Chapter 8
Robinson's Theorem

In trying to decide which orthogonal designs to look for, it would be useful to formulate, and hopefully prove valid, some general principles of the sort, "All orthogonal designs of a certain type exist in certain orders." The Hadamard conjecture, the skew-Hadamard conjecture, the weighing matrix conjecture, and other conjectures that have been made, and extensively verified, provide some solid information which must be dealt with in order to state such principles. We have seen singularly unsuccessful in formulating correct principles of a general nature; some conjectures that we have made in the light of those principles have proved to be false.

One principle we had bandied about for awhile was: "In order n, if k is much smaller than $\rho(n)$, then all orthogonal designs on k variables exist in order n." Of course, the key to focusing on this principle is to decide what "much smaller" should mean.

In orders n where n is odd or $n = 2t$, t odd, algebraic conditions appear immediately in deciding if orthogonal designs exist, and so, in terms of deciding what "much smaller" should mean, we put those cases aside. When $n = 4t$, t odd, the situation is different. The algebraic theory says nothing about one-variable designs, i.e., weighing matrices. This fact, coupled with a fair bit of evidence for the weighing matrix conjecture in orders $4t$, t odd, led us to formulate a "sub-principle" for the phase "much smaller": to wit, we proposed the following: "If, in order n, the algebraic theory imposes *no* restrictions on any possible k-variable design in order n, than all k-variable designs exist in order n."

If this principle was a sound one, it would say, for example, that whenever $n = 16t$, $t \geq 1$, any orthogonal design on ≤ 7 variables exists. (See Proposition 3.34 and what follows it.) The principle, unfortunately (depending on your point of view), is far from correct. Peter J. Robinson decisively settled that issue and many other alternative ones with the following remarkable theorem.

Using the orthogonal designs $AOD(24;1,1,1,1,1,2,17)$ from Lemma A.7, $OD(32;1,1,1,1,1,12,15)$, $OD(32;1,1,1,1,1,9,9,9)$ and $OD(40;1,1,1,1,1,35)$

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found by Kharaghani and Tayfeh-Rezaie [122] given in Tables 8.1, 8.2 and 8.3 respectively we have using theorem 8.2

Theorem 8.1 (Robinson). *An* $OD(n;1,1,1,1,1,n-5)$ *exists if and only if* $n = 8$*,* 16*,* 24*,* 32*,* 40*.*

Theorem 8.2 (Robinson). *If* $n > 40$*, there is no orthogonal design of type* $(1,1,1,1,1,n-5)$ *in order* n*.*

We first note that:

$OD(8;1,1,1,1,1,3)$	See: Section 4.2
$OD(16;1,1,1,1,1,11)$	Appendix F.2
$OD(24;1,1,1,1,1,19)$	Table 8.1
$OD(32;1,1,1,1,1,27)$	Table 8.2
$OD(40;1,1,1,1,1,35)$	Table 8.3

do exist.

Proof. The proof is a very careful analysis of what such a design would have to look like, and we have expanded Robinson's proof so as to make the verification a bit easier for the reader. □

With no loss of generality we may assume the first 4×4 diagonal block of the orthogonal design is

$$\begin{bmatrix} x_1 & x_2 & x_3 & a_1 \\ -x_2 & x_1 & a_1 & -x_3 \\ -x_3 & -a_1 & x_1 & x_2 \\ -a_1 & x_3 & -x_2 & x_1 \end{bmatrix}$$

Either $a_1 = \pm x_4$, or not.

If $a_1 = \pm x_4$, we proceed to obtain the following 8×8 diagonal block:

$$\begin{bmatrix} x_1 & x_2 & x_3 & a_1 & x_5 & & & \\ -x_2 & x_1 & a_1 & -x_3 & & -x_5 & & * \\ -x_3 & -a_1 & x_1 & x_2 & & & -x_5 & \\ -a_1 & x_3 & -x_2 & x_1 & & * & & x_5 \\ -x_5 & & & & x_1 & x_2 & x_3 & b_1 \\ & x_5 & * & & -x_2 & x_1 & b_1 & -x_3 \\ * & & x_5 & & -x_3 & -b_1 & x_1 & x_2 \\ & & & -x_5 & -b_1 & x_3 & -x_2 & x_1 \end{bmatrix}, \tag{8.1}$$

and $b_1 = -a_1$.

In case $a_1 \neq \pm x_4$, we may proceed to make the following 8×8 diagonal block.

$$\begin{bmatrix} x_1 & x_2 & x_3 & a_1 & x_4 & & & \\ -x_2 & x_1 & a_1 & & & -x_4 & \ast & \\ -x_3 & -a_1 & x_1 & x_2 & \ast & & -x_4 & \\ -a_1 & x_3 & -x_2 & x_1 & & & & x_4 \\ -x_4 & & & & x_1 & x_2 & x_3 & b_1 \\ & x_4 & \ast & & -x_2 & x_1 & b_1 & -x_3 \\ & \ast & x_4 & & -x_3 & -b_1 & x_1 & x_2 \\ & & & -x_4 & -b_1 & x_3 & -x_2 & x_1 \end{bmatrix}, \tag{8.2}$$

and $b_1 = -a_1$.

We continue the process and end up with 8×8 blocks of type (8.1) or (8.2) on the diagonal.

Claim. There is at most one diagonal block of type (8.1).

Proof of Claim. Let X denote the orthogonal design of type $(1,1,1,1,1,n-5)$ in order n, and write $X = A_1x_1 + A_2x_2 + A_3x_3 + \cdots + A_6x_6$. As we have already seen, we may assume

$$A_1 = I_n, A_2 = \bigoplus_{n/2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$A_3 = \bigoplus_{\frac{n}{4}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Since X is an orthogonal design, we must have $A_iA_j^\top + A_jA_i^\top = 0$, $1 \le i \ne j \le 6$. The patient reader will then discover than these conditions force A_4, A_5, A_6 to each be skew-symmetric and partitioned into 4×4 blocks, where if $(p\,q\,r\,s)$ is the first row of a block, then the block looks like:

$$\begin{bmatrix} p & q & r & s \\ q & -p & s & -r \\ r & -s & -p & q \\ -s & -r & q & p \end{bmatrix}.$$

Now, assume that there are two 8×8 diagonal blocks like (8.1), with $a_1 = \pm x_4, b_1 = -a_1, a_2 = \pm x_4, b_2 = -a_2$; we shall obtain a contradiction (we shall first consider the case $a_1 = a_2 = x_4$ and leave the remaining three possibilities for the reader to check). The contradiction shall be obtained by looking at the 8×8 off-diagonal blocks which are at the juncture of the two diagonal positions. See Figure 8.1.

