Representations of Lie Superalgebras

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Abstract Abstract In these notes we give an introduction to representation theory of simple finite-dimensional Lie superalgebras. We concentrate on so called basic superalgebras. Those are superalgebras which have even reductive part and admit an invariant form. We start with structure theory of basic superalgebras emphasizing abstract properties of roots and then proceed to representations, trying to demonstrate the variety of methods: Harish-Chandra homomorphism, support variety, translation functors, Borel-Weil-Bott theory and localization.

Keywords Atypicality • Blocks • BorelŰWeilŰBott theorem • Harish-Chandra homomorphism • Lie superalgebras • Supermanifold • Translation functors

1 Introduction

In these notes we give an introduction to representation theory of simple finite-dimensional Lie superalgebras. We concentrate on so called basic superalgebras. Those are superalgebras which have even reductive part and admit an invariant form. Representation theory of these superalgebras was initiated in 1978 by V. Kac, see [23]. It turned out that finite-dimensional representations of basic superalgebras are not easy to describe completely and questions which arise in this theory are analogous to similar questions in positive characteristic.

We start with structure theory of basic superalgebras emphasizing abstract properties of roots and then proceed to representations, trying to demonstrate the variety of methods: Harish-Chandra homomorphism, support variety, translation functors, Borel-Weil-Bott theory and localization.

We assume from the reader the thorough knowledge of representation theory of reductive Lie algebras (in characteristic zero) and rudimentary knowledge of algebraic geometry.

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Let me mention several monographs related to the topic of these lectures: [32] and [4] on Lie superalgebras and [3] on supermanifolds. The reader can find some details in these books.

2 Preliminaries

2.1 Superalgebras in General

In supermathematics we study $\mathbf{Z}_2$-graded objects. The word super means simply "$\mathbf{Z}_2$-graded", whenever it is used (superalgebra, superspace etc.).

We denote by k the ground field and assume that $char(k) \neq 2$.

Definition 1 An *associative superalgebra* is a $\mathbf{Z}_2$ graded algebra $A = A_0 \oplus A_1$. If $a \in A_i$ is a homogeneous element, then $\bar{a}$ will denote the parity of a, that is $\bar{a} = 0$ if $a \in A_0$ or $\bar{a} = 1$ if $a \in A_1$.

All modules over an associative superalgebra A are also supposed to be $\mathbf{Z}_2$-graded. Thus, an A-module M has a grading $M = M_0 \oplus M_1$ such that $A_i M_j \subset M_{i+j}$.

In particular, a vector superspace is a $\mathbf{Z}_2$-graded vector space. The associative algebra $\mathrm{End}_k(V)$ of all k-linear transformation of a vector superspace V has a natural structure of a superalgebra with the $\mathbf{Z}_2$-grading given by:

$$\mathrm{End}_k(V)_0 = \{\phi \mid \phi(V_i) \subset V_i\}, \quad \mathrm{End}_k(V)_1 = \{\phi \mid \phi(V_i) \subset V_{i+1}\},$$

If $e_1, \dots, e_m$ is a basis of V_0 and $e_{m+1}, \dots, e_{m+n}$ is a basis of V_1, then we can identify $\mathrm{End}_k(V)$ with block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and

$$\mathrm{End}_k(V)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad \mathrm{End}_k(V)_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

All formulas are written for homogeneous elements only and then extended to all objects by linearity. Every term has a sign coefficient, which is determined by following the *sign rule*:

If one term is obtained from another by swapping adjacent symbols x and y we put the coefficient $(-1)^{\bar{x}\bar{y}}$.

Example 1 Consider the commutator $[x, y]$. In the classical world it is defined by $[x, y] = xy - yx$. In superworld we write instead:

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx.$$

The sign rule has its roots in the tensor category theory. More precisely, the category *SVect* of supervector spaces is an abelian rigid symmetric tensor category with brading $s : V \otimes W \to W \otimes V$ given by the sign rule

$$s(v \otimes w) = (-1)^{\bar{v}\bar{w}} w \otimes v.$$

All objects, which can be defined in context of tensor category: affine schemes, algebraic groups etc. can be generalized to superschemes, supergroups etc. if we work in the category *SVect* instead of the category *Vect* of vector spaces. We refer the reader to [9] for details in this approach. We will follow the sign rule naively and see that it always gives the correct answer.

Definition 2 We say that a superalgebra A is *supercommutative* if

$$xy = (-1)^{\bar{x}\bar{y}} yx$$

for all homogeneous $x, y \in A$.

Exercise Show that a free supercommutative algebra with odd generators $\xi_1, \dots, \xi_n$ is the exterior (Grassmann) algebra $\Lambda(\xi_1, \dots \xi_n)$.

All the morphisms between superalgebras, modules etc. have to preserve parity. In this way if A is a superalgebra then the category of A-modules is an abelian category. This category is equipped with the *parity change functor* Π. If $M = M_0 \oplus M_1$ is an A-module we set $\Pi M := M$ with new grading $(\Pi M)_0 = M_1, (\Pi M)_1 = M_0$ and the obviuos A-action. It is clear that Π is an autoequivalence of the abelian category of A-modules.

Exercise Let V be a finite dimensional vector superspace and V^* be the dual vector space with $\mathbf{Z}_2$-grading defined in the obvious way. Consider a linear operator $X : V \longrightarrow V$. We would like to define the adjoint operator $X^* : V^* \longrightarrow V^*$ following the sign rule. For $\phi \in V^*$ and $v \in V$ we set

$$< X^*\phi, v >=< \phi, (-1)^{\bar{X}\bar{\phi}} Xv >,$$

where $< \cdot, \cdot >: V^* \otimes V \to k$ is the natural pairing. Let $\{e_i\}$, $i = 1, \dots, m+n$ be a homogeneous basis of V as above and $\{f_i\}$ be the dual basis of V^* in the sense that $< f_j, e_i >= \delta_{i,j}$. Show that if the matrix of X in the basis $\{e_i\}$ is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then the matrix of X^* in the basis $\{f_i\}$ equals $X^{st} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}$. The operation $X \mapsto X^{st}$ is called the *supertransposition* and it satisfies the identity

$$(XY)^{st} = (-1)^{\bar{X}\bar{Y}} Y^{st} X^{st}.$$

Our next example is the *supertrace*. To define it we use the canonical identification $V \otimes V^* \cong \mathrm{End}_k(V)$ given by

$$v \otimes \phi(w) =< \phi, w > v \text{ for all } v, w \in V, \phi \in V^*.$$

The supertrace str : $\mathrm{End}_k(V) \to k$ is the composition

$$\mathrm{str} : V \otimes V^* \xrightarrow{s} V^* \otimes V \xrightarrow{<\cdot,\cdot>} k.$$

Exercise Prove that if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then

(a) $\mathrm{str}(X) = \mathrm{tr}\,(A) - \mathrm{tr}\,D$,
(b) $\mathrm{str}([X, Y]) = 0$.

The *superdimension* $\mathrm{sdim}\, V$ of a superspace V is by definition the supertrace of the identity operator in V. It follows from the above exercise that $\mathrm{sdim}\, V = \dim V_0 - \dim V_1$. It is important sometimes to remember both even and odd dimension of V. So we set $\dim V = (\dim V_0 | \dim V_1) = (m|n)$ be an element $m + n\varepsilon$ in the ring $\mathbf{Z}(\varepsilon)/(\varepsilon^2 - 1)$.

Exercise Show that

(a) $\mathrm{sdim}(V \oplus W) = \mathrm{sdim}\, V + \mathrm{sdim}\, W$ and $\dim(V \oplus W) = \dim V + \dim W$,
(b) $\mathrm{sdim}(V \otimes W) = \mathrm{sdim}\, V \,\mathrm{sdim}\, W$ and $\dim(V \otimes W) = \dim V \dim W$,
(c) $\mathrm{sdim}(\Pi V) = -\,\mathrm{sdim}\, V$ and $\dim(\Pi V) = \varepsilon \dim V$.

2.2 Lie Superalgebras

Definition 3 A *Lie superalgebra* $\mathfrak{g}$ is a vector superspace with a bilinear even map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that:

1. $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$,
2. $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$.

Example 2 If A is an associative superalgebra, one can make it into a Lie superalgebra $\mathrm{Lie}(A)$ by defining the bracket:

$$[a, b] = ab - (-1)^{\bar{a}\bar{b}} ba.$$

For example if $A = \mathrm{End}(V)$, $\dim(V) = (m|n)$, then $Lie(A)$ is the Lie superalgebra which we denote by $\mathfrak{gl}(m|n)$.

Definition 4 If A is an associative superalgebra, $d : A \longrightarrow A$ is a derivation of A if:

$$d(ab) = d(a)b + (-1)^{\bar{d}\bar{a}} a d(b).$$

Exercise

(a) Check that the space $Der(A)$ of all derivations of A with bracket given by the supercommutator is a Lie superalgebra.
(b) Consider $A = \Lambda(\xi_1, \dots \xi_n)$. Then $Der(A)$ is a finite-dimensional superalgebra denoted by $W(0|n)$. Show that its dimension is $(2^{n-1}n|2^{n-1}n)$.

Exercise Show that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with bracket $[\cdot, \cdot]$ is a Lie superalgebra if and only if

1. $\mathfrak{g}_0$ is a Lie algebra;
2. $[\cdot, \cdot] : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$ equips $\mathfrak{g}_1$ with the structure of a $\mathfrak{g}_0$-module;
3. $[,] : S^2\mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is a homomorphism of $\mathfrak{g}_0$-modules;
4. for all $x \in \mathfrak{g}_1$, $[x, [x, x]] = 0$.

Example 3 Let us introduce the "smallest" simple Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2)$ of dimension $(3|2)$. Take $\mathfrak{g}_0 = \mathfrak{sl}(2)$ and $\mathfrak{g}_1 = V$, where V is the two dimensional irreducible representation of $\mathfrak{sl}(2)$. The isomorphims $S^2V \simeq \mathfrak{sl}(2)$ of $\mathfrak{sl}(2)$-modules defines the bracket $S^2\mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$. One can easily check that $[x, [x, x]] = 0$ for all $x \in \mathfrak{g}_1$ and hence by the previous exercise these data define a Lie superalgebra structure.

Example 4 (Bernstein) Consider a symplectic manifold M, with symplectic form $\omega \in \Omega^2 M$. Consider the following operators acting on the de Rham complex $\Omega(M)$:

- $\omega : \Omega^i(M) \longrightarrow \Omega^{i+2}(M)$, given by $\wedge\omega$,
- $i_\omega : \Omega^i(M) \longrightarrow \Omega^{i-2}(M)$, given by contraction with bivector ω^*,
- grading operator $h : \Omega^i(M) \longrightarrow \Omega^i(M)$.

It is a well known fact that ω, h, i_ω form an sl(2)-triple.

Assume now that $\mathcal{L}$ is a line bundle on M with a connection ∇. Assume further that the curvature of ∇ equals $t\omega$ for some non-zero t. Recall that ∇ is an operator of degree 1 on the sheaf $\mathcal{L} \otimes \Omega(M)$ of differential forms with coefficients in $\mathcal{L}$

$$\nabla : \mathcal{L} \otimes \Omega^i \longrightarrow \mathcal{L} \otimes \Omega^{i+1}.$$

On the other hand, ω, h, i_ω act on $\mathcal{L} \otimes \Omega$ in the same manner as before. Set $\nabla^* := [\nabla, i_\omega]$. One can check that ∇, ∇^*, together with ω, h, i_ω span the superalgebra isomorphic to $\mathfrak{osp}(1|2)$.

The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is the associative superalgebra which satisfies the natural universality property in the category of superalgebras. It can be defined as the quotient of the tensor superalgebra $T(\mathfrak{g})$ by the ideal generated by $XY - (-1)^{\bar{X}\bar{Y}}YX - [X, Y]$ for all homogeneous $X, Y \in \mathfrak{g}$. The PBW theorem holds in the supercase, i.e. $Gr\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$. However, $S(\mathfrak{g})$ is a free commutative superalgebra. From the point of view of the usual tensor algebra we have an isomorphism $S(\mathfrak{g}) \simeq S(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$.

3 Basic Lie Superalgebras

3.1 Simple Lie Superalgebras

A Lie superalgebra is *simple* if it does not have proper non-trivial ideals (ideals are of course $\mathbf{Z}_2$-graded).

Exercise Prove that if a Lie superalgebra $\mathfrak{g}$ is simple, then $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$ and $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$.

In 1977 Kac classified simple Lie superalgebras over an algebraically closed field k of characteristic zero, [22]. He divided simple Lie superalgebras into three groups:

- *basic*: classical and exceptional,
- *strange*: $P(n)$, $Q(n)$,
- *Cartan type*: $W(0|n) = Der\Lambda(\xi_1, \dots \xi_n)$ and some subalgebras of it.

Basic and strange Lie superalgebras have a reductive even part. Cartan type superalgebras have a non-reductive $\mathfrak{g}_0$.

Definition 5 A simple Lie superalgebra $\mathfrak{g}$ is *basic* if $\mathfrak{g}_0$ is reductive and if $\mathfrak{g}$ admits a non-zero invariant even symmetric form $(\cdot, \cdot)$, i. e. the form satisfying the condition

$$([x, y], z) + (-1)^{\bar{x}\bar{y}}(y, [x, z]) = 0, \quad \text{for all} \quad x, y, z \in \mathfrak{g},$$

or, equivalently,

$$([x, y], z) = (x, [y, z]).$$

and $(x, y) \neq 0$ implies $\bar{x} = \bar{y}$.

Exercise 1 Let V be a finite-dimensional $\mathfrak{g}$-module. Then the form

$$(x, y) := \mathrm{str}_V(yx)$$

is an invariant even symmetric form.
In this section we describe the basic Lie superalgebras. We start with classical Lie superalgebras. The invariant symmetric form is given by the supertrace in the natural module V.

Special linear Lie Superalgebra $\mathfrak{sl}(m|n)$ is the subalgebra of $\mathfrak{gl}(m|n)$ of matrices with supertrace zero. It is not hard to verify that $\mathfrak{sl}(m|n)$ is simple if $m \neq n$ and $m + n \geqslant 2$. What happens when $m = n$? In this case the supertrace of the identity matrix is zero and therefore $\mathfrak{sl}(n|n)$ has a one-dimensional center $\mathfrak{z}$ consisting of all scalar matrices. We define $\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\mathfrak{z}$.

Exercise Check that $\mathfrak{psl}(n|n)$ is simple if $n \geqslant 2$.

Look at the case $n = 1$. Then $\mathfrak{sl}(1|1) =< x, y, z >$, where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the commutators are:

$$[x, z] = [y, z] = 0, \quad [x, y] = z,$$

and we see that $\mathfrak{sl}(1|1)$ is a nilpotent $(1|2)$-dimensional Lie superalgebra, which is the superanalogue of the Heisenberg algebra. Furthermore, $\mathfrak{psl}(1|1)$ is an abelian $(0|2)$-dimensional superalgebra.

We have $\mathfrak{sl}(m|n)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid \mathrm{tr}\,(A) = \mathrm{tr}\,(D) \right\}$. Hence

$$\mathfrak{sl}(m|n)_0 \cong \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus k.$$

Note also that $\mathfrak{g} = \mathfrak{sl}(m|n)$ has a compatible $\mathbf{Z}$-grading[1]:

$$\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$$

with $\mathfrak{g}_0 = \mathfrak{g}(0)$ and

$$\mathfrak{g}(1) = V_0 \otimes V_1^*, \qquad \mathfrak{g}(-1) = V_0^* \otimes V_1.$$

The Orthosymplectic Lie Superalgebra $\mathfrak{osp}(m|n)$ is also a subalgebra of $\mathfrak{gl}(m|n)$. Let V be a vector superspace of dimension $(m|n)$ equipped with an even non-degenerate bilinear symmetric form $(\cdot, \cdot)$, i.e. for all homogeneous $v, w \in V$ we have

$$(v, w) = (-1)^{\bar{v}\bar{w}}(w, v), \quad (v, w) \neq 0 \implies \bar{v} = \bar{w}.$$

Note that $(\cdot, \cdot)$ is symmetric on V_0 and symplectic on V_1. Hence the dimension of V_1 must be even, $n = 2l$. We define:

$$\mathfrak{osp}(m|n) := \{X \in \mathfrak{gl}(m|n) \mid (Xv, w) + (-1)^{\bar{X}\bar{v}}(v, Xw) = 0\}.$$

It is easy to see that $\mathfrak{g}_0 = \mathfrak{so}(m) \oplus \mathfrak{sp}(2l)$. So the two classical series, orthogonal and symplectic, come together in the superalgebra theory. One can see also that $\mathfrak{g}_1$ is isomorphic to $V_0 \otimes V_1$ as a $\mathfrak{g}_0$-module. Furthermore it is easy to check that $\mathfrak{osp}(m|2l)$ is simple for all $m, l > 0$.

[1] A grading $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is compatible if $\mathfrak{g}(2j) \subset \mathfrak{g}_0$ and $\mathfrak{g}(2j + 1) \subset \mathfrak{g}_1$.

Lemma 1 *Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra over an algebraically closed field k. Then the center of $\mathfrak{g}_0$ is at most one dimensional.*

Proof Assume the opposite. Let z_1, z_2 be two linearly independent elements in the center of $\mathfrak{g}_0$. For all $a, b \in k$ set

$$\mathfrak{g}(a,b) = \{x \in \mathfrak{g}_1 \mid (\mathrm{ad}_{z_1} - a)^{\dim \mathfrak{g}_1} x = 0,\ (\mathrm{ad}_{z_2} - b)^{\dim \mathfrak{g}_1} x = 0\}.$$

Then we have

1. $\mathfrak{g}_1 = \bigoplus \mathfrak{g}(a,b)$;
2. $[\mathfrak{g}_0, \mathfrak{g}(a,b)] \subset \mathfrak{g}(a,b)$;
3. $[\mathfrak{g}(a,b), \mathfrak{g}(c,d)] \neq 0$ implies $a = -c, b = -d$.

