Dirac Cohomology in Representation Theory

Jing-Song Huang

Abstract This article is an introduction to Dirac cohomology for reductive Lie groups, reductive Lie algebras and rational Cherednik algebras. We also survey recent results focusing particularly on Dirac cohomology of unitary representations and its connection with Lie algebra cohomology.

Keywords Category $\mathcal{O}$ • Dirac cohomology • Harish-Chandra module • Rational Cherednik algebra • Reductive Lie group and Lie algebra

1 Introduction

Consider a possibly indefinite inner product $\langle x, y\rangle = \sum_i \epsilon_i x_i y_i$, for $x, y \in \mathbb{R}^n$ with $n \geqslant 2$ and $\epsilon_i = \pm 1$. Let $\Delta = \sum_i \epsilon_i \partial_i^2$ be the corresponding Laplace operator. We look for a first order differential operator D such that $D^2 = \Delta$. If we write $D = \sum_i e_i \partial_i$ for some scalars e_i, then $D^2 = \sum_i e_i^2 \partial_i^2 + \sum_{i<j}(e_i e_j + e_j e_i)\partial_i \partial_j$. It leads to require the relations

$$e_i^2 = \epsilon_i \qquad \text{and} \qquad e_i e_j + e_j e_i = 0 \quad \text{for} \quad i \neq j.$$

This is clearly impossible for real or complex scalars e_i's. Nevertheless, we can consider an algebra generated by $e_1, \ldots, e_n$, satisfying the same relations. If we allow e_i's to be in the Clifford algebra, then we do get a Dirac operator D which squares to Δ.

In representation theory Dirac operators were employed in 1970s by Parthasarathy [50] and Atiyah-Schmid [5] for purpose of constructing the discrete series representations [20]. It turns out that they can be constructed as kernels of Dirac operators acting on certain spin bundles on the symmetric space G/K. In 1990s, Vogan made a conjecture on the property of the Dirac operator in the setting of a reductive Lie algebra and its associated Clifford algebra [53]. This property

J.-S. Huang (✉)
Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong SAR, China
e-mail: mahuang@ust.hk

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implies that the standard parameter of the infinitesimal character of a Harish-Chandra module X and the infinitesimal character of its Dirac cohomology $H_D(X)$ are conjugate under the Weyl group. Vogan's conjecture was consequently verified in [26], and it has been playing a key role in the theory of Dirac cohomology. Dirac cohomology offers new perspectives for understanding irreducible unitary representations and proofs of some classical theorems. It is a basic invariant related to $(\mathfrak{g}, K)$-cohomology, $\mathfrak{u}$-cohomology, the K-characters and the global characters. It has interesting applications in harmonic analysis such as branching laws and endoscopy. We summarize some recent results here.

1. Dirac cohomology provides a new point of view for understanding classic theory. The geometric construction of discrete series representations initially did not use Dirac cohomology and Vogan's conjecture, but using Dirac cohomology makes some of the proofs easier [28]. Dirac cohomology is further used for geometric quantization [11, 22]. Simpler proofs of the generalized Weyl character formula [39] and generalized Bott-Borel-Weil theorem [40] are given in [28]. Moreover, Dirac cohomology is used to extend the Langlands formula on dimensions of automorphic forms [45] to a slightly more general setting [28].
2. The Dirac cohomology of several families of Harish-Chandra modules has been determined. These modules include finite-dimensional modules and irreducible unitary $A_{\mathfrak{q}}(\lambda)$-modules [25]. It was proved that if X is a unitary Harish-Chandra module, then

$$H^*(\mathfrak{g}, K; X \otimes F^*) \cong \mathrm{Hom}(H_D(F), H_D(X))$$

   for any irreducible finite-dimensional module F. More precisely, Dirac cohomology determines the $(\mathfrak{g}, K)$-cohomology when the latter exists, and can be thought of as a generalization of $(\mathfrak{g}, K)$ cohomology when the latter no longer exists. It is evident that unitary representations with nonzero Dirac cohomology are closely related to automorphic representations [54].
3. Another aspect of Dirac cohomology is its connection with $\mathfrak{u}$-cohomology. Kostant has extended Vogan's conjecture to the setting of the cubic Dirac operator and proved a nonvanishing result on Dirac cohomology for highest weight modules in the most general setting [41]. He also determined the Dirac cohomology of finite-dimensional modules in the equal rank case. The Dirac cohomology for all irreducible highest weight modules was determined in [34] in terms of coefficients of Kazhdan–Lusztig polynomials. It is proved Dirac cohomology and $\mathfrak{u}$-cohomology are isomorphic up to a one-dimensional character for irreducible highest weight modules [34].
4. Dirac cohomology, or rather its Euler characteristic, or the Dirac index, gives the K-characters of representations. It leads to a generalization of certain classical branching formulas due to Littlewood [32] which describe how a finite dimensional representation of $GL(n, \mathbb{C})$ decomposes under orthogonal or symplectic subgroups. We also generalize some of the other classical branching rules in [23]. When G is Hermitian symmetric and $\mathfrak{u}$ is unipotent radical of

a parabolic subalgebra with Levi subgroup K, [30] showed that for a unitary representation its Dirac cohomology is isomorphic to its $\mathfrak{u}$-cohomology up to a twist of a one-dimensional character. In particular, Enright's calculation of $\mathfrak{u}$-cohomology [16] gives the Dirac cohomology of the irreducible unitary highest weight modules. The Dirac cohomology of unitary lowest weight modules of scalar type is calculated more explicitly in [31]. Dirac cohomology of more families of unitary representation are determined in [6, 7] and [48].

