On Some Applications of Williamson's Transform in Copula Theory

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Abstract. We show several interesting examples of connection between distribution of a positively valued random variable and an Archimedean copula through Williamson's transformation (and Laplace transform), especially when arranged in a sequence. Naturally, there appears a question: how can we use statistical properties of distance functions to draw statistical properties of copulas, and vice versa? This question is formulated in two open problems.

Keywords: Archimedean copula · Williamson's transform · Laplace transform

1 Introduction

Copulas [5,14,17] are particular functions describing the dependence stucture of random vectors. Not going into details, recall that one of the prominent copula classes important for numerous applications is the class of Archimedean copulas. Formally, for $n \geq 2$, a function $C\colon [0,1]^n \to [0,1]$ is an n-ary Archimedean copula whenever it is a Post associative n-ary copula (i.e., for any $(x_1, \ldots, x_{2n-1}) \in [0,1]^{2n-1}$ it holds $C\left(C(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}\right) = C\left(x_1, C(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}\right) = \ldots = C\left(x_1, \ldots, x_{n-1}, C(x_n, \ldots, x_{2n-1})\right)$ and $C(x, \ldots, x) < x$ for any $x \in]0,1[$, see [18]. Due to [9] we have next representation of n-ary Archimedean copulas.

Theorem 1. *Let $f\colon [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $f(1) = 0$ (i.e., f is an additive generator of a continuous Archimedean t-norm, see [7]). Then the n-ary function $C\colon [0,1]^n \to [0,1]$ given by*

$$C(x_1, \ldots, x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(x_i)\right). \tag{1}$$

(where $f^{(-1)}\colon [0,\infty] \to [0,1]$ given by $f^{(-1)}(u) = f^{-1}\big(\min(u, f(0))\big)$ is the pseudo-inverse of f) is an n-ary copula if and only if the function $g\colon [-\infty, 0] \to [0,1]$ given by $g(u) = f^{(-1)}(-u)$ is $(n-2)$-times differentiable with non-negative derivatives $g', \ldots, g^{(n-2)}$ on $]-\infty, 0[$ (or equivalently, $(-1)^n (f^{(-1)})^{(n)}(u) \geq 0$), and $g^{(n-2)}$ is a convex function (Fig. 1).

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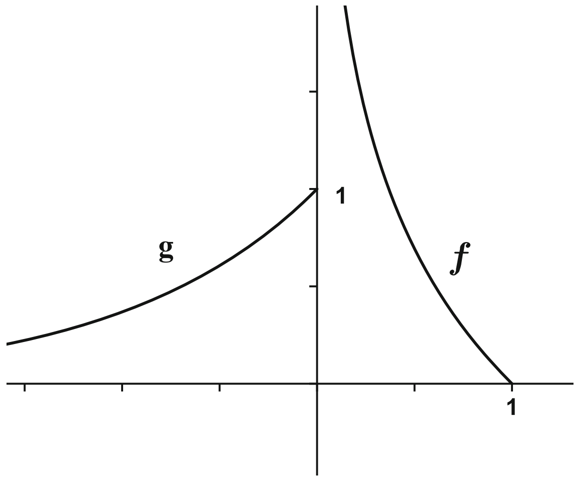

Fig. 1. Illustration of a generator f and its corresponding function g

We denote by $\mathcal{F}_n$ the class of all additive generators that generate n-ary copulas as characterized in Theorem 1.

Additive generators, which generate an n-ary copula for any $n \geq 2$, are called universal generators. Due to Theorem 1, we have the next result, see [6,9].

Corollary 1. *Let $f\colon [0,1] \to [0,\infty]$ be an additive generator of a binary copula $C\colon [0,1]^2 \to [0,1]$. Then the n-ary extension $C\colon [0,1]^n \to [0,1]$ given by (1) is an n-ary copula for each $n \geq 2$ if and only if the function $g\colon [-\infty,0] \to [0,1]$ given by $g(u) = f^{(-1)}(-u)$ is absolutely monotone, i.e., $g^{(k)}$ exists and is non-negative for each $k \in N = \{1,2,\ldots\}$.*

The class of all universal additive generators will be denoted by $\mathcal{F}_\infty$. It is not difficult to check that $\mathcal{F}_2 \supset \mathcal{F}_3 \supset \ldots \supset \mathcal{F}_\infty$.

For any $n \geq 2$, there is an important link between the additive generators of n-ary Archimedean copulas and distance functions $F\colon [0,\infty[\;\to\; [0,1]$, i.e., distribution functions of positive random variables restricted to $[0,\infty[$. Observe that then $F(0) = 0$, F is monotone non-decreasing right-continuous and $\lim_{x\to\infty} F(x) = 1$. We denote the class of all distance functions as $\mathcal{D}$.

