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Perspectives in Lie Theory

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Preface

The present volume contains contributions related to the INdAM intensive research period which took place in the Centro De Giorgi in Pisa in the period December 2014–February 2015.

The volume is divided in two parts. Part I contains the lectures notes of the four minicourses delivered during the trimester. Part II contains papers contributed by participants.

Here is a brief description of the contents of the lecture notes. The first one by Victor Kac is devoted to giving an introduction to the theory of vertex algebras and Poisson vertex algebras. After a quick review of the basic notions related to vertex algebras, the notes give special emphasis to their classical limit, Poisson vertex algebras, and their applications to the classification of integrable Hamiltonian PDE's and their conservation laws. This starts with the famous paper of Drinfeld and Sokolov from 1985 and the notes contain an up to date report of the state of the art in this subject illustrating results by Kac himself, De Sole and others.

The second set of notes, due to Fyodor Malikov, gives an introduction to the theory of chiral differential operators. In the notes one starts with the, by now classical, localization result of Beilinson and Bernstein which is illustrated in the case of the group SL_2. Inspired by this, via the notion of Chiral Algebroid, the author gives the general notion of algebra of chiral differential operators and illustrates it with a wealth of interesting examples.

The third set of notes is by Vera Serganova. It gives a comprehensive introduction, with plenty of examples, of the theory of finite dimensional representations of basic Lie superalgebras. The notion of a basic Lie superalgebras is recalled and, among other things the author introduces analogues of many of the typical constructions which one performs in the finite dimensional representation theory of semisimple Lie algebras.

Finally, the last set of notes is by Tomoyuki Arakawa. The notes give an introduction to the theory of finite and affine W-algebras and their representation theory. In particular, they provide an outline of the proof of the conjecture of Frenkel, Kac and Wakimoto on the existence and construction of the so called minimal models of W-algebras, which gives rise to rational conformal field theories as in the case of the integrable representations of affine Kac-Moody algebras and the minimal models of the Virasoro algebra.

Part II contains 13 papers which cover a variety of subjects. Some of them are related to the theory of Lie algebras and their representations, affine algebras and vertex algebras. Others relate to the study of braid groups, the topology of hyperplane arrangements, and various applications.

The INdAM intensive period has been organized with the contribution of INdAM, Sapienza Università di Roma, CRM "Ennio de Giorgi", Foundation Compositio Mathematica, Università di Pisa, Università di Bologna, FIRB "Perspectives in Lie Theory", PRIN "Spazi di Moduli e Teoria di Lie" Dipartimento di Matematica di Roma Tor Vergata, Università di Padova, NSF, EMS.

The organizers acknowledge the generous support of INdAM and warmly thanks the CRM De Giorgi for their hospitality.

Pisa, Italy — Filippo Callegaro
Padova, Italy — Giovanna Carnovale
Bologna, Italy — Fabrizio Caselli
Roma, Italy — Corrado De Concini
Roma, Italy — Alberto De Sole
February 2017

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About the Editors

Filippo Callegaro received his PhD in Mathematics from the Scuola Normale Superiore of Pisa in 2007 under the direction of Mario Salvetti. In 2008 he took on a teaching position at the Scuola Normale, before moving to the University of Pisa in 2013. His work focuses on the topology of configuration spaces, cohomology of braid groups and generalizations, and hyperplane and toric arrangements.

Giovanna Carnovale received her PhD from the University of Utrecht in 1999 under the direction of Tom Koornwinder. She has held post-doc positions in Cergy-Pontoise, Paris VI, Antwerp, and Rome Tor Vergata. In 2001 she took on a position in Padua, where she has been an associate professor since 2015. Her research interests are in algebra, especially Lie theory, Hopf algebra theory, and representation theory, as well as their the interplay with geometry.

Fabrizio Caselli received his PhD from the Università di Roma La Sapienza in 2003 under the direction of Francesco Brenti. He has held post-doc positions in Lyon and Rome. In 2006 he took on a position in Bologna, where he has been an associate professor since 2014. His main research interests are in algebra and combinatorics, in particular Kazhdan-Luszig theory, Coxeter groups, complex reflection groups and their representations.

Corrado De Concini received his PhD in Mathematics from the University of Warwick in 1975 under the direction of George Lusztig. He has been a professor of algebra since 1981. Since 1996 he has been a professor at the University of Roma, La Sapienza. He has been a visiting researcher at numerous institutions, including Brandeis University, Mittag Leffler Institute, Tata Institute of Fundamental Research, Harvard University, and MIT. In addition, he has made valuable contributions in several areas of algebra and algebraic geometry, including invariant theory, commutative algebra, algebraic and quantum group theory, the Schottky problem, and hyperplane arrangements.

Alberto De Sole received his PhD in Mathematics from the Massachusetts Institute of Technology in 2003 under the supervision of Victor Kac. He has been a B.P. assistant professor at Harvard University's Department of Mathematics, a researcher at the Department of Mathematics of the University of Rome La Sapienza, and, since 2012, an associate professor at the same University. He has been a visiting researcher at MIT (Boston), IHP (Paris), IHES (Bure sur Yvette, France), and ESI (Vienna). His primary research interests are in algebra, particularly Lie theory, vertex algebras, W-algebras, and their applications to integrable systems, as well as statistical mechanics in physics.

Part I
Lecture Notes

Introduction to Vertex Algebras, Poisson Vertex Algebras, and Integrable Hamiltonian PDE

Victor Kac

Abstract These lectures were given in Session 1: "Vertex algebras, W-algebras, and applications" of INdAM Intensive research period "Perspectives in Lie Theory" at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy, December 9, 2014–February 28, 2015.

Keywords Hamiltonian PDE • Lenard-Magri scheme • Lie conformal algebra • Poisson vertex algebra • Pre-vertex algebra • Quantum field • Variational complex • Vertex algebra • Zhu algebra

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Notation

- $U[z]$: polynomials with coefficients in a vector space U.
- $U[z, z^{-1}]$: Laurent polynomials.
- $U[[z]]$: formal power series.
- $U((z))$: formal Laurent series.
- $U[[z, z^{-1}]]$: bilateral series.
- $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.
- $\mathbb{F}$: the base field, a field of characteristic 0. All vector spaces are considered over $\mathbb{F}$.

About the LaTeX'ing of These Notes

These lecture notes were typeset by Vidas Regelskis (Lectures 1 and 6), Tamás F. Görbe (2), Xiao He (3), Biswajit Ransingh (4) and Laura Fedele (5 and 6).

The author would like to thank all the above mentioned scribes for their work, especially Laura Fedele and Vidas Regelskis for many corrections to the edited manuscript.

1 Lecture 1 (December 9, 2014)

In the first lecture we give the definition of a vertex algebra and explain calculus of formal distributions. We end the lecture by giving two examples of non-commutative vertex algebras: the free boson and the free fermion.

1.1 Definition of a Vertex Algebra

From a physicist's point of view, a vertex algebra can be understood as an algebra of chiral fields of a 2-dimensional conformal field theory. This point of view is explained in my book [17].

From a mathematician's point of view a vertex algebra can be understood as a natural "infinite" analogue of a unital commutative associative differential algebra. Recall that a differential algebra is an algebra V with a derivation T. A simple, but important remark is that a unital algebra V is commutative and associative if and only if

$$\hat{a}\hat{b} = \hat{b}\hat{a}, \quad a, b \in V, \tag{1}$$

where $\hat{a}$ denotes the operator of left multiplication by $a \in V$.

Exercise 1 Prove this remark.

A vertex algebra is roughly a unital differential algebra with a product, depending on a parameter z, satisfying a locality axiom, similar to (1). To be more precise, let me first introduce the notion of a z-algebra. (Sorry for the awkward name, but I was unable to find a better one.)

Definition 1 A *z-algebra* is a vector space V endowed with a bilinear (over $\mathbb{F}$) product, valued in $V((z)), a \otimes b \mapsto a(z)b$, endowed with a derivation T of this product:

(i) $T(a(z)b) = (Ta)(z)b + a(z)Tb$,
  such that the following consistency property holds:
(ii) $(Ta)(z) = \partial_z a(z)$.

Here and further on we denote by $a(z)$ the operator of left multiplication by $a \in V$ in the z-algebra V. Using the standard notation

$$a(z)b = \sum_{n\in\mathbb{Z}} (a_{(n)}b)z^{-n-1}, \tag{2}$$

we can write

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}, \text{ where } a_{(n)} \in \operatorname{End} V. \tag{3}$$

The bilinear (over $\mathbb{F}$) product $a_{(n)}b$ is called the *nth product*. Note that $a(z)$ is an End V*-valued distribution*, i.e., an element of $(\operatorname{End} V)[[z, z^{-1}]]$. Moreover, $a(z)$ is a *quantum field*, i.e., $a_{(n)}b = 0$ for $b \in V$ and sufficiently large n (depending on b).

Remark 1 Axioms (i) and (ii) of a z-algebra imply the following *translation covariance* property of $a(z)$:

$$[T, a(z)] = \partial_z a(z), \quad \text{i.e.,} \quad [T, a_{(n)}] = -na_{(n-1)}, \quad \forall n \in \mathbb{Z}. \tag{4}$$

Moreover, the translation covariance of $a(z)$ and either of the axioms (i) or (ii) in Definition 1 imply the other axiom.

Next, we define a unital z-algebra.

Definition 2 A *unit element* of a z-algebra V is a non-zero vector $1 \in V$, such that

$$1(z)a = a, \text{ and } a(z)1 = a \mod zV[[z]].$$

Lemma 1 *Let V be a vector space, let $1 \in V$ and $T \in \operatorname{End} V$ be such that $T1 = 0$. Then*

(a) For any translation covariant (with respect to T) quantum field $a(z)$, we have $a(z)1 \in V[[z]]$.
(b)

$$a(z)\mathbf{1} = e^{zT}a \ \ (= \sum_{n=0}^{\infty} \frac{z^n}{n!} T^n(a)), \ \textit{where } a = a_{(-1)}1. \tag{5}$$

Proof For (a) we have to prove that $a_{(n)}1 = 0$ for all $n \in \mathbb{Z}_+$. Since $a(z)$ is a quantum field, $a_{(n)}1 = 0$ for $n \geqslant N$ with some $N \in \mathbb{Z}_+$. Also by translation covariance we have $[T, a_{(n)}] = -na_{(n-1)}$ for all $n \in \mathbb{Z}$. Apply both sides of the last equality to 1:

$$[T, a_{(n)}]1 = Ta_{(n)}1 - a_{(n)}T1 = Ta_{(n)}1 = -na_{(n-1)}1. \tag{6}$$

Therefore $a_{(n)}1 = 0$ for $n > 0$ implies $a_{(n-1)}1 = 0$. Hence $a_{(n)}1 = 0$ for all $n \in \mathbb{Z}_+$ and $a(z)1 \in V[[z]]$.

