Some Semi-Direct Products with Free Algebras of Symmetric Invariants

Oksana Yakimova

Abstract Let $\mathfrak{g}$ be a complex reductive Lie algebra and V the underlying vector space of a finite-dimensional representation of $\mathfrak{g}$. Then one can consider a new Lie algebra $\mathfrak{q} = \mathfrak{g}\ltimes V$, which is a semi-direct product of $\mathfrak{g}$ and an Abelian ideal V. We outline several results on the algebra $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ of symmetric invariants of $\mathfrak{q}$ and describe all semi-direct products related to the defining representation of $\mathfrak{sl}_n$ with $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ being a free algebra.

Keywords Coadjoint representation • Non-reductive Lie algebras • Polynomial rings • Regular invariants

1 Introduction

Let Q be a connected complex algebraic group. Set $\mathfrak{q} = \operatorname{Lie} Q$. Then $\mathcal{S}(\mathfrak{q}) = \mathbb{C}[\mathfrak{q}^*]$ and $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}[\mathfrak{q}^*]^{Q}$. We will call the latter object the *algebra of symmetric invariants* of $\mathfrak{q}$. An important property of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is that it is isomorphic to $Z\mathbf{U}(\mathfrak{q})$ as an algebra by a classical result of M. Duflo (here $Z\mathbf{U}(\mathfrak{q})$ is the centre of the universal enveloping algebra of $\mathfrak{q}$).

Let $\mathfrak{g}$ be a reductive Lie algebra. Then by the Chevalley restriction theorem $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[H_1, \ldots, H_{\operatorname{rk}\mathfrak{g}}]$ is a polynomial ring (in $\operatorname{rk}\mathfrak{g}$ variables). A quest for non-reductive Lie algebras with a similar property has recently become a trend in invariant theory. Here we consider finite-dimensional representations $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ and the corresponding semi-direct products $\mathfrak{q} = \mathfrak{g}\ltimes V$. The Lie bracket on $\mathfrak{q}$ is defined by

$$[\xi + v, \eta + u] = [\xi, \eta] + \rho(\xi)u - \rho(\eta)v \qquad (1)$$

for all $\xi, \eta \in \mathfrak{g}$, $v, u \in V$. Let G be a connected simply connected Lie group with $\operatorname{Lie} G = \mathfrak{g}$. Then $\mathfrak{q} = \operatorname{Lie} Q$ with $Q = G\ltimes \exp(V)$.

It is easy to see that $\mathbb{C}[V^*]^G \subset \mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ and therefore $\mathbb{C}[V^*]^G$ must be a polynomial ring if $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is, see [10, Section 3]. Classification of the representations

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of complex simple algebraic groups with free algebras of invariants was carried out by Schwarz [7] and independently by Adamovich and Golovina [1]. One such representation is the spin-representation of Spin_7, which leads to $Q = \mathrm{Spin}_7 \ltimes \mathbb{C}^8$. Here $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring in three variables generated by invariants of bi-degrees $(0, 2)$, $(2, 2)$, $(6, 4)$ with respect to the decomposition $\mathfrak{q} = \mathfrak{so}_7 \oplus \mathbb{C}^8$, see [10, Proposition 3.10].

In this paper, we treat another example, $G = \mathrm{SL}_n$, $V = m(\mathbb{C}^n)^* \oplus k\mathbb{C}^n$ with $n \geqslant 2$, $m \geqslant 1$, $m \geqslant k$. Here $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring in exactly the following three cases:

- $k = 0$, $m \leqslant n + 1$, and $n \equiv t \pmod m$ with $t \in \{-1, 0, 1\}$;
- $m = k$, $k \in \{n - 2, n - 1\}$;
- $n \geqslant m > k > 0$ and $m - k$ divides $n - m$.

We also briefly discuss semi-direct products arising as $\mathbb{Z}_2$-contractions of reductive Lie algebras.

2 Symmetric Invariants and Generic Stabilisers

Let $\mathfrak{q} = \mathrm{Lie}\, Q$ be an algebraic Lie algebra, Q a connected algebraic group. The index of $\mathfrak{q}$ is defined as

$$\mathrm{ind}\, \mathfrak{q} = \min_{\gamma \in \mathfrak{q}^*} \dim \mathfrak{q}_\gamma,$$

where $\mathfrak{q}_\gamma$ is the stabiliser of γ in $\mathfrak{q}$. In view of Rosenlicht's theorem, $\mathrm{ind}\, \mathfrak{q} = \mathrm{tr.deg}\, \mathbb{C}(\mathfrak{q}^*)^Q$. In case $\mathrm{ind}\, \mathfrak{q} = 0$, we have $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}$. For a reductive $\mathfrak{g}$, $\mathrm{ind}\, \mathfrak{g} = \mathrm{rk}\, \mathfrak{g}$. Recall that $(\dim \mathfrak{g} + \mathrm{rk}\, \mathfrak{g})/2$ is the dimension of a Borel subalgebra of $\mathfrak{g}$. For $\mathfrak{q}$, set $\mathbf{b}(\mathfrak{q}) := (\mathrm{ind}\, \mathfrak{q} + \dim \mathfrak{q})/2$.

Let $\{\xi_i\}$ be a basis of $\mathfrak{q}$ and $\mathcal{M}(\mathfrak{q}) = ([\xi_i, \xi_j])$ the structural matrix with entries in $\mathfrak{q}$. This is a skew-symmetric matrix of rank $\dim \mathfrak{q} - \mathrm{ind}\, \mathfrak{q}$. Let us take Pfaffians of the principal minors of $\mathcal{M}(\mathfrak{q})$ of size $\mathrm{rk}\, \mathcal{M}(\mathfrak{q})$ and let $\mathbf{p} = \mathbf{p}_{\mathfrak{q}}$ be their greatest common divisor. Then $\mathbf{p}$ is called the *fundamental semi-invariant* of $\mathfrak{q}$. The zero set of $\mathbf{p}$ is the maximal divisor in the so called *singular set*

$$\mathfrak{q}^*_{\mathrm{sing}} = \{\gamma \in \mathfrak{q}^* \mid \dim \mathfrak{q}_\gamma > \mathrm{ind}\, \mathfrak{q}\}$$

of $\mathfrak{q}$. Since $\mathfrak{q}^*_{\mathrm{sing}}$ is clearly a Q-stable subset, $\mathbf{p}$ is indeed a semi-invariant, $Q{\cdot}\mathbf{p} \subset \mathbb{C}\mathbf{p}$. One says that $\mathfrak{q}$ has the "codim-2" property (satisfies the "codim-2" condition), if $\dim \mathfrak{q}^*_{\mathrm{sing}} \leqslant \dim \mathfrak{q} - 2$ or equivalently if $\mathbf{p} = 1$.

