Chapter 7
Applications

Abstract Applications of the formalism of finite quantum systems, to angle and angular momentum operators, interferometry, orbital angular momentum states, etc, are briefly discussed.

In this chapter we discuss applications of the formalism into the area of Quantum Optics and Quantum Information, and also into other areas. Each of these applications is a subject in its own right, and here we briefly define the basic quantities and guide the reader through the literature.

7.1 Angle States and Angular Momentum States

In this section we apply the general formalism of finite quantum systems, to a system with angular momentum j. In this case $d = 2j + 1$ where j is an integer ('Bose case'), and the variables take values in $\mathbb{Z}(2j + 1)$. The relevant Hilbert space is $H[\mathbb{Z}(2j + 1)]$, which in this chapter we denote for simplicity $H(2j + 1)$.

The analogue of the momentum states are here the usual angular momentum states, which we denote as $|J; j\ m\rangle$. The extra J to the usual notation is not a variable, but it simply indicates angular momentum states. The analogue of position states are the angle states [1], which we denote as $|\theta; j\ m\rangle$, and which are defined through Fourier transform below.

The angular momentum operators J_z, J_+, J_-, form the $SU(2)$ algebra

$$[J_z, J_+] = J_+; \qquad [J_z, J_-] = -J_-; \qquad [J_+, J_-] = 2J_z. \tag{7.1}$$

The Casimir operator is

$$J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = j(j+1)\mathbf{1}. \tag{7.2}$$

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Then

$$\begin{aligned} J_+|J;\, j\ m\rangle &= [j(j+1) - m(m+1)]^{1/2}|J;\, j\ m+1\rangle \\ J_-|J;\, j\ m\rangle &= [j(j+1) - m(m-1)]^{1/2}|J;\, j\ m-1\rangle \\ J_z|J;\, j\ m\rangle &= m|J;\, j\ m\rangle \\ J^2|J;\, j\ m\rangle &= j(j+1)|J;\, j\ m\rangle. \end{aligned} \tag{7.3}$$

The Fourier transform in the present context is

$$F = \frac{1}{\sqrt{2j+1}} \sum_{m,n} \omega(mn)|J;\, j\ m\rangle\langle J;\, j\ n|; \quad F^4 = \mathbf{1}. \tag{7.4}$$

Acting with it on the angular momentum states, we get angle states:

$$|\theta;\, j\ m\rangle = F^\dagger|J;\, j\ m\rangle = \frac{1}{\sqrt{2j+1}} \sum_n \omega(-mn)|J;\, j\ n\rangle \tag{7.5}$$

Also acting with it on the angular momentum operators we get the angle operators

$$F^\dagger J_z F = \theta_z; \quad F^\dagger J_+ F = \theta_+; \quad F^\dagger J_- F = \theta_- \tag{7.6}$$

which form the $SU(2)$ algebra

$$[\theta_z, \theta_+] = \theta_+; \quad [\theta_z, \theta_-] = -\theta_-; \quad [\theta_+, \theta_-] = 2\theta_z \tag{7.7}$$

The corresponding Casimir operator is

$$\theta^2 = \theta_z^2 + \frac{1}{2}(\theta_+\theta_- + \theta_-\theta_+) = j(j+1)\mathbf{1}. \tag{7.8}$$

Relations analogous to Eqs. (7.3), also hold for angle operators and angle states (because we have performed a Fourier transform, which is a unitary transform):

$$\begin{aligned} \theta_+|\theta;\, j\ m\rangle &= [j(j+1) - m(m+1)]^{1/2}|\theta;\, j\ m+1\rangle \\ \theta_-|\theta;\, j\ m\rangle &= [j(j+1) - m(m-1)]^{1/2}|\theta;\, j\ m-1\rangle \\ \theta_z|\theta;\, j\ m\rangle &= m|\theta;\, j\ m\rangle \\ \theta^2|\theta;\, j\ m\rangle &= j(j+1)|\theta;\, j\ m\rangle \end{aligned} \tag{7.9}$$