Now we prove (b). By (a), the LHS of (5) lies in $V[[z]]$. Both sides are solutions of the differential equation

$$\frac{df}{dz} = Tf(z), \quad f(z) \in V[[z]]. \tag{7}$$

For the RHS it is obvious, and for the LHS it follows from (4) and $T1 = 0$:

$$\partial_z a(z)1 = Ta(z)1 - a(z)T1 = Ta(z)1. \tag{8}$$

Both sides obviously satisfy the same initial condition $f(0) = a$, hence they are equal. □

Since, $1_{(-1)}1 = 1$ and T is a derivation of nth products, we have in a unital z-algebra:

$$T1 = 0. \tag{9}$$

Note that Lemma 1(a) implies that $a(z)1 \in V[[z]]$, and by Lemma 1(b) one actually has (5). Lemma 1(b) implies that

$$Ta = a_{(-2)}1, \tag{10}$$

so that the derivation T is "built in" the product of a unital z-algebra.

Now we can define a vertex algebra.

Definition 3

(a) A z-algebra is called *local* if

$$(z-w)^{N_{ab}}a(z)b(w)$$
$$= (z-w)^{N_{ab}}b(w)a(z), \text{ for some } N_{ab} \in \mathbb{Z}_+ (\text{depending on } a, b \in V). \quad (11)$$

(b) A *vertex algebra* is a local unital z-algebra.

A frequently asked question is: why one cannot cancel $(z-w)^{N_{ab}}$ on both sides of (11)? As we will see in a moment, the answer is: due to the existence of the delta function. In fact, the case $N_{ab} = 0$ for all $a, b \in V$ is not very interesting, since all such vertex algebras correspond bijectively to unital commutative associate differential algebras, as Exercise 2 below demonstrates.

Example 1 A commutative vertex algebra, i.e., $[a(z), b(w)] = 0$ for all $a, b \in V$, can be constructed as follows. Take V to be a unital commutative associative algebra with a derivation T. Then V is a commutative vertex algebra with the product $a(z)b = (e^{zT}a)b$.

Exercise 2 Check that the above example is indeed a commutative vertex algebra. Using Lemma 1, prove that all commutative vertex algebras are of the form given in Example 1.

Remark 2 A unital z-algebra V is a vector space with unit element 1 and bilinear products $a_{(n)}b$, $n \in \mathbb{Z}$. (Recall that T is obtained by (10).) Through these bilinear products we can naturally define z-algebra homomorphisms/isomorphisms, and subalgebras/ideals. Namely, a *homomorphism* between two z-algebras V and V' is a linear map f such that $f(1) = 1$ and $f(a)_{(n)}f(b) = f(a_{(n)}b)$, $\forall a, b \in V$ and $\forall n \in \mathbb{Z}$. It is an *isomorphism* if it is a homomorphism of z-algebras and also an isomorphism as vector spaces. A *subalgebra* is a subspace W of V which contains 1, such that $a_{(n)}b \in W$, $\forall a, b \in W$ and $\forall n \in \mathbb{Z}$. And an *ideal* is a subspace I such that $a_{(n)}b, b_{(n)}a \in I$, $\forall a \in V$, $\forall b \in I$ and $\forall n \in \mathbb{Z}$. Note that both a subalgebra and an ideal are T-invariant due to (9), and if an ideal I contains 1, then it must be the whole vertex algebra V.

Now I will give another definition of a vertex algebra, in the spirit of quantum field theory, using language closer to physics: a unit element is called a vacuum vector, element of a vector space is called a state, etc.

Definition 4 A vertex algebra is a vector space V (the space of states) with a non-zero vector $|0\rangle$ (the vacuum vector) and a linear map from V to the space of End V-valued quantum fields (the state-field correspondence) $a \mapsto a(z)$, satisfying the following axioms:

vacuum axiom: $|0\rangle(z) = I_V$, $a(z)|0\rangle = a + (Ta)z + \dots$, where $T \in \text{End}\, V$;
translation covariance axiom (4);
locality axiom (11).

Remark 1 demonstrates that a vertex algebra defined in the spirit of differential algebra is a vertex algebra defined in the spirit of quantum field theory. However, in order to prove the converse, one has to show that axiom (ii) of Definition 1 holds. This will follow from the proof of the Extension theorem in Lecture 3.

Definition 5 Given a vertex algebra V, the map of the space of its quantum fields to V, defined by

$$fs\colon a(z) \mapsto a(z)|0\rangle_{z=0} = a_{(-1)}|0\rangle = a, \tag{12}$$

is called the *field-state correspondence*. This map is obviously surjective. If this map is also injective, then the inverse map

$$sf\colon a \mapsto a(z) \tag{13}$$

is called the *state-field correspondence*.

The first fundamental theorem, which allows one to construct non-commutative vertex algebras, is the so-called Extension theorem.

Theorem 1 (Extension Theorem) *Let V be a vector space, $|0\rangle \in V$ a non-zero vector, $T \in \operatorname{End} V$ and*

$$\mathcal{F} = \left\{ a^j(z) = \sum_{n\in\mathbb{Z}} a^j_{(n)} z^{-n-1} \right\}_{j\in J} \tag{14}$$

a collection of End V*-valued quantum fields indexed by a set J. Suppose that the following properties hold:*

(i) (vacuum axiom) $T|0\rangle = 0$,
(ii) (translation covariance) $[T, a^j(z)] = \partial_z a^j(z)$ *for all* $j \in J$,
(iii) (locality) $(z-w)^{N_{ij}}[a^i(z), a^j(w)] = 0$ *for all* $i, j \in J$ *with some* $N_{ij} \in \mathbb{Z}_+$,
(iv) (completeness) $\operatorname{span}\{a^{j_1}_{(n_1)} \cdots a^{j_s}_{(n_s)}|0\rangle \mid j_i \in J,\ n_i \in \mathbb{Z},\ s \in \mathbb{Z}_+\} = V$.

Let $\mathcal{F}_{\max}$ denote the set of all translation covariant quantum fields $a(z)$ such that $a(z)$, $a^j(z)$ is a local pair for all $j \in J$. Then the map

$$fs\colon \mathcal{F}_{\max} \to V, \quad a(z) \mapsto a(z)|0\rangle_{z=0} \tag{15}$$

is bijective and the inverse map $sf\colon V \to \mathcal{F}_{\max}$ endows V with a structure of a vertex algebra (in the sense of Definition 4) with vacuum vector $|0\rangle$ and translation operator T.

Remark 3 By conditions (ii) and (iii) we have $\mathcal{F} \subset \mathcal{F}_{\max}$, hence the name Extension theorem.

Some historical remarks:

- Vertex algebras first appeared implicitly in the paper of Belavin et al. [4] in 1984.

- The first definition of vertex algebras was given by Borcherds [5] in 1986.
- The Extension Theorem was proved in [6]. In [17] a weaker version was given.
- Connection to physics (Wightman axioms of a quantum field theory [20] in the 1950s) is discussed e.g. in [17].

Remark 4 (Super Version) A vertex superalgebra V is a local unital z-superalgebra V, cf. Definition 3. Namely V is a vector superspace

$$V = V_{\bar{0}} \oplus V_{\bar{1}}, \ \{\bar{0}, \bar{1}\} = \mathbb{Z}/2\mathbb{Z}, \tag{16}$$

$a(z)b \in V_{\alpha+\beta}((z))$ if $a \in V_\alpha, b \in V_\beta$, and $TV_\alpha \subset V_\alpha$, $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. An element $a \in V$ has *parity* $p(a) = \alpha$ if $a \in V_\alpha$. Finally, the locality axiom (11) is written as $(z-w)^{N_{ab}}[a(z), b(w)] = 0$, where the commutator is understood in the "super" sense, i.e.

$$[a(z), b(w)] = a(z)b(w) - (-1)^{p(a)p(b)} b(w)a(z).$$

All the identities in the "super" case are obtained from the respective identities in the purely even case by the Koszul-Quillen rule: there is a sign change if the order of two odd elements is reversed; no change otherwise. It is a general convention to drop the adjective "super" in the case of vertex superalgebras.

1.2 Calculus of Formal Distributions

Definition 6 Let U be a vector space. A *U-valued formal distribution* $a(z)$ is an element of $U[[z, z^{-1}]]$:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \quad a_n \in U. \tag{17}$$

The *residue* of $a(z)$ is

$$\operatorname{Res} a(z)dz = a_{-1}. \tag{18}$$

Most often one uses a different indexing of coefficients:

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad \text{so that} \quad a_{(n)} = \operatorname{Res} a(z) z^n dz. \tag{19}$$

Note that $a(z)$ is a linear function on the space of test functions $\mathbb{F}[z, z^{-1}]$:

$$\langle a(z), \varphi(z)\rangle = \operatorname{Res} a(z)\varphi(z)dz \in U, \quad \forall \varphi(z) \in \mathbb{F}[z, z^{-1}], \tag{20}$$

and it is easy to see that one thus gets all linear functions on $\mathbb{F}[z, z^{-1}]$.

A formal distribution in two variables z and w is an element $a(z, w) \in U[[z, z^{-1}, w, w^{-1}]]$.

Definition 7 A formal distribution $a(z, w)$ is called *local* if $(z - w)^N a(z, w) = 0$ for some $N \in \mathbb{Z}_+$.