If we number the rows of figure 8.1 from 1 to 16, then we obtain (by taking the inner product of rows 1 and 9) a summand $2sx_4$ which we cannot eliminate since x_4 appears only once in each column and row.

(Note: We have used here the fact that $s \ne 0$, but we would get the same result if *any* of p, q, r, or s were $\ne 0$ or if any of k, l, m, v were $\ne 0$. Thus, so far,

Fig. 8.1 Contradiction of off-diagonal blocks at juncture of two diagonal positions

$$
\begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & & & & \\
-x_2 & x_1 & x_4 & -x_3 & & & & \\
-x_3 & -x_4 & x_1 & x_2 & & * & & \\
-x_4 & x_3 & -x_2 & x_1 & & & & \\
 & & & & x_1 & x_2 & x_3 & -x_4 \\
 & & & & -x_2 & x_1 & -x_4 & -x_3 \\
 & * & & & -x_3 & x_4 & x_1 & x_2 \\
 & & & & x_4 & x_3 & -x_3 & x_1
\end{bmatrix}
\leftrightsquigarrow
\begin{bmatrix}
p & q & r & s & b & c & d & e \\
q & -p & s & -r & c & -b & e & d \\
r & -s & -p & q & d & -e & -b & c \\
-s & -r & q & p & -e & -d & c & b \\
f & g & h & j & k & \ell & m & v \\
g & -f & j & -h & \ell & -k & v & -m \\
h & -j & -f & g & m & -v & -k & \ell \\
-j & -h & g & f & -v & -m & \ell & k
\end{bmatrix}
$$

$$\updownarrow$$

$$
\begin{bmatrix}
-p & -q & -r & s & -f & -g & -h & j \\
-q & p & s & r & -g & f & j & h \\
-r & -s & p & -q & -h & -j & f & -g \\
-s & r & -q & p & -j & h & -g & -f \\
-b & -c & -d & e & -k & -\ell & -m & v \\
-c & b & e & d & -\ell & k & v & m \\
-d & -e & b & -c & -m & -v & k & -\ell \\
-e & d & -c & -b & -v & m & -\ell & -k
\end{bmatrix}
\leftrightsquigarrow
\begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & & & & \\
-x_2 & x_1 & x_4 & -x_3 & & & & \\
-x_3 & -x_4 & x_1 & x_2 & & * & & \\
-x_4 & x_3 & -x_2 & x_1 & & & & \\
 & & & & x_1 & x_2 & x_3 & -x_4 \\
 & & & & -x_2 & x_1 & -x_4 & -x_3 \\
 & * & & & -x_3 & x_4 & x_1 & x_2 \\
 & & & & x_4 & x_3 & -x_3 & x_1
\end{bmatrix}
$$

we could be discussing orthogonal designs of type $(1,1,1,1,1,n-k)$, $5 \le k \le 8$, and the conclusions that are drawn would still hold; i.e., no two diagonal blocks of type (8.1).)

We have now seen that there can only be one diagonal block of type (8.1) and that all other diagonal blocks are of type (8.2).

We now seek to discover where in the orthogonal design the x_5's are located. If there is a diagonal block of type (8.1), then we know where the x_5's in the rows and columns controlled by that diagonal block are, namely, in the diagonal block.

Claim. The x_5's are always in the diagonal blocks.

Proof. To prove this, it would be enough to show that there is no x_5 in an off-diagonal block which is above **and** across from a diagonal block of type Equation (8.2). Thus, we have the following in Figure 8.2.

Now, by checking inner products (just using x_4's), we find that

$$
\begin{aligned}
bx_4 - fx_4 &= 0, \quad \text{i.e., } b = f; \\
-cx_4 - gx_4 &= 0, \quad \text{i.e., } c = -g; \\
-dx_4 - hx_4 &= 0, \quad \text{i.e., } d = -h; \\
ex_4 - jx_4 &= 0, \quad \text{i.e., } e = j.
\end{aligned}
$$

Similarly,

Fig. 8.2 No x_5 in off-diagonal block above and across from a diagonal block of type Equation (8.2)

$$\begin{bmatrix} x_1 & x_2 & x_3 & \alpha & x_4 & & & \\ -x_2 & x_1 & \alpha & -x_3 & & -x_4 & & \\ -x_3 & -\alpha & x_1 & x_2 & & & -x_4 & \\ -\alpha & x_3 & -x_2 & x_1 & & & & x_4 \\ -x_4 & & & & x_1 & x_2 & x_3 & -\alpha \\ & x_4 & & & -x_2 & x_1 & -\alpha & -x_3 \\ & & x_4 & & -x_3 & \alpha & x_1 & x_2 \\ & & & x_4 & \alpha & x_3 & -x_2 & x_1 \end{bmatrix} \leftrightsquigarrow \begin{bmatrix} p & q & r & s & b & c & d & e \\ q & -p & s & -r & c & -b & e & -d \\ r & -s & -p & q & d & -e & -b & c \\ -s & -r & q & p & -e & -d & c & b \\ f & g & h & j & k & \ell & m & v \\ g & -f & j & -h & \ell & -k & v & -m \\ h & -j & -f & g & m & -v & -k & \ell \\ -j & -h & g & f & -v & -m & \ell & k \end{bmatrix} = Y$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$-Y^\top = \begin{bmatrix} -p & -q & -r & s & -f & -g & -h & j \\ -q & p & s & r & -g & f & j & h \\ -r & -s & p & -q & -h & -j & f & -g \\ -s & r & -q & p & -j & h & -g & -f \\ -b & -c & -d & e & -k & -\ell & -m & v \\ -c & b & e & d & -\ell & k & v & m \\ -d & -e & b & -c & -m & -v & k & -\ell \\ -e & d & -c & -b & -v & m & -\ell & -k \end{bmatrix} \leftrightsquigarrow \begin{bmatrix} x_1 & x_2 & x_3 & \beta & x_4 & & & \\ -x_2 & x_1 & \beta & -x_3 & & x_4 & & \\ -x_3 & -\beta & x_1 & x_2 & & & x_4 & \\ -\beta & x_3 & -x_2 & x_1 & & & & x_4 \\ -x_4 & & & & x_1 & x_2 & x_3 & -\beta \\ & x_4 & & & -x_2 & x_1 & \beta & -x_3 \\ & & x_4 & & -x_3 & \beta & x_1 & x_2 \\ & & & -x_4 & \beta & x_3 & -x_2 & x_1 \end{bmatrix}$$