5. Dirac index and the K-character are intimately related to the global characters on the set of elliptic elements. Dirac cohomology is employed as a tool to study a class of irreducible unitary representations, called elliptic representations [24]. More precisely, Harish-Chandra showed that the characters of irreducible or more generally admissible representations are locally integrable functions and smooth on the open dense subset of regular elements [19]. An elliptic representation has a global character that does not vanish on the elliptic elements in the set of regular elements. It is proved that an irreducible admissible (not necessarily unitary) representation is elliptic if and only if its Dirac index is nonzero [24, Theorem 8.3]. Dirac index is nonzero implies that Dirac cohomology is nonzero. Note that under the condition of regular infinitesimal character, the Dirac index is zero if and only if the Dirac cohomology is zero [24, Theorem 10.1]. This equivalence is conjectured to hold in general without the regularity condition [24, Conjecture 10.3]. In particular, an irreducible tempered elliptic representation has nonzero Dirac cohomology, and therefore it is a discrete series or a limit of discrete series representation [15, Theorem 7.5]. The characters of the irreducible tempered elliptic representations are associated in a natural way to the supertempered distributions defined by Harish-Chandra [21].
6. Better understanding of the endoscopic transfer factor for real groups [47] is the first of the 'problems for real groups' raised by Arthur [4]. It is observed [24] there is a connection between Labesse's calculation [44] of the endoscopic transfer of pseudo-coefficients of discrete series [43] and the calculation of the characters of the Dirac index of discrete series. This offers a new point of view for understanding the endoscopic transfer in the framework of Dirac cohomology and the Dirac index.
7. Vogan's conjecture has been extended to several other settings by many authors as follows:

   (i) Kostant considered the case when the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is replaced by any reductive subalgebra $\mathfrak{r}$ such that the form B remains nondegenerate when restricted to $\mathfrak{r}$. The appropriate analogue of D is then Kostant's cubic Dirac operator. He generalized Vogan's conjecture to this setting of the cubic Dirac operator [41].

   (ii) Alekseev and Meinrenken proved a version of Vogan's conjecture in their study of *Lie theory and the Chern–Weil homomorphism* [2].

   (iii) Kumar proved a similar version of Vogan's conjecture in *Induction functor in non-commutative equivariant cohomology and Dirac cohomology* [42].

(iv) Pandžić and I defined an analogue of D and prove an analogue of Vogan's conjecture in case when $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a basic classical Lie superalgebra. We extended Vogan's conjecture to the symplectic Dirac operator in Lie superalgebras [27].

(v) Kac, Möseneder Frajria and Papi extended Vogan's conjecture to the affine cubic Dirac operator in affine Lie algebras [37].

(vi) Barbasch, Ciubotaru and Trapa extended Vogan's conjecture to the setting of Luszting's graded affine Hecke algebras [8]. They also found applications of Dirac cohomology to unitary representations of p-adic groups.

(vii) Ciubotaru and Trapa proved a version of Vogan's conjecture for studying Weyl group representations in connection with Springer theory [13].

Mostly recently, Ciubotaru extended the definition of Dirac operator and Vogan's conjecture to the setting of Drinfeld's graded Hecke algebras including symplectic reflection algebras [17] and particularly rational Cherednik algebras [12]. Many results on Dirac cohomology and Lie algebra cohomology for Hermitian symmetric Lie groups have analogues for rational Cherednik algebras [33].

2 Dirac Cohomology of Harish-Chandra Modules

A complex Lie algebra $\mathfrak{g}$ is called reductive if its adjoint representation is completely reducible [35]. A Lie group G is called reductive if the complexification of the Lie algebra of G is reductive. Typical examples of reductive Lie groups include various matrix groups, i.e., closed subgroups of the general linear group $GL(n, \mathbb{C})$, for instance, $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $U(p, q)$, $O(p, q)$, $Sp(p, q)$ and $Sp(2n, \mathbb{R})$. Each reductive Lie group G comes with a Cartan involution Θ. In the above matrix examples, one can take Θ to be the transpose inverse of the complex conjugate matrix, i.e., $\Theta(g) = (\bar{g}^{-1})^t$. In what follows we assume that the group $K = G^{\Theta}$ of fixed points of Θ is a maximal compact subgroup of G. The involution Θ induces a decomposition of the complexified Lie algebra of G, called the Cartan decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{1}$$

where $\mathfrak{k}$ is the complexified Lie algebra of K, and $\mathfrak{p}$ is the (-1)-eigenspace for the differential of Θ.

A topological vector space $\mathcal{H}$ over $\mathbb{C}$ is a representation of G if there is a continuous action of G on $\mathcal{H}$ by linear operators. Assume now that G is reductive with Cartan involution Θ and maximal compact subgroup $K = G^{\Theta}$. Then we can consider the subspace $\mathcal{H}_K$ of the representation space $\mathcal{H}$ consisting of K-finite vectors, i.e., vectors $h \in \mathcal{H}$ such that the subspace of $\mathcal{H}$ spanned by $K \cdot h$ is finite-dimensional. One can show that the G-action on $\mathcal{H}$ induces an action of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}_K$. Thus $\mathcal{H}_K$ becomes an example of a *Harish-Chandra* *module* for the pair $(\mathfrak{g}, K)$, which is by definition a vector space with a Lie algebra action of $\mathfrak{g}$ and a finite action of the group K, with certain natural compatibility conditions. The space $\mathcal{H}_K$ can be decomposed into a direct sum of irreducible (finite-dimensional) representations of K, each appearing with certain multiplicity. If all these multiplicities are finite, then $\mathcal{H}_K$ and $\mathcal{H}$ are called admissible. An important special class of representations of G consists of *unitary representations*, for which the space $\mathcal{H}$ is a Hilbert space, and G acts on $\mathcal{H}$ by unitary operators. Harish-Chandra showed that irreducible unitary representations are automatically admissible. Irreducible admissible representations were classified by Langlands [46]. We refer to [52] and [38] for the theory of representations of real reductive Lie groups.

Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, which restricts to the Killing form on the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ the Clifford algebra of $\mathfrak{p}$ with respect to B. Then one can consider the following version of the Dirac operator:

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p});$$

here $Z_1, \ldots, Z_n$ is an orthonormal basis of $\mathfrak{p}$ with respect to the symmetric bilinear form B. It follows that D is independent of the choice of the orthonomal basis $Z_1, \ldots, Z_n$ and it is invariant under the diagonal adjoint action of K.