We next introduce a polar decomposition of the 'Cartesian operators' J_+ and J_- in terms of the 'radial operator' J_r and the 'exponential of the phase operator' Z:

$$\begin{aligned}
&J_+ = J_r Z; \qquad J_- = Z^\dagger J_r \\
&J_r = (J_+ J_-)^{1/2} = [j(j+1)\mathbf{1} - J_z^2 + J_z]^{1/2}; \quad [J_r, J_z] = 0 \\
&Z = \sum_m |J; j\ m+1\rangle\langle J; j\ m|
\end{aligned} \tag{7.10}$$

The dual relations to them are

$$\begin{aligned}
&\theta_+ = \theta_r X; \qquad \theta_- = X^\dagger \theta_r \\
&\theta_r = (\theta_+ \theta_-)^{1/2} = [j(j+1)\mathbf{1} - \theta_z^2 + \theta_z]^{1/2}; \quad [\theta_r, \theta_z] = 0 \\
&X = \sum_m |\theta; j\ m+1\rangle\langle \theta; j\ m|.
\end{aligned} \tag{7.11}$$

We can show that the X, Z obey Proposition 4.2, with the following correspondence:

$$|X; m\rangle \ \to \ |\theta; j\ m\rangle; \quad |P; m\rangle \ \to \ |J; j\ m\rangle. \tag{7.12}$$

Also the analogue of Eq. (4.18), is here

$$X = \exp\left[-i\frac{2\pi}{d}J_z\right]; \quad Z = \exp\left[i\frac{2\pi}{d}\theta_z\right]. \tag{7.13}$$

Therefore all the formalism in Chap. 4, can be used here also.

7.1.1 *The Schwinger Representation*

We consider a two-mode harmonic oscillator with Hilbert space $\mathscr{H}_1 \times \mathscr{H}_2$. Let $a_1^\dagger, a_1$ and $a_2^\dagger, a_2$ be the creation and annihilation operators for the two modes, and $|N_1, N_2\rangle$ the number eigenstates:

$$a_1^\dagger a_1 |N_1, N_2\rangle = N_1 |N_1, N_2\rangle; \quad a_2^\dagger a_2 |N_1, N_2\rangle = N_2 |N_1, N_2\rangle. \tag{7.14}$$

In the Schwinger representation of $SU(2)$ [2], the angular momentum operators are expressed as

$$J_+ = a_1^\dagger a_2; \qquad J_- = a_1 a_2^\dagger; \qquad J_z = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right). \tag{7.15}$$

The Casimir operator is

$$\begin{aligned}
&J^2 = n_s(n_s+1); \quad n_s = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2) \\
&[n_s, J_+] = [n_s, J_-] = [n_s, J_z] = 0.
\end{aligned} \tag{7.16}$$

The number eigenstates play the role of the angular momentum states, as follows:

$$|N_1, N_2\rangle \leftrightarrow |J; j\, m\rangle; \quad j = \frac{1}{2}(N_1 + N_2); \quad m = \frac{1}{2}(N_1 - N_2) \tag{7.17}$$

With this correspondence, we can easily show that the standard angular momentum relations in Eq. (7.3) hold. Here the $(2j + 1)$-dimensional Hilbert space $H(2j + 1)$, contains superpositions of the states

$$H(2j + 1) = \{|N, 2j + 1 - N\rangle \mid N = 0, ..., 2j + 1\} \tag{7.18}$$

Then the Hilbert space $\mathscr{H}_1 \times \mathscr{H}_2$ can be written as the direct sum:

$$\begin{aligned} &\mathscr{H}_1 \times \mathscr{H}_2 = \mathscr{H}_B \oplus \mathscr{H}_F \\ &\mathscr{H}_B = \bigoplus_j H(2j + 1); \quad j = 0, 1, 2, ... \\ &\mathscr{H}_F = \bigoplus_j H(2j + 1); \quad j = \frac{1}{2}, \frac{3}{2}, ... \end{aligned} \tag{7.19}$$

$\mathscr{H}_B$ is the Bose Hilbert space (the direct sum of spaces with integer j), and $\mathscr{H}_F$ is the Fermi Hilbert space (the direct sum of spaces with half-integer j). $\mathscr{H}_B$ is spanned by number eigenstates with an odd total number of photons in the two modes. $\mathscr{H}_F$ is spanned by number eigenstates with an even total number of photons in the two modes.