$$\begin{aligned} k &= -p, \\ \ell &= q, \\ m &= r, \\ v &= -s; \end{aligned}$$

i.e., we have

$$Y = \left[\begin{array}{cccc|cccc} p & q & r & s & b & c & d & e \\ & \text{etc.} & & & & \text{etc.} & & \\ \hline & \text{etc.} & & & & \text{etc.} & & \\ -e & d & -c & b & s & -r & q & -p \end{array}\right]$$

If we consider the inner product between the two rows and recall that none of p, q, r, s, b, c, d, or $e = 0$, we find that $\pm x_5 \notin \{p, q, r, s, b, c, d, e\}$.

Now, the i^{th} diagonal block of type (8.2) has four (as yet) undetermined entries, and we have seen that one of them must be $\pm x_5$. If there are four (or more) blocks of type (8.2), then in two of them x_5 (up to sign) must occupy the same position. We shall assume that occurs in the i^{th} and j^{th} diagonal blocks and write them (along with the $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ off-diagonal blocks) in Figure 8.3.

Now, suppose β_1 and β_2 are each $\pm x_5$. Considering rows 1 and 9 of Figure 8.3 we find $\beta_1 c + \beta_2 c = 0$, i.e., $\beta_1 = -\beta_2$, but considering rows 1 and 11 we find $-\beta_1 e + \beta_2 e = 0$, i.e., $\beta_1 = \beta_2$. This contradiction establishes that β_1, β_2 cannot

Fig. 8.3 Four (or more) blocks of type (8.2), x_5 must occupy the same position in two of them

$$
\begin{bmatrix}
x_1 & x_2 & x_3 & \alpha_1 & x_4 & \beta_1 & \gamma_1 & \delta_1 \\
-x_2 & x_1 & \alpha_1 & -x_3 & \beta_1 & -x_4 & \delta_1 & -\gamma_1 \\
-x_3 & -\alpha_1 & x_1 & x_2 & \gamma_1 & -\delta_1 & -x_4 & \beta_1 \\
-\alpha_1 & x_3 & -x_2 & x_1 & -\delta_1 & -\gamma_1 & \beta_1 & x_4 \\
-x_4 & -\beta_1 & -\gamma_1 & \delta_1 & x_1 & x_2 & x_3 & -\alpha_1 \\
-\beta_1 & x_4 & \delta_1 & \gamma_1 & -x_2 & x_1 & -\alpha_1 & -x_3 \\
-\gamma_1 & -\delta_1 & x_4 & -\beta_1 & -x_3 & \alpha_1 & x_1 & x_2 \\
-\delta_1 & \gamma_1 & -\beta_1 & -x_4 & \alpha_1 & x_3 & -x_2 & x_1
\end{bmatrix}
\leftrightsquigarrow
\begin{bmatrix}
p & q & r & s & b & c & d & e \\
q & -p & s & -r & c & -b & e & -d \\
r & -s & -p & q & d & -e & -b & c \\
-s & -r & q & p & -e & -d & c & b \\
b & -c & -d & e & -p & q & r & -s \\
-c & -b & e & d & q & p & -s & -r \\
-d & -e & -b & -c & r & s & p & q \\
-e & d & -c & b & s & -r & q & -p
\end{bmatrix}
$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$
\begin{bmatrix}
-p & -q & -r & s & -b & c & d & e \\
-q & p & s & r & c & b & e & -d \\
-r & -s & p & -q & d & -e & b & c \\
-s & r & -q & p & -e & -d & c & -b \\
-b & -c & -d & e & p & -q & -r & -s \\
-c & b & e & d & -q & -p & -s & r \\
-d & -e & b & -c & -r & s & -p & -q \\
-e & d & -c & -b & s & r & -q & p
\end{bmatrix}
\leftrightsquigarrow
\begin{bmatrix}
x_1 & x_2 & x_3 & \alpha_2 & x_4 & \beta_2 & \gamma_2 & \delta_2 \\
-x_2 & x_1 & \alpha_2 & -x_3 & \beta_2 & -x_4 & \delta_2 & -\gamma_2 \\
-x_3 & -\alpha_2 & x_1 & x_2 & \gamma_2 & -\delta_2 & -x_4 & \beta_2 \\
-\alpha_2 & x_3 & -x_2 & x_1 & -\delta_2 & -\gamma_2 & \beta_2 & x_4 \\
-x_4 & -\beta_2 & -\gamma_2 & \delta_2 & x_1 & x_2 & x_3 & -\alpha_2 \\
-\beta_2 & x_4 & \delta_2 & \gamma_2 & -x_2 & x_1 & -\alpha_2 & -x_3 \\
-\gamma_2 & -\delta_2 & x_4 & -\beta_2 & -x_3 & \alpha_2 & x_1 & x_2 \\
-\delta_2 & \gamma_2 & -\beta_2 & -x_4 & \alpha_2 & x_3 & -x_2 & x_1
\end{bmatrix}
$$

both have absolute value x_5. A similar argument gives the same conclusion for Y_1 and Y_2, Δ_1 and Δ_2, and α_1 and α_2. This then completes the proof. □

It is possible to use Robinson's Theorem to obtain many other non-existence results. We give just two illustrations.

Corollary 8.1. *There do not exist amicable orthogonal designs of type* $AOD((1,1,m-2);(1,m-1))$ *in any order* $m > 10$.

