowski utilized RDM (*relative distance measure*) interval arithmetics [15]. Furthermore, Stupnanova combined fuzzy operations with probabilistic modeling [20]. Such methods should be compared to the OFN model in a more detailed way. However, we should remember that all of them aim at better controlling rather than eliminating/reversing fuzziness during calculations.

3.4 Decomposition of Fuzzy Memberships

The remaining sections include some basic observations and suggestions on how to change standard representation and meaning of fuzzy numbers. We start by recalling

the original concept of a fuzzy set [22]. Then we concentrate on the nature of shapes of standard fuzzy numbers. Finally, we introduce a new representation based on the already-mentioned inversion of the roles between the domains and the sets of values for particular components of those shapes.

As we know, a fuzzy set A over a space X is defined as a set of pairs, namely $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \rightarrow [0, 1]$ denotes a fuzzy membership function that assigns to each element $x \in X$ its degree of membership to A . In Sect. 3.5, we refer to a more general understanding of fuzzy sets. However, for now let us follow the above formulation and consider a fuzzy number as a fuzzy set over the space of real numbers \mathbb{R} ¹:

$$A = \{(x, \mu_A(x)) : x \in \mathbb{R}\} \quad (3.1)$$

As pointed out in Chap. 1, each fuzzy number A is supposed to be a normal fuzzy set; that is, there exists $x \in \mathbb{R}$ such that $\mu_A(x) = 1$. Moreover, its support should be bounded: that is, there exists interval (s_A, e_A) such that $\mu_A(x) > 0$, if and only if $x \in (s_A, e_A)$.² Finally, A is supposed to be a convex fuzzy set and its membership function should be piecewise continuous.

The convexity of A corresponds to the strict quasi-concavity of function μ_A or, equivalently, the strict quasi-convexity of function $-\mu_A$ [4]. Detailed properties of strict quasi-convexity can be found, for example, in [13]. Saying that function $-\mu_A$ is strictly quasi-convex (and μ_A is strictly quasi-concave) means:

$$\forall_{x,y,z \in \mathbb{R}} (x < y < z \wedge \mu_A(x) \neq \mu_A(z)) \Rightarrow (\mu_A(y) > \min(\mu_A(x), \mu_A(z))) \quad (3.2)$$

Figure 3.1 illustrates a difference between functions that are convex/concave and strictly quasi-convex/quasi-concave.

According to one of the theorems proved in [13], the fuzzy membership function μ_A is strictly quasi-concave within a convex set X , if and only if any segment $(x_1, x_2] \subseteq X$ can be divided into three sections such that μ_A is increasing in the first, constant in the second, and decreasing in the third section. Moreover, any one or two of these sections may be empty or degenerated to a single point. Thus, one can conclude that for a given fuzzy number A we have values $1_A^-, 1_A^+ \in (s_A, e_A)$ such that μ_A has an increasing part defined over $(s_A, 1_A^-)$ and a decreasing part over $(1_A^+, e_A)$. There is also a constant part equal to 1 over the interval (or a single point) $[1_A^-, 1_A^+]$.

The above kind of piecewise representation goes well together with some other approaches to model fuzzy arithmetics, for example, by means of so-called L-R numbers [5]. For more details on this methodology, let us refer to [2], where L-R numbers are thoroughly compared to the OFN model. From our perspective, it is especially interesting that both increasing and decreasing parts of strictly quasi-concave fuzzy

¹In the literature, fuzzy numbers are often denoted by \tilde{A} . However, as in our case it does not lead to any misunderstanding, we simply use notation A .

² s_A and e_A can be intuitively regarded as a *start* and *end* of A . Analogous notation is used in Chap. 4, along with 1_A^- and 1_A^+ . However, inequalities $s_A < 1_A^- \leq 1_A^+ < e_A$ that hold for standard fuzzy numbers will not need to be true in the OFN model.