These conditions imply that $[\mathfrak{g}(a,b), \mathfrak{g}(-a,-b)] + \mathfrak{g}(a,b) + \mathfrak{g}(-a,-b)$ is an ideal in $\mathfrak{g}$. Therefore by simplicity of $\mathfrak{g}$ we obtain that for some $a, b \in k$, $\mathfrak{g} = [\mathfrak{g}(a,b), \mathfrak{g}(-a,-b)] + \mathfrak{g}(a,b) + \mathfrak{g}(-a,-b)$. Set $z = bz_1 - az_2$ if $a \neq 0$ and $z = z_1$ if $a = 0$. Then ad_z acts nilpotently on $\mathfrak{g}_1$. But $\mathfrak{g}_0 \oplus [z, \mathfrak{g}_1]$ is an ideal in $\mathfrak{g}$. Hence $z = 0$ and we obtain a contradiction.

Lemma 2 *Let $\mathfrak{g}$ be a basic Lie superalgebra and $\mathfrak{g}_1 \neq 0$. Then one of the following holds.*

1. *There is a* $\mathbf{Z}$*-grading* $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$*, such that* $\mathfrak{g}(0) = \mathfrak{g}_0$ *and* $\mathfrak{g}(\pm 1)$ *are irreducible* $\mathfrak{g}_0$*-modules.*
2. *The even part* $\mathfrak{g}_0$ *is semisimple and* $\mathfrak{g}_1$ *is an irreducible* $\mathfrak{g}_0$*-module.*

Proof Consider the restriction of the invariant form $(\cdot,\cdot)$ on $\mathfrak{g}_1$. Let $M, N \subset \mathfrak{g}_1$ be two $\mathfrak{g}_0$ submodules such that $(M, N) = 0$. Then by invariance of the form we have $([M,N], \mathfrak{g}_0) = (M, [\mathfrak{g}_0, N]) = 0$. Hence $[M,N] = 0$. In particular, let $M \subset \mathfrak{g}_1$ be an irreducible $\mathfrak{g}_0$ submodule. Then the restriction of $(\cdot,\cdot)$ on M is either non-degenerate or zero.

In the first case, let $N = M^{\perp}$ and $I = M \oplus [M,M]$. Then $[N, I] = 0$ and $[\mathfrak{g}_0, I] \subset I$. Hence I is an ideal of $\mathfrak{g}$, which implies $N = 0$, $M = \mathfrak{g}_1$ and $\mathfrak{g}$ satisfies 2. It follows from the proof of Lemma 1 that $\mathfrak{g}_0$ has trivial center.

In the second case there exists an irreducible isotropic submodule $M' \subset \mathfrak{g}_1$ such that $(\cdot,\cdot)$ defines a $\mathfrak{g}_0$-invariant non-degenerate pairing $M \times M' \to k$. By the same argument as in the previous case we have $\mathfrak{g}_1 = M \oplus M'$, $[M,M] = [M',M'] = 0$. Thus, we can set

$$\mathfrak{g}(1) = M, \mathfrak{g}(-1) = M', \mathfrak{g}(0) = \mathfrak{g}_0.$$

Hence $\mathfrak{g}$ satisfies 1.

We say that a basic $\mathfrak{g}$ is of *type 1* (resp. of *type 2*) if it satisfies 1 (resp. 2). Note that if $\mathfrak{g}$ is of type 1, then $\mathfrak{g}(1)$ and $\mathfrak{g}(-1)$ are dual $\mathfrak{g}_0$-modules.

Exercise Check that $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(m|m)$ and $\mathfrak{osp}(2|2n)$ are of type 1, and $\mathfrak{osp}(m|2n)$ is of type 2 if $m \neq 2$.

In contrast with simple Lie algebras, simple Lie superalgebras can have non-trivial central extensions, derivations and deformations. Besides, finite-dimensional representations of simple Lie superalgebras are not completely reducible.

Example 5 Consider the short exact sequence of Lie superalgebras:

$$0 \longrightarrow k \longrightarrow \mathfrak{sl}(2|2) \longrightarrow \mathfrak{psl}(2|2) \longrightarrow 0.$$

One can see that this sequence does not split. In other words, a simple Lie superalgebra $\mathfrak{psl}(2|2)$ has a non-trivial central extension. The dual of this example implies that a finite-dimensional representation of a simple Lie algebra may be not completely reducible, just look at the representation of $\mathfrak{psl}(2|2)$ in $\mathfrak{pgl}(2|2)$ and the exact sequence

$$0 \longrightarrow \mathfrak{psl}(2|2) \longrightarrow \mathfrak{pgl}(2|2) \longrightarrow k \longrightarrow 0.$$

The next example will show that sometimes simple Lie superalgebras have non-trivial deformations.

Example 6 Let $\mathfrak{g} = \mathfrak{osp}(4|2)$. We have

$$\mathfrak{g}_0 = \mathfrak{so}(4) \oplus \mathfrak{sl}(2) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2).$$

In fact, this is the only example of a classical Lie superalgebra whose even part has more then two simple ideals. If V denotes the irreducible 2-dimensional representation of $\mathfrak{sl}(2)$, then $\mathfrak{g}_1$ is isomorphic to $V \boxtimes V \boxtimes V$ as a $\mathfrak{g}_0$-module.

We will construct a one parameter deformation of this superalgebra by deforming the bracket $S^2\mathfrak{g}_1 \to \mathfrak{g}_0$. Let $\psi : S^2V \to \mathfrak{sl}(2)$ and $\omega : \Lambda^2V \to \mathfrak{sl}(2)$ be non-trivial $\mathfrak{sl}(2)$-equivariant maps. Define the bracket between two odd elements by the formula

$$\begin{aligned}&[v_1 \otimes v_2 \otimes v_3, w_1 \otimes w_2 \otimes w_3]\\ &\quad = (t_1\omega(v_2, w_2)\omega(v_3, w_3)\psi(v_1, w_1), t_2\omega(v_1, w_1)\omega(v_3, w_3)\psi(v_2, w_2),\\ &\quad t_3\omega(v_1, w_1)\omega(v_2, w_2)\psi(v_3, w_3))\end{aligned}$$

for some $t_1, t_2, t_3 \in k$.

Exercise The Jacobi identity holds if and only if $t_1 + t_2 + t_3 = 0$.

When $t_1+t_2+t_3 = 0$ we obtain a new Lie superalgebra structure on $\mathfrak{g}$: we denote the corresponding Lie superalgebra by $D(2, 1|t_1, t_2, t_3)$. We see immediately that

$$D(2, 1|t_1, t_2, t_3) \cong D(2, 1|t_{s(1)}, t_{s(2)}, t_{s(3)}) \cong D(2, 1|ct_1, ct_2, ct_3)$$

for all $c \in k^*$ and $s \in S_3$. One can check that $D(2, 1|1, 1, -2) \cong \mathfrak{osp}(4|2)$ and that $D(2, 1|t_1, t_2, t_3)$ is simple whenever $t_1t_2t_3 \neq 0$. By setting $a = \frac{t_2}{t_1}$ one obtains a one-parameter family $D(2,1,a)$ of Lie superalgebras. One can consider a as a local coordinate in $\mathbf{P}^1 \setminus \{0,-1,\infty\}$.

Exercise Prove that, if $a = 0$, then $D(2,1,a)$ has the ideal J isomorphic to $\mathfrak{psl}(2|2)$ with the quotient $D(2,1,a)/J$ isomorphic to $\mathfrak{sl}(2)$. Use this to prove that the superalgebra of derivations of $\mathfrak{psl}(2|2)$ is isomorphic to $D(2,1,0)$.

Consider now the following general construction of a basic Lie superalgebra of type 2. Let

$$\mathfrak{g}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2, \quad \mathfrak{g}_1 = M \otimes N$$

where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are simple Lie algebras, M is a simple $\mathfrak{l}_1$-module and N a simple $\mathfrak{l}_2$-module. Assume in addition that M has an $\mathfrak{l}_1$-invariant skewsymmetric form ω, while N has an $\mathfrak{l}_2$-invariant symmetric form γ. Then we have isomorphisms $S^2M \simeq \mathfrak{sp}(M)$ and $\Lambda^2N \simeq \mathfrak{so}(N)$. Hence $\mathfrak{l}_1$ is a submodule in S^2M and $\mathfrak{l}_2$ is a submodule in Λ^2N. Let $\phi : S^2M \longrightarrow \mathfrak{l}_1$, $\psi : \wedge^2N \longrightarrow \mathfrak{l}_2$ denote the projections on the corresponding submodules. For some $t \in k$ and all $x, x' \in M$, $y, y' \in N$ we set

$$[x \otimes y, x' \otimes y'] := \omega(x,x')\psi(y \wedge y') + t\gamma(y,y')\phi(x \cdot x')$$

If for a suitable $t \in k$ we have $[X,[X,X]] = 0$ for all $X \in \mathfrak{g}_1$, then $\mathfrak{g}$ is a Lie superalgebra. For instance, this construction works for $\mathfrak{osp}(m|2n)$ with $\mathfrak{l}_1 = \mathfrak{sp}(2n)$, $\mathfrak{l}_2 = \mathfrak{so}(m)$ and M, N being the standard modules.

This construction also works for exceptional Lie superalgebras: G_3 and F_4 (in Kac's notation). We prefer to use the notation AG_2 and AB_3 to avoid confusion with Lie algebras.

- $\mathfrak{g} = AG_2$ with $\mathfrak{l}_1 = \mathfrak{sl}(2)$, $\mathfrak{l}_2 = G_2$, M is the 2-dimensional irreducible $\mathfrak{sl}(2)$-module and N is the smallest irreducible G_2-module of dimension 7. One can easily see that $\dim AG_2 = (17|14)$.
- $\mathfrak{g} = AB_3$ with $\mathfrak{l}_1 = \mathfrak{sl}(2)$, $\mathfrak{l}_2 = \mathfrak{so}(7)$, M is again the 2-dimensional irreducible $\mathfrak{sl}(2)$-module, N is the spinor representation of $\mathfrak{so}(7)$. Clearly, $\dim AB_3 = (24|16)$.

Theorem 1 (Kac, [22]) *Let k be an algebraically closed field of characteristic zero and $\mathfrak{g}$ be a basic Lie superalgebra over k with nontrivial $\mathfrak{g}_1$. Then $\mathfrak{g}$ is isomorphic to one of the following superalgebras:*

- $\mathfrak{sl}(m|n)$, $1 \leqslant m < n$;
- $\mathfrak{psl}(n|n)$, $n \geqslant 2$;
- $\mathfrak{osp}(m|2n)$, $m, n \geqslant 1$, $(m,n) \neq (2,1), (4,1)$;
- $D(2,1,a)$, $a \in (\mathbf{P}^1 \setminus \{0,-1,\infty\})/S_3$;
- AB_3;
- AG_2.

For the proof of Theorem 1 we refer the reader to the original paper of Kac. Some hints can be also found in the next Section.

Exercise Show that $\mathfrak{sl}(1|2)$ and $\mathfrak{osp}(2|2)$ are isomorphic Lie superalgebras. Check that the list in Theorem 1 does not contain isomorphic superalgebras.

3.2 Roots Decompositions of Basic Lie Superalgebras

From now on we will always assume that k is algebraically closed of characteristic zero.

Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and W denote the Weyl group of $\mathfrak{g}_0$. If $\mathfrak{g}$ is of type 1 but $\mathfrak{g}_0$ is semisimple it will be convenient to consider a bigger superalgebra $\tilde{\mathfrak{g}}$ by adding to $\mathfrak{g}$ the grading element z (if $\mathfrak{g} = \mathfrak{psl}(n|n)$, then $\tilde{\mathfrak{g}} = \mathfrak{pgl}(n|n)$). In this case we set $\tilde{\mathfrak{h}}_0 := \mathfrak{h}_0 + kz$, otherwise $\tilde{\mathfrak{h}}_0 := \mathfrak{h}_0$. Let $\mathfrak{h}$ be the centralizer of $\tilde{\mathfrak{h}}_0$ in $\mathfrak{g}$.

Lemma 3 *We have* $\mathfrak{h} = \mathfrak{h}_0$ *and* $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_0$.

Proof If $\mathfrak{g}$ is of type 1, the statement is trivial. If $\mathfrak{g}$ is of type 2, then $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module which admits invariant symplectic form. Then such representation does not have zero weight, see [34, Chap. 4.3, Exercise 13].

Lemma 3 implies that $\tilde{\mathfrak{h}}$ acts semisimply on $\mathfrak{g}$. Hence we have a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \text{for all } h \in \tilde{\mathfrak{h}}\}.$$

Here Δ is a finite subset of non-zero vectors in $\tilde{\mathfrak{h}}^*$, whose elements are called *roots*. The subalgebra $\mathfrak{h}$ is called a *Cartan subalgebra* of $\mathfrak{g}$.

The following conditions are straightforward

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \neq 0$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.
- The invariant form $(\cdot, \cdot)$ defines a non-degenerate pairings $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to k$ for all $\alpha \in \Delta$ and $\mathfrak{h} \times \mathfrak{h} \to k$.
- $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a one-dimensional subspace in $\mathfrak{h}$. That follows from the first two properties and the identity $([x, y], h) = \alpha(h)(x, y)$ for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}, h \in \mathfrak{h}_0$.

We can define the non-degenerate symmetric form on $(\cdot|\cdot)$ on $\tilde{\mathfrak{h}}^*$ as the pull back of $(\cdot, \cdot)$ with respect to $\tilde{\mathfrak{h}}^* \xrightarrow{p} \mathfrak{h}^* \xrightarrow{s} \mathfrak{h}$, where p is the canonical projection and $s : \mathfrak{h}^* \to \mathfrak{h}$ is an isomorphism induced by $(\cdot, \cdot)$. For any two roots $\alpha, \beta \in \Delta$

$$\beta(\mathfrak{h}_\alpha) = 0 \quad \text{if and only if} \quad (\alpha, \beta) = 0. \tag{1}$$

Lemma 4 *Let* $\alpha \in \Delta$ *be a root.*

1. $\dim(\mathfrak{g}_\alpha)_0 \leqslant 1$*;*
2. If $(\mathfrak{g}_\alpha)_0 \neq 0$*, then* $(\mathfrak{g}_\alpha)_1 = 0$.

Proof Since $\mathfrak{g}_0$ is reductive 1 is trivial. To prove 2 consider the root $\mathfrak{sl}(2)$-subalgebra $\{x_\alpha, h_\alpha, y_\alpha\} \subset \mathfrak{g}_0$. Let $x \in (\mathfrak{g}_\alpha)_1$ and $x \neq 0$. Then from representation theory of $\mathfrak{sl}(2)$ we know that $[y_\alpha, x] \neq 0$. But $[y_\alpha, x] \in \mathfrak{h}_1 = 0$. Contradiction.

We call $\alpha \in \Delta$ even (resp. odd) if $(\mathfrak{g}_\alpha)_1 = 0$, (resp. $(\mathfrak{g}_\alpha)_0 = 0$). We denote by Δ_0 (resp. Δ_1) the set of even (resp. odd roots). The preceding lemma implies that Δ is the disjoint union of Δ_0 and Δ_1.

Lemma 5

1. *If* $\alpha \in \Delta_0$, *then* $(\alpha|\alpha) \neq 0$.
2. *If* $\alpha \in \Delta_1$ *and* $(\alpha|\alpha) \neq 0$, *then for any non-zero* $x \in \mathfrak{g}_\alpha$, $[x, x] \neq 0$. *Hence* $2\alpha \in \Delta_0$.
3. *If* $\alpha \in \Delta_1$ *and* $(\alpha|\alpha) \neq 0$, *then* $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \{-1, 0, 1\}$ *for any* $\beta \in \Delta_0$.
4. *If* $\alpha \in \Delta_1$ *and* $(\alpha|\alpha) = 0$, *then* $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \{-2, -1, 0, 1, 2\}$ *for any* $\beta \in \Delta_0$.

Proof 1 is the property of root decomposition of reductive Lie algebras. To show 2 let $y \in \mathfrak{g}_{-\alpha}$ be such that $(x, y) \neq 0$. Then $h = [y, x] \neq 0$ and by (1) we obtain

$$[y, [x, x]] = 2[h, x] = 2\alpha(h)x \neq 0.$$

To prove the last two statements we consider the root $\mathfrak{sl}(2)$-triple $\{x_\beta, h_\beta, y_\beta\}$. Then from the representation theory of $\mathfrak{sl}(2)$ we obtain that $\frac{2(\alpha|\beta)}{(\beta|\beta)} = \alpha(h_\beta)$ must be an integer.

To show 3 we use the fact that 2α is an even root. We know from the structure theory of reductive Lie algebras that

$$\frac{2(2\alpha|\beta)}{(\beta, \beta)} \in \{-3, -2 - 1, 0, 1, 2, 3\}.$$

Taking into account that $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \mathbf{Z}$, we obtain the assertion.

Finally, let us prove 4. Without loss of generality we may assume that $k = \alpha(h_\beta) > 1$. Then we claim that $y_\beta(\mathfrak{g}_\alpha) \neq 0$, hence $\alpha - \beta$ is a root. Moreover

$$(\alpha - \beta|\alpha - \beta) = (\beta|\beta)(1 - k) \neq 0.$$

Therefore $\gamma := 2(\alpha - \beta)$ is an even root and we have

$$\frac{2(\beta|\gamma)}{(\gamma|\gamma)} = \frac{k/2 - 1}{1 - k} \in \mathbf{Z},$$

which implies $k = 2$.

Exercise An odd root α is called *isotropic* if $(\alpha|\alpha) = 0$. Show that if $\mathfrak{g}$ is of type 1, then all odd roots are isotropic.

It is clear that W acts on Δ and preserves the parity and the scalar products between roots.

Lemma 6

(a) If $\mathfrak{g}$ *is of type* 1 *then* W *has two orbits in* Δ_1*, the roots of* $\mathfrak{g}(1)$ *and the roots of* $\mathfrak{g}(-1)$.
(b) If $\mathfrak{g}$ *is of type* 2*, then* W *acts transitively on the set of isotropic and the set of non-isotropic odd roots.*

Proof If all roots of $\mathfrak{g}$ are isotropic, then it follows from the proof of Lemma 5 (4) that $\alpha(h_\beta) = \pm 1$ or 0 for any odd root α and even root β. In particular, if we fix positive roots in Δ_0 and consider a highest weight α in $\mathfrak{g}_1$ (or $\mathfrak{g}(\pm 1)$ in type 1 case), the above condition implies that $\mathfrak{g}_1$ (resp. $\mathfrak{g}(\pm 1)$) is a minuscule representation of $\mathfrak{g}_0$.

