On the Construction of Associative, Commutative and Increasing Operations by Paving

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Abstract. Bodjanova, Kalina and Král' recently introduced a construction method, called paving, which enables to define a new associative, commutative and increasing operation from a given one and a discrete representable partial operation. As a matter of fact, not every discrete t-norm is representable, i.e. it can not always be generated by some additive generator, and this also holds for t-conorms and uninorms. Inspired by this fact and the method of paving, we construct some new associative, commutative and increasing operations on the unit interval from a t-norm on the unit interval and a discrete t-norm, t-superconorm, t-conorm or uninorm. Because of the duality between t-norms and t-conorms, we also define some operations from a t-conorm and a discrete t-norm, t-subnorm, t-conorm or uninorm.

Keywords: Associative operations · Uninorms · T-norms · T-conorms · Paving

1 Introduction

The associativity models the independence of the aggregation on the grouping of input values and it allows to investigate binary aggregation operators only (as far as their n-ary extensions are then determined uniquely). It is needless to emphasize the key role of associative operations (t-norms, t-conorms, uninorms, nullnorms, etc.) not only in fuzzy set theory, but also in many areas of application, especially in decision-making under uncertainty [5], image processing [1,6], fuzzy neural networks [7] and so on. The most important classes of associative, commutative, increasing operations in the framework of fuzzy sets is that of uninorms ([4,5,18]), which includes t-norms [10,17] and t-conorms [10] as two special classes. A large number of methods to construct uninorms (including t-norms and t-conorms) are introduced: Klement et al. [10], Schweizer and Sklar [17], Jenei [8], Ling [13], Maes and De Baets [11], Mas et al. [12], Mesiarová-Zemánková [14–16] and so on.

V. Torra et al. (eds.), *Aggregation Functions in Theory and in Practice*,
Advances in Intelligent Systems and Computing 581, DOI 10.1007/978-3-319-59306-7_24

Kalina et al. [2,9] introduced a construction method called paving. The main idea is as follows: the unit interval is split into countably many disjoint subintervals $(I_i)_{i\in J_n}$ with J_n an index-set and with the help of an appropriate operation $*'$ on J_n and a family of increasing transformations $\varphi_i : I_i \to [0,1]$, a new operation $\oplus$ is defined by

$$x \oplus y = \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), \quad x \in I_i,\ y \in I_j. \tag{1}$$

Unfortunately, Kalina et al. only consider discrete representable associative operations as operation $*'$, which is rather restrictive. For instance, not every discrete t-norm can be generated by some additive generator, and this applies to t-conorms and uninorms. Moreover, the operation $*'$ in [2] is not always internal on J_n. In this paper, we will consider a general discrete associative operation as operation $*'$ on J_n, to construct some new associative, commutative and increasing operations. The graphical schema of paving is depicted in Fig. 1