The Dirac operator D is a square root of the Laplace operator associated to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. To explain this, we start with a Lie algebra map

$$\alpha : \mathfrak{k} \to C(\mathfrak{p}),$$

which is defined by the adjoint map $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ composed with the embedding of $\mathfrak{so}(\mathfrak{p})$ into $C(\mathfrak{p})$ using the identification $\mathfrak{so}(\mathfrak{p}) \simeq \bigwedge^2 \mathfrak{p}$. The explicit formula for α is (see [28, §2.3.3])

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j. \tag{2}$$

Using α we can embed the Lie algebra $\mathfrak{k}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, by

$$X \mapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{k})$. We denote the image of $\mathfrak{k}$ by $\mathfrak{k}_\Delta$, and then the image of $U(\mathfrak{k})$ is the enveloping algebra $U(\mathfrak{k}_\Delta)$ of $\mathfrak{k}_\Delta$.

Let $\Omega_\mathfrak{g}$ be the Casimir operator for $\mathfrak{g}$, given by $\Omega_\mathfrak{g} = \sum Z_i^2 - \sum W_j^2$, where W_j is an orthonormal basis for $\mathfrak{k}_0$ with respect to the inner product $-B$, where B is the Killing form. Let $\Omega_\mathfrak{k} = -\sum W_j^2$ be the Casimir operator for $\mathfrak{k}$. The image of $\Omega_\mathfrak{k}$ under Δ is denoted by $\Omega_{\mathfrak{k}_\Delta}$. Fix a positive root system $\Delta^+(\mathfrak{g})$ for $\mathfrak{t}$ in $\mathfrak{g}$. Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Write $\rho = \rho(\Delta^+(\mathfrak{g}))$, $\rho_c = \rho(\Delta^+(\mathfrak{k}))$ and $\rho_n = \rho - \rho_c$. Then

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + (||\rho_c||^2 - ||\rho||^2)1 \otimes 1. \tag{3}$$

The Parthasarathy's Dirac inequality for unitary Harish-Chandra modules is an important criteria for irreducible unitary representations of reductive Lie groups. Let X be an irreducible Harish-Chandra module with infinitesimal character Λ. Consider the action of the Dirac operator D on $X \otimes S$, with S the spinor module for the Clifford algebra $C(\mathfrak{p})$. If X is unitary, then D is self-adjoint with respect to a natural Hermitian inner product on $X \otimes S$. Let E_μ be any $\widetilde{K}$-module occurring in $X \otimes S$ with a highest weight $\mu \in \mathfrak{t}^*$, then

$$\langle \mu + \rho_c, \mu + \rho_c \rangle \geq \langle \Lambda, \Lambda \rangle.$$

The Dirac cohomology are defined to be those E_μ so that the equality holds, namely $H_D(X) = \operatorname{Ker} D = \operatorname{Ker} D^2$.

For better understanding these E_μ in $\operatorname{Ker} D$, Vogan formulated a conjecture saying that every element $z \otimes 1$ of $Z(\mathfrak{g}) \otimes 1 \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$ can be written as

$$\zeta(z) + Da + bD$$

where $\zeta(z)$ is in $Z(\mathfrak{k}_\Delta)$, and $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Vogan's conjecture implies a refinement of Parthasarathy's Dirac inequality, namely the equality holds if and only if conjugate of Λ is equal to $\mu + \rho_c$.

A main result in [26] is introducing a differential d on the K-invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by a super bracket with D, and determining the cohomology of this differential complex. As a consequence, Pandžić and I proved the following theorem. In the following we denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ so that $\mathfrak{t}^*$ is embedded into $\mathfrak{h}^*$, and by W and W_K the Weyl groups of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively.

Theorem 1 ([26]) *Let $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}) \cong Z(\mathfrak{k}_\Delta)$ be the algebra homomorphism that is determined by the following commutative diagram:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{k}) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta_{\mathfrak{k}}} \\ P(\mathfrak{h}^*)^W & \xrightarrow{\text{Res}} & P(\mathfrak{t}^*)^{W_K}, \end{array}$$

where P denotes the polynomial algebra, and vertical maps η and $\eta_{\mathfrak{k}}$ are Harish-Chandra isomorphisms. Then for each $z \in Z(\mathfrak{g})$ one has

$$z \otimes 1 - \zeta(z) = Da + aD, \textit{ for some } a \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

The Dirac cohomology is defined as follows:

$$H_D(X) := \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D.$$

It follows from the identity (3) that $H_D(X)$ is a finite-dimensional module for the spin double cover $\widetilde{K}$ of K. As a consequence of the above theorem, we have that $H_D(X)$, if nonzero, determines the infinitesimal character of X.

Theorem 2 ([26]) *Let X be an admissible $(\mathfrak{g}, K)$-module with standard infinitesimal character parameter $\Lambda \in \mathfrak{h}^*$. Suppose that $H_D(X)$ contains a representation of $\widetilde{K}$ with infinitesimal character λ. Then Λ and $\lambda \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$ are conjugate under W.*

The above theorem is proved in [26] for a connected semisimple Lie group G. It is straightforward to extend the result to a possibly disconnected reductive Lie group in Harish-Chandra's class [15].

Let G be a connected reductive algebraic group over a local field F of characteristic 0. Arthur [3] studied a subset $\Pi_{\text{temp, ell}}(G(F))$ of tempered representations of $G(F)$, namely elliptic tempered representations. The set of tempered representations $\Pi_{\text{temp}}(G(F))$ includes the discrete series and in general the irreducible constituents of representations induced from the discrete series. These are exactly the representations which occur in the Plancherel formula for $G(F)$.

In Harish-Chandra's theory, the character of an infinite-dimensional representation π is defined as a distribution

$$\Theta(\pi,f) = \operatorname{tr}\Big(\int_{G(F)} f(x)\pi(x)dx \Big), \quad f \in C_c^\infty(G(F)),$$

which can be identified with a function on $G(F)$. In other words,

$$\Theta(\pi,f) = \int_{G(F)} f(x)\Theta(\pi,x)dx, \quad f \in C_c^\infty(G(F)),$$

where $\Theta(\pi,x)$ is a locally integrable function on $G(F)$ that is smooth on the open dense subset $G_{\text{reg}}(F)$ of regular elements. A representation π is called elliptic if $\Theta(\pi,x)$ does not vanish on the set of elliptic elements in $G_{\text{reg}}(F)$. Elliptic representations are precisely those representations with nonzero Dirac index (see 5. in Sect. 1).