Proof. If there did, we could use Theorem 6.1 (i.e., a product design of type $PD(4:1,1,1,;1,1,1;1))$ to obtain an orthogonal design in order $4m > 40$ of type $(1,1,m-2,1,1,1,m-1,m-1,m-1)$, contradicting Theorem 8.2. □

Note. This shows how difficult it is to obtain "full" amicable orthogonal designs which have several 1's in their types.

Corollary 8.2. *There is no product design of type* $PD(n;1,1,n-3;1,n-2;1)$ *in any order* $n > 20$.

Proof. If there were, Construction 6.1 would contradict Robinson's Theorem. □

Remark 8.1. This corollary shows how special the product designs constructed in Examples 6.2, 6.3 and Theorem 6.1 really are.

Table 8.1 An $OD(32;1,1,1,1,1,12,15)$ [a]

$abcsdts\bar{e}$	$\bar{s}s\bar{t}\bar{t}\bar{s}stt$	$ssss\bar{t}t\bar{t}t$	$\bar{t}\bar{s}\bar{t}\bar{t}\bar{t}st\bar{t}$
$\bar{b}a\bar{s}ct\bar{d}es$	$sst\bar{t}ss\bar{t}t$	$s\bar{s}\bar{s}stt\bar{t}\bar{t}$	$\bar{s}tt\bar{t}sttt$
$\bar{c}sa\bar{b}s\bar{e}\bar{d}\bar{t}$	$\bar{t}\bar{t}s\bar{s}tts\bar{s}$	$ss\bar{s}\bar{s}\bar{t}tt\bar{t}$	$\bar{t}\bar{t}tst\bar{t}t\bar{s}$
$\bar{s}\bar{c}baes\bar{t}d$	$t\bar{t}\bar{s}\bar{s}\bar{t}t\bar{s}\bar{s}$	$\bar{s}s\bar{s}s\bar{t}\bar{t}\bar{t}\bar{t}$	$t\bar{t}s\bar{t}tt\bar{s}\bar{t}$
$\bar{d}\bar{t}\bar{s}\bar{e}abc\bar{s}$	$\bar{s}\bar{s}\bar{t}tss\bar{t}t$	$\bar{t}\bar{t}tt\bar{s}ss\bar{s}$	$t\bar{t}s\bar{t}\bar{t}\bar{t}st$
$\bar{t}de\bar{s}\bar{b}asc$	$\bar{s}s\bar{t}\bar{t}s\bar{s}\bar{t}\bar{t}$	$\bar{t}t\bar{t}tssss$	$\bar{t}\bar{t}ts\bar{t}t\bar{t}s$
$\bar{s}\bar{e}dt\bar{c}\bar{s}a\bar{b}$	$\bar{t}tss\bar{t}t\bar{s}\bar{s}$	$tttts\bar{s}s\bar{s}$	$s\bar{t}\bar{t}tsttt$
$e\bar{s}t\bar{d}s\bar{c}ba$	$\bar{t}\bar{t}s\bar{s}\bar{t}\bar{t}\bar{s}s$	$\bar{t}tt\bar{t}ss\bar{s}\bar{s}$	$tstt\bar{t}st\bar{t}$
$s\bar{s}t\bar{t}sstt$	$abc\bar{s}d\bar{s}e\bar{t}$	$\bar{t}\bar{t}t\bar{t}\bar{t}\bar{s}s\bar{t}$	$\bar{t}\bar{s}\bar{t}\bar{s}stst$
$\bar{s}\bar{s}tts\bar{s}\bar{t}t$	$\bar{b}asc\bar{s}\bar{d}te$	$\bar{t}ttt\bar{s}tts$	$\bar{s}ts\bar{t}t\bar{s}\bar{t}s$
$t\bar{t}\bar{s}stt\bar{s}\bar{s}$	$\bar{c}\bar{s}a\bar{b}e\bar{t}\bar{d}s$	$t\bar{t}tts\bar{t}ts$	$\bar{t}\bar{s}tsst\bar{s}\bar{t}$
$ttss\bar{t}t\bar{s}s$	$s\bar{c}batesd$	$ttt\bar{t}tss\bar{t}$	$s\bar{t}s\bar{t}\bar{t}s\bar{t}s$
$s\bar{s}\bar{t}t\bar{s}\bar{s}tt$	$\bar{d}s\bar{e}\bar{t}abcs$	$\bar{t}s\bar{s}\bar{t}t\bar{t}tt$	$s\bar{t}s\bar{t}t\bar{s}t\bar{s}$
$\bar{s}\bar{s}\bar{t}\bar{t}\bar{s}s\bar{t}t$	$sdt\bar{e}\bar{b}a\bar{s}c$	$stt\bar{s}\bar{t}\bar{t}\bar{t}t$	$\bar{t}\bar{s}ts\bar{s}\bar{t}st$