If $\mathfrak{g}$ is of type 2 and the highest weight α is isotropic, then we have $\alpha(h_\beta) = \pm 1, \pm 2$ or 0 for any positive β. That implies the existence of two orbits. Finally if α is not isotropic, then $\mathfrak{g}_1$ is minuscule, hence there is one W-orbit consisting of non-isotropic roots.

Corollary 1 *For any root* $\alpha \in \Delta$ *the root space* $\mathfrak{g}_\alpha$ *has dimension* $(1|0)$ *or* $(0|1)$.

Proof We need to prove the statement only for odd α. If $\mathfrak{g}$ is of type 1 or of type 2 with only isotropic or only non-isotropic odd roots, then the statement follows from Lemma 6 since the multiplicity of the highest weight is 1. If $\mathfrak{g}$ contains both isotropic and non-isotropic roots, we have to show only that $\dim\mathfrak{g}_\alpha = (0|1)$ for a non-isotropic odd root α, which easily follows from Lemma 5 (2).

Remark 1 Note that if we do not extend $\mathfrak{psl}(2|2)$ to $\mathfrak{pgl}(2|2)$, then Corollary 1 does not hold since the dimension of $\mathfrak{g}_\alpha$ equals $(0|2)$ for any odd α.

Example 7 Let $\mathfrak{g} = \mathfrak{sl}(m|n)$. We take as our Cartan subalgebra $\mathfrak{h}$ the subalgebra of diagonal matrices. Let us denote by $\epsilon_1, \dots \epsilon_m, \delta_1, \dots \delta_n$ the roots in $\mathfrak{h}^*$ ($\epsilon_i(\mathrm{diag}(a_1, \dots a_m)) = a_i$ and similarly for δ_j). We have:

$$\Delta_0 = \{\epsilon_i - \epsilon_j,\ 1 \leqslant i \neq j \leqslant m\} \cup \{\delta_i - \delta_j,\ 1 \leqslant i \neq j \leqslant n\}, \qquad \Delta_1 = \{\pm(\epsilon_i - \delta_j)\}.$$

The invariant form is:

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\epsilon_i, \delta_j) = 0, \quad (\delta_i, \delta_j) = -\delta_{ij},$$

All odd roots are isotropic.

Example 8 Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. $\mathfrak{g}_0 = \mathfrak{sp}(2n)$.

$$\Delta_0 = \{\pm(\epsilon_i \pm \epsilon_j),\ \pm 2\epsilon_i \,|\, i, j = 1 \dots n,\ i \neq j\}, \qquad \Delta_1 = \{\pm\epsilon_i \,|\, i = 1 \dots n\}.$$

This is the only example of a basic superalgebra such that all odd roots are non-isotropic.

The above implies that we have in general three types of roots:

1. $\alpha \in \Delta_0$. In this case the root spaces $\mathfrak{g}_{\pm\alpha}$ generate a $\mathfrak{sl}(2)$ subalgebra (white node in a Dynkin diagram).

2. $\alpha \in \Delta_1$, $(\alpha, \alpha) \neq 0$. Then the root spaces $\mathfrak{g}_{\pm\alpha}$ generate a subalgebra isomorphic to $\mathfrak{osp}(1|2)$ (black node in a Dynkin diagram).
3. $\alpha \in \Delta_1$, $(\alpha, \alpha) = 0$. The roots spaces $\mathfrak{g}_{\pm\alpha}$ generate a subalgebra isomorphic to $\mathfrak{sl}(1|1)$ (grey node in a Dynkin diagram).

Definition 6 Let E be a vector space (over k) equipped with non-degenerate scalar product $(\cdot|\cdot)$. A finite subset $\Delta \subset E \setminus \{0\}$ is called a *generalized root system* if the following conditions hold:

- if $\alpha \in \Delta$, then $-\alpha \in \Delta$;
- if $\alpha, \beta \in \Delta$ and $(\alpha|\alpha) \neq 0$, then $k_{\alpha,\beta} = \frac{2(\alpha|\beta)}{(\alpha|\alpha)}$ is an integer and $\beta - k_{\alpha,\beta}\alpha \in \Delta$;
- if $\alpha \in \Delta$ and $(\alpha|\alpha) = 0$, then there exists an invertible map $r_\alpha : \Delta \to \Delta$ such that

$$r_\alpha(\beta) = \begin{cases} \beta \text{ if } (\alpha|\beta) = 0 \\ \beta \pm \alpha \text{ if } (\alpha|\beta) \neq 0 \end{cases}.$$

Exercise Check that if $\mathfrak{g}$ is a basic Lie superalgebra, then the set of roots Δ is a generalized root system.

Indecomposable generalized root systems are classified in [39]. In fact, they coincide with root systems of basic Lie superalgebras. That gives an approach to the proof of Theorem 1.

Exercise Let Q_0 be the lattice generated by Δ_0 and Q be the lattice generated by Q. Check that

- If $\mathfrak{g}$ is of type 1, then Q_0 is a sublattice of corank 1 in Q.
- If $\mathfrak{g}$ is of type 2, then Q_0 is a finite index subgroup in Q.

3.3 Bases and Odd Reflections

As in the case of simple Lie algebras we can represent Δ as a disjoint union $\Delta^+ \coprod \Delta^-$ of positive and negative roots (by dividing $\tilde{\mathfrak{h}}^*$ in two half-spaces).

We are going to use the *triangular decomposition*:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \text{where} \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha,$$

The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called a *Borel subalgebra* of $\mathfrak{g}$.

We call $\alpha \in \Delta^+$ indecomposable if it is not a sum of two positive roots. We call the set of indecomposable roots $\alpha_1, \ldots \alpha_n \in \Delta^+$ *simple roots* or a *base* as in the Lie algebra case. Clearly, W action on Δ permutes bases. However, not all bases can be obtained from one by the action of W.

Example 9 The Weyl group of $\mathfrak{gl}(2|2)$ is isomorphic to $S_2 \times S_2$. One can see that the following two bases are not conjugate by the action of W: $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \delta_1, \delta_1 - \delta_2\}$, $\Pi' = \{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \epsilon_2 - \delta_2\}$.

Since W does not act transitively on the set of bases, more than one Dynkin diagram may be associated to the same Lie superalgebra. The existence of several Dynkin diagrams implies existence of several non conjugate Borel subalgebras, which in turn implies that there are several non isomorphic flag supervarieties.

To every base Π we associate the *Cartan matrix* in the following way. Take $X_i \in \mathfrak{g}_{\alpha_i}$, $Y_i \in \mathfrak{g}_{-\alpha_i}$, and set $H_i := [X_i, Y_i]$ and $a_{ij} := \alpha_j(H_i)$. In the classical theory of Kac-Moody algebras Cartan matrices are normalized so that the diagonal entries are equal to 2. In the supercase we can do the same for non-isotropic simple roots. It is not difficult to see that H_i, X_i, Y_i for $i = 1, \dots, n$ generate $\mathfrak{g}$ and satisfy the relations

$$[H_i, X_j] = a_{ij} X_j, \qquad [H_i, Y_j] = -a_{ij} Y_j, \qquad [X_i, Y_j] = \delta_{ij} H_i, \qquad [H_i, H_j] = 0.$$

Let $\bar{\mathfrak{g}}$ be the free Lie superalgebra with above generators and relations. We define the Kac-Moody superalgebra $\mathfrak{g}(A)$ as the quotient of $\bar{\mathfrak{g}}$ by the maximal ideal which intersects trivially the Cartan subalgebra. In this way we recover basic finite dimensional Lie superalgebras. In contrast with Lie algebra case we may get a finite-dimensional Kac-Moody superalgebra even if $det(A) = 0$, for example, $\mathfrak{g}(A) = \mathfrak{gl}(n|n)$. Note that in this case $\mathfrak{g}(A)$ is not simple but a non-trivial central extension of the corresponding simple superalgebra. In many applications, it is better to consider $\mathfrak{g}(A)$ instead of the corresponding quotient, which essentially means that in what follows we rather discuss representations and structure theory of $\mathfrak{gl}(n|n)$ instead of $\mathfrak{psl}(n|n)$.

Definition 7 Let Π be a base (set of simple roots) and let $\alpha \in \Pi$ be an isotropic odd root. We define an *odd reflection* $r_\alpha : \Pi \to \Pi'$ by

$$r_\alpha(\beta) = \begin{cases} \beta + \alpha \text{ if } (\alpha|\beta) \neq 0 \\ \beta \text{ if } (\alpha|\beta) = 0,\ \beta \neq \alpha \\ -\alpha \text{ if } \beta = \alpha \end{cases}$$

Exercise Check that $\Pi' = r_\alpha(\Pi)$ is a base.

Notice that if $(\alpha|\alpha) \neq 0$ we can define the usual reflection $r_\alpha(x) := x - \frac{2(x|\alpha)}{(\alpha|\alpha)}\alpha$, which is an orthogonal linear transformation of $\mathfrak{h}^*$. In fact, since $r_\alpha = r_{2\alpha}$, one can see that these reflections generate W. Though the odd reflections are defined on simple roots only, one can show that they may be extended (uniquely) to permutations of all roots. However, in most cases such extension can not be further extended to a linear map of the root lattice.

Proposition 1 *Let $\mathfrak{g}$ be a basic Lie superalgebra.*

1. *If Π and Π' are two bases, then Π' can be obtained from Π by application of odd and even reflections.*
2. *If Π and Π' are bases such that $\Delta^+ \cap \Delta_0 = (\Delta')^+ \cap \Delta_0$, then Π' can be obtained from Π by application of odd reflections.*

Go back to the example of $\mathfrak{gl}(2|2)$. The Cartan matrix associated with Π is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The odd reflection r_α associated with the root $\alpha = \epsilon_2 - \delta_1 \in \Pi$ maps Π to Π'. Indeed, we have:

$$\begin{aligned} r_\alpha(\epsilon_1 - \epsilon_2) &= \epsilon_1 - \delta_1 = (\epsilon_1 - \epsilon_2 + \epsilon_2 - \delta_1) \\ r_\alpha(\epsilon_2 - \epsilon_1) &= \delta_1 - \epsilon_2 \\ r_\alpha(\delta_1 - \delta_2) &= \epsilon_2 - \delta_2 = (\epsilon_2 - \delta_1 + \delta_1 - \delta_2). \end{aligned}$$

The Cartan matrix associated with Π' is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Exercise Use odd reflections to get all bases of AG_2.

Remark 2 Let $\mathfrak{g}$ be of type 1 and let us fix a Borel subalgebra $\mathfrak{b}_0 \subset \mathfrak{g}_0$. We have two especially convenient Borel subalgebras:

$$\mathfrak{b}_d = \mathfrak{b}_0 \oplus \mathfrak{g}(1), \qquad \mathfrak{b}_{ad} = \mathfrak{b}_0 \oplus \mathfrak{g}(-1).$$

We call them *distinguished* and *antidistinguished*, respectively.

4 Representations of Basic Superalgebras

4.1 Highest Weight Theory

We assume in this section that $\mathfrak{g}$ is a basic superalgebra or its Kac Moody extension (in the case of $\mathfrak{gl}(n|n)$). Let us fix a triangular decomposition: $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Define the *Verma module*:

$$M_{\mathfrak{b}}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} C_\lambda,$$

where C_λ is the one-dimensional $\mathfrak{b}$-module with trivial action of $\mathfrak{n}^+$ and weight λ. One can prove exactly as in the Lie algebra case that $M_{\mathfrak{b}}(\lambda)$ has a unique simple quotient which we denote by $L_{\mathfrak{b}}(\lambda)$.

We say that λ is *integral dominant* if $L_{\mathfrak{b}}(\lambda)$ is finite dimensional.

Exercise Prove that if λ is integral dominant, then $M_{\mathfrak{b}}(\lambda)$ has the unique maximal finite dimensional quotient $K_{\mathfrak{b}}(\lambda)$. If $\mathfrak{g}$ is of type 1 and $\mathfrak{b}$ is distinguished, then $K_{\mathfrak{b}}(\lambda)$ is isomorphic to the induced module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}(1))} L_{\mathfrak{b}_0}(\lambda)$, where $L_{\mathfrak{b}_0}(\lambda)$ is the simple $\mathfrak{g}_0$-module with trivial action of $\mathfrak{g}(1)$. In this case it is called a *Kac module*.

Proposition 2 *Any finite-dimensional simple* $\mathfrak{g}$*-module is isomorphic to* $L_{\mathfrak{b}}(\lambda)$ *for some integral dominant* λ.

Proof Any finite dimensional simple module M is semisimple over $\mathfrak{h}$ and hence has a finite number of weights. Let λ be a weight such that $\lambda + \alpha$ is not a weight for all positive roots α. Then, by Frobenius reciprocity, M is a quotient of $M_{\mathfrak{b}}(\lambda)$.

Remark 3 Let $\mathcal{O}$ be the category of finitely generated $\mathfrak{h}$-semisimple $\mathfrak{g}$-modules with locally nilpotent action of $\mathfrak{n}^+$. Note that this definition depends on the choice of a Borel subalgebra $\mathfrak{b}$. In fact, it depends only on the choice of $\mathfrak{b}_0$, since the local nilpotency of $\mathfrak{n}_0^+$ implies the local nilpotency of $\mathfrak{n}^+$.

How do we check whether λ is dominant integral with respect to a particular Borel subalgebra $\mathfrak{b}$? If $\mathfrak{g}$ is of type 1 and $\mathfrak{b}$ is distinguished or antidistinguished, it is sufficient to check that λ is integral dominant with respect to $\mathfrak{b}_0$, i.e. $\lambda(h_\alpha) \in \mathbb{N}$ for all simple even roots α. In general, the condition of dominance is more complicated.

Exercise

(a) If $\mathfrak{b}$ and $\mathfrak{b}'$ are two Borel subalgebras of $\mathfrak{g}$ with the same even part, then we must have an isomorphism $L_{\mathfrak{b}}(\lambda) \simeq L_{\mathfrak{b}'}(\lambda')$ for some weights λ and λ'. Let $\mathfrak{b}'$ be obtained from $\mathfrak{b}$ by an odd reflection r_α. Check that

$$\lambda' = \begin{cases} \lambda - \alpha \text{ if } (\lambda, \alpha) \neq 0 \\ \lambda \text{ if } (\lambda, \alpha) = 0. \end{cases} \tag{2}$$

(b) Fix a base Π and the corresponding Borel subalgebra $\mathfrak{b}$. Let Π_0 denote the base of Δ_0^+. Prove that $L_{\mathfrak{b}}(\lambda)$ is finite-dimensional if and only if for any $\beta \in \Pi_0$ and a base Π' obtained from Π by odd reflections such that $\beta \in \Pi'$ or $\frac{\beta}{2} \in \Pi'$ we have $\frac{2(\lambda|\beta)}{(\beta|\beta)} \in \mathbb{N}$. (*Hint*: you just have to check that $y_\beta \in \mathfrak{g}_{-\beta}$ acts locally nilpotently on $L_{\mathfrak{b}}(\lambda)$.)

4.2 Typicality

We define the *Weyl* vector $\rho_{\mathfrak{b}} \in \mathfrak{h}^*$ by:

$$\rho_{\mathfrak{b}} := \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha - \frac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha.$$

If $\mathfrak{b}$ is fixed and clear we simplify notation by setting $\rho = \rho_{\mathfrak{b}}$.

Exercise Let Π be the base corresponding to $\mathfrak{b}$. Show that

$$(\rho|\alpha) = \begin{cases} \frac{1}{2}(\alpha|\alpha) \text{ if } \alpha \in \Pi \cap \Delta_0 \\ (\alpha|\alpha) \text{ if } \alpha \in \Pi \cap \Delta_1 \end{cases}.$$

Definition 8 A weight λ is called *typical* if $(\lambda + \rho, \alpha) \neq 0$ for all isotropic roots $\alpha \in \Delta$.

Exercise Check that the definition of typicality does not depend on the choice of $\mathfrak{b}$. To show this assume that $\mathfrak{b}'$ is obtained from $\mathfrak{b}$ by an odd reflection r_α and λ is typical. Then $\rho'_{\mathfrak{b}} = \rho_{\mathfrak{b}} + \alpha$ and $L_{\mathfrak{b}}(\lambda) = L'_{\mathfrak{b}}(\lambda')$, where $\lambda + \rho_{\mathfrak{b}} = \lambda' + \rho_{\mathfrak{b}'}$.

4.3 Characters of Simple Finite-Dimensional Modules

If M is in the category $\mathcal{O}$, then, by definition, M is $\mathfrak{h}$-semisimple, and therefore has weight decomposition $M = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mu$. The character $\operatorname{ch} M$ is the generating function

$$\operatorname{ch} M := \sum \operatorname{sdim}(M_\mu)e^\mu.$$

Exercise Show, using Corollary 1, that if M is generated by one weight vector, in particular, if M is simple then every weight space M_μ is either purely even or purely odd.

Theorem 2 ([23]) *If λ is a typical integral dominant weight then*

$$\operatorname{ch} L_{\mathfrak{b}}(\lambda) = \frac{D_1}{D_0}\sum_{w\in W} sgn(w)e^{w(\lambda+\rho_{\mathfrak{b}})}, \tag{3}$$

where W is the Weyl group of the even part $\mathfrak{g}_0$ and

$$D_1 = \prod_{\alpha\in\Delta_1^+}(e^{\alpha/2} - e^{-\alpha/2}), \qquad D_0 = \prod_{\alpha\in\Delta_0^+}(e^{\alpha/2} - e^{-\alpha/2}).$$

Exercise Using the isomorphism of $\mathfrak{h}$-modules $\mathcal{U}(\mathfrak{n}^-) \simeq S(\mathfrak{n}^-)$ show that

$$\operatorname{ch} \mathcal{U}(\mathfrak{n}^-) = \prod_{\alpha \in \Delta_1} (1 - e^{-\alpha}) / \prod_{\alpha \in \Delta_0} (1 - e^{-\alpha}),$$

and

$$\operatorname{ch} M_{\mathfrak{b}}(\lambda)) = e^{\lambda+\rho} \frac{D_1}{D_0}.$$

Remark 4

- If $\mathfrak{g} = \mathfrak{g}_0$ then we get the usual Weyl character formula.
- The formula (3) is invariant with respect to the change of Borel subalgebra.
- The formula (3) can be rewritten in the form

$$\operatorname{ch} L_{\mathfrak{b}}(\lambda) = \sum_{w \in W} sgn(w) \operatorname{ch} M_{\mathfrak{b}}(w \cdot \lambda),$$

where $w \cdot \lambda := w(\lambda + \rho) - \rho$ is the shifted action.

