Chapter 10
Complex, Quaternion and Non Square Orthogonal Designs

10.1 Introduction

A detailed study of complex, quaternion and non-square orthogonal designs is beyond the scope of this book. We give just a small taste to highlight the deep and practical nature of these almost unstudied algebraic structures.

A multiple antenna system has been used to solve bandwidth limitation and channel fading problems in a wireless communication system. Space-time block codes from real and complex orthogonal designs, have attracted considerable attention lately, since they can approach the potential huge capacity of multiple antenna systems and have a simple decoupled maximum-likelihood (ML) decoding scheme [208]. Space-time block codes have been adopted in the newly proposed standard for wireless LANs IEEE 802.11n [147]. Multi-path fading in a wireless channel can cause severe degradation of transmission performance. In order to overcome the fading problem, some diversity techniques are used, e.g. space-time coding scheme combines space diversity and time diversity. We expect that additional forms of diversity, i.e. polarization diversity and frequency diversity, should be considered with space and time diversity to improve capacity.

It has been shown that polarization diversity, together with other forms of diversity, can add to the performance improvements offered by other diversity techniques. Isaeva and Sarytchev [113] showed that the utilization of polarization diversity with other forms of diversity can be modelled by means of quaternions since two orthogonal complex constellations form a quaternion. This motivated the study of orthogonal designs over the quaternion domain for future applications in signal processing as space-time-polarization block codes [28, 60, 184, 257].

We give general construction techniques to build amicable orthogonal designs of quaternions, which we believe can be used for constructing quaternion orthogonal designs, just like the applications of amicable orthogonal de-

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signs(AODs) for complex space-time codes, e.g. our previous work in [186,212].

10.2 Complex orthogonal designs

Complex orthogonal design is a complex analog of orthogonal designs and was first studied by A.V. Geramita and J.M. Geramita in [76]. The coefficient matrices of complex orthogonal designs are over the complex domain and can be used in the study of complex weighing matrices.

Seberry and Adams [181] noted that quaternion orthogonal designs (QODs) were introduced as a mathematical construct with the potential for applications in wireless communications. The potential applications require new methods for constructing QODs, as most of the known methods of construction do not produce QODs with the exact properties required for implementation in wireless systems. Real amicable orthogonal designs and the Kronecker product may be used to construct new families of QODs. Their Amicable-Kronecker Construction can be applied to build quaternion orthogonal designs of a variety of sizes and types. Although it has not yet been simulated whether the resulting designs are useful for applications, their properties look promising for the desired implementations. Furthermore, the construction itself is interesting because it uses a simple family of real amicable orthogonal designs and the Kronecker product as building blocks, opening the door for future construction algorithms using other families of amicable designs and other matrix products.

The exposition of the bulk of this chapter is due to Zhao, Seberry, Xia, Wysocki, Wysocki [257], Chun Le Tran [186,212], and Sarah Spence Adams [2,181,184,185].

There are many possible definitions for COD. Signal processing encourages us to consider matrices with complex entries $a+\boldsymbol{i}b$, rather than a and/or $\boldsymbol{i}b$, a,b real.

Definition 10.1. A *complex orthogonal design*, *COD*, of order n and type $(s_1,s_2,\ldots,s_u)$, denoted $COD(n;s_1,\ s_2,\ldots,s_u)$, is an $n\times n$ matrix A with entries in the set of complex variables $y_i+\boldsymbol{i}z_i$ where y_i, z_i are in the set of real commuting variables $x_1,x_2,\ldots,x_u$ satisfying

$$A^HA=AA^H=\left(\sum_{h=1}^{u}s_hx_h^2\right)I_n,$$

where $(.)^H$ denotes the Hermitian transpose. We note this is a different definition of COD from that which we have previously used.

Example 10.1. The matrix $\begin{bmatrix}\boldsymbol{i}x_1 & x_2\\ x_2 & \boldsymbol{i}x_1\end{bmatrix}$, where x_1 and x_2 are real commuting variables, is a $COD(2;1,1)$.

In [254], Yuen, Guan and Tjhung defined an amicable complex orthogonal design which is a complex extension of amicable orthogonal design.

Definition 10.2. Two complex orthogonal designs, A and B, with complex coefficient matrices, are said to be *amicable* if $AB^H = BA^H$ or $A^H B = B^H A$. We write $ACOD(n; w_1, w_2, \ldots, w_u; z_1, z_2, \ldots, z_v)$ to denote that two designs $COD(n; w_1, w_2, \ldots, w_u)$ and $COD(n; z_1, z_2, \ldots, z_v)$ are *complex amicable.*

Example 10.2. Let $A = \left[\begin{smallmatrix} a & b \\ -\boldsymbol{i}b & \boldsymbol{i}a \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} c & d \\ \boldsymbol{i}d & -\boldsymbol{i}c \end{smallmatrix}\right]$, where $a, b, c, d \in \mathbb{R}$. A and B are amicable complex orthogonal designs $ACOD(2; 1, 1; 1, 1)$.

Yuen et al [254] also concluded that the maximum total number of variables of an ACOD is equal to the maximum total number of variables in an AOD of same order.

10.3 Amicable orthogonal designs of quaternions

Definition 10.3. A *quaternion variable* **a** is defined in the form $\boldsymbol{a} = a_1 + a_2\boldsymbol{i} + a_3\boldsymbol{j} + a_4\boldsymbol{k}$, where a_p, $p = 1, \ldots, 4$ are real numbers and the elements $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ satisfy $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$.

A quaternion variable is a non-commutative extension of the complex variables since we can also write $\boldsymbol{a} = (a_1 + a_2\boldsymbol{i}) + (a_3 + a_4\boldsymbol{i})\boldsymbol{j}$.

The *quaternion conjugate* is given by $\boldsymbol{a}^Q = a_1 - a_2\boldsymbol{i} - a_3\boldsymbol{j} - a_4\boldsymbol{k}$.

The *quaternion norm* is therefore defined by

$$\sqrt{\boldsymbol{a}\boldsymbol{a}^Q} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}\,.$$

Given a matrix $A = (\boldsymbol{a}_{\ell,m})$, where $\boldsymbol{a}_u$ are quaternion variables or numbers, we define its *quaternion transform* by $A^Q = (\boldsymbol{a}^Q_{m,\ell})$.

The following definitions of orthogonal design of quaternions and restricted quaternion orthogonal design were originally given in [184].

Definition 10.4. An *orthogonal design of quaternions*, *ODQ*, of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $ODQ(n; s_1, s_2, \ldots, s_u)$, on the commuting real variables $x_1, x_2, \ldots, x_u$ is a square matrix A of order n with entries from $\{0, \boldsymbol{q}_1 x_1, \boldsymbol{q}_2 x_2, \ldots, \boldsymbol{q}_u x_u\}$, where each $\boldsymbol{q}_j \in \{\pm\boldsymbol{1}, \pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\}$ such that

$$A^Q A = A A^Q = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n\,,$$

where $(.)^Q$ denotes quaternion transform. We can extend this definition to include *rectangular* designs that satisfy

$$A^Q A = \left(\sum_{h=1}^{u} s_h x_h^2 \right) I_n \,.$$

Example 10.3. Consider $A = \begin{bmatrix} -x_1 & x_2\boldsymbol{i} \\ -x_2\boldsymbol{j} & x_1\boldsymbol{k} \end{bmatrix}$, where x_1, x_2 are real, commuting variables. Then,

$$\begin{aligned} A^Q A &= \begin{bmatrix} -x_1 & x_2\boldsymbol{j} \\ -x_2\boldsymbol{i} & -x_1\boldsymbol{k} \end{bmatrix} \quad \begin{bmatrix} -x_1 & x_2\boldsymbol{i} \\ -x_2\boldsymbol{j} & x_1\boldsymbol{k} \end{bmatrix} \\ &= \begin{bmatrix} x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 \end{bmatrix} \end{aligned}$$

so A is an $ODQ(2;1,1)$.

Definition 10.5. A *restricted quaternion orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$, denoted $RQOD(n; s_1, s_2, \ldots, s_u)$, on the complex variables z_1, z_2, $\ldots$, z_u is an $n \times n$ matrix A with entries from $\{0, \boldsymbol{q}_1 z_1, \boldsymbol{q}_1 z_1^*, \boldsymbol{q}_2 z_2, \boldsymbol{q}_2 z_2^*, \ldots, \boldsymbol{q}_u z_u, \boldsymbol{q}_u z_u^*\}$, where each $\boldsymbol{q}_p$ is a linear combination of $\{\pm\mathbf{1}, \pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\}$ such that

$$A^Q A = A A^Q = \left(\sum_{h=1}^{u} s_h |z_h|^2 \right) I_n \,.$$

This definition can be extended to include *rectangular* designs that satisfy $A^Q A = (\sum_{h=1}^{u} s_h |z_h|^2) I_n$.