Based on the results of Williamson [19], we recall the next important result.

Theorem 2 (McNeil and Nešlehová [9], Corollary 3.1). *The following claims are equivalent for an arbitrary $n \in \{2,3,\ldots\}$:*

(i) $f \in \mathcal{F}_n$
(ii) *Under the notation of Theorem 1, the function $F\colon [0,\infty[\to [0,1]$ given by $F(0) = 0$ and for $x > 0$,*

$$F(x) = 1 - \sum_{k=0}^{n-2} \frac{(-1)^k x^k (f^{(-1)})^{(k)}(x)}{k!} - \frac{(-1)^{n-1} x^{n-1} (f^{(-1)})_+^{(n-1)}(x)}{(n-1)!}, \quad (2)$$

is a distance function from $\mathcal{D}$*, where* $\cdot_{+}^{(n-1)}$ *denotes the right-derivative of order* $n-1$.

Note that due to [19], if F is a positive distance function, i.e., a distribution function of a positive random variable X, then for a fixed $n \in \{2, 3, \ldots\}$ the Williamson n-transform provides an inverse transformation to (2),

$$f^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{n-1} dF(t) = \begin{cases} \max\left(0, E\left[1 - \frac{x}{X}\right]^{n-1}\right), & x > 0 \\ 1 - F(0), & x = 0, \end{cases} \tag{3}$$

where $x \in [0, \infty[$ and $f^{(-1)}(\infty) = 0$.

Note that a similar relationship can be shown between additive generators from $\mathcal{F}_\infty$ and positive distance functions, based on the Laplace transform, i.e.

$$f^{(-1)}(x) = \int_0^\infty e^{-xt} dF(t). \tag{4}$$

For more and interesting details we recommend [9].

Note that if $F_X \in \mathcal{D}$ is a distance function linked to a positive random vector X, then for any positive real constant c, also $F_{cX} \in \mathcal{D}$, and for the related additive generators (independently of $n \geq 2$), $f_{cX}(x) = cf_X(x)$. However, both f_X and f_{cX} generate the same (n-ary) Archimedean copula.

The aim of this paper is to discuss some applications of the introduced link between additive generators and distance functions in the copula theory. The paper is organised as follows. In Sect. 2, some examples are given. In Sect. 3, we introduce and discuss particular sequences of additive generators (distance functions) related to a fixed distance function (additive generator). In Sect. 4, we open several interesting problems dealing with relations between classes $\mathcal{F}_n$ and $\mathcal{F}_m$ for $n \neq m$ and between distance functions related to classes $\mathcal{F}_n$ and $\mathcal{F}_m$, respectively. Finally, some concluding remarks are given.

2 Examples

Example 1. Let F be equal to a Dirac function δ_a focused at point $a > 0$,

$$F(x) = \delta_a(x) = \begin{cases} 0 & x < a \\ 1 & a \leq x \end{cases},$$

then, as is also shown in [9], by the Williamson n-transform we get generator $f_n(x) = a\left(1 - x^{\frac{1}{n-1}}\right)$ of the weakest n-dimensional Archimedean copula, i.e., the non-strict Clayton copula with parameter $\lambda = \frac{-1}{n-1}$, see Fig. 2. By rescaling generator to $\tilde{f}_n(x) = \frac{f(x)}{f(1/2)}$, $x \in [0, 1]$, the copula would not change, yet such a generator is fixed to the value $\tilde{f}_n(\frac{1}{2}) = 1$.

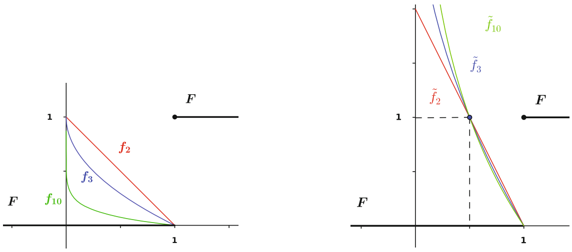

Fig. 2. Dirac function F, the corresponding generators f_n for different n and rescaled generators $\tilde{f}_n$.

