Chapter 11
A Quantum System with Positions in the Profinite Group $\mathbb{Z}_p$

Abstract Quantum systems with positions in $\mathbb{Z}_p$ and momenta in $\mathbb{Q}_p/\mathbb{Z}_p$, are discussed. The Schwartz-Bruhat space of wavefunctions in these systems, is presented. The Heisenberg-Weyl group as a locally compact and totally disconnected topological group, is discussed. Wigner and Weyl functions in this context, are also discussed.

In a mathematical context, there is a lot of work on functional analysis on p-adic numbers[1–6], and on wavelets with p-adic numbers [7–12]. There is also a lot of work on various problems in mathematical physics with p-adic numbers [13–32]. Work on condensed matter with p-adic numbers is discussed in [33–36], on particle physics and string theory in [37–41], and on path-integrals in [22, 42]. The use of p-adic numbers in classical computation is discussed in [43].

In this chapter we discuss the quantum system $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, with positions in the profinite group $\mathbb{Z}_p$, and momenta in its Pontryagin dual group $\mathbb{Q}_p/\mathbb{Z}_p$. Intuitively $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ is the system $\Sigma[\mathbb{Z}(p^e)]$, with $e = \infty$. All finite systems $\Sigma[\mathbb{Z}(p^e)]$, are subsystems of $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$.

The set of the systems $\Sigma[\mathbb{Z}(p^e)]$ where $e \in \mathbb{N}$, with the order subsystem, is a chain. This chain is not complete, but when we add the 'top element' $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, it becomes complete.

This chapter belongs to the general area of p-adic physics, but we approach this area from a novel angle. We use inverse and direct limits and profinite groups, to provide a rigorous approach to study of the systems $\Sigma[\mathbb{Z}(p^e)]$, with very large e.

11.1 Locally Constant Functions with Compact Support

We define the concepts of locally constant functions (at small distances) and functions with compact support (at large distances). They reduce the integrals into finite sums, and ensure convergence.

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Definition 11.1 A complex function $f_p(a_p)$ with $a_p \in \mathbb{Q}_p$, is locally constant with degree n, if $f_p(a_p + b_p) = f_p(a_p)$ for all $|b_p|_p \leq p^{-n}$. Such a function is effectively defined on $\mathbb{Q}_p/p^n\mathbb{Z}_p$. We denote this as

$$\mathbf{LC}[f_p(a_p)] = n. \tag{11.1}$$

Definition 11.2 A complex function $f_p(a_p)$ with $a_p \in \mathbb{Q}_p$, has compact support with degree k, if $f_p(a_p) = 0$ for all $|a_p|_p > p^k$. Such a function is effectively defined on $p^{-k}\mathbb{Z}_p$. We denote this as

$$\mathbf{CS}[f_p(a_p)] = k. \tag{11.2}$$

Notation 11.1 *We use the notation $\mathscr{A}_p(k, n)$ for the set of functions*

$$\mathscr{A}_p(k, n) = \{f_p(a_p) \mid \mathbf{CS}[f_p(a_p)] \leq k \ \text{ and } \ \mathbf{LC}[f_p(a_p)] \leq n\}. \tag{11.3}$$

We also use the notation

$$\mathscr{A}_p(k, *) = \bigcup_n \mathscr{A}_p(k, n); \quad \mathscr{A}_p(*, n) = \bigcup_k \mathscr{A}_p(k, n); \quad \mathscr{A}_p = \bigcup_{k,n} \mathscr{A}_p(k, n). \tag{11.4}$$

The star in $\mathscr{A}_p(k, *)$ indicates that n can take any finite value, and similarly for $\mathscr{A}_p(*, n)$.

Clearly $\mathscr{A}_p(k_1, n_1) \subseteq \mathscr{A}_p(k_2, n_2)$ if $k_1 \leq k_2$ and $n_1 \leq n_2$.

Remark 11.1
- All functions $f_p(\mathfrak{a}_p)$ with $\mathfrak{a}_p \in \mathbb{Q}_p/\mathbb{Z}_p$ have $\mathbf{LC}[f_p(a_p)] = 0$, and therefore they belong to $\mathscr{A}_p(*, 0)$. These functions obey the relation $f_p(\mathfrak{a}_p) = f_p(\mathfrak{a}_p + 1)$.
- All functions $f_p(a)$ with $a_p \in \mathbb{Z}_p$ have $\mathbf{CS}[f_p(a_p)] = 0$, and therefore they belong to $\mathscr{A}_p(0, *)$.
- A function with $\mathbf{LC}[f_p(a_p)] = n$ and $\mathbf{CS}[f_p(a_p)] = k$, is effectively defined on $p^{-k}\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}(p^{n+k})$, and it can be represented as a p^{n+k}-dimensional vector.

11.2 Integrals of Complex Functions on $\mathbb{Q}_p$

Integrals of complex functions over $\mathbb{Q}_p$ use the Haar measure, with the normalization:

$$\int_{\mathbb{Z}_p} da_p = 1. \tag{11.5}$$

The integral over $\mathbb{Q}_p$, of a function $f_p(a_p) \in \mathscr{A}_p(k, n)$, is given by

$$\int_{\mathbb{Q}_p} f_p(a_p)da_p = p^{-n}\sum f_p(\overline{a}_{-k}p^{-k} + ... + \overline{a}_{n-1}p^{n-1}). \tag{11.6}$$

The sum is over all $\overline{a}_{-k}, ..., \overline{a}_{n-1}$. It is a finite sum with p^{n+k} terms, and therefore it converges. The fact that $\mathbf{LC}[f_p(a_p)] = n$, ensures that if we truncate the sum at $n + m \geq n$, we get the same result:

$$\begin{aligned} &p^{-(n+m)}\sum f_p(\overline{a}_{-k}p^{-k} + ... + \overline{a}_{n+m-1}p^{n+m-1}) \\ &= p^{-n}\sum f_p(\overline{a}_{-k}p^{-k} + ... + \overline{a}_{n-1}p^{n-1}). \end{aligned} \tag{11.7}$$

The fact that $\mathbf{CS}[f_p(a_p)] = k$, ensures that if we truncate the sum at $k + m > k$, we get the same result :

$$\begin{aligned} &p^{-n}\sum f_p(\overline{a}_{-(k+m)}p^{-(k+m)} + ... + \overline{a}_{n+m-1}p^{n-1}) \\ &= p^{-n}\sum f_p(\overline{a}_{-k}p^{-k} + ... + \overline{a}_{n-1}p^{n-1}). \end{aligned} \tag{11.8}$$

The following proposition is helpful if we want to change variables.

Proposition 11.1 *Let $f_p(a_p)$ where $a_p \in \mathbb{Q}_p$, be a complex function in $\mathscr{A}_p(k, n)$. Also let $F_p(a_p) = f_p(\lambda a_p)$, where $|\lambda|_p = p^s$. Then:*

(1) The function $F_p(a_p)$ belongs to $\mathscr{A}_p(k - s, n + s)$.
(2)

$$\int_{\mathbb{Q}_p} f_p(a_p)da_p = p^s \int_{\mathbb{Q}_p} F_p(a_p)da_p \tag{11.9}$$

If we call $a'_p = \lambda a_p$, then we can express this as

$$da'_p = |\lambda|_p da_p. \tag{11.10}$$

If λ, p are coprime then $da'_p = da_p$. If $\lambda = p$, then $d(pa_p) = p^{-1}da_p$.

