Centralizers of Nilpotent Elements and Related Problems, a Survey

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Abstract Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero. This note is a survey on several results, obtained jointly with Jean-Yves Charbonnel, concerning the centralizer $\mathfrak{g}^e$ of a nilpotent element e of $\mathfrak{g}$. First, we take interest in a famous conjecture by Elashvili on the index of $\mathfrak{g}^e$. Second, we study the question of whether the algebra $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ of symmetric invariants of $\mathfrak{g}^e$ is a polynomial algebra. Our main result stipulates that if for some homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$, the initial homogeneous components of their restrictions to $e + \mathfrak{g}^f$ are algebraically independent, with (e, h, f) an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. As applications, we pursue the investigations of Panyushev et al. (J Algebra 313:343–391, 2007) and produce new examples of nilpotent elements that verify the above polynomiality condition.

We also present a recent result of Arakawa-Premet related to the above problems.

Keywords Centralizer • Elashvili conjecture • Slodowy grading • Symmetric invariant

This note is a survey on several results, obtained jointly with Jean-Yves Charbonnel, on centralizers of elements in a reductive Lie algebra. These results are mostly based on the articles [5, 6].

1 Elashvili's Conjecture and Consequences

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. The *index* of $\mathfrak{g}$, denoted by $\operatorname{ind}\mathfrak{g}$, is the minimal dimension of the stabilizers of linear forms on $\mathfrak{g}$ for the coadjoint representation, [11]:

$$\operatorname{ind}\mathfrak{g} := \min\{\dim\mathfrak{g}^{\xi} \ ; \ \xi \in \mathfrak{g}^*\}$$

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where $\mathfrak{g}^{\xi} = \{x \in \mathfrak{g} \ ; \ \xi([x, \mathfrak{g}]) = 0\}$. The notion of the index is important in representation theory and invariant theory. By Rosenlicht [28], if $\mathfrak{g}$ is algebraic, i.e., $\mathfrak{g}$ is the Lie algebra of some algebraic linear group G, then the index of $\mathfrak{g}$ is the transcendence degree of the field of G-invariant rational functions on $\mathfrak{g}^*$. The index of a reductive algebra is equal to its rank. In general, computing the index of an arbitrary Lie algebra is a wild problem. However, there are a large number of interesting results for several classes of non-reductive subalgebras of reductive Lie algebras. For example, the centralizers of elements form an interesting class of subalgebras (cf. e.g.,[13, 22, 32]). This topic is closely related to the theory of integrable Hamiltonian systems [2, 3]. Let us make this link precise.

The symmetric algebra $\mathrm{S}(\mathfrak{g})$ carries a natural Poisson structure. Let $\xi \in \mathfrak{g}^*$ and consider the *Mishchenko-Fomenko subalgebra* $\mathcal{A}_{\xi}$ of $\mathrm{S}(\mathfrak{g}) \cong \Bbbk[\mathfrak{g}^*]$, constructed by the so-called *argument shift method*, [20]. It is generated by the ξ-shifts of p for p in the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants of $\mathrm{S}(\mathfrak{g})$, that is, $\mathcal{A}_{\xi}$ is generated by all the derivatives $D_{\xi}^{i}(p)$ for $p \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and $i \in \{0, \dots, \deg p - 1\}$, where

$$D_{\xi}^{i}(p)(x) := \frac{\mathrm{d}^i}{\mathrm{d}t} p(x + t\xi)|_{t=0}, \qquad x \in \mathfrak{g}^*.$$

It is well-known that $\mathcal{A}_{\xi}$ is a Poisson-commutative subalgebra of $S(\mathfrak{g})$.

Let $\mathfrak{g}^*_{\mathrm{sing}}$ be the set of nonregular linear forms $x \in \mathfrak{g}^*$, i.e.,

$$\mathfrak{g}^*_{\mathrm{sing}} := \{x \in \mathfrak{g}^* \mid \dim \mathfrak{g}^x > \operatorname{ind} \mathfrak{g}\}.$$

If $\mathfrak{g}^*_{\mathrm{sing}}$ has codimension at least 2 in $\mathfrak{g}^*$, we say that $\mathfrak{g}$ is *nonsingular*.

Theorem 1 (Bolsinov, [3, Theorem 2.1 and 3.2]) *Assume that* $\mathfrak{g}$ *is* nonsingular, *and let* $x \in \mathfrak{g}^*$. *For some* $\xi \in \mathfrak{g}^*$, *there is a Mishchenko-Fomenko subalgebra* $\mathcal{A}_{\xi}$ *in* $S(\mathfrak{g}) \cong \Bbbk[\mathfrak{g}^*]$ *such that its restriction to* Gx *contains* $\frac{1}{2}\dim(Gx)$ *algebraically independent functions if and only if* $\operatorname{ind} \mathfrak{g}^x = \operatorname{ind} \mathfrak{g}$.

If $\mathfrak{g}$ is reductive, then $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic as Lie algebras and its index $\operatorname{ind} \mathfrak{g}$ is equal to its rank, denoted by ℓ. The stabilizer of an element $x \in \mathfrak{g}^*$ identifies with the centralizer $\mathfrak{g}^x$ of x viewed as an element of $\mathfrak{g}$ through this isomorphism. Motivated by the preceding result of Bolsinov, Elashvili formulated a conjecture:

Conjecture 1 *Assume that* $\mathfrak{g}$ *is reductive. Then the index of* $\mathfrak{g}^x$ *is equal to* $\operatorname{ind} \mathfrak{g} = \ell$ *for any* $x \in \mathfrak{g}$.