Example 2. Let F be a uniform probability distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}.$$

Then for dimension $n = 2$ we get

$$f_2^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{2-1} F'(t)dt = \begin{cases} \int_x^1 \left(1 - \frac{x}{t}\right) dt & 0 \leq x < 1 \\ \int_x^\infty \left(1 - \frac{x}{t}\right) 0dt & 1 \leq x \end{cases} =$$
$$= \begin{cases} [t - x \log t]_x^1 = 1 - x + x \log(x) & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases}$$

(where F' denotes the density related to F) from which the corresponding generator can be obtained only numerically, and so is the case also with the higher dimensions, e.g.,

$$f_3^{(-1)}(x) = \begin{cases} 1 + 2x \log x - x^2 & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases}.$$

We continue with the examples of constructing generators of non-strict Archimedean copulas while restricting the support of univariate distribution in the unit interval. By applying a suitable increasing transformation (such as power function) to a positive distance function on $[0, 1]$ we obtain a new distribution.

Example 3. Consider a positive distance function $F(x) = min(1, x^2)$ and the corresponding density $F'(x) = 2x$ on $[0, 1]$. Then

$$f_2^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{2-1} dF(t) = \begin{cases} \int_x^1 (t - x) \frac{2t}{t} dt = (1 - x)^2 & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases} =$$
$$= \max(1 - x, 0)^2.$$

Then the generator $f_2(x) = 1-\sqrt{x}$, $x \in [0,1]$, is the generator of Clayton copula for parameter $\lambda = -\frac{1}{2}$. Nevertheless, in higher dimensions, $n \geq 3$, the generator has no closed form, e.g., $f_3^{(-1)}(x) = 1 - 4x + x^2(3 - 2\log x)$ for $x \in [0,1]$ and 0 otherwise (Fig. 3).

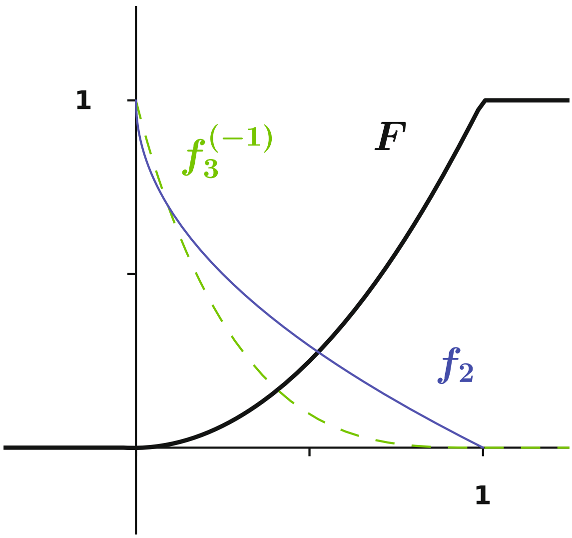

Fig. 3. Illustration of Example 3 with $a = 1$

Example 4. For any distance function $F \in \mathcal{D}$ related to a random variable X a shifted random variable $a + X$, $a \geq 0$, generates a distance function $F_a \in \mathcal{D}$ given by

$$F_a(x) = \begin{cases} 0 & x \leq a \\ F(x-a) & \text{otherwise} \end{cases}.$$

This observation allows to introduce parametric families of n-ary Archimedean copulas. Continuing in Example 2, distance functions F_a are just distribution functions of random variables uniformly distributed on $[a, a+1]$ and the related pseudo-inverses of additive generators are given by

$$f_2^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{2-1} F'(t)dt = \begin{cases} \int_a^{a+1} \left(1-\frac{x}{t}\right) dt & x < a \\ \int_x^{a+1} \left(1-\frac{x}{t}\right) dt & a \leq x < a+1 \\ \int_x^\infty \left(1-\frac{x}{t}\right) 0dt & a+1 \leq x \end{cases} =$$

$$= \begin{cases} [t - x\log t]_a^{a+1} = 1 - x\log\left(\frac{a+1}{a}\right) & x < a \\ [t - x\log t]_x^{a+1} = a + 1 - x - x\log\left(\frac{a+1}{x}\right) & a \leq x < a+1 \\ 0 & a+1 \leq x \end{cases}$$

and

$$f_3^{(-1)}(x) = \begin{cases} 1 - 2x\log\left(\frac{a+1}{a}\right) + \frac{x^2}{a(a+1)} & x < a \\ a+1-2x\log\left(\frac{a+1}{x}\right) - \frac{x^2}{a+1} & a \le x < a+1 \\ 0 & a+1 \le x \end{cases} .$$

displayed in Fig. 4.

Remark 1. Note that considering a random variable X uniformly distributed on $[a, b] \in [0, \infty[$, the random variable $Y = \frac{X}{b-a}$ is uniformly distributed on $[\frac{a}{b-a}, \frac{b}{b-a}] = [c, c+1]$ with $c = \frac{a}{b-a}$. Hence the additive generators of Archimedean copulas discussed in Example 4 covers all cases related to uniformly distributed random variables.