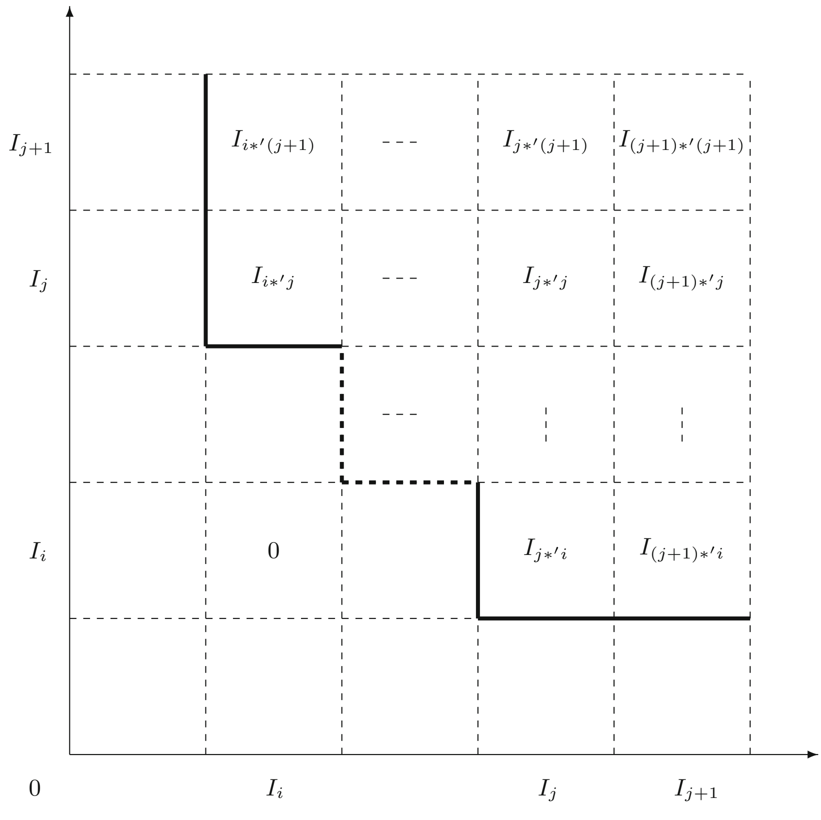

Fig. 1. The structure of $\oplus$, where the thick line is the boundary between $\{(i,j) \mid i *' j = 0\}$ and $\{(i,j) \mid i *' j > 0)\}$. Inside the blocks it is shown in which sub-interval the operation $\oplus$ takes its values.

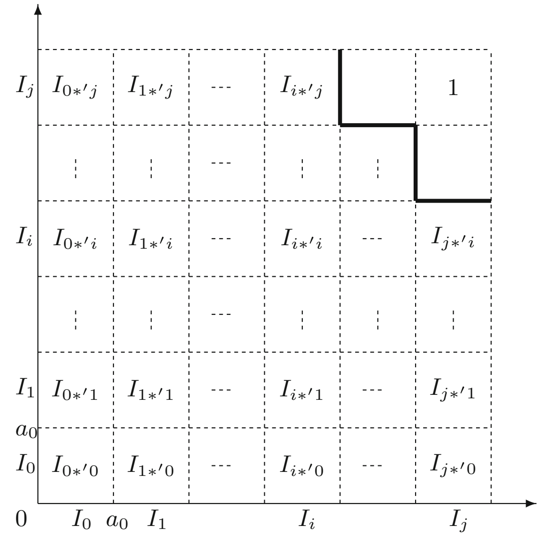

Fig. 2. The structure of $\oplus$, where the thick line is the boundary between $\{(i,j) \mid i *' j = n\}$ and $\{(i,j) \mid i *' j < n)\}$. Inside the blocks it is shown in which sub-interval the operation $\oplus$ takes its values.

(which depicts the construction of a conjunctive operation $\oplus$) and Fig. 2 (which depicts the construction of a disjunctive operation $\oplus$).

The paper is organized as follows. In Sect. 2, we present some preliminary notions and results that are necessary for the rest of the paper. Starting from Eq. (1), when $*$ is a t-norm and $*'$ is a discrete t-norm, t-superconorm, t-conorm or uninorm, we construct some new associative, commutative and increasing operations in Sect. 3. At the same time, all the dual constructions when $*$ is a t-conorm are also listed in Sect. 3.

2 Preliminaries

In this section we recall some basic notions and facts that are necessary for the understanding of what follows.

Definition 1 [10]**.** *A decreasing function $N : [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$ and $N(1) = 0$. Moreover, a fuzzy negation N is called strong if it is involutive, i.e., if $N(N(x)) = x$ for all $x \in [0,1]$.*

Definition 2 [18]**.** *A binary operation $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is associative, commutative, increasing and has a neutral element $e \in [0,1]$, i.e., $U(x,e) = x$ for all $x \in [0,1]$.*

A uninorm with neutral element $e = 1$ is a *t-norm* [10,17] and a uninorm with neutral element $e = 0$ is a *t-conorm* [10]. We say that a uninorm U is *proper* if $e \in]0,1[$. If $U(1,0) = 0$, then U is called *conjunctive*. If $U(1,0) = 1$, then U is called *disjunctive*. Conjunctive and disjunctive uninorms are dual to each other. For an arbitrary disjunctive uninorm U and a strong negation N, its N-dual conjunctive uninorm is given by

$$U_N^d(x,y) = N(U(N(x), N(y))). \tag{2}$$

For an overview of basic properties of uninorms, we refer to [3].