We note that a real reductive group $G(\mathbb{R})$ has elliptic elements if and only if it is of equal rank with $K(\mathbb{R})$. We also assume this equal rank condition. Induced representations from proper parabolic subgroups are not elliptic. Consider the quotient of the Grothendieck group of the category of finite length Harish-Chandra modules by the subspace generated by induced representations. Let us call this quotient group the elliptic Grothendieck group. Arthur [3] found an orthonormal basis of this elliptic Grothendieck group in terms of elliptic tempered (possibly virtual) characters. Those characters are the supertempered distributions defined by Harish-Chandra [21].

For a real reductive algebraic group $G(\mathbb{R})$, the Harish-Chandra modules of irreducible elliptic unitary representations with regular infinitesimal characters are showed to be strongly regular (in the sense of [51]) and hence they are $A_{\mathfrak{q}}(\lambda)$-modules.

An irreducible tempered representation is either elliptic or induced from an elliptic tempered representation by parabolic induction. If $G(\mathbb{R})$ is not of equal rank, then there is no elliptic representation for $G(\mathbb{R})$. Still, we know that $G(\mathbb{R})$ has representations with nonzero Dirac cohomology.

Conjecture 1 ([24]) A unitary representation either has nonzero Dirac cohomology or is induced from a unitary representation with nonzero Dirac cohomology by parabolic induction (including complement series).
This conjecture holds for $GL(n, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the twofolds covering group of $GL(n, \mathbb{R})$. A recent preprint of Adams–van Leeuwen–Trapa–Vogan [1] gives an algorithm to determine the irreducible unitary representations. The above conjecture means that one may regard unitary representations with nonzero Dirac cohomology as 'cuspidal' ones. Classification of irreducible unitary representations with nonzero Dirac cohomology remains to be an open problem.

3 Dirac Cohomology in Category $\mathcal{O}$

Let $\mathfrak{g}$ be a complex reductive Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ in a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. The category $\mathcal{O}$ introduced by Bernstein, et al. [9, 36] is the category of all $\mathfrak{g}$-modules, which are finitely generated, locally $\mathfrak{b}$-finite and semisimple under the $\mathfrak{h}$-action. Kostant proved a nonvanishing result on Dirac cohomology for highest weight modules in the most general setting. His theorem implies that for the equal rank case all highest weight modules have nonzero Dirac cohomology. He also determined the Dirac cohomology of finite-dimensional modules in this case. The connection of Dirac cohomology of $(\mathfrak{g}, K)$-modules and that of highest weight modules was studied in [14] using the Jacquet functor. In [34] we determined the Dirac cohomology of all irreducible highest weight modules in terms of Kazhdan–Lusztig polynomials.

We first recall the definition of Kostant's cubic Dirac operator and the basic properties of the corresponding Dirac cohomology. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Killing form B. Let $\mathfrak{r} \subset \mathfrak{g}$ be a reductive Lie subalgebra such that $B|_{\mathfrak{r}\times\mathfrak{r}}$ is nondegenerate. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be the orthogonal decomposition with respect to B. Then the restriction $B|_{\mathfrak{s}}$ is also nondegenerate. Denote by $C(\mathfrak{s})$ the Clifford algebra of $\mathfrak{s}$ with

$$uu' + u'u = -2B(u, u')$$

for all $u, u' \in \mathfrak{s}$. The above choice of sign is the same as in [28], but different from the definition in [39], as well as in [30]. The two different choices of signs make no essential difference since the two bilinear forms are equivalent over $\mathbb{C}$. Now fix an orthonormal basis $Z_1, \ldots, Z_m$ of $\mathfrak{s}$. Kostant [39] defines the cubic Dirac operator D by

$$D = \sum_{i=1}^{m} Z_i \otimes Z_i + 1 \otimes v \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Here $v \in C(\mathfrak{s})$ is the image of the fundamental 3-form $w \in \bigwedge^3(\mathfrak{s}^*)$,

$$w(X, Y, Z) = \frac{1}{2} B(X, [Y, Z]),$$

under the Chevalley map $\bigwedge(\mathfrak{s}^*) \to C(\mathfrak{s})$ and the identification of $\mathfrak{s}^*$ with $\mathfrak{s}$ by the Killing form B. Explicitly,

$$v = \frac{1}{2} \sum_{1 \leqslant i < j < k \leqslant m} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k.$$

The cubic Dirac operator has a good square in analogy with the Dirac operator associated with the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ in Sect. 2. We have a similar Lie algebra map

$$\alpha : \mathfrak{r} \to C(\mathfrak{s})$$

which is defined by the adjoint map $\mathrm{ad} : \mathfrak{r} \to \mathfrak{so}(\mathfrak{s})$ composed with the embedding of $\mathfrak{so}(\mathfrak{s})$ into $C(\mathfrak{s})$ using the identification $\mathfrak{so}(\mathfrak{s}) \simeq \bigwedge^2 \mathfrak{s}$. The explicit formula for α is (see [28, §2.3.3])

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j, \qquad X \in \mathfrak{r}. \tag{4}$$

Using α we can embed the Lie algebra $\mathfrak{r}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, by

$$X \mapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{r})$. We denote the image of $\mathfrak{r}$ by $\mathfrak{r}_\Delta$, and then the image of $U(\mathfrak{r})$ is the enveloping algebra $U(\mathfrak{r}_\Delta)$ of $\mathfrak{r}_\Delta$. Let $\Omega_\mathfrak{g}$ (resp. $\Omega_\mathfrak{r}$) be the Casimir elements for $\mathfrak{g}$ (resp. $\mathfrak{r}$). The image of $\Omega_\mathfrak{r}$ under Δ is denoted by $\Omega_{\mathfrak{r}_\Delta}$.

Let $\mathfrak{h}_\mathfrak{r}$ be a Cartan subalgebra of $\mathfrak{r}$ which is contained in $\mathfrak{h}$. It follows from Kostant's calculation ([39, Theorem 2.16]) that

$$D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{r}_\Delta} - (\|\rho\|^2 - \|\rho_\mathfrak{r}\|^2) 1 \otimes 1, \tag{5}$$

where $\rho_{\mathfrak{r}}$ denotes the half sum of positive roots for $(\mathfrak{r}, \mathfrak{h}_{\mathfrak{r}})$. We also note the sign difference with Kostant's formula due to our choice of bilinear form for the definition of the Clifford algebra $C(\mathfrak{s})$.