Elashvili's conjecture has attracted the interest of many invariant theorists (e.g. [10, 22, 26, 32]). Thanks to Jordan decomposition, the conjecture reduces to the case where $x \in \mathfrak{g}$ is a nilpotent element. Also, it reduces to the case where $\mathfrak{g}$ is simple. Then the conjecture was proven by Yakimova for $\mathfrak{g}$ a simple Lie algebra of classical type, [32], and checked by a computer program by De Graaf for $\mathfrak{g}$ a simple Lie algebra of exceptional type, [10]. Before that, the result was established for some particular classes of nilpotent elements by Panyushev, [22, 23].

In a joint work with Jean-Yves Charbonnel, [5], we gave an almost case-free proof of Elashvili's conjecture using Bolsinov's criterion (cf. Theorem 1). Our approach was totally different from the previous ones. In more detail, this criterion is used to reduce the conjecture to the case of *rigid nilpotent elements*, that is those whose nilpotent G-orbit cannot be properly induced in the sense of Lusztig-Spaltenstein, [18]. For the rigid nilpotent elements, we have developed other methods that cover all cases except one case in type $\mathbf{E}_7$ and six cases in type $\mathbf{E}_8$. These remaining cases have been dealt with the computer program `GAP4`.

To summarize, we can state:

Theorem 2 ([5, Theorem 1.2]) *Assume that* $\mathfrak{g}$ *is reductive. Then the index of* $\mathfrak{g}^x$ *is equal to* $\operatorname{ind}\mathfrak{g} = \ell$ *for any* $x \in \mathfrak{g}$.

Assume from now on that $\mathfrak{g}$ is simple of rank ℓ. Denote by G its adjoint group and by $\langle .\,,.\rangle$ the Killing form of $\mathfrak{g}$.

Elashvili's conjecture also appears in invariant theory through the following interesting question, first raised by Premet:

Question 1 *Let* $x \in \mathfrak{g}$. *Is* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ *a polynomial algebra in* ℓ *variables?*

In order to answer this question, we can assume that x is nilpotent. Besides, if $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial for some $x \in \mathfrak{g}$, then it is so for any element in the adjoint orbit Gx of x. If $x = 0$, it is well-known since Chevalley that $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is polynomial in ℓ variables. At the extreme opposite, if x is a regular nilpotent element of $\mathfrak{g}$, then $\mathfrak{g}^x$ is abelian of dimension ℓ, [12], and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)$ is polynomial in ℓ variables too.

A positive answer to Question 1 was suggested in [26, Conjecture 0.1] for any simple $\mathfrak{g}$ and any $x \in \mathfrak{g}$. Yakimova has since discovered a counter-example in type $\mathbf{E}_8$, [33], disconfirming the conjecture. More precisely, the elements of the minimal nilpotent orbit in $\mathbf{E}_8$ do not satisfy the polynomiality condition. The present note contains another counter-example in type $\mathbf{D}_7$ (cf. Example 6). Question 1 still remains interesting and has a positive answer for a large number of nilpotent elements $e \in \mathfrak{g}$ as it is explained below.

Elashvili's conjecture (cf. Theorem 2) is deeply related to Question 1. First of all, it implies that if $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, it is so in ℓ variables. Further, according to Theorem 2, the main results of [26], that we summarize below, can be applied (see Theorem 3).

Let e be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov Theorem, e is embedded into an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$. Denote by $\mathcal{S}_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with* e. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $(\mathfrak{g}^e)^*$ with $\mathfrak{g}^f$, through the Killing form $\langle .\,,.\rangle$. For p in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}^*] \simeq \Bbbk[\mathfrak{g}]$, denote by ${}^e p$ the initial homogenous component of its restriction to $\mathcal{S}_e$. According to [26, Proposition 0.1], if p is in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Theorem 3 (Panyushev-Premet-Yakimova, [26, Theorem 0.3]) *Suppose that the following two conditions are satisfied:*

(1) for some homogenous generators $q_1, \ldots, q_\ell$ *of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$*, the polynomial functions* ${}^e q_1, \ldots, {}^e q_\ell$ *are algebraically independent,*
(2) $\mathfrak{g}^e$ *is nonsingular.*

Then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is a polynomial algebra with generators* ${}^e q_1, \ldots, {}^e q_\ell$*.*

As a consequence of Theorem 3, if $\mathfrak{g}$ is simple of type $\mathbf{A}_\ell$ or $\mathbf{C}_\ell$, then all nilpotent elements of $\mathfrak{g}$ verify the polynomiality condition, cf. [26, Theorem 4.2 and 4.4]. The result for the type $\mathbf{A}_\ell$ was independently obtained by Brown and Brundan, [4]. In [26], the authors also provide some examples of nilpotent elements satisfying the polynomiality condition in the simple Lie algebras of types $\mathbf{B}_\ell$ and $\mathbf{D}_\ell$, and a few ones in the simple exceptional Lie algebras. At last, note that the analogue question to Question 1 for the positive characteristic was dealt with by Topley for the simple Lie algebras of types $\mathbf{A}_\ell$ and $\mathbf{C}_\ell$, [31].

2 Characterization of Good Elements

We now summarize the main result of [6], which continues the investigations of [26]. The following definition will be central:

Definition 1 An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some graded sequence $(p_1, \ldots, p_\ell)$ in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$.

For example, regular nilpotent elements are good. Indeed, for e in the regular nilpotent orbit of $\mathfrak{g}$ and $(q_1, \ldots q_\ell)$ a homogenous generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, it is well-known that ${}^e q_i = \mathrm{d}_e q_i$ for $i = 1, \ldots, \ell$ and that $(\mathrm{d}_e q_1, \ldots, \mathrm{d}_e q_\ell)$ forms a basis of $\mathfrak{g}^e$, [17]. Hence e is good.