$\bar{t}t\bar{s}sttss$	$\bar{e}\bar{t}d\bar{s}\bar{c}sa\bar{b}$	$\bar{s}\bar{t}t\bar{s}tt\bar{t}t$	$s\bar{t}\bar{s}tt\bar{s}\bar{t}s$
$\bar{t}\bar{t}ss\bar{t}ts\bar{s}$	$t\bar{e}\bar{s}\bar{d}\bar{s}\bar{c}ba$	$t\bar{s}\bar{s}\bar{t}\bar{t}ttt$	$tstsstst$
$\bar{s}\bar{s}\bar{s}stt\bar{t}t$	$tt\bar{t}\bar{t}t\bar{s}s\bar{t}$	$abcsdes\bar{t}$	$ts\bar{s}t\bar{s}\bar{t}t\bar{s}$
$\bar{s}s\bar{s}\bar{s}t\bar{t}\bar{t}\bar{t}$	$t\bar{t}t\bar{t}\bar{s}tts$	$\bar{b}a\bar{s}ce\bar{d}ts$	$s\bar{t}\bar{t}\bar{s}\bar{t}sst$
$\bar{s}sss\bar{t}t\bar{t}\bar{t}$	$\bar{t}\bar{t}\bar{t}\bar{t}s\bar{t}\bar{t}s$	$\bar{c}sa\bar{b}s\bar{t}\bar{d}\bar{e}$	$\bar{s}t\bar{t}\bar{s}t\bar{s}st$
$\bar{s}\bar{s}s\bar{s}\bar{t}\bar{t}\bar{t}t$	$t\bar{t}\bar{t}ttsst$	$\bar{s}\bar{c}bats\bar{e}d$	$\bar{t}\bar{s}\bar{s}tstt\bar{s}$
$t\bar{t}tts\bar{s}\bar{s}\bar{s}$	$ts\bar{s}\bar{t}\bar{t}t\bar{t}t$	$\bar{d}\bar{e}\bar{s}\bar{t}abc\bar{s}$	$\bar{t}\bar{s}\bar{s}t\bar{s}\bar{t}\bar{t}s$
$\bar{t}\bar{t}\bar{t}t\bar{s}\bar{s}s\bar{s}$	$s\bar{t}t\bar{s}tt\bar{t}\bar{t}$	$\bar{e}dt\bar{s}\bar{b}asc$	$\bar{s}t\bar{t}\bar{s}\bar{t}s\bar{s}\bar{t}$
$tt\bar{t}t\bar{s}\bar{s}\bar{s}s$	$\bar{s}\bar{t}\bar{t}\bar{s}\bar{t}tt\bar{t}$	$\bar{s}\bar{t}de\bar{c}\bar{s}a\bar{b}$	$\bar{s}tts\bar{t}sst$
$\bar{t}ttts\bar{s}ss$	$t\bar{s}\bar{s}t\bar{t}\bar{t}\bar{t}\bar{t}$	$t\bar{s}e\bar{d}s\bar{c}ba$	$\bar{t}\bar{s}s\bar{t}\bar{s}\bar{t}t\bar{s}$
$tst\bar{t}\bar{t}t\bar{s}\bar{t}$	$tst\bar{s}\bar{s}t\bar{s}\bar{t}$	$\bar{t}\bar{s}sttsst$	$abc\bar{d}e\bar{s}t\bar{s}$
$s\bar{t}ttttt\bar{s}$	$s\bar{t}sttst\bar{s}$	$\bar{s}t\bar{t}ss\bar{t}\bar{t}s$	$\bar{b}adc\bar{s}\bar{e}st$
$t\bar{t}\bar{t}\bar{s}\bar{s}\bar{t}t\bar{t}$	$t\bar{s}\bar{t}\bar{s}\bar{s}\bar{t}s\bar{t}$	$sttsst\bar{t}\bar{s}$	$\bar{c}\bar{d}a\bar{b}t\bar{s}\bar{e}s$
$tt\bar{s}tt\bar{s}\bar{t}\bar{t}$	$st\bar{s}tt\bar{s}\bar{t}\bar{s}$	$\bar{t}ss\bar{t}\bar{t}s\bar{s}t$	$d\bar{c}bastse$
$t\bar{s}\bar{t}\bar{t}tt\bar{s}t$	$\bar{s}\bar{t}\bar{s}t\bar{t}s\bar{t}\bar{s}$	$st\bar{t}\bar{s}stts$	$\bar{e}s\bar{t}\bar{s}abcd$
$\bar{s}\bar{t}t\bar{t}t\bar{t}\bar{t}\bar{s}$	$\bar{t}s\bar{t}\bar{s}sts\bar{t}$	$t\bar{s}s\bar{t}t\bar{s}\bar{s}t$	$ses\bar{t}\bar{b}a\bar{d}c$
$\bar{t}\bar{t}\bar{t}s\bar{s}t\bar{t}\bar{t}$	$\bar{s}tst\bar{t}\bar{s}t\bar{s}$	$\bar{t}\bar{s}s\bar{t}ts\bar{s}\bar{t}$	$\bar{t}\bar{s}e\bar{s}\bar{c}da\bar{b}$
$t\bar{t}st\bar{t}\bar{s}\bar{t}t$	$\bar{t}\bar{s}t\bar{s}s\bar{t}\bar{s}\bar{t}$	$s\bar{t}\bar{t}s\bar{s}t\bar{t}s$	$s\bar{t}\bar{s}\bar{e}\bar{d}\bar{c}ba$