We denote by W the Weyl group associated to the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ and $W_{\mathfrak{r}}$ the Weyl group associated to the root system $\Delta(\mathfrak{r}, \mathfrak{h}_{\mathfrak{r}})$. The following theorem due to Kostant is an extension of Vogan's conjecture on the symmetric pair case which is proved in [26]. (See also [41, Theorems 4.1 and 4.2] or [28, Theorem 4.1.4]).

Theorem 3 *There is an algebra homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{r}) \cong Z(\mathfrak{r}_\Delta)$ such that for any $z \in Z(\mathfrak{g})$ one has*

$$z \otimes 1 - \zeta(z) = Da + aD \text{ for some } a \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Moreover, ζ is determined by the following commutative diagram:

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{r}) \\ \eta \downarrow & & \eta_{\mathfrak{r}} \downarrow \\ P(\mathfrak{h}^*)^W & \xrightarrow{\text{Res}} & P(\mathfrak{h}_{\mathfrak{r}}^*)^{W_{\mathfrak{r}}}. \end{array}$$

Here the vertical maps η and $\eta_{\mathfrak{r}}$ are Harish-Chandra isomorphisms.

Let S be a spin module of $C(\mathfrak{s})$. Consider the action of D on $V \otimes S$

$$D : V \otimes S \to V \otimes S \qquad (6)$$

with $\mathfrak{g}$ acting on V and $C(\mathfrak{s})$ on S. The *Dirac cohomology* of V is defined to be the $\mathfrak{r}$-module

$$H_D(V) := \operatorname{Ker} D / \operatorname{Ker} D \cap \operatorname{Im} D.$$

The following theorem is a consequence of the above theorem.

Theorem 4 ([28, 41]) *Let V be a $\mathfrak{g}$-module with $Z(\mathfrak{g})$ infinitesimal character χ_Λ. Suppose that an $\mathfrak{r}$-module N is contained in the Dirac cohomology $H_D(V)$ and has $Z(\mathfrak{r})$ infinitesimal character χ_λ. Then $\lambda = w\Lambda$ for some $w \in W$.*

Suppose that V_λ is a finite-dimensional representation with highest weight $\lambda \in \mathfrak{h}^*$. Kostant [40] calculated the Dirac cohomology of V_λ with respect to any equal rank quadratic subalgebra $\mathfrak{r}$ of $\mathfrak{g}$. Assume that $\mathfrak{h} \subset \mathfrak{r} \subset \mathfrak{g}$ is the Cartan subalgebra for both $\mathfrak{r}$ and $\mathfrak{g}$. Define $W(\mathfrak{g}, \mathfrak{h})^1$ to be the subset of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ defined by

$$W(\mathfrak{g}, \mathfrak{h})^1 = \{w \in W(\mathfrak{g}, \mathfrak{h}) \mid w(\rho) \text{ is } \Delta^+(\mathfrak{r}, \mathfrak{h})\text{-dominant}\}.$$

This is the same as the subset of elements $w \in W(\mathfrak{g}, \mathfrak{h})$ that map the positive Weyl $\mathfrak{g}$-chamber into the positive $\mathfrak{r}$-chamber. There is a bijection

$$W(\mathfrak{r}, \mathfrak{h}) \times W(\mathfrak{g}, \mathfrak{h})^1 \to W(\mathfrak{g}, \mathfrak{h})$$

given by $(w, \tau) \mapsto w\tau$. Kostant proved [40] that

$$H_D(V_\lambda) = \bigoplus_{w \in W(\mathfrak{g}, \mathfrak{h})^1} E_{w(\lambda+\rho)-\rho_{\mathfrak{r}}}.$$

This result has been extended to the unequal rank case by Mehdi and Zierau [49]. Dirac cohomology of a simple highest weight module of possibly infinite dimension and its relation with nilpotent Lie algebra cohomology are determined in [34].

4 Rational Cherednik Algebras

Ciubotaru has extended Dirac cohomology and Vogan's conjecture to very general setting for Drinfeld's graded Hecke algebras including symplectic reflection algebras [17] and particularly rational Cherednik algebras [12]. The case for rational Cherednik algebras is particularly interesting to us, since it has Lie algebra cohomology defined by half Dirac operators [33].

Let W be a finite complex reflection group acting on a complex vector space $\mathfrak{h}$, i.e., W is a finite group generated by the pseudo-reflections $s \in \mathcal{R}$ fixing a hyperplane $H_s \in \mathfrak{h}$. Let $\alpha_s \in \mathfrak{h}^*$ be a non-zero vector so that the W-invariant symmetric pairing $\langle\ ,\ \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$ gives $\langle y, \alpha_s\rangle = 0$ for all $y \in H_s$. Similarly, we define $\alpha_s^\vee \in \mathfrak{h}$ corresponding to the action of s on $\mathfrak{h}^*$. Set $V = \mathfrak{h} \oplus \mathfrak{h}^*$.

The *rational Cherednik algebra* $\mathbf{H}_{t,c}$ associated to $\mathfrak{h}$, W, with parameters $t \in \mathbb{C}$ and W-invariant functions $c : \mathcal{R} \to \mathbb{C}$ is defined as the quotient of $S(V) \rtimes \mathbb{C}[W]$ by the relation

$$[y, x] = t\langle y, x\rangle - \sum_{s \in \mathcal{R}} c(s) \frac{\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} s$$

for all $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$.