Remark 1. Note that, for a strong negation N, the N-dual operation to a t-norm T defined by $S(x,y) = N(T(N(x), N(y)))$ is a t-conorm. For more information, see, e.g., [10].

Definition 3 [8]**.** *(i) A binary operation $\widetilde{T} : [0,1]^2 \to [0,1]$ is called a triangular subnorm (t-subnorm, for short), if it is associative, commutative, increasing and fulfills the condition $\widetilde{T}(x,y) \le \min(x,y)$ for all $(x,y) \in [0,1]^2$.*

(ii) A binary operation $\widetilde{S} : [0,1]^2 \to [0,1]$ is called a triangular superconorm (t-superconorm, for short), if it is associative, commutative, increasing and fulfills the condition $\widetilde{S}(x,y) \ge \max(x,y)$ for all $(x,y) \in [0,1]^2$.

Definition 4. *Let $* : [0,1]^2 \to [0,1]$ be a commutative operation. Fix a value $a \in [0,1]$. We say that $x \in [0,1]$, $x \ne a$, is an a-divisor if there exists $y \in [0,1]$, $y \ne a$, such that*

$$x * y = a. \tag{3}$$

3 Construction of New Operations

The main idea of our construction method is described in Eq. (1) with the help of a discrete associative operation $*'$. For the rest of this paper, we adopt the following notations.

Let $\mathbb{N}$ be the set of all positive integers. We consider an index-set

$$J_n = \{0, 1, 2, \ldots, n\}$$

for some $n \in \mathbb{N}$.

We will split the interval $[0,1]$ into $n+1$ sub-intervals by choosing the end-points of the system of sub-intervals

$$0 = a_{-1} < a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = 1.$$

Because of this partition, we will use half-open intervals, i.e., either left-open or right-open. We will use indexing of the chosen sub-intervals in accordance with the right end-point. For the case of left-open sub-intervals, $I_i =]a_{i-1}, a_i]$; for the case of right-open sub-intervals, $I_i = [a_{i-1}, a_i[$.

For a fixed system of right-open sub-intervals $(I_i)_{i=0}^n$, $\varphi_i : I_i \to [0,1[$ are increasing bijections. For a fixed system of left-open sub-intervals $(I_i)_{i=0}^n$, $\chi_i : I_i \to]0,1]$ are increasing bijections.

Remark 2 [2]. In order not to get out of the range of the transformations χ_i when using left-open sub-intervals, the starting operation $*$ (the basic paving stone) must be without zero-divisors. Similarly, when using right-open sub-intervals, $*$ must be without one-divisors.

Here, we consider to construct new associative, commutative and increasing operations from a given one $*$, and two certain cases of associative, commutative and increasing operations will be taken into account: the case that $*$ is a t-norm and the case that $*$ is a t-conorm.

3.1 The Case that $*$ Is a T-Norm

In this subsection, we construct some new associative, commutative and increasing operations on the unit interval from a t-norm on the unit interval and a discrete t-norm/t-superconorm/t-conorm/uninorm.

Firstly, we construct a new operation $\oplus$ from a t-norm $*$ and a discrete t-norm $*'$ in Eq. (1). Because of the partition of unit interval, we distinguish two cases: when right-open sub-intervals of $[0,1[$ and left-open sub-intervals of $]0,1]$.

Proposition 1. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-norm,* $(I_i)_{i=0}^n$ *be a partition of* $[0,1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete t-norm on* $J_n = \{0,\ldots,n\}$ *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$*. Then the operation* $\oplus_1$ *defined by*

$$x \oplus_1 y = \begin{cases} \varphi_{i*'j}^{-1}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

is a t-norm.

In fact, $\oplus_1$ is not always increasing without the condition that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J,\ i *' j > 0\}$.