Example 2 The *formal delta function* $\delta(z, w)$, defined by

$$\delta(z, w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n, \tag{21}$$

is an example of an $\mathbb{F}$-valued formal distribution in two variables. It is local since $(z - w)\delta(z, w) = 0$. In fact, one can write $\delta(z, w)$ as

$$\delta(z, w) = i_{z,w} \frac{1}{z - w} - i_{w,z} \frac{1}{z - w}, \tag{22}$$

where $i_{z,w}$ denotes the expansion in the domain $|z| > |w|$ and $i_{w,z}$ denotes the expansion in the domain $|z| < |w|$, i.e.

$$i_{z,w} \frac{1}{z - w} = z^{-1} \frac{1}{1 - \frac{w}{z}} = \sum_{n \geqslant 0} z^{-n-1} w^n, \quad \text{and}$$

$$i_{w,z} \frac{1}{z - w} = -w^{-1} \frac{1}{1 - \frac{z}{w}} = -\sum_{n < 0} z^{-n-1} w^n. \tag{23}$$

The following formula, which is derived by differentiating (21) and (22) $n \in \mathbb{Z}_+$ times, will be useful:

$$\frac{\partial_w^n \delta(z, w)}{n!} = i_{z,w} \frac{1}{(z - w)^{n+1}} - i_{w,z} \frac{1}{(z - w)^{n+1}} = \sum_{j \in \mathbb{Z}} \binom{j}{n} w^{j-n} z^{-j-1}. \tag{24}$$

Let us list some properties of the formal delta function, which are straightforward by (24):

1. $(z - w)^m \frac{\partial_w^n \delta(z, w)}{n!} = \begin{cases} \frac{\partial_w^{n-m} \delta(z, w)}{(n - m)!} & \text{if } n \geqslant m \geqslant 0, \\ 0, & \text{if } m > n, \end{cases}$
2. $\delta(z, w) = \delta(w, z)$,
3. $\partial_z \delta(z, w) = -\partial_w \delta(z, w)$,
4. $a(z)\delta(z, w) = a(w)\delta(z, w)$, where $a(z)$ is any formal distribution,
5. $\operatorname{Res} a(z)\delta(z, w)dz = a(w)$.

Theorem 2 (Decomposition Theorem) *Any local formal distribution $a(z, w)$ can be uniquely decomposed as a finite sum of derivatives of the formal delta function* *with formal distributions in w as coefficients:*

$$a(z,w) = \sum_{j\geq 0} c^j(w)\frac{\partial_w^j \delta(z,w)}{j!}. \tag{25}$$

Moreover

$$c^j(w) = \operatorname{Res} a(z,w)(z-w)^j dz. \tag{26}$$

Proof Multiply both sides of (25) by $(z-w)^j$ and take residues. Using properties of the delta function listed above we obtain (26). To show (25) we set

$$b(z,w) = a(z,w) - \sum_{j\geq 0} c^j(w)\frac{\partial_w^j \delta(z,w)}{j!} \tag{27}$$

with $c^j(w)$ given by (26). It is immediate that

$$\operatorname{Res} b(z,w)(z-w)^j dz = 0 \quad \text{for all} \quad j \in \mathbb{Z}_+, \tag{28}$$

hence

$$b(z,w) = \sum_{n\geq 0} b_n(w) z^n. \tag{29}$$

By definition $b(z,w)$ is local, therefore (29) implies that $b(z,w) = 0$. □

Remark 5 If we have a local pair $a(z), b(z) \in \mathfrak{g}[[z,z^{-1}]]$, where $\mathfrak{g}$ is a Lie (super)algebra (i.e. $[a(z), b(w)]$ is a local formal distribution in z and w), then, by the Decomposition theorem, we have:

$$[a(z), b(w)] = \sum_{j\geq 0} (a(w)_{(j)} b(w))\frac{\partial_w^j \delta(z,w)}{j!}, \tag{30}$$

where the sum over j is finite, and

$$\mathfrak{g}[[w,w^{-1}]] a(w)_{(j)} b(w) := \operatorname{Res}(z-w)^j [a(z), b(w)] dz \ \ (= c^j(w)). \tag{31}$$

Using (24) and comparing the coefficients of $z^m w^n$ on both sides of (30), we find

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}, \quad \forall m, n \in \mathbb{Z}. \tag{32}$$

1.3 Free Boson and Free Fermion Vertex Algebras

Example 3 (Free Boson) Let $B = \mathbb{F}[x_1, x_2, \ldots]$, $|0\rangle = 1$, $T = \sum_{j\geqslant 2} jx_j \frac{\partial}{\partial x_{j-1}}$ and

$$\mathcal{F} = \left\{ a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \right\}, \quad \text{where} \quad a_{(n)} = \begin{cases} \dfrac{\partial}{\partial x_n}, & \text{if } n > 0, \\ -nx_{-n}, & \text{if } n < 0, \\ 0, & \text{if } n = 0, \end{cases} \tag{33}$$

so that

$$[a_{(m)}, a_{(n)}] = m\delta_{m,-n}, \quad \forall m, n \in \mathbb{Z}. \tag{34}$$

The quantum field $a(z)$ is called the *free boson* field. Since (34) is equivalent to

$$[a(z), a(w)] = \partial_w \delta(z, w), \tag{35}$$

we have

$$(z-w)^2 [a(z), a(w)] = 0, \tag{36}$$

i.e., $a(z)$ is local with itself.

The translation covariance of the free boson field $a(z)$, that is $[T, a_{(n)}] = -na_{(n-1)}$, $\forall n \in \mathbb{Z}$, can be verified directly. Vacuum axiom and completeness are obviously satisfied. Locality is (36). So, by the Extension theorem, B carries a vertex algebra structure.

Example 4 (Free Fermion) Let $F = \Lambda[\xi_1, \xi_2, \ldots]$ be a Grassmann superalgebra, i.e.,

$$\xi_i \xi_j = -\xi_j \xi_i, \ \ p(\xi_i) = \bar{1}\,.$$

Let $|0\rangle = 1$ and $T = \sum_{j\geqslant 1} j\xi_{j+1} \frac{\partial}{\partial \xi_j}$, where $\frac{\partial}{\partial \xi_j}$ is an odd derivation of the superalgebra F (i.e. $\frac{\partial (ab)}{\partial \xi_j} = \frac{\partial a}{\partial \xi_j} b + (-1)^{p(a)} \frac{\partial b}{\partial \xi_j}$), such that

$$\frac{\partial}{\partial \xi_j} \xi_i = \delta_{ij}. \tag{37}$$

Set

$$\mathcal{F} = \left\{ \varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_{(n)} z^{-n-1} \right\}, \quad \text{where} \quad \varphi_{(n)} = \begin{cases} \dfrac{\partial}{\partial \xi_{n+1}}, & \text{if } n \geqslant 0, \\ \xi_{-n}, & \text{if } n < 0, \end{cases} \tag{38}$$

then

$$[\varphi_{(m)}, \varphi_{(n)}] = \delta_{m,-n-1}, \quad \forall m, n \in \mathbb{Z}. \tag{39}$$

The odd quantum field $\varphi(z)$ is called the *free fermion* field. Since (39) is equivalent to

$$[\varphi(z), \varphi(w)] = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n = \delta(z, w), \tag{40}$$

$\varphi(z)$ is local to itself. As in Example 3, the vacuum axiom and completeness are immediate. Translation covariance follows from the exercise below.

Exercise 3 Show that the free fermion field is translation covariant, i.e.,

$$[T, \varphi_{(n)}] = -n\varphi_{(n-1)}, \quad \forall n \in \mathbb{Z}. \tag{41}$$

2 Lecture 2 (December 11, 2014)

In the first lecture we discussed the two simplest examples of non-commutative vertex algebras (see Examples 3 and 4). In this lecture we will consider further important examples, among them a generalization of those two mentioned previously. First, we need to introduce the necessary notions.

2.1 Formal Distribution Lie Algebras and Their Universal Vertex Algebras

Definition 8 A *formal distribution Lie (super)algebra* is a pair $(\mathfrak{g}, \mathcal{F})$, where $\mathfrak{g}$ is a Lie (super)algebra and $\mathcal{F}$ is a collection of pairwise local $\mathfrak{g}$-valued formal distributions $a^j(z) = \sum_{n \in \mathbb{Z}} a^j_{(n)} z^{-n-1}$, $j \in J$, such that the coefficients $\{a^j_{(n)} \mid j \in J,\ n \in \mathbb{Z}\}$ span $\mathfrak{g}$. A formal distribution Lie (super)algebra $(\mathfrak{g}, \mathcal{F})$ is called *regular* if:

(i) the $\mathbb{F}[\partial_z]$-span of $\mathcal{F}$ is closed under all *nth products* for $n \in \mathbb{Z}_+$,

$$a(z)_{(n)} b(z) := \operatorname{Res}(w - z)^n [a(w), b(z)] dw, \tag{42}$$

i.e., if $a(z)$ and $b(z)$ are elements of the form $\sum_{j \in J} f_j(\partial_z) a^j(z)$, where $f_j(\partial_z) \in \mathbb{F}[\partial_z]$, and only finitely many $f_j(\partial_z) \neq 0$, then their nth product for $n \in \mathbb{Z}_+$ is still an element of the same form.

(ii) there exists a derivation $T \in \operatorname{Der} \mathfrak{g}$ such that

$$T(a^j(z)) = \partial_z a^j(z), \text{ i.e., } T(a^j_{(n)}) = -na^j_{(n-1)} \quad \forall j \in J. \tag{43}$$

The *annihilation subalgebra* of $\mathfrak{g}$ is $\mathfrak{g}_- = \operatorname{span}\{a^j_{(n)} \mid j \in J,\ n \in \mathbb{Z}_+\}$.

Exercise 4 Show that $\mathfrak{g}_-$ is a T-invariant subalgebra of $\mathfrak{g}$. (*Hint*: use the commutation formulas (32) and (43).)

The following theorem allows one to construct vertex algebras via the Extension theorem. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

Theorem 3 *Let $(\mathfrak{g}, \mathcal{F}_0)$ be a regular formal distribution Lie algebra, and let $\mathfrak{g}_-$ be the annihilation subalgebra. Let $V = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}_-$ (also known as the induced $\mathfrak{g}$-module $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{g}_-}(\mathbb{F})$) and let π be the representation of $\mathfrak{g}$ in V induced via the left multiplication. Let $|0\rangle = \bar{1}$ be the image of 1 in V and $T \in \operatorname{End} V$ be the endomorphism of V induced by the derivation of $\mathfrak{g}$. Let $\mathcal{F}$ be the collection of $\operatorname{End} V$-valued formal distributions*

$$\mathcal{F} = \left\{ \pi\left(a^j(z)\right) = \sum_{n\in\mathbb{Z}} \pi\left(a^j_{(n)}\right) z^{-n-1} \,\middle|\, a^j(z) \in \mathcal{F}_0,\ j \in J \right\}. \tag{44}$$

Then $\mathcal{F}$ consists of quantum fields and $(V, |0\rangle, T, \mathcal{F})$ satisfies the conditions of the Extension theorem, hence V is a vertex algebra, which we denote by $V(\mathfrak{g}, \mathcal{F}_0)$.