Also, by Panyushev et al. [26, Theorem 5.4], all nilpotent elements of a simple Lie algebra of type $\mathbf{A}_\ell$ are good. Moreover, according to [34, Corollary 8.2], *even*[1] nilpotent elements without odd (respectively even) Jordan blocks of $\mathfrak{g}$ are good if $\mathfrak{g}$ is of type $\mathbf{C}_\ell$ (respectively $\mathbf{B}_\ell$ or $\mathbf{D}_\ell$). We generalize these results (cf. Proposition 3).

Our first result is the following:

Proposition 1 ([6]) *Let* x *be a good element of* $\mathfrak{g}$*. Then* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ *is a polynomial algebra and* $\mathrm{S}(\mathfrak{g}^x)$ *is a free extension of* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$*.*

Furthermore, we show that x is good if and only if so is its nilpotent component in the Jordan decomposition. As a consequence, we can restrict the study to the case of nilpotent elements.

[1] i.e., this means that the Dynkin grading of $\mathfrak{g}$ associated with the nilpotent element has no odd term.

Our main result is the following theorem whose proof is outlined in Sect. 3:

Theorem 4 ([6]) *Suppose that for some homogeneous generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent. Then e is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, $({}^e q_1, \ldots, {}^e q_\ell)$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

In other words, Theorem 4 says that Condition (1) of Theorem 3 is sufficient to ensure the polynomiality of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. However, if only Condition (1) of Theorem 3 is satisfied, the (polynomial) algebra $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not necessarily generated by the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$. As a matter of fact, there are nilpotent elements e satisfying Condition (1) and for which $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not generated by some ${}^e q_1, \ldots, {}^e q_\ell$, for any choice of homogenous generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ (cf. Remark 1).

Theorem 4 can be applied to a great number of nilpotent orbits in the simple classical Lie algebras (cf. Sect. 4), and for some nilpotent orbits in the exceptional Lie algebras (cf. Sect. 5). For example, according to [25, Proposition 6.3] and Theorem 4, the elements of the subregular nilpotent orbit of $\mathfrak{g}$ are good.

All examples of good elements we encounter satisfy the hypothesis of Theorem 4. In fact, we have recently proven that the converse of Theorem 4 is true (see [7]).

Theorem 5 *Let e be a nilpotent of $\mathfrak{g}$. If e is good then for some graded generating sequence $(q_1, \ldots, q_\ell)$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over $\Bbbk$. In other words, the converse implication of Theorem 4 holds.*

Notice that it may happen that for some $r_1, \ldots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent over $\Bbbk$, but e is not good. This is the case for instance for the nilpotent elements in $\mathfrak{so}_{12}(\Bbbk)$ associated with the partition $(5, 3, 2^2)$, see Example 5.

In fact, according to [26, Corollary 2.3], for any nilpotent e of $\mathfrak{g}$, there exist $r_1, \ldots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent over $\Bbbk$.

3 Outline of the Proof of Theorem 4

Let $q_1, \ldots, q_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \ldots, d_\ell$ respectively. The sequence $(q_1, \ldots, q_\ell)$ is ordered so that $d_1 \leqslant \cdots \leqslant d_\ell$. Assume that the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent.

According to Proposition 1, it suffices to show that e is good, and so it suffices to show that the nullvariety of ${}^e q_1, \ldots, {}^e q_\ell$ in $\mathfrak{g}^f$ has codimension ℓ since ${}^e q_1, \ldots, {}^e q_\ell$ are invariant homogeneous polynomials. To this end, it suffices to prove that

$$S := \mathrm{S}(\mathfrak{g}^e)$$

is a free extension of the $\Bbbk$-algebra generated by ${}^e q_1, \ldots, {}^e q_\ell$. We are led to find a subspace V_0 of S such that the linear map

$$V_0 \otimes_\Bbbk \Bbbk[{}^e q_1, \ldots, {}^e q_\ell] \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism. We explain below the construction of the subspace V_0.

Let $x_1, \ldots, x_r$ be a basis of $\mathfrak{g}^e$ such that for $i = 1, \ldots, r$, $[h, x_i] = n_i x_i$ for some nonnegative integer n_i. For $\mathbf{j} = (j_1, \ldots, j_r)$ in $\mathbb{N}^r$, set:

$$|\mathbf{j}| := j_1 + \cdots + j_r, \qquad |\mathbf{j}|_e := j_1 n_1 + \cdots + j_r n_r + 2|\mathbf{j}|, \qquad x^{\mathbf{j}} = x_1^{j_1} \cdots x_r^{j_r}.$$

The algebra S has two gradations: the standard one and the *Slodowy gradation*. For all $\mathbf{j}$ in $\mathbb{N}^r$, $x^{\mathbf{j}}$ is homogeneous with respect to these two gradations. It has standard degree $|\mathbf{j}|$ and, by definition, it has Slodowy degree $|\mathbf{j}|_e$. For m nonnegative integer, denote by $S^{[m]}$ the subspace of S of Slodowy degree m.