Suppose that $F_1, \ldots, F_r \in \mathcal{S}(\mathfrak{q})$ are homogenous algebraically independent polynomials. The *Jacobian locus* $\mathcal{J}(F_1, \ldots, F_r)$ of these polynomials consists of all $\gamma \in \mathfrak{q}^*$ such that the differentials $d_\gamma F_1, \ldots, d_\gamma F_r$ are linearly dependent. In other words, $\gamma \in \mathcal{J}(F_1, \ldots, F_r)$ if and only if $(dF_1 \wedge \ldots \wedge dF_r)_\gamma = 0$. The set $\mathcal{J}(F_1, \ldots, F_r)$ is a proper Zariski closed subset of $\mathfrak{q}^*$. Suppose that $\mathcal{J}(F_1, \ldots, F_r)$ does not contain divisors. Then by the characteristic zero version of a result of

Skryabin, see [5, Theorem 1.1], $\mathbb{C}[F_1,\dots,F_r]$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{q})$, each $H \in \mathcal{S}(\mathfrak{q})$ that is algebraic over $\mathbb{C}(F_1,\dots,F_r)$ is contained in $\mathbb{C}[F_1,\dots,F_r]$.

Theorem 1 (cf. [3, Section 5.8]) *Suppose that* $\mathbf{p}_\mathfrak{q} = 1$ *and suppose that* $H_1,\dots,H_r \in \mathcal{S}(\mathfrak{q})^\mathfrak{q}$ *are homogeneous algebraically independent polynomials such that* $r = \operatorname{ind}\mathfrak{q}$ *and* $\sum_{i=1}^r \deg H_i = \mathbf{b}(\mathfrak{q})$. *Then* $\mathcal{S}(\mathfrak{q})^\mathfrak{q} = \mathbb{C}[H_1,\dots,H_r]$ *is a polynomial ring in r generators.*

Proof Under our assumptions $\mathcal{J}(H_1,\dots,H_r) = \mathfrak{q}^*_{\rm sing}$, see [5, Theorem 1.2] and [9, Section 2]. Therefore $\mathbb{C}[H_1,\dots,H_r]$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{q})$ by [5, Theorem 1.1]. Since tr.deg $\mathcal{S}(\mathfrak{q})^\mathfrak{q} \leqslant r$, each symmetric $\mathfrak{q}$-invariant is algebraic over $\mathbb{C}[H_1,\dots,H_r]$ and hence is contained in it. □

For semi-direct products, we have some specific approaches to the symmetric invariants. Suppose now that $\mathfrak{g} = \operatorname{Lie} G$ is a reductive Lie algebra, no non-zero ideal of $\mathfrak{g}$ acts on V trivially, G is connected, and $\mathfrak{q} = \mathfrak{g}\ltimes V$, where V is a finite-dimensional G-module.

The vector space decomposition $\mathfrak{q} = \mathfrak{g}\oplus V$ leads to $\mathfrak{q}^* = \mathfrak{g}\oplus V^*$, where we identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Each element $x \in V^*$ is considered as a point of $\mathfrak{q}^*$ that is zero on $\mathfrak{g}$. We have $\exp(V){\cdot}x = \operatorname{ad}^*(V){\cdot}x + x$, where each element of $\operatorname{ad}^*(V){\cdot}x$ is zero on V. Note that $\operatorname{ad}^*(V){\cdot}x \subset \operatorname{Ann}(\mathfrak{g}_x) \subset \mathfrak{g}$ and $\dim(\operatorname{ad}^*(V){\cdot}x)$ is equal to $\dim(\operatorname{ad}^*(\mathfrak{g}){\cdot}x) = \dim\mathfrak{g} - \dim\mathfrak{g}_x$. Therefore $\operatorname{ad}^*(V){\cdot}x = \operatorname{Ann}(\mathfrak{g}_x)$.

The decomposition $\mathfrak{q} = \mathfrak{g}\oplus V$ defines also a bi-grading on $\mathcal{S}(\mathfrak{q})$ and clearly $\mathcal{S}(\mathfrak{q})^\mathfrak{q}$ is a bi-homogeneous subalgebra, cf. [10, Lemma 2.12].

A statement is true for a "generic x" if and only if this statement is true for all points of a non-empty open subset.

Lemma 1 *A function* $F \in \mathbb{C}[\mathfrak{q}^*]$ *is a* V*-invariant if and only if* $F(\xi + \operatorname{ad}^*(V){\cdot}x, x) = F(\xi, x)$ *for generic* $x \in V^*$ *and any* $\xi \in \mathfrak{g}$.

Proof Condition of the lemma guaranties that for each $v \in V$, $\exp(v){\cdot}F = F$ on a non-empty open subset of $\mathfrak{q}^*$. Hence F is a V-invariant. □

For $x \in V^*$, let $\varphi_x\colon \mathbb{C}[\mathfrak{q}^*]^Q \to \mathbb{C}[\mathfrak{g} + x]^{G_x\ltimes\exp(V)}$ be the restriction map. By [10, Lemma 2.5] $\mathbb{C}[\mathfrak{g} + x]^{G_x\ltimes\exp(V)} \cong \mathcal{S}(\mathfrak{g}_x)^{G_x}$. Moreover, if we identify $\mathfrak{g} + x$ with $\mathfrak{g}$ choosing x as the origin, then $\varphi_x(F) \in \mathcal{S}(\mathfrak{g}_x)$ for any $\mathfrak{q}$-invariant F [10, Section 2]. Under certain assumptions on G and V the restriction map φ_x is surjective, more details will be given shortly.

There is a non-empty open subset $U \subset V^*$ such that the stabilisers G_x and G_y are conjugate in G for any pair of points $x, y \in U$ see e.g. [8, Theorem 7.2]. Any representative of the conjugacy class $\{hG_xh^{-1} \mid h \in G, x \in U\}$ is said to be a *a generic stabiliser* of the G-action on V^*.

There is one easy to handle case, $\mathfrak{g}_x = 0$ for a generic $x \in V^*$. Here $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G$, see e.g. [10, Example 3.1], and $\xi + y \in \mathfrak{q}^*_{\rm sing}$ only if $\mathfrak{g}_y \neq 0$, where $\xi \in \mathfrak{g}$, $y \in V^*$. The case $\operatorname{ind}\mathfrak{g}_x = 1$ is more involved.

Lemma 2 *Assume that G has no proper semi-invariants in* $\mathbb{C}[V^*]$. *Suppose that* $\operatorname{ind}\mathfrak{g}_x = 1$, $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x} \neq \mathbb{C}$, *and the map* φ_x *is surjective for generic* $x \in V^*$. *Then*

$\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}[V^*]^G[F]$, *where* F *is a bi-homogeneous preimage of a generator of* $\mathcal{S}(\mathfrak{g}_x)^{G_x}$ *that is not divisible by any non-constant* G*-invariant in* $\mathbb{C}[V^*]$.