As an application of this we consider a two-mode system described by the following Hamiltonian, which is used for the description of frequency converters in Quantum Optics:

$$\begin{aligned} \mathfrak{H} &= E_1 a_1^\dagger a_1 + E_2 a_2^\dagger a_2 + \lambda a_1^\dagger a_2 + \lambda^* a_1 a_2^\dagger \\ &= (E_1 + E_2) n_s + (E_1 - E_2) J_z + \lambda J_+ + \lambda^* J_- \end{aligned} \tag{7.20}$$

Systems with this Hamiltonian can be studied with the above formalism.

7.1.2 Angle States and Angular Momentum States in $\mathscr{H}_B$

Let α, β be spherical coordinates describing the points on a two-dimensional sphere S_2, with radius one. We define the following angular momentum states in $\mathscr{H}_B$:

$$|J; \alpha, \beta\rangle = \sum_{j,m} Y_{jm}^*(\alpha, \beta)|J; j\, m\rangle; \quad 0 \le \alpha \le \pi; \quad 0 \le \beta < 2\pi \tag{7.21}$$

$Y_{jm}(\alpha, \beta)$ are the usual spherical harmonics. We also introduce the 'dual spherical harmonics' [3] which are related to the usual spherical harmonics through a finite Fourier transform:

$$X_{jn}(\alpha, \beta) = \frac{1}{\sqrt{2j+1}} \sum_m Y_{jm}(\alpha, \beta)\omega(nm) \tag{7.22}$$

We define angle states in $\mathscr{H}_B$, as:

$$|\theta; \alpha, \beta\rangle = \sum_{j,m} Y^*_{jm}(\alpha, \beta)|\theta; j\ m\rangle = \sum_{j,m} X^*_{jm}(\alpha, \beta)|J; j\ m\rangle. \tag{7.23}$$

The states $|\theta; \alpha, \beta\rangle$ and also the states $|J; \alpha, \beta\rangle$ form orthonormal bases in $\mathscr{H}_B$.

$$\int |\theta; \alpha, \beta\rangle\langle\theta; \alpha, \beta| d\cos\alpha d\beta = \int |J; \alpha, \beta\rangle\langle J; \alpha, \beta| d\cos\alpha d\beta = \mathbf{1}. \tag{7.24}$$

An arbitrary state $|f\rangle$ in $\mathscr{H}_B$, can be represented with the functions

$$f_J(\alpha, \beta) = \langle J; \alpha, \beta|f\rangle; \quad f_\theta(\alpha, \beta) = \langle\theta; \alpha, \beta|f\rangle. \tag{7.25}$$

7.1.3 Area Preserving Diffeomorphisms on a Sphere

Above we discussed angle and angular momentum operators based on the $SU(2)$ group. The $SU(2)$ is locally isomorphic to $SO(3)$ which describes rotations of a solid sphere.

A more general group is the $SDiff(S_2)$ of area preserving diffeomorphisms on a sphere S_2. They describe general transformations of a perfect liquid on a sphere. Since rotations of a solid sphere are a very special case of these transformations, we expect that this more general formalism will lead to the standard angular momentum operators plus many other operators. Such groups for a sphere and also other surfaces, have been studied in the context of string theory [4–10].

We consider the following transformations from $(\cos\alpha, \beta)$ to

$$\begin{aligned} &\cos\gamma = \mathscr{A}(\cos\alpha, \beta); \quad \delta = \mathscr{B}(\cos\alpha, \beta) \\ &\frac{\partial(\cos\gamma, \delta)}{\partial(\cos\alpha, \beta)} = \frac{\partial\cos\gamma}{\partial\cos\alpha}\frac{\partial\delta}{\partial\beta} - \frac{\partial\delta}{\partial\cos\alpha}\frac{\partial\cos\gamma}{\partial\beta} = 1. \end{aligned} \tag{7.26}$$

Since the Jacobian is equal to one, the area is preserved under these transformations.