Example 10.4. Consider $A = \begin{bmatrix} \boldsymbol{i}z_1 & \boldsymbol{i}z_2 \\ -\boldsymbol{j}z_2^* & \boldsymbol{j}z_1^* \end{bmatrix}$, where z_1, z_2 are complex commuting variables. Then,

$$\begin{aligned} A^Q A &= \begin{bmatrix} -z_1^*\boldsymbol{i} & z_2\boldsymbol{j} \\ -z_2^*\boldsymbol{i} & -z_1\boldsymbol{j} \end{bmatrix} \quad \begin{bmatrix} \boldsymbol{i}z_1 & \boldsymbol{i}z_2 \\ -\boldsymbol{j}z_2^* & \boldsymbol{j}z_1^* \end{bmatrix} \\ &= \begin{bmatrix} |z_1|^2 + |z_2|^2 & 0 \\ 0 & |z_1|^2 + |z_2|^2 \end{bmatrix} \end{aligned}$$

so A is an $RQOD(2;1,1)$. To illustrate why this is called a *restricted* QOD, we replace complex variables in A using $z_i = x_i + y_i\boldsymbol{i}$, where the x_i, y_i are real variables. This gives

$$A = \begin{bmatrix} -y_1 + \boldsymbol{i}x_1 & -y_2 + \boldsymbol{i}x_2 \\ -\boldsymbol{j}x_2 - \boldsymbol{k}y_2 & \boldsymbol{j}x_1 + \boldsymbol{k}y_1 \end{bmatrix} .$$

We now can see that the entries of A are quaternion variables such that certain components of the variables are *restricted* to zero.

Definition 10.6. Two orthogonal designs of quaternions, A and B, are said to be *amicable* if $AB^Q = BA^Q$ or $A^Q B = B^Q A$. We write

$$AODQ\,(n; w_1, w_2, \ldots, w_u; z_1, z_2, \ldots, z_v)$$

to denote that two designs $ODQ(n; w_1, w_2, \ldots, w_u)$ and $ODQ(n; z_1, z_2, \ldots, z_v)$ are amicable.

Example 10.5. Let

$$A = \begin{bmatrix} -x_1 & x_2\boldsymbol{i} \\ -x_2\boldsymbol{j} & x_1\boldsymbol{k} \end{bmatrix} \text{ and } B = \begin{bmatrix} y_1 & y_2\boldsymbol{i} \\ y_2\boldsymbol{j} & y_1\boldsymbol{k} \end{bmatrix}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. A and B are amicable orthogonal designs of quaternions of type $AODQ(2;1,1;1,1)$.

Proof. The proof that A and B are orthogonal designs of quaternions is straight-forward. We show A and B are amicable.

$$\begin{aligned} AB^Q &= \begin{bmatrix} -x_1 & x_2\boldsymbol{i} \\ -x_2\boldsymbol{j} & x_1\boldsymbol{k} \end{bmatrix} \quad \begin{bmatrix} y_1 & -y_2\boldsymbol{j} \\ -y_2\boldsymbol{i} & -y_1\boldsymbol{k} \end{bmatrix} \\ &= \begin{bmatrix} -x_1y_1 + x_2y_2 & x_1y_2\boldsymbol{j} + x_2y_1\boldsymbol{j} \\ -x_2y_1\boldsymbol{j} - x_1y_2\boldsymbol{j} & -x_2y_2 + x_1y_1 \end{bmatrix} \\ BA^Q &= \begin{bmatrix} y_1 & y_2\boldsymbol{i} \\ y_2\boldsymbol{j} & y_1\boldsymbol{k} \end{bmatrix} \quad \begin{bmatrix} -x_1 & x_2\boldsymbol{j} \\ -x_2\boldsymbol{i} & -x_1\boldsymbol{k} \end{bmatrix} \\ &= \begin{bmatrix} -x_1y_1 + x_2y_2 & x_1y_2\boldsymbol{j} + x_2y_1\boldsymbol{j} \\ -x_2y_1\boldsymbol{j} - x_1y_2\boldsymbol{j} & -x_2y_2 + x_1y_1 \end{bmatrix} \\ &= AB^Q \end{aligned}$$

Hence A and B are amicable orthogonal designs of quaternions. □

Let X and Y be amicable orthogonal designs of quaternions of type $AODQ(n; u_1, \ldots, u_s; v_1, \ldots, v_t)$. Write

$$X = \sum_{i=1}^{s} A_i x_i\,, \quad Y = \sum_{j=1}^{t} B_j y_j\,,$$

we then have:

(i) $A_i * A_\ell = 0, \quad 1 \le i \ne \ell \le s;$
$B_j * B_k = 0, \quad 1 \le j \ne k \le t;$
(ii) $A_i A_i^Q = u_i I_n, \quad 1 \le i \le s;$
$B_j B_j^Q = v_j I_n, \ 1 \le j \le t;$
(iii) $A_i A_\ell^Q + A_\ell A_i^Q = 0, \quad 1 \le i \ne \ell \le s;$
$B_j B_k^Q + B_k B_j^Q = 0, \quad 1 \le j \ne k \le t;$
(iv) $A_i B_j^Q = B_j A_i^Q, \quad 1 \le i \le s, \ 1 \le j \le t,$

where A_i, B_j are all $\{0,\pm1,\pm i,\pm j,\pm k\}$ quaternion matrices. It is clear that conditions (i)–(iv) are necessary and sufficient for the existence of amicable orthogonal designs of quaternions $AODQ(n;u_1,\ldots,u_s;v_1,\ldots,v_t)$.

Problem 10.1 (Research Problem 4). Investigate the algebra which corresponds to the properties (i), (ii), (iii) and (iv) of the proof of Example 10.5.

Proposition 10.1. *A necessary and sufficient condition that there exist amicable orthogonal designs of quaternions X and Y of type $AODQ(n;u_1,\ldots,u_s;v_1,\ldots,v_t)$ is that there exists a family of matrices of $\{A_1,\ldots,A_s;B_1,\ldots,B_t\}$ of order n satisfying (i)–(iv) above.*

Proof. Let X and Y be such an amicable pair and write $X=A_1x_1+\cdots+A_sx_s$ and $Y=B_1y_1+\cdots+B_ty_t$ as linear monomials in the $x_i,y_i\in\mathbb{R}$. By definition, the proof of (i) and (ii) is straight-forward. Since we have

$$\begin{aligned} XX^Q &= (A_1x_1+\cdots+A_sx_s)(A_1^Qx_1+\cdots+A_s^Qx_s)\\ &= \sum_{j=1}^{s}\left(A_jA_j^Qx_j^2\right)+\sum_{j\neq k}\left(A_jA_k^Q+A_kA_j^Q\right)x_jx_k\\ &= \left(\sum_{j=1}^{s}u_jx_j^2\right)I_n, \end{aligned}$$

hence, conditions in (iii) are satisfied. Condition (iv) can be proved by comparing coefficient matrices of $XY^Q=YX^Q$ on both sides. Conversely, if we have $\{A_1,\ldots,A_s;B_1,\ldots,B_t\}$ of order n satisfying (i)–(iv), then it is obvious that $X=A_1x_1+\cdots+A_sx_s$ and $Y=B_1y_1+\cdots+B_ty_t$ are an AODQ with required type. □

Definition 10.7. An *amicable family of quaternions*(AFQ) of type $(u_1,\ldots,u_s;v_1,\ldots,v_t)$ in order n is a collection of quaternion matrices $\{A_1,\ldots A_s;B_1,\ldots,B_t\}$ satisfying (ii), (iii), (iv) above.

The definition of amicable family of quaternions(AFQ) is analogous to the definition of amicable family of orthogonal designs given in [80]. However, the upper bound on the total number of variables of an AODQ, i.e. $s+t$, is an unsolved problem.

10.4 Construction techniques

In this section, we present several construction techniques for building amicable orthogonal designs over the real and quaternion domain. There are some existing methods for generating real amicable orthogonal designs. We can extend these techniques to build designs over the quaternion domain. However, due to the non-commutativity of the quaternions, we need to modify existing techniques to make them suitable for designs over the quaternion domain.

10.4.1 Amicable orthogonal designs

We recall from Chapter 5:

Definition 10.8. A symmetric conference matrix N of order n is a square $(0,1,-1)$ matrix satisfying $N = N^T$ and $NN^T = (n-1)I_n$. It is shown in [39] that if such a matrix exists, one may assume it has zero diagonal.