Proof (1) The function $f_p(a_p)$ has $\mathbf{LC}[f_p(a_p)] = n$, and therefore

$$F_p(a_p + b_p) = f_p(\lambda a_p + \lambda b_p) = f_p(\lambda a_p) \text{ if } |\lambda b_p|_p \leq p^{-n}. \tag{11.11}$$

This gives $|b_p|_p \leq p^{-n-s}$, and therefore the function $F_p(a_p)$ has $\mathbf{LC}[F_p(a_p)] = n + s$.
The function $f_p(a_p)$ has $\mathbf{CS}[f_p(a_p)] = k$, and therefore

$$F_p(a_p) = f_p(\lambda a_p) = 0 \text{ if } |\lambda a_p|_p > p^k. \tag{11.12}$$

This gives $|a_p|_p > p^{k-s}$. Therefore the function $F_p(a_p)$ has $\mathbf{LC}[F_p(a_p)] = k - s$.

(2) The integral for the function $f_p(a_p) \in \mathscr{A}_p(k, n)$, has the prefactor is p^{-n} in Eq.(11.6). The integral for the function $F_p(a_p) \in \mathscr{A}_p(k - s, n + s)$, has the prefactor is p^{-n-s}, and it needs to be 'corrected' with multiplication by p^s. It is seen that the p^s in Eq.(11.9) (or the $|\lambda|_p$ in Eq.(11.10)), compensate the change in the degrees of local constancy and compact support in the function $F_p(a_p)$, which affects the prefactor in Eq.(11.6).
If λ, p are coprime then $|\lambda|_p = 1$.

Example 11.1 For $p = 3$, we consider the following function in $\mathscr{A}_3(0, 2)$:

$$\begin{aligned}
&f_p(0 + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 1 - i \\
&f_p(1 + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 2 \\
&f_p(2 + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 2 + i \\
&f_p(0 + p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = -1 \\
&f_p(1 + p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 0 \\
&f_p(2 + p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 1 - i \\
&f_p(0 + 2p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 1 - i \\
&f_p(1 + 2p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 2 \\
&f_p(2 + 2p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 2 + i
\end{aligned} \tag{11.13}$$

Also for $\overline{a}_{-k} \neq 0$ with $k > 0$, we get

$$f_p(\overline{a}_{-k} p^{-k} + \overline{a}_{-k+1} p^{-k+1} + \cdots) = 0. \tag{11.14}$$

In this case

$$\int_{\mathbb{Q}_p} f_p(a_p) da_p = \frac{1}{9}(10 - i). \tag{11.15}$$

11.3 Integrals of Complex Functions on $\mathbb{Q}_p/\mathbb{Z}_p$ and Weil Transforms

Let $g_p(\mathfrak{p}_p)$ be a complex function of $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$, with $\mathbf{CS}[g_p(\mathfrak{p}_p)] = k$. We have explained earlier that such a function belongs to $\mathscr{A}_p(k, 0)$. Its integral over $\mathbb{Q}_p/\mathbb{Z}_p$ is

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} g_p(\mathfrak{p}_p) d\mathfrak{p}_p = \sum g_p\left(\overline{p}_{-k} p^{-k} + \overline{p}_{-k+1} p^{-k+1} + \ldots + \overline{p}_{-1} p^{-1}\right). \tag{11.16}$$

The counting measure is used here.

The $\mathfrak{p}_p$ are cosets and we represented them with the element that has zero integer part. If we represent them with elements that have non-zero integer part, we get the same result. Indeed, let $c_p = \mathfrak{p}_p + b_p \in \mathbb{Q}_p$, where $b_p \in \mathbb{Z}_p$. The function $g_p(\mathfrak{p}_p)$ assigns a single complex value to each coset $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$, and this implies that $g_p(\mathfrak{p}_p + b_p) = g_p(\mathfrak{p}_p)$. We rewrite the above integral as an integral over $\mathbb{Q}_p$, as

$$\begin{aligned}\int_{\mathbb{Q}_p} dc_p g_p(c_p) &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \int_{\mathbb{Z}_p} db_p\ g_p(\mathfrak{p}_p + b_p) \\ &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p\ g_p(\mathfrak{p}_p) \int_{\mathbb{Z}_p} db_p = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p\ g_p(\mathfrak{p}_p). \end{aligned} \tag{11.17}$$

The counting measure for integration over $\mathbb{Q}_p/\mathbb{Z}_p$ ensures that this relation holds.

More generally let $g_p(\mathfrak{p}_p)$ be a complex function of $\mathfrak{p}_p \in \mathbb{Q}_p/p^{-s}\mathbb{Z}_p$, with $\mathbf{CS}[g_p(\mathfrak{p}_p)] = k$. Such a function belongs to $\mathscr{A}_p(k, -s)$, and its integral over $\mathbb{Q}_p/p^{-s}\mathbb{Z}_p$, is

$$\begin{aligned}&\int_{\mathbb{Q}_p/p^{-s}\mathbb{Z}_p} g_p(\mathfrak{p}_p) d\mathfrak{p}_p \\ &= p^s \sum g_p\left(\overline{p}_{-k}p^{-k} + \overline{p}_{-k+1}p^{-k+1} + ... + \overline{p}_{-s-1}p^{-s-1}\right) \\ &= \sum g_p\left(\overline{p}_{-k}p^{-k} + \overline{p}_{-k+1}p^{-k+1} + ... + \overline{p}_{-s-1}p^{-s-1} + ... + \overline{p}_{-1}p^{-1}\right) \end{aligned} \tag{11.18}$$

Here the function g_p does not depend on $\overline{p}_{-s}, ..., \overline{p}_{-1}$ and this gives the prefactor p^s in the second expression.

Proposition 11.2 *Change of the variable $\mathfrak{p}_p$ into $\mathfrak{p}'_p = \lambda \mathfrak{p}_p$, is performed with the relation*

$$|\lambda|_p \int_{\mathbb{Q}_p/|\lambda|_p\mathbb{Z}_p} g_p(\lambda \mathfrak{p}_p) d\mathfrak{p}_p = \int_{\mathbb{Q}_p/\mathbb{Z}_p} g_p(\mathfrak{p}'_p) d\mathfrak{p}'_p. \tag{11.19}$$

Therefore

$$d\mathfrak{p}'_p = |\lambda|_p d\mathfrak{p}_p. \tag{11.20}$$

If λ, p are coprime then $|\lambda|_p = 1$. If $\lambda = p$, then $d(p\mathfrak{p}_p) = p^{-1} d\mathfrak{p}_p$.

Proof We first point out that if $\mathfrak{p}_p \in \mathbb{Q}_p/|\lambda|_p\mathbb{Z}_p$, then $\mathfrak{p}'_p = \lambda \mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$. Therefore the domain of integration changes.

Equation (11.19) follows from Eq.(11.18). The $|\lambda|_p$ 'corrects' the prefactor, as already discussed in the proof of Proposition 11.1.