An infinitesimal version of these transformations is

$$\cos\gamma = \cos\alpha + A(\cos\alpha, \beta)\varepsilon; \quad \delta = \beta + B(\cos\alpha, \beta)\varepsilon$$
$$\frac{\partial A}{\partial \cos\alpha} + \frac{\partial B}{\partial \beta} = 0. \tag{7.27}$$

ε is an infinitesimal parameter. The last equation comes from the fact that the Jacobian is equal to one, and for topologically trivial manifolds like a sphere, implies the existence of a function $g(\alpha, \beta)$ such that

$$A = -\frac{\partial g}{\partial \beta}; \quad B = \frac{\partial g}{\partial \cos\alpha}. \tag{7.28}$$

We consider two bases $|J; \alpha, \beta\rangle$ and $|J; \gamma, \delta\rangle$, where γ, δ are related to α, β through the infinitesimal transformations in Eq. (7.27). We represent an arbitrary state $|f\rangle$ in $\mathscr{H}_B$, with the functions

$$f(\alpha, \beta) = \langle J; \alpha, \beta|f\rangle; \quad f(\gamma, \delta) = \langle J; \gamma, \delta|f\rangle. \tag{7.29}$$

Then

$$\frac{f(\gamma, \delta) - f(\alpha, \beta)}{\varepsilon} \approx \frac{\partial(g(\alpha, \beta), f(\alpha, \beta))}{\partial(\cos\alpha, \beta)}. \tag{7.30}$$

This leads to the following definition.

Definition 7.1 The operator J_g acts on $f_J(\alpha, \beta)$, as follows:

$$J_g f(\alpha, \beta) = \langle J; \alpha, \beta|J_g|f\rangle = \frac{\partial(g(\alpha, \beta), f(\alpha, \beta))}{\partial(\cos\alpha, \beta)}. \tag{7.31}$$

In analogous way we define the operators θ_g. The following proposition describes some properties of J_g.

Proposition 7.1 *(1) The commutator of J_g and J_h, is given in terms of the Poisson bracket of g, h (with respect to* $\cos\alpha, \beta$*), by*

$$[J_g, J_h] = J_{\{g,h\}}; \quad \{g, h\} = \frac{\partial g}{\partial \cos\alpha}\frac{\partial h}{\partial \beta} - \frac{\partial h}{\partial \beta}\frac{\partial g}{\partial \cos\alpha}. \tag{7.32}$$

(2) J_g acts on the sum of two functions as follows:

$$J_g[\mu_1 f_1(\alpha, \beta) + \mu_2 f_2(\alpha, \beta)] = \mu_1 J_g f_1(\alpha, \beta) + \mu_2 J_g f_2(\alpha, \beta). \tag{7.33}$$

(3) J_g acts on the product of two functions as follows:

$$J_g[f_1(\alpha, \beta) f_2(\alpha, \beta)] = f_1(\alpha, \beta) J_g f_2(\alpha, \beta) + f_2(\alpha, \beta) J_g f_1(\alpha, \beta). \tag{7.34}$$

(4) The exponential of J_g acts on the sum of two functions as follows:

$$\exp(\lambda J_g)[\mu_1 f_1(\alpha,\beta) + \mu_2 f_2(\alpha,\beta)] = \mu_1 \exp(\lambda J_g) f_1(\alpha,\beta) + \mu_2 \exp(\lambda J_g) f_2(\alpha,\beta). \quad (7.35)$$

(5) The exponential of J_g acts on the product of two functions as follows:

$$\exp(\lambda J_g)[f_1(\alpha,\beta) f_2(\alpha,\beta)] = [\exp(\lambda J_g) f_1(\alpha,\beta)][\exp(\lambda J_g) f_2(\alpha,\beta)]. \quad (7.36)$$

Proof For the proof we refer to Ref. [11].

We expand the function $g(\alpha,\beta)$ in terms of spherical harmonics, as

$$g(\alpha,\beta) = \sum_{j,m} g_{jm} Y_{jm}(\alpha,\beta). \quad (7.37)$$

Then

$$J_g = \sum_{j,m} g_{jm} J_{jm}; \quad J_{jm} f(\alpha,\beta) = \frac{\partial(Y_{jm}(\alpha,\beta), f(\alpha,\beta))}{\partial(\cos\alpha,\beta)}. \quad (7.38)$$

In particular

$$J_{jm} Y_{\ell n}(\alpha,\beta) = \langle J;\alpha,\beta|J_{jm}|J;\ell\ n\rangle = \frac{\partial(Y_{jm}(\alpha,\beta), Y_{\ell n}(\alpha,\beta))}{\partial(\cos\alpha,\beta)}. \quad (7.39)$$