Proof of Theorem 2 We will give the proof for type 1 superalgebras, i.e. assuming a compatible grading $\mathfrak{g} = \mathfrak{g}(-1) + \mathfrak{g}(0) + \mathfrak{g}(1)$. By Remark 4 it suffices to prove the formula for the distinguished $\mathfrak{b} = \mathfrak{b}_d$.

Note that the Kac module $K_{\mathfrak{b}}(\lambda)$ is isomorphic to

$$\mathcal{U}(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda) = \Lambda(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda)$$

as a $\mathfrak{g}_0 + \mathfrak{g}(-1)$-module. Therefore

$$\operatorname{ch} K_{\mathfrak{b}}(\lambda) = \operatorname{ch} \Lambda(\mathfrak{g}(-1)) \operatorname{ch} L_{\mathfrak{b}_0}(\lambda) = \prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha}) \operatorname{ch} L_{\mathfrak{b}_0}(\lambda).$$

Furthermore, if $\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_i} \alpha$, for $i = 0, 1$, then

$$\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha}) = e^{\rho_1} D_1, \quad \operatorname{ch} L_{\mathfrak{b}_0}(\lambda) = \frac{1}{D_0} \sum_{w \in W} sgn(w) e^{w(\lambda+\rho_0)}.$$

Note also that $w(\rho_1) = \rho_1$ for all $w \in W$. Therefore $\operatorname{ch} K_{\mathfrak{b}}(\lambda)$ is given by (3). Thus, it remains to show that $K_{\mathfrak{b}}(\lambda) = L_{\mathfrak{b}}(\lambda)$.

One can see easily that any submodule of $K_{\mathfrak{b}}(\lambda)$ contains a simple $\mathfrak{g}_0$-submodule

$$\Lambda^{top}(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda).$$

Hence $K_{\mathfrak{b}}(\lambda)$ has a unique simple submodule isomorphic to $L_{\mathfrak{b}}(\mu)$ for some μ.

Next we observe that

$$\lambda' := \lambda - \sum_{\alpha \in \Delta_1^+} \alpha$$

is the highest weight of $L_{\mathfrak{b}}(\mu)$ with respect to the anti-distinguished Borel $\mathfrak{b}_{ad}$, since λ' is the $\mathfrak{b}_0$-highest weight in $\Lambda^{top}(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda)$ and

$$\mathfrak{g}(-1)\Lambda^{top}(\mathfrak{g}(-1)) = 0.$$

Therefore we have

$$L_{\mathfrak{b}}(\mu) = L_{\mathfrak{b}_{ad}}(\lambda').$$

Applying (2) several times to move from $\mathfrak{b}$ to $\mathfrak{b}_{ad}$ and using the typicality of λ we obtain $\lambda = \mu$. Hence $K_{\mathfrak{b}}(\lambda) = L_{\mathfrak{b}}(\lambda)$.

4.4 The Center of $\mathcal{U}(\mathfrak{g})$

Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In the supersetting the Duflo theorem states that there exists an isomorphism of supercommutative rings

$$S(\mathfrak{g})^{\mathfrak{g}} \simeq \mathcal{Z}(\mathfrak{g}).$$

For the proof in the supercase see [26].

Recall that if $\mathfrak{g}$ is a reductive Lie algebra then $\mathcal{Z}(\mathfrak{g})$ is a polynomial ring, see, for example, [10]. This fact follows from so called Harish-Chandra homomorphism. One can generalize the Harish-Chandra homomorphism for basic superalgebras, however, as we will see, $\mathcal{Z}(\mathfrak{g})$ is not Noetherian.

Choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, then by PBW theorem we have the decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^+).$$

The Harish-Chandra map

$$HC : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) = k[\mathfrak{h}^*]$$

is the projection with kernel $\mathfrak{n}^-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$. The restriction

$$HC : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h}) = k[\mathfrak{h}^*]$$

is a homomorphism of rings.

For any $w \in W$ and $\lambda \in \mathfrak{h}^*$ we set $w \cdot \lambda := w(\lambda + \rho) - \rho$.

Theorem 3 *The homomorphism* $HC : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ *is injective and* $f \in k[\mathfrak{h}^*]$ *belongs to* $HC(\mathcal{Z}(\mathfrak{g}))$ *if and only if*

- $f(w \cdot \lambda) = f(\lambda)$*, for any* $\lambda \in \mathfrak{h}^*, w \in W$*;*
- *if* $(\lambda + \rho|\alpha) = 0$ *for some isotropic root* α *then* $f(\lambda + t\alpha) = f(\lambda)$ *for all* $t \in k$*.*

The proof of this Theorem can be found in [24, 45] or [16]. One of the consequences of the above theorem is that the supercommutative ring $\mathcal{Z}(\mathfrak{g})$ has trivial odd part and hence is in fact a usual commutative ring.

The proof in [45] makes use of the superanalogue of the Chevalley restriction theorem. Since $\mathfrak{g}$ is basic, then the adjoint representation is self-dual. Thus, we can identify the invariant polynomials on $\mathfrak{g}$ and $\mathfrak{g}^*$:

$$k[\mathfrak{g}]^{\mathfrak{g}} \simeq k[\mathfrak{g}^*]^{\mathfrak{g}}.$$

If $F : k[\mathfrak{g}]^{\mathfrak{g}} \to k[\mathfrak{h}]$ denotes the restriction map induced by the embedding $\mathfrak{h} \subset \mathfrak{g}$, then the image of F consists of W-invariant polynomials on $\mathfrak{h}$ satisfying the additional condition:

if $(\lambda|\alpha) = 0$ for some isotropic root α then $f(\lambda + t\alpha) = f(\lambda)$ for all $t \in k$.

Example 10 Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. The ring $S(\mathfrak{g}^*)^{\mathfrak{g}}$ is generated by $str(X^s)$ $s = 1, 2, 3 \ldots$. After restriction to the diagonal subalgebra they become polynomials in $P_1, P_2, \dots \in k[x_1, \dots, x_m, y_1, \dots, y_n]$ given by the formula Set

$$P_s := x_1^s + \dots x_m^s - y_1^s - \dots - y_n^s.$$

One can see that the subring in $k[x_1, \dots, x_m, y_1, \dots, y_n]$ generated by P_s is not a Noetherian ring.

If *Specm* stands for the spectrum of maximal ideals, then HC induces the map $\theta : Specm(k[\mathfrak{h}^*]) = \mathfrak{h}^* \longrightarrow Specm(\mathcal{Z}(\mathfrak{g}))$. In other words we associate with every weight $\lambda \in \mathfrak{h}^*$ the central character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to k$ by setting $\chi_\lambda(z) := HC(z)(\lambda)$. We would like to describe the fibers of θ. The following corollary implies that every fiber is a union of finitely many affine subspaces of the same dimension.

Corollary 2 *Let* $\lambda \in \mathfrak{h}^*$ *and let* $\{\alpha_1, \dots, \alpha_k\}$ *be a maximal set of mutually orthogonal linearly independent isotropic roots such that* $(\lambda + \rho|\alpha_i) = 0$*. If* $\chi = \chi_\lambda$*, then*

$$\theta^{-1}(\chi) = \bigcup_{w \in W} w \cdot (\lambda + \sum_{i=1}^{k} k\alpha_i).$$

Example 11 If $\mathfrak{g} = \mathfrak{sl}(1|2)$, then $\dim\mathfrak{h} = 2$ and the image of the Harish Chandra homomorphism in $k[x, y]$ consists of polynomials $k[x, y^2]$ which are constant on the cross $y = \pm x$.

Corollary 3 *If λ is typical then* $(\theta)^{-1}(\chi_\lambda) = W \cdot \lambda$.

Corollary 4 *If λ is dominant integral and typical, then* $\mathrm{Ext}^1(L_{\mathfrak{b}}(\lambda), L_{\mathfrak{b}}(\mu)) = 0$ *for any integral dominant $\mu \neq \lambda$. Hence $L_{\mathfrak{b}}(\lambda)$ is projective in the category $\mathcal{F}$ of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$.*

Proof If λ is dominant integral and typical, then $W \cdot \lambda$ does not contain any other integral dominant weight. Therefore $L_{\mathfrak{b}}(\lambda)$ and $L_{\mathfrak{b}}(\mu)$ admit different central characters. Hence $\mathrm{Ext}^1(L_{\mathfrak{b}}(\lambda), L_{\mathfrak{b}}(\mu)) = 0$. Semisimplicity over $\mathfrak{g}_0$ ensures that $\mathrm{Ext}^1_{\mathcal{F}}(L_{\mathfrak{b}}(\lambda), L_{\mathfrak{b}}(\lambda)) = 0$.

Remark 5 If $\mathfrak{g}$ is of type 2, then any finite-dimensional $\mathfrak{g}$-module is semisimple over $\mathfrak{g}_0$. In type 1 case, $L_{\mathfrak{b}}(\lambda)$ is not projective in the category of *all* finite-dimensional $\mathfrak{g}$-modules since it has non-trivial self-extension.

Definition 9 (Kac–Wakimoto) The dimension of $\theta^{-1}(\chi)$ is called the *atypicality degree* of χ. We will denote it by $\mathrm{at}(\chi)$. It follows from Corollary 2 that if $\chi_\lambda = \chi$, then $\mathrm{at}(\chi)$ is the maximal number of mutually orthogonal linearly independent isotropic roots α such that $(\lambda + \rho|\alpha) = 0$. We also use the notation $\mathrm{at}(\lambda) = \mathrm{at}(\chi_\lambda)$. The central character χ is typical (resp. atypical) if $\mathrm{at}(\chi) = 0$ (resp. $f(\chi) > 0$).

The *defect* $\mathrm{def}\,\mathfrak{g}$ of $\mathfrak{g}$ is the maximal number of mutually orthogonal linearly independent isotropic roots, i.e. the maximal dimension of the fiber of θ.

Exercise Show that

$$\mathrm{def}\,\mathfrak{gl}(m|n) = \mathrm{def}\,\mathfrak{osp}(2m|2n) = \mathrm{def}\,\mathfrak{osp}(2m+1|2n) = \min(m, n).$$

Check that the defect of the exceptional superalgebras AG_2, AB_3 and $D(1, 2; a)$ is 1.

Note that $\mathfrak{osp}(1|2n)$ is the only basic superalgebra with defect zero. Hence we have the following proposition.

Proposition 3 *All finite-dimensional representations of $\mathfrak{osp}(1|2n)$ are completely reducible and the character of any irreducible finite-dimensional representation of $\mathfrak{osp}(1|2n)$ is given by (3).*

Finally, let us formulate without proof the following general result which allows to reduce many questions about typical representations (finite or infinite-dimensional) to the same questions for the even part $\mathfrak{g}_0$.

Theorem 4 ([15, 36]) *Suppose that $\chi = \chi_\lambda$ is a typical central character such that $(\lambda + \rho|\beta) \neq 0$ for any non-isotropic root β. Let $\mathcal{U}_\chi(\mathfrak{g}) := \mathcal{U}(\mathfrak{g})/(Ann(\chi))$. Then there exists a central character χ_0 of $\mathcal{Z}(\mathfrak{g}_0)$ such that $\mathcal{U}_\chi(\mathfrak{g})$ is Morita equivalent to $\mathcal{U}_{\chi_0}(\mathfrak{g}_0) := \mathcal{U}(\mathfrak{g}_0)/(Ann(\chi_0))$.*

Remark 6 If $\mathfrak{g}$ is of type 1, then $\mathcal{U}_\chi(\mathfrak{g})$ is isomorphic to the matrix algebra over $\mathcal{U}_{\chi_0}(\mathfrak{g}_0)$.

5 Associated Variety

5.1 Self-Commuting Cone

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra. The self-commuting cone X is the subvariety of $\mathfrak{g}_1$ defined by

$$X = \{x \in \mathfrak{g}_1 \mid [x, x] = 0\}$$

This cone was studied first in [17] for applications to Lie superalgebras cohomology.

Example 12 Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Then

$$X = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid AB = 0 = BA \right\}.$$

We discuss geometry of X for basic classical $\mathfrak{g}$. Let G_0 be a connected, reductive algebraic group such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$ and let B_0 be a Borel subgroup of G_0. It is clear that X is G_0-stable with respect to the adjoint action of G_0 on $\mathfrak{g}_1$. Denote by X/B_0 (resp. X/G_0) the set of B_0 (resp. G_0)-orbits in X. We will see that both sets are finite.

Denote by S_p the set of all p-tuples of linearly independent and mutually orthogonal isotropic roots and set

$$S := \coprod_{p=0}^{\operatorname{def}\mathfrak{g}} S_p, \quad \text{where} \quad S_0 = \{\emptyset\}.$$

Let $u = \{\alpha_1, \ldots, \alpha_p\} \in S_p$, choose non-zero $x_i \in \mathfrak{g}_{\alpha_i}$ and set

$$x_u := x_1 + \cdots + x_p.$$

Then $x_u \in X$ and it is not hard to see that a different choice of the x_i-s produces an element in the same H-orbit, where H is the maximal torus in G_0 with Lie algebra $\mathfrak{h}$. Therefore we have a well-defined map

$$\Phi : S \to X/B_0.$$

Furthermore, the Weyl group W acts on S and clearly $x_{w(u)}$ and x_u belong to the same G_0-orbit. Therefore we also have a map

$$\Psi : S/W \to X/G_0.$$

Theorem 5 *Both maps Φ and Ψ are bijections.*

The proof that Ψ is a bijection can be found in [11] and it is done by case by case inspection. It would be interesting to find a conceptual proof, using for example only properties of the root decomposition. For the proof that Φ is a bijection we refer the reader to [7]. It uses the result about Ψ and the Bruhat decomposition of G_0. It is possible that a conceptual proof of Theorem 5 is related to the following analogue of the Jacobson–Morozov theorem.

Theorem 6 *Let* $\mathfrak{g}$ *be a basic classical Lie superalgebra and* $x \in \mathfrak{g}_1$ *be an odd element such that* $[x,x]$ *is nilpotent. Then*

1. *If* $[x,x]=0$*, then* x *can be embedded into an* $\mathfrak{sl}(1|1)$*-subalgebra of* $\mathfrak{g}$*.*
2. *If* $[x,x]\neq 0$ *then* x *can be embedded into an* $\mathfrak{osp}(1|2)$*-subalgebra of* $\mathfrak{g}$*.*

As a consequence of Theorem 5 we know that every $x \in X$ is G_0-conjugate to x_u for $u \in S_p$. We call the number p the rank of x. If $\mathfrak{g}=\mathfrak{gl}(m|n)$, then the rank coincides with the usual rank of the matrix. We denote by X_p the set of all elements in X of rank p. In this way we define the stratification

$$X = \coprod_{p=0}^{\operatorname{def}\mathfrak{g}} X_p,$$

where $X_0=\{0\}$. Clearly, the Zariski closure of X_p is the disjoint union of X_q for all $q \leqslant p$.

Proposition 4 *The closure of every stratum* X_p *is an equidimensional variety or, equivalently, if* $x,y \in X$ *have the same rank, then* $\dim G_0 x = \dim G_0 y$*. Furthermore, if* $u=\{\alpha_1,\dots,\alpha_p\}\in S_p$ *and*

$$u^{\perp} := \{\beta \in \Delta_1 \mid (\beta|\alpha_i)=0,\ i=1,\dots,p\},$$

then

$$\dim G_0 x_u = \frac{1}{2}|\Delta_1 \setminus u^{\perp}| + p.$$

Proof We start with proving the second assertion. For any $x \in \mathfrak{g}_1$ consider the odd analogue of the Kostant-Kirillov form:

$$\omega(y,z) = (x,[y,z]).$$

This is an odd skew-symmetric form. It is easy to see that $\ker(\omega)=\ker(ad_x)$. Using the isomorphism $[x,\mathfrak{g}] \simeq \mathfrak{g}/\ker(ad_x)$ we can push forward ω to $[x,\mathfrak{g}]$, where it becomes non-degenerate. Since ω is odd, we obtain

$$\dim G_0 x = \dim [x,\mathfrak{g}_0] = \dim [x,\mathfrak{g}_1] = \frac{1}{2}\dim [x,\mathfrak{g}].$$

We compute $\dim[x, \mathfrak{g}]$. Let $x = x_u = x_1 + \cdots + x_p$. Fix some $y_i \in \mathfrak{g}_{-\alpha}$ and let $h_i := [x, y_i] \in \mathfrak{h}_{\alpha_i}$. Consider a generic linear combination $y = c_1 y_1 + \cdots + c_p y_p$ and set $h = [x, y]$. Then x, h, y span an $\mathfrak{sl}(1|1)$-subalgebra $\mathfrak{l}$. Let $\mathfrak{g}'$ be the direct sum of all eigenspaces of ad_h with non-zero eigenvalue and $\mathfrak{g}^h$ denote the centralizer of h. Clearly, $\mathfrak{g}'$ and $\mathfrak{g}^h$ are $\mathfrak{l}$-stable. Furthermore, it is easy to see that

$$\operatorname{sdim}\mathfrak{g}' = 0, \quad [x, \mathfrak{g}'] = \mathfrak{g}' \cap \ker \mathrm{ad}_x \quad \text{hence} \quad \dim[x, \mathfrak{g}'] = \frac{1}{2}\dim\mathfrak{g}' = \dim\mathfrak{g}'_1.$$

For generic $c_1, \ldots, c_p$ we have

$$\mathfrak{g}'_1 = \bigoplus_{\beta \in \Delta_1 \setminus u^{\perp}} \mathfrak{g}_{\beta}.$$

Therefore we obtain

$$\dim[x, \mathfrak{g}'] = |\Delta_1 \setminus u^{\perp}|.$$

On the other hand, a simple calculation shows that

$$[\mathfrak{g}^h, x] = [\mathfrak{l}, x] \oplus [\mathfrak{h}, x] = \bigoplus_{i \leqslant p} (kx_i \oplus kh_i).$$

Therefore $\dim[\mathfrak{g}^h, x] = 2p$.

$$\dim G_0 x = \frac{1}{2}(\dim[x, \mathfrak{g}'] + \dim[x, \mathfrak{g}^h]) = \frac{1}{2}|\Delta_1 \setminus u^{\perp}| + p.$$

The first assertion follows from the fact that for any two $u, u' \in S_p$ there exists $w \in W$ such that $wu' \subset u \cup -u$. This fact is established by case by case inspection.