Proof The only non-obvious part is to check that all $\pi\left(a^j(z)\right)$ are quantum fields, i.e., $\pi\left(a^j(z)\right)v \in V((z))$ for each $v \in V$. Due to the *PBW* theorem, it is sufficient to check it for vectors of the following form (we use the same notation for elements in $U(\mathfrak{g})$ and their images in V):

$$v = a^{j_1}_{(n_1)} \cdots a^{j_s}_{(n_s)}|0\rangle, \text{ where } j_1, \ldots, j_s \in J. \tag{45}$$

We argue by induction on s. For $s = 0$ we have $v = |0\rangle$, hence

$$\pi\left(a^j(z)\right)|0\rangle = \sum_{n\in\mathbb{Z}} \pi\left(a^j_{(n)}\right) z^{-n-1}|0\rangle = \sum_{n<0} \pi\left(a^j_{(n)}\right) z^{-n-1} \in V[[z]]. \tag{46}$$

The last equality follows from the fact that $a^j_{(n)}|0\rangle = 0$ for $n \geqslant 0$. We proceed by proving the induction step:

$$\pi\left(a^j(z)\right) a^{j_1}_{(n_1)} \cdots a^{j_s}_{(n_s)}|0\rangle = [a^j(z), a^{j_1}_{(n_1)}] a^{j_2}_{(n_2)} \cdots a^{j_s}_{(n_s)}|0\rangle + a^{j_1}_{(n_1)} a^j(z) a^{j_2}_{(n_2)} \cdots a^{j_s}_{(n_s)}|0\rangle. \tag{47}$$

By assumption of induction, the second term in the right-hand side is in $V((z))$, so we only need to show that the first term is also in $V((z))$. Now recall the commutation formula (32). We have

$$[a^j(z), a^{j_1}_{(n_1)}] = \sum_{m\in\mathbb{Z}} \sum_{k\geqslant 0} \binom{m}{k} (a^j_{(k)} a^{j_1})_{(m+n_1-k)} z^{-m-1}, \tag{48}$$

where $(a^j_{(k)} a^{j_1})_{(m+n_1-k)}$ is the Fourier coefficient of the formal distribution $a^j(z)_{(k)} a^{j_1}(z)$. By the regularity property, we know that $a^j(z)_{(k)} a^{j_1}(z)$ is contained in the $\mathbb{F}[\partial_z]$-span of $\mathcal{F}$, thus we can assume that

$$a^j(z)_{(k)} a^{j_1}(z) = \sum_{l\in J} f_l^k(\partial_z) a^l(z). \tag{49}$$

Since $a^j(z), a^{j_1}(z)$ is a local pair, we know that there exists an integer $N \in \mathbb{Z}_+$ such that $a^j(z)_{(k)} a^{j_1}(z) = 0$ for $k \geqslant N$. This allows us to rewrite formula (48) as follows,

$$[a^j(z), a^{j_1}_{(n_1)}] = \sum_{0\leqslant k\leqslant N} \sum_{m\in\mathbb{Z}} \binom{m}{k} \Big(\sum_{l\in J} f_l^k(\partial_z) a^l(z)\Big)_{(m+n_1-k)} z^{-m-1}. \tag{50}$$

By assumption of induction, for each k,

$$\sum_{m\in\mathbb{Z}} \Big(\sum_{l\in J} f_l^k(\partial_z) a^l(z)\Big)_{(m+n_1-k)} z^{-m-1} a^{j_2}_{(n_2)} \cdots a^{j_s}_{(n_s)} |0\rangle \in V((z)) \tag{51}$$

thus the first term in the right-hand side of (47) is also in $V((z))$. □

Remark 6 Recall that by the Decomposition theorem for any local pair $a(z), b(w)$ we have

$$[a(z), b(w)] = \sum_{j\geqslant 0} (a(w)_{(j)} b(w)) \frac{\partial_w^j \delta(z,w)}{j!}, \tag{52}$$

which is equivalent to the commutator formula

$$[a_{(m)}, b_{(n)}] = \sum_{j\geqslant 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}, \quad \forall m, n \in \mathbb{Z}, \tag{53}$$

where $(a_{(j)} b)(w) = a(w)_{(j)} b(w)$ is given by (31). This, along with the obvious formula

$$\big(\partial_w a(w)\big)_{(n)} = -n a(w)_{(n-1)}, \tag{54}$$

allows us to convert the commutator formula into the decomposition formula, thereby establishing locality.

Let us now discuss the next important example of a non-commutative vertex algebra.

Example 5 Let $\mathfrak{g} = \mathrm{Vir}$ be the Virasoro algebra with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}C, \quad [C, L_m] = 0, \quad \forall m, n \in \mathbb{Z}. \tag{55}$$

Consider the formal distribution

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{56}$$

so that $L_{(n)} = L_{n-1}$. Then the commutation relations (55) can be written in the equivalent form

$$[L(z), L(w)] = \partial_w L(w)\delta(z,w) + 2L(w)\partial_w\delta(z,w) + \frac{C}{2}\partial_w^3\delta(z,w). \tag{57}$$

Indeed, by (57) we have: $L_{(0)}L = \partial L$, $L_{(1)}L = 2L$, $L_{(3)}L = \frac{C}{2}$, and $L_{(j)}L = 0$ for all other $j \geqslant 0$. Hence by (53) and (54), (57) is equivalent to (55). It follows that $L(z)$ is local with itself, hence $(\mathrm{Vir}, \{L(z), C\})$ is a formal distribution Lie algebra. Furthermore, it is regular. There are two conditions (1) and (2) we need to check: (1) is obvious, for (2) take $T = \mathrm{ad}L_{-1}$, then $[L_{-1}, L_n] = (-1-n)L_{n-1}$, which gives (43). The annihilation subalgebra is

$$\mathrm{Vir}_- = \sum_{n\geqslant -1} \mathbb{F}L_n. \tag{58}$$

So, by Theorem 3 and the Extension theorem, we get the associated vertex algebra

$$V(\mathrm{Vir}, \{L(z), C\}), \tag{59}$$

called the *universal Virasoro vertex algebra*. One can make it slightly smaller by taking $c \in \mathbb{F}$ and factorizing by the ideal generated by $(C - c)$. Let V^c stand for the corresponding factor vertex algebra, which is called the *universal Virasoro vertex algebra with central charge c*.

Remark 7 V^c can be non-simple for certain values of c. Namely, V^c is non-simple if and only if [16]

$$c = 1 - \frac{6(p-q)^2}{pq}, \quad \text{with} \quad p, q \in \mathbb{Z}_{\geqslant 2} \text{ coprime}. \tag{60}$$

Exercise 5 The vertex algebra V^c has a unique maximal ideal J^c.

Let $V_c = V^c/J^c$. Since c in (60) is symmetric in p and q we may assume that $p < q$. The smallest example $p = 2$, $q = 3$ gives $c = 0$; V_0 is the one-dimensional vertex algebra. The next example is $p = 3$, $q = 4$ when $c = 1/2$; the vertex algebra $V_{\frac{1}{2}}$ is related to the Ising model. The simple vertex algebras V_c with c of the form (60) are called *discrete series vertex algebras*. They play a fundamental role in conformal field theory [4].

Example 6 Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a non-degenerate symmetric invariant bilinear form $(.|.)$. Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{F}K$ be the associated Kac-Moody affinization, with commutation relations

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a|b)K, \quad [K, at^m] = 0, \tag{61}$$

where $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Let $a(z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$ and $\mathcal{F} = \{a(z)\}_{a\in\mathfrak{g}} \cup \{K\}$ be an (infinite) collection of formal distributions. The commutation relations (61) are equivalent to

$$[a(z), b(w)] = [a, b](w)\delta(z, w) + (a|b)\partial_w\delta(z, w)K, \quad [K, a(z)] = 0. \tag{62}$$

Hence $\mathcal{F}$ is a local family. So $(\widehat{\mathfrak{g}}, \mathcal{F})$ is a formal distribution Lie algebra. The annihilation subalgebra is $\widehat{\mathfrak{g}}_- = \mathfrak{g}[t]$.

Exercise 6 Show that the formal distribution Lie algebra $(\widehat{\mathfrak{g}}, \mathcal{F})$ defined above is regular with $T = -\partial_t$.

The associated vertex algebra $V(\widehat{\mathfrak{g}}, \mathcal{F})$ is called the *universal affine vertex algebra associated to* $\big(\mathfrak{g}, (.|.)\big)$. Again, it can be made a little smaller by taking $k \in \mathbb{F}$ and considering

$$V^k(\mathfrak{g}) = V(\widehat{\mathfrak{g}}, \mathcal{F})/(K - k)V(\widehat{\mathfrak{g}}, \mathcal{F}), \tag{63}$$

which is called the *universal affine vertex algebra of level k*. There are certain values of k for which $V^k(\mathfrak{g})$ is non-simple (it is a known set of rational numbers [16]).