Let $\{y_1, \ldots, y_n\}$ be a basis of $\mathfrak{h}$, and $\{x_1, \ldots, x_n\}$ be the corresponding dual basis of $\mathfrak{h}^*$. Set

$$\mathbf{h} := \sum_i (x_i y_i + y_i x_i) = 2\sum_i x_i y_i + nt - \sum_{s \in \mathcal{R}} c(s)s \in \mathbf{H}_{t,c}^W,$$

where $\mathbf{H}_{t,c}^W$ denotes the W-invariants in $\mathbf{H}_{t,c}$. Clearly, $\mathbf{h}$ does not depend on choice of bases. Denote by

$$\Omega_{\mathbf{H}_{t,c}} := \mathbf{h} - \sum_{s\in\mathcal{R}} c(s)\frac{1+\lambda_s}{1-\lambda_s}s = 2\sum_i x_i y_i + nt - \sum_{s\in\mathcal{R}} \frac{2c(s)}{1-\lambda_s}s,$$

where $\lambda_s = \det_{\mathfrak{h}}(s) \in \mathbb{C}$. Then $\Omega_{\mathbf{H}_{t,c}}$ is in $\mathbf{H}_{t,c}^W$ and it satisfies (see [12] (4.12))

$$[\Omega_{\mathbf{H}_{t,c}}, x] = 2tx,\ [\Omega_{\mathbf{H}_{t,c}}, y] = -2ty, \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

Let $\langle\ ,\ \rangle$ be a W-invariant bilinear product on V given by $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = 0$, $\langle x_i, y_j\rangle = \delta_{ij}$. The *Clifford algebra* $C(V)$ with respect to $\langle\ ,\ \rangle$ is the tensor algebra of V subject to the relations

$$x_i x_j + x_j x_i = y_i y_j + y_j y_i = 0,\ x_i y_j + y_j x_i = -2\delta_{ij}.$$

The *spinor module* S corresponding to the Clifford algebra $C(V)$ can be realized as $S \cong \wedge^\bullet \mathfrak{h}$ as vector spaces. The $C(V)$ action on S is defined by

$$x(y_{i_1} \wedge \cdots \wedge y_{i_p}) = 2\sum_j (-1)^j \langle x, y_{i_j}\rangle y_{i_1} \wedge \cdots \wedge \widehat{y_{i_j}} \wedge \cdots \wedge y_{i_p},\quad x \in \mathfrak{h}^*;$$

$$y(y_{i_1} \wedge \cdots \wedge y_{i_p}) = y \wedge y_{i_1} \wedge \cdots \wedge y_{i_p},\quad y \in \mathfrak{h}.$$

We denote by $O(V) = O(V, \langle\ ,\ \rangle)$ the complex orthogonal group preserving the symmetric form $\langle\ ,\ \rangle$. Let $\widetilde{W}$ be the twofolds cover of W defined by the pull back of the covering map $p : \mathrm{Pin}(V) \to O(V)$ via $W \hookrightarrow O(V)$. Then one has

$$\widetilde{W} \hookrightarrow Pin(V) \hookrightarrow C(V)^\times.$$

We note that the covering map $p : \widetilde{W} \to W$ factors through

$$W < GL(\mathfrak{h}) \hookrightarrow O(V) \text{ and } W < GL(\mathfrak{h}^*) \hookrightarrow O(V).$$

There is a well-defined genuine character

$$\chi\colon \widetilde{W} \to \mathbb{C},\ \text{such that } \chi^2(\tilde{w}) = \det_{\mathfrak{h}^*}(p(\tilde{w})).$$

We have the $\widetilde{W}$-module isomorphism

$$S \cong \wedge^\bullet \mathfrak{h} \otimes \chi,$$

where $\widetilde{W}$-action on $\wedge^\bullet \mathfrak{h}$ factors through the natural action of W on $\mathfrak{h}$.

We define the half Dirac operators D_x, D_y and the Dirac operator D by

$$D_x = \sum_i x_i \otimes y_i,\ D_y = \sum_i y_i \otimes x_i \text{ and } D = D_x + D_y \ \in \mathbf{H}_{t,c} \otimes C(V).$$

Clearly, these definitions are independent of the choice of bases.

Proposition 1 (Proposition 4.9 [12]) *We have*

(i) *Let $\Delta : \mathbb{C}[\widetilde{W}] \to \mathbf{H}_{t,c} \otimes C(V)$ be the diagonal embedding $\widetilde{w} \mapsto p(\widetilde{w}) \otimes \widetilde{w}$. Then D, D_x and D_y commute with $\Delta(\mathbb{C}[\widetilde{W}])$.*
(ii) $D_x^2 = D_y^2 = 0$.
(iii) *Let $\Omega_{\widetilde{W},c} \in \mathbb{C}[\widetilde{W}]$ be the Casimir element of $\mathbb{C}[\widetilde{W}]$ defined by (2.3.12) in [12].*

Then $\Delta(\Omega_{\widetilde{W},c}) \in (\mathbf{H}_{t,c} \otimes C(V))^{\widetilde{W}}$, and

$$D^2 = \widetilde{\Omega}_{\mathbf{H}_{t,c}} - \Delta(\Omega_{\widetilde{W},c}),$$

where $\widetilde{\Omega}_{\mathbf{H}_{t,c}} = -\Omega_{\mathbf{H}_{t,c}} \otimes 1 + 1 \otimes \frac{t}{2}(\sum_i x_i y_i + n) \in (\mathbf{H}_{t,c} \otimes C(V))^{\widetilde{W}}$.
For a $\mathbf{H}_{t,c}$-module M, the action of D (and D_x and D_y) on $M \otimes S$ is given by

$$D(m \otimes s)(= D_x(m \otimes s) + D_y(m \otimes s)) = \sum_i x_i \cdot m \otimes y_i s + \sum_j y_j \cdot m \otimes x_j s.$$

The *Dirac cohomology* $H_D(M)$ of M is defined by

$$H_D(M) = \ker D/(\ker D \cap \operatorname{im} D).$$

Regarding $\mathfrak{h}$ and $\mathfrak{h}^*$ as Abelian Lie algebras, one can define the $\mathfrak{h}^*$-cohomology $H^\bullet(\mathfrak{h}^*, M)$ and $\mathfrak{h}$-homology $H_\bullet(\mathfrak{h}, M)$ as W-modules [33]. By the above identification $S = \wedge^\bullet \mathfrak{h} \otimes \chi$ and the differentials with the action of D_x and D_y on the complexes, we have $\widetilde{W}$-module isomorphisms:

$$\ker D_x/\operatorname{im} D_x \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi \text{ and } \ker D_y/\operatorname{im} D_y \cong H_\bullet(\mathfrak{h}, M) \otimes \chi.$$