[a] Kharaghani and Tayfeh-Rezaie [122, p317-324]

Table 8.2 An $OD(32;1,1,1,1,1,9,9,9)$ [a]

$$\begin{bmatrix}
abcsdut\bar{e} & \bar{s}s\bar{s}\bar{s}\bar{t}ttt & utut\bar{t}u\bar{t}u & \bar{s}\bar{u}\bar{u}\bar{s}\bar{s}uu\bar{s} \\
\bar{b}a\bar{s}cu\bar{d}et & sss\bar{s}tt\bar{t}t & t\bar{u}\bar{t}uut\bar{u}\bar{t} & \bar{u}ss\bar{u}ussu \\
\bar{c}sa\bar{b}t\bar{e}\bar{d}\bar{u} & \bar{s}\bar{s}s\bar{s}ttt\bar{t} & ut\bar{u}\bar{t}\bar{t}ut\bar{u} & \bar{u}\bar{s}suu\bar{s}s\bar{u} \\
\bar{s}\bar{c}baet\bar{u}d & s\bar{s}\bar{s}\bar{s}\bar{t}t\bar{t}\bar{t} & \bar{t}u\bar{t}u\bar{u}\bar{t}\bar{u}\bar{t} & s\bar{u}u\bar{s}su\bar{u}\bar{s} \\
\bar{d}\bar{u}\bar{t}\bar{e}abc\bar{s} & \bar{t}\bar{t}\bar{t}tss\bar{s}s & \bar{t}\bar{u}tu\bar{u}tu\bar{t} & s\bar{u}u\bar{s}\bar{s}\bar{u}us \\
\bar{u}de\bar{t}\bar{b}asc & \bar{t}t\bar{t}\bar{t}s\bar{s}\bar{s}\bar{s} & \bar{u}t\bar{u}ttutu & \bar{u}\bar{s}su\bar{u}s\bar{s}u \\
\bar{t}\bar{e}du\bar{c}\bar{s}a\bar{b} & \bar{t}ttt\bar{s}s\bar{s}\bar{s} & tutuu\bar{t}u\bar{t} & u\bar{s}\bar{s}uussu \\
e\bar{t}u\bar{d}s\bar{c}ba & \bar{t}\bar{t}t\bar{t}\bar{s}\bar{s}\bar{s}s & \bar{u}tu\bar{t}tu\bar{t}\bar{u} & suus\bar{s}uu\bar{s} \\
s\bar{s}s\bar{s}tttt & abc\bar{s}d\bar{t}e\bar{u} & \bar{u}\bar{s}s\bar{u}\bar{s}\bar{u}u\bar{s} & \bar{u}\bar{u}\bar{t}\bar{t}uutt \\
\bar{s}\bar{s}sst\bar{t}\bar{t}t & \bar{b}asc\bar{t}\bar{d}ue & \bar{s}uus\bar{u}ssu & \bar{u}ut\bar{t}u\bar{u}\bar{t}t \\
s\bar{s}\bar{s}stt\bar{t}\bar{t} & \bar{c}\bar{s}a\bar{b}e\bar{u}\bar{d}t & s\bar{u}usu\bar{s}su & \bar{t}\bar{t}uutt\bar{u}\bar{u} \\
ssss\bar{t}t\bar{t}t & s\bar{c}bauetd & uss\bar{u}suu\bar{s} & t\bar{t}u\bar{u}\bar{t}t\bar{u}u \\
t\bar{t}\bar{t}t\bar{s}\bar{s}ss & \bar{d}t\bar{e}\bar{u}abcs & \bar{s}u\bar{u}\bar{s}u\bar{s}su & t\bar{t}u\bar{u}\bar{t}tu\bar{u} \\
\bar{t}\bar{t}\bar{t}\bar{t}\bar{s}s\bar{s}s & tdu\bar{e}\bar{b}a\bar{s}c & uss\bar{u}\bar{s}\bar{u}\bar{u}s & \bar{t}\bar{t}uu\bar{t}\bar{t}uu \\
\bar{t}t\bar{t}tssss & \bar{e}\bar{u}d\bar{t}\bar{c}sa\bar{b} & \bar{u}\bar{s}s\bar{u}su\bar{u}s & u\bar{u}\bar{t}tu\bar{u}\bar{t}t \\
\bar{t}\bar{t}tt\bar{s}ss\bar{s} & u\bar{e}\bar{t}\bar{d}\bar{s}\bar{c}ba & s\bar{u}\bar{u}\bar{s}\bar{u}ssu & uuttuutt \\
\bar{u}\bar{t}\bar{u}ttu\bar{t}u & us\bar{s}\bar{u}s\bar{u}u\bar{s} & abctdes\bar{u} & st\bar{t}s\bar{t}\bar{s}s\bar{t} \\
\bar{t}u\bar{t}\bar{u}u\bar{t}\bar{u}\bar{t} & s\bar{u}u\bar{s}\bar{u}\bar{s}su & \bar{b}a\bar{t}ce\bar{d}us & t\bar{s}\bar{s}\bar{t}\bar{s}tts \\
\bar{u}tut\bar{t}u\bar{t}\bar{u} & \bar{s}\bar{u}\bar{u}\bar{s}u\bar{s}\bar{s}u & \bar{c}ta\bar{b}s\bar{u}\bar{d}\bar{e} & \bar{t}s\bar{s}\bar{t}s\bar{t}ts \\
\bar{t}\bar{u}t\bar{u}\bar{u}\bar{t}\bar{u}t & u\bar{s}\bar{s}usuus & \bar{t}\bar{c}baus\bar{e}d & \bar{s}\bar{t}\bar{t}stss\bar{t} \\
t\bar{u}tuu\bar{t}\bar{u}\bar{t} & su\bar{u}\bar{s}\bar{u}s\bar{s}u & \bar{d}\bar{e}\bar{s}\bar{u}abc\bar{t} & \bar{s}\bar{t}\bar{t}s\bar{t}\bar{s}\bar{s}t \\
\bar{u}\bar{t}\bar{u}t\bar{t}\bar{u}t\bar{u} & u\bar{s}s\bar{u}su\bar{u}\bar{s} & \bar{e}du\bar{s}\bar{b}atc & \bar{t}s\bar{s}\bar{t}\bar{s}t\bar{t}\bar{s} \\
tu\bar{t}u\bar{u}\bar{t}\bar{u}t & \bar{u}\bar{s}\bar{s}\bar{u}\bar{s}uu\bar{s} & \bar{s}\bar{u}de\bar{c}\bar{t}a\bar{b} & \bar{t}sst\bar{s}tts \\
\bar{u}tutt\bar{u}tu & s\bar{u}\bar{u}s\bar{u}\bar{s}\bar{s}\bar{u} & u\bar{s}e\bar{d}t\bar{c}ba & \bar{s}\bar{t}t\bar{s}\bar{t}\bar{s}s\bar{t} \\
suu\bar{s}\bar{s}u\bar{u}\bar{s} & uut\bar{t}\bar{t}t\bar{u}\bar{u} & \bar{s}\bar{t}tsstts & abc\bar{d}e\bar{u}t\bar{s} \\
u\bar{s}suuss\bar{u} & u\bar{u}ttttu\bar{u} & \bar{t}s\bar{s}tt\bar{s}\bar{s}t & \bar{b}adc\bar{u}\bar{e}st \\
u\bar{s}\bar{s}\bar{u}\bar{u}\bar{s}s\bar{u} & t\bar{t}\bar{u}\bar{u}\bar{u}\bar{u}t\bar{t} & tsstts\bar{s}\bar{t} & \bar{c}\bar{d}a\bar{b}t\bar{s}\bar{e}u \\
su\bar{u}ss\bar{u}\bar{u}\bar{s} & tt\bar{u}uu\bar{u}\bar{t}\bar{t} & \bar{s}tt\bar{s}\bar{s}t\bar{t}s & d\bar{c}bastue \\
s\bar{u}\bar{u}\bar{s}su\bar{u}s & \bar{u}\bar{u}t\bar{t}\bar{t}t\bar{u}\bar{u} & ts\bar{s}\bar{t}tsst & \bar{e}u\bar{t}\bar{s}abcd \\
\bar{u}\bar{s}s\bar{u}u\bar{s}\bar{s}\bar{u} & \bar{u}u\bar{t}\bar{t}\bar{t}tu\bar{u} & s\bar{t}t\bar{s}s\bar{t}\bar{t}s & ues\bar{t}\bar{b}a\bar{d}c \\
\bar{u}\bar{s}\bar{s}u\bar{u}s\bar{s}\bar{u} & \bar{t}tuu\bar{u}\bar{u}t\bar{t} & \bar{s}\bar{t}\bar{t}\bar{s}st\bar{t}\bar{s} & \bar{t}\bar{s}e\bar{u}\bar{c}da\bar{b} \\
s\bar{u}us\bar{s}\bar{u}\bar{u}s & \bar{t}\bar{t}u\bar{u}u\bar{u}\bar{t}\bar{t} & t\bar{s}\bar{s}t\bar{t}s\bar{s}t & s\bar{t}\bar{u}\bar{e}\bar{d}\bar{c}ba
\end{bmatrix}$$