Example 7 Let A be a finite dimensional vector superspace with a non-degenerate skewsymmetric bilinear form $\langle .|. \rangle$:

$$\langle a|b\rangle = -(-1)^{p(a)p(b)}\langle b|a\rangle, \quad a, b \in A. \tag{64}$$

Take the associated Clifford affinization

$$\hat{A} = A[t, t^{-1}] + \mathbb{F}K, \tag{65}$$

with commutation relations

$$[at^m, bt^n] = \delta_{m,-n-1}\langle a|b\rangle K, \quad [K, at^m] = 0, \ a, b \in A. \tag{66}$$

Consider the formal distributions

$$a(z) = \sum_{n\in\mathbb{Z}} (at^n) z^{-n-1}, \quad a \in A, \tag{67}$$

and define $\mathcal{F}$ to be

$$\mathcal{F} = \{a(z)\}_{a\in A} \cup \{K\}. \tag{68}$$

Then the commutation relations (66) are equivalent to

$$[a(z), b(w)] = \langle a|b\rangle \delta(z, w) K, \ [K, a(z)] = 0. \tag{69}$$

Hence $\mathcal{F}$ is a local family, and $(\hat{A}, \mathcal{F})$ is a formal distribution Lie superalgebra. Its annihilation subalgebra is $\hat{A}_- = A[t]$. Furthermore, $(\hat{A}, \mathcal{F})$ is regular with $T = -\partial_t$ and

$$F(A) = V(\hat{A}, \mathcal{F})/(K-1)V(\hat{A}, \mathcal{F}) \tag{70}$$

is the associated vertex algebra called the *vertex algebra of free superfermions*.

Exercise 7

(1) If A is a 1-dimensional odd superspace we get the free fermion vertex algebra $F = F(A)$.
(2) If $\mathfrak{g}$ is the 1-dimensional Lie algebra $\mathbb{F}$, with bilinear form $(a|b) = ab$ and level $k = 1$, then we get the free boson vertex algebra $B = V^1(\mathbb{F})$.

Exercise 8 Show that the vertex algebra $F(A)$ is always simple.

2.2 *Formal Cauchy Formulas and Normally Ordered Product*

We proceed by proving some statements which are analogous to the Cauchy formula and are true for any formal distribution. Let U be a vector space and $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ be a U-valued formal distribution. We call

$$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1} \tag{71}$$

the *creation part* or *"positive" part* of $a(z)$ and

$$a(z)_- = \sum_{n\geqslant 0} a_{(n)} z^{-n-1} \tag{72}$$

the *annihilation part* or *"negative" part* of $a(z)$. Note that $\partial_z\big(a(z)_\pm\big) = \big(\partial_z a(z)\big)_\pm$.

Proposition 1 *Formal Cauchy formulas can be written as follows:*

(a) For the "positive" and "negative" parts of $a(z)$ *we have*

$$a(w)_+ = \operatorname{Res} a(z) i_{z,w} \frac{1}{z-w} dz, \qquad -a(w)_- = \operatorname{Res} a(z) i_{w,z} \frac{1}{z-w} dz. \tag{73}$$

(b) For the derivatives of $a(z)_\pm$ *we have*

$$\begin{aligned} \frac{1}{n!}\,\partial_w^n a(w)_+ &= \operatorname{Res} a(z) i_{z,w} \frac{1}{(z-w)^{n+1}} dz, \\ -\frac{1}{n!}\,\partial_w^n a(w)_- &= \operatorname{Res} a(z) i_{w,z} \frac{1}{(z-w)^{n+1}} dz. \end{aligned} \tag{74}$$

Proof Use property (5) of the delta function and (22) to get

$$a(w) = \operatorname{Res} a(z)\delta(z,w)dz = \operatorname{Res} a(z)\left(i_{z,w}\frac{1}{z-w} - i_{w,z}\frac{1}{z-w}\right)dz. \tag{75}$$

Collect the (non-negative) powers of w on both sides to get (a). Differentiating (a) by w n times gives (b). □

Multiplying two quantum fields naïvely would lead to divergences. The next definition is introduced to circumvent this problem.

Definition 9 The *normally ordered product* of End V-valued quantum fields $a(z)$ and $b(z)$ is defined by

$$: a(z)b(z) : = a(z)_+ b(z) + (-1)^{p(a)p(b)} b(z)a(z)_-. \tag{76}$$

It must be proved that $: a(z)b(z) :$ is an "honest" quantum field, i.e., all the divergences are removed.

Proposition 2 *If* $a(z)$ *and* $b(z)$ *are quantum fields then so is* $: a(z)b(z) :$.

Proof Apply $: a(z)b(z) :$, defined by (76), to any vector $v \in V$:

$$: a(z)b(z) : v = a(z)_+ b(z)v + (-1)^{p(a)p(b)} b(z)a(z)_- v. \tag{77}$$

Since $b(z)$ is assumed to be a quantum field, $b(z)v$ in the first term of the right-hand side of (77) is a Laurent series by definition. The creation part $a(z)_+$ has only non-negative powers of z, therefore $a(z)_+ b(z)v$ is still a Laurent series. In the second term $a(z)_- v$ consists of finitely many terms with negative powers, i.e., it is a Laurent polynomial. Now $b(z)a(z)_- v$ is a Laurent series multiplied by a Laurent polynomial which is still a Laurent series. Hence we proved that $: a(z)b(z) :$ is a sum (or a difference) of two Laurent series, thus it is a Laurent series. □

Exercise 9 Let $a(z)$ and $b(z)$ be quantum fields. Show that their nth product $a(z)_{(n)}b(z)$, $n \in \mathbb{Z}_+$ and derivatives $\partial_z a(z)$, $\partial_z b(z)$ are also quantum fields.

On the space of quantum fields we have defined $a(w)_{(n)}b(w)$ for $n \geqslant 0$. Introduce

$$a(w)_{(-n-1)}b(w) = \frac{1}{n!} : \partial_w^n a(w) b(w) :, \tag{78}$$

so that $a(w)_{(-1)}b(w) =: a(w)b(w) :$. Thus for each $n \in \mathbb{Z}$ we have the nth product $a(w)_{(n)}b(w)$. Using the formal Cauchy formulas above, we get the *unified formulas for all nth products* of quantum fields

$$a(w)_{(n)}b(w) = \mathrm{Res}\left(a(z)b(w)i_{z,w}(z-w)^n - (-1)^{p(a)p(b)}b(w)a(z)i_{w,z}(z-w)^n\right)dz, \quad n \in \mathbb{Z}. \tag{79}$$

Remark 8 For a local pair of quantum fields physicists write

$$a(z)b(w) = \sum_{n\in\mathbb{Z}} \frac{a(w)_{(n)}b(w)}{(z-w)^{n+1}}. \tag{80}$$

This way of writing is useful but might be confusing, since different parts of it are expanded in different domains. Therefore it is worth giving a rigorous interpretation of (80) by writing

$$a(z)b(w) = \sum_{n\geqslant 0} a(w)_{(n)}b(w)\, i_{z,w}\frac{1}{(z-w)^{n+1}} + : a(z)b(w) : \tag{81}$$

and

$$(-1)^{p(a)p(b)}b(w)a(z) = \sum_{n\geqslant 0} a(w)_{(n)}b(w)\, i_{w,z}\frac{1}{(z-w)^{n+1}} + : a(z)b(w) : \tag{82}$$

By taking the difference (81)–(82) we get

$$[a(z), b(w)] = \sum_{j\in\mathbb{Z}_+} \left(a(w)_{(j)}b(w)\right)\frac{\partial_w^j \delta(z,w)}{j!}. \tag{83}$$

Conversely, by separating the negative (resp. non-negative) powers of z in (83) we get (81) [resp. (82)]. We still need to explain (80) for negative n. By Taylor's formula in the domain $|z-w| < |w|$ ([17], (2.4.3)), we have

$$: a(z)b(w) := \sum_{n\geqslant 0} : \partial_w^n a(w)b(w) : \frac{(z-w)^n}{n!} = \sum_{n\geqslant 0} \left(a(w)_{(-n-1)}b(w)\right)(z-w)^n, \tag{84}$$

i.e., the nth products for negative n are "contained" in the normally ordered product.

2.3 Bakalov's Formula and Dong's Lemma

Locality of the pair $a(z), b(z)$ of End V-valued quantum fields means that

$$(z-w)^N a(z)b(w) = (-1)^{p(a)p(b)}(z-w)^N b(w)a(z) \text{ for some } N \in \mathbb{Z}_+. \tag{85}$$

Denote either side of this equality by $F(z,w)$. Then for each $k \in \mathbb{Z}_+$ we have

$$\begin{aligned}\operatorname{Res} F(z,w)\frac{\partial_w^k \delta(z,w)}{k!}dz &= \operatorname{Res} F(z,w)\, i_{z,w}\frac{1}{(z-w)^{k+1}}dz \\ &\quad - \operatorname{Res} F(z,w)\, i_{w,z}\frac{1}{(z-w)^{k+1}}dz.\end{aligned} \tag{86}$$

The first term of the left-hand side of (86) is

$$\operatorname{Res} a(z)b(w)\, i_{z,w}(z-w)^{N-k-1}dz, \tag{87}$$

while the second term of the right-hand side of (86) is

$$-\operatorname{Res} a(z)b(w)\, i_{w,z}(z-w)^{N-k-1}dz. \tag{88}$$

Applying the unified formula (79) the sum of (87) and (88) can be written as

$$a(w)_{(N-k-1)}b(w). \tag{89}$$

Hence we obtain *Bakalov's formula*

$$a(w)_{(N-k-1)}b(w) = \operatorname{Res} F(z,w)\frac{\partial_w^k \delta(z,w)}{k!}dz = \frac{1}{k!}(\partial_z^k F(z,w))|_{z=w}, \tag{90}$$

which holds for each non-negative integer k and sufficiently large positive integer N. The second equality follows from the first one by properties (3) and (5) of the formal delta function.

Remark 9 Since $a(z)$ and $b(z)$ are quantum fields, it follows from (85) that $F(z,w)v$ lies in the space $V[[z,w]][z^{-1},w^{-1}]$ for each $v \in V$. Hence (90) makes sense.

Remark 10 It follows from (85) that if we replace $a(z)$ in this equation by $\partial_z^k a(z)$ for some positive integer k, then it still holds with N replaced by $N+k$.