A symmetric conference matrix is a special type of weighing matrix which has been long studied in order to design experiments to weight n objects whose weights are small compared with the weights of the moving parts of the balance being used [80]. In Chapter 5 we have studied the application of symmetric conference matrices for constructing amicable orthogonal designs.

Lemma 10.1. *Let N be a symmetric conference matrix in order n and x, y real commuting variables. Then there is a complex orthogonal design $COD(n;1,n-1)$.*

Proof. Let $Y = xI_n\boldsymbol{i} + yN$; then Y is easily proved to be the required COD. □

Lemma 10.2 below improves results of Theorem 2 given in [177].

Lemma 10.2. *Let N be a symmetric conference matrix in order n. Then there exist pairs of amicable orthogonal designs:*

a) $AOD(2n;n,n;n,n)$,
b) $AOD(2n;n,n;2,2(n-1))$,
c) $AOD(2n;n,n;1,n-1)$,
d) $AOD(2n;2,2(n-1);1,n-1)$.

Proof. Let a, b, c and d be real commuting variables. Then the required designs are:

$$\text{for a)} \begin{bmatrix} aI_n + bN & bI_n - aN \\ bI_n - aN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n + dN & dI_n - cN \\ -dI_n + cN & cI_n + dN \end{bmatrix},$$

$$\text{for b)} \begin{bmatrix} aI_n + bN & bI_n - aN \\ bI_n - aN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n + dN & cI_n - dN \\ -cI_n + dN & cI_n + dN \end{bmatrix},$$

$$\text{for c)} \begin{bmatrix} aI_n + bN & bI_n - aN \\ bI_n - aN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n & dN \\ -dN & cI_n \end{bmatrix},$$

$$\text{for d)} \begin{bmatrix} aI_n + bN & aI_n - bN \\ aI_n - bN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n & dN \\ -dN & cI_n \end{bmatrix}. \square$$

Corollary 10.1. *Let n be the order of the symmetric conference matrices, we then have a number of amicable orthogonal designs of order $2n$ of different types. For example, for $n-1 \equiv 1 \pmod 4$, where $n-1$ is a prime power, there exist*

a) $AOD(2n;n,n;n,n)$,
b) $AOD(2n;n,n;2,2(n-1))$,
c) $AOD(2n;n,n;1,n-1)$,
d) $AOD(2n;2,2(n-1);1,n-1)$.

Example 10.6. For $n=6$ and $n=10$, there exist

a) $AOD(12;6,6;6,6)$, a') $AOD(20;10,10;10,10)$,
b) $AOD(12;6,6;2,10)$, b') $AOD(20;10,10;2,18)$,
c) $AOD(12;6,6;1,5)$, c') $AOD(20;10,10;1,9)$,
d) $AOD(12;2,10;1,5)$, d') $AOD(20;2,18;1,9)$,

separately.

We recall the oft quoted:

Lemma 10.3. *For $p \equiv 3 \pmod 4$ be a prime power. Then there exists a pair of amicable orthogonal designs $AOD(p+1;1,p;1,p)$.*

Proof. Almost straightforward verification since $aI+bS$ is type 1 and $(cI+dS)R$ is type 2 matrix. □

Example 10.7. For $p=3$, we define type 1 matrix $S = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix}$ and the back diagonal matrix $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then, we construct

$$A = \begin{bmatrix} a & b & b & b \\ -b & a & b & -b \\ -b & -b & a & b \\ -b & b & -b & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -c & d & d & d \\ d & -d & d & c \\ d & d & c & -d \\ d & c & -d & d \end{bmatrix}.$$

A and B is a pair of amicable orthogonal design $AOD(4;1,3;1,3)$.

10.5 Amicable orthogonal design of quaternions

Theorem 10.1. *If there exists a pair of amicable orthogonal designs of quaternions, $AODQ(n;a_1,\cdots,a_s;b_1,\cdots,b_t)$ and a pair of amicable orthogonal designs $AOD(m;c_1,\cdots,c_u;d_1,\cdots,d_v)$, then there exists a pair of amicable orthogonal designs of quaternions*
$AODQ(nm;b_1c_1,\ldots,b_1c_{u-1},a_1c_u,\ldots,a_sc_u;b_1d_1,\ldots,b_1d_v,b_2c_u,\ldots,b_tc_u)$.

Proof. Let $X = \sum_{i=1}^{s} A_i x_i$ and $Y = \sum_{j=1}^{t} B_j y_j$ be the amicable orthogonal designs of quaternions in order n and let $Z = \sum_{k=1}^{u} C_k z_k$ and $W = \sum_{l=1}^{v} D_l w_l$ be the amicable orthogonal designs in order m.
Construct the matrices

$$P = \sum_{i=1}^{u-1} (B_1 \otimes C_i)\, p_i + \sum_{j=1}^{s} (A_j \otimes C_u)\, p_{j+u-1}$$

$$Q = \sum_{i=1}^{v} (B_1 \otimes D_i)\, q_i + \sum_{j=2}^{t} (B_j \otimes C_u)\, q_{j+v-1}$$

where the p_i's and q_i's are real commuting variables and $\otimes$ denotes Kronecker product. □

The above theorem is similar to Wolfe's theorem [247] which gave a general construction method for amicable orthogonal designs. The only change in Theorem 10.1 is that X and Y are amicable orthogonal designs of quaternions (AODQ). It is important to note that Z and W must be amicable orthogonal designs over the **real** domain, otherwise the non-commutative property of quaternions can not guarantee the amicability of the results.

Example 10.8. Let $A = \begin{bmatrix} -x_1 & x_2\boldsymbol{i} \\ -x_2\boldsymbol{j} & x_1\boldsymbol{k} \end{bmatrix}$ and $B = \begin{bmatrix} y_1 & y_2\boldsymbol{i} \\ y_2\boldsymbol{j} & y_1\boldsymbol{k} \end{bmatrix}$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. A and B are amicable orthogonal designs of quaternions $AODQ(2;1,1;1,1)$. Another pair of amicable orthogonal designs is given as $Z = \begin{bmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{bmatrix}$, where $z_1, z_2, w_1, w_2 \in \mathbb{R}$. Theorem 10.1 gives

$$P = (B_1 \otimes C_1)p_1 + (A_1 \otimes C_2)p_2 + (A_2 \otimes C_2)p_3\,,$$
$$Q = (B_1 \otimes D_1)q_1 + (B_1 \otimes D_2)q_2 + (B_2 \otimes C_2)q_3\,.$$

The quaternion coefficient matrices for P and Q are:

$$P_1 = B_1 \otimes C_1 = \begin{bmatrix} 1 & 0 \\ 0 & \boldsymbol{k} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boldsymbol{k} & 0 \\ 0 & 0 & 0 & \boldsymbol{k} \end{bmatrix},$$

$$P_2 = A_1 \otimes C_2 = \begin{bmatrix} -1 & 0 \\ 0 & \boldsymbol{k} \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{k} \\ 0 & 0 & -\boldsymbol{k} & 0 \end{bmatrix},$$

$$P_3 = A_2 \otimes C_2 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{j} & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{i} \\ 0 & 0 & -\boldsymbol{i} & 0 \\ 0 & -\boldsymbol{j} & 0 & 0 \\ \boldsymbol{j} & 0 & 0 & 0 \end{bmatrix},$$

$$Q_1 = B_1 \otimes D_1 = \begin{bmatrix} 1 & 0 \\ 0 & \boldsymbol{k} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \boldsymbol{k} & 0 \\ 0 & 0 & 0 & -\boldsymbol{k} \end{bmatrix},$$

$$Q_2 = B_1 \otimes D_2 = \begin{bmatrix} 1 & 0 \\ 0 & \boldsymbol{k} \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{k} \\ 0 & 0 & \boldsymbol{k} & 0 \end{bmatrix},$$

$$Q_3 = B_2 \otimes C_2 = \begin{bmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{j} & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{i} \\ 0 & 0 & -\boldsymbol{i} & 0 \\ 0 & \boldsymbol{j} & 0 & 0 \\ -\boldsymbol{j} & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$P = \begin{bmatrix} p_1 & -p_2 & 0 & p_3\boldsymbol{i} \\ p_2 & p_1 & -p_3\boldsymbol{i} & 0 \\ 0 & -p_3\boldsymbol{j} & p_1\boldsymbol{k} & p_2\boldsymbol{k} \\ p_3\boldsymbol{j} & 0 & -p_2\boldsymbol{k} & p_1\boldsymbol{k} \end{bmatrix} \text{ and } Q = \begin{bmatrix} q_1 & q_2 & 0 & q_3\boldsymbol{i} \\ q_2 & -q_1 & -q_3\boldsymbol{i} & 0 \\ 0 & q_3\boldsymbol{j} & q_1\boldsymbol{k} & q_2\boldsymbol{k} \\ -q_3\boldsymbol{j} & 0 & q_2\boldsymbol{k} & -q_1\boldsymbol{k} \end{bmatrix}$$

are are amicable orthogonal designs of quaternions $AODQ(4;1,1,1;1,1,1)$ since they both are ODQs and satisfy $PQ^Q = QP^Q$.