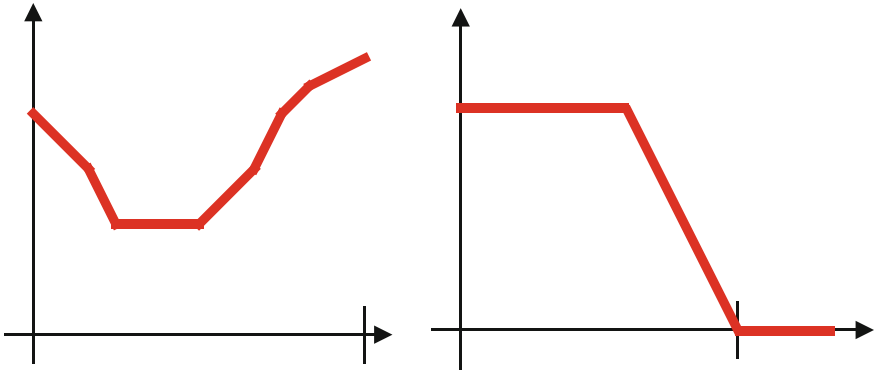


Fig. 3.1 Examples of functions that are strictly quasi-convex but not convex (*left*) and strictly quasi-concave but not concave (*right*)

membership functions are invertible. Such characteristics are hereinafter referred to as quasi-invertibility. This enables us to define operations on inverted parts of fuzzy numbers' membership functions.

1. Inverted increasing parts of μ_A and μ_B are added to each other.
2. Inverted decreasing parts of μ_A and μ_B are added to each other.
3. After reinverting both obtained sums, we obtain a function that is treated as membership μ_C of fuzzy number $C = A + B$ (see Fig. 3.2).

The above mechanism was studied in the literature as an alternative way of thinking about fuzzy arithmetics [11]. Although conceptually it does not change the standard model, it turns out to be very simple to implement in practice. It is also the best starting point for explaining the OFN model.

3.5 Idea of Ordered Fuzzy Numbers

The quasi-invertibility-based addition procedure recalled in Sect. 3.4 could be alternatively rewritten by representing fuzzy numbers A and B as unordered pairs of piecewise continuous monotonic functions $f_A, g_A : [0, 1] \rightarrow \mathbb{R}$ and $f_B, g_B : [0, 1] \rightarrow \mathbb{R}$. In the first step, we would then need to check which of those functions are increasing/decreasing in order to obtain a valid result. However, it would not yet solve the issue of propagation of imprecision in computations, which was one of our main motivations while searching for a new model of fuzzy arithmetics.

Let us go back for a while to the discussion in Sect. 3.3. Let us note that one of the requirements to limit imprecision corresponds to a need of introducing opposite fuzzy numbers, that is, for each A , defining fuzzy number $-A$ such that $A + (-A) = A - A = \mathbf{0}$, where $\mathbf{0}$ denotes the fuzzy number representation of crisp 0.