The Poisson bracket of $Y_{j_1 m_1}(\alpha,\beta)$ and $Y_{j_2 m_2}(\alpha,\beta)$, is given by

$$\{Y_{j_1 m_1}, Y_{j_2 m_2}\} = \sum_{\ell,n} \tau(j_1,m_1;\ j_2,m_2|\ell,n) Y_{\ell n}. \quad (7.40)$$

The structure constants $\tau(j_1,m_1;\ j_2,m_2|\ell,n)$ are given in [5]. Consequently

$$[J_{j_1 m_1}, J_{j_2 m_2}] = \sum_{\ell,n} \tau(j_1,m_1;\ j_2,m_2|\ell,n) J_{\ell n}. \quad (7.41)$$

The J_{jm} are generalizations of the angular momentum operators. The J_{1m} are simply the standard angular momentum operators J_+, J_z, J_- (with a different normalization).

This formalism has been used in string theory, but it might also be useful in the general area of quantum optics and quantum information, because it generalizes the angular momentum formalism.

7.2 Interferometry in Multimode Systems

In this section we use the formalism of finite quantum systems, in the context of interferometry that involves d harmonic oscillators. The overall Hilbert space in this problem is $H_{\rm osc} \otimes ... \otimes H_{\rm osc}$, where $H_{\rm osc}$ is the infinite-dimensional Hilbert space of the harmonic oscillator. The mode index is the 'position' in this problem, and it takes values in $\mathbb{Z}(d)$. Through a finite Fourier transform of the d modes, we get a dual mode index which plays the role of 'momentum', and which also takes values in $\mathbb{Z}(d)$. So in this context, the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ is a 'mode phase space'.

The formalism has important applications in metrology, because it leads to resolutions below the standard quantum limit [12]. It has been studied extensively both with photons and also with Bose-Einstein condensates. Here we present briefly the link between this area, and the formalism of finite quantum systems studied in Chap. 4. We refer to the literature for more details, and for practical applications of these devices [13–23].

We consider a system comprised of d harmonic oscillators. The creation and annihilation operators corresponding to the m-th mode, are:

$$a_m^\dagger = \mathbf{1} \otimes ... \otimes a^\dagger \otimes ... \otimes \mathbf{1}; \quad a_m = \mathbf{1} \otimes ... \otimes a \otimes ... \otimes \mathbf{1}$$
$$[a_m, a_n^\dagger] = \delta(m, n); \quad m, n \in \mathbb{Z}(d). \tag{7.42}$$

Let Λ be a $d \times d$ Hermitian matrix, and U the unitary operator

$$U = \exp\left[i \sum_{m,n} a_m^\dagger \Lambda_{mn} a_n\right]. \tag{7.43}$$

It is known (e.g. [24]) that

$$b_m = U a_m U^\dagger = \sum_n V_{mn} a_n; \quad b_m^\dagger = U a_m^\dagger U^\dagger = \sum_n V_{mn}^* a_n^\dagger$$
$$V = \exp(-i\Lambda); \qquad VV^\dagger = \mathbf{1}. \tag{7.44}$$

The vacuum state remains invariant under these transformations. Also the total average number of photons in a state remains invariant under the U transformations:

$$U|0, ..., 0\rangle = |0, ..., 0\rangle; \quad \sum_m b_m^\dagger b_m = \sum_m a_m^\dagger a_m. \tag{7.45}$$

7.2.1 *Fourier Interferometry and Applications to Metrology*

A special case of the formalism above, is the Fourier transform of the modes:

$$U_F = \exp\left[i\sum_{m,n} a_m^\dagger \Lambda_{mn} a_n\right]; \quad \Lambda = i\ln F; \quad ; \quad (U_F)^4 = \mathbf{1}, \tag{7.46}$$

where F is the $d\times d$ Fourier matrix, in Eq. (4.2). Then

$$\begin{aligned} b_m &= U_F a_m U_F^\dagger = \frac{1}{\sqrt{d}}\sum_n \omega(mn)a_n \\ b_m^\dagger &= U_F a_m^\dagger U_F^\dagger = \frac{1}{\sqrt{d}}\sum_n \omega(-mn)a_n^\dagger \end{aligned} \tag{7.47}$$

The dual mode index related to $b_m, b_m^\dagger$ plays the role of momentum. So in the present context position and momentum is the mode index related to the $a_m, a_m^\dagger$ and $b_m, b_m^\dagger$, correspondingly. Experiments that use beam splitters to implement these transforms have been discussed in [14]. The use of the factorization in Sect. 4.9 reduces the number of beam splitters, as discussed in [23].