Corollary 10.2. *If there exists a pair of amicable orthogonal designs of quaternions $AODQ(n;a_1,...,a_s;b_1,...,b_t)$, then there exists a pair of amicable orthogonal designs of quaternions of type*

a) $AODQ(2n;a_1,a_1,2a_2...,2a_s;2b_1,...,2b_t)$,
b) $AODQ(2n;a_1,a_1,a_2...,a_s;b_1,...,b_t)$.

Proof. Let $X = \sum_{i=1}^{s} A_i x_i$ and $Y = \sum_{j=1}^{t} B_j y_j$ be amicable designs of quaternions in order n.
a) Let $M = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $N = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ be **real** weighing matrices and construct the matrices

$$P = (A_1 \otimes I_2)\, p_1 + (A_1 \otimes M)\, p_2 + \sum_{i=2}^{s} (A_i \otimes N)\, p_{i+1}$$

and

$$Q = \sum_{j=1}^{t} (B_j \otimes N)\, q_j$$

b) Same as a), only set $N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

It's obvious that all the quaternion matrices P_i's and Q_i's satisfy the conditions (i)-(iv) because the weighing matrices M, N have the following properties: $M = -M^\top$, $N = N^\top$, and $MN^\top = NM^\top$, where $(.)^\top$ denotes matrix transpose. □

Example 10.9. Consider a pair of $AODQ(2;1,1;1,1)$ given in Example 10.5, we construct a new $AODQ(4;1,1,2;2,2)$ using Corollary 10.2(a):

$$P = \begin{bmatrix} -p_1 & -p_2 & p_3\boldsymbol{i} & p_3\boldsymbol{i} \\ p_2 & -p_1 & p_3\boldsymbol{i} & -p_3\boldsymbol{i} \\ -p_3\boldsymbol{j} & -p_3\boldsymbol{j} & p_1\boldsymbol{k} & p_2\boldsymbol{k} \\ -p_3\boldsymbol{j} & p_3\boldsymbol{j} & -p_2\boldsymbol{k} & p_1\boldsymbol{k} \end{bmatrix} \quad Q = \begin{bmatrix} q_1 & q_1 & q_2\boldsymbol{i} & q_2\boldsymbol{i} \\ q_1 & -q_1 & q_2\boldsymbol{i} & -q_2\boldsymbol{i} \\ q_2\boldsymbol{j} & q_2\boldsymbol{j} & q_1\boldsymbol{k} & q_1\boldsymbol{k} \\ q_2\boldsymbol{j} & -q_2\boldsymbol{j} & q_1\boldsymbol{k} & -q_1\boldsymbol{k} \end{bmatrix}$$

In Theorem 10.1, we can also replace the amicable orthogonal designs $AOD(m;c_1,\dots,c_u;d_1,\dots,d_v)$ by an amicable family to get more amicable orthogonal designs of quaternions.

Example 10.10. Consider a pair of $AODQ(2;1,1;1,1)$ given in Example 10.5, let

$$C_1 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{and } D_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

be an amicable family $\{C_1, C_2; D_1, D_2\}$. We construct

$$P = (B_1 \otimes C_1)p_1 + (A_1 \otimes C_2)p_2 + (A_2 \otimes C_2)p_3\,,$$
$$Q = (B_1 \otimes D_1)q_1 + (B_1 \otimes D_2)q_2 + (B_2 \otimes C_2)q_3\,.$$

The new amicable orthogonal designs of quaternions are:

$$P = \begin{bmatrix} -p_1 - p_2 & p_1 - p_2 & p_3\boldsymbol{i} & p_3\boldsymbol{i} \\ p_1 - p_2 & p_1 + p_2 & p_3\boldsymbol{i} & -p_3\boldsymbol{i} \\ -p_3\boldsymbol{j} & -p_3\boldsymbol{j} & -p_1\boldsymbol{k} + p_2\boldsymbol{k} & p_1\boldsymbol{k} + p_2\boldsymbol{k} \\ -p_3\boldsymbol{j} & p_3\boldsymbol{j} & p_1\boldsymbol{k} + p_2\boldsymbol{k} & p_1\boldsymbol{k} - p_2\boldsymbol{k} \end{bmatrix}$$

and

$$Q = \begin{bmatrix} q_1 + q_2 & -q_1 + q_2 & q_3\boldsymbol{i} & q_3\boldsymbol{i} \\ q_1 - q_2 & q_1 + q_2 & q_3\boldsymbol{i} & -q_3\boldsymbol{i} \\ q_3\boldsymbol{j} & q_3\boldsymbol{j} & q_1\boldsymbol{k} + q_2\boldsymbol{k} & -q_1\boldsymbol{k} + q_2\boldsymbol{k} \\ q_3\boldsymbol{j} & -q_3\boldsymbol{j} & q_1\boldsymbol{k} - q_2\boldsymbol{k} & q_1\boldsymbol{k} + q_2\boldsymbol{k} \end{bmatrix}.$$

In this design, some entries are linear combinations of two variables which may make it unsuitable for real applications in communications. To normalize the above design, we set new variables $a_1 = p_1 + p_2$, $a_2 = p_1 - p_2$, $a_3 = p_3$, and $b_1 = q_1 + q_2$, $b_2 = q_1 - q_2$, $b_3 = q_3$, then we get

$$P = \begin{bmatrix} -a_1 & a_2 & a_3\boldsymbol{i} & a_3\boldsymbol{i} \\ a_2 & a_1 & a_3\boldsymbol{i} & -a_3\boldsymbol{i} \\ -a_3\boldsymbol{j} & -a_3\boldsymbol{j} & -a_2\boldsymbol{k} & a_1\boldsymbol{k} \\ -a_3\boldsymbol{j} & a_3\boldsymbol{j} & a_1\boldsymbol{k} & a_2\boldsymbol{k} \end{bmatrix} \quad Q = \begin{bmatrix} b_1 & -b_2 & b_3\boldsymbol{i} & b_3\boldsymbol{i} \\ b_2 & b_1 & b_3\boldsymbol{i} & -b_3\boldsymbol{i} \\ b_3\boldsymbol{j} & b_3\boldsymbol{j} & b_1\boldsymbol{k} & -b_2\boldsymbol{k} \\ b_3\boldsymbol{j} & -b_3\boldsymbol{j} & b_2\boldsymbol{k} & b_1\boldsymbol{k} \end{bmatrix}.$$

This is an $AODQ(4;1,1,2;1,1,2)$ design without zero entries and no linear processing.

In [255], Yuen et al gave a construction method for amicable complex orthogonal designs. We can also apply it in constructing amicable orthogonal designs of quaternions.

Lemma 10.4. *If there exists a pair of amicable orthogonal designs of quaternions $AODQ(n;a_1,\dots,a_s;b_1,\dots,b_t)$, then there exists a pair of amicable orthogonal designs of quaternions of type $AODQ(4n;a_1,a_1,a_1,b_2,\dots,b_t;b_1,b_1,b_1,a_2,\dots,a_s)$.*

Proof. Let $X = \sum_{i=1}^{s} A_i x_i$ and $Y = \sum_{j=1}^{t} B_j y_j$ be the amicable orthogonal designs of quaternions in order n and define following **real** weighing matrices:

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad N_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad N_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Construct the matrices

$$P = \sum_{i=1}^{3} (A_1 \otimes N_i) p_i + \sum_{j=2}^{t} (B_j \otimes I_4) p_{2+j}$$

$$Q = \sum_{i=1}^{3} (B_1 \otimes M_i) q_i + \sum_{j=2}^{s} (A_j \otimes I_4) q_{2+j}\,.$$

All the quaternion matrices P_i's and Q_i's satisfy the conditions (i)-(iv) because the weighing matrices $\{M_i\}$ and $\{N_i\}$ are skew-symmetric and they also form an amicable family. □