Example 11.2 For $p = 2$, we consider the following function in $\mathscr{A}_2(2, 0)$, which is described with 4 complex values:

$$\begin{aligned}
&g_p(p^{-2} + \overline{a}_0 + \overline{a}_1 p + ...) = 1 - i \\
&g_p(p^{-2} + p^{-1} + \overline{a}_0 + \overline{a}_1 p + ...) = 2 \\
&g_p(p^{-1} + \overline{a}_0 + \overline{a}_1 p + ...) = 2 - i \\
&g_p(\overline{a}_0 + \overline{a}_1 p + ...) = -1
\end{aligned} \tag{11.21}$$

Also for $\overline{a}_{-k} \neq 0$ with $k > 2$, we get

$$g_p(\overline{a}_{-k} p^{-k} + \overline{a}_{-k+1} p^{-k+1} + ...) = 0. \tag{11.22}$$

In this example

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} g_p(\mathfrak{p}_p) d\mathfrak{p}_p = 4 - 2i. \tag{11.23}$$

11.3.1 Weil Transforms

Given a function $F_p(c_p)$ with $c_p \in \mathbb{Q}_p$, which is locally constant and has compact support, we express c_p as $c_p = \mathfrak{p}_p + b_p$, where $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$ and $b_p \in \mathbb{Z}_p$. The Weil transform [44], maps the function $F_p(c_p)$ in $\mathbb{Q}_p$, into the following function in $\mathbb{Q}_p/\mathbb{Z}_p$:

$$f(\mathfrak{p}_p) = \int_{\mathbb{Z}_p} F_p(\mathfrak{p}_p + b_p) db_p. \tag{11.24}$$

We note that for any $e_p \in \mathbb{Z}_p$, we get $f(\mathfrak{p}_p) = f(\mathfrak{p}_p + e_p)$. Then

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} f(\mathfrak{p}_p) d\mathfrak{p}_p = \int_{\mathbb{Q}_p} F_p(c_p) dc_p. \tag{11.25}$$

Example 11.3 For $p = 2$, we consider the following function on $\mathbb{Q}_p$, which belongs to $\mathscr{A}_2(1, 2)$:

$$\begin{aligned}
&F_p(p^{-1} + 1 + p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 1 \\
&F_p(0p^{-1} + 1 + p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 2 - i \\
&F_p(p^{-1} + 0 + p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 3
\end{aligned}$$

$$
\begin{aligned}
&F_p(p^{-1} + 1 + 0p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 1 - i \\
&F_p(0p^{-1} + 0 + p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = 1 + i \\
&F_p(0p^{-1} + 1 + 0p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = i \\
&F_p(p^{-1} + 0 + 0p + \overline{a}_2 p^2 + \overline{a}_3 p^3 + \cdots) = -i \\
&F_p(0p^{-1} + 0 + 0p + \overline{a}_2 p^2 + \overline{a}_3 p^3 +) = 0
\end{aligned} \tag{11.26}
$$

Also for $\overline{a}_{-k} \neq 0$ with $k > 1$, we get

$$
F_p(\overline{a}_{-k} p^{-k} + \overline{a}_{-k+1} p^{-k+1} + ...) = 0. \tag{11.27}
$$

The Weil transform of this function is the following function on $\mathbb{Q}_p/\mathbb{Z}_p$, which belongs to $\mathscr{A}_2(1, 0)$:

$$
\begin{aligned}
f_p(p^{-1}) = \int_{\mathbb{Z}_p} F_p(p^{-1} + b_p) db_p &= \frac{1}{4}[F_p(p^{-1} + 1 + p) \\
&+ F_p(p^{-1} + 0 + p) + F_p(p^{-1} + 1 + 0p) + F_p(p^{-1} + 0 + 0p)] \\
&= \frac{1}{4}[5 - 2i],
\end{aligned} \tag{11.28}
$$

and

$$
\begin{aligned}
f_p(0) = \int_{\mathbb{Z}_p} F_p(p^{-1} + b_p) db_p &= \frac{1}{4}[F_p(0p^{-1} + 1 + p) \\
&+ F_p(0p^{-1} + 0 + p) + F_p(0p^{-1} + 1 + 0p) + F_p(0p^{-1} + 0 + 0p)] \\
&= \frac{1}{4}[3 + i].
\end{aligned} \tag{11.29}
$$

In this case

$$
\int_{\mathbb{Q}_p/\mathbb{Z}_p} f(\mathfrak{p}_p) d\mathfrak{a}_p = \int_{\mathbb{Q}_p} F_p(c_p) dc_p = \frac{1}{4}[8 - i]. \tag{11.30}
$$

11.3.2 Delta Functions

Delta function in the present context, is a function $\delta_p(x_p)$ where $x_p \in \mathbb{Z}_p$, such that

$$
\int_{\mathbb{Z}_p} dx_p f_p(x_p) \delta_p(x_p - a_p) = f_p(a_p). \tag{11.31}
$$

It is a generalized function. It does not belong to $\mathscr{A}_p$ because it is not locally constant.

We also introduce the following function $\Delta_p(\mathfrak{p}_p)$ where $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$:

$$\begin{aligned} &\Delta_p(0) = 1 \\ &\Delta_p(\mathfrak{p}_p) = 0 \;\text{ if }\; \mathfrak{p}_p \neq 0. \end{aligned} \tag{11.32}$$

Then

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p F_p(\mathfrak{p}_p)\Delta_p(\mathfrak{p}_p - \mathfrak{a}_p) = F_p(\mathfrak{a}_p). \tag{11.33}$$

The following relations are useful later:

$$\begin{aligned} &\int_{\mathbb{Z}_p} dx_p \chi_p(x_p\mathfrak{p}_p) = \Delta_p(\mathfrak{p}_p) \\ &\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \chi_p(x_p\mathfrak{p}_p) = \delta_p(x_p). \end{aligned} \tag{11.34}$$

11.4 The Quantum System $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$

We define the Schwartz-Bruhat space $\mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, which is the space of complex wavefunctions for the quantum system $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$. The definition [4–6] aims to ensure convergence of the scalar products of the wavefunctions. Below we usually use fraktur letters for elements of $\mathbb{Q}_p/\mathbb{Z}_p$.

Definition 11.3 The Schwartz-Bruhat space $\mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ consists of functions $f_p(x_p) \in \mathscr{A}_p(0, *)$ where $x_p \in \mathbb{Z}_p$, or equivalently of functions $F_p(\mathfrak{p}_p) \in \mathscr{A}_p(*, 0)$ where $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$. The scalar product is given by

$$(f, g) = \int_{\mathbb{Z}_p} dx_p f_p(x_p)g_p(x_p); \quad (F, G) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p F_p(\mathfrak{p}_p)G_p(\mathfrak{p}_p). \tag{11.35}$$

The Fourier transform in this space, is defined as follows:

Definition 11.4 The Fourier transform of a function $f_p(x_p) \in \mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ where $x_p \in \mathbb{Z}_p$ (such a function belongs to $\mathscr{A}_p(0, *)$), is the function

$$(\mathfrak{F}_p f_p)(\mathfrak{p}_p) = \widetilde{f}_p(\mathfrak{p}_p) = \int_{\mathbb{Z}_p} dx_p f_p(x_p)\chi_p(-x_p\mathfrak{p}_p), \tag{11.36}$$

which is defined on $\mathbb{Q}_p/\mathbb{Z}_p$, and belongs to the set $\mathscr{A}_p(*, 0)$.

Proposition 11.3 *(1) The inverse Fourier transform of a complex function $\widetilde{f}_p(\mathfrak{p}_p) \in \mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, where $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$ (such a function belongs to $\mathscr{A}_p(*, 0)$), is*

$$(\mathfrak{F}_p^{-1}\widetilde{f}_p)(x_p) = f_p(x_p) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \widetilde{f}_p(\mathfrak{p}_p)\chi_p(x_p\mathfrak{p}_p). \tag{11.37}$$

It is a complex function on $\mathbb{Z}_p$, which belong to the set $\mathscr{A}_p(0, *)$.