Proof If we have a Lie algebra of index 1, in our case $\mathfrak{g}_x$, then the algebra of its symmetric invariants is a polynomial ring. There are many possible explanations of this fact. One of them is the following. Suppose that two non-zero homogeneous polynomials f_1, f_2 are algebraically dependent. Then $f_1^a = cf_2^b$ for some coprime integers $a, b > 0$ and some $c \in \mathbb{C}^\times$. If f_1 is an invariant, then so is a polynomial function $\sqrt[b]{f_1} = \sqrt[ab]{c}\sqrt[a]{f_2}$.

Since $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x} \neq \mathbb{C}$, it is generated by some homogeneous f. The group G_x has finitely many connected components, hence $\mathcal{S}(\mathfrak{g}_x)^{G_x}$ is generated by a suitable power of f, say $\mathbf{f} = f^d$.

Let $F \in \mathbb{C}[\mathfrak{q}^*]^Q$ be a preimage of $\mathbf{f}$. Each its bi-homogeneous component is again a $\mathfrak{q}$-invariant. Without loss of generality we may assume that F is bi-homogenous. Also if F is divisible by some non-scalar $H \in \mathbb{C}[V^*]^G$, then we replace F with F/H and repeat the process as long as possible.

Whenever G_y (with $y \in V^*$) is conjugate to G_x and $\varphi_y(F) \neq 0$, $\varphi_y(F)$ is a G_y-invariant of the same degree as $\mathbf{f}$ and therefore is a generator of $\mathcal{S}(\mathfrak{g}_y)^{G_y}$. Clearly $\mathbb{C}(V^*)^G[F] \subset \mathbb{C}[\mathfrak{q}^*]^Q \otimes_{\mathbb{C}[V^*]^G} \mathbb{C}(V^*)^G =: \mathcal{A}$ and $\mathcal{A} \subset \mathcal{S}(\mathfrak{g}) \otimes \mathbb{C}(V^*)^G$. If $\mathcal{A}$ contains a homogeneous in $\mathfrak{g}$ polynomial T that is not proportional (over $\mathbb{C}(V^*)^G$) to a power of F, then $\varphi_u(T)$ is not proportional to a power of $\varphi_u(F)$ for generic $u \in V^*$. But $\varphi_u(T) \in \mathcal{S}(\mathfrak{g}_u)^{G_u}$. This implies that $\mathcal{A} = \mathbb{C}(V^*)^G[F]$. It remains to notice that $\mathbb{C}(V^*)^G = \operatorname{Quot}\mathbb{C}[V^*]^G$, since G has no proper semi-invariants in $\mathbb{C}[V^*]$, and by the same reason $\mathbb{C}(V^*)^G[F] \cap \mathbb{C}[\mathfrak{q}] = \mathbb{C}[V^*]^G[F]$ in case F is not divisible by any non-constant G-invariant in $\mathbb{C}[V^*]$. □

It is time to recall the Raïs' formula [6] for the index of a semi-direct product:

$$\operatorname{ind}\mathfrak{q} = \dim V - (\dim\mathfrak{g} - \dim\mathfrak{g}_x) + \operatorname{ind}\mathfrak{g}_x \ \text{ with } x \in V^* \text{ generic.} \tag{2}$$

Lemma 3 *Suppose that* $H_1, \ldots, H_r \in \mathcal{S}(\mathfrak{q})^Q$ *are homogenous polynomials such that* $\varphi_x(H_i)$ *with* $i \leqslant \operatorname{ind}\mathfrak{g}_x$ *freely generate* $\mathcal{S}(\mathfrak{g}_x)^{G_x} = \mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ *for generic* $x \in V^*$ *and* $H_j \in \mathbb{C}[V^*]^G$ *for* $j > \operatorname{ind}\mathfrak{g}_x$*; and suppose that* $\sum_{i=1}^{\operatorname{ind}\mathfrak{g}_x} \deg_{\mathfrak{g}} H_i = \mathbf{b}(\mathfrak{g}_x)$. *Then* $\sum_{i=1}^{r} \deg H_i = \mathbf{b}(\mathfrak{q})$ *if and only if* $\sum_{i=1}^{r} \deg_V H_i = \dim V$.

Proof In view of the assumptions, we have $\sum_{i=1}^{r} \deg H_i = \mathbf{b}(\mathfrak{g}_x) + \sum_{i=1}^{r} \deg_V H_i$. Further, by Eq. (2)

$$\begin{aligned} \mathbf{b}(\mathfrak{q}) &= (\dim\mathfrak{q} + \dim V - (\dim\mathfrak{g} - \dim\mathfrak{g}_x) + \operatorname{ind}\mathfrak{g}_x)/2 = \\ &= \dim V + (\dim\mathfrak{g}_x + \operatorname{ind}\mathfrak{g}_x)/2 = \mathbf{b}(\mathfrak{g}_x) + \dim V. \end{aligned}$$

The result follows. □

From now on suppose that G is semisimple. Then both G and Q have only trivial characters and hence cannot have proper semi-invariants. In particular, the fundamental semi-invariant is an invariant. We also have tr.deg $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \operatorname{ind}\mathfrak{q}$. Set $r = \operatorname{ind}\mathfrak{q}$ and let $x \in V^*$ be generic. If $\mathbb{C}[\mathfrak{q}^*]^Q$ is a polynomial ring, then there are bi-homogenous generators $H_1, \ldots, H_r$ such that H_i with $i > \operatorname{ind}\mathfrak{g}_x$ freely generate $\mathbb{C}[V^*]^G$ and the invariants H_i with $i \leqslant \operatorname{ind}\mathfrak{g}_x$ are *mixed*, they have positive degrees in $\mathfrak{g}$ and V.

Theorem 2 ([3, Theorem 5.7] and [10, Proposition 3.11]) *Suppose that G is semisimple and $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring with homogeneous generators $H_1, \ldots, H_r$. Then*

(i) $\sum_{i=1}^{r} \deg H_i = \mathbf{b}(\mathfrak{q}) + \deg \mathbf{p}_{\mathfrak{q}}$*;*
(ii) for generic $x \in V^*$, the restriction map $\varphi_x\colon \mathbb{C}[\mathfrak{q}^*]^Q \to \mathbb{C}[\mathfrak{g}+x]^{G_x \ltimes V} \cong \mathcal{S}(\mathfrak{g}_x)^{G_x}$ is surjective, $\mathcal{S}(\mathfrak{g}_x)^{G_x} = \mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$, and $\mathcal{S}(\mathfrak{g}_x)^{G_x}$ is a polynomial ring in $\operatorname{ind}\mathfrak{g}_x$ variables.

It is worth mentioning that φ_x is also surjective for stable actions. An action of G on V is called *stable* if generic G-orbits in V are closed, for more details see [8, Sections 2.4 and 7.5]. By [10, Theorem 2.8] φ_x is surjective for generic $x \in V^*$ if the G-action on V^* is stable.