Corollary 5 *X is an equidimensional variety.*

5.2 Functor F_x

Let $\mathfrak{g}$ be an arbitrary superalgebra and $x \in \mathfrak{g}_1$ satisfy $[x, x] = 2x^2 = 0$. For any $\mathfrak{g}$-module M we have $x^2 M = 0$ and therefore can define the cohomology

$$M_x := \ker x / xM.$$

Lemma 7

1. $(M \oplus N)_x = M_x \oplus N_x$.
2. $\operatorname{sdim}(M_x) = \operatorname{sdim}(M)$ *(superdimension).*
3. $M_x^* \simeq (M_x)^*$.
4. *We have a canonical isomorphism* $(M \otimes N)_x \simeq M_x \otimes N_x$.

Proof 1, 2 and 3 are straightforward. To prove 4 consider M as a $k[x]/(x^2)$-module. We have the obvious map $M_x \otimes N_x \to (M \otimes N)_x$. On the other hand, we have decompositions $M = M_x \oplus F$ and $N = N_x \oplus F'$, where F and F' are free $k[x]/(x^2)$-modules.

$$M \times N \simeq M_x \otimes N_x \oplus (F \otimes N \oplus M \otimes F').$$

Since a tensor product of any $k[x]/(x^2)$-module with a free $k[x]/(x^2)$-module is free we obtain the isomorphism $(M \otimes N)_x \simeq M_x \otimes N_x$.

Applying the above construction to the adjoint representations we get

$$\mathfrak{g}_x = \ker(ad_x)/[x, \mathfrak{g}] = \mathfrak{g}^x/[x, \mathfrak{g}].$$

Exercise Check that $[x, \mathfrak{g}]$ is an ideal in $\mathfrak{g}^x$. Hence $\mathfrak{g}_x$ is a Lie superalgebra.

Let M be a $\mathfrak{g}$-module. Then we have a canonical $\mathfrak{g}_x$-module structure on M_x. Indeed, it is easy to check that both $\ker x$ and xM are $\mathfrak{g}^x$-stable, For any $y \in \mathfrak{g}$ we have $[x, y]m = xym \in [\mathfrak{g}, x]m$. Therefore $[\mathfrak{g}, x] \ker x \subset xM$ and the induced action of $[\mathfrak{g}, x]$ on M_x is trivial. Thus, we obtain the following proposition.

Proposition 5 *Let $\mathfrak{g}$ be a superalgebra and x be an odd self-commuting element. The assignment $M \to M_x$ induces a tensor functor F_x from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules.*

Remark 7 F_x is neither left nor right exact.
Note that if x, y lie in the same orbit of G_0 then $\mathfrak{g}_x$ and $\mathfrak{g}_y$ are isomorphic Lie superalgebras. Moreover, if $\mathfrak{g}$ is basic, then $\mathfrak{g}_x$ is constant on each stratum $X_p \subset X$.

Lemma 8 *Let $\mathfrak{g}$ be a basic Lie superalgebra, then $\mathfrak{g}_x \simeq \mathfrak{g}_y$ if $x, y \in X_p$.*

Proof Let $x = x_u = x_1 + \cdots + x_p$, y_i and h_i be as in the proof of Proposition 4. Let $\mathfrak{k}$ be the subalgebra generated by x_i, y_i, h_i for all $i \leqslant p$. Then it follows from the proof of Proposition 4 that $\mathfrak{g}_x$ is the quotient of the centralizer of $\mathfrak{k}$ by the center of $\mathfrak{k}$. Note that by the last remark in the same proof we know that y is G_0-conjugate to x_v for some $v \in u \cup -u$. It follows that $\mathfrak{g}_{x_u} = \mathfrak{g}_{x_v}$. Hence the statement.

Exercise Let $\mathfrak{g}$ be one of the basic superalgebras and $x \in X_p$, check that $\mathfrak{g}_x$ is the following:

- $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{g}_x = \mathfrak{gl}(m-p|n-p)$;
- $\mathfrak{g} = \mathfrak{osp}(m|2n)$, $\mathfrak{g}_x = \mathfrak{osp}(m-2p|2n-2p)$;
- $\mathfrak{g} = AG_2, p = 1$, $\mathfrak{g}_x = \mathfrak{sl}_2$;
- $\mathfrak{g} = AB_3, p = 1$, $\mathfrak{g}_x = \mathfrak{sl}_3$;
- $\mathfrak{g} = D(2, 1; a), p = 1$, $\mathfrak{g}_x = \mathfrak{sl}_2$.

Consider $\mathcal{U}(\mathfrak{g})$ as the adjoint $\mathfrak{g}$-module. Then it is not difficult to see that $(\mathcal{U}(\mathfrak{g}))_x \simeq \mathcal{U}(\mathfrak{g}_x)$, hence we have a projection $f_x : \mathcal{U}(\mathfrak{g})^{ad(x)} \longrightarrow \mathcal{U}(\mathfrak{g}_x)$. Note that $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})^{ad(x)}$ and the restriction of f_x to $\mathcal{Z}(\mathfrak{g})$ defines a homomorphism $\phi_x : \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{g}_x)$.

We are interested in the dual map.

$$\check{\phi}_x : \mathrm{Hom}(\mathcal{Z}(\mathfrak{g}_x), k) \longrightarrow \mathrm{Hom}(\mathcal{Z}(\mathfrak{g}), k).$$

Theorem 7 *Let* $\psi \in \mathrm{Hom}(\mathcal{Z}(\mathfrak{g}_x), k)$, $x \in X_p$, *then*

1. $\mathrm{at}(\check{\phi}_x(\psi)) = p + \mathrm{at}(\psi)$.
2. *The image of* $\check{\phi}_x$ *consists of all central characters of atypicality degree greater or equal than* p.
3. *If* $\mathrm{at}(\chi) \geqslant p$, *then the fiber* $\check{\phi}_x^{-1}(\chi)$ *consists of one or two points.*

Proof Let $x = x_u$ where $u = \{\alpha_1, \ldots, \alpha_p\}$. It is always possible to find a triangular decomposition such that $\alpha_1, \ldots, \alpha_p$ are simple roots. We consider the Harish-Chandra map $HC : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ related to this particular triangular decomposition and the analogous map $HC_x : \mathcal{Z}(\mathfrak{g}_x) \longrightarrow S(\mathfrak{h}_x)$ with dual map denoted by θ_x. Let

$$\mathfrak{h}_u := \bigcap_{i=1}^{p} \ker \alpha_i,$$

from the proof of Lemma 8 we have

$$\mathfrak{h}_x = \mathfrak{h}_u / \operatorname{span}\{h_1, \ldots, h_p\}.$$

Let $i_x : \mathfrak{h}_x^* \to \mathfrak{h}_u^*$ be the map dual to the natural projection. We claim the existence of the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_x^* & \xrightarrow{\theta_x} & Specm\,\mathcal{Z}(\mathfrak{g}_x) \\ \Big\downarrow{\scriptstyle i_x} & & \Big\downarrow{\scriptstyle \check{\phi}_x} \\ \mathfrak{h}_u^* & \xrightarrow{\theta} & Specm\,\mathcal{Z}(\mathfrak{g}_x) \end{array}$$

Indeed, for any $\mu \in \mathfrak{h}_x^*$ let $\lambda = i_x(\mu)$ and $M = L_{\mathfrak{b}}(\lambda)$ be the irreducible module with highest weight λ (may be infinite-dimensional). The highest weight vector of this module belongs to M_x and therefore M_x contains a $\mathfrak{g}_x$-submodule which admits central character χ_μ while M admit central character χ_λ. That implies $\check{\phi}_x(\chi_\mu) = \chi_\lambda$.

2 is a direct consequence of 1 and 3 is obtained by case by case inspection using Corollary 2.

Exercise If a $\mathfrak{g}$-module M admits central character χ, then M_x is a sum of modules which admit central characters in $\check{\phi}_x^{-1}(\chi)$.

Corollary 6 *Assume that M admits central character χ with atypicality degree p.*

(a) $F_x(M) = 0$ *for any* $x \in X_q$ *such that* $q > p$. *In particular, if* χ *is typical, then* $F_x(M) = 0$ *for any* $x \neq 0$.

(b) If $x \in X_p$, then $F_x(M)$ is a direct sum of $\mathfrak{g}_x$-modules with typical central character.

Conjecture 1 Let $\mathfrak{g}$ be a basic Lie superalgebra. If M is a finite dimensional simple $\mathfrak{g}$-module, then M_x is a semisimple $\mathfrak{g}_x$-module.
By Corollary 6 Conjecture 1 is true when the rank of x equals the atypicality degree of M. In particular, it holds if the rank of x equals the defect of $\mathfrak{g}$. In this case $\mathfrak{g}_x$ is either a Lie algebra or $\mathfrak{osp}(1|2k)$. For general x the conjecture is proven for $\mathfrak{g} = \mathfrak{gl}(m|n)$ in [21].

5.3 Associated Variety

Definition 10 Let $\mathfrak{g}$ be a Lie superalgebra, X self-commuting cone and M a $\mathfrak{g}$-module. The *associated variety* of M is

$$X_M = \{x \in X \mid M_x \neq 0\}.$$

Exercise In general X_M may be not closed, see [7]. Prove that if M is finite dimensional then X_M is a closed G_0 invariant subvariety of X. If M is an object of the category $\mathcal{O}$, then X_M is B_0-invariant.

The following properties of X_M follow immediately from the corresponding properties of F_x

1. $X_{M\oplus N} = X_M \cup X_N$.
2. $X_{M\otimes N} = X_M \cap X_N$.
3. $X_{M^*} = X_M$.

Note also that Corollary 6 implies the following:

Proposition 6 *Let $\mathfrak{g}$ be a basic superalgebra. If M admits a central character χ of atypicality degree p, then X_M belongs to the Zariski closure of X_p.*

The following result has a rather complicated proof which can be found in [42] for classical superalgebras and in [14, 29] for exceptional.

Theorem 8 *Let $\mathfrak{g}$ be a classical Lie superalgebra and L be a finite dimensional simple $\mathfrak{g}$-module of atypicality degree p. Then the associated variety X_L coincides with the Zariski closure of X_p.*

Finally, let us mention that to every $\mathfrak{g}$-module M integrable over G_0 we can associate a G_0-equivariant coherent sheaf $\mathcal{M}$ on X in the following way. Let $k[X]$ denote the ring of regular functions on X and $k[X] \otimes M$ be a free $k[X]$-module. Define $\partial : k[X] \otimes M \to k[X] \otimes M$ by setting

$$\partial f(x) = xf(x) \quad \text{for every} \quad x \in X.$$

Then $\partial^2 = 0$ and the cohomology of ∂ is a $k[X]$-module $\mathcal{M}$. It is clear that $\mathrm{supp}\mathcal{M} \subset X_M$ and it is proven in [11] that $\mathrm{supp}\mathcal{M} = X$ if $X_M = X$.

Conjecture 2 $\mathrm{supp}\mathcal{M} = X$.

5.4 Some Applications

Conjecture 3 (Kac–Wakimoto,[25]) Let $\mathfrak{g}$ be a basic Lie superalgebra and L be a simple finite-dimensional $\mathfrak{g}$-module. Then $\mathrm{sdim}\, L \neq 0$ if and only the degree of atypicality of L equals the defect of $\mathfrak{g}$.

Kac-Wakimoto conjecture was verified for classical superalgebras in [42] and for exceptional in [29]. Here we can give a simple proof in one direction. Since F_x preserves superdimension, Corollary 6 (a) implies the following statement.

Corollary 7 *Let M be a finite-dimensional $\mathfrak{g}$-module which admits central character χ. If* $\mathrm{at}(\chi) < \mathrm{def}\, \mathfrak{g}$ *then* $\mathrm{sdim}\, M = 0$.

Let $k = \mathbf{C}$, M be a finite dimensional $\mathfrak{g}$-module, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Define a function p_M on $\mathfrak{h}$ by setting

$$p_M(h) = str_M(e^h).$$

It is clear that p_M is analytic. Consider the Taylor series for p_M at $h = 0$

$$p_M(h) = \sum_{i=0}^{\infty} p_i(h),$$

where p_i is a homogeneous polynomial of degree i. The order of zero is the minimal i such that $p_i \neq 0$.

The following result can be considered as a generalization of the Kac-Wakimoto conjecture.

Theorem 9 ([11]) *Assume that $\mathfrak{g}$ does not have non-isotropic odd roots and let M be simple. Then the order of $p_M(h)$ equals the codimension of X_M in X.*

6 Classification of Blocks

6.1 General Results

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. Recall that we denote by $\mathcal{F}$ the category of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$.

Lemma 9 *Let $\mathfrak{g}_0$ be reductive and $\mathfrak{g}_1$ be a semisimple $\mathfrak{g}$-module. Then the category $\mathcal{F}$ has enough projective and injective objects. Moreover, $\mathcal{F}$ is a Frobenius category, i.e. every projective module is injective and vice versa.*

Proof To prove the first assertion note that if M is a simple $\mathfrak{g}_0$-module, then by Frobenius reciprocity the induced module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M$ is projective in $\mathcal{F}$ and the coinduced module $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), M)$ is injective. For the second assertion use the following.

Exercise Show the isomorphism of $\mathfrak{g}$-modules

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M \simeq \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), M \otimes \Lambda^{top}\mathfrak{g}_1).$$

From now on we assume that $\mathfrak{g}$ is basic. For a central character $\chi : \mathcal{Z}(\mathfrak{g}) \to k$ let $\mathcal{F}_\chi$ be the subcategory of $\mathcal{F}$ consisting of modules which admit generalized central character χ.

Lemma 10

(a) We have a decomposition of $\mathcal{F}$ into a direct sum of subcategories

$$\mathcal{F} = \bigoplus_{\chi} \mathcal{F}_\chi.$$

(b) For every χ with non-empty $\mathcal{F}_\chi$ we have a decomposition

$$\mathcal{F}_\chi = \mathcal{F}_\chi^+ \oplus \mathcal{F}_\chi^-$$

such that $\mathcal{F}_\chi^- = \Pi\mathcal{F}_\chi^+$. (Recall that Π is the change of parity functor.)

Proof

(a) If M is finite-dimensional, then $\mathcal{Z}(\mathfrak{g})$ acts locally finitely on M, so M decomposes into the direct sum of generalized weight spaces of $\mathcal{Z}(\mathfrak{g})$.
(b) Every module $M \in \mathcal{F}$ is $\mathfrak{h}$-semisimple. Thus, M has a weight decomposition $M = \oplus M_\mu$. One can define a function $p : \mathfrak{h}^* \to \mathbf{Z}_2$ such that $p(\lambda + \alpha) = p(\lambda)$ for any even root α and $p(\lambda + \alpha) = p(\lambda) + 1$ for any odd root α. Set

$$M_\mu^+ := \begin{cases} (M_\mu)_0 \text{ if } p(\mu) = 0 \\ (M_\mu)_1 \text{ if } p(\mu) = 1 \end{cases}, \quad M_\mu^- := \begin{cases} (M_\mu)_1 \text{ if } p(\mu) = 1 \\ (M_\mu)_1 \text{ if } p(\mu) = 0 \end{cases},$$

and let $M^\pm := \oplus M_\mu^\pm$. Then $M^\pm$ are submodules of M and M is the direct sum $M^+ \oplus M^-$. Therefore we can define $\mathcal{F}_\chi^\pm$ as the full subcategory of $\mathcal{F}_\chi$ consisting of modules M such that $M^\mp = 0$.

We call *principal block* the subcategory $\mathcal{F}_{\chi_0}^+$ which contains the trivial module.

Theorem 10

1. *The subcategories $\mathcal{F}_\chi^{\pm}$ are indecomposable.*
2. *If $\mathfrak{g} = \mathfrak{gl}(m|n)$ (resp. $\mathfrak{osp}(2m+1|2n)$), and $p = \mathrm{at}\chi$, then $\mathcal{F}_\chi^{\pm}$ is equivalent to the principal block of $\mathfrak{gl}(p|p)$ (resp. $\mathfrak{osp}(2p+1|2p)$).*
3. *If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ then $\mathcal{F}_\chi^{\pm}$ is equivalent to the principal block of $\mathfrak{osp}(2p|2p)$ or $\mathfrak{osp}(2p+2|2p)$.*
4. *For exceptional superalgebras $D(2,1,a)$ AG_2 or AB_3 $\mathcal{F}_\chi^{\pm}$ with atypical χ is equivalent to the principal block of $\mathfrak{gl}(1|1)$ or $\mathfrak{osp}(3|2)$.*

In these notes we give the proof for $\mathfrak{g} = \mathfrak{gl}(m|n)$. One can find the proof for all classical superalgebras in [19] and for exceptional in [14] and [29].

Remark 8 If χ is typical, then $\mathcal{F}_\chi^{\pm}$ is semisimple and has one up to isomorphism simple object.

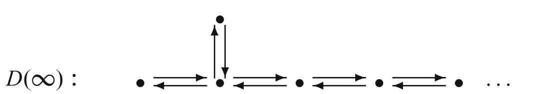
Remark 9 The problem of classifying blocks in the category $\mathcal{O}$ is still open. In contrast with $\mathcal{F}$, there are infinitely many non-equivalent blocks of given atypicality degree, [7].

6.2 Tame Blocks

Using general approach, see [12], every block is equivalent to the category of finite-dimensional representations of a certain quiver with relations. This approach for Lie superalgebras was initiated by J. Germoni, [13]. In this method an important role is played by the dichotomy: wild vs tame categories. Roughly speaking, in tame categories, we can describe indecomposable modules by a finite number of parameters, while in wild categories it is impossible.

The following statement was originally conjectured by Germoni and now follows from Theorem 10 and results in [14, 18, 29] and [33].