(2)

$$\mathbf{LC}[\widetilde{f}_p(\mathfrak{p}_p)] = \mathbf{CS}[f_p(x_p)]; \quad \mathbf{CS}[\widetilde{f}_p(\mathfrak{p}_p)] = \mathbf{LC}[f_p(x_p)]. \tag{11.38}$$

Therefore if $f_p(x_p) \in \mathscr{A}_p(k, n)$, then its Fourier transform $\widetilde{f}_p(\mathfrak{p}_p) \in \mathscr{A}_p(n, k)$.

(3)

$$\mathfrak{F}_p^4 = \mathbf{1}. \tag{11.39}$$

(4) Parceval's theorem holds:

$$\int_{\mathbb{Z}_p} dx_p f_p(x_p)g_p(x_p) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \widetilde{f}_p(\mathfrak{p}_p)\widetilde{g}_p(\mathfrak{p}_p) \tag{11.40}$$

Proof (1) We prove that Eqs.(11.36),(11.37) are compatible using Eq.(11.34).

(2) If the function $f_p(x_p)$ has $\mathbf{LC}[f_p(x_p)] = n$, then for all $|\alpha_p|_p \leq p^{-n}$ we get $f_p(x_p + a_p) - f_p(x_p) = 0$ and we rewrite this as

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \; \chi_p(x_p\mathfrak{p}_p)\widetilde{f}_p(\mathfrak{p}_p)[1 - \chi_p(\alpha_p\mathfrak{p}_p)] = 0. \tag{11.41}$$

It is seen that the Fourier transform of $\widetilde{f}_p(\mathfrak{p}_p)[1 - \chi_p(\alpha_p\mathfrak{p}_p)]$ is zero. Consequently, $\widetilde{f}_p(\mathfrak{p}_p)[1 - \chi_p(\alpha_p\mathfrak{p}_p)] = 0$. But for $|\alpha_p|_p \leq p^{-n}$ and $|\mathfrak{p}_p| > p^n$ the $1 - \chi_p(\alpha_p\mathfrak{p}_p) \neq 0$ and therefore $\widetilde{f}_p(\mathfrak{p}_p) = 0$. This proves that $\mathbf{CS}[\widetilde{f}_p(\mathfrak{p}_p)] = n$. In a similar way we prove the other relation.

(3) The proof of this is based on Eq.(11.34).

(4) The proof of this is based on Eq.(11.34).

Remark 11.2 Equation (11.34) can be interpreted as follows:

- the Fourier transform of the function $f_p(x_p) = 1$ on $\mathbb{Z}_p$, is the function $\Delta_p(\mathfrak{p}_p)$ on $\mathbb{Q}_p/\mathbb{Z}_p$.
- the Fourier transform of the function $\widetilde{f}_p(\mathfrak{p}_p) = 1$ on $\mathbb{Q}_p/\mathbb{Z}_p$, is the function $\delta_p(x_p)$ on $\mathbb{Z}_p$.

11.5 The Heisenberg-Weyl Group $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$

The phase space of the system $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ is $\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)$, and we define displacement operators and the Heisenberg-Weyl group [27].

Definition 11.5 The displacement operators $D_p(\mathfrak{a}_p, b_p)$ where $b_p \in \mathbb{Z}_p$ and $\mathfrak{a}_p \in \mathbb{Q}_p/\mathbb{Z}_p$, are defined by one of the following ways, which are equivalent to each other:

(1) They act on the wavefunctions $f_p(x_p) \in \mathscr{A}_p(0, *)$ where $x_p \in \mathbb{Z}_p$, as follows:

$$[D_p(\mathfrak{a}_p, b_p) f_p](x_p) = \chi_p\left(-\mathfrak{a}_p b_p + 2\mathfrak{a}_p x_p\right) f_p(x_p - b_p). \tag{11.42}$$

(2) They act on the wavefunctions $F_p(\mathfrak{p}_p) \in \mathscr{A}(*, 0)$, where $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$ as follows:

$$[D_p(\mathfrak{a}_p, b_p) F_p](\mathfrak{p}_p) = \chi_p\left(\mathfrak{a}_p b_p - b_p \mathfrak{p}_p\right) F_p(\mathfrak{p}_p - 2\mathfrak{a}_p). \tag{11.43}$$

The equivalence of the definitions, is easily proved with a Fourier transform.

Proposition 11.4 *The displacement operators $D_p(\mathfrak{a}_p, b_p)\chi_p(\mathfrak{c}_p)$ form a representation of the Heisenberg-Weyl group $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ (the notation indicates the sets in whch the variables $\mathfrak{a}_p, b_p, \mathfrak{c}_p$ belong).*

Proof Using the definition in Eq.(11.42), we prove the multiplication rule

$$\begin{aligned} &D_p(\mathfrak{a}_p, b_p) D_p(\mathfrak{a}'_p, b'_p) \\ &= D_p(\mathfrak{a}_p + \mathfrak{a}'_p, b_p + b'_p)\chi_p(\mathfrak{a}_p b'_p - \mathfrak{a}'_p b_p). \end{aligned} \tag{11.44}$$

Taking into account the Definition 4.2, we conclude that the $D_p(\mathfrak{a}_p, b_p)\chi_p(\mathfrak{c}_p)$ form a representation of the Heisenberg-Weyl group.

11.5.1 $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ *as a Locally Compact and Totally Disconnected Topological Group*

We define the following subgroups of $HW(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$:

$$\begin{aligned} HW_1(\mathbb{Q}_p/\mathbb{Z}_p) &= \{D_p(\mathfrak{a}_p, 0) \mid \mathfrak{a}_p \in \mathbb{Q}_p/\mathbb{Z}_p\} \cong \mathbb{Q}_p/\mathbb{Z}_p \\ HW_2(p^e\mathbb{Z}_p) &= \{D_p(0, b_p) \mid b_p \in p^e\mathbb{Z}_p\} \cong p^e\mathbb{Z}_p \\ HW_3(\mathbb{Q}_p/\mathbb{Z}_p) &= \{\chi_p(\mathfrak{c}_p) \mid \mathfrak{c}_p \in \mathbb{Q}_p/\mathbb{Z}_p\} \cong \mathbb{Q}_p/\mathbb{Z}_p. \end{aligned} \tag{11.45}$$

If $e_1 \leq e_2$ then $HW_2(p^{e_2}\mathbb{Z}_p) \prec HW_2(p^{e_1}\mathbb{Z}_p)$. The set

$$\mathfrak{N}_p = \{HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]\} \cup \{HW_2(p^e\mathbb{Z}_p) \mid e \in \mathbb{Z}_0^+\} \quad (11.46)$$

with the order subgroup, is a chain.

If A, B are subsets of a group G and $g \in G$, we use the notation:

$$gA = \{ga|a \in A\}; \qquad gAg^{-1} = \{gag^{-1}|a \in A\}$$
$$AB = \bigcup_{a\in A} aB; \qquad A^{-1} = \{a^{-1}|a \in A\}. \quad (11.47)$$

Proposition 11.5 *We regard the set $\mathfrak{N}_p$ in Eq.(11.46), as a fundamental system of open neighborhoods of the identity of $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$. Then the $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ becomes a topological group, which is totally disconnected and locally compact.*

Proof We prove that the elements of $\mathfrak{N}_p$, satisfy the following properties of a fundamental system of open neighborhoods of the identity (e.g. Sect. III.1.2 in [45]). These properties ensure compatibility between the group structure and the topology.