Lemma 2 (Dong) *If $a(z)$, $b(z)$ and $c(z)$ are pairwise mutually local quantum fields, then $a(z)_{(n)}b(z)$, $c(z)$ is a local pair for any $n \in \mathbb{Z}$.*

Proof [1] It suffices to prove that for N and k as in (90) we have for some $M \in \mathbb{Z}_+$:

$$(z_2-z_3)^M(a(z_2)_{(N-k-1)}b(z_2))c(z_3) = \pm(z_2-z_3)^M c(z_3)a(z_2)_{(N-k-1)}b(z_2), \tag{91}$$

where $\pm$ is the Koszul-Quillen sign, if (85) holds for all three pairs (a, b), (a, c) and (b, c). We let $M = 2N + k$. By Bakalov's formula (90), the left-hand side of (91) is equal to

$$\frac{1}{k!}(z_2 - z_3)^{2N+k} \left(\partial_{z_1}^k((z_1 - z_2)^N a(z_1)b(z_2)c(z_3))\right)\Big|_{z_1=z_2}$$
$$=(z_2 - z_3)^{2N+k} \sum_{i=0}^{k} \binom{N}{i} (z_1 - z_2)^{N-i} \left(\frac{\partial_{z_1}^{k-i}}{(k-i)!} a(z_1)\right) b(z_2)c(z_3)\Big|_{z_1=z_2}$$
$$= \sum_{i=0}^{k} \binom{N}{i} (z_2 - z_3)^N (z_1 - z_3)^{N+k} (z_1 - z_2)^{N-i} \left(\frac{\partial_{z_1}^{k-i}}{(k-i)!} a(z_1)\right) b(z_2)c(z_3)\Big|_{z_1=z_2}.$$

Due to (85) for the pair (b, c), we can permute $c(z_3)$ with $b(z_2)$ (up to the Koszul-Quillen sign), and after that similarly permute $c(z_3)$ and the $(k - i)$th derivative of $a(z_1)$, using Remark 10. We thus obtain the right-hand side of (91). □

3 Lecture 3 (December 16, 2014)

In this lecture, we will prove the Extension theorem, the Borcherds identity and the skewsymmetry. We will also introduce the concepts of conformal vector, conformal weight and Hamiltonion operators. In the end, we give some properties of the Formal Fourier Transform.

3.1 Proof of the Extension Theorem

First of all, let us give a name for the data which appeared in the Extension theorem.

Definition 10 A *pre-vertex algebra* is a quadruple $\{V, |0\rangle, T, \mathcal{F} = \{a^j(z) = \sum_{n\in\mathbb{Z}} a^j_{(n)} z^{-n-1}\}_{j\in J}\}$, where V is a vector space with a non-zero element $|0\rangle$, $T \in \operatorname{End} V$ and $\mathcal{F}$ is a collection of quantum fields with values in $\operatorname{End} V$ satisfying the following conditions:

(i) (vacuum axiom) $T|0\rangle = 0$,
(ii) (translation covariance) $[T, a^j(z)] = \partial_z a^j(z)$ for all $j \in J$,
(iii) (locality) $(z - w)^{N_{ij}}[a^i(z), a^j(w)] = 0$ for all $i, j \in J$ with some $N_{ij} \in \mathbb{Z}_+$,
(iv) (completeness) $\operatorname{span}\{a^{j_1}_{(n_1)} \cdots a^{j_s}_{(n_s)}|0\rangle \mid j_i \in J,\ n_i \in \mathbb{Z},\ s \in \mathbb{Z}_+\} = V$.

Let $\{V, |0\rangle, T, \mathcal{F}\}$ be a pre-vertex algebra. Define

$$\mathcal{F}_{\min} = \text{span}\left\{ \left(a^{j_1}(z)_{(n_1)}\left(a^{j_2}(z)_{(n_2)} \cdots \left(a^{j_s}(z)_{(n_s)} I_V\right) \cdots\right) \mid n_i \in \mathbb{Z},\ j_i \in J,\ s \in \mathbb{Z}_+ \right\}, \tag{92}$$

where I_V is the constant field equal to the identity operator I_V on V. Let, as in Lecture 1, $\mathcal{F}_{\max}$ be the set of all translation covariant quantum fields $a(z)$, such that $a(z), a^j(z)$ is a local pair for all $j \in J$. The following is a more precise version of the Extension theorem, stated in Lecture 1.

Theorem 4 (Extension Theorem) *For a pre-vertex algebra $\{V, |0\rangle, T, \mathcal{F}\}$, let $\mathcal{F}_{\min}, \mathcal{F}_{\max}$ be defined as above, then we have,*

(a) $\mathcal{F}_{\min} = \mathcal{F}_{\max}$,
(b) The map

$$fs\ :\ \mathcal{F}_{\max} \longrightarrow V, \quad a(z) \longmapsto a(z)|0\rangle\big|_{z=0} \tag{93}$$

is well-defined and bijective. Denote by sf the inverse map.
(c) The z-product $a(z)b := sf(a)b$ endows V with a vertex algebra structure, which extends the pre-vertex algebra structure.

Remark 11

(1) The map fs is called the *field-state correspondence* since it sends a field to a vector in V, called a "state" in physics. Its inverse map, called the *state-field correspondence*, is denoted by

$$sf\ :\ V \to \mathcal{F}_{\max}, \quad a \mapsto a(z). \tag{94}$$

(2) Denote by $\mathcal{F}_{\text{tc}} = \{a(z) \mid [T, a(z)] = \partial_z a(z)\}$ the space of translation covariant quantum fields. By Lemma 1, $a(z)|0\rangle \in V[[z]]$ for $a(z) \in \mathcal{F}_{\text{tc}}$, hence $fs(a(z)) \in V$ is well-defined.

Lemma 3 *$\mathcal{F}_{\text{tc}}$ contains I_V, it is ∂_z-invariant and is closed under all nth product, i.e., $a(z)_{(n)}b(z) \in \mathcal{F}_{\text{tc}}$ for any $n \in \mathbb{Z}$ if $a(z), b(z) \in \mathcal{F}_{\text{tc}}$.*

Proof Since $[T, I_V] = 0 = \partial_z I_V$, we have $I_V \in \mathcal{F}_{tc}$. Now if $a(z)$ is translation covariant, we need to show that $[T, \partial_z a(z)] = \partial_z \partial_z a(z)$ and so $\partial_z a(z)$ is also translation covariant. But

$$\begin{aligned}[T, \partial_z a(z)] &= [T, \sum_{n\in\mathbb{Z}}(-n-1)a_{(n)}z^{-n-2}] = \sum_{n\in\mathbb{Z}}(-n-1)[T, a_{(n)}]z^{-n-2} \\ &= \sum_{n\in\mathbb{Z}}(-n-1)(-n)a_{(n-1)}z^{-n-2}\end{aligned} \tag{95}$$

and

$$\partial_z \partial_z a(z) = \partial_z (\sum_{n\in\mathbb{Z}} (-n-1) a_{(n)} z^{-n-2}) = \sum_{n\in\mathbb{Z}} (-n-1)(-n-2) a_{(n)} z^{-n-3}$$
$$= \sum_{n\in\mathbb{Z}} (-n-1)(-n) a_{(n-1)} z^{-n-2}. \tag{96}$$

For the last part of this lemma, let us recall the definition of the nth product,

$$a(w)_{(n)} b(w) = \operatorname{Res} \big(a(z) b(w) i_{z,w} (z-w)^n - b(w) a(z) i_{w,z} (z-w)^n \big) dz, \quad n \in \mathbb{Z}. \tag{97}$$

We want to prove $[T, a(w)_{(n)} b(w)] = \partial_w (a(w)_{(n)} b(w))$. Both T and ∂_w commute with Res, moreover, ∂_w commutes with $i_{z,w}$ and $i_{w,z}$. So we have

$$\begin{aligned}\partial_w (a(w)_{(n)} b(w)) &= \operatorname{Res} \big(\partial_w (a(z) b(w) i_{z,w} (z-w)^n) - \partial_w (b(w) a(z) i_{w,z} (z-w)^n) \big) dz \\ &= \operatorname{Res} \big(a(z) (\partial_w b(w)) i_{z,w} (z-w)^n - (\partial_w b(w)) a(z) i_{w,z} (z-w)^n \big) dz \\ &+ \operatorname{Res} \big(a(z) b(w) i_{z,w} (\partial_w (z-w)^n) - b(w) a(z) i_{w,z} (\partial_w (z-w)^n) \big) dz.\end{aligned} \tag{98}$$

Note that $\partial_w (z-w)^n = -\partial_z (z-w)^n$ and $-\operatorname{Res} a(z) i_{w,z} \partial_z (z-w)^n dz = \operatorname{Res} (\partial_z a(z)) i_{w,z} (z-w)^n dz$. So

$$\partial_w (a(w)_{(n)} b(w)) = a(w)_{(n)} \partial_w b(w) + (\partial_w a(w))_{(n)} b(w). \tag{99}$$

This shows that ∂_w is a derivation for the nth product. Now

$$\begin{aligned}[T, a(w)_{(n)} b(w)] &= \operatorname{Res} \big(T a(z) b(w) i_{z,w} (z-w)^n - a(z) b(w) T i_{z,w} (z-w)^n \\ &- T b(w) a(z) i_{w,z} (z-w)^n + b(w) a(z) T i_{w,z} (z-w)^n \big) dz \\ &= \operatorname{Res} \big(T a(z) b(w) i_{z,w} (z-w)^n - a(z) T b(w) i_{z,w} (z-w)^n \\ &+ a(z) T b(w) i_{z,w} (z-w)^n - a(z) b(w) T i_{z,w} (z-w)^n \\ &- T b(w) a(z) i_{w,z} (z-w)^n + b(w) T a(z) i_{w,z} (z-w)^n \big) dz \\ &- b(w) T a(z) i_{w,z} (z-w)^n + b(w) a(z) T i_{w,z} (z-w)^n \big) dz \\ &= \operatorname{Res} \big([T, a(z)] b(w) i_{z,w} (z-w)^n - b(w) [T, a(z)] i_{w,z} (z-w)^n \\ &+ \operatorname{Res} \big(a(z) [T, b(w)] i_{z,w} (z-w)^n - [T, b(w)] a(z) i_{w,z} (z-w)^n.\end{aligned} \tag{100}$$

Since both $a(z), b(z)$ are translation covariant, we have

$$[T, a(w)_{(n)} b(w)] = a(w)_{(n)} \partial_w b(w) + (\partial_w a(w))_{(n)} b(w). \tag{101}$$

This completes the proof. □

We have inclusions

$$\mathcal{F} \subset \mathcal{F}_{\min} \subset \mathcal{F}_{\max} \subset \mathcal{F}_{\mathrm{tc}}. \tag{102}$$

The first inclusion is because for any $a(z) \in \mathcal{F}$, we have $a(z)_{(-1)}I_V = a(z) \in \mathcal{F}_{\min}$. The second inclusion is by Lemma 3 and Dong's Lemma (locality). The last inclusion is by definition.