Proposition 7 *A block $\mathcal{F}_\chi^{\pm}$ is tame if and only if* $\mathrm{at}(\chi) \leqslant 1$. *An atypical tame block is equivalent to the category of finite-dimensional representations of one of the following two quivers:*



with relations $ba = cd$, $ac = 0 = db$ *for any subquiver isomorphic to:*

$$\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet \underset{d}{\overset{c}{\rightleftarrows}} \bullet$$

Remark 10 It follows from Corollary 6 that for any $x \in X$ the functor F_x maps a block $\mathcal{F}_\chi^\pm$ to

$$\bigoplus_{\tau \in \check{\phi}_x^{-1}} \mathcal{F}_\tau.$$

There is some evidence that a more subtle relation is true, namely

$$F_x(\mathcal{F}_\chi^\pm) = \bigoplus_{\tau \in \check{\phi}_x^{-1}} \mathcal{F}_\tau^\pm.$$

In the case of the most atypical block it is possible to show that the superdimension is constant on a Zariski open subset of simple modules in the block.

6.3 Proof of Theorem 10 for $\mathfrak{gl}(m|n)$

In this subsection $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{b} = \mathfrak{b}_d$ is the distinguished Borel, and we skip the low index in the notation for simple, Kac and projective modules. For instance $L(\lambda) := L_\mathfrak{b}(\lambda)$. The weight

$$\lambda = c_1\epsilon_1 + \dots + c_m\epsilon_m + d_1\delta_1 + \dots + d_n\delta_n = (c_1, \dots, c_m \,|\, d_1, \dots, d_n)$$

is integral dominant if and only if $c_i - c_{i+1} \in \mathbf{Z}_+$, $d_j - d_{j+1} \in \mathbf{Z}_+$ for all $i \leqslant m-1$, $j \leqslant n-1$. We assume in addition that $c_i, d_j \in \mathbf{Z}$.[2]

For the Weyl vector we use

$$\rho = (m-1, \dots, 1, 0 \,|\, 0, -1, \dots, -n).$$

In [2] Brundan and Stroppel introduced an extremely useful way to represent weights by the so called *weight diagrams*.

Let λ be a dominant integral weight, and

$$\lambda + \rho = (a_1, \dots, a_m \,|\, b_1, \dots, b_n), \quad a_i > a_{i+1},\ b_j > b_{j+1}.$$

[2]This assumption is not essential and can be dropped. It is here only for convenience of notations.

The weight diagram f_λ is the map $\mathbf{Z} \to \{\circ, >, <, \times\}$ defined as follows

$$f_\lambda(t) = \begin{cases} \circ \text{ if } \ a_i \neq t, b_j \neq -t \ \text{ for all } \ i = 1, \ldots, m, j = 1, \ldots n; \\ > \ \text{ if } \ a_i = t \ \text{ for some } \ i, \quad b_j \neq -t \ \text{ for all } j = 1, \ldots, n; \\ < \ \text{ if } \ b_i = -t \text{ for some } i, \quad a_j \neq t \ \text{ for all } j = 1, \ldots, m; \\ \times \text{ if } \ a_i = t, b_j = -t \ \text{ for some } \ i, j. \end{cases}$$

We represent f_λ by a picture on the number line with position $t \in \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ filled with $f_\lambda(t)$. We consider $\circ$ as a placeholder for an empty position. The *core* diagram $\bar{f}_\lambda$ is obtained from f_λ by removing all $\times$. We call $>$ and $<$ core symbols.

Example 13 Take the adjoint representation of $\mathfrak{gl}(2|3)$. Then

$$\lambda = (1, 0 \,|\, 0, 0, -1), \quad \lambda + \rho = (2, 0 \,|\, 0, -1, -3)$$

and f_λ can be represented by the picture $\circ \ \times < > < \circ$ ——
where all negative positions and all positions $t > 3$ are empty. The core diagram is

$\circ \ \circ < > < \circ$ ——

Exercise Check that

- The degree of atypicality of λ equals the number of $\times$-s in the weight diagram f_λ.
- Core diagrams parametrize blocks, namely, $\chi_\lambda = \chi_\mu$ if and only if $\bar{f}_\lambda = \bar{f}_\mu$.

The above exercise implies that blocks $\mathcal{F}^+_\chi$ can be parametrized by weight diagrams without $\times$-s. We use the notation $f_\chi := \bar{f}_\lambda$ for any λ such that $\chi = \chi_\lambda$.

Definition 11 We define the following operations on a weight diagram:

- *Left simple move*: Move $>$ one position to the right or move $<$ one position to the left.
- *Right simple move*: Move $>$ one position to the left or move $<$ one position to the right.

In this definition we assume that $\times$ is the union $><$, and we can split it or join $><$ into $\times$.

Example 14 Let f be as in the previous example: $\circ \ \times < > < \circ$ ——
Then the following are possible right simple moves

1. Moving the rightmost $<$ one position right: $\circ \ \times < > \circ <$ ——
2. Moving the leftmost $<$ one position right (new $\times$ in position 2 appears):

   $\circ \ \times \circ \times \circ <$ ——

3. Moving $>$ one position left (new $\times$ in position 1 appears): $\circ \ \times \times \circ \circ <$ ——
4. Splitting $\times$. Here we can not move $<$ to the right since it does not produce a valid diagram. But we can move $>$ to the left. $> < < > < \circ$ ——

Let V and V^* denote the natural and conatural representations respectively.

Lemma 11 *If $K(\lambda)$ is the Kac module with highest weight λ, then $K(\lambda) \otimes V$ (resp. $K(\lambda) \otimes V^*$) has a filtration by Kac modules $K(\mu)$ for all f_μ obtained from f_λ by a left (resp. right) simple move.*

Proof Recall that $K(\lambda) = U(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}(1))} L_{\mathfrak{b}_0}(\lambda)$. Hence

$$K(\lambda) \otimes V \simeq U(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}(1))} (L_{\mathfrak{b}_0}(\lambda) \otimes V).$$

Since the weights of V are $\{\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n\}$, then $K(\lambda) \otimes V$ has a filtration by $K(\mu)$ for all dominant μ in $\{\lambda + \epsilon_1, \dots, \lambda + \epsilon_m, \lambda + \delta_1, \dots, \lambda + \delta_n\}$. The corresponding weight diagrams are exactly those obtained from f_λ by a right simple move. The case of $K(\lambda) \otimes V^*$ is similar.

Next step is to define translation functors inspired by translation functors in classical category $\mathcal{O}$. For every M in $\mathcal{F}$ we denote by $(M)_\tau$ the projection on the block $\mathcal{F}_\tau^+$. Then the translation functors between $\mathcal{F}_\chi^+$ and $\mathcal{F}_\tau^+$ are defined by

$$T_{\chi,\tau} : \mathcal{F}_\chi^+ \longrightarrow \mathcal{F}_\tau^+, \qquad M \mapsto (M \otimes V)_\tau$$

$$T^*_{\tau,\chi} : \mathcal{F}_\tau^+ \longrightarrow \mathcal{F}_\chi^+, \qquad M \mapsto (M \otimes V^*)_\chi$$

Exercise Show that:

1. The functors $T_{\chi,\tau}$, $T^*_{\tau,\chi}$ are exact.
2. $T^*_{\tau,\chi}$ is left and right adjoint to $T_{\chi,\tau}$.
3. $T_{\chi,\tau}$, $T^*_{\tau,\chi}$ map projective modules to projective modules.
4. Assume that $T_{\chi,\tau}$ and $T^*_{\tau,\chi}$ establish a bijection between simple modules in both blocks, then they establish an equivalence $\mathcal{F}_\chi^+ \cong \mathcal{F}_\tau^+$ of abelian categories.
5. If $T_{\chi,\tau}$ and $T^*_{\tau,\chi}$ establish a bijection between Kac modules in both blocks, they also establish a bijection between simple modules.

Proposition 8 *Assume that* $\mathrm{at}(\chi) = \mathrm{at}(\tau)$ *and f_τ is obtained from f_χ by a left (resp. right) simple move, then $T_{\chi,\tau} : \mathcal{F}_\chi \to \mathcal{F}_\tau$ (resp. $T^*_{\chi,\tau} : \mathcal{F}_\chi \to \mathcal{F}_\tau$) is an equivalence of abelian categories.*

Proof Without loss of generality we do the proof in the case of a left move. Using Lemma 11 one can easily check that $T_{\chi,\tau}$ and $T^*_{\tau,\chi}$ provide a bijection between Kac modules in both blocks. Hence the statement follows from the preceding exercise.

Definition 12 A weight λ is *stable* if all $\times$-s in the weight diagram f_λ stay to the left of $<$ and $>$.

Introduce an order on the set of weights in the same block by setting $\nu \leqslant \mu$ if $\mu - \nu$ is a sum of positive roots. One can easily see that $\nu < \mu$ if ν is obtained from μ by moving some $\times$ to the left. Therefore if μ is stable and $\nu < \mu$, then ν is also stable. We denote by $\mathcal{F}_\chi^\mu$ the full subcategory of $\mathcal{F}_\chi^+$ whose simple constituents $L(\lambda)$ satisfy $\lambda \leqslant \mu$. We call $\mathcal{F}_\chi^\mu$ a *truncated* block.

Proposition 9 *Let μ be a stable weight of atypicality degree p, $\chi = \chi_\mu$. Let $s \in \mathbf{Z}$ be minimal such that $f_\chi(s) \neq \circ$. Let ν be the weight of the principal block of $\mathfrak{gl}(p|p)$ with weight diagram*

$$f_\nu = \begin{cases} \times \text{ if } s-p \leqslant t \leqslant s-1 \\ \circ \text{ otherwise} \end{cases} .$$

Then $\mathcal{F}^\mu_\chi$ is equivalent to the truncation $\mathcal{F}^\nu_0$ of the principal block of $\mathfrak{gl}(p|p)$.

Proof (Sketch) We just explain how to define the functors establishing the equivalence. Let $\mu = (c_1, \ldots, c_m \,|\, d_1, \ldots, d_n)$. Start with defining the parabolic subalgebra

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta'} \mathfrak{g}_\alpha,$$

where

$$\Delta' = \Delta^+ \cup \{\epsilon_i - \epsilon_j \,|\, m-p < j < i \leqslant m\} \cup \{\delta_i - \delta_j \,|\, 1 \leqslant j < i \leqslant p\} \\ \cup \{\delta_i - \epsilon_j \,|\, 1 \leqslant i \leqslant p,\ m-p < j \leqslant m\}.$$

in other words $\mathfrak{p}$ consists of block matrices of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

where the middle square block has size $p|p$. Set

$$\mathfrak{l} := \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\}, \quad \mathfrak{m} := \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Clearly, $\mathfrak{p}$ is a semi-direct product of the subalgebra $\mathfrak{l} \simeq \mathfrak{gl}(p|p) \oplus k^{m+n-2p}$ and the nilpotent ideal $\mathfrak{m}$. Consider the functor $R : \mathcal{F}^\mu_\chi \to \mathcal{F}^\nu_0$ defined by $R(M) = M^{\mathfrak{m}}$. Then its left adjoint $I : \mathcal{F}^\nu_0 \to \mathcal{F}^\mu_\chi$ maps a $\mathfrak{gl}(p|p)$-module N to the maximal finite-dimensional quotient of the parabolically induced module

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (N \boxtimes C_\mu),$$

where C_μ is the one-dimensional representation of k^{m+n-2p} with weight

$$\mu := (c_1, \ldots, c_{m-p} \,|\, d_{p+1}, \ldots, d_n).$$

It suffices to show that R and I are exact and establish the bijection between simple modules. Indeed, the exactness of R can be proven by noticing that R picks up the eigenspace of k^{m+n-2p} with weight μ. Furthermore, if $L(\lambda)$ is a simple module in $\mathcal{F}_\chi^\mu$, then

$$\lambda = (c_1, \dots, c_{m-p}, t_1, \dots, t_p \mid -t_p, \dots, -t_1, d_{p+1}, \dots, d_n)$$

for some $t_1, \dots, t_p$. It is easy to see that $R(L(\lambda)) = L(\lambda')$, where $\lambda' = (t_1, \dots, t_p \mid -t_p, \dots, -t_1)$ and that $I(L(\lambda')) = L(\lambda)$. The exactness of I can be now proven by induction on the length of a module.

The following combinatorial lemma is straightforward.

Lemma 12 *For any weight diagram f_μ there exists a stable weight diagram $f_{\mu'}$ obtained from f_μ by a sequence of simple moves which do not change the degree of atypicality.*

Now we are ready to prove Theorem 10. Indeed, let $\mathcal{F}_\chi^+$ be a block with atypicality degree p. Lemma 12 and Proposition 8 imply that any truncated block $\mathcal{F}_\chi^\mu$ is equivalent to a stable truncated block of the same atypicality. Hence by Proposition 9 $\mathcal{F}_\chi^\mu$ is equivalent to some truncation of a principal block of $\mathfrak{gl}(p|p)$. Taking the direct limit of $\mathcal{F}_\chi^\mu$ we obtain equivalence between $\mathcal{F}_\chi^+$ and the principal block of $\mathfrak{gl}(p|p)$.

It remains to prove the indecomposability of the principal block of $\mathfrak{gl}(p|p)$. Note that $f_{\nu'}$ is obtained from f_ν by moving a $\times$ one position left, then $[K(\nu) : L(\nu')] = 1$. Since $K(\nu)$ is indecomposable, $L(\nu)$ lies in the indecomposable block containing $L(\nu')$. Since any diagram in the principal block can be obtained from the fixed one by repeatedly moving $\times$-s one position left or right, the statement follows.

6.4 Calculating the Kazhdan-Lusztig Multiplicities

We would like to mention without proof other applications of weight diagrams and translation functors. We still assume that $\mathfrak{g} = \mathfrak{gl}(m|n)$. Then the category $\mathcal{F}$ is a highest weight category, [47], where standard objects are Kac modules. In particular, we have BGG reciprocity for the multiplicities:

$$[K(\lambda) : L(\mu)] = [P(\mu) : K(\lambda)],$$

where $P(\mu)$ denotes the projective cover of $L(\mu)$. It is useful to compute these multiplicities. It was done in [40] and in [1] by different methods. The answer is very easy to formulate in terms of weight diagrams.

Let f be some weight diagram. We decorate it with caps by the following rule:

- Every cap has left end at $\times$ and right end at $\circ$.
- Every $\times$ is engaged in some cap, so the number of caps equals the number of crosses.

- There are no $\circ$ under a cap.
- Caps do not cross.

We say that f' is *adjacent* to f if f' is obtained from f by moving one $\times$ from the left end of its cap to the right end. We say that f' is *adjoint* to f if f' is obtained from f by moving several $\times$ from the left end of its cap to the right end. We assume that f is adjoint to itself. If f has p $\times$-s, then it has exactly p adjacent diagrams and 2^p adjoint diagrams

Theorem 11 ([1, 33])

$$Ext^1_{\mathcal{F}}(L(\lambda), L(\mu)) = \begin{cases} k \text{ if } f_\lambda \text{ is adjacent to } f_\mu \text{ or } f_\mu \text{ is adjacent to } f_\lambda \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 12 ([1])

$$[P(\lambda) : K(\mu)] = \begin{cases} 1 \text{ if } f_\mu \text{ is adjoint to } f_\lambda \\ 0 \text{ otherwise.} \end{cases}$$

7 Supergeometry and Borel–Weil–Bott Theorem

7.1 Supermanifolds

The notion of supermanifold exists in three flavors: smooth, analytic and algebraic. We concentrate here on the algebraic version. The main idea is the same: we define first superdomains and then glue them together.

By a *superdomain* we understand a pair $(U_0, \mathcal{O}_U)$, where U_0 is an affine manifold and $\mathcal{O}_U$ is the sheaf of superalgebras isomorphic to

$$\Lambda(\xi_1, \dots, \xi_n) \otimes \mathcal{O}_{U_0},$$

$\mathcal{O}_{U_0}$ denotes the structure sheaf on U_0. The dimension of U is $(m|n)$ where $m = \dim U_0$.

For example, the affine superspace $\mathbb{A}^{m|n}$ is a pair $(\mathbb{A}^m, \mathcal{O}_{\mathbb{A}^{m|n}})$. The ring of global sections of $\mathcal{O}(\mathbb{A}^{m|n})$ is a free supercommutative ring $k[x_1 \dots x_m, \xi_1, \dots \xi_n]$. If we work in local coordinates, then we use roman letters for even variables, greek letters for odd ones.

Definition 13 A supermanifold is a pair $(X_0, \mathcal{O}_X)$ where X_0 is a manifold and $\mathcal{O}_X$ is a sheaf locally isomorphic to $(U_0, \mathcal{O}_U)$ for a superdomain U. The manifold X_0 is called the *underlying manifold* of X and $\mathcal{O}_X$ is called the *structure sheaf*.

One way to define a supermanifold is by introducing local charts U_i and then gluing them together.

Example 15 Consider two copies of $\mathbb{A}^{1|2}$ with coordinates (x, ξ_1, ξ_2) and (y, η_1, η_2). We give the gluing by setting:

$$y = x^{-1} + \xi_1\xi_2, \quad \eta_1 = x^{-1}\xi_1, \quad \eta_2 = \xi_2.$$

Example 16 Let X_0 be a manifold, $\mathcal{V}$ be a vector bundle on X_0 and $\mathcal{O}_X$ is the sheaf of sections of the exterior algebra bundle $\Lambda(\mathcal{V})$. In particular, X_0 with the sheaf of forms Ω_{X_0} is a supermanifold.

Given the supermanifold X, we have the canonical embedding $X_0 \to X$ and the corresponding morphism of structure sheaves $\mathcal{O}_X \to \mathcal{O}_{X_0}$. Denote by I_{X_0} the kernel of this map. It is not difficult to see that I_{X_0} is the nilpotent ideal generated by all odd sections of $\mathcal{O}_X$. Consider the filtration

$$\mathcal{O}_X \supset I_{X_0} \supset I_{X_0}^2 \supset \dots .$$

Then $Gr(X) := (X_0, Gr\mathcal{O}_X)$ is again a supermanifold. One can identify $Gr(X)$ with $(X_0, \Gamma(\Lambda(N^*_{X_0}X))$, where $N^*_{X_0}X$ denotes the conormal bundle for $X_0 \subset X$.