[a] Kharaghani and Tayfeh-Rezaie [122, p317-324] ©Elsevier

Table 8.3 An $OD(40;1,1,1,1,1,35)$ [a]

$abcsdss\bar e$	$\bar s sss\bar s s\bar s\bar s$	$s\bar s s\bar s ssss$	$\bar s ss\bar s\bar s\bar s\bar s\bar s$	$\bar s\bar s\bar s\bar s s\bar s\bar s s$
$\bar b a\bar s cs\bar d es$	$ss\bar s ssss\bar s$	$\bar s\bar s ssss\bar s\bar s s$	$ssss\bar s ss\bar s$	$\bar s ss\bar s\bar s\bar s\bar s\bar s$
$\bar c sa\bar b s\bar e\bar d\bar s$	$sss\bar s\bar s\bar s s\bar s$	$s\bar s\bar s ssss\bar s\bar s$	$s\bar s s\bar s\bar s\bar s ss$	$\bar s\bar s ss\bar s s\bar s s$
$\bar s\bar c baess\bar s d$	$\bar s s\bar s\bar s s\bar s\bar s\bar s$	$ssss\bar s s\bar s s$	$ss\bar s\bar s s\bar s s\bar s$	$s\bar s s\bar s\bar s\bar s ss$
$\bar d\bar s\bar s\bar e abc\bar s$	$\bar s\bar s s\bar s sss\bar s$	$s\bar s ss\bar s\bar s ss$	$ss\bar s\bar s\bar s s\bar s s$	$sssss\bar s\bar s s$
$\bar s de\bar s\bar b asc$	$\bar s ssss\bar s ss$	$\bar s\bar s\bar s\bar s\bar s s\bar s s$	$s\bar s s\bar s ss\bar s\bar s$	$s\bar s\bar s s\bar s\bar s\bar s\bar s$
$\bar s\bar e ds\bar c\bar s a\bar b$	$s\bar s sss\bar s\bar s\bar s$	$\bar s s\bar s sssss$	$\bar s\bar s\bar s\bar s\bar s ss\bar s$	$ss\bar s\bar s\bar s s\bar s s$
$e\bar s s\bar d s\bar c ba$	$sss\bar s ss\bar s s$	$\bar s\bar s ss\bar s ss\bar s$	$s\bar s\bar s s\bar s\bar s\bar s\bar s$	$\bar s s\bar s s\bar s\bar s ss$
$s\bar s\bar s ssss\bar s\bar s$	$abc\bar s d\bar s e\bar s$	$\bar s\bar s s\bar s\bar s\bar s s\bar s$	$\bar s\bar s\bar s\bar s ssss$	$\bar s\bar s\bar s\bar s\bar s\bar s\bar s\bar s$
$\bar s\bar s\bar s\bar s s\bar s s\bar s$	$\bar b asc\bar s\bar d se$	$\bar s sss\bar s sss$	$\bar s ss\bar s s\bar s\bar s s$	$\bar s ss\bar s\bar s ss\bar s$
$\bar s s\bar s s\bar s\bar s\bar s\bar s$	$\bar c sa\bar b e\bar s\bar d s$	$s\bar s sss\bar s ss$	$\bar s\bar s ssss\bar s\bar s$	$\bar s\bar s ss\bar s\bar s ss$
$\bar s\bar s sss\bar s\bar s s$	$s\bar c basesd$	$sss\bar s sss\bar s$	$s\bar s s\bar s\bar s s\bar s s$	$s\bar s s\bar s s\bar s s\bar s$
$s\bar s s\bar s\bar s\bar s\bar s\bar s$	$\bar d s\bar e\bar s abcs$	$\bar s s\bar s\bar s s\bar s ss$	$s\bar s s\bar s s\bar s s\bar s$	$\bar s ss\bar s s\bar s\bar s s$
$\bar s\bar s ss\bar s ss\bar s$	$sds\bar e\bar b a\bar s c$	$sss\bar s\bar s\bar s\bar s s$	$\bar s\bar s ss\bar s\bar s ss$	$ssss\bar s\bar s\bar s\bar s$
$s\bar s\bar s s\bar s\bar s ss$	$\bar e\bar s d\bar s\bar c sa\bar b$	$\bar s\bar s s\bar s ss\bar s s$	$s\bar s\bar s ss\bar s\bar s s$	$s\bar s s\bar s\bar s s\bar s s$
$sssss\bar s s\bar s$	$s\bar e\bar s\bar d\bar s\bar c ba$	$s\bar s\bar s\bar s\bar s sss$	$ssssssss$	$ss\bar s\bar s\bar s\bar s ss$
$\bar s s\bar s\bar s\bar s sss$	$ss\bar s\bar s s\bar s s\bar s$	$abcsdess\bar s$	$\bar s\bar s ssss\bar s\bar s s$	$ss\bar s\bar s s\bar s\bar s s$
$sss\bar s sss\bar s s$	$s\bar s s\bar s\bar s\bar s ss$	$\bar b a\bar s ce\bar d ss$	$\bar s s\bar s s\bar s\bar s\bar s\bar s$	$s\bar s s\bar s\bar s\bar s\bar s\bar s$
$\bar s\bar s s\bar s sss\bar s$	$\bar s\bar s s\bar s s\bar s\bar s s$	$\bar c sa\bar b s\bar s\bar d\bar e$	$ssss\bar s s\bar s s$	$\bar s\bar s\bar s\bar s\bar s s\bar s s$
$s\bar s\bar s\bar s\bar s s\bar s\bar s$	$s\bar s\bar s sssss$	$\bar s\bar c bass\bar e d$	$\bar s ss\bar s\bar s\bar s ss$	$s\bar s\bar s s\bar s\bar s ss$