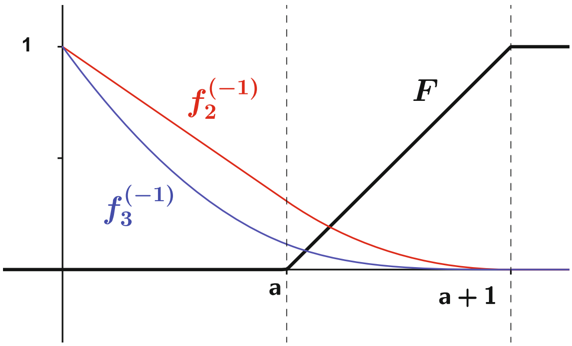

Fig. 4. Uniform $U(a, a+1)$ probability distribution function F and pseudo-inverses of the corresponding generators f_n.

Example 5. Generalizing Examples 2 and 3 such that $F(x) = \min(1, x^p)$, $p \in\,]0, \infty[$, we get

$$f_2^{(-1)}(x) = \begin{cases} 1 - \frac{px - x^p}{p-1} & 0 \le x \le 1 \\ 0 & 1 < x \end{cases} \quad \text{for } p \ne 1,$$

(with special case for $p = 1$ given in Example 2) whose corresponding generator for most values of p can be obtained only numerically, and the copulas C_p it generates span from M $(p \to 0)$ to W $(p \to \infty)$ excluding Π. Kendall's correlation coefficient as a function of parameter p can be expressed in the closed form $\tau_2(p) = 1 - 4\frac{p}{2(p+1)}$. Higher order Williamson transforms, e.g.,

$$f_3^{(-1)}(x) = \begin{cases} 1 - \frac{2x^p - p(p-1)x^2 + 2p(p-2)x}{(p-1)(p-2)} & 0 \le x \le 1 \\ 0 & 1 < x \end{cases} \quad \text{for } p \ne 1, 2,$$

(with special cases for $p = 1, 2$ given in Examples 2 and 3, respectively) neither provide convenience of generator in closed form, nor the full span of dependence range, e.g. $\tau_3(p) = 1 - 4\frac{p(p+3)}{3(p+1)(p+2)}$, see Fig. 5.

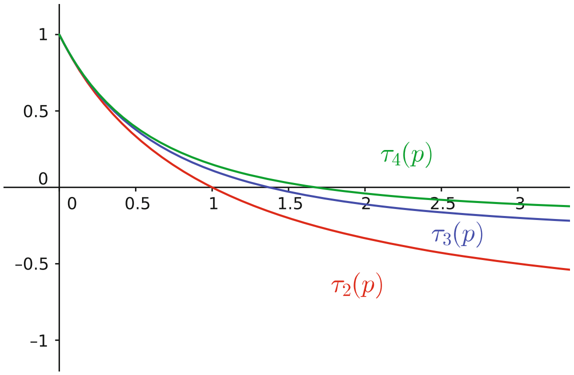

Fig. 5. Kendall's tau τ_n related to copula family generated by Williamson's transform of $F(x) = \min(1, x^p)$ distance function.

3 Williamson's Transforms and Sequences of Additive Generators/Distance Functions

Example 6. Take a generator of the product copula $f(x) = -\frac{1}{p}\log x$ with constant $p > 0$ and inverse $f^{-1}(x) = \exp(-px)$. From (2) for $n = 2$ we get $F(x) = 1 - \exp(-px)(1 - px)$. By comparing the density $\frac{\partial F(x)}{\partial x} = p^2 x \exp(-px)$ and the convolution of two exponential distribution $\mathcal{D}_p$ densities with parameter $p > 0$, $\int_0^x p\exp(-pt)p\exp(-p(x-t))dt = p^2 x\exp(-px)$ it becomes clear that the resulting distribution is a distribution of the random variable $Y = X_1 + X_2$, where $X_1, X_2 \sim \mathcal{D}_p$ are independent (and identically distributed) random variables. The relation holds for any $n \geq 2$, thus (2) yields a cumulative distribution function of the sum of i.i.d. random variables $X_1, \ldots, X_n \sim \mathcal{D}_p$, $F_{X_1+\ldots+X_n}(x) = 1 - \exp(-px)\sum_{i=1}^{n} \frac{(px)^{i-1}}{(i-1)!}$ with $p > 0$ which defines the Erlang distribution with rate parameter p and shape parameter n.