- Given any $U \in \mathfrak{N}_p$ there exists $V \in \mathfrak{N}_p$ such that $VV \prec U$. This holds because for $U = HW_2(p^n\mathbb{Z}_p)$, all the $V = HW_2(p^k\mathbb{Z}_p)$ with $k \geq n$ satisfy this.
- Given any $U \in \mathfrak{N}_p$ there exists $V \in \mathfrak{N}_p$ such that $V^{-1} \prec U$. This holds because for $U = HW_2(p^n\mathbb{Z}_p)$ all the $V^{-1} = V = HW_2(p^k\mathbb{Z}_p)$ with $k \geq n$ satisfy this.
- Given any element $D(\mathfrak{a}, b)\chi_p(\mathfrak{c})$ of $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ and any $U \in \mathfrak{N}_p$, there exists $V \in \mathfrak{N}_p$ such that

$$V \prec \left[D(\mathfrak{a}, b)\chi_p(\mathfrak{c})\right] U \left[D(-\mathfrak{a}, -b)\chi_p(-\mathfrak{c})\right]. \quad (11.48)$$

  This holds because for $U = HW_2(p^n\mathbb{Z}_p)$, we get

$$D(\mathfrak{a}, b)\chi_p(\mathfrak{c})\ D(0, b')\ D(-\mathfrak{a}, -b)\chi_p(-\mathfrak{c}) = D(0, b')\chi_p(\mathfrak{a}b'); \quad (11.49)$$

  where $b' \in p^n\mathbb{Z}_p$. Any subgroup $V = HW_2(p^k\mathbb{Z}_p)$ with $k \geq \max(n, -\mathrm{ord}(\mathfrak{a}))$ satisfies Eq.(11.48).

Therefore $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ is a topological group.

We next show that it is totally disconnected and locally compact. $HW_1(\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Q}_p/\mathbb{Z}_p$ is a discrete locally compact topological group. $HW_2(\mathbb{Z}_p) \cong \mathbb{Z}_p$ is a profinite group, i.e. a totally disconnected compact topological group. Since both of these groups are totally disconnected and locally compact, it follows that the $HW_1(\mathbb{Q}_p/\mathbb{Z}_p) \times HW_2(\mathbb{Z}_p)$ with the product topology, is a totally disconnected and locally compact topological group.

$HW_3(\mathbb{Q}_p/\mathbb{Z}_p)$ is a normal subgroup of $HW(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$. We consider the quotient group

$$\begin{aligned} &HW(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)/HW_3(\mathbb{Q}_p/\mathbb{Z}_p) \\ &\cong HW_1(\mathbb{Q}_p/\mathbb{Z}_p) \times HW_2(\mathbb{Z}_p). \end{aligned} \tag{11.50}$$

Both the $HW_1(\mathbb{Q}_p/\mathbb{Z}_p) \times HW_2(\mathbb{Z}_p)$ and $HW_3(\mathbb{Q}_p/\mathbb{Z}_p)$ are totally disconnected and locally compact topological groups. Consequently, $HW(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$ is a totally disconnected and locally compact topological group.

Remark 11.3 For completeness we define another representation of the Heisenberg-Weyl group, although it is not relevant for Physics. This is the profinite Heisenberg-Weyl group $HW(\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p)$, and is different from the $HW[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ (which is not profinite).

The $HW(\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p)$ is defined as the inverse limit of the finite Heisenberg-Weyl groups $HW[\mathbb{Z}(p^e), \mathbb{Z}(p^e), \mathbb{Z}(p^e)]$. For $k \le n$ we define the homomorphisms

$$\widetilde{\varphi}_{kn}: \quad HW[\mathbb{Z}(p^k), \mathbb{Z}(p^k), \mathbb{Z}(p^k)] \leftarrow HW[\mathbb{Z}(p^n), \mathbb{Z}(p^n), \mathbb{Z}(p^n)], \tag{11.51}$$

where

$$\begin{aligned} &\widetilde{\varphi}_{kn}[D(\alpha_{p^n}, \beta_{p^n})\omega_{p^n}(\gamma_{p^n})] = D(\alpha_{p^k}, \beta_{p^k})\omega_{p^k}(\gamma_{p^k}) \\ &\alpha_{p^k} = \varphi_{kn}(\alpha_{p^n}); \qquad \beta_{p^k} = \varphi_{kn}(\beta_{p^n}); \qquad \gamma_{p^k} = \varphi_{kn}(\gamma_{p^n}). \end{aligned} \tag{11.52}$$

The map φ_{kn} has been defined in Eq.(10.43). The $\widetilde{\varphi}_{kn}$ are compatible, and the $\{HW[\mathbb{Z}(p^n), \mathbb{Z}(p^n), \mathbb{Z}(p^n)], \widetilde{\varphi}_{kn}\}$ is an inverse system, whose inverse limit we denote as $HW(\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p)$. The elements of this group are $\mathfrak{D}_p(a_p, b_p)\chi_p(c_p)$ where

$$\begin{aligned} &\mathfrak{D}_p(a_p, b_p) = (D(\alpha_p, \beta_p), D(\alpha_{p^2}, \beta_{p^2}), ...) \\ &\chi_p(c_p) = (\omega_p(\gamma_p), \omega_{p^2}(\gamma_{p^2}), ...); \quad a_p, b_p, c_p \in \mathbb{Z}_p \\ &a_p = (\alpha_p, \alpha_{p^2}, ...); \quad b_p = (\beta_p, \beta_{p^2}, ...); \quad c_p = (\gamma_p, \gamma_{p^2}, ...). \end{aligned} \tag{11.53}$$

We stress that the $\mathfrak{D}_p(a_p, b_p)$ where $a_p, b_p \in \mathbb{Z}_p$, is very different from the $D_p(\mathfrak{a}_p, b_p)$ where $\mathfrak{a}_p \in \mathbb{Q}_p/\mathbb{Z}_p$ and $b_p \in \mathbb{Z}_p$ (see also Remark 10.2).

Multiplication of these elements is componentwise, and obeys the rule in the Definition 4.2. Therefore we have a representation of the Heisenberg-Weyl group. But the Pontryagin dual group to $\mathbb{Z}_p$ does not appear here. Consequently, the $HW(\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p)$ cannot be associated with displacements of dual quantum variables, and it is not relevant to quantum mechanics. Pontryagin duality of the groups of positions and momenta is an essential feature of quantum mechanics.