For any subspace V of S, set:

$$V[t] := \Bbbk[t] \otimes_\Bbbk V, \qquad V[[t]] := \Bbbk[[t]] \otimes_\Bbbk V, \qquad V((t)) := \Bbbk((t)) \otimes_\Bbbk V.$$

For V a subspace of $S[[t]]$, denote by $V(0)$ the image of V by the quotient morphism

$$S[[t]] \longrightarrow S, \qquad a(t) \longmapsto a(0).$$

The Slodowy grading of S induces a grading of the algebra $S((t))$ with t having degree 0. Let τ be the morphism of algebras

$$S \longrightarrow S[t], \qquad x_i \mapsto t x_i, \quad i = 1, \ldots, r.$$

The morphism τ is a morphism of graded algebras. Denote by $\delta_1, \ldots, \delta_\ell$ the standard degrees of ${}^e q_1, \ldots, {}^e q_\ell$ respectively, and set for $i = 1, \ldots, \ell$:

$$Q_i := t^{-\delta_i} \tau(\kappa(q_i)) \quad \text{with} \quad \kappa(q_i)(x) := q_i(e + x), \quad \forall\, x \in \mathfrak{g}^f.$$

Let A be the subalgebra of $S[t]$ generated by $Q_1, \ldots, Q_\ell$. Then $A(0)$ is the subalgebra of S generated by ${}^e q_1, \ldots, {}^e q_\ell$. For $(j_1, \ldots, j_\ell)$ in $\mathbb{N}^\ell$, $\kappa(q_1^{j_1}) \cdots \kappa(q_\ell^{j_\ell})$ and ${}^e q_1^{j_1} \cdots {}^e q_\ell^{j_\ell}$ are Slodowy homogenous of Slodowy degree $2d_1 j_1 + \cdots + 2d_\ell j_\ell$ (cf. e.g [26, 27]). Hence, A and $A(0)$ are graded subalgebras of $S[t]$ and S respectively. Denote by $A(0)_+$ the augmentation ideal of $A(0)$, and let V_0 be a graded complement to $SA(0)_+$ in S. The main properties of our data A and $A(0)$ are the following ones:

(1) A is a graded polynomial algebra,
(2) the canonical morphism $A \to A(0)$ is a homogenous isomorphism,
(3) the algebra $S[t, t^{-1}]$ is a free extension of A,
(4) the ideal $S[t, t^{-1}]A_+$ of $S[t, t^{-1}]$ is radical.

With these properties we first obtain that $S[[t]]$ is a free extension of A and that $S[[t]]$ is a free extension of the subalgebra $\tilde{A}$ of $S[[t]]$ generated by $\Bbbk[[t]]$ and A. From these results, we deduce that the linear map

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism, as expected.

4 Consequences of Theorem 4 for the Simple Classical Lie Algebras

By Panyushev et al. [26, Theorem 4.2 and 4.4], the first consequence of Proposition 1 and Theorem 4 is the following.

Proposition 2 *Assume that $\mathfrak{g}$ is simple of type $\mathbf{A}_\ell$ or $\mathbf{C}_\ell$. Then all the elements of $\mathfrak{g}$ are good.*

To state our results for the simple Lie algebras of types $\mathbf{B}_\ell$ and $\mathbf{D}_\ell$, let us introduce some more notations. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V}) \subset \mathfrak{gl}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $n = 2\ell + 1$ or $n = 2\ell$. For an endomorphism x of $\mathbb{V}$ and for $i \in \{1, \dots, n\}$, denote by $Q_i(x)$ the coefficient of degree $n - i$ of the characteristic polynomial of x. Then for any x in $\mathfrak{g}$, $Q_i(x) = 0$ whenever i is odd. Define a generating family $q_1, \dots, q_\ell$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell - 1$, set $q_i := Q_{2i}$. If $n = 2\ell + 1$, set $q_\ell := Q_{2\ell}$, and if $n = 2\ell$, let q_ℓ be the Pfaffian that is a homogenous element of degree ℓ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$.

Following the notations of Sects. 2 and 3, denote by ${}^e q_i$ the initial homogeneous component of the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto q_i(e + x)$, and by δ_i the degree of ${}^e q_i$.[2] According to [26, Theorem 2.1], ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent if and only if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0.$$

In that event, by Theorem 4, e is good and we say that e is *very good*.

The very good nilpotent elements of $\mathfrak{g}$ can be characterized in term of their associated partitions of n as follows. Assume that $\lambda = (\lambda_1, \dots, \lambda_k)$, with $\lambda_1 \geqslant \cdots \geqslant \lambda_k$, is the partition of n associated with the nilpotent orbit Ge. Then the even integers of λ have an even multiplicity, [9, §5.1]. Thus k and n have the same parity. Consider the following conditions on a sequence $\mu = (\mu_1, \dots, \mu_j)$ with $\mu_1 \geqslant \cdots \geqslant \mu_j$:

1) μ_{j-1} and μ_j are odd,
2) μ_{j-1} and μ_j are even,
3) $j > 3$, μ_1 and μ_j are odd and μ_i is even for any $i \in \{2, \dots, j - 1\}$.

[2] The sequence of the degrees $(\delta_1, \dots, \delta_\ell)$ is described by [26, Remark 4.2].

Lemma 1

(i) *If n is odd, then λ is very good if and only if λ_1 is odd and if $(\lambda_2, \ldots, \lambda_k)$ is a concatenation of sequences verifying Conditions* (1) *or* (2) *with $j = 2$.*
(ii) *If n is even, then λ is very good if and only if λ is a concatenation of sequences verifying Condition* (3) *or Condition* (1) *with $j = 2$.*

For example, the partitions $(5, 3^2, 2^2)$ and $(7, 5^2, 4^2, 3, 1^2)$ of 15 and 30 respectively are very good. In particular, by Lemma 1, all even nilpotent elements in type $\mathbf{B}_\ell$, or in type $\mathbf{D}_\ell$ with odd rank ℓ, correspond to very good partitions and so are good.