A supermanifold X is called *split* if it is isomorphic to $Gr(X)$. In the category of smooth supermanifolds all supermanifolds are split but this is not true for algebraic supermanifolds.

Exercise Show that any supermanifold of dimension $(m|1)$ is split. Is the supermanifold defined in Example 15 split?

Another way to define a supermanifold is to use the functor of points, which is a functor from the category (Salg) of commutative superalgebras to the category (Sets). For general definitions see [3]. Let us illustrate this approach with the following example.

Example 17 We define the projective superspace $X = \mathbf{P}^{1|1}$ as follows. For a commutative superalgebra $\mathcal{A}$ the set of $\mathcal{A}$-points is the set of all submodules $\mathcal{A}^{1|0} \subset \mathcal{A}^{2|1}$. This is the set of all triples (z_1, z_2, ζ) with $z_1, z_2 \in \mathcal{A}_0$ and $\zeta \in \mathcal{A}_1$, such that at least one of z_1, z_2 is invertible, modulo rescaling by an invertible element of $\mathcal{A}_0$. This supermanifold has two affine charts $\{(1, x, \xi)\}$ and $\{(y, 1, \eta)\}$ with gluing functions $\xi = x^{-1}\eta$, $y = x^{-1}$.

Exercise Check that in Example 17 $X_0 = \mathbf{P}^1$ and $\mathcal{O}_X \simeq \mathcal{O} \oplus \Pi\mathcal{O}(-1)$.

8 Algebraic Supergroups

An affine supermanifold G equipped with morphisms $m : G \times G \to G$, $i : D \to G$ and $e : \{point\} \to G$ satisfying usual group axioms is called an affine algebraic supergroup. We skip the word "affine" in what follows.

The ring $\mathcal{O}(G)$ of global sections of $\mathcal{O}_G$ has a structure of Hopf superalgebra. In fact, one can start with a Hopf superalgebra $\mathcal{O}(G)$ and define a supergroup as a functor:

$$G : (\text{Salg}) \longrightarrow \{\text{Groups}\}, \quad G(\mathcal{A}) = \text{Hom}(\mathcal{O}(G), \mathcal{A}).$$

Properties of Hopf algebras allow one to define the group structure on $G(\mathcal{A})$.

The ideal I generated by the odd elements in $\mathcal{O}(G)$ is an Hopf ideal. The quotient Hopf algebra $\mathcal{O}(G)/I$ is the Hopf algebra of regular functions on the underlying algebraic group G_0.

Exercise GL$(m|n)$.

$$\text{GL}(m|n)(\mathcal{A}) = \left\{ Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$$

satisfying the following conditions

- the entries on A and D are even elements in $\mathcal{A}$, while the the entries of B and C are odd;
- Y is invertible.

Show that GL$(m|n)$ is representable and construct the corresponding Hopf superalgebra.

Example 18 (Exercise) Consider the functor

$$\text{Ber} : \text{GL}(m|n) \longrightarrow \text{GL}(1), \qquad \text{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BDC)/\det(D).$$

Check that Ber is a homomorphism. Hint: Write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}.$$

We define SL$(m|n)$ by imposing the condition Ber $= 1$.

Show that GL$(m|n)_0 = \text{GL}(m) \times \text{GL}(n)$ and

$$\text{SL}(m|n)_0 = \{(A, D) \in \text{GL}(m) \times \text{GL}(n) \mid \det A = \det D\}.$$

Definition 14 Lie(G) is the Lie superalgebra of left invariant derivations of $\mathcal{O}(G)$ and can be identified with $T_e(G)$.

Exercise Lie(GL$(m|n)) = \mathfrak{gl}(m|n)$, Lie(SL$(m|n)) = \mathfrak{sl}(m|n)$.

A useful approach to algebraic supergroups is via the so called Harish-Chandra pairs. In the case of Lie groups it is due to Koszul and Kostant, [27, 28], for complex analytic category it is done in [46], for algebraic groups see [31].

We call an HC pair the following data

- a finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$;
- an algebraic group G_0 such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$;
- a G_0-module structure on $\mathfrak{g}_1$ with differential equal to the superbracket $\mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$.

Theorem 13 *The category of HC pairs is equivalent to the category of algebraic supergroups.*

Let us comment on the proof. It is clear that every supergroup G defines uniquely a HC pair $(\mathfrak{g}, G_0)$. The difficult part is to go back: given an HC pair $(\mathfrak{g}, G_0)$, define a Hopf superalgebra $\mathcal{O}(G)$. One way to approach this problem is to set

$$R = \mathcal{O}(G) := \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0)).$$

Define a multiplication map $m : R \otimes R \to R$ by

$$m(f_1, f_2)(X) := m_0((f_1 \otimes f_2)(\Delta_U(X))),$$

where m_0 is the multiplication in $\mathcal{O}(G_0)$ and Δ_U is the comultiplication in $\mathcal{U}(\mathfrak{g})$:

$$\Delta_u(x) = x \otimes 1 + 1 \otimes x, \quad x \in \mathfrak{g}.$$

It is easy to see that R is a commutative superalgebra isomorphic to $S(\mathfrak{g}_1^*) \otimes \mathcal{O}(G_0)$, [28]. In particular, this implies that an algebraic group is a split supermanifold.

Next define the comultiplication $\Delta : R \to R \otimes R$. For $g, h \in G_0$ and $x, y \in \mathcal{U}(\mathfrak{g})$ we set

$$\Delta f(x, y)_{g,h} = f(\mathrm{Ad}(h^{-1})(x)y)_{gh}.$$

The counit map $\epsilon : R \to k$ is defined by

$$\epsilon f := \epsilon_0 \circ f(1),$$

where ϵ_0 is the counit in $\mathcal{O}(G_0)$. Finally, define the antipode $s : R \to R$ by setting for all $g \in G_0$, $x \in \mathcal{U}(\mathfrak{g})$

$$sf(X)_g = f(\mathrm{Ad}(g)s_U(X))_{g^{-1}},$$

where s_U is the antipode in $\mathcal{U}(\mathfrak{g})$.

Theorem 14 ([31]) *The category of representations of G is equivalent to the category of $(\mathfrak{g}, G_0)$-modules.*

We now concentrate on the case of reductive G_0. By the above Theorem the category $Rep(G)$ of finite-dimensional representations of G is a full subcategory of $\mathcal{F}$. Therefore we immediately obtain the following.

Corollary 8 *Let G_0 be reductive.*

- *Then $Rep(G)$ has enough projective and injective objects.*
- *Every injective G-module is projective.*

Exercise Assume that G_0 is reductive. Check that

$$\mathcal{O}(G) \simeq \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0))$$

is an isomorphism of $(\mathfrak{g}, G_0)$-modules and use it prove that

$$\mathcal{O}(G) = \oplus P(L)^{\dim(L_0)}$$

where L runs the set of irreducible representations of G and $P(L)$ is the projective cover of L. *Hint:* Use Frobenius reciprocity and the structure of $\mathcal{O}(G_0)$ as a G_0-module.

9 Geometric Induction

9.1 General Construction

Let $H \subset G$ be a subsupergroup. It is possible to show that G/H is a supermanifold, see [30]. The space of global sections of the structure sheaf is given by

$$\mathcal{O}(G/H) := \mathcal{O}(G)^H,$$

where H-invariants are defined with respect to the right action of H on G. Furthermore, if M is a representation of H, then $G \times_H M$ is a G-equivariant vector bundle on G/H. We define:

$$\mathcal{O}(G/H, M) = (\mathcal{O}(G) \otimes M)^H = \{f : G \to M \,|\, f(gh) = h^{-1}f(g),\ h \in H\}.$$

Thus, we associated in functorial way to every representation of H a representation of G, namely, the space of global sections of $G \times_H M$. The corresponding functor $\Gamma : Rep(H) \longrightarrow Rep(G)$ is left exact. The right derived functor is given by the cohomology

$$R^i\Gamma(M) = H^i(G/H, G \times_H M).$$

It is a little bit more convenient to us to work with dual functors $\Gamma_i(G/H, \cdot)$ defined by

$$\Gamma_i(G/H, M) := H^i(G/H, G \times_H M^*)^*.$$

The following statement is the Frobenius reciprocity for geometric induction and the proof is the same as for algebraic groups.

Proposition 10 *For any H-module M and G-module N we have a canonical isomorphism*

$$\mathrm{Hom}_G(\Gamma_0(G/H, M), N) \simeq \mathrm{Hom}_H(M, N).$$

Exercise If $G = G_0$, then $\Gamma_i(M) = 0$ for $i > 0$ and $\Gamma_0(M) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M$.

9.2 The Borel-Weil-Bott Theorem

Let G be an algebraic supergroup with basic Lie superalgebra $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ and denote by $B \subset G$ and $H \subset B$ the corresponding subgroups. The supermanifold G/B is called a *flag supermanifold*. Its underlying manifold G_0/B_0 is a classical flag manifold.

Recall that in the Lie algebra case flag manifolds play a crucial role in the representation theory of $\mathfrak{g}$. In particular, all the irreducible representations of a reductive algebraic group can be realized as global sections of line bundles on the flag variety by the Borel–Weil–Bott theorem. Let us see what happens in the supercase.

Consider the H-weight lattice Λ in $\mathfrak{h}^*$. Every $\lambda \in \Lambda$ defines a unique one-dimensional representation of B which we denote by c_λ. We are interested in computing $\Gamma_i(G/B, c_\lambda) = 0$. The Frobenius reciprocity (Proposition 10) implies the following

Corollary 9 *$\Gamma_i(G/B, c_\lambda)$ is isomorphic to the maximal finite-dimensional quotient $K_\mathfrak{b}(\lambda)$ of the Verma module $M_\mathfrak{b}(\lambda)$.*

Lemma 13 *Assume that the defect of $\mathfrak{g}$ is positive. Then the flag supervariety G/B is split if and only if $\mathfrak{g}$ is type 1 and $\mathfrak{b}$ is distinguished or antidistinguished.*

Proof First, let us assume that G/B is split. Then we have a projection $\pi : G/B \to G_0/B_0$ and the pull back map

$$\pi^* : G_0 \times_{B_0} c_{-\lambda} \to G \times_B c_{-\lambda}$$

which induces the embedding

$$H^0(G_0/B_0, G_0 \times_{B_0} c_{-\lambda}) \to H^0(G/B, G \times_B c_{-\lambda}).$$

After dualizing we obtain a surjection

$$\Gamma_0(G/B, c_\lambda) \to \Gamma_0(G_0/B_0, c_\lambda).$$

If λ is a G_0-dominant weight, then $\Gamma_0(G_0/B_0, c_\lambda) = L_{\mathfrak{b}_0}(\lambda) \neq 0$. By Corollary 9 $K_{\mathfrak{b}}(\lambda) \neq 0$. Hence λ is G-dominant. Thus, every dominant G_0-weight is G dominant and this is possible only for distinguished Borel or for $\mathfrak{osp}(1|2n)$.

Now let $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{g}(\pm 1)$ be a distinguished or antidistinguished Borel subalgebra. Then it is easy to see that

$$\mathcal{O}_{G/B} = \mathcal{O}_{G_0/B_0} \otimes \Lambda(\mathfrak{g}(\pm 1)^*).$$

The following result is a generalization of Borel–Weil–Bott theorem in the case of typical λ. We call a weight μ regular (resp. singular) if it has trivial (resp. non-trivial) stabilizer in W. We denote by $\Lambda^+ \subset \Lambda$ the set of all $\mu \in \Lambda$ such that $\frac{2(\mu|\alpha)}{(\alpha|\alpha)} \in \mathbf{Z}_+$ for all even positive roots α. It follows from Sect. 4.1 that a typical λ is dominant if and only if $\lambda + \rho \in \Lambda^+$.

Theorem 15 ([35]) *Let $\lambda \in \Lambda$ be typical.*

1. *If $\lambda + \rho$ is singular then $\Gamma_i(G/B, c_\lambda) = 0$ for all i.*
2. *If $\lambda + \rho$ is regular there exists a unique $w \in W$ such that $w(\lambda + \rho) \in \Lambda^+$. Let l be the length of w. Then*

$$\Gamma_i(G/B, c_\lambda) = \begin{cases} 0 & \text{if } i \neq l, \\ L(w \cdot \lambda), & \text{if } i = l. \end{cases}$$

Proof We give here just the outline, see details in [35]. First, if λ is dominant then by Corollary 9 $\Gamma_0(G/B, c_\lambda) = K_{\mathfrak{b}}(\lambda)$ and by typicality of λ we have $K_{\mathfrak{b}}(\lambda) = L_{\mathfrak{b}}(\lambda)$.

If α or $\frac{1}{2}\alpha$ is a simple root of B, then one can show using the original Demazure argument, that

$$\Gamma_i(G/B, c_\mu) \simeq \Gamma_{i+1}(G/B, c_{r_\alpha \cdot \mu}), \tag{4}$$

if $\frac{2(\mu|\alpha)}{(\alpha|\alpha)} > 0$. Furthermore, if $\frac{2(\mu|\alpha)}{(\alpha|\alpha)} = 0$, then

$$\Gamma_i(G/B, c_\mu) = 0 \tag{5}$$

for all i.

However, not every simple root of $\mathfrak{b}_0$ is a simple root of $\mathfrak{b}$ and therefore we need to involve odd reflections and change of Borel subalgebras.

Let α be an isotropic simple root and $\mathfrak{b}'$ be obtained from $\mathfrak{b}$ by the odd reflection r_α. Then we claim that

$$\Gamma_i(G/B, c_\lambda) \simeq \Gamma_i(G/B', c_{\lambda'}), \tag{6}$$

where $\lambda+\rho = \lambda'+\rho'$. To show this we consider the parabolic subalgebra $\mathfrak{p} = \mathfrak{b}+\mathfrak{b}'$. Then we have two projections

$$p : G/B \to G/P, \quad p' : G/B' \to G/P,$$

the fiber of both projections is a $(0|1)$-dimensional affine space and we have

$$p_*(G \times_B c_{-\lambda}) = p'_*(G \times_{B'} c_{-\lambda'}) = G \times_P V_\lambda,$$

where V_λ is the two-dimensional simple P-module with weights $-\lambda$ and $-\lambda'$. Note that here we use that $(\lambda + \rho, \alpha) \neq 0$ by the typicality of λ. This implies

$$H^i(G/B, G \times_B c_{-\lambda}) \simeq H^i(G/P, G \times_P V_\lambda) \simeq H^i(G/B', G \times_{B'} c'_{-\lambda}).$$

After dualization we obtain (6).

Let us assume again that λ is dominant and consider the Borel subalgebra $\mathfrak{b}'$ opposite to $\mathfrak{b}$. Combining (4) and (6) we obtain

$$\Gamma_i(G/B, c_\lambda) = \Gamma_{i+d}(G/B', c_{w_0\cdot\lambda}),$$

where w_0 is the longest element of W and its length d equals $\dim G_0/B_0$. That implies the second statement of the theorem for dominant λ. Using (4) and (6) we can reduce the case of arbitrary regular $\lambda + \rho$ to the dominant case.

If $\lambda + \rho$ is singular, then there is a simple root α of $\mathfrak{b}_0$ such that $(\lambda + \rho, \alpha) = 0$. Using odd reflections and (6) we can change the Borel subgroup B to B' and λ so that α or $\frac{1}{2}\alpha$ is a simple root of B'. Then the vanishing of cohomology follows from (5).

Computing $\Gamma_i(G/B, c_\lambda)$ for atypical λ is an open question. The main reason why the proof in this case does not work is the absence of (6). It is known from examples that $\Gamma_i(G/B, c_\lambda)$ may not vanish for several i.

Finally let us formulate the following analogue of Bott's reciprocity relating Γ_i with Lie superalgebra cohomology. The proof is straightforward using the definition of the derived functor (see [20]).

Proposition 11 *For any finite-dimensional B-module M and any dominant weight λ, we have*

$$\left[H^i(G/B, G \times_B M) : L_{\mathfrak{b}}(\lambda)\right] = \dim \operatorname{Ext}^i_B(P_{\mathfrak{b}}(\lambda), M) = \dim H^i(\mathfrak{n}^+, P^*_{\mathfrak{b}}(\lambda) \otimes M)^{\mathfrak{h}},$$

where $P_{\mathfrak{b}}(\lambda)$ denotes the projective cover of $L_{\mathfrak{b}}(\lambda)$.

After dualizing and setting $M = c_{-\nu}$ we obtain the following

Corollary 10

$$[\Gamma_i(G/B, c_\nu) : L_{\mathfrak{b}}(\lambda)] = \dim \operatorname{Hom}_{\mathfrak{h}}\left(c_\nu, H^i(\mathfrak{n}^+, P_{\mathfrak{b}}(\lambda))\right).$$

9.3 Application to Characters

Although we do not know $\Gamma_i(G/B, c_\lambda)$ for atypical λ, we can calculate the character of the Euler characteristic.

Theorem 16 *The character of the Euler characteristic is given by the typical character formula, i.e.*

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} \Gamma_i(G/B, c_\lambda) = \frac{D_1}{D_0} \sum_{w \in W} sgn(w) e^{w(\lambda+\rho)}.$$

Proof Consider the associated split manifold $Gr(G/B)$ and the associated graded $\mathcal{L} = Gr(\Gamma)$ of the sheaf Γ of sections of $G \times_B c_{-\lambda}$. Since Euler characteristic is preserved after going to the associated graded sheaf we have

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} H^i(G/B, G \times_B c_{-\lambda}) = \sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} H^i(Gr(G/B), \mathcal{L}).$$

Note that $\mathcal{L}$ is a G_0-equivariant vector bundle on G_0/B_0, and the classical Borel–Weil–Bott theorem allows us to calculate the right hand side of the above equality. Indeed, if $\mathcal{N}$ denotes the conormal bundle to G_0/B_0, then

$$\mathcal{L} \simeq \Lambda(\mathcal{N}) \otimes (G_0 \times_{B_0} c_{-\lambda}) = G_0 \times_{B_0} \left(c_{-\lambda} \otimes \Lambda^*(\mathfrak{g}_1/\mathfrak{b}_1)\right),$$

and

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} H^i(G/B, \mathcal{L}) = \frac{1}{D_0} \sum_{w \in W} sgn(w) w(e^{\lambda+\rho_0} \prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})),$$

which is equivalent to the typical character formula.