3 $\mathbb{Z}/2\mathbb{Z}$-contractions

The initial motivation for studying symmetric invariants of semi-direct products was related to a conjecture of D. Panyushev on $\mathbb{Z}_2$-contractions of reductive Lie algebras. The results of [10], briefly outlined in Sect. 2, have settled the problem.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a *symmetric decomposition*, i.e., a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$. A semi-direct product, $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ is an Abelian ideal, can be seen as a *contraction*, in this case a $\mathbb{Z}_2$*-contraction*, of $\mathfrak{g}$. For example, starting with a symmetric pair $(\mathfrak{so}_{n+1}, \mathfrak{so}_n)$, one arrives at $\tilde{\mathfrak{g}} = \mathfrak{so}_n \ltimes \mathbb{C}^n$. In [4], it was conjectured that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring (in rk $\mathfrak{g}$ variables).

Theorem 3 ([4, 9, 10]) *Let $\tilde{\mathfrak{g}}$ be a $\mathbb{Z}_2$-contraction of a reductive Lie algebra $\mathfrak{g}$. Then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring (in* rk $\mathfrak{g}$ *variables) if and only if the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ is surjective.*

If we are in one of the "surjective" cases, then one can describe the generators of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$. Let $H_1, \ldots, H_r$ be suitably chosen homogeneous generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and let $H_i^\bullet$ be the bi-homogeneous (w.r.t. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$) component of H_i of the highest $\mathfrak{g}_1$-degree. Then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by the polynomials $H_i^\bullet$ (of course, providing the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ is surjective) [4, 9].

Unfortunately, this construction of generators cannot work if the restriction homomorphism is not surjective, see [4, Remark 4.3]. As was found out by Helgason [2], there are four "non-surjective" irreducible symmetric pairs, namely, (E_6, F_4),

$(E_7, E_6\oplus\mathbb{C})$, $(E_8, E_7\oplus\mathfrak{sl}_2)$, and $(E_6, \mathfrak{so}_{10}\oplus\mathfrak{so}_2)$. The approach to semi-direct products developed in [10] showed that Panyushev's conjecture does not hold for them. Next we outline some ideas of the proof.

Let $G_0 \subset G$ be a connected subgroup with $\operatorname{Lie} G_0 = \mathfrak{g}_0$. Then G_0 is reductive, it acts on $\mathfrak{g}_1 \cong \mathfrak{g}_1^*$, and this action is stable. Let $x \in \mathfrak{g}_1$ be a generic element and $G_{0,x}$ be its stabiliser in G_0. The groups $G_{0,x}$ are reductive and they are known for all symmetric pairs. In particular, $\mathcal{S}(\mathfrak{g}_{0,x})^{G_{0,x}}$ is a polynomial ring. It is also known that $\mathbb{C}[\mathfrak{g}_1]^{G_0}$ is a polynomial ring. By [4] $\tilde{\mathfrak{g}}$ has the "codim-2" property and $\operatorname{ind}\tilde{\mathfrak{g}} = \operatorname{rk}\mathfrak{g}$.

Making use of the surjectivity of φ_x one can show that if $\mathbb{C}[\tilde{\mathfrak{g}}^*]^{\tilde{\mathfrak{g}}}$ is freely generated by some $H_1, \ldots, H_r$, then necessary $\sum_{i=1}^{r} \deg H_i > \mathbf{b}(\tilde{\mathfrak{g}})$ for $\tilde{\mathfrak{g}}$ coming from one of the "non-surjective" pairs [10]. In view of some results from [3] this leads to a contradiction.

Note that in case of $(\mathfrak{g}, \mathfrak{g}_0) = (E_6, F_4)$, $\mathfrak{g}_0 = F_4$ is simple and $\tilde{\mathfrak{g}}$ is a semi-direct product of F_4 and $\mathbb{C}^{26}$, which, of course, comes from one of the representations in Schwarz's list [7].

4 Examples Related to the Defining Representation of $\mathfrak{sl}_n$

Form now assume that $\mathfrak{g} = \mathfrak{sl}_n$ and $V = m(\mathbb{C}^n)^* \oplus k\mathbb{C}^n$ with $n \geqslant 2$, $m \geqslant 1$, $m \geqslant k$. According to [7] $\mathbb{C}[V]^G$ is a polynomial ring if either $k = 0$ and $m \leqslant n+1$ or $m \leqslant n$, $k \leqslant n-1$. One finds also the description of the generators of $\mathbb{C}[V^*]^G$ and their degrees in [7]. In this section, we classify all cases, where $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring and for each of them give the fundamental semi-invariant.

Example 1 Suppose that either $m \geqslant n$ or $m = k = n-1$. Then $\mathfrak{g}_x = 0$ for generic $x \in V^*$ and therefore $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G$, i.e., $\mathbb{C}[\mathfrak{q}^*]^Q$ is a polynomial ring if and only if $\mathbb{C}[V^*]^G$ is. The latter takes place for $(m, k) = (n+1, 0)$, for $m = n$ and any $k < n$, as well as for $m = k = n-1$. Non-scalar fundamental semi-invariants appear here only for

- $m = n$, where $\mathbf{p}$ is given by $\det(v)^{n-1-k}$ with $v \in n\mathbb{C}^n$;
- $m = k = n-1$, where $\mathbf{p}$ is the sum of the principal $2k{\times}2k$-minors of

$$\left(\begin{array}{c|c} 0 & v \\ \hline w & 0 \end{array}\right) \text{ with } v \in k\mathbb{C}^n, w \in k(\mathbb{C}^n)^*.$$

In the rest of the section, we assume that $\mathfrak{g}_x \neq 0$ for generic $x \in V^*$.

4.1 The Case $k = 0$

Here the ring of G-invariants on V^* is generated by

$$\{\Delta_I \mid I \subset \{1, \dots, m\}, |I| = n\} \ \text{[8, Section 9]},$$

where each $\Delta_I(v)$ is the determinant of the corresponding submatrix of $v \in V^*$. The generators are algebraically independent if and only if $m \leqslant n + 1$, see also [7].

We are interested only in m that are smaller than n. Let $n = qm + r$, where $0 < r \leqslant m$, and let $I \subset \{1, \dots, m\}$ be a subset of cardinality r. By choosing the corresponding r columns of v we get a matrix $w = v_I$. Set

$$F_I(A, v) := \det\left(v|Av|\dots|A^{q-1}v|A^q w\right), \quad \text{where } A \in \mathfrak{g}, v \in V^*. \tag{3}$$

Clearly each F_I is an SL_n-invariant. Below we will see that they are also V-invariants. If $r = m$, then there is just one invariant, $F = F_{\{1,\dots,m\}}$. If r is either 1 or $m - 1$, we get m invariants.