A $\mathbf{H}_{t,c}$-module M is said to be $\Omega_{\mathbf{H}_{t,c}}$*-admissible* if M can be decomposed into a direct sum of generalized $\Omega_{\mathbf{H}_{t,c}}$-eigenspaces, i.e.

$$M = \bigoplus_\lambda M_\lambda,\ M_\lambda = \{m \in M |\ (\Omega_{\mathbf{H}_{t,c}} - \lambda)^n m = 0\}$$

with each generalized $\Omega_{\mathbf{H}_{t,c}}$-eigenspace M_λ being finite-dimensional. Let M be a $\mathbf{H}_{t,c}$-module that is $\Omega_{\mathbf{H}_{t,c}}$-admissible. Then the Dirac cohomology $H_D(M)$ is a finite-dimensional $\widetilde{W}$-module (see Lemma 3.13 of [12]).

Etingof and Stoica [18] define and study unitary $\mathbf{H}_{t,c}$-modules with respect to a star operation $*$. Let M be such a unitary module. It follows that we have on $M \otimes S$

$$D_x^* = -D_y, D_y^* = -D_x \text{ and } D^* = -D.$$

The following theorem is the analogue of the Hodge decomposition theorem for Dirac cohomology of unitary representations of a reductive Lie group of Hermitian symmetric type [30]. We do not assume that M is $\Omega_{\mathbf{H}_{t,c}}$-admissible in the following theorem.

Theorem 5 ([33]) *Let M be a unitary $\mathbf{H}_{t,c}$-module. Then*

(i) $H_D(M) = \ker D = \ker D^2$.
(ii) $M \otimes S = \ker D \oplus \operatorname{im} D_x \oplus \operatorname{im} D_y$.
(iii) $\ker D_x = \ker D \oplus \operatorname{im} D_x$, $\ker D_y = \ker D \oplus \operatorname{im} D_y$. *Consequently,*

$$H_D(M) \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi \cong H_\bullet(\mathfrak{h}, M) \otimes \chi.$$

We note that for $t \neq 0$ the center of $\mathbf{H}_{t,c}$ consists of scalar $\mathbb{C}$ only. One can however consider a larger commutative subalgebra $\mathcal{B} \subset \mathbf{H}_{t,c} \otimes C(V)$ (see sect. 5.5 [12]) for the extension of Vogan's conjecture in this case. We also refer to Theorem 5.8 [12] for the case $\mathbf{H}_{t,c}$ with $t = 0$. An extension of Vogan's conjecture to more general setting of Drinfeld's Hecke algebras is proved in Theorem 3.5 and Theorem 3.14 [12].

An analogue of the Casselman-Osborne Lemma [10] is proved by generalizing Vogan's conjecture to the setting of half Dirac operators D_x and D_y [33]. This is based on the ideas for the similar results for reductive Lie algebras in [29].

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References

1. J. Adams, M. van Leeuwen, P. Trapa, D.A. Vogan Jr., Unitary representations of real reductive groups. arXiv:1212:2192 (2012 preprint)
2. A. Alekseev, E. Meinrenken, Lie theory and the Chern-Weil homomorphism. Ann. Scient. Éc. Norm. Sup., 4^e série, t. **38**, 303–338 (2005)
3. J. Arthur, On elliptic tempered characters. Acta Math. **171**, 73–138 (1993)
4. J. Arthur, Problems for real groups, in *Representation Theory of Real Reductive Lie Groups*. Contemporary Mathematics, vol. 472 (American Mathematical Society, Providence, RI, 2008), pp. 39–62
5. M. Atiyah, W. Schmid, A geometric construction of the discrete series for semisimple Lie groups. Invent. Math. **42**, 1–62 (1977); **54**, 189–192 (1979)

6. D. Barbasch, P. Pandžić, Dirac cohomology and unipotent representations of complex groups, in *Noncommutative Geometry and Global Analysis*, ed. by A. Connes, A. Gorokhovsky, M. Lesch, M. Pflaum, B. Rangipour. Contemporary Mathematics, vol. 546 (American Mathematical Society, Providence, 2011), pp. 1–22
7. D. Barbasch, P. Pandžić, Dirac cohomology of unipotent representations of $Sp(2n, \mathbb{R})$ and $U(p, q)$. J. Lie Theory **25**, 185–213 (2015)
8. D. Barbasch, D. Ciubotaru, P. Trapa, Dirac cohomology for graded Hecke algebras. Acta Math. **209**, 197–227 (2012)
9. J. Bernstein, I. Gelfand, S. Gelfand, Category of $\mathfrak{g}$-modules. Funct. Anal. Appl. **10**, 87–92 (1976)
10. W. Casselman, M. Osborne, The $\mathfrak{n}$-cohomology of the representations with an infinitesimal characters. Compos. Math. **31**, 219–227 (1975)
11. M.-K. Chuah, J.-S. Huang, Dirac cohomology and geometric quantization. J. Reine Angew. Math. **720**, 33–50 (2016) doi:10.1515/crelle-2014-0050
12. D. Ciubotaru, Dirac cohomology for symplectic reflection groups. Sel. Math. **22**, 111–144 (2016)
13. D. Ciubotaru, P. Trapa, Characters of Springer representations on elliptic conjugacy classes. Duke Math. J. **162**, 201–223 (2013)
14. C.-P. Dong, J.-S. Huang, Jacquet modules and Dirac cohomology. Adv. Math. **226**, 2911–2934 (2011)
15. C.-P. Dong, J.-S. Huang, Dirac cohomology of cohomologically induced modules. Am. J. Math. **139**, 37–60 (2015)
16. T. Enright, Analogues of Kostant's $\mathfrak{u}$-cohomology formulas for unitary highest weight modules. J. Reine Angew. Math. **392**, 27–36 (1988)
17. P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. Invent. Math. **147**(2), 243–348 (2002)
18. P. Etingof, E. Stoica, Unitary representations of rational Cherednik algebras. Represent. Theory **13**, 349–370 (2009)
19. Harish-Chandra, The characters of semisimple Lie groups. Trans. Am. Math. Soc. **83**, 98–163 (1956)
20. Harish-Chandra, Discrete series for semisimple Lie groups, I and II. Acta Math. **113**, 242–318 (1965); **116**, 1–111 (1966)
21. Harish-Chandra, Supertempered distributions on real reductive groups, *Studies in Applied Mathematics, Advances in Mathematics Supplementary Studies*, vol. 8 (Academic Press, New York, NY, 1983), pp. 139–153 (It is contained in Harish-Chandra's collected Collected Papers, Volume IV, 447–461.)
22. J.-S. Huang, Dirac cohomology and Dirac induction. Sci. China. Math. **54**, 2373–2381 (2011)
23. J.-S. Huang, Dirac cohomology and generalization of classical branching rules, in *Developments and Retrospectives in Lie Theory, Algebraic Methods*, ed. by G. Mason, I. Penkov, J. Wolf. Developments in Mathematics, vol. 38 (Springer, Berlin/Heidelberg, 2014), pp. 207–228
24. J.-S. Huang, Dirac cohomology, elliptic representations and endoscopy, in *Representations of Reductive Groups: In Honor of the 60th Birthday of David A. Vogan Jr.*, ed. by M. Newins, P.E. Trapa (Springer, Berlin/Heidelberg, 2015). doi:10.1007/978-3-319-23443-4_9
25. J.-S. Huang, Y.-F. Kang, P. Pandžić, Dirac cohomology of some Harish-Chandra modules. Transform. Groups **14**(1), 163–173 (2009)
26. J.-S. Huang, P. Pandžić, Dirac cohomology, unitary representations and a proof of a conjecture of Vogan. J. Am. Math. Soc. **15**, 185–202 (2002)
27. J.-S. Huang, P. Pandžić, Dirac cohomology for Lie Superalgebras. Transform. Group **10**, 201–209 (2005)
28. J.-S. Huang, P. Pandžić, *Dirac Operators in Representation Theory*. Mathematics: Theory & Applications (Birkhäuser, Basel, 2006)
29. J.-S. Huang, P. Pandžić, D. Renard, Dirac operarors and Lie algebra cohomology. arXiv:math/0503582 (2005)