$\bar s\bar s\bar s sss\bar s s$	$ss\bar s\bar s\bar s s\bar s s$	$\bar d\bar e\bar s\bar s abc\bar s$	$\bar s ss\bar s ss\bar s\bar s$	$ssss\bar s s\bar s s$
$\bar s s\bar s\bar s s\bar s\bar s\bar s$	$s\bar s s\bar s ss\bar s\bar s$	$\bar e ds\bar s\bar b asc$	$sssss\bar s s\bar s$	$s\bar s\bar s ssss\bar s\bar s$
$\bar s sss\bar s s\bar s\bar s$	$\bar s\bar s\bar s\bar s\bar s ss\bar s$	$\bar s\bar s de\bar c\bar s a\bar b$	$s\bar s s\bar s\bar s\bar s\bar s\bar s$	$ss\bar s\bar s\bar s ss\bar s$
$\bar s\bar s s\bar s\bar s\bar s\bar s s$	$s\bar s\bar s s\bar s\bar s\bar s\bar s$	$s\bar s e\bar d s\bar c ba$	$ss\bar s\bar s s\bar s\bar s s$	$\bar s s\bar s s\bar s\bar s\bar s\bar s$
$s\bar s\bar s\bar s\bar s\bar s s\bar s$	$sss\bar s\bar s s\bar s\bar s$	$ss\bar s ss\bar s\bar s\bar s$	$abc\bar d es\bar s\bar s$	$s\bar s\bar s\bar s\bar s ss\bar s$
$\bar s\bar s s\bar s\bar s sss$	$s\bar s sssss\bar s$	$s\bar s\bar s\bar s\bar s\bar s s\bar s$	$\bar b adcs\bar e s\bar s$	$\bar s\bar s s\bar s\bar s sss$
$\bar s\bar s\bar s ss\bar s ss$	$s\bar s\bar s\bar s\bar s\bar s s\bar s$	$\bar s s\bar s\bar s\bar s\bar s ss$	$\bar c\bar d a\bar b\bar s\bar s\bar e\bar s$	$\bar s\bar s\bar s ss\bar s ss$
$s\bar s sssss\bar s$	$ss\bar s ss\bar s\bar s\bar s$	$\bar s\bar s\bar s ss\bar s ss$	$d\bar c bas\bar s\bar s e$	$s\bar s sssss\bar s$
$sss\bar s s\bar s ss$	$\bar s\bar s\bar s s\bar s s\bar s\bar s$	$\bar s sss\bar s\bar s s\bar s$	$\bar e\bar s s\bar s abcd$	$s\bar s sss\bar s\bar s\bar s s$
$s\bar s ss\bar s\bar s\bar s s$	$\bar s s\bar s\bar s sss\bar s$	$ss\bar s s\bar s sss$	$\bar s ess\bar b a\bar d c$	$\bar s\bar s\bar s s\bar s s\bar s\bar s$
$s\bar s\bar s\bar s ss\bar s s$	$\bar s sss\bar s\bar s s\bar s$	$sss\bar s s\bar s ss$	$s\bar s es\bar c da\bar b$	$ss\bar s s\bar s sss$
$ss\bar s s\bar s sss$	$\bar s\bar s s\bar s s\bar s\bar s\bar s$	$\bar s s\bar s\bar s ssss\bar s$	$sss\bar e\bar d\bar c ba$	$\bar s sss\bar s\bar s s\bar s$
$sss\bar s\bar s\bar s\bar s s$	$sss\bar s s\bar s\bar s\bar s$	$\bar s\bar s s\bar s\bar s\bar s\bar s s$	$\bar s ss\bar s\bar s s\bar s s$	$abc\bar e dsss$
$s\bar s ss\bar s s\bar s\bar s$	$s\bar s ss\bar s\bar s s\bar s$	$\bar s sss\bar s s\bar s\bar s$	$ssssss\bar s\bar s$	$\bar b aecs\bar d\bar s s$
$s\bar s\bar s\bar s\bar s sss$	$s\bar s\bar s\bar s\bar s\bar s\bar s s$	$s\bar s ss\bar s sss$	$s\bar s s\bar s\bar s ss\bar s$	$\bar c\bar e a\bar b ss\bar d\bar s$
$ss\bar s s\bar s\bar s s\bar s$	$ss\bar s ss\bar s ss$	$sss\bar s\bar s\bar s s\bar s$	$ss\bar s\bar s\bar s\bar s\bar s\bar s$	$e\bar c ba\bar s s\bar s d$
$\bar s sss\bar s sss$	$sss\bar s\bar s sss$	$\bar s ssss\bar s ss$	$ss\bar s\bar s ssss$	$\bar d\bar s\bar s sabce$
$ss\bar s ssss\bar s s$	$s\bar s ssss\bar s s$	$ss\bar s s\bar s\bar s\bar s s$	$s\bar s s\bar s s\bar s\bar s s$	$\bar s d\bar s\bar s\bar b a\bar e c$
$sss\bar s sss\bar s$	$s\bar s\bar s\bar s sss\bar s$	$sss\bar s ss\bar s s$	$\bar s\bar s\bar s\bar s ss\bar s\bar s$	$\bar s sds\bar c ea\bar b$
$\bar s s\bar s\bar s\bar s s\bar s\bar s$	$ss\bar s s\bar s s\bar s\bar s$	$\bar s s\bar s\bar s\bar s sss$	$s\bar s s\bar s ss\bar s s$	$\bar s\bar s s\bar d\bar e\bar c ba$

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