Lemma 4 *Each F_I defined by Eq. (3) is a V-invariant.*

Proof According to Lemma 1 we have to show that $F_I(\xi + \mathrm{ad}^*(V){\cdot}x, x) = F(\xi, x)$ for generic $x \in V^*$ and any $\xi \in \mathfrak{sl}_n$. Since $m < n$, there is an open SL_n-orbit in V^* and we can take x as E_m. Let $\mathfrak{p} \subset \mathfrak{gl}_n$ be the standard parabolic subalgebra corresponding to the composition $(m, n - m)$ and let $\mathfrak{n}_-$ be the nilpotent radical of the opposite parabolic. Each element (matrix) $\xi \in \mathfrak{gl}_n$ is a sum $\xi = \xi_- + \xi_p$ with $\xi_- \in \mathfrak{n}_-$, $\xi_p \in \mathfrak{p}$. In this notation $F_I(A, E_m) = \det\left(A_-|(A^2)_-|\dots|(A^{q-1})_-|(A^q)_{-,I}\right)$.

Let $\alpha = \alpha_A$ and $\beta = \beta_A$ be $m{\times}m$ and $(n - m){\times}(n - m)$-submatrices of A standing in the upper left and lower right corner, respectively. Then $(A^{s+1})_- = \sum_{t=0}^{s} \beta^t A_- \alpha^{s-t}$. Each column of $A_-\alpha$ is a linear combination of columns of A_- and each column of $\beta^t A_- \alpha^{j+1}$ is a linear combination of columns of $\beta^t A_- \alpha^j$. Therefore

$$\begin{aligned} F_I(A, E_m) &= \det\left(A_-|\dots|(A^{q-1})_-|(A^q)_{-,I}\right) = \\ &= \det\left(A_-|\beta A_-|\dots|\beta^{q-2}A_-|\beta^{q-1}A_{-,I}\right). \end{aligned} \tag{4}$$

Notice that $\mathfrak{g}_x \subset \mathfrak{p}$ and the nilpotent radical of $\mathfrak{p}$ is contained in $\mathfrak{g}_x$ (with $x = E_m$). Since $\mathrm{ad}^*(V){\cdot}x = \mathrm{Ann}(\mathfrak{g}_x) = \mathfrak{g}_x^\perp \subset \mathfrak{g}$ (after the identification $\mathfrak{g} \cong \mathfrak{g}^*$), $A_- = 0$ for any $A \in \mathfrak{g}_x^\perp$; and we have $\beta_A = cE_{n-m}$ with $c \in \mathbb{C}$ for this A. An easy observation is that

$$\begin{aligned} &\det\left(\xi_-|(\beta_\xi + cE_{n-m})\xi_-|\dots|(\beta_\xi + cE_{n-m})^{q-1}\xi_{-,I}\right) = \\ &\qquad = \det\left(\xi_-|\beta_\xi\xi_-|\dots|\beta_\xi^{q-1}\xi_{-,I}\right). \end{aligned}$$

Hence $F_I(\xi + A, E_m) = F_I(\xi, E_m)$ for all $A \in \mathrm{ad}^*(V){\cdot}E_m$ and all $\xi \in \mathfrak{sl}_n$. □

Theorem 4 *Suppose that* $\mathfrak{q} = \mathfrak{sl}_n \ltimes m(\mathbb{C}^n)^*$. *Then* $\mathbb{C}[\mathfrak{q}^*]^Q$ *is a polynomial ring if and only if* $m \leqslant n+1$ *and* m *divides either* $n-1$, n *or* $n+1$. *Under these assumptions on* m, $\mathbf{p}_{\mathfrak{q}} = 1$ *exactly then, when* m *divides either* $n-1$ *or* $n+1$.

Proof Note that the statement is true for $m \geqslant n$ by Example 1. Assume that $m \leqslant n-1$. Suppose that $n = mq + r$ as above. A generic stabiliser in $\mathfrak{g}$ is $\mathfrak{g}_x = \mathfrak{sl}_{n-m} \ltimes m\mathbb{C}^{n-m}$. On the group level it is connected. Notice that $\operatorname{ind} \mathfrak{g}_x = \operatorname{tr.deg} \mathcal{S}(\mathfrak{g}_x)^{G_x}$, since G_x has no non-trivial characters. Note also that $\mathbb{C}[V^*]^G = \mathbb{C}$, since $m < n$. If $\mathbb{C}[\mathfrak{q}^*]^Q$ is a polynomial ring, then so is $\mathbb{C}[\mathfrak{g}_x^*]^{G_x}$ by Theorem 2(ii) and either $n - m = 1$ or, arguing by induction, $n - m \equiv t \pmod{m}$ with $t \in \{-1, 0, 1\}$.

Next we show that the ring of symmetric invariants is freely generated by the polynomials F_I for the indicated m. Each element $\gamma \in \mathfrak{g}_x^*$ can be presented as $\gamma = \beta_0 + A_-$, where $\beta_0 \in \mathfrak{sl}_{n-m}$. Each restriction $\varphi_x(F_I)$ can be regarded as an element of $\mathcal{S}(\mathfrak{g}_x)$. Equation (4) combined with Lemma 4 and the observation that $\mathfrak{g}_x^* \cong \mathfrak{g}/\operatorname{Ann}(\mathfrak{g}_x)$ shows that $\varphi_x(F_I)$ is either Δ_I of $\mathfrak{g}_x$ (in case $q = 1$, where $F_I(A, E_m) = \det A_{-,I}$) or F_I of $\mathfrak{g}_x$. Arguing by induction on n, we prove that the restrictions $\varphi_x(F_I)$ freely generate $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ for $x = E_m$ (i.e., for a generic point in V^*). Notice that $n - m = (q-1)m + r$.

The group SL_n acts on V^* with an open orbit $\mathrm{SL}_n{\cdot}E_m$. Therefore the restriction map φ_x is injective. By the inductive hypothesis it is also surjective and therefore is an isomorphism. This proves that the polynomials F_I freely generate $\mathbb{C}[\mathfrak{q}^*]^Q$.

If m divides n, then $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[F]$ and the fundamental semi-invariant is a power of F. As follows from the equality in Theorem 2(i), $\mathbf{p} = F^{m-1}$.

Suppose that m divides either $n-1$ or $n+1$. Then we have m different invariants F_I. By induction on n, $\mathfrak{g}_x$ has the "codim-2" property, therefore the sum of $\deg \varphi_x(F_I)$ is equal to $\mathbf{b}(\mathfrak{g}_x)$ by Theorem 2(i). The sum of V-degrees is $m \times n = \dim V$ and hence by Lemma 3 $\sum \deg F_I = \mathbf{b}(\mathfrak{q})$. Thus, $\mathfrak{q}$ has the "codim-2" property. □

Remark 1 Using induction on n one can show that the restriction map φ_x is an isomorphism for all $m < n$. Therefore the polynomials F_I generate $\mathbb{C}[\mathfrak{q}^*]^Q$ for all $m < n$.