Summarizing, we see that the sequence $(F_n)_{n=1}^{\infty}$ of the Erlang distribution functions (with either fixed or variable parameter p) is related via Williamson's transforms with the product copula. Observe that when considering the Laplace transform (4), then the product copula is related to Dirac function δ_p, $p \in]0, \infty[$.

Similarly, one can consider any other Archimedean copula for which each n-ary version is an n-ary copula, i.e., possessing a universal additive generator.

Example 7. Consider the generator of the Ali-Mikhail-Haq copula $f(x) = \frac{1}{x} - 1$ corresponding to the parameter $\lambda = 1$ and denote by F_n, $n = 2, 3, \ldots$, a positive distance function related to f through (2). Then $F_n(x) = 1 - \frac{1}{1+x} - \frac{x}{(1+x)^2} - \ldots - \frac{x^{n-1}}{(1+x)^n} = \left(\frac{x}{1+x}\right)^n$ which can be viewed as a parametric subfamily of all positive valued distribution functions F_p with any positive parameter p.

Observe that when considering the Laplace transform (4), then the discussed Ali-Mikhail-Haq copula is related to the exponential distribution with the distance function $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$

On the other hand, fixing a distance function F, one can introduce related n-ary copulas (universal copula) by means of (3) (of (4)).

Example 8. Starting with positive distance function of

- discrete random variable with probability mass concentrated in $\lambda > 0$, i.e. Dirac function $F(x) = 0$ for $x < \lambda$ and 1 otherwise, then the sequence from Example 1 is completed by the Laplace transform (4) that leads through $f^{-1}(x) = \exp(\lambda x)$ to the product copula Π.
- exponential distribution $F(x) = 1 - \exp(-\lambda x)$, $\lambda > 0$, by (4) we get $f^{-1}(x) = \frac{\lambda}{x+\lambda}$ and $f(x) = \lambda\left(\frac{1}{x} - 1\right)$ which generates the same copula (Clayton and Ali-Mikhail-Haq copula, both with parameter equal to 1) regardless of the choice of λ.
- distribution from Example 5, that is $F(x) = \min(1, x^p)$, $p \in]0, \infty[$ with Kendall's tau for $n = 1, 2, 3$ shown on Fig. 5, although no explicit form of universal generator inverse can be drawn, one can observe sequence of the lower bounds for Kendall's correlation coefficient, $\{\inf[\tau_n(p)]\}_{n=2}^{\infty} = \{-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, \ldots\} = \left\{-\frac{1}{2n-3}\right\}_{n=2}^{\infty}$.

4 Some Open Problems

In Sect. 3 we have indicated some interesting consequences of the discussed links between additive generators of Archimedean copulas (of dimension $n = 2, 3, \ldots$ and universal) and distance functions via Williamson's transforms. Now we formulate some interesting arisen open problems explicitly.

Problem 1. Are there some statistical links between Archimedean copulas of dimensions n and m, $n \neq m$, related to the same distance function? Recall, for example, that fixing $F = \delta_p$ for some $p \in]0, \infty[$, the corresponding n-ary (universal) Archimedean copulas are the smallest n-ary (universal) Archimedean copulas.

Problem 2. For any fixed $n, m \geq 2$, $n \neq m$, one can define a transform $\varphi_{n,m} \colon \mathcal{D} \to \mathcal{D}$ on distance functions obtained as follows: for a fixed distance function $F \in \mathcal{D}$ one can define by means of (3) a pseudo-inverse $f_n^{(-1)}$ considering n in transform (3). Then, taking into account that $f_n^{(-1)}$ generates an

m-dimensional Archimedean copula for any $2 \leq m < n$, applying the transform (2), a new distance function $F_{n,m}$ is obtained. Now we put $\varphi_{n,m}(F) = F_{n,m}$. It is not difficult to check that if $2 \leq k < m < n$, then $\varphi_{m,k} \circ \varphi_{n,m} = \varphi_{n,k}$. Are there some interesting properties of transforms $\varphi_{n,m}$? For example, does this transform preserve the expected value, $E(F) = E(F_{n,m})$?

5 Conclusion

We have shown several interesting examples of connection between distributions of positively valued random variables (represented by distance function) and Archimedean copulas (represented by generator and it's inverse) through Williamson's transformation and Laplace transform, especially when arranged in a sequence. For instance, Williamson's n-transform ($n = 2, 3, \ldots$) links the product copula with distribution of sum of n exponentially distributed independent random variables while the Laplace transformation links it to the most elementary distance function, the Dirac function. Naturally there appears a question: how can we use statistical properties of distance functions to draw statistical properties of copulas, and vice versa? This question was itemized into two open problems, but surely more such problems could be formulated.

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