Example 10.11. Consider a pair of $AODQ(2;1,1;1,1)$ given in Example 10.5, we apply Lemma 10.4 to construct the following $AODQ(8;1,1,1,1;1,1,1,1)$:

$$P = \begin{bmatrix} 0 & -p_1 & -p_2 & -p_3 & p_4\boldsymbol{i} & 0 & 0 & 0 \\ p_1 & 0 & p_3 & -p_2 & 0 & p_4\boldsymbol{i} & 0 & 0 \\ p_2 & -p_3 & 0 & p_1 & 0 & 0 & p_4\boldsymbol{i} & 0 \\ p_3 & p_2 & -p_1 & 0 & 0 & 0 & 0 & p_4\boldsymbol{i} \\ p_4\boldsymbol{j} & 0 & 0 & 0 & 0 & p_1\boldsymbol{k} & p_2\boldsymbol{k} & p_3\boldsymbol{k} \\ 0 & p_4\boldsymbol{j} & 0 & 0 & -p_1\boldsymbol{k} & 0 & -p_3\boldsymbol{k} & p_2\boldsymbol{k} \\ 0 & 0 & p_4\boldsymbol{j} & 0 & -p_2\boldsymbol{k} & p_3\boldsymbol{k} & 0 & -p_1\boldsymbol{k} \\ 0 & 0 & 0 & p_4\boldsymbol{j} & -p_3\boldsymbol{k} & -p_2\boldsymbol{k} & p_1\boldsymbol{k} & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & q_1 & q_2 & q_3 & q_4\boldsymbol{i} & 0 & 0 & 0 \\ -q_1 & 0 & q_3 & -q_2 & 0 & q_4\boldsymbol{i} & 0 & 0 \\ -q_2 & -q_3 & 0 & q_1 & 0 & 0 & q_4\boldsymbol{i} & 0 \\ -q_3 & q_2 & -q_1 & 0 & 0 & 0 & 0 & q_4\boldsymbol{i} \\ -q_4\boldsymbol{j} & 0 & 0 & 0 & 0 & q_1\boldsymbol{k} & q_2\boldsymbol{k} & q_3\boldsymbol{k} \\ 0 & -q_4\boldsymbol{j} & 0 & 0 & -q_1\boldsymbol{k} & 0 & q_3\boldsymbol{k} & -q_2\boldsymbol{k} \\ 0 & 0 & -q_4\boldsymbol{j} & 0 & -q_2\boldsymbol{k} & -q_3\boldsymbol{k} & 0 & q_1\boldsymbol{k} \\ 0 & 0 & 0 & -q_4\boldsymbol{j} & -q_3\boldsymbol{k} & q_2\boldsymbol{k} & -q_1\boldsymbol{k} & 0 \end{bmatrix}.$$

Although we only give examples of $AODQ$ of orders 2, 4 and 8 in this chapter, there actually exist many designs of order other than powers of 2. We know that symmetric conference matrices exist for orders $n = q+1$, $q \equiv 1 \pmod 4$ a prime power, e.g., $n = 6$. Applying Theorem 10.1 on AODQ(2;1,1;1,1) and AODs from Corollary 10.1 gives us the following corollary.

Corollary 10.3. *Let* $n \equiv 2 \pmod 4$ *be the order of the symmetric conference matrices, then there exist*

a) $AODQ(4n;n,n,n;n,n,n)$,
b) $AODQ(4n;n,n,n;2,2(n-1),n)$,
c) $AODQ(4n;n,n,n;1,n-1,n)$,
d) $AODQ(4n;2,2(n-1),2(n-1);1,n-1,2(n-1))$,

An example is that for $n=6$, we have $AODQ(24;6,6,6;6,6,6)$, $AODQ(24;6,6,6;2,10,6)$, etc.

Corollary 10.4. *For* $q \equiv 3 \pmod 4$ *a prime power, there exist* $AODQ(2(q+1);1,q,q;1,q,q)$.

Proof. This corollary follows by applying Theorem 10.1 on $AODQ(2;1,1;1,1)$ and AODs from Lemma 10.3. □

The above corollary also gives an example of $AODQ(24;1,11,11;1,11,11)$ when $q=11$.

10.6 Combined Quaternion Orthogonal Designs from Amicable Designs

In [184], Seberry et al gave a technique named *combined quaternion orthogonal designs* from real and complex orthogonal designs. This combined design uses the property that if AB^H is a symmetric matrix, where A and B are matrices with complex entries, so that $AB^H\boldsymbol{q} = \boldsymbol{q}BA^H$ for $\boldsymbol{q} \in \{\pm\boldsymbol{j}, \pm\boldsymbol{k}\}$, to construct new $RQOD$. There is a connection between the combined design and amicable designs, in that the form of AB^H are examined. For amicable orthogonal designs of quaternions, the condition that AB^Q is a symmetric matrix can be relaxed since we have $AB^Q = BA^Q$ for A and B. In the case of combined design from amicable orthogonal design of quaternions, we also need to be careful about what quaternion appears as entries of AB^Q. We illustrate this with the following example:

Example 10.12. Consider the $AODQ(2;1,1;1,1)$ designs A and B from Example 10.5. We have

$$
\begin{aligned}
A^Q B &= \begin{bmatrix} -x_1 & x_2\boldsymbol{j} \\ -x_2\boldsymbol{i} & -x_1\boldsymbol{k} \end{bmatrix} \begin{bmatrix} y_1 & y_2\boldsymbol{i} \\ y_2\boldsymbol{j} & y_1\boldsymbol{k} \end{bmatrix} \\
&= \begin{bmatrix} -x_1y_1 - x_2y_2 & (-x_1y_2 + x_2y_1)\boldsymbol{i} \\ (x_1y_2 - x_2y_1)\boldsymbol{i} & x_1y_1 + x_2y_2 \end{bmatrix} \\
&= B^Q A.
\end{aligned}
$$

Let $D = A + B\boldsymbol{q}$, $\boldsymbol{q} \in \{\pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\}$ be a new design for which we have

$$\begin{aligned} D^Q D &= (A^Q - \boldsymbol{q}B^Q)(A + B\boldsymbol{q}) \\ &= A^Q A + A^Q B\boldsymbol{q} - \boldsymbol{q}B^Q A - \boldsymbol{q}B^Q B\boldsymbol{q} \\ &= (A^Q A + B^Q B) + (A^Q B)\boldsymbol{q} - \boldsymbol{q}(B^Q A), \end{aligned}$$

where $A^Q B = B^Q A$ for the amicability of A and B, we also notice that all entries in $A^Q B$ are either real or products with quaternion $\boldsymbol{i}$. Thus $A^Q B\boldsymbol{i} = \boldsymbol{i}B^Q A$, and we have $D^Q D = A^Q A + B^Q B = (x_1^2 + x_2^2 + y_1^2 + y_2^2)I_2$. The new design $D = A + B\boldsymbol{i}$ is of the form:

$$D = \begin{bmatrix} -x_1 + y_1\boldsymbol{i} & x_2\boldsymbol{i} - y_2 \\ -x_2\boldsymbol{j} - y_2\boldsymbol{k} & x_1\boldsymbol{k} + y_1\boldsymbol{j} \end{bmatrix}.$$

Let complex symbols $z_i = x_i + \boldsymbol{i}y_i$, for $1 \le i \le 2$, then we can write above D as

$$D = \begin{bmatrix} -z_1^* & \boldsymbol{i}z_2 \\ -\boldsymbol{j}z_2^* & \boldsymbol{k}z_1 \end{bmatrix}.$$

The above design satisfies $D^Q D = (|z_1|^2 + |z_2|^2)I_2$ and hence is an $RQOD(2;1,1)$ on complex variables z_1 and z_2. The new RQOD in Example 10.12 has no zero entries, which may have practical advantages when used in wireless communication since there is no need to switch antennas off and back on during transmission.

We now provide an example constructing an $RQOD$ with order 4, which has no zero entries but with linear processing.