4.2 The Case m = k

Here $\mathbb{C}[V^*]^G$ is a polynomial ring if and only if $k \leqslant n-1$; a generic stabiliser is $\mathfrak{sl}_{n-k}$, and the G-action on $V \cong V^*$ is stable. We assume that $\mathfrak{g}_x \neq 0$ for generic $x \in V^*$ and therefore $k \leqslant n-2$.

For an $N \times N$-matrix C, let $\Delta_i(C)$ with $1 \leqslant i \leqslant N$ be coefficients of its characteristic polynomial, each Δ_i being a homogeneous polynomial of degree i. Let $\gamma = A + v + w \in \mathfrak{q}^*$ with $A \in \mathfrak{g}$, $v \in k\mathbb{C}^n$, $w \in k(\mathbb{C}^n)^*$. Having these objects we form an $(n+k) \times (n+k)$-matrix

$$Y_\gamma := \left(\begin{array}{c|c} A & v \\ \hline w & 0 \end{array}\right)$$

and set $F_i(\gamma) = \Delta_i(Y_\gamma)$ for each $i \in \{2k+1, 2k+2, 2k+3, \ldots, n+k\}$. Each F_i is an $\mathrm{SL}_n \times \mathrm{GL}_k$-invariant. Unfortunately, these polynomials are not V-invariants.

Remark 2 If we repeat the same construction for $\tilde{\mathfrak{q}} = \mathfrak{gl}_n \ltimes V$ with $k \leqslant n-1$, then $\mathbb{C}[\tilde{\mathfrak{q}}^*]^{\tilde{Q}} = \mathbb{C}[V^*]^{\mathrm{GL}_n}[\{F_i \mid 2k+1 \leqslant i \leqslant n+k\}]$ and it is a polynomial ring in $\operatorname{ind}\tilde{\mathfrak{q}} = n-k+k^2$ generators.

Theorem 5 *Suppose that $m = k \leqslant n-1$. Then $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring if and only if $k \in \{n-2, n-1\}$. In case $k = n-2$, $\mathfrak{q}$ has the "codim-2" property.*

Proof Suppose that $k = n-2$. Then a generic stabiliser $\mathfrak{g}_x = \mathfrak{sl}_2$ is of index 1 and since the G-action on V is stable, $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ has to be a polynomial ring by [10, Example 3.6]. One can show that the unique mixed generator is of the form $F_{2k+2}H_{2k} - F_{2k+1}^2$, where H_{2k} is a certain $\mathrm{SL}_n \times \mathrm{GL}_k$-invariant on V of degree $2k$ and then see that the sum of degrees is $\mathbf{b}(\mathfrak{q})$.

More generally, $\mathfrak{q}$ has the "codim-2" property for all $k \leqslant n-2$. Here each G-invariant divisor in V^* contains a G-orbit of maximal dimension, say Gy. Set $u = n-k-1$. If G_y is not SL_{n-k}, then $\mathfrak{g}_y = \mathfrak{sl}_u \ltimes (\mathbb{C}^u \oplus (\mathbb{C}^u)^* \oplus \mathbb{C})$ is a semi-direct product with a Heisenberg Lie algebra. Following the proof of [4, Theorem 3.3], one has to show that $\operatorname{ind}\mathfrak{g}_y = u$ in order to prove that $\mathfrak{q}$ has the "codim-2" property. This is indeed the case, $\operatorname{ind}\mathfrak{g}_y = 1 + \operatorname{ind}\mathfrak{sl}_u$.

Suppose that $0 < k < n-2$ and assume that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a polynomial ring. Then there are bi-homogeneous generators $\mathbf{h}_2, \ldots, \mathbf{h}_{n-k}$ of $\mathbb{C}[\mathfrak{q}^*]^{Q}$ over $\mathbb{C}[V^*]^G$ such that their restrictions to $\mathfrak{g} + x$ form a generating set of $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ for a generic x (with $\mathfrak{g}_x \cong \mathfrak{sl}_{n-k}$), see Theorem 2(ii). In particular, $\deg_{\mathfrak{g}}\mathbf{h}_t = t$.

Take $\tilde{\mathfrak{g}} = (\mathfrak{sl}_n \oplus \mathfrak{gl}_k) \ltimes V$, which is a $\mathbb{Z}_2$-contraction of $\mathfrak{sl}_{n+k}$. Then $\mathfrak{q}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$. Note that GL_k acts on $\mathfrak{q}$ via automorphisms and therefore we may assume that the $\mathbb{C}$-linear span of $\{\mathbf{h}_t\}$ is GL_k-stable. By degree considerations, each $\mathbf{h}_t$ is an SL_k-invariant as well. The Weyl involution of SL_n acts on V and has to preserve each line $\mathbb{C}\mathbf{h}_t$. Since this involution interchanges $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$, each $\mathbf{h}_t$ is also a GL_k-invariant. Thus,

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{q})^{\tilde{\mathfrak{g}}} = \mathcal{S}(\mathfrak{q}) \cap \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}.$$

Since $\tilde{\mathfrak{g}}$ is a "surjective" $\mathbb{Z}_2$-contraction, its symmetric invariants are known [4, Theorem 4.5]. The generators of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ are $\Delta_j^\bullet$ with $2 \leqslant j \leqslant n+k$. Here $\deg \Delta_j^\bullet = j$ and the generators of $(\mathfrak{sl}_n \oplus \mathfrak{gl}_k)$-degrees $2, 3, \ldots, n-k$ are $\Delta_{2k+2}^\bullet, \Delta_{2k+3}^\bullet, \ldots, \Delta_{n+k}^\bullet$. As the restriction to $\mathfrak{sl}_n \oplus \mathfrak{gl}_k + x$ shows, none of the generators $\Delta_j^\bullet$ with $j \geqslant 2k+2$ lies in $\mathcal{S}(\mathfrak{q})$. This means that $\mathbf{h}_t$ cannot be equal or even proportional over $\mathbb{C}[V^*]^G$ to $\Delta_{2k+t}^\bullet$ and hence has a more complicated expression. More precisely, a product $\Delta_{2k+1}^\bullet \Delta_{2k+t-1}^\bullet$ necessary appears in $\mathbf{h}_t$ with a non-zero coefficient from $\mathbb{C}[V^*]^G$ for $t \geqslant 2$. Since $\deg_V \Delta_{2k+1}^\bullet = 2k$, we have $\deg_V \mathbf{h}_t \geqslant 4k$ for every $t \geqslant 2$. The ring $\mathbb{C}[V^*]^G$ is freely generated by k^2 polynomials of degree two. Therefore, the total sum of degrees over all generators of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is greater than or equal to

$$\mathbf{b}(\mathfrak{sl}_{n-k}) + 4k(n-k-1) + 2k^2 = \mathbf{b}(\mathfrak{q}) + 2k(n-k-2).$$

This contradicts Theorem 2(i) in view of the fact that $\mathbf{p}_{\mathfrak{q}} = 1$.