Theorem 4 also allows to obtain examples of good, but not very good, nilpotent elements of $\mathfrak{g}$. For them, there are a few more work to do. Let us state one result as an illustration:

Proposition 3 ([6])

(i) *Assume that for some $k' \in \{2, \ldots, k\}$, λ_i is even for all $i \leqslant k'$, that $(\lambda_{k'+1}, \ldots, \lambda_k)$ is very good and that $\lambda_1 = \cdots = \lambda_{k'}$. Then λ is not very good, but e is good.*
(ii) *Assume that $k = 4$ and that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even. Then e is good.*

For example, $(6^4, 5, 3)$ satisfies the hypothesis of (i), and $(6^2, 4^2)$ satisfies the hypothesis of (ii).

There are also examples of elements that verify the polynomiality condition but that are not good; see Examples 4 and 5. To deal with them, we use different techniques, more similar to those used in [26] and that we present in Sect. 6.

As a result of all this, we observe for example that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^n)$, with $n \leqslant 8$, are good, and that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^n)$, with $n \leqslant 13$, verify the polynomiality condition.

Our results do not cover all nilpotent orbits in type $\mathbf{B}_\ell$ and $\mathbf{D}_\ell$ for larger ℓ (cf. Example 6).

Remark 1 Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$, with $\dim \mathbb{V} = 12$, and that $\lambda = (5^2, 1^2)$. Then the degrees of ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$ are $1, 1, 2, 2, 2, 2$ respectively. Since $10 = 1 + 1 + 2 + 2 + 2 + 2 = (\dim \mathfrak{g}^e + \ell)/2$, the polynomial functions ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$ are algebraically independent, and by Theorem 4, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial. One can verify that ${}^eq_5 = z^2$ for some z in the center $\mathfrak{z}(\mathfrak{g}^e)$ of $\mathfrak{g}^e$. Since $\mathfrak{z}(\mathfrak{g}^e)$ has dimension 3, for any other choice of homogenous generators $q_1, \ldots, q_\ell$ of $S(\mathfrak{g})^{\mathfrak{g}}$, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ cannot be generated by the elements ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$ for degree reasons.

This shows that Condition (2) of Theorem 3 cannot be removed to ensure that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra in the variables ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$. However one can sometimes, as in this example, provide explicit generators.

5 Examples in Simple Exceptional Lie Algebras

We give in this section examples of good nilpotent elements in simple exceptional Lie algebras of type $\mathbf{E}_6$, $\mathbf{F}_4$ or $\mathbf{G}_2$ which are not covered by Panyushev et al. [26]. These examples are all obtained through Theorem 4.

According to [26, Theorem 0.4] and Theorem 4, the elements of the minimal nilpotent orbit of $\mathfrak{g}$, for $\mathfrak{g}$ not of type $\mathbf{E}_8$, are good. In addition, as it is explained in Sect. 2, the elements of the regular, or subregular, nilpotent orbit of $\mathfrak{g}$ are good. So we do not consider here these cases.

Example 1 Suppose that $\mathfrak{g}$ has type $\mathbf{E}_6$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 in the notation of Bourbaki. It has dimension 27 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{27}(\mathbb{k})$. For x in $\mathfrak{sl}_{27}(\mathbb{k})$ and for $i = 2, \ldots, 27$, let $p_i(x)$ be the coefficient of T^{27-i} in $\det(T-x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then $(q_2, q_5, q_6, q_8, q_9, q_{12})$ is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, [19]. Note that (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{27}(\mathbb{k})$. We denote by ${}^e p_i$ the initial homogeneous component of the restriction to $e + \tilde{\mathfrak{g}}^f$ of p_i where $\tilde{\mathfrak{g}}^f$ is the centralizer of f in $\mathfrak{sl}_{27}(\mathbb{k})$. For $i = 2, 5, 6, 8, 9, 12$,

$$\deg {}^e p_i \leqslant \deg {}^e q_i.$$

On the other hand,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} \leqslant \frac{1}{2}(\dim \mathfrak{g}^e + 6),$$

with equality if and only if ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent. So if the sum

$$\Sigma := \deg {}^e p_2 + \deg {}^e p_5 + \deg {}^e p_6 + \deg {}^e p_8 + \deg {}^e p_9 + \deg {}^e p_{12}$$

is equal to

$$\Sigma' := \frac{1}{2}(\dim \mathfrak{g}^e + 6),$$

then we can directly conclude that e is good. Otherwise, from the knowledge of the maximal eigenvalue $\nu_{\max}$ of the restriction of $\mathrm{ad}\, h$ to $\mathfrak{g}$ and the $\mathrm{ad}\, h$-weight of ${}^e p_i$, it is sometimes possible to deduce that $\deg {}^e p_i < \deg {}^e q_i$ and that e is good. We list in Table 1 the cases where we able to conclude in this way. The details are omitted. In Table 1, the fourth column gives the partition of 27 corresponding to the nilpotent element e of $\mathfrak{sl}_{27}(\mathbb{k})$, and the sixth one gives the $\mathrm{ad}\, h$-weights of ${}^e p_2$, ${}^e p_5$, ${}^e p_6$, ${}^e p_8$, ${}^e p_9$, ${}^e p_{12}$.

In conclusion, there remain nine unsolved nilpotent orbits in type $\mathbf{E}_6$.