Example 1. Assume that $J_7 = \{0,1,2,\ldots,7\}$, $(I_i = [i/8,(i+1)/8[)_{i=0}^7$ is a partition of $[0,1[$. Let $*$ be the t-norm $T_M(x,y) = \min(x,y)$ on $[0,1]$, $*'$ be the discrete t-norm $T_M(i,j) = \min(i,j)$ on J_7, $\varphi_i(x) = \frac{x-a_{i-1}}{a_i-a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi_{\min(i,j)}^{-1}(\min(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i,\ y \in I_j, \text{ and } \min(i,j) > 0, \\ \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{3}{16}$, $y = \frac{3}{16}$ and $z = \frac{1}{4}$, then we have that

$$x \oplus y = \varphi_1^{-1}(\frac{1}{2}) = \frac{3}{16} > \frac{1}{8} = \varphi_1^{-1}(0) = x \oplus z. \tag{5}$$

That is, $\oplus$ is not increasing.

By (4), we can see that for any t-norm $*$, its values on the upper right boundary of the unit square $[0,1]^2$ have no impact on the properties of $\oplus_1$. Moreover, It is obvious that associativity, commutativity and monotonicity of $\oplus_1$ are determined by the corresponding properties of $*$, respectively. Thus, we can easily obtain that Proposition 1 holds for t-subnorm instead of t-norm.

Example 2. Assume that $J_n = \{0, 1, 2, \ldots, n\}$, $(I_i)_{i=0}^n$ is a partition of $[0, 1[$ consisting of right-open sub-intervals. Let $*$ be the t-subnorm $\widetilde{T} = \max(\min(x, \frac{1}{2}) + \min(y, \frac{1}{2}) - \frac{3}{4}, 0)$ on $[0, 1]$, $*$ be the discrete t-norm $T_L(i, j) = \max(0, i + j - n)$ on J_n, $\varphi_i(x) = \frac{x - a_{i-1}}{a_i - a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi^{-1}_{i+j-n}(\widetilde{T}(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i,\ y \in I_j \text{ and } i + j > n, \\ \min(x, y), & \text{if } \max(x, y) = 1, \\ 0, & \text{otherwise}, \end{cases} \quad (6)$$

is a t-norm.

As stated earlier, $*$ must be a t-norm without zero-divisors when left-open sub-intervals are taken into account. Similar to Proposition 1, the following proposition can be obtained:

Proposition 2. *Let $* : [0, 1]^2 \to [0, 1]$ be a t-norm without zero-divisors, $(I_i)_{i=0}^n$ be a partition of $]0, 1]$ consisting of left-open sub-intervals. Assume that $*'$ is a discrete t-norm on J_n such that $*'$ is strictly increasing on the domain $\{(i, j) \mid i, j \in J_n,\ i *' j > 0\}$. Then the operation $\oplus_2$ defined by*

$$x \oplus_2 y = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1, \\ \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i \setminus \{1\},\ y \in I_j \setminus \{1\} \text{ and } i *' j > 0, \\ 0, & \text{otherwise}, \end{cases} \quad (7)$$

is a t-norm.

Next, we discuss the construction when $*$ is a t-norm and $*'$ is a discrete t-superconorm. Analogously, two cases of right-open sub-intervals of $[0, 1[$ and left-open sub-intervals of $]0, 1]$ are taken into account. We start with the case of the right-open sub-intervals.

Proposition 3. *Let $* : [0, 1]^2 \to [0, 1]$ be a t-norm, $(I_i)_{i=0}^n$ be a partition of $[0, 1[$ consisting of right-open sub-intervals. Assume that $*'$ is a discrete t-superconorm on J_n such that $*'$ is strictly increasing and $i *' j > \max(i, j)$ on the domain $\{(i, j) \mid i, j \in J_n,\ i *' j < n\}$. Then the operation $\oplus_3$ defined by*

$$x \oplus_3 y = \begin{cases} \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ 1, & \text{otherwise}, \end{cases} \quad (8)$$

is a t-superconorm.

Without the condition that $*'$ is strictly increasing on the domain $\{(i, j) \mid i, j \in J_n,\ i *' j < n\}$, $\oplus_3$ is not always increasing. We have the following counterexample.