Example 10.13. Consider the $AODQ(4;1,1,2;1,1,2)$ designs A and B in Example 10.10 with variables a_1, a_2, a_3 and $b_1, b_2, b_3 \in \mathbb{R}$. We have $X = A^Q B$

$$\begin{aligned} &= \begin{bmatrix} -a_1 & a_2 & a_3\boldsymbol{j} & a_3\boldsymbol{j} \\ a_2 & a_1 & a_3\boldsymbol{j} & -a_3\boldsymbol{j} \\ -a_3\boldsymbol{i} & -a_3\boldsymbol{i} & a_2\boldsymbol{k} & -a_1\boldsymbol{k} \\ -a_3\boldsymbol{i} & a_3\boldsymbol{i} & -a_1\boldsymbol{k} & -a_2\boldsymbol{k} \end{bmatrix} \begin{bmatrix} b_1 & -b_2 & b_3\boldsymbol{i} & b_3\boldsymbol{i} \\ b_2 & b_1 & b_3\boldsymbol{i} & -b_3\boldsymbol{i} \\ b_3\boldsymbol{j} & b_3\boldsymbol{j} & b_1\boldsymbol{k} & -b_2\boldsymbol{k} \\ b_3\boldsymbol{j} & -b_3\boldsymbol{j} & b_2\boldsymbol{k} & b_1\boldsymbol{k} \end{bmatrix} \\ &= \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12}^Q & X_{22} & X_{23} & X_{24} \\ X_{13}^Q & X_{23}^Q & X_{33} & X_{34} \\ X_{14}^Q & X_{24}^Q & X_{34}^Q & X_{44} \end{bmatrix} \\ &= B^Q A, \end{aligned}$$

Where

$X_{11} = -a_1b_1 + a_2b_2 - 2a_3b_3\,,$ $X_{12} = a_1b_2 + a_2b_1\,,$
$X_{13} = (-a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2)\boldsymbol{i}\,,$ $X_{14} = (-a_1b_3 - a_2b_3 + a_3b_1 - a_3b_2)\boldsymbol{i}\,,$
$X_{22} = a_1b_1 - a_2b_2 - 2a_3b_3\,,$ $X_{23} = (a_1b_3 + a_2b_3 + a_3b_1 - a_3b_2)\boldsymbol{i}\,,$
$X_{24} = (-a_1b_3 + a_2b_3 - a_3b_1 - a_3b_2)\boldsymbol{i}\,,$ $X_{33} = a_1b_2 - a_2b_1 + 2a_3b_3\,,$
$X_{34} = a_1b_1 + a_2b_2$ and $X_{44} = -a_1b_2 + a_2b_1 + 2a_3b_3\,.$

Since only quaternion $\boldsymbol{i}$ appears in X, we then set $D = A + B\boldsymbol{i}$ as the new design:

$$D = \begin{bmatrix} -a_1 + b_1\boldsymbol{i} & a_2 - b_2\boldsymbol{i} & a_3\boldsymbol{i} - b_3 & a_3\boldsymbol{i} - b_3 \\ a_2 + b_2\boldsymbol{i} & a_1 + b_1\boldsymbol{i} & a_3\boldsymbol{i} - b_3 & -a_3\boldsymbol{i} + b_3 \\ -a_3\boldsymbol{j} - b_3\boldsymbol{k} & -a_3\boldsymbol{j} - b_3\boldsymbol{k} & -a_2\boldsymbol{k} + b_1\boldsymbol{j} & a_1\boldsymbol{k} - b_2\boldsymbol{j} \\ -a_3\boldsymbol{j} - b_3\boldsymbol{k} & a_3\boldsymbol{j} + b_3\boldsymbol{k} & a_1\boldsymbol{k} + b_2\boldsymbol{j} & a_2\boldsymbol{k} + b_1\boldsymbol{j} \end{bmatrix}.$$

Let complex symbols $z_i = a_i + \boldsymbol{i}b_i$, for $1 \le i \le 3$, then we can write above D as

$$D = \begin{bmatrix} -z_1^* & z_2^* & \boldsymbol{i}z_3 & \boldsymbol{i}z_3 \\ z_2 & z_1 & \boldsymbol{i}z_3 & -\boldsymbol{i}z_3 \\ -\boldsymbol{j}z_3^* & -\boldsymbol{j}z_3^* & -\boldsymbol{k}(a_2 - b_1\boldsymbol{i}) & \boldsymbol{k}(a_1 - b_2\boldsymbol{i}) \\ -\boldsymbol{j}z_3^* & \boldsymbol{j}z_3^* & \boldsymbol{k}(a_1 + b_2\boldsymbol{i}) & \boldsymbol{k}(a_2 + b_1\boldsymbol{i}) \end{bmatrix}.$$

The above design satisfies $D^Q D = (|z_1|^2 + |z_2|^2 + 2|z_3|^2)I_4$ and hence is an $RQOD(4;1,1,2)$ on the complex variables z_1, z_2 and z_3. Note that if an entry in the orthogonal design is a linear combination of variables from the given domain, the design is said to be **with linear processing**. Obviously, the new $RQOD$ design has the property of no zero entries but with linear processing on some entries, i.e the position $(3,3)$ is the quaternion combination of real part of symbol z_2 and imaginary part of symbol z_1.

The following Lemma shows construction of orthogonal designs of quaternions by using symmetric conference matrices.

Lemma 10.5. *Suppose a, b, c, d are real commuting variables. Let N be a symmetric conference matrix of order n and I identity matrix of same order. Then, $X = aI\boldsymbol{i} + bN$ and $Y = cI\boldsymbol{j} + dN\boldsymbol{k}$ are orthogonal designs of quaternions $ODQ(n;1,n-1)$, and $XY^Q + YX^Q = 0$, so X and Y are $AAODQ(n;1,n-1;1,n-1)$ (anti-amicable orthogonal design of quaternions). Hence $\left[\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right]$ is a $ODQ(2n;1,1,n-1,n-1)$.*

The proof for Lemma 10.5 is straightforward.

Example 10.14. For a symmetric conference matrix N of order 6, we construct the following matrices:

$$X = \begin{bmatrix} a\boldsymbol{i} & b & b & b & b & b \\ b & a\boldsymbol{i} & b & -b & -b & b \\ b & b & a\boldsymbol{i} & b & -b & -b \\ b & -b & b & a\boldsymbol{i} & b & -b \\ b & -b & -b & b & a\boldsymbol{i} & b \\ b & b & -b & -b & b & a\boldsymbol{i} \end{bmatrix} \quad Y = \begin{bmatrix} c\boldsymbol{j} & d\boldsymbol{k} & d\boldsymbol{k} & d\boldsymbol{k} & d\boldsymbol{k} & d\boldsymbol{k} \\ d\boldsymbol{k} & c\boldsymbol{j} & d\boldsymbol{k} & -d\boldsymbol{k} & -d\boldsymbol{k} & d\boldsymbol{k} \\ d\boldsymbol{k} & d\boldsymbol{k} & c\boldsymbol{j} & d\boldsymbol{k} & -d\boldsymbol{k} & -d\boldsymbol{k} \\ d\boldsymbol{k} & -d\boldsymbol{k} & d\boldsymbol{k} & c\boldsymbol{j} & d\boldsymbol{k} & -d\boldsymbol{k} \\ d\boldsymbol{k} & -d\boldsymbol{k} & -d\boldsymbol{k} & d\boldsymbol{k} & c\boldsymbol{j} & d\boldsymbol{k} \\ d\boldsymbol{k} & d\boldsymbol{k} & -d\boldsymbol{k} & -d\boldsymbol{k} & d\boldsymbol{k} & c\boldsymbol{j} \end{bmatrix}.$$

X and Y both are $ODQ(6;1,5)$. They also form a pair of $AAODQ(6;1,5;1,5)$.

Corollary 10.5. *Let $p \equiv 1 \pmod 4$ be a prime power. Then there exist orthogonal designs of quaternions $ODQ(p+1;1,p)$ and $ODQ(2(p+1);1,p,1,p)$, also a pair of anti-amicable orthogonal designs of quaternions $AAODQ(p+1;1,p;1,p)$.*

Corollary 10.5 follows directly from Lemma 10.5.