Example 2 Suppose that $\mathfrak{g}$ is simple of type $\mathbf{F}_4$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_4 in the notation of Bourbaki. Then $\mathbb{V}$ has

Table 1 Data for $\mathbf{E}_6$

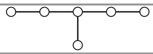
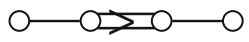
Label		$\dim \mathfrak{g}^e$	Partition	$\deg {}^e p_i$	Weights	$\nu_{\max}$	Σ	Σ'
D_5	2 0 2 0 2 2	10	(11,9,5,1,1)	1,1,1,1,1,1	2,8,10,14,16,22	14	6	8
$E_6(a_3)$	2 0 2 0 2 0	12	$(9, 7, 5^2, 1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	9
$D_5(a_1)$	1 1 0 1 1 2	14	(8,7,6,3,2,1)	1,1,1,1,2,2	2,8,10,14,14,20	10	8	10
A_5	2 1 0 1 2 1	14	$(9, 6^2, 5, 1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	10
$A_4 + A_1$	1 1 0 1 1 1	16	$(7, 6, 5, 4, 3, 2)$	1,1,1,2,2,2	2,8,10,12,14,20	8	9	11
D_4	0 0 2 0 0 2	18	$(7^3, 1^6)$	1,1,1,2,2,2	2,8,10,12,14,20	10	9	12
$D_4(a_1)$	0 0 2 0 0 0	20	$(5^3, 3^3, 1^3)$	1,1,2,2,2,3	2,8,8,12,14,18	6	11	13
$2A_2 + A_1$	1 0 1 0 1 0	24	$(5, 4^2, 3^3, 2^2, 1)$	1,1,2,2,2,3	2,8,8,12,14,18	5	11	15

Table 2 Data for $\mathbf{F}_4$

Label		$\dim \mathfrak{g}^e$	Partition	$\deg {}^e p_i$	Weights	$\nu_{\max}$	Σ	Σ'
$F_4(a_2)$	0 2 0 2	8	$(9, 7, 5^2)$	1,1,1,2	2,10,14,20	10	5	6
C_3	1 0 1 2	10	$(9, 6^2, 5)$	1,1,1,2	2,10,14,20	10	5	7
B_3	2 2 0 0	10	$(7^3, 1^5)$	1,1,2,2	2,10,12,20	10	6	7
$F_4(a_3)$	0 2 0 0	12	$(5^3, 3^3, 1^2)$	1,2,2,3	2,8,12,18	6	8	8
$C_3(a_1)$	1 0 1 0	14	$(5^2, 4^2, 3, 2^2, 1)$	1,2,2,3	2,8,12,18	6	8	9
$\tilde{A}_2 + A_1$	0 1 0 1	16	$(5, 4^2, 3^3, 2^2)$	1,2,2,3	2,8,12,18	5	8	10

dimension 26 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{26}(\Bbbk)$. For x in $\mathfrak{sl}_{26}(\Bbbk)$ and for $i = 2, \ldots, 26$, let $p_i(x)$ be the coefficient of T^{26-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then (q_2, q_6, q_8, q_{12}) is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, [19]. As in Example 1, in some cases, it is possible to prove that e is good. These cases are listed in Table 2, indexed as in Example 1.

In conclusion, there remain six unsolved nilpotent orbits in type $\mathbf{F}_4$.

Example 3 Suppose that $\mathfrak{g}$ is simple of type $\mathbf{G}_2$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 in the notation of Bourbaki. Then $\mathbb{V}$ has dimension 7 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_7(\Bbbk)$. For x in $\mathfrak{sl}_7(\Bbbk)$ and for $i = 2, \ldots, 7$, let $p_i(x)$ be the coefficient of T^{7-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then q_2, q_6 is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, [19]. There is only one nonzero nilpotent orbit which is neither regular, subregular or minimal. For e in it, we can show that e is good from Table 3, indexed as in Example 1.

In conclusion, all elements are good in type $\mathbf{G}_2$.

Table 3 Data for $\mathbf{G}_2$

Label		$\dim \mathfrak{g}^e$	Partition	$\deg {}^e p_i$	Weights	$\nu_{\max}$	Σ	Σ'
$\tilde{A}_1$	1 0	6	$(3, 2^2)$	1,3	2,6	3	4	4

6 A Result of Joseph-Shafrir and Applications

In this section, we provide, in a different way, examples of nilpotent elements which verify the polynomiality condition but that are not good, using techniques developed by Joseph-Shafrir, [16]. We also obtain an example of nilpotent element in type $\mathbf{D}_7$ which does not verify the polynomiality condition.

Let $\eta_0 \in \mathfrak{g}^e \otimes_{\Bbbk} \bigwedge^2 \mathfrak{g}^f$ be the bivector defining the Poisson bracket on $\mathrm{S}(\mathfrak{g}^e)$ induced from the Lie bracket. According to the main theorem of [27], $\mathrm{S}(\mathfrak{g}^e)$ is the graded algebra relative to the Kazhdan filtration of the *finite W-algebra* associated with e so that $\mathrm{S}(\mathfrak{g}^e)$ inherits another Poisson structure. The graded algebra structure so-obtained is the Slodowy graded algebra structure. Let $\eta \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^2 \mathfrak{g}^f$ be the bivector defining this other Poisson structure. According to [27, Prop. 6.3] (see also [26, §2.4]), η_0 is the initial homogeneous component of η. Denote by r the dimension of $\mathfrak{g}^e$ and set:

$$\omega := \eta^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge{}^{r-\ell} \mathfrak{g}^f, \quad \omega_0 := \eta_0^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge{}^{r-\ell} \mathfrak{g}^f.$$

Then ω_0 is the initial homogeneous component of ω. This fact is the key point in the proof of the results we state now.