11.6 Wigner and Weyl Functions

In this section we discuss Wigner and Weyl functions in the present context[27, 32]. We point out from the outset, that there are differences between the two cases $p = 2$ and $p \neq 2$. Some of the integrals have domain of integration $\mathbb{Q}_p/|2|_p\mathbb{Z}_p$, and also the prefactor $|2|_p$. This is related to change of variables using Eq.(11.19), and it is analogous to our comment in the context of finite quantum systems earlier, that there are technical differences in the two cases of even or odd dimension. We recall that

$$|2|_p = 1 \text{ if } p \neq 2$$
$$|2|_2 = \frac{1}{2}. \tag{11.54}$$

We consider an operator $\theta(x_p, y_p)$ where $x_p, y_p \in \mathbb{Z}_p$, and let

$$\widetilde{\theta}(\mathfrak{p}_p, \mathfrak{p}'_p) = \int_{\mathbb{Z}_p} dx_p \int_{\mathbb{Z}_p} dy_p \theta(x_p, y_p)\chi_p(-x_p\mathfrak{p}_p + y_p\mathfrak{p}'_p), \tag{11.55}$$

where $\mathfrak{p}_p, \mathfrak{p}'_p \in \mathbb{Q}_p/\mathbb{Z}_p$. θ acts on a function $f_p(x_p)$, and its Fourier transform $\widetilde{f}_p(\mathfrak{p}_p)$, as follows:

$$(\theta f_p)(x_p) = \int_{\mathbb{Z}_p} dy_p \theta(x_p, y_p) f_p(y_p)$$
$$(\theta \widetilde{f}_p)(\mathfrak{p}_p) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p \widetilde{\theta}(\mathfrak{p}_p, \mathfrak{p}'_p) \widetilde{f}_p(\mathfrak{p}'_p) \tag{11.56}$$

The trace of θ is given by

$$\mathrm{tr}\theta = \int_{\mathbb{Z}_p} dx_p \theta(x_p, x_p) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \widetilde{\theta}(\mathfrak{p}_p, \mathfrak{p}_p) \tag{11.57}$$

Definition 11.6 The parity operator with respect to the point $(\mathfrak{a}_p, b_p)$ in the phase space $(\mathbb{Q}_p/\mathbb{Z}_p) \times \mathbb{Z}_p$, is

$$P_p(\mathfrak{a}_p, b_p) = [D_p(\mathfrak{a}_p, b_p)]^\dagger\, \mathfrak{F}_p^2\, D_p(\mathfrak{a}_p, b_p)$$
$$= [D_p(2\mathfrak{a}_p, 2b_p)]^\dagger\, \mathfrak{F}_p^2 = \mathfrak{F}_p^2\, D_p(2\mathfrak{a}_p, 2b_p) \tag{11.58}$$

In particular the parity operator with respect to the point (0, 0) is $P_p(0, 0) = \mathfrak{F}_p^2$.

Proposition 11.6 *(1) The parity operator acts on wavefunctions in $\mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, as follows:*

$$P_p(\mathfrak{a}_p, b_p) f_p(x_p) = \chi_p(-4\mathfrak{a}_p b_p - 4\mathfrak{a}_p x_p) f_p(-x_p - 2b_p)$$
$$P_p(\mathfrak{a}_p, b_p) \widetilde{f}_p(\mathfrak{p}_p) = \chi_p(4\mathfrak{a}_p b_p + 2\mathfrak{p}_p b_p) \widetilde{f}_p(-\mathfrak{p}_p - 4\mathfrak{a}_p). \tag{11.59}$$

(2)

$$[P_p(\mathfrak{a}_p, b_p)]^2 = \mathbf{1}; \quad P_p\left(\mathfrak{a}_p + \frac{1}{4}, b_p\right) = P_p(\mathfrak{a}_p, b_p). \tag{11.60}$$

Proof (1) This is proved using Eqs.(11.42), (11.43).
(2) Using Eq.(11.59) we easily prove that $[P_p(\mathfrak{a}_p, b_p)]^2 = \mathbf{1}$. Also using the second of Eqs.(11.59), we prove that $P_p\left(\mathfrak{a}_p + \frac{1}{4}, b_p\right) = P_p(\mathfrak{a}_p, b_p)$.

Remark 11.4 For $p \neq 2$, the $\frac{1}{4} \in \mathbb{Z}_p$, and for $p = 2$, the $\frac{1}{4} \in 2^{-2}\mathbb{Z}_p$. For any $c_p \in \mathbb{Z}_p$, we get $P_p(\mathfrak{a}_p + c_p, b_p) = P_p(\mathfrak{a}_p, b_p)$. Therefore the $P_p(\mathfrak{a}_p + \frac{1}{4}, b_p) = P_p(\mathfrak{a}_p, b_p)$ is a new result, only for $p = 2$.

Definition 11.7 The Weyl function of an operator θ, is defined as:

$$\widetilde{W}(\mathfrak{a}_p, b_p; \theta) = \mathrm{tr}[D_p(-\mathfrak{a}_p, -b_p)\theta]; \quad \mathfrak{a}_p \in \mathbb{Q}_p/\mathbb{Z}_p; \quad b_p \in \mathbb{Z}_p. \tag{11.61}$$

The Wigner function of an operator θ, is defined as:

$$W(\mathfrak{a}_p, b_p; \theta) = \mathrm{tr}[\theta P_p(\mathfrak{a}_p, b_p)]; \quad \mathfrak{a}_p \in \mathbb{Q}_p/\mathbb{Z}_p; \quad b_p \in \mathbb{Z}_p. \tag{11.62}$$

Proposition 11.7 *(1) The Weyl function is given by*

$$\widetilde{W}(\mathfrak{a}_p, b_p; \theta) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \ \chi_p\left(\mathfrak{a}_p b_p + \mathfrak{p}_p b_p\right) \widetilde{\theta}(\mathfrak{p}_p + 2\mathfrak{a}_p, \mathfrak{p}_p) \tag{11.63}$$

(2) The Wigner function is given by

$$W(\mathfrak{a}_p, b_p; \theta) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \ \chi_p(-4\mathfrak{a}_p b_p - 2\mathfrak{p}_p b_p)\widetilde{\theta}(\mathfrak{p}_p, -\mathfrak{p}_p - 4\mathfrak{a}_p). \tag{11.64}$$

Proof (1) We act with $D_p(-\mathfrak{a}_p, -b_p)$ on the kernel $\widetilde{\theta}(\mathfrak{p}_p, \mathfrak{p}'_p)$ of the operator θ and we get

$$[D_p(-\mathfrak{a}_p, -b_p)\widetilde{\theta}](\mathfrak{p}_p, \mathfrak{p}'_p) = \chi_p\left(\mathfrak{a}_p b_p + \mathfrak{p}_p b_p\right) \widetilde{\theta}(\mathfrak{p}_p + 2\mathfrak{a}_p, \mathfrak{p}'_p) \tag{11.65}$$

Therefore its trace, which is the Weyl function, is

$$\begin{aligned} \widetilde{W}(\mathfrak{a}_p, b_p; \theta) &= \mathrm{tr}[D_p(-\mathfrak{a}_p, -b_p)\theta] \\ &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \ \chi_p\left(\mathfrak{a}_p b_p + \mathfrak{p}_p b_p\right) \widetilde{\theta}(\mathfrak{p}_p + 2\mathfrak{a}_p, \mathfrak{p}_p) \end{aligned} \tag{11.66}$$

(2) We act with $P_p(\mathfrak{a}_p, b_p)$ on the kernel $\widetilde{\theta}(\mathfrak{p}_p, \mathfrak{p}'_p)$ of the operator θ and we get

$$[P_p(\mathfrak{a}_p, b_p)\widetilde{\theta}](\mathfrak{p}_p, \mathfrak{p}'_p) = \chi_p\left(4\mathfrak{a}_p b_p + 2\mathfrak{p}_p b_p\right) \widetilde{\theta}(-\mathfrak{p}_p - 4\mathfrak{a}_p, \mathfrak{p}'_p) \tag{11.67}$$