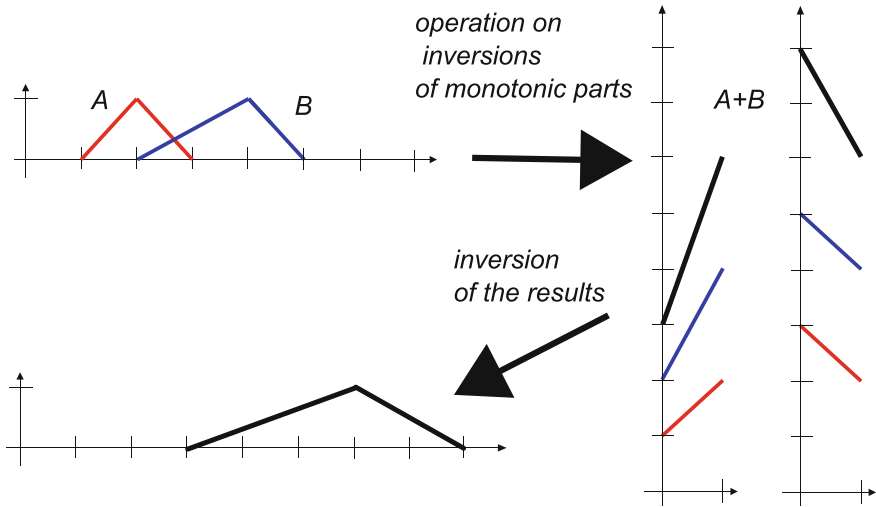


Fig. 3.2 Example of adding standard fuzzy numbers using quasi-invertibility

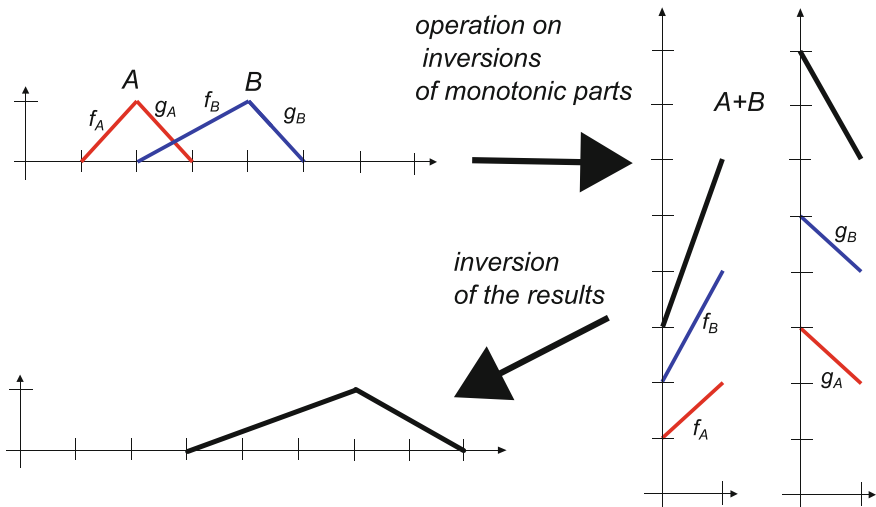


Fig. 3.3 Example of adding two OFNs. The result is comparable to that obtained in Fig. 3.2

As expressing opposite numbers is hardly possible in standard fuzzy arithmetics, let us consider a slightly revised notion of fuzzy number. The following definition includes an imposed order between components representing fuzzy numbers. This order may be regarded as an additional aspect of information – a kind of fuzzy number’s context – which is independent of the values of fuzzy memberships. Let us also note that fuzzy number components are now defined in an inverted way when compared to standard fuzzy numbers.

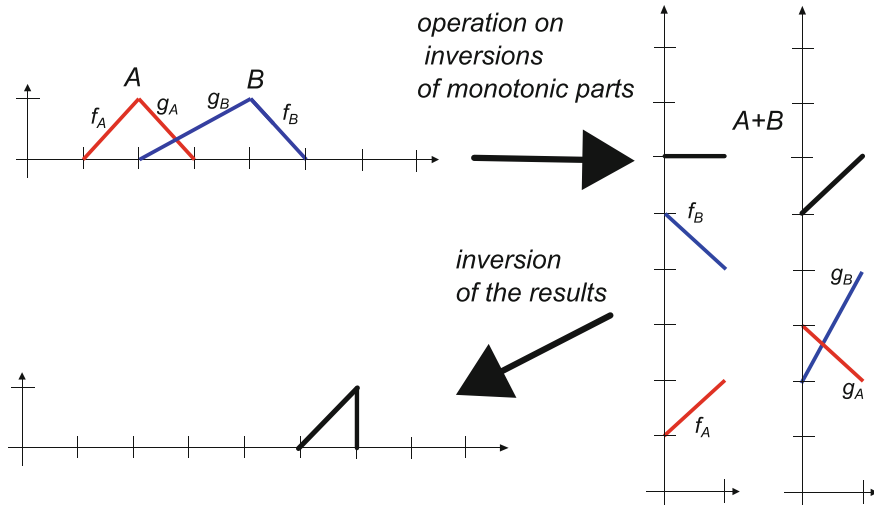


Fig. 3.4 Example of adding two OFNs, where B has a reversed order compared to Fig. 3.3

Definition 1 (*Ordered Fuzzy Number*) $A = (f_A, g_A)$ is an ordered pair of continuous functions $f_A, g_A : [0, 1] \rightarrow \mathbb{R}$, called the **up part**, and the **down part** of A , respectively.³

The shape related to the pair (f_A, g_A) is not different from the case of (g_A, f_A) . However, these are two different OFNs (unless $f_A = g_A$). They differ by something that can be interpreted as *direction*, in some papers also called *orientation*. Actually, interpretation related to *direction* inspired some researchers to propose renaming the presented model as *directed fuzzy numbers* or *fuzzy numbers with direction*. Moreover, in papers [16–18], the name *Kosiński’s Fuzzy Numbers* is used to honor the contribution of Witold Kosiński in a development of the considered model. Although in this book we keep the original OFN terminology, we believe that the discussion about the most appropriate name is not over.