Lemma 10.6. *For a pair of $AAODQ(n;1,n-1;1,n-1)$ X and Y given in Lemma 10.5, then $D = X + Y\boldsymbol{i}$ is an $RQOD(n;1,n-1)$.*

Proof. We have

$$\begin{aligned} D^Q D &= (X^Q - \boldsymbol{i}Y^Q)(X + Y\boldsymbol{i}) \\ &= X^Q X + X^Q Y\boldsymbol{i} - \boldsymbol{i}Y^Q X - \boldsymbol{i}Y^Q Y\boldsymbol{i} \\ &= (X^Q X + Y^Q Y) + (X^Q Y)\boldsymbol{i} - \boldsymbol{i}(Y^Q X). \end{aligned}$$

For $X = xI\boldsymbol{i} + bN$ and $Y = cI\boldsymbol{j} + dN\boldsymbol{k}$, where N is a conference matrix of order n and I is the identity matrix with same order, we have

$$\begin{aligned} X^Q Y &= (-aI\boldsymbol{i} + bN^T)(cI\boldsymbol{j} + dN\boldsymbol{k}) \\ &= -acI\boldsymbol{k} + adN\boldsymbol{j} + bcN\boldsymbol{j} + bdNN^T\boldsymbol{k} \\ &= -Y^Q X, \end{aligned}$$

since only quaternions $\boldsymbol{k}$ and $\boldsymbol{j}$ appear in $X^Q Y$, we have $(X^Q Y)\boldsymbol{i} = \boldsymbol{i}(Y^Q X)$. Hence,

$$D^Q D = X^Q X + Y^Q Y = (a^2 + (n-1)b^2 + c^2 + (n-1)d^2)I_n,$$

i.e. D is an $RQOD(n;1,n-1)$. □

Example 10.15. Consider a pair of $AAODQ(6;1,5;1,5)$ given in Example 10.14, we have the following $D = X + Y\boldsymbol{i}$:

$$\begin{bmatrix} \boldsymbol{i}(a-c\boldsymbol{j}) & b+d\boldsymbol{j} & b+d\boldsymbol{j} & b+d\boldsymbol{j} & b+d\boldsymbol{j} & b+d\boldsymbol{j} \\ b+d\boldsymbol{j} & \boldsymbol{i}(a-c\boldsymbol{j}) & b+d\boldsymbol{j} & -(b+d\boldsymbol{j}) & -(b+d\boldsymbol{j}) & b+d\boldsymbol{j} \\ b+d\boldsymbol{j} & b+d\boldsymbol{j} & \boldsymbol{i}(a-c\boldsymbol{j}) & b+d\boldsymbol{j} & -(b+d\boldsymbol{j}) & -(b+d\boldsymbol{j}) \\ b+d\boldsymbol{j} & -(b+d\boldsymbol{j}) & b+d\boldsymbol{j} & \boldsymbol{i}(a-c\boldsymbol{j}) & b+d\boldsymbol{j} & -(b+d\boldsymbol{j}) \\ b+d\boldsymbol{j} & -(b+d\boldsymbol{j}) & -(b+d\boldsymbol{j}) & b+d\boldsymbol{j} & \boldsymbol{i}(a-c\boldsymbol{j}) & b+d\boldsymbol{j} \\ b+d\boldsymbol{j} & b+d\boldsymbol{j} & -(b+d\boldsymbol{j}) & -(b+d\boldsymbol{j}) & b+d\boldsymbol{j} & \boldsymbol{i}(a-c\boldsymbol{j}) \end{bmatrix}.$$

In above design D, if we replace quaternion element $\boldsymbol{j}$ by $\boldsymbol{i}$, $\boldsymbol{i}$ by an undecided quaternion element $\boldsymbol{q}$, and let complex variables $z_1 = a + c\boldsymbol{i}$ and $z_2 = b + d\boldsymbol{i}$, then we have D:

$$\begin{bmatrix} \boldsymbol{q}z_1^* & z_2 & z_2 & z_2 & z_2 & z_2 \\ z_2 & \boldsymbol{q}z_1^* & z_2 & -z_2 & -z_2 & z_2 \\ z_2 & z_2 & \boldsymbol{q}z_1^* & z_2 & -z_2 & -z_2 \\ z_2 & -z_2 & z_2 & \boldsymbol{q}z_1^* & z_2 & -z_2 \\ z_2 & -z_2 & -z_2 & z_2 & \boldsymbol{q}z_1^* & z_2 \\ z_2 & z_2 & -z_2 & -z_2 & z_2 & \boldsymbol{q}z_1^* \end{bmatrix}.$$

$\boldsymbol{q}$ in above D can be chosen from the set $\{\pm\boldsymbol{k}, \pm\boldsymbol{j}\}$ since $\boldsymbol{q}z_1^* z_2^* = z_2 z_1 \boldsymbol{q}$ for any $\boldsymbol{q} \in \{\pm\boldsymbol{k}, \pm\boldsymbol{j}\}$. It is easy to prove $D^Q D = (|z_1|^2 + 5|z_2|^2)I_6$. Hence, D is a restricted quaternion orthogonal design $RQOD(6;1,5)$ with no zero entries.

10.7 Le Tran's Complex Orthogonal Designs of Order Eight

Square, Complex Orthogonal Space-Time Block Codes (CO STBCs) are known for the relatively simple receiver structure and minimum processing delay in the case of complex signal constellations. One of the methods to construct square CO STBCs is based on amicable orthogonal designs (AODs). The simplest CO STBC is the Alamouti code [3] for two transmitter (Tx) antennas, which is based on an amicable orthogonal pair of order-2 matrices. The Alamouti code achieves the transmission rate of one for 2 TX antennas, while the CO STBCs for more than 2 Tx antennas cannot provide the rate of one (see [214, Section 2.3] or [148, 149]). However they can still achieve the full diversity for the given number of Tx antennas.

The construction of CO STBCs follows directly from complex orthogonal designs (CODs).

Definition 10.9. A square COD $Z = X + iY$ of order n is an $n \times n$ matrix on the complex indeterminates $s_1, \ldots, s_p$, with entries chosen from $0, \pm s_1, \ldots, \pm s_p$, their conjugates $\pm s_1^*, \ldots, \pm s_p^*$, or their products with $i = \sqrt{-1}$ such that:

$$Z^H Z = \left(\sum_{k=1}^{p} |s_k|^2 \right) I_n \tag{10.1}$$

where Z^H denotes the Hermitian transpose of Z and I_n is the identity matrix of order n.

For the matrix Z to satisfy (10.1), the matrices X and Y must be a pair of AODs, which implies that both X and Y are orthogonal designs themselves and $XY^\top = YX^\top$, where $(.)^\top$ denotes matrix transposition.

It has been shown in [80] that, for $n = 8$, the total number of different variables in the amicable pair X and Y cannot exceed eight.

It has been shown in [203], that the construction of CODs can be faciliated by representing Z as

$$Z = \sum_{j=1}^{p} A_j s_j^R + i \sum_{j=1}^{p} B_j s_j^I \tag{10.2}$$

where s_j^R and s_j^I denote the real and imaginary parts of the complex variables $s_j = s_j^R + i s_j^I$ and A_j and B_j are the real coefficient matrices for s_j^R and s_j^I, respectively. To satisfy (10.1), the matrices $\{A_j\}$ and $\{B_j\}$ of order n must satisfy the following conditions:

$$\begin{aligned} &A_j A_j^\top = I,\ B_j B_j^\top = I,\ \forall j = 1,\ldots,p \\ &A_k A_j^\top = -A_j A_j^\top,\ B_k B_j^\top = -B_j B_k^\top,\ k \neq j \\ &A_k B_j^\top = B_j A_k^\top,\ \forall k,\ j = 1,\ldots,p \end{aligned} \tag{10.3}$$

The conditions in (10.3) are necessary and sufficient for the existence of AODs of order n. Thus, the problem of finding CODs is connected to the theory of AODs.

From the perspective of constructing CO STBCs, the most promising case is that in which both X and Y have four variables. This case has been considered in the conventional, order-8 CO STBCs, corresponding to $COD(8;1,1,1,1)$ with all four variables appearing *once* in each column of Z. An example is given in Fig. 10.1, (see [209, 210], or [214, Eq.(2.34)]).

Fig. 10.1 A conventional COD of order eight [a]

$$\mathbf{Z}_1 = \begin{bmatrix} s_1 & s_2 & s_3 & 0 & s_4 & 0 & 0 & 0 \\ -s_2^* & s_1^* & 0 & -s_3 & 0 & -s_4 & 0 & 0 \\ -s_3^* & 0 & s_1^* & s_2 & 0 & 0 & -s_4 & 0 \\ 0 & s_3^* & -s_2^* & s_1 & 0 & 0 & 0 & s_4 \\ -s_4^* & 0 & 0 & 0 & s_1^* & s_2 & s_3 & 0 \\ 0 & s_4^* & 0 & 0 & -s_2^* & s_1 & 0 & -s_3 \\ 0 & 0 & s_4^* & 0 & -s_3^* & 0 & s_1 & s_2 \\ 0 & 0 & 0 & -s_4^* & 0 & s_3^* & -s_2^* & s_1^* \end{bmatrix}$$

[a] Tran, Wysocki, Mertins, and Seberry [213, p75] ©Springer

These conventional codes contain numerous zero entries which are undesirable. Note that we use the similar notation to that mentioned in [80], i.e. $COD(8;1,1,1,1)$, to denote a square, order-8 COD containing four complex variables and each variable appearing once in each column. Readers may refer to [80] for more details.

In [186, 212, 256], two new codes of order eight are introduced where some variables appear more often than others (more than once in each column), i.e., codes based on $COD(8;1,1,2,2)$ and $COD(8;1,1,1,4)$. These codes, namely Z_2 and Z_3, are given in Fig. 10.2 and 10.3, respectively. It is easy to check that these codes satisfy the conditions (10.1).