Exercise 10 Show that the constant field T is translation covariant, but is not local to any non-constant field.

Lemma 4 *Let $a(z), b(z) \in \mathcal{F}_{\mathrm{tc}}$, and $a = fs(a(z)), b = fs(b(z))$. Then:*

(a) $fs(I_V) = |0\rangle$,
(b) $fs(\partial_z a(z)) = Ta$,
(c) $fs(a(z)_{(n)}b(z)) = a_{(n)}b$. Here we write $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$.

Proof (a) is obvious. For (b), since $a(z)|0\rangle = e^{zT}a = a + (Ta)z + \frac{T^2 a}{2}z^2 + o(z^2)$ we have $\partial_z a(z)|0\rangle = Ta + T^2 az + o(z)$, so $fs(\partial_z a(z)) = \partial_z a(z)|0\rangle_{z=0} = Ta$. For (c), by definition, we have

$$fs(a(z)_{(n)}b(z)) = a(z)_{(n)}b(z)|0\rangle\big|_{z=0}, \tag{103}$$

and the right hand side, by definition of the nth product, is equal to

$$\operatorname{Res}\left(a(w)b(z)i_{w,z}(w-z)^n|0\rangle - b(z)a(w)i_{z,w}(w-z)^n|0\rangle dw\right)\Big|_{z=0}. \tag{104}$$

Now, since $a(w)|0\rangle \in V[[w]]$ and $i_{z,w}(w-z)^n$ has only non-negative powers of w, we have

$$\operatorname{Res} b(z)a(w)i_{z,w}(w-z)^n dw|0\rangle = 0.$$

For the first term, since $b(z)|0\rangle \in V[[z]]$, we can let $z = 0$ before we calculate the residue, which gives

$$\operatorname{Res} a(w)b(z)i_{w,z}(w-z)^n|0\rangle)dw\Big|_{z=0} = \operatorname{Res} a(w)bw^n dw = a_{(n)}b. \tag{105}$$

This completes the proof. □

Lemma 5 *Let $a(z) \in \mathcal{F}_{\mathrm{tc}}$. Then $e^{wT}a(z)e^{-wT} = i_{z,w}a(z+w)$.*

Proof Both sides are in $(\operatorname{End} V)[[z, z^{-1}]][[w]]$, and both satisfy the differential equation $\frac{df(w)}{dw} = (\operatorname{ad} T)f(w)$ with the initial condition $f(0) = a(z)$. □

Lemma 6 (Uniqueness Lemma) *Let $\mathcal{F}' \subset \mathcal{F}_{\mathrm{tc}}$ and let $a(z)$ be some quantum field in $\mathcal{F}_{\mathrm{tc}}$. Assume that*

(i) $fs(a(z)) = 0$,

(ii) $a(z)$ *is local with any element in* $\mathcal{F}'$,
(iii) $fs(\mathcal{F}') = V$.

Then $a(z) = 0$.

Proof Let $b(z) \in \mathcal{F}'$. By the locality of $a(z)$ and $b(z)$, we have $(z - w)^N[a(z), b(w)] = 0$ for some $N \in \mathbb{Z}_+$. Apply both sides to $|0\rangle$. We get

$$(z-w)^N a(z)b(w)|0\rangle = \pm(z-w)^N b(w)a(z)|0\rangle. \tag{106}$$

By the property (i) we have $a_{(-1)}|0\rangle = 0$ and $a(z)$ is translation covariant, hence by Lemma 1(b), $a(z)|0\rangle = 0$. Now, by Lemma 1(a), $b(w)|0\rangle \in V[[w]]$, so we can let $w = 0$ and get $z^N a(z)b = 0$, which means $a_{(n)}b = 0$ for any $n \in \mathbb{Z}$. This is true for any $b \in V$ by the property (iii). So in fact, we have $a(z) = 0$. □

Proof of the Extension Theorem We have the following two properties of the map fs:

(i) the map $fs : \mathcal{F}_{\min} \to V$ defined by $fs(a(z)) = a(z)|0\rangle\big|_{z=0}$ is given by

$$(a^{j_1}(z)_{(n_1)}(a^{j_2}(z)_{(n_2)}\cdots(a^{j_s}(z)_{(n_s)}I_V)\cdots) \mapsto a^{j_1}_{(n_1)}a^{j_2}_{(n_2)}\cdots a^{j_s}_{(n_s)}|0\rangle, \tag{107}$$

and it is surjective, by (a), (c) of Lemma 4 and the completeness axiom;
(ii) $fs : \mathcal{F}_{\max} \to V$ is injective using the Uniqueness Lemma with $\mathcal{F}' = \mathcal{F}_{\min}$.

Recall the inclusion $\mathcal{F}_{\min} \subset \mathcal{F}_{\max}$. We now have that $fs\colon \mathcal{F}_{\min} \to V$ is surjective and $fs\colon \mathcal{F}_{\max} \to V$ is injective, so we can conclude that it is in fact bijective and $\mathcal{F}_{\min} = \mathcal{F}_{\max}$. This proves (a) and (b) in the Extension Theorem. For (c), we need to show that $a(z)$ is translation covariant $\forall a \in V$ and that each pair $a(z), b(w)$ $\forall a, b \in V$ is a local pair. But translation covariance comes from Lemma 3 and locality comes from Dong's lemma. □

Corollary 1 (of the Proof)

(a) $sf(a^{j_1}_{(n_1)}a^{j_2}_{(n_2)}\cdots a^{j_s}_{(n_s)}|0\rangle) = (a^{j_1}(z)_{(n_1)}(a^{j_2}(z)_{(n_2)}\cdots(a^{j_s}(z)_{(n_s)}I_V)\cdots)$.
(b) $(Ta)(z) = \partial_z a(z)$.
(c) $(a_{(n)}b)(z) = a(z)_{(n)}b(z)$, *which is called the* nth product identity.

Proof (a) is by definition since sf is the inverse of fs, while fs is given by (107). Letting $s = 1, n_1 = -2$ in (a) we get (b). Letting $s = 2, n_1 = n, n_2 = -1$ in (a) we get (c). □

Remark 12 Due to Corollary 1(b) and Remark 1, the Definitions 3 and 5 of a vertex algebra are equivalent.

Remark 13 (Special Case of (a) in the Corollary) For $n_1, \ldots, n_s \in \mathbb{Z}_+$, we have,

$$sf(a^{j_1}_{(-n_1-1)}a^{j_2}_{(-n_2-1)}\cdots a^{j_s}_{(-n_s-1)}|0\rangle) = \frac{:\partial_z^{n_1}a^{j_1}(z)\partial_z^{n_2}a^{j_2}(z)\cdots\partial_z^{n_s}a^{j_s}(z):}{n_1!n_2!\cdots n_s!}. \tag{108}$$

Corollary 2 (of the Proof) $\mathrm{Lie}_V := \mathrm{span}\{a_{(n)} | a \in V, n \in \mathbb{Z}\} \subset \mathrm{End}\, V$ *is a subalgebra of the Lie superalgebra* End V *with the commutator formula*

$$[a(z), b(w)] = \sum_{j\geqslant 0} (a(w)_{(j)} b(w)) \frac{\partial_w^j \delta(z,w)}{j!}, \tag{109}$$

which is equivalent to each of the following two expressions

$$[a_{(m)}, b(z)] = \sum_{j\geqslant 0} \binom{m}{j} (a_{(j)} b)(z) z^{m-j}, \tag{110}$$

$$[a_{(m)}, b_{(n)}] = \sum_{j\geqslant 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}. \tag{111}$$

Moreover, Lie_V *is a regular formal distribution Lie algebra with the data* $(\mathrm{Lie}_V, \mathcal{F} = \{a(z)\}_{a\in V}, \mathrm{ad}\, T)$.

3.2 Borcherds Identity and Some Other Properties

Proposition 3 (Borcherds Identity) *For* $n \in \mathbb{Z}$, $a, b \in V$, *where* V *is a vertex algebra, we have*

$$a(z)b(w)i_{z,w}(z-w)^n - (-1)^{p(a)p(b)} b(w)a(z)i_{w,z}(z-w)^n = \sum_{j\in\mathbb{Z}_+} (a_{(n+j)}b)(w) \frac{\partial_w^j \delta(z,w)}{j!}. \tag{112}$$

Proof The left hand side of (112) is a local formal distribution in z and w. Apply to it the Decomposition theorem to get that it is equal to

$$\sum_{j\in\mathbb{Z}_+} c^j(w) \partial_w^j \delta(z,w)/j!\,, \tag{113}$$

where

$$\begin{aligned} c^j(w) &= \mathrm{Res}\left(a(z)b(w)i_{z,w}(z-w)^n - (-1)^{p(a)p(b)} b(w)a(z)i_{w,z}(z-w)^n\right)(z-w)^j dz \\ &= \mathrm{Res}\left(a(z)b(w)i_{z,w}(z-w)^{n+j} - (-1)^{p(a)p(b)} b(w)a(z)i_{w,z}(z-w)^{n+j}\right) dz \\ &= a(w)_{(n+j)} b(w) \\ &= (a_{(n+j)}b)(w). \end{aligned} \tag{114}$$

The last equality follows from the nth product formula, all other equalities are just by definition. □

Exercise 11 Prove that a unital z-algebra satisfying the Borcherds identity is a vertex algebra.

Proposition 4 (Skewsymmetry) *For $a, b \in V$, where V is a vertex algebra, we have:*

$$a(z)b = (-1)^{p(a)p(b)} e^{zT} b(-z)a. \tag{115}$$

Proof By locality, we know that, there exists $N \in \mathbb{Z}$, such that

$$(z-w)^N a(z)b(w) = (-1)^{p(a)p(b)}(z-w)^N b(w)a(z).$$

Apply both sides to $|0\rangle$; by Lemma 1(b) we get

$$(z-w)^N a(z)e^{wT} b = (-1)^{p(a)p(b)}(z-w)^N b(w)e^{zT} a. \tag{116}$$

Now use Lemma 5:

$$RHS = (-1)^{p(a)p(b)}(z-w)^N e^{zT} e^{-zT} b(w)e^{zT} a = (-1)^{p(a)p(b)}(z-w)^N e^{zT} i_{w,z} b(w-z)a. \tag{117}$$

For $N \gg 0$, this is a formal power series in $(z-w)$, so we can set $w = 0$ and get

$$LHS = z^N a(z)b = (-1)^{p(a)p(b)} e^{zT} z^N b(-z)a = RHS, \tag{118}$$

which proves the proposition. □

Proposition 5 *T is a derivation for all nth products, i.e.,*

$$T(a_{(n)}b) = (Ta)_{(n)}b + a_{(n)}(Tb), \quad \forall n \in \mathbb{Z}. \tag{119}$$

Proof It follows from Remark 12. □

In view of the nth product identity, we let : $ab := a_{(-1)}b$ and call this the normally ordered product of two elements of a vertex algebra.