Let us notice that it is now possible to define A ’s opposite as $-A = (f_{-A}, g_{-A})$, where $f_{-A} = -f_A$ and $g_{-A} = -g_A$. In a standard model, while adding such numbers represented as unordered pairs $f_A, g_A : [0, 1] \rightarrow \mathbb{R}$ and $f_{-A}, g_{-A} : [0, 1] \rightarrow \mathbb{R}$, we would need to combine f_A with g_{-A} and g_A with f_{-A} . However, in the OFN model – as formally introduced in Chap. 4 – we follow the ordering of component functions rather than their increasing/decreasing characteristics. Figures 3.3 and 3.4 illustrate more examples. We can see that the ordering of components defining a fuzzy number

³Notation $A = (f_A, g_A)$ reflects the original way of referring to the up and down parts introduced in [10]. Surely, one could also think about a more intuitive naming, for example, \uparrow_A and \downarrow_A instead of f_A and g_A , respectively. One could also think about denoting OFNs in a different way, such as using sign \rightsquigarrow above A (by analogy to \bar{A} ; see the footnote in Sect. 3.5). Nevertheless, by writing $A = (f_A, g_A)$ we wish to keep consistency with the previous materials.

can have a huge influence on results, including the opportunity to reverse fuzziness, that is, to have outputs that are crisper than inputs.

In our previous works, we paid special attention to direction-related interpretation of OFNs, that is, the above-mentioned new aspect of information that enables us to distinguish between pairs (f_A, g_A) and (g_A, f_A) . One possibility is to refer here to a *trend* of fuzzy observation or measurement [9]. Indeed, decomposition of a fuzzy number's membership function onto two ordered components establishes an interesting background for representation of a trend by means of the up part, which is a natural beginning, and the down part, which is a natural end of observation. For example, by reversing the ordering of the OFN's components one might specify whether a given observed imprecise value is generally likely to increase or decrease. Surely, one could claim that such information is expressible also in classical fuzzy logic by adding new trend-related linguistic variables. However, that would result in a more complex fuzzy-rule-based representation, leading towards a less intuitive framework for conducting arithmetic calculations on measurements.

We refer to Chap. 4 for further details on possible interpretations of information represented by the OFN model. For now, let us add that by introducing an order of components (f_A, g_A) and, this way, letting A 's up and down parts be potentially both increasing and decreasing, we enter a far richer space of outcomes of arithmetic calculations. In Chap. 4 we show that operations on such ordered pairs may lead towards results that are not interpretable as standard fuzzy numbers, as some elements of \mathbb{R} correspond to multiple memberships. One could think of it as a special case of some extensions of fuzzy set theory [14]. One could also refer to original ideas of Lotfi A. Zadeh who, in his paper [22], stated that "The concept in question is that of fuzzy set, that is a 'class' with a continuum of grades of membership." Thus, the OFN model could be interpreted as a new way of assigning real numbers with the *continuum of grades of membership*. Certainly, further theoretical studies in this respect are necessary as well.

3.6 Summary

In this chapter, we recalled the roots of the Ordered Fuzzy Number (OFN) model [10]. We outlined disadvantages of standard fuzzy arithmetics and discussed how to look at fuzzy numbers in an alternative way, by representing them as ordered pairs of functions that encode the shapes of fuzzy memberships. In this way, we obtained a mathematical framework that extends the standard approach including a new type of information referred to as a direction of a fuzzy number [18]. We showed a kind of evolution of our way of thinking about fuzzy arithmetics, starting from the classical approach, via quasi-invertible representation of convex fuzzy numbers, and finishing with formal definition of OFNs. We also discussed how the obtained model lets us better manage degrees of imprecision during calculations and how one could interpret fuzzy numbers' direction.

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