□

4.3 The Case $0 < k < m$

Here $\mathbb{C}[V^*]^G$ is a polynomial ring if and only if $m \leqslant n$, [7]. If $n = m$, then $\mathfrak{g}_x = 0$ for generic $x \in V^*$. For $m < n$, our construction of invariants is rather intricate.

Let $\pi_1, \ldots, \pi_{n-1}$ be the fundamental weights of $\mathfrak{sl}_n$. We use the standard convention, $\pi_i = \varepsilon_1 + \ldots + \varepsilon_i$, $\varepsilon_n = -\sum_{i=1}^{n-1} \varepsilon_i$. Recall that for any t, $1 \leqslant t < n$, $\Lambda^t\mathbb{C}^n$ is irreducible with the highest weight π_t. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{C}^n$ such that each e_i is a weight vector and $\ell_t := e_1 \wedge \ldots \wedge e_t$ is a highest weight vector of $\Lambda^t\mathbb{C}^n$. Clearly $\Lambda^t\mathbb{C}^n \subset \mathcal{S}^t(t\mathbb{C}^n)$. Write $n-k = d(m-k)+r$ with $0 < r \leqslant (m-k)$. Let $\varphi : m\mathbb{C}^n \to \Lambda^m\mathbb{C}^n$ be a non-zero m-linear G-equivariant map. Such a map is unique up to a scalar and one can take φ with $\varphi(v_1 + \ldots + v_m) = v_1 \wedge \ldots \wedge v_m$. In case $r \neq m-k$, for any subset $I \subset \{1, \ldots, m\}$ with $|I| = k+r$, let $\varphi_I : m\mathbb{C}^n \to (k+r)\mathbb{C}^n \to \Lambda^{k+r}\mathbb{C}^n$ be the corresponding (almost) canonical map. By the same principle we construct $\tilde{\varphi} : k(\mathbb{C}^n)^* \to \Lambda^k(\mathbb{C}^n)^*$.

Let us consider the tensor product $\mathbb{W} := (\Lambda^m\mathbb{C}^n)^{\otimes d} \otimes \Lambda^{k+r}\mathbb{C}^n$ and its weight subspace $\mathbb{W}_{d\pi_k}$. One can easily see that $\mathbb{W}_{d\pi_k}$ contains a unique up to a scalar non-zero highest weight vector, namely

$$w_{d\pi_k} = \sum_{\sigma \in S_{n-k}} \mathrm{sgn}(\sigma)(\ell_k \wedge e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(m)}) \otimes \ldots \otimes (\ell_k \wedge e_{\sigma(n-r+1)} \ldots \wedge e_{\sigma(n)}).$$

This means that $\mathbb{W}$ contains a unique copy of $V_{d\pi_k}$, where $V_{d\pi_k}$ is an irreducible $\mathfrak{sl}_n$-module with the highest weight $d\pi_k$. We let ρ denote the representation of $\mathfrak{gl}_n$ on $\Lambda^m\mathbb{C}^n$ and ρ_r the representation of $\mathfrak{gl}_n$ on $\Lambda^{k+r}\mathbb{C}^n$. Let $\xi = A + v + w$ be a point in $\mathfrak{q}^*$. (It is assumed that $A \in \mathfrak{sl}_n$.) Finally let (,) denote a non-zero $\mathfrak{sl}_n$-invariant scalar product between $\mathbb{W}$ and $\mathcal{S}^d(\Lambda^k(\mathbb{C}^n)^*)$ that is zero on the $\mathfrak{sl}_n$-invariant complement of $V_{d\pi_k}$ in $\mathbb{W}$. Depending on r, set

$$\mathbf{F}(\xi) := (\varphi(v) \otimes \rho(A)^{m-k}\varphi(v) \otimes \rho(A^2)^{m-k}\varphi(v) \otimes \ldots \otimes \rho(A^d)^{m-k}\varphi(v), \tilde{\varphi}(w)^d)$$
$$\text{for } r = m-k;$$
$$\mathbf{F}_I(\xi) := (\varphi(v) \otimes \rho(A)^{m-k}\varphi(v) \otimes \ldots \otimes \rho(A^{d-1})^{m-k}\varphi(v) \otimes \rho_r(A^d)^r\varphi_I(v), \tilde{\varphi}(w)^d)$$

for each I as above in case $r < m - k$. By the constructions the polynomials $\mathbf{F}$ and $\mathbf{F}_I$ are SL_n-invariants.

Lemma 5 *The polynomials* $\mathbf{F}$ *and* $\mathbf{F}_I$ *are* V*-invariants.*

Proof We restrict $\mathbf{F}$ and $\mathbf{F}_I$ to $\mathfrak{g}^* + x$ with $x \in V^*$ generic. Changing a basis in V if necessary, we may assume that $x = E_m + E_k$. If $r < m - k$, some of the invariants $\mathbf{F}_I$ may become linear combinations of such polynomials under the change of basis, but this does not interfere with V-invariance. Now $\varphi(v)$ is a vector of weight π_m and $\tilde{\varphi}(w)^d$ of weight $-d\pi_k$. Notice that $dm + (k + r) = n + kd$. If $\sum_{i=1}^{n+kd} \lambda_i = d\sum_{i=1}^{k} \varepsilon_i$ and each λ_i is one of the ε_j, $1 \leqslant j \leqslant n$, then in the sequence $(\lambda_1, \dots, \lambda_{n+kd})$ we must have exactly one ε_j for each $k < j \leqslant n$ and $d + 1$ copies of each ε_i with $1 \leqslant i \leqslant k$. Hence the only summand of $\rho(A^s)^{m-k}\varphi(E_m)$ that plays any rôle in $\mathbf{F}$ or $\mathbf{F}_I$ is $\ell_k \wedge A^s e_{k+1} \wedge \dots \wedge A^s e_m$. Moreover, in $A^s e_{k+1} \wedge \dots \wedge A^s e_m$ we are interested only in vectors lying in $\Lambda^{m-k}\mathrm{span}(e_{k+1}, \dots, e_n)$.