Note that Theorems 16 and 15 imply Theorem 2.

Definition 15 Let λ be a weight of atypicality degree p. It is called *tame* with respect to the Borel subalgebra $\mathfrak{b}$ if there exists isotropic mutually orthogonal *simple* roots $\alpha_1, \ldots, \alpha_p$ such that

$$(\lambda + \rho|\alpha_1) = \cdots = (\lambda + \rho|\alpha_p) = 0.$$

Conjecture 4 (Kac–Wakimoto, [25]) If λ is dominant and tame with respect to $\mathfrak{b}$, then

$$\operatorname{ch} L_{\mathfrak{b}}(\lambda) = \frac{D_1}{D_0} \sum_{w \in W} sgn(w) w \left(\frac{e^{\lambda+\rho}}{\prod_{i=1}^{p}(1 - e^{-\alpha_i})} \right). \tag{7}$$

The right hand side of formula (7) is the character of the Euler characteristic

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} \Gamma_i(G/Q, c_\lambda),$$

where Q is the parabolic subgroup with Lie superalgebra

$$\mathfrak{q} := \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{-\alpha_p}.$$

Hence one way to prove Conjecture 4 is to prove the following

Conjecture 5 If λ is tame with respect to $\mathfrak{b}$, then $\Gamma_i(G/Q, c_\lambda) = 0$ if $i > 0$ and $\Gamma_0(G/Q, c_\lambda) = L_{\mathfrak{b}}(\lambda)$.
For classical Lie superalgebras Conjecture 5 is proven in [5].

9.4 Weak BGG Reciprocity

Let $\mathcal{K}(G)$ denote the Grothendieck group of the category *Rep*(G) and $[M]$ denote the class of a G-module M. Clearly $[L_{\mathfrak{b}}(\lambda)]$, for all dominant $\lambda \in \Lambda$, is a basis of $\mathcal{K}(G)$. Set

$$[\mathcal{E}_{\mathfrak{b}}(\lambda)] = \sum_i (-1)^i [\Gamma_i(G/B, c_\lambda)].$$

As we already mentioned in Sect. 6.4, if $\mathfrak{g}$ is of type 1 then *Rep*(G) is a highest weight category. For type 2 superalgebras this is not true. Nevertheless one can use virtual modules $\mathcal{E}_{\mathfrak{b}}(\lambda)$ instead of $K_{\mathfrak{b}}(\lambda)$ and obtain the following weak BGG reciprocity.

Theorem 17 ([20]) *Let $\lambda \in \Lambda$ be dominant and $\mu \in \Lambda$ be such that $\mu + \rho \in \Lambda^+$. There exists unique $a_{\lambda,\mu} \in \mathbf{Z}$ such that*

$$[\mathcal{E}_{\mathfrak{b}}(\mu)] = \sum a_{\lambda,\mu}[L_{\mathfrak{b}}(\lambda)]$$

and

$$[P_{\mathfrak{b}}(\lambda)] = \sum a_{\lambda,\mu}[\mathcal{E}_{\mathfrak{b}}(\mu)].$$

9.5 $\mathcal{D}$-Modules

In this subsection we discuss briefly possible generalizations of the Beilinson-Bernstein localization theorem for basic classical Lie superalgebras. The basics on

$\mathcal{D}$-modules on supermanifold can be found in [36]. The main result there is that if X is a supermanifold with underlying manifold X_0 then Kashiwara extension functor provides the equivalence between categories of $\mathcal{D}_{X_0}$-modules and $\mathcal{D}_X$-modules.

This fact is easy to explain in the case when X is a superdomain. Indeed, in this case

$$\mathcal{O}(X) = \mathcal{O}(X_0) \otimes \Lambda(\xi_1, \ldots, \xi_n),$$

and this implies an isomorphism

$$\mathcal{D}(X) = \mathcal{D}(X_0) \otimes D(\Lambda(\xi_1, \ldots, \xi_n)),$$

where $D(\Lambda(\xi_1, \ldots, \xi_n)$ is the superalgebra of the differential operators on $(0|n)$-dimensional supermanifold $\mathbb{A}^{(0|n)}$. Since $\Lambda(\xi_1, \ldots, \xi_n)$ is finite-dimensional, the superalgebra $D(\Lambda(\xi_1, \ldots, \xi_n))$ coincides with the superalgebra $\mathrm{End}_k(\Lambda(\xi_1, \ldots, \xi_n))$. This immediately implies the Morita equivalence of $\mathcal{D}(X)$ and $\mathcal{D}(X_0)$.

Let λ be a weight of $\mathfrak{g}$ and $X = G/B$ be a flag supermanifold. As in the usual case one can define the sheaf of twisted differential operators D_X^λ. Let $\mathcal{U}^\lambda(\mathfrak{g})$ denote the quotient of $\mathcal{U}(\mathfrak{g})$ by the ideal generated by the kernel of the central character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to k$. The embedding of the Lie superalgebra $\mathfrak{g}$ to the Lie superalgebra of vector fields on X induces the homomorphism of superalgebras

$$p_\lambda : \mathcal{U}^\lambda(\mathfrak{g}) \to D^\lambda(X).$$

Recall that it is an isomorphism if $\mathfrak{g}$ is a reductive Lie algebra. Moreover, for dominant λ the localization functor provides equivalence of categories of $\mathcal{U}^\lambda(\mathfrak{g})$-modules and $\mathcal{D}_X^\lambda$-modules. In the supercase, the similar result is true for generic typical λ, see [36].

Theorem 18 *Let λ be a generic typical weight such that $\frac{2(\lambda|\alpha)}{(\alpha|\alpha)} \notin \mathbf{Z}_{<0}$ for all even positive roots α. Then the functors of localization and global sections establish equivalence of categories of $\mathcal{U}^\lambda(\mathfrak{g})$-modules and $\mathcal{D}_X^\lambda$-modules.*

Note that essentially this theorem is equivalent to Theorem 4. In fact Theorem 18 was used by Penkov for the proof of Theorem 4. If λ is not typical, then the homomorphism p_λ is neither surjective nor injective. On the other hand, it is not difficult to see that for atypical λ the superalgebra $\mathcal{U}^\lambda(\mathfrak{g})$ has a non-trivial Jacobson radical, see [41]. There is an evidence that the following conjecture may hold.

Conjecture 6 Let λ be a regular weight, tame with respect to $\mathfrak{b}$, and let $\bar{\mathcal{U}}^\lambda(\mathfrak{g})$ denote the quotient of $\mathcal{U}^\lambda(\mathfrak{g})$ by the Jacobson radical. Let $\mathcal{Z}$ denote the center of $\bar{\mathcal{U}}^\lambda(\mathfrak{g})$. Let $Q \supset B$ be the maximal parabolic subgroup of G such that its Lie superalgebra $\mathfrak{q}$ admits one-dimensional representation c_λ. Finally let $Y := G/Q$.

If $\tau : \mathcal{Z} \to k$ is a generic central character and $\bar{\mathcal{U}}_\tau^\lambda(\mathfrak{g})$ is the quotient of $\bar{\mathcal{U}}^\lambda(\mathfrak{g})$ by the ideal $(\ker \tau)$, then the categories of $\bar{\mathcal{U}}_\tau^\lambda(\mathfrak{g})$-modules and $\mathcal{D}_Y^\lambda$-modules are equivalent.

10 Direct Limits of Lie Algebras and Superalgebras

The goal of this section is to say few words about representations of direct limits of classical Lie superalgebras. We will discuss here only the case of $\mathfrak{gl}(\infty|\infty)$ and refer to [43] for the case of $\mathfrak{osp}(\infty|\infty)$. Surprisingly, for some class of representations the difference between the Lie superalgebra $\mathfrak{gl}(\infty|\infty)$ and the Lie algebra $\mathfrak{gl}(\infty)$ disappears.

10.1 Category of Tensor Modules

Let V, W be countable-dimensional vector spaces (resp. superspaces) with non-degenerate even pairing $\langle\cdot,\cdot\rangle : W \times V \to k$. It is known that one can choose a pair of dual bases in V and W. The tensor product $V \otimes W$ is a Lie algebra (resp. superalgebra) $\mathfrak{g}$ with the following bracket:

$$[v_1 \otimes w_1, v_2 \otimes w_2] = \langle w_1, v_2\rangle v_1 \otimes w_2 - (-1)^{(\bar{v}_1+\bar{w}_1)(\bar{v}_2+\bar{w}_2)}\langle w_2, v_1\rangle v_2 \otimes w_1.$$

We denote this (super)algebra $\mathfrak{gl}(\infty)$ in the even case and $\mathfrak{gl}(\infty|\infty)$ in the supercase. Note that both V and W are $\mathfrak{g}$-modules and $\mathfrak{g}$ acts on V and W by linear operators of finite rank. It is not difficult to see that $\mathfrak{g}$ can be identified with infinite matrices with finitely many non-zero entries and hence

$$\mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n), \quad \mathfrak{gl}(\infty|\infty) = \varinjlim \mathfrak{gl}(m|n).$$

Let $T^{p,q} = V^{\otimes p} \otimes W^{\otimes q}$. We would like to understand the structure of $\mathfrak{g}$-module on $T^{p,q}$. It is clear that the product of symmetric groups $S_p \times S_q$ acts on $T^{p,q}$ and this action commutes with the action of $\mathfrak{g}$. Irreducible representations of $S_p \times S_q$ are parametrized by bipartitions (λ, μ) such that $|\lambda| = p$, $|\mu| = q$. The following result is a classical Schur–Weyl duality. In the supercase its proof is due to Sergeev, [44].

Theorem 19 *Let $\mathfrak{g} = \mathfrak{gl}(\infty)$ or $\mathfrak{gl}(\infty|\infty)$. Then we have the following decomposition*

$$T^{p,q} = \bigoplus_{|\lambda|=p, |\mu|=q} S_\lambda(V) \otimes S_\mu(W) \otimes Y_{\lambda,\mu},$$

where $S_\lambda(V)$ and $S_\mu(W)$ are simple $\mathfrak{g}$-modules and $Y_{\lambda,\mu}$ is the irreducible representation of $S_p \times S_q$ associated with a bipartition (λ, μ).

Let $\mathfrak{g} = \mathfrak{gl}(\infty)$. It is proven in [37] that $S_\lambda(V) \otimes S_\mu(W)$ is an indecomposable $\mathfrak{g}$-module of finite length with simple socle $V(\lambda, \mu)$. Denote by $\mathrm{Trep}\,\mathfrak{g}$ the abelian category of $\mathfrak{g}$-modules generated by finite direct sums of $T^{p,q}$ and all their subquotients. This is a symmetric monoidal category which in the case of $\mathfrak{g} = \mathfrak{gl}(\infty)$ was studied in [8] and [38].

Theorem 20 ([8]) *Let* $\mathfrak{g} = \mathfrak{gl}(\infty)$. *Any simple object of* Trep$\mathfrak{g}$ *is isomorphic to* $V(\lambda, \mu)$ *for some bipartition* (λ, μ) *and* $S_\lambda(V) \otimes S_\mu(W)$ *is the injective hull of* $V(\lambda, \mu)$. *In particular, the category* Trep$\mathfrak{g}$ *has enough injective objects. Moreover, any object in* Trep$\mathfrak{g}$ *has a finite injective resolution.*

It is also proven in [8] that Trep$\mathfrak{g}$ is a Koszul self-dual category.

Let us consider the case $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$. We start by constructing two functors F_l and F_r from the category Trep$\mathfrak{g}$ to the category Trep$\mathfrak{gl}(\infty)$. Observe that the even part $\mathfrak{gl}(\infty|\infty)_0$ is a direct sum $\mathfrak{g}_l \oplus \mathfrak{g}_r$ with both $\mathfrak{g}_l = V_0 \otimes W_0$ and $\mathfrak{g}_r = V_1 \otimes W_1$ isomorphic to $\mathfrak{gl}(\infty)$. For any $M \in$ Trep$\mathfrak{g}$ we set

$$F_l(M) := M^{\mathfrak{g}_r}, \quad F_r(M) := M^{\mathfrak{g}_l}.$$

Theorem 21 ([43]) *Let* $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$.

(a) F_l *and* F_r *are exact tensor functors, i.e.* $F_l(M \otimes N) = F_l(M) \otimes F_l(N)$ *and the same for* F_r.
(b) F_l *and* F_r *have left adjoint functors which we denote by* R_l *and* R_r *respectively.*
(c) F_l *and* R_l *(resp.* F_r *and* R_r*) are mutually inverse equivalences of tensor categories* Trep$\mathfrak{g}$ *and* Trep$\mathfrak{g}_l$ *(resp.* Trep$\mathfrak{g}_r$*).*

Remark 11 The compositions $F_r \circ R_l$ and $F_l \circ R_r$ provide an autoequivalence of Trep$\mathfrak{gl}(\infty)$ which sends a simple module $V(\lambda, \mu)$ to the simple module $V(\lambda', \mu')$, where ν' stands for the partition conjugate to ν.

Remark 12 The corresponding construction works as well for the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(\infty|\infty)$. Here $\mathfrak{g}_l = \mathfrak{so}(\infty)$ and $\mathfrak{g}_r = \mathfrak{sp}(\infty)$. In particular, we establish equivalence of tensor categories Trep$\mathfrak{so}(\infty)$ and Trep$\mathfrak{sp}(\infty)$.

Remark 13 The category Trep$\mathfrak{g}$ contains a semisimple subcategory Trep$^+\mathfrak{g}$ consisting of modules appearing in $T^{p,0}$, $p \in \mathbb{N}$.

10.2 Equivalences for Parabolic Category $\mathcal{O}$

In this subsection we will show how functors F_r and F_l help to prove equivalence of certain parabolic category $\mathcal{O}$ for $\mathfrak{gl}(\mathfrak{m}|\infty)$ and $\mathfrak{gl}(\infty)$. This result is originally proven in [6] by using infinite chain of odd reflections.

Let $\mathfrak{g}' = \mathfrak{gl}(\infty)$, $\mathfrak{g}'' = \mathfrak{gl}(m|\infty)$ and $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$. We fix the embeddings $\mathfrak{g}'$ and $\mathfrak{g}''$ into $\mathfrak{g}$ in the following way. Realize $\mathfrak{g}$ as matrices with finitely many non-zero entries written in the block form

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix},$$

where $A_{1,1}$ has size $m \times m$, $A_{1,2}$ and $A_{1,3}$ have size $m \times \infty$, $A_{2,1}$ and $A_{3,1}$ have size $\infty \times m$ and $A_{2,2}$ and $A_{3,3}$ have size $\infty \times \infty$. The even part $\mathfrak{g}_0$ consists of matrices of the form

$$\begin{pmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} & 0 \\ 0 & 0 & A_{3,3} \end{pmatrix},$$

and the odd part $\mathfrak{g}_1$ of matrices of the form

$$\begin{pmatrix} 0 & 0 & A_{1,3} \\ 0 & 0 & A_{2,3} \\ A_{3,1} & A_{3,2} & 0 \end{pmatrix}.$$

Then $\mathfrak{g}'$ consists of matrices

$$\begin{pmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\mathfrak{g}''$ of matrices

$$\begin{pmatrix} A_{1,1} & 0 & A_{1,3} \\ 0 & 0 & 0 \\ A_{3,1} & 0 & A_{3,3} \end{pmatrix}.$$

Let $\mathfrak{k}'$ and $\mathfrak{k}''$ be subalgebras of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{3,3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Then it is not hard to see that $\mathfrak{g}'$ is the centralizer of $\mathfrak{k}'$ and $\mathfrak{g}''$ is the centralizer of $\mathfrak{g}''$.

Next we consider the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ consisting of matrices

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ 0 & A_{2,2} & A_{2,3} \\ 0 & A_{3,2} & A_{3,3} \end{pmatrix},$$

with abelian ideal $\mathfrak{m}$

$$\begin{pmatrix} 0 & A_{1,2} & A_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the Levi subalgebra $\mathfrak{l}$

$$\begin{pmatrix} A_{1,1} & 0 & 0 \\ 0 & A_{2,2} & A_{2,3} \\ 0 & A_{3,2} & A_{3,3} \end{pmatrix},$$

isomorphic to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(\infty|\infty)$.

Finally we set $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ and $\mathfrak{p}'' := \mathfrak{p} \cap \mathfrak{g}''$. Note that $\mathfrak{p}' \subset \mathfrak{g}'$ and $\mathfrak{p}'' \subset \mathfrak{g}''$ are parabolic subalgebras. Now we consider the category $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ consisting of all $\mathfrak{g}$-modules M satisfying the following conditions

- M is finitely generated;
- M is semisimple over the diagonal subalgebra of $\mathfrak{g}$ with integral weights;
- M is an integrable $\mathfrak{p}$-module and the restriction to the subalgebra $\mathfrak{gl}(\infty|\infty) \subset \mathfrak{p}$ belongs to the inductive completion of $\mathrm{Trep}^{+}\mathfrak{gl}(\infty|\infty)$.

In a similar way we define the categories $\mathcal{O}(\mathfrak{g}', \mathfrak{p}')$ and $\mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$ for algebras $\mathfrak{g}'$ and $\mathfrak{g}''$ respectively. As in the previous subsection we define the functors

$$F' : \mathcal{O}(\mathfrak{g}, \mathfrak{p}) \to \mathcal{O}(\mathfrak{g}', \mathfrak{p}'), \quad F'' : \mathcal{O}(\mathfrak{g}, \mathfrak{p}) \to \mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$$

by setting

$$F'(M) = M^{\mathfrak{k}'}, \quad F''(M) = M^{\mathfrak{k}''}.$$

Then we have the following analogue of Theorem 21.

Theorem 22

(a) F' and F'' have left adjoint functors which we denote by R' and R'' respectively.
(b) F' and R' (resp. F'' and R'') are mutually inverse equivalences of abelian categories $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ and $\mathcal{O}(\mathfrak{g}', \mathfrak{p}')$ (resp. $\mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$).
(c) The composite functors $F'' \circ R'$ and $F' \circ R''$ are mutually inverse equivalences of abelian categories $\mathcal{O}(\mathfrak{g}', \mathfrak{p}')$ and $\mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$.

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