Example 3. Assume that $J_7 = \{0, 1, 2, \ldots, 7\}$, $(I_i = [i/8, (i+1)/8[)_{i=0}^{7}$ is a partition of $[0, 1[$. Let $*$ be the t-norm $T_M(x, y) = \min(x, y)$ on $[0, 1]$, $*'$ be the discrete t-superconorm $\widetilde{S} = \min(n, \max(i, j) + 4)$ on J_7, $\varphi_i(x) = \frac{x - a_{i-1}}{a_i - a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi^{-1}_{\widetilde{S}(i,j)}(\min(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ 1, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{1}{16}$, $y = \frac{1}{8}$ and $z = \frac{3}{16}$, then we have that

$$x \oplus z = \varphi_5^{-1}\big(\frac{1}{2}\big) = \frac{11}{16} > \frac{5}{8} = \varphi_5^{-1}(0) = y \oplus z. \tag{9}$$

Obviously, $\oplus$ is not increasing.

In Eq. (8), let $x \oplus_3 y = \max(x, y)$ on the domain $\{(x, y) \mid x, y \in [0, 1], \min(x, y) = 0\}$. We can easily prove that the operation $\oplus_3$ is a t-conorm by simple calculations.

Similarly, when left-open sub-intervals are taken into account, $*$ must be a t-norm without zero-divisors. Then, the following proposition can be obtained:

Proposition 4. *Let* $* : [0, 1]^2 \to [0, 1]$ *be a t-norm without zero-divisors,* $(I_i)_{i=0}^{n}$ *be a partition of* $]0, 1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-superconorm on* J_n *such that* $*'$ *is strictly increasing and* $i *' j > \max(i, j)$ *on the domain* $\{(i, j) \mid i, j \in J_n,\ i *' j < n\}$*. Then the operation* $\oplus_4$ *defined by*

$$x \oplus_4 y = \begin{cases} \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ \max(x, y), & \text{if } \min(x, y) = 0, \\ 1, & \text{otherwise,} \end{cases} \tag{10}$$

is a t-conorm.

In what follows, we construct a new operation from a t-norm $*$ and a discrete uninorm $*'$.

Proposition 5. *Let* $* : [0, 1]^2 \to [0, 1]$ *be a t-norm,* $(I_i)_{i=0}^{n}$ *be a partition of* $[0, 1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete uninorm on* J_n *with neutral element* h *such that* $*'$ *is strictly increasing on the domain* $\{(i, j) \mid i, j \in J_n,\ \max(i, j) \le h, i *' j > 0\}$ *and* $\{(i, j) \mid i, j \in J_n,\ \min(i, j) \ge h, i *' j < n\}$*. Then the operation* $\oplus_5$ *defined by*

$$x \oplus_5 y = \begin{cases} a_i, & \text{if } \min(x, y) < a_h \text{ and } a_h \le a_i \le \max(x, y) < a_{i+1}, \\ \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i, y \in I_j, \max(i, j) \le h \text{ and } i *' j > 0, \\ & \quad \text{or } h < \min(i, j) \text{ and } i *' j < n, \\ 1, & \text{if } x \in I_i, y \in I_j,\ h < \min(i, j) \text{ and } i *' j = n, \\ & \quad \text{or } \max(x, y) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

is associative, commutative and increasing.

In fact, the similar proposition does not hold when $(I_i)_{i=0}^n$ is a partition of $]0,1]$ consisting of left-open sub-intervals. A counterexample is as follows:

Example 4. Assume that $J_4 = \{0,1,2,3,4\}$, $(I_i =]i/5,(i+1)/5])_{i=0}^4$ is a partition of $]0,1]$. Let $*$ be the t-norm $T_M(x,y) = \min(x,y)$ on $[0,1]$, $*'$ be the discrete uninorm U with neutral element 2:

$$U(i,j) = \begin{cases} T_L(i,j), & \text{if } 0 \le i,j \le 2, \\ S_L(i,j), & \text{if } 2 \le i,j \le 4, \\ \min(i,j), & \text{otherwise,} \end{cases}$$

where $T_L(i,j) = \max(0, i+j-2)$, $S_L(i,j) = \min(4, i+j-2)$.