Let us choose blocks α, U, β of A as shown in Fig. 1. Then up to a non-zero scalar $\mathbf{F}(A, E_m + E_k)$ is the determinant of

$$(U | \beta U + U\alpha | P_2(\alpha, U, \beta) | \dots | P_{d-1}(\alpha, U, \beta)),$$

$$\text{where } P_s(\alpha, U, \beta) = \sum_{t=0}^{s} \beta^t U \alpha^{s-t}.$$

Each column of $U\alpha$ is a linear combination of the columns of U, a similar relation exists between $\beta^t U\alpha^{s+1}$ and $\beta^t U\alpha^s$. Therefore

$$\mathbf{F}(A, E_m + E_k) = \det\left(U | \beta U | \beta^2 U | \dots | \beta^{d-1} U\right). \tag{5}$$

We have to check that $\mathbf{F}(\xi + A, x) = \mathbf{F}(\xi, x)$ for any $A \in \mathrm{ad}^*(V) \cdot x$ and any $\xi \in \mathfrak{g}$, see Lemma 1. Recall that $\mathrm{ad}^*(V) \cdot x = \mathrm{Ann}(\mathfrak{g}_x) = \mathfrak{g}_x^{\perp} \subset \mathfrak{g}$. In case $x = E_m + E_k$, U is zero in each $A \in \mathfrak{g}_x^{\perp}$ and β corresponding to such A is a scalar matrix. Therefore $\mathbf{F}(\xi + \mathrm{ad}^*(V) \cdot x, x) = \mathbf{F}(\xi, x)$.

The case $r < m - k$ is more complicated. If $\{1, \dots, k\} \subset I$, then $I = \tilde{I} \sqcup \{1, \dots, k\}$. Let $U_{\tilde{I}}$ be the corresponding submatrix of U and $\alpha_{\tilde{I} \times \tilde{I}}$ of α. One just has to replace

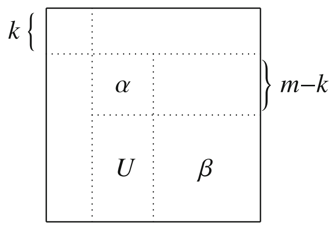

Fig. 1 Submatrices of $A \in \mathfrak{sl}_n$

U by $U_{\tilde{I}}$ and α by $\alpha_{\tilde{I}\times\tilde{I}}$ in the last polynomial $P_{d-1}(\alpha, U, \beta)$ obtaining

$$\mathbf{F}_I(A,x) = \det\left(U|\beta U|\beta^2 U|\dots|\beta^{d-2}U|\beta^{d-1}U_{\tilde{I}}\right).$$

These are $\binom{m-k}{r}$ linearly independent invariants in $\mathcal{S}(\mathfrak{g}_x)$.

Suppose that $\{1,\dots,k\} \not\subset I$. Then $\rho_I(A^d)^r$ has to move more than r vectors e_i with $k+1 \leqslant i \leqslant m$, which is impossible. Thus, $\mathbf{F}_I(A,x) = 0$ for such I. □

Theorem 6 *Suppose that* $0 < k < m < n$ *and* $m-k$ *divides* $n-m$*, then* $\operatorname{ind}\mathfrak{g}_x = 1$ *for generic* $x \in V^*$ *and* $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G[\mathbf{F}]$ *is a polynomial ring, the fundamental semi-invariant is equal to* $\mathbf{F}^{m-k-1}$.

Proof A generic stabiliser $\mathfrak{g}_x$ is $\mathfrak{sl}_{n-m}\ltimes(m-k)\mathbb{C}^{n-m}$. Its ring of symmetric invariants is generated by $F = \varphi_x(\mathbf{F})$, see Theorem 4 and Eq. (5). We also have $\operatorname{ind}\mathfrak{g}_x = 1$. It remains to see that $\mathbf{F}$ is not divisible by a non-constant G-invariant polynomial on V^*. By the construction, $\mathbf{F}$ is also invariant with respect to the action of $\mathrm{SL}_m\times\mathrm{SL}_k$. The group $L = \mathrm{SL}_n\times\mathrm{SL}_m\times\mathrm{SL}_k$ act on V^* with an open orbit. As long as $\operatorname{rk} w = k$, $\operatorname{rk} v = m$, the L-orbit of $y = v + w$ contains a point $v' + E_k$, where also $\operatorname{rk} v' = m$. If in addition the upper $k\times m$-part of v has rank k, then $L{\cdot}y$ contains $x = E_m + E_k$. Here $\mathbf{F}$ is non-zero on $\mathfrak{g} + y$. Since the group L is semisimple, the complement of $L{\cdot}(E_m + E_k)$ contains no divisors and $\mathbf{F}$ is not divisible by any non-constant G-invariant in $\mathbb{C}[V^*]$. This is enough to conclude that $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G[\mathbf{F}]$, see Theorem 2.

The singular set $\mathfrak{q}^*_{\rm sing}$ is L-stable. And therefore $\mathbf{p}_{\mathfrak{q}}$ is also an $\mathrm{SL}_m\times\mathrm{SL}_k$-invariant. Hence $\mathbf{p}$ is a power of $\mathbf{F}$. In view of Theorem 2(i), $\mathbf{p} = \mathbf{F}^{m-k-1}$. □

Theorem 7 *Suppose that* $0 < k < m < n$ *and* $m-k$ *does not divide* $n-m$*, then* $\mathbb{C}[\mathfrak{q}^*]^Q$ *is not a polynomial ring.*

Proof The reason for this misfortune is that $\binom{m}{k+r} > \binom{m-k}{r}$ for $r < m-k$. One could prove that each $\mathbf{F}_I$ must be in the set of generators and thereby show that $\mathbb{C}[\mathfrak{q}^*]^Q$ is not a polynomial ring. But we present a different argument.

Assume that the ring of symmetric invariants is polynomial. It is bi-graded and SL_m acts on it preserving the bi-grading. Since SL_m is reductive, we can assume that there is a set $\{H_1,\dots,H_s\}$ of bi-homogeneous mixed generators such that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}[V^*]^G[H_1,\dots,H_s]$ and the $\mathbb{C}$-linear span $\mathcal{H} := \operatorname{span}(H_1,\dots,H_s)$ is SL_m-stable. The polynomiality implies that a generic stabiliser $\mathfrak{g}_x = \mathfrak{sl}_{n-m}\ltimes(m-k)\mathbb{C}^{n-m}$ has a free algebra of symmetric invariants, see Theorem 2(ii), and by the same statement φ_x is surjective. This means that r is either 1 or $m-k-1$, see Theorem 4, $s = m-k$, and φ_x is injective on $\mathcal{H}$. Taking our favourite (generic) $x = E_m + E_k$, we see that there is SL_{m-k} embedded diagonally into $G\times\mathrm{SL}_m$, which acts on $\varphi_x(\mathcal{H})$ as on $\Lambda^r\mathbb{C}^{m-k}$. The group SL_{m-k} acts on $\mathcal{H}$ in the same way. Since $m-k$ does not divide $n-m$, we have $m-k \geqslant 2$. The group SL_m cannot act on an irreducible module $\Lambda^r\mathbb{C}^{m-k}$ of its non-trivial subgroup SL_{m-k}, this is especially obvious in our two cases of interest, $r = 1$ and $r = m-k-1$. A contradiction. □

Conjecture 1 It is very probable that $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}[V^*]^G[\{\mathbf{F}_l\}]$ for all $n > m > k \geqslant 1$.

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