Fig. 10.2 Code Z_2 [a]

$$\mathbf{Z}_2 = \begin{bmatrix} s_1 & s_2 & \frac{s_3}{\sqrt{2}} & \frac{s_3}{\sqrt{2}} & 0 & 0 & \frac{s_4}{\sqrt{2}} & \frac{s_4}{\sqrt{2}} \\ -s_2^* & s_1^* & \frac{s_3}{\sqrt{2}} & -\frac{s_3}{\sqrt{2}} & 0 & 0 & \frac{s_4}{\sqrt{2}} & -\frac{s_4}{\sqrt{2}} \\ \frac{s_3^*}{\sqrt{2}} & \frac{s_3^*}{\sqrt{2}} & -s_1^R + is_2^I & -s_2^R + is_1^I & \frac{s_4}{\sqrt{2}} & \frac{s_4}{\sqrt{2}} & 0 & 0 \\ \frac{s_3^*}{\sqrt{2}} & -\frac{s_3^*}{\sqrt{2}} & s_2^R + is_1^I & -s_1^R - is_2^I & \frac{s_4}{\sqrt{2}} & -\frac{s_4}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{s_4^*}{\sqrt{2}} & \frac{s_4^*}{\sqrt{2}} & s_1 & s_2 & -\frac{s_3^*}{\sqrt{2}} & -\frac{s_3^*}{\sqrt{2}} \\ 0 & 0 & \frac{s_4^*}{\sqrt{2}} & -\frac{s_4^*}{\sqrt{2}} & -s_2^* & s_1^* & -\frac{s_3^*}{\sqrt{2}} & \frac{s_3^*}{\sqrt{2}} \\ \frac{s_4^*}{\sqrt{2}} & \frac{s_4^*}{\sqrt{2}} & 0 & 0 & -\frac{s_3^*}{\sqrt{2}} & -\frac{s_3^*}{\sqrt{2}} & -s_1^R + is_2^I & -s_2^R + is_1^I \\ \frac{s_4^*}{\sqrt{2}} & -\frac{s_4^*}{\sqrt{2}} & 0 & 0 & -\frac{s_3^*}{\sqrt{2}} & \frac{s_3^*}{\sqrt{2}} & s_2^R + is_1^I & -s_1^R - is_2^I \end{bmatrix}$$

[a] Tran, Wysocki, Mertins, and Seberry [213, p76] ©Springer

Fig. 10.3 Code Z_3 [a]

$$\mathbf{Z}_3 = \begin{bmatrix} s_1 & 0 & s_3^R + is_2^I & s_2^R + is_3^I & \frac{s_4}{2} & \frac{s_4}{2} & \frac{s_4}{2} & \frac{s_4}{2} \\ 0 & s_1 & -s_2^R + is_3^I & s_3^R - is_2^I & \frac{s_4}{2} & -\frac{s_4}{2} & \frac{s_4}{2} & -\frac{s_4}{2} \\ -s_3^R + is_2^I & s_2^R + is_3^I & s_1^* & 0 & \frac{s_4}{2} & \frac{s_4}{2} & -\frac{s_4}{2} & -\frac{s_4}{2} \\ -s_2^R + is_3^I & -s_3^R - is_2^I & 0 & s_1^* & \frac{s_4}{2} & \frac{s_4}{2} & -\frac{s_4}{2} & \frac{s_4}{2} \\ -\frac{s_4^*}{2} & -\frac{s_4^*}{2} & -\frac{s_4^*}{2} & -\frac{s_4^*}{2} & s_1^R - is_3^I & s_2^* & s_3^R - is_1^I & 0 \\ -\frac{s_4^*}{2} & \frac{s_4^*}{2} & -\frac{s_4^*}{2} & \frac{s_4^*}{2} & -s_2 & s_1^R + is_3^I & 0 & s_3^R - is_1^I \\ -\frac{s_4^*}{2} & -\frac{s_4^*}{2} & \frac{s_4^*}{2} & \frac{s_4^*}{2} & -s_3^R - is_1^I & 0 & s_1^R + is_3^I & -s_2^* \\ -\frac{s_4^*}{2} & \frac{s_4^*}{2} & \frac{s_4^*}{2} & -\frac{s_4^*}{2} & 0 & -s_3^R - is_1^I & s_2 & s_1^R - is_3^I \end{bmatrix}$$

[a] Tran, Wysocki, Mertins, and Seberry [213, p77] ©Springer

All the CO STBCs proposed here achieve the *maximum* code rate for order-8, *square* CO STBCs, which is equal to $\frac{1}{2}$. We would like to recall that, according to Liang's paper [148], the maximum achievable rate of CO STBCs for $n = 2m-1$ or $n = 2m$ Tx antennas is $R_{\max} = \frac{(m+1)}{2m}$. Particularly, for $n = 8$, i.e., $m = 4$, the maximum achievable rate of CO STBCs is $\frac{5}{8}$.

However, this maximum rate is only achievable for *non-square* constructions. For *square* constructions of orders $n = 2^a(2b+1)$, the maximum achievable rate is $R_{\max} = \frac{(a+1)}{[2^a(2b+1)]}$. For $n = 8$, i.e., $a = 3$ and $b = 0$, the maximum achievable rate of *square* CO STBCs is only $\frac{1}{2}$.

The vague statement on the maximum achievable rate of CO STBCs in Liang's paper [148], which easily makes readers confused, has been pointed out in [214, Remark 2.3.2.1]

A question that could be raised is why *square* CO STBCs are of particular interest. It is because, *square* CO STBCs have a great advantage over *non-square* CO STBCs that they require a much smaller length of the codes, i.e., much smaller processing delay, though, the maximum rate of the former may be smaller than that of the later.

Let us consider CO STBCs for $n = 8$ Tx antennas as an example (also see [214, Example 2.3.2.1]). The *non-square* CO STBC that achieves the maximum rate 5/8 requires the length of 112 STSs as shown by Table 2.6 in [214, p.40]. The [112,8,70] CO STBC given in Appendix E in Liang's paper [55] is an example for this case. As opposed to *non-square* CO STBCs, *square* CO STBCs only require the length of 8 STSs to achieve the maximum rate 1/2, which is slightly smaller than the maximum rate of *non-square* CO STBCs. Clearly, *square* CO STBCs require a much shorter length, especially for a large number of Tx antennas, with the consequence of a slightly lower code rate. For this reason, *square* CO STBCs are of our particular interest.

Apart from having the maximum rate, our proposed CO STBCs Z_2 and Z_3, (see Figures 10.2 and 10.3) have fewer zero entries (compared to the conventional codes) or even no zero entries in the code matrices. This property results in a more uniform transmission power distribution between Tx antennas. Intuitively, due to this property, our proposed CO STBCs require a lower peak power per Tx antenna to achieve the same bit error performance as the conventional CO STBCs containing numerous zeros. Equivalently, with the same peak power at Tx antennas, our proposed codes provide a better bit error performance than the conventional CO STBCs.

In addition, our codes are more amenable to practical implementation than the conventional code, since, transmitter antennas are turned off less frequently or even are not required to be turned off during transmission unlike with the conventional codes.

10.8 Research Problem

Thus we have some methods for building amicable orthogonal designs over the real and quaternion domain, e.g. the way to construct amicable orthogonal designs of quaternions (AODQ) by using Kronecker product with **real** amicable orthogonal designs or **real** weighing matrices from an amicable family.

This construction ensures that, for any existing **real** amicable orthogonal design generated by using the Kronecker product, we can easily find an AODQ with same order and type. We also showed that if A and B forms a pair of AODQ, then the combined design $A+B\boldsymbol{q}$ for $\boldsymbol{q} \in \{\pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\}$is an RQOD by carefully choosing $\boldsymbol{q}$. Our newly constructed AODQs and RQODs, especially those with no zero entries, could have applications as orthogonal space-time-polarization block codes.

However, there are still some problems which need to be solved:

Problem 10.2 (Research Problem 5). Do there exist any new amicable orthogonal designs of quaternions for which there are no such real or complex designs.

Problem 10.3 (Research Problem 6). Determine the maximum number of variables in an AODQ.

It is known that finding the maximum number of variables in an AOD is equivalent to finding the number of members in a *Hurwitz-Radon family* of corresponding type [80], which also implies that the so-called Clifford algebras [29] have a matrix representation of the same order.

Problem 10.4 (Research Problem 7). How can we find a set of anti-commuting real, complex and quaternion matrices representation to determine the maximum number of variables in an AODQ.