Besides, $\varphi_i(x) = \frac{x-a_{i-1}}{a_i-a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} a_{i+1}, & \text{if } \frac{3}{5} < \max(x,y) \text{ and } a_i < \min(x,y) \le a_{i+1} \le \frac{3}{5}, \\ \varphi_{U(i,j)}^{-1}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i, y \in I_j, \max(i,j) \le 2 \text{ and } i *' j > 0, \\ & \text{ or } 2 < \min(i,j) \text{ and } i *' j < 4, \\ 0, & \text{if } x \in I_i, y \in I_j,\ \max(i,j) \le 2 \text{ and } i *' j = 0, \\ & \text{ or } \min(x,y) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{1}{2}$, $y = \frac{1}{2}$ and $z = \frac{4}{5}$, then we have that

$$(x \oplus y) \oplus z = a_{U(2,2)} = a_2 = \frac{3}{5} \ne \frac{1}{2} = x \oplus a_2 = x \oplus (y \oplus z). \tag{12}$$

Obviously, $\oplus$ is not associative.

When $*$ is a t-norm and $*'$ is a discrete t-conorm, we can construct some proper uninorms.

Proposition 6. *Let $* : [0,1]^2 \to [0,1]$ be a t-norm, $(I_i)_{i=0}^n$ be a partition of $[0,1[$ consisting of right-open sub-intervals. Assume that $*'$ is a discrete t-conorm on J_n such that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$. Then the operation $\oplus_6$ defined by*

$$x \oplus_6 y = \begin{cases} \varphi_{i*'j}^{-1}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i \setminus \{a_0\},\ y \in I_j \setminus \{a_0\} \text{ and } i *' j < n, \\ & \text{ or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ y, & \text{if } x = a_0, \\ x, & \text{if } y = a_0, \\ 1, & \text{otherwise,} \end{cases} \tag{13}$$

is a proper disjunctive uninorm with neutral element a_0 if and only if $*$ has no zero-divisors.

In what follows, we give an example to illustrate that $*'$ must be strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$.

Example 5. Assume that $J_4 = \{0,1,2,3,4\}$, $(I_i = [i/5,(i+1)/5[)_{i=0}^{4}$ is a partition of $[0,1[$. Let $*$ be the t-norm $T_M(x,y) = \min(x,y)$ on $[0,1]$, $*'$ be the discrete t-conorm $S_M = \max(i,j)$ on J_4, $\varphi_i(x) = \frac{x-a_{i-1}}{a_i-a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi^{-1}_{\max(i,j)}(\min(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i \setminus \{\frac{1}{5}\},\ y \in I_j \setminus \{\frac{1}{5}\} \text{ and } \max(i,j) < 4, \\ & \text{or } \min(x,y) \in [0,\frac{1}{5}[,\ \max(x,y) \in [\frac{4}{5},1[, \\ y, & \text{if } x = \frac{1}{5}, \\ x, & \text{if } y = \frac{1}{5}, \\ 1, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{3}{10}$, $y = \frac{2}{5}$ and $z = \frac{1}{2}$, then we have that

$$x \oplus z = \varphi_2^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} > \frac{2}{5} = \varphi_2^{-1}(0) = y \oplus z. \tag{14}$$

Obviously, $\oplus$ is not increasing.

Similar to Proposition 6, when the left-open sub-intervals are taken into account, we have the following result:

Proposition 7. *Let $* : [0,1]^2 \to [0,1]$ be a t-norm without zero-divisors, $(I_i)_{i=0}^{n}$ be a partition of $]0,1]$ consisting of left-open sub-intervals. Assume that $*'$ is a discrete t-conorm on J_n such that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$. Then the operation $\oplus_7$ defined by*

$$x \oplus_7 y = \begin{cases} \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ & \text{or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ 0, & \text{if } \min(x,y) = 0, \\ 1, & \text{otherwise,} \end{cases} \tag{15}$$

is a proper conjunctive uninorm with neutral element a_0.

3.2 The Case that $*$ Is a T-Conorm

Taking into account the duality between t-norms and t-conorms, the results in the case that $*$ is a t-conorm are easily obtained.