There are various applications of these devices. As an example, we consider the case where the input is a number state with N photons in the m-th mode, and vacuum in the other modes:

$$|s\rangle = |0, ..., 0, N, 0, ...0\rangle \tag{7.48}$$

Then in the large d limit, the phase uncertainty in the m-th output is [20]

$$\Delta\theta_m \sim \frac{\sqrt{d}}{N}. \tag{7.49}$$

This is below the standard quantum limit and can have applications in metrology.

It is seen that the formalism of finite quantum systems presented in this monograph, can also be used for the study of interferometry in multimode systems (with a finite number of modes).

7.2.2 Other Types of Interferometry

Here we consider other special cases of the general operators U in Eq. (7.43). The first one, is:

$$U_X = \exp\left[i \sum_{m,n} a_m^\dagger \Lambda_{mn} a_n\right]; \quad \Lambda = i \ln X; \quad (U_X)^d = \mathbf{1}. \tag{7.50}$$

where X is the $d \times d$ matrix, in Eq. (4.19). Then

$$\begin{aligned} b_m &= U_X a_m U_X^\dagger = a_{m+1} \\ b_m^\dagger &= U_X a_m^\dagger U_X^\dagger = a_{m+1}^\dagger \end{aligned} \tag{7.51}$$

This shifts the modes by one place (and the last mode becomes first). In other words, it shifts the modes in the 'mode-position' direction, in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ mode phase space.

Another special case is

$$U_Z = \exp\left[i \sum_{m,n} a_m^\dagger \Lambda_{mn} a_n\right]; \quad \Lambda = i \ln Z; \quad (U_Z)^d = \mathbf{1}. \tag{7.52}$$

where Z is the $d \times d$ matrix, in Eq. (4.19). Then

$$\begin{aligned} b_m &= U_Z a_m U_Z^\dagger = a_m \omega(m) \\ b_m^\dagger &= U_Z a_m^\dagger U_Z^\dagger = a_m \omega(-m). \end{aligned} \tag{7.53}$$

This multiplies each mode a_m by $\omega(m)$, i.e., it shifts the modes in the 'mode-momentum' direction, in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ mode phase space.

We next divide the Hilbert space $H_{\rm osc} \otimes ... \otimes H_{\rm osc}$, into d 'sectors':

$$\begin{aligned} &H_{\rm osc} \otimes ... \otimes H_{\rm osc} = \bigoplus_{n=0}^{d-1} \mathscr{H}_n \\ &\mathscr{H}_n = {\rm span}\{|N_0, ..., N_{d-1}\rangle \mid N_0 + ... + N_{d-1} = n ({\rm mod}\ d)\}; \quad n \in \mathbb{Z}(d). \end{aligned} \tag{7.54}$$

The sector $\mathscr{H}_n$ is spanned by number eigenstates, with a total number of photons equal to $n({\rm mod}\ d)$. We call π_n the projector to $\mathscr{H}_n$. It can be shown that π_n commutes with both U_X, U_Z, and we define the:

$$U_{Xn} = U_X \pi_n; \quad U_X = \sum_{n=0}^{d-1} U_{Xn}; \quad [U_X, \pi_n] = 0$$
$$U_{Zn} = U_Z \pi_n; \quad U_Z = \sum_{n=0}^{d-1} U_{Zn}; \quad [U_Z, \pi_n] = 0. \tag{7.55}$$

Then the U_{Xn}, U_{Zn} form a Heisenberg-Weyl group within $\mathscr{H}_n$, which has been studied in [21]:

$$U_{Xn}^{\alpha} U_{Zn}^{\beta} = U_{Zn}^{\beta} U_{Xn}^{\alpha} \omega(-n\alpha\beta); \quad \alpha, \beta \in \mathbb{Z}(d). \tag{7.56}$$

So apart from the Fourier interferometry devices, there are many other devices which can have various applications in Quantum Optics and Quantum Information.