Theorem 6 ([6]) *Let $q_1, \dots, q_\ell$ be some homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, and let $r_1, \dots, r_\ell$ be algebraically independent homogeneous elements of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent.*

(i) *For some homogeneous element p of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,*

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell = p\, \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_\ell$$

and we have,

$$\sum_{i=1}^{\ell} \deg {}^e r_i = \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

(ii) *If ${}^e p$ is a greatest divisor of $\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^{\ell} \mathfrak{g}^e$, then $\mathfrak{g}^e$ is nonsingular.*

(iii) *Assume that for some homogeneous polynomials $p_1, \dots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, ${}^e r_1, \dots, {}^e r_\ell$ are in $\Bbbk[p_1, \dots, p_\ell]$ and that*

$$\deg p_1 + \cdots + \deg p_\ell = d + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

where d is the degree of a greatest divisor of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell$ *in* $\mathrm{S}(\mathfrak{g}^e)$. *Then* $\mathfrak{g}^e$ *is nonsingular.*

The following proposition is a particular case of [16, §5.7].

Proposition 4 (Joseph-Shafrir, [16]) *Suppose that* $\mathfrak{g}^e$ *is nonsingular.*

(i) *If there exist algebraically independent homogeneous polynomials* $p_1, \ldots, p_\ell$ *in* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *such that*

$$\deg p_1 + \cdots + \deg p_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell),$$

then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is a polynomial algebra generated by* $p_1, \ldots, p_\ell$.

(ii) *Suppose that the semiinvariant elements of* $\mathrm{S}(\mathfrak{g}^e)$ *are invariant. If* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is a polynomial algebra, then it is generated by homogeneous polynomials* $p_1, \ldots, p_\ell$ *such that*

$$\deg p_1 + \cdots + \deg p_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

To produce new examples, our general strategy is the following: We first apply Theorem 6,(ii), in order to prove that $\mathfrak{g}^e$ is nonsingular. Next, we search for independent homogeneous polynomials $p_1, \ldots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ satisfying the conditions of Theorem 6,(iii), with $d = 0$. Then we can apply Proposition 4,(i).

Proposition 4,(ii), is useful to construct counter-examples (cf. Example 6).

Example 4 Let e be a nilpotent element of $\mathfrak{so}_{10}(\Bbbk)$ associated with the partition $(3^2, 2^2)$. Then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra but e is not good as explained below. Let $q_1, \ldots, q_5$ be as in Sect. 4. The degrees of ${}^e q_1, \ldots, {}^e q_5$ are $1, 2, 2, 3, 2$ respectively. Using the computer program Maple, we get the algebraic relation:

$${}^e q_4^2 - 4\, {}^e q_3\, {}^e q_5^2 = 0.$$

Set:

$$r_i := \begin{cases} q_i & \text{if } i = 1, 2, 3, 5 \\ q_4^2 - 4 q_3 q_5^2 & \text{if } \quad i = 4. \end{cases}$$

The polynomials $r_1, \ldots, r_5$ are algebraically independent over $\Bbbk$ and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_5 = 2\, q_4\, \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_5$$

Moreover, ${}^e r_4$ has degree at least 7, ${}^e r_1, \ldots, {}^e r_5$ are algebraically independent, and ${}^e r_4$ has degree 7.

A precise computation shows that ${}^e r_3 = p_3^2$ for some p_3 in the center of $\mathfrak{g}^e$, and that ${}^e r_4 = p_4\, {}^e r_5$ for some polynomial p_4 of degree 5 in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i := {}^e r_i$ for

$i = 1, 2, 5$, the polynomials $p_1, \dots, p_5$ are algebraically independent homogeneous polynomials of degree 1, 2, 1, 5, 2 respectively. Furthermore, the greatest divisors of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_5$ in $\mathrm{S}(\mathfrak{g}^e)$ have degree 0, and p_4 is in the ideal of $\mathrm{S}(\mathfrak{g}^e)$ generated by p_3 and p_5. So, by Theorem 6,(iii), $\mathfrak{g}^e$ is nonsingular, and by Proposition 4,(i), $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1, \dots, p_5$.

At last, e is not good since the nullvariety of $p_1, \dots, p_5$ in $(\mathfrak{g}^e)^*$ has codimension at most 4.

Example 5 In the same way, for the nilpotent element e of $\mathfrak{so}_{11}(\mathbb{k})$ associated with the partition $(3^2, 2^2, 1)$, we can prove that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by polynomials of degree 1, 1, 2, 2, 7, $\mathfrak{g}^e$ is nonsingular but e is not good.

Remark 2 Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or 2ℓ and let $e \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$. Our results imply that if $\ell \leqslant 6$, then either e is good, or e is not good but $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathfrak{g}^e$ is nonsingular.

In particular, there are good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1. Indeed, by Panyushev et al. [26, §3.9], for some nilpotent element e' in $\mathbf{B}_3$, the codimension of $(\mathfrak{g}^{e'})^*_{\mathrm{sing}}$ in $(\mathfrak{g}^{e'})^*$ is one but, in $\mathbf{B}_3$, all nilpotent elements are good. For such nilpotent elements, note that [26, Theorem 0.3] (cf. Theorem 3) cannot be applied.

Example 6 From the rank 7, there are elements that do no satisfy the polynomiality condition. Let e be a nilpotent element of $\mathfrak{so}_{14}(\mathbb{k})$ associated with the partition $(3^2, 2^4)$. Then $\ell = 7$ and the degrees of ${}^e q_1, \dots, {}^e q_7$ are 1, 2, 2, 3, 4, 5, 3 respectively, with $q_1, \dots, q_7$ as in Sect. 4. Using Proposition 4,(ii), we can prove that e does not satisfy the polynomiality condition.

7 A Result of Arakawa-Premet

Let us mention a recent result of Arakawa and Premet, [1], which is related to the problems addressed in the previous sections.