30. J.-S. Huang, P. Pandžić, D. Renard, Dirac operators and Lie algebra cohomology. Represent. Theory **10**, 299–313 (2006)
31. J.-S. Huang, P. Pandžić, V. Protsak, Dirac cohomology of Wallach representations. Pac. J. Math. **250**, 163–190 (2011)
32. J.-S. Huang, P. Pandžić, F.-H. Zhu, Dirac cohomology, K-characters and branching laws. Am. J. Math. **135**, 1253–1269 (2013)
33. J.-S. Huang, D.K. Wong, *Lie Algebra Cohomology for Rational Cherednik Algebras* (to appear in Transform Group)
34. J.-S. Huang, W. Xiao, Dirac cohomology of highest weight modules. Sel. Math. **18**, 803–824 (2012)
35. J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics, vol. 9 (Springer, Berlin/Heidelberg, 1972). (Third printing, revised, 1980.)
36. J.E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*. Graduate Studies in Mathematics, vol. 94 (American Mathematical Society, Providence, 2008)
37. V.G. Kac, P. Möseneder Frajria, P. Papi, Multiplets of representations, twisted Dirac operators and Vogan's conjecture in affine setting. Adv. Math. **217**, 2485–2562 (2008)
38. A.W. Knapp, D.A. Vogan Jr., *Cohomological Induction and Unitary Representations* (Princeton University Press, Princeton, 1995)
39. B. Kostant, A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups. Duke Math. J. **100**, 447–501 (1999)
40. B. Kostant, A generalization of the Bott-Borel-Weil theorem and Euler number multiplets of representations. Lett. Math. Phys. **52**, 61–78 (2000)
41. B. Kostant, *Dirac Cohomology for the Cubic Dirac Operator, Studies in Memory of Issai Schur*. Progress in Mathematics, vol. 210 (Birkhäuser, Boston, 2003), pp. 69–93
42. S. Kumar, Induction functor in non-commutative equivariant cohomology and Dirac cohomology. J. Algebra **291**, 187–207 (2005)
43. J.-P. Labesse, Pseudo-coefficients très cuspidaux et K-théorie. Math. Ann. **291**, 607–616 (1991)
44. J.-P. Labesse, Introduction to endoscopy, in *Representation Theory of Real Reductive Lie Groups*. Contemporary Mathematics, vol. 472 (American Mathematical Society, Providence, RI, 2008), pp. 175–213
45. R.P. Langlands, The dimension of spaces of automorphic forms. Am. J. Math. **85**, 99–125 (1963)
46. R.P. Langlands, *On the Classification of Irreducible Representations of Real Algebraic Groups*. Mathematical Surveys and Monographs, vol. 31 (American Mathematical Society, Providence, RI, 1989), pp. 101–170
47. R.P. Langlands, D. Shelstad, On the definition of transfer factors. Math. Ann. **278**, 219–271 (1987)
48. S. Mehdi, R. Parthasarathy, Cubic Dirac cohomology for generalized Enright-Varadarajan modules. J. Lie Theory **21**, 861–884 (2011)
49. S. Mehdi, R. Zierau, The Dirac cohomology of a finite dimensional representation. Proc. Am. Math. Soc. **142**, 1507–1512 (2014)
50. R. Parthasarathy, Dirac operator and the discrete series. Ann. Math. **96**, 1–30 (1972)
51. S.A. Salamanca-Riba, On the unitary dual of real reductive Lie groups and the $A_{\mathfrak{q}}(\lambda)$ modules: the strongly regular case. Duke Math. J. **96**, 521–546 (1998)
52. D.A. Vogan Jr., *Representations of Real Reductive Lie Groups*. Progress in Mathematics, vol. 15 (Birkhäuser, Boston, 1981)
53. D.A. Vogan Jr., *Dirac operator and unitary representations*, 3 talks at MIT Lie groups seminar, Fall of 1997
54. D.A. Vogan Jr., G.J. Zuckerman, Unitary representations with non-zero cohomology. Compos. Math. **53**, 51–90 (1984)