7.3 Orbital Angular Momentum States

The paraxial wave equation in cylindrical coordinates, leads to the Laguerre-Gauss modes

$$u_{nm}(r, \phi) \sim r^{|m|} L_n^{|m|} \left(\frac{2r^2}{w^2} \right) \exp\left(-\frac{r^2}{w^2} \right) \exp(-im\phi) \tag{7.57}$$

Here $L_n^{|m|}$ are Laguerre polynomials, and n, m are the radial quantum number, and the orbital angular momentum quantum number, correspondingly. The physical meaning of the radial quantum number n is discussed in [25]. w describes the width of the beam. Photons in these beams have angular momentum m.

These solutions describe the orbital angular momentum states or twisted light [26–29], and they are an important tool in modern quantum optical technologies. They are created experimentally by imposing $\exp(im\phi)$ phase structure on a laser beam. There is currently much work on the generation of orbital angular momentum states and their applications (e.g., [30–34]). They are robust in noisy environments (e.g., [35]), and therefore important for quantum communications.

In our context, they are important because they provide an experimental implementation of a quantum system with a finite dimensional Hilbert space. The whole formalism of this monograph can be used in the context of orbital angular momentum states. Mutually unbiased bases with orbital angular momentum states have been studied in [36, 37], and entanglement in [38]. Applications to quantum cryptography have been discussed in [39].

7.4 Other Applications

We discussed above applications in the area of quantum optics and quantum information. Applications in other areas include quantum maps [40–45], two-dimensional electron system in a uniform magnetic field and the magnetic translation group [46–50], and the quantum Hall effect [51, 52].

All these ideas are also used in the context of Signal Processing, where the dual variables position and momentum become time and frequency [54, 55]. For example, the factorization discussed in Sect. 4.9, is inspired by Ref. [56] on fast Fourier transforms, in the context of Signal Processing.

Work related to the formalism of finite quantum systems, in the context of Applied Mathematics is summarized in [57].

References

1. Vourdas, A. (1990). *Physical Review A*, *41*, 1653.
2. Schwinger, J. (1965). In L.C. Biedenharn, H. van Dam (Eds.) *Quantum Theory of Angular Momentum*. New York: Academic.
3. Vourdas, A. (1991). *Physical Review A*, *43*, 1564.
4. Arnold, V. I. (1966). *Annales- Institut Fourier*, *16*, 319.
5. Arakelyan, T., & Savvidy, G. K. (1988). *Physics Letters B*, *214*, 350.
6. Floratos, E., & Iliopoulos, J. (1988). *Physics Letters B*, *201*, 237.
7. Pope, C. N., & Stelle, K. (1989). *Physics Letters B*, *226*, 257.
8. de Witt, B., Hoppe, J., Nicolai, H. (1988). *Nuclear Physics*, B305 [FS23], 545 (1988)
9. Marquard, V., & Nicolai, H. (1990). *Communications in Mathematical Physics*, *128*, 39.
10. Dowker, J. S. (1990). *Classical and Quantum Gravity*, *7*, 1241.
11. Deprit, A. (1969). *Cel Mech*, *1*, 12.
12. Holland, M. J., & Burnett, K. (1993). *Physical Review Letters*, *71*, 1355.
13. Lane, A. S., Braunstein, S., & Caves, C. M. (1993). *Physical Review*, *47*, 1667.
14. Reck, M., Zeillinger, A., Bernstein, H. J., & Bertani, P. (1994). *Physical Review Letters*, *73*, 58.
15. Jex, I., Stenholm, S., & Zeillinger, A. (1995). *Optics Communication*, *117*, 95.
16. Torma, P., Stenholm, S., & Jex, I. (1995). *Physical Review A*, *52*, 4853.
17. Dunningham, J., & Burnett, K. (2000). *Physical Review*, *61*, 065601.
18. Dunningham, J., Burnett, K., & Burnett, S. M. (2002). *Physical Review Letters*, *89*, 150401.
19. Campos, R. A., Gerry, C., & Benmoussa, A. (2003). *Physical Review A*, *68*, 023810.
20. Vourdas, A., & Dunningham, J. (2005). *Physical Review A*, *71*, 013809.
21. Vourdas, A. (2005). *Physical Review A*, *71*, 043821.
22. Dunnigham, J., & Vourdas, A. (2006). *Journal of Physics B: Atomic, Molecular and Optical Physics*, *39*, 1579.
23. Zhang, S., Lei, C., Vourdas, A., & Dunnigham, J. (2006). *Journal of Physics B: Atomic, Molecular and Optical Physics*, *39*, 1625.
24. Ma, X., & Rhodes, W. (1990). *Physical Review A*, *41*, 4625.
25. Plick, W. N., & Krenn, M. (2015). *Physical Review A*, *92*, 063841.
26. Allen, L., Beijersbergen, M., Spreeuw, R., & Woerdman, J. (1992). *Physical Review A*, *45*, 8185.
27. Allen, L., Barnett, S. M., & Padgett, M. J. (Eds.). (2003). *Optical Angular Momentum*. Bristol: Institute of Physics.