Proposition 8. *Let $* : [0,1]^2 \to [0,1]$ be a t-conorm without one-divisors, $(I_i)_{i=0}^{n}$ be a partition of $[0,1[$ consisting of right-open sub-intervals. Assume that $*'$ is a discrete t-conorm on J_n such that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$. Then the operation $\oplus^1$ defined by*

$$x \oplus^1 y = \begin{cases} \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i \setminus \{0\},\ y \in I_j \setminus \{0\} \text{ and } i *' j < n, \\ \max(x,y), & \text{if } \min(x,y) = 0, \\ 1, & \text{otherwise,} \end{cases}$$

is a t-conorm.

Similar to the case that $*$ is a t-norm, Proposition 8 holds for t-superconorm instead of t-conorm.

Proposition 9. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-conorm on* J_n *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i*'j < n\}$. *Then the operation* $\oplus^2$ *defined by*

$$x \oplus^2 y = \begin{cases} \max(x,y), & \text{if } \min(x,y) = 0, \\ \chi_{i*'j}^{-1}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ 1, & \text{otherwise,} \end{cases}$$

is a t-conorm.

Proposition 10. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm without one-divisors,* $(I_i)_{i=0}^n$ *be a partition of* $[0,1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete t-subnorm on* J_n *such that* $*'$ *is strict increasing and* $i *' j < \min(i,j)$ *on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$. *Then operation* $\oplus^3$ *defined by*

$$x \oplus^3 y = \begin{cases} \varphi_{i*'j}^{-1}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is a t-norm.

Proposition 11. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-subnorm on* J_n *such that* $*'$ *is strictly increasing and* $i *' j < \min(i,j)$ *on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$. *Then the operation* $\oplus^4$ *defined by*

$$x \oplus^4 y = \begin{cases} \chi_{i*'j}^{-1}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a t-subnorm.

Proposition 12. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm without one-divisors,* $(I_i)_{i=0}^n$ *be a partition of* $[0,1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete t-norm on* J_n *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$. *Then the operation* $\oplus^5$ *defined by*

$$x \oplus^5 y = \begin{cases} \varphi_{i*'j}^{-1}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ & \quad \text{or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ 1, & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is a proper disjunctive uninorm with neutral element a_{n-1}.

Proposition 13. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete uninorm on* J_n *with neutral element* h *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ \max(i,j) \le h, i *' j > 0\}$ *and* $\{(i,j) \mid i,j \in J_n,\ \min(i,j) \ge h, i *' j < n\}$*. Then the operation* $\oplus^6$ *defined by*

$$x \oplus^6 y = \begin{cases} a_{i+1}, & \text{if } \max(x,y) > a_{h-1} \text{ and } a_i < \min(x,y) \le a_{i+1} \le a_{h-1}, \\ \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i, y \in I_j, \max(i,j) \le h-1 \text{ and } i *' j > 0, \\ & \ \text{or } h-1 < \min(i,j) \text{ and } i *' j < n, \\ 0, & \text{if } x \in I_i, y \in I_j, \max(i,j) \le h-1 \text{ and } i *' j = 0, \\ & \ \text{or } \min(x,y) = 0, \\ 1, & \text{otherwise}, \end{cases}$$

is associative, commutative and increasing.

Proposition 14. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-norm on* J_n *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$*. Then the operation* $\oplus^7$ *defined by*

$$x \oplus^7 y = \begin{cases} \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i \setminus \{a_{n-1}\},\ y \in I_j \setminus \{a_{n-1}\} \text{ and } i *' j > 0, \\ & \text{or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ y, & \text{if } x = a_{n-1}, \\ x, & \text{if } y = a_{n-1}, \\ 0, & \text{otherwise}, \end{cases}$$

is a proper conjunctive uninorm with neutral element a_{n-1} *if and only if* $*$ *has no one-divisors.*

Results

Inspired by the construction method of paving, we construct some new associative, commutative and increasing operations on the unit interval from a t-norm on the unit interval and a discrete t-norm/t-superconorm/t-conorm/uninorm. Similarly, we present the dual constructions from a t-conorm and a discrete t-norm/t-subnorm/t-conorm/uninorm.

Acknowledgements. This work is supported by the National Natural Science Foundation of China No. 61573211. The author Wenwen Zong is supported by the China Scholarship Council under Grant No. 201606220121. The author Yong Su is supported by the China Scholarship Council under Grant No. 201506220039.

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