We assume in this section that $\mathbb{k}$ is the field of complex numbers $\mathbb{C}$. Let $\xi \in (\mathfrak{g}^e)^*$ and denote by $\mathcal{A}_{e,\xi}$ be the Mishchenko-Fomenko subalgebra of $\mathrm{S}(\mathfrak{g}^e)$ generated by the derivatives $D_\xi^i({}^e p)$ for $p \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and $i \in \{0, \dots, \deg {}^e p - 1\}$.

Theorem 7 (Panyushev-Yakimova, [24]) *Suppose that the conditions (1) and (2) of Theorem 3 are satisfied and that $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ has codimension $\geqslant 3$ in $(\mathfrak{g}^e)^*$. Then for a regular element $\xi \in (\mathfrak{g}^e)^*$, $\mathcal{A}_{e,\xi}$ is a polynomial algebra in the variables $D_\xi^j({}^e q_i)$, for $i \in \{1, \dots, \ell\}$ and $j \in \{0, \dots, \deg {}^e q_i - 1\}$. Moreover, $\mathcal{A}_{e,\xi}$ is a maximal Poisson-commutative subalgebra of* $\mathrm{S}(\mathfrak{g}^e)$.

Arakawa and Premet proved the following.

Theorem 8 (Arakawa-Premet, [1]) *Under the assumption of Theorem 7, there exists a maximal commutative subalgebra $\hat{\mathcal{A}}_{e,\xi}$ of $U(\mathfrak{g})$ such that* $\mathrm{gr}\, \hat{\mathcal{A}}_{e,\xi} \cong \mathcal{A}_{e,\xi}$.

Theorem 8 was known in the case where $e = 0$. It has been proven by Tarasov [30], and independently by Cherednik, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, by Nazarov-Olshanski [21] for classical $\mathfrak{g}$, and by Rybnikov [29], Chervov-Falqui-Rybnikov [8] and Feigin-Frenkel-Toledano-Laredo [15] for an arbitrary $\mathfrak{g}$.

The main step to prove Theorem 8 is to establish a chiralization of Theorem 3. Namely, Arakawa and Premet proved the following.

Let $\hat{\mathfrak{g}}^e$ be the affine Kac-Moody algebra associated with $\mathfrak{g}^e$ and a certain invariant bilinear form $\langle\cdot\,,\cdot\rangle_e$ on $\mathfrak{g}^e$, that is $\hat{\mathfrak{g}}_e$ is the central extension of the Lie algebra $\mathfrak{g}^e((t)) = \mathfrak{g}^e \otimes \mathbb{C}((t))$ by the one-dimensional center $\mathbb{C}\mathbf{1}$ with commutation relations:

$$[x(m), y(n)] = [x, y](m+n) + m\langle x, y\rangle_e \delta_{m+n,0}\mathbf{1},$$

where $x(m) = x \otimes t^m$ for $m \in \mathbb{Z}$. For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}^e) := U(\hat{\mathfrak{g}}^e) \otimes_{U(\mathfrak{g}^e[t]\oplus\mathbb{C}\mathbf{1})} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}^e[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{g}^e[t]$ acts trivially and $\mathbf{1}$ acts as multiplication by k. The space $V^k(\mathfrak{g}^e)$ is naturally a vertex algebra, and it is called the *universal affine vertex algebra associated with* $\mathfrak{g}^e$ *at level* k. By the PBW theorem, $V^k(\mathfrak{g}^e) \cong U(\mathfrak{g}^e[t^{-1}]t^{-1})$ as $\mathbb{C}$-vector spaces and there is a natural filtration on $V^k(\mathfrak{g}^e)$ such that $\operatorname{gr} V^k(\mathfrak{g}^e) = \mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$. For $a \in V^k(\mathfrak{g}^e)$, denote by $\sigma(a) \in \mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ its symbol. We regard $\mathrm{S}(\mathfrak{g}^e)$ as a subring of $\mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ via the embedding defined by $x \mapsto x \otimes t^{-1}$, $x \in \mathfrak{g}^e$. The translation operator T on the Vertex Poisson algebra $\mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ is the derivation of the ring $\mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ defined by

$$Tx(-m) = mx(-m-1), \qquad T1 = 0.$$

Assume from now that $k = cri$ is the critical level cri, and let $Z(V^{cri}(\mathfrak{g}^e))$ be the center of $V^{cri}(\mathfrak{g}^e)$.

Theorem 9 (Arakawa-Premet, [1]) *Assume that Conditions (1) et (2) of Theorem 3 are satisfied. Then there exist homogeneous elements* $^e\hat{q}_1, \ldots, {}^e\hat{q}_\ell$ *in* $Z(V^{cri}(\mathfrak{g}^e))$ *such that* $\sigma(\hat{q}_i) = {}^eq_i \in \mathrm{S}(\mathfrak{g}^e) \subset \mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$. *Moreover,* $Z(V^{cri}(\mathfrak{g}^e))$ *is a polynomial algebra in the variables* $T^j({}^e\hat{q}_i)$ *with* $j \in \{1, \ldots, \ell\}$ *and* $j \in \{0, 1, \ldots\}$.

The particular case where $e = 0$ is an old result of Feigin-Frenkel, [14]. Arakawa and Premet have used *affine W-algebras* to prove the general case.

Remark 3 It would be interesting to extend the results of Arakawa and Premet to the setting of Theorem 4, that is to the case where only Condition (1) of Theorem 3 is satisfied. We can hope such a generalization at least in the case where we have explicit generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}^e}$, not necessarily of the form ${}^eq_1, \ldots, {}^eq_\ell$ for some generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, as in Remark 1.

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