28. Torres, J. P., & Torner, L. (Eds.). (2011). *Twisted photons: applications of light with orbital angular momentum*. Bristol: Wiley.
29. Andrews, D. L., & Babiker, M. (Eds.). (2012). *The angular momentum of light*. Cambridge: Cambridge University Press.
30. Mair, A., Vaziri, A., Weihs, G., & Zeillinger, A. (2001). *Nature*, *412*, 313.
31. Gibson, G., et al. (2004). *Optics Express*, *12*, 5448.
32. Marrucci, L., Manzo, C., & Paparo, D. (2006). *Physical Review Letters*, *96*, 163905.
33. Wang, J., et al. (2012). *Nature Photonics*, *6*, 488.
34. Bozinovic, N., et al. (2013). *Science*, *340*, 1545.
35. Krenn, M., et al. (2014). *New Journal of Physics*, *16*, 113028.
36. Giovannini, D., et al. (2013). *Physical Review Letters*, *110*, 143601.
37. D'Ambrosio, V., et al. (2013). *Science Reports*, *3*, 2726.
38. Jack, B., et al. (2010). *Physical Review A*, *81*, 043844.
39. Mirhosseini, M., et al. (2015). *New Journal of Physics*, *17*, 033033.
40. Berry, M. V. (1987). *Proceedings of the Royal Society*, *A473*, 183.
41. Balazs, N. L., & Voros, A. (1986). *Physics Reports*, *C143*, 109.
42. Leboeuf, P., & Voros, A. (1990). *Journal of Physics A*, *23*, 1765.
43. Leboeuf, P., Kurchan, J., Feingold, M., & Arovas, D. P. (1992). *Chaos*, *2*, 125.
44. Vivaldi, F. (1994). *Nonlinearity 5*, 133 (1992); Keating, J.P. *Journal of Physics*, *A27*, 6605.
45. Athanasiu, G. G., Floratos, E., & Nicolis, S. (1996). *Journal of Physics A*, *29*, 6737.
46. Brown, E. (1964). *Physical Review A*, *133*, 1038.
47. Zak, J. (1964). *Physical Review A*, *134*, 1602.
48. Zak, J. (1989). *Physical Review B*, *39*, 694.
49. Dubrovin, B. A., & Novikov, S. P. (1980). *Soviet Mathematics Doklady*, *22*, 240.
50. Novikov, S. P. (1980). *Soviet Mathematics Doklady*, *23*, 298.
51. Wen, X. G., & Niu, Q. (1990). *Physical Review B*, *41*, 9377.
52. Martinez, J., & Stone, M. (1993). *International Journal of Modern Physics B*, *7*, 4389.
53. Abarbanel, H., & Rouhi, A. (1994). *Physical Review E*, *48*, 3643.
54. Grochenig, K. (2001). *Foundations of time-frequency analysis*. Boston: Birkhauser.
55. Cohen, L. (1995). *Time-frequency analysis*. New Jersey: Prentice-Hall.
56. Good, I. J., & Trans, I. E. E. E. (1971). *Computers*, *C20*, 310.
57. Terras, A. (1999). *Fourier analysis on finite groups and applications*. Cambridge: Cambridge University Press.