Therefore its trace, which is the Wigner function, is

$$W(\mathfrak{a}_p, b_p; \theta) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p \ \chi_p\left(4\mathfrak{a}_p b_p + 2\mathfrak{p}_p b_p\right)$$
$$\times \widetilde{\theta}(-\mathfrak{p}_p - 4\mathfrak{a}, \mathfrak{p}'_p)\Delta_p(\mathfrak{p}_p + \mathfrak{p}'_p + 4\mathfrak{a}_p)$$
$$= \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \ \widetilde{\theta}(\mathfrak{p}_p, -\mathfrak{p}_p - 4\mathfrak{a}_p)\chi_p(-4\mathfrak{a}_p b_p - 2\mathfrak{p}_p b_p). \quad (11.68)$$

Proposition 11.8 *The parity operators are related to the displacement operators through a Fourier transform:*

$$P_p(\mathfrak{a}_p, b_p) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}'_p \int_{\mathbb{Z}_p} db'_p \ D_p(\mathfrak{a}'_p, b'_p)\chi_p(2\mathfrak{a}'_p b_p - 2\mathfrak{a}_p b'_p). \quad (11.69)$$

Also the Wigner function is related to the Weyl function through a Fourier transform:

$$W(\mathfrak{a}_p, b_p; \theta) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}'_p \int_{\mathbb{Z}_p} db'_p \ \widetilde{W}(-\mathfrak{a}'_p, -b'_p)\chi_p(2\mathfrak{a}'_p b_p - 2\mathfrak{a}_p b'_p). \quad (11.70)$$

Proof We act with the right hand side of Eq.(11.69) on an arbitrary function $F_p(\mathfrak{p}_p)$, and we get

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}'_p \ \chi_p(2\mathfrak{a}'_p b_p)F_p(\mathfrak{p}_p - 2\mathfrak{a}'_p) \int_{\mathbb{Z}_p} db'_p \ \chi_p[b'_p(\mathfrak{a}'_p - \mathfrak{p}_p - 2\mathfrak{a}_p)]$$
$$= \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}'_p \ \chi_p(2\mathfrak{a}'_p b_p)F_p(\mathfrak{p}_p - 2\mathfrak{a}'_p)\Delta_p(\mathfrak{a}'_p - \mathfrak{p}_p - 2\mathfrak{a}_p)$$
$$= \chi_p(4b_p\mathfrak{a}_p + 2b_p\mathfrak{p}_p) \ F_p(-\mathfrak{p}_p - 4\mathfrak{a}_p) = P_p(\mathfrak{a}, b)F_p(\mathfrak{p}_p). \quad (11.71)$$

From this we prove Eq.(11.70), using the definitions for the Wigner and Weyl functions.

Proposition 11.9 *Let θ be a trace class operator acting on functions in $\mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$. Then*

(1)

$$|2|_p \int_{\mathbb{Q}_p/|2|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Z}_p} db_p \ D_p(\mathfrak{a}_p, b_p) \ \theta \ [D_p(\mathfrak{a}_p, b_p)]^\dagger = \mathbf{1}\mathrm{tr}\theta. \quad (11.72)$$

(2) θ can be expanded in terms of displacement operators, with the Weyl function as coefficients:

$$\theta = |2|_p \int_{\mathbb{Q}_p/|2|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Z}_p} db_p \ D_p(\mathfrak{a}_p, b_p)\widetilde{W}(\mathfrak{a}_p, b_p; \theta). \quad (11.73)$$

Proof (1) We act with $D_p(\mathfrak{a}_p, b_p)\ \theta\ [D_p(\mathfrak{a}_p, b_p)]^\dagger$ on a function $F_p(\mathfrak{p}_p) \in \mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ and we get

$$\left[D_p(\mathfrak{a}_p, b_p)\ \theta\ [D_p(\mathfrak{a}_p, b_p)]^\dagger F_p\right](\mathfrak{p}_p)$$
$$= \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p\ \chi_p(2\mathfrak{a}_p b_p - \mathfrak{p}_p b_p + \mathfrak{p}'_p b_p)$$
$$\times\theta(\mathfrak{p}_p - 2\mathfrak{a}_p, \mathfrak{p}'_p)\ F_p(\mathfrak{p}'_p + 2\mathfrak{a}_p). \tag{11.74}$$

The scalar product of this with an arbitrary function $G_p(\mathfrak{p}_p) \in \mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, gives

$$|2|_p \int_{\mathbb{Q}_p/|2|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Z}_p} db_p\ \left(G_p, D_p(\mathfrak{a}_p, b_p)\ \theta\ [D_p(\mathfrak{a}_p, b_p)]^\dagger F_p\right)$$
$$= |2|_p \int_{\mathbb{Q}_p/|2|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Z}_p} db_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p\ [G_p(\mathfrak{p}_p)]^*$$
$$\times\chi_p(2\mathfrak{a}_p b_p - \mathfrak{p}_p b_p + \mathfrak{p}'_p b_p)\theta(\mathfrak{p}_p - 2\mathfrak{a}_p, \mathfrak{p}'_p)\ F_p(\mathfrak{p}'_p + 2\mathfrak{a}_p)$$
$$= |2|_p \int_{\mathbb{Q}_p/|2|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p\ [G_p(\mathfrak{p}_p)]^*$$
$$\times\Delta_p(2\mathfrak{a}_p - \mathfrak{p}_p + \mathfrak{p}'_p)\theta(\mathfrak{p}_p - 2\mathfrak{a}_p, \mathfrak{p}'_p)\ F_p(\mathfrak{p}'_p + 2\mathfrak{a}_p) \tag{11.75}$$

We now change the variable $2\mathfrak{a}_p$ into $\mathfrak{a}'_p$, taking into account Eq.(11.19). We get:

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p\ [G_p(\mathfrak{p}_p)]^*\ F_p(\mathfrak{p}_p) \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p\ \theta(\mathfrak{p}'_p, \mathfrak{p}'_p) = (G_p, F_p)\mathrm{tr}(\theta). \tag{11.76}$$

This proves the proposition.

(2) The operator in Eq.(11.73) acts on an arbitrary function $F_p(\mathfrak{p}_p) \in \mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, as follows (use Eq.(11.63)):

$$|2|_p \int_{\mathbb{Q}_p/|2|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Z}_p} db_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p\ \chi_p\left(\mathfrak{a}_p b_p + \mathfrak{p}_p b_p\right)$$
$$\times\widetilde{\theta}(\mathfrak{p}_p + 2\mathfrak{a}_p, \mathfrak{p}_p)\chi_p\left(\mathfrak{a}_p b_p - \mathfrak{p}'_p b_p\right) F_p(\mathfrak{p}'_p - 2\mathfrak{a}_p)$$
$$= |2|_p \int_{\mathbb{Q}_p/|2|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p\ \widetilde{\theta}(\mathfrak{p}_p + 2\mathfrak{a}_p, \mathfrak{p}_p)$$
$$\times\Delta_p(2\mathfrak{a}_p + \mathfrak{p}_p - \mathfrak{p}'_p)F_p(\mathfrak{p}'_p - 2\mathfrak{a}_p) \tag{11.77}$$

We now change the variable $2\mathfrak{a}_p$ into $\mathfrak{a}'_p$, taking into account Eq.(11.19). We get

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p\ \widetilde{\theta}(\mathfrak{p}'_p, \mathfrak{p}_p)F_p(\mathfrak{p}_p) \tag{11.78}$$

This proves the proposition.

Proposition 11.10 *Let θ be a trace class operator acting on functions in the Schwartz-Bruhat space $\mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$. Then*

(1)

$$|2|_p^3 \int_{\mathbb{Q}_p/|4|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{|2|_p\mathbb{Z}_p} db_p \; P_p(\mathfrak{a}_p, b_p)\; \theta \; P_p(\mathfrak{a}_p, b_p) = \mathbf{1}\mathrm{tr}\theta. \quad (11.79)$$

(2) θ can be expanded in terms of parity operators, with the Wigner function as coefficients:

$$\theta = |2|_p^3 \int_{\mathbb{Q}_p/|4|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{|2|_p\mathbb{Z}_p} db_p \; P_p(\mathfrak{a}_p, b_p) W(\mathfrak{a}_p, b_p; \theta). \quad (11.80)$$

Proof (1) We substitute $\mathfrak{a}_p$ with $2\mathfrak{a}_p$ and b_p with $2b_p$ in Eq.(11.72), and change accordingly the domains of integration. We get

$$\begin{aligned} &|2|_p^3 \int_{\mathbb{Q}_p/|4|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{|2|_p\mathbb{Z}_p} db_p \; D_p(2\mathfrak{a}_p, 2b_p)\; \theta \;[D_p(2\mathfrak{a}_p, 2b_p)]^\dagger \\ &= \mathbf{1}\mathrm{tr}\theta. \end{aligned} \quad (11.81)$$

Then we multiply each side with $\mathfrak{F}_p^2$ on the left and with $(\mathfrak{F}_p^2)^\dagger$ on the right, and we prove the statement.

(2) We substitute Eq.(11.64) on the right hand side of Eq.(11.80), and act on an arbitrary function $F_p(\mathfrak{p}_p)$, in order to prove that this is the operator θ acting on $F_p(\mathfrak{p}_p)$:

$$\begin{aligned} &(|2|_p)^3 \int_{\mathbb{Q}_p/|4|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{|2|_p\mathbb{Z}_p} db_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p \; \widetilde{\theta}(\mathfrak{p}'_p, -\mathfrak{p}'_p - 4\mathfrak{a}_p) \\ &\times \chi_p(-4\mathfrak{a}_p b_p - 2\mathfrak{p}'_p b_p)\chi(4\mathfrak{a}_p b_p + 2\mathfrak{p}_p b_p) F_p(-\mathfrak{p}_p - 4\mathfrak{a}_p) \end{aligned} \quad (11.82)$$

Integration over $2b_p$ gives,

$$\begin{aligned} &|4|_p \int_{\mathbb{Q}_p/|4|_p\mathbb{Z}_p} d\mathfrak{a}_p \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}'_p \; \widetilde{\theta}(\mathfrak{p}'_p, -\mathfrak{p}'_p - 4\mathfrak{a}_p) \\ &\times \Delta_p(\mathfrak{p}_p - \mathfrak{p}'_p) F_p(-\mathfrak{p}_p - 4\mathfrak{a}_p) \\ &= |4|_p \int_{\mathbb{Q}_p/|4|_p\mathbb{Z}_p} d\mathfrak{a}_p \; \widetilde{\theta}(\mathfrak{p}_p, -\mathfrak{p}_p - 4\mathfrak{a}_p) F_p(-\mathfrak{p}_p - 4\mathfrak{a}_p) \end{aligned} \quad (11.83)$$

Now we change the variable $-\mathfrak{p}_p - 4\mathfrak{a}_p$ into $\mathfrak{q}_p$, taking into account Eq.(11.19). We prove that the right hand side of Eq.(11.80), is equal to the operator θ.

11.7 The Complete Chain of Subsystems of $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$

In sect. 10.8 we studied the complete chain $\mathbb{N}_S^{(G,\widetilde{G})}(p)$ which contains pairs of groups $(\mathbb{Z}(p^k), C(p^k))$ which are Pontryagin dual to each other. To each of these pairs corresponds a quantum system, as follows:

$$\begin{aligned}(\mathbb{Z}(p^k), C(p^k)) &\rightarrow \Sigma[\mathbb{Z}(p^k)]; \quad k \in \mathbb{N}\\ (\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)) &\rightarrow \Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]\end{aligned} \tag{11.84}$$

We denote as $\mathbb{N}_S^Q(p)$, the set of these quantum systems (the superfix Q indicates quantum systems). It is a complete chain

$$\Sigma[\mathbb{Z}(p)] \prec \Sigma[\mathbb{Z}(p^2)] \prec ... \prec \Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)], \tag{11.85}$$

with the order subsystem [32]. $\mathbb{N}_S^Q(p)$ is order isomorphic to $\mathbb{N}_S^{(G,\widetilde{G})}(p)$ and also to $\mathbb{N}_S(p)$:

$$\mathbb{N}_S^Q(p) \cong \mathbb{N}_S^{(G,\widetilde{G})}(p) \cong \mathbb{N}_S(p). \tag{11.86}$$

Below we give some technical details related to the fact that $\Sigma[\mathbb{Z}(p^k)]$ is a subsystem of $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$. We define a subspace $\mathscr{B}[\mathbb{Z}(p^k)]$ of $\mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, and show that it is isomorphic to the space $H[\mathbb{Z}(p^k)]$, which describes the system $\Sigma[\mathbb{Z}(p^k)]$.

Definition 11.8 The subspace $\mathscr{B}[\mathbb{Z}(p^k)]$ of $\mathscr{B}[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ is defined by one of the following ways, which are equivalent to each other:

(1) It contains functions $f_p(x_p) \in \mathscr{A}(0, k)$, where $x_p \in \mathbb{Z}_p$. These functions can be regarded as functions $f(m)$ where $m \in \mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}(p^k)$. The scalar product of Eq.(11.35) reduces to

$$(f, g) = \frac{1}{p^n} \sum_{m\in\mathbb{Z}(p^k)} [f(m)]^* g(m). \tag{11.87}$$

(2) It contains functions $F_p(\mathfrak{p}_p) \in \mathscr{A}(k, 0)$, where $\mathfrak{p}_p \in \mathbb{Q}_p/\mathbb{Z}_p$. These functions can be regarded as functions $F(n)$ where $n \in p^{-k}\mathbb{Z}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^k)$. In this case the scalar product of Eq.(11.35) reduces to

$$(F, G) = \sum_{n\in\mathbb{Z}(p^k)} [F(n)]^* G(n). \tag{11.88}$$

In the subspace $\mathscr{B}[\mathbb{Z}(p^k)]$, the Fourier transform of Eq.(11.36), reduces to the finite Fourier transform used in the space $H[\mathbb{Z}(p^k)]$ for the quantum system $\Sigma[\mathbb{Z}(p^n)]$:

$$F(n) = \frac{1}{p^n} \sum_{m \in \mathbb{Z}(p^n)} f(m)\omega_{p^n}(-mn); \quad m, n \in \mathbb{Z}(p^n). \tag{11.89}$$

Therefore the subspace $\mathscr{B}[\mathbb{Z}(p^k)]$ is isomorphic to the space $H[\mathbb{Z}(p^k)]$, that describes the quantum system $\Sigma[\mathbb{Z}(p^n)]$.

The systems $\Sigma[\mathbb{Z}(p^e)]$ are subsystems of $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$. The chain of all $\Sigma[\mathbb{Z}(p^e)]$ with $e \in \mathbb{N}$ is not complete. By adding the 'top element' $\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ (which describes regorously the case $e = \infty$), we make it complete.

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