Proposition 6 *The nth products for negative n are expressed via the normally ordered product: $a_{(-n-1)}b =: \frac{T^n a}{n!} b :$.*

Proof We have $(a_{(-n-1)}b)(z) = a(z)_{(-n-1)}b(z) =: \frac{\partial_z^n a(z)}{n!} b(z) :$, where the first equality is the nth product identity and the second equality is (78). But we also have $T(a)(z) = \partial_z a(z)$, hence by induction we have $: \frac{\partial_z^n a(z)}{n!} b(z) : =: \frac{(T^n a)(z)}{n!} b(z) : = (\frac{(T^n a)}{n!}{}_{(-1)} b)(z)$, and by the bijection of the state-field correspondence, we have $a_{(-n-1)}b = \frac{(T^n a)}{n!}{}_{(-1)} b =: \frac{T^n a}{n!} b :$. □

Now we take care of the nth products $a_{(n)}b$ for $n \in \mathbb{Z}_+$. For this we define the *λ-bracket*

$$[a_\lambda b] = \sum_{j\geqslant 0} \frac{\lambda^j}{j!}(a_{(j)}b) \in V[\lambda], \quad \text{for} \quad a, b \in V. \tag{120}$$

Thus we get a quadruple $(V, T, :ab:, [a_\lambda b])$, which will be shown in the next lecture to have a very similar structure to a Poisson Vertex Algebra (PVA).

3.3 Conformal Vector and Conformal Weight, Hamiltonian Operator

Definition 11 A vector L of a vertex algebra V is called a *conformal vector* if

(i) $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, such that,

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}cI_V \tag{121}$$

for some $c \in \mathbb{F}$, which is called the *central charge*,

(ii) $L_{-1} = T$,

(iii) L_0 acts diagonalizably on V, its eigenvalues are called *conformal weights*.

Since $L_{n-1} = L_{(n)}$, using the commutator formula (109), we get

$$[L(z), a(w)] = \sum_{j\geqslant 0}(L_{j-1}a)(w)\partial_w^j\delta(z,w)/j!, \tag{122}$$

which is equivalent to [cf. (111)]

$$[L_{(m)}, a_{(n)}] = \sum_{j\geqslant 0}\binom{m}{j}(L_{j-1}a)_{(m+n-j)}. \tag{123}$$

So we have

$$[L_\lambda a] = \sum_{j\geqslant 0}\frac{\lambda^j}{j!}L_{j-1}a = Ta + \lambda\Delta_a a + o(\lambda). \tag{124}$$

Here we assume that a is an eigenvector of L_0 with the eigenvalue Δ_a. We call L_0 the energy operator. It is a Hamiltonian operator by the definition below and (123) for $m = 0$.

Definition 12 A diagonalizable operator H is called a *Hamiltonian operator* if it satisfies the equation

$$[H, a(z)] = (z\partial_z + \Delta_a)a(z) \quad \Longleftrightarrow \quad [H, a_{(n)}] = (\Delta_a - n - 1)a_{(n)} \tag{125}$$

for any eigenvector a of H with eigenvalue Δ_a.

If we write $a(z) = \sum_{n\in -\Delta_a+\mathbb{Z}} a_n z^{-n-\Delta_a}$, then due to the equality $a_{(n)} = a_{n-\Delta_a+1}$, we have:

$$[H, a_n] = -na_n. \tag{126}$$

This is an equivalent definition of a Hamiltonian operator.

Proposition 7 *If H is a Hamiltonian operator, then we have:*

(a) $\Delta_{|0\rangle} = 0$,
(b) $\Delta_{Ta} = \Delta_a + 1$,
(c) $\Delta_{a_{(n)}b} = \Delta_a + \Delta_b - n - 1$.

Proof To prove (a), we just need to know that $|0\rangle(z) = I_V$, and we use (125) with $a = |0\rangle$. Since $Ta = a_{(-2)}|0\rangle$, (b) follows from (a) and (c) with $b = |0\rangle$, $n = -2$. For (c), we have

$$\begin{aligned} H(a_{(n)}b) &= [H, a_{(n)}]b + a_{(n)}Hb \\ &= (\Delta_a - n - 1)a_{(n)}b + \Delta_b a_{(n)}b \\ &= (\Delta_a + \Delta_b - n - 1)a_{(n)}b. \end{aligned} \tag{127}$$

□

Remark 14

(a) For a conformal vector L, we have $[L_\lambda L] = (T + 2\lambda)L + \frac{\lambda^3}{2}c|0\rangle$, which implies $\Delta_L = 2$. That is why we write $L(z)$ in the form $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.
(b) Conformal weight is a good "book-keeping device", if we let $\Delta_\lambda = \Delta_T = 1$. Then all summands in the λ-bracket $[a_\lambda b] = \sum_{j\geqslant 0} \frac{\lambda^j}{j!}(a_{(j)}b)$ have the same conformal weight $\Delta_a + \Delta_b - 1$.

Remark 15 The translation covariance (4) of the quantum field $a(z)$ is equivalent to the following "global" translation covariance:

$$e^{\epsilon T}a(z)e^{-\epsilon T} = i_{z,\epsilon}a(z+\epsilon).$$

Likewise, the property (125) of $a(z)$ is equivalent to the following "global" scale covariance:

$$\gamma^H a(z)\gamma^{-H} = (\gamma^{\Delta_a}a)(\gamma z), \text{ where } Ha = \Delta_a a.$$

The more general property (122) is called the conformal invariance. It is the basic symmetry of conformal field theory.

3.4 Formal Fourier Transform

Definition 13 The *Formal Fourier Transform* is the map $F_z^\lambda\colon U[[z,z^{-1}]] \mapsto U[[\lambda]]$ defined by

$$F_z^\lambda a(z) = \operatorname{Res} e^{\lambda z} a(z) dz. \tag{128}$$

Proposition 8

(a) $F_z^\lambda \partial_z a(z) = -\lambda F_z^\lambda a(z)$,
(b) $F_z^\lambda \partial_w^k \delta(z,w) = e^{\lambda w}\lambda^k$,
(c) $F_z^\lambda a(-z) = -F_z^{-\lambda} a(z)$,
(d) $F_z^\lambda (e^{zT} a(z)) = F_z^{\lambda+T} a(z)$, *where* $T \in \operatorname{End} U$, *provided that* $a(z) \in U((z))$.

Proof

(a) Assume $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, then $\partial_z a(z) = \sum_{n\in\mathbb{Z}} (-n-1) a_{(n)} z^{-n-2}$. Now

$$\begin{aligned} F_z^\lambda a(z) &= \operatorname{Res} e^{\lambda z} a(z) dz \\ &= \operatorname{Res}\Big(\sum_{i\in\mathbb{Z}_+} \frac{\lambda^i z^i}{i!}\Big)\Big(\sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}\Big) dz \\ &= \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!} a_{(n)}, \end{aligned} \tag{129}$$

$$\begin{aligned} F_z^\lambda \partial_z a(z) &= \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!} (-n) a_{(n-1)} \\ &= -\lambda \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!} a_{(n)}. \end{aligned}$$

(b) Recall that $\frac{\partial_w^k \delta(z,w)}{k!} = \sum_{j\in\mathbb{Z}} \binom{j}{k} w^{j-k} z^{-j-1}$, so

$$\begin{aligned} F_z^\lambda \partial_w^k \delta(z,w) &= \operatorname{Res} e^{\lambda z} k! \sum_{j\in\mathbb{Z}_+} \binom{j}{k} w^{j-k} z^{-j-1} dz \\ &= \sum_{j\in\mathbb{Z}_+} \frac{\lambda^j}{j!} k! \frac{j!}{k!(j-k)!} w^{j-k} \\ &= \lambda^k \sum_{j-k\in\mathbb{Z}_+} \frac{\lambda^{j-k}}{(j-k)!} w^{j-k} \\ &= e^{\lambda w}\lambda^k. \end{aligned} \tag{130}$$

(c) By definition

$$
\begin{aligned}
F_z^{\lambda} a(-z) &= \operatorname{Res}\Big(\sum_{i\in\mathbb{Z}_+} \frac{\lambda^i z^i}{i!}\Big)\Big(\sum_{n\in\mathbb{Z}} a_{(n)}(-z)^{-n-1}\Big)dz \\
&= \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}(-1)^{n+1} a_{(n)} \\
&= -\sum_{n\in\mathbb{Z}_+} \frac{(-\lambda)^n}{n!} a_{(n)} \\
&= -F_z^{-\lambda} a(z).
\end{aligned}
\tag{131}
$$

(d) Since $a(z) \in U((z))$, $e^{zT}a(z) \in U((z))$ is well defined. Now

$$
\begin{aligned}
F_z^{\lambda}(e^{zT}a(z)) &= \operatorname{Res} e^{\lambda z} e^{zT} a(z)dz \\
&= \operatorname{Res} e^{(\lambda+T)z} a(z)dz \\
&= F_z^{\lambda+T} a(z).
\end{aligned}
\tag{132}
$$

□

Similarly, we can define the Formal Fourier Transform in two variables.

Definition 14 The Formal Fourier Transform in two variables is the map

$$
F_{z,w}^{\lambda} : U[[z, z^{-1}, w, w^{-1}]] \to U[[w, w^{-1}]][[\lambda]], \tag{133}
$$

defined by

$$
F_{z,w}^{\lambda} a(z, w) = \operatorname{Res} e^{\lambda(z-w)} a(z, w)dz = e^{-\lambda w} F_z^{\lambda}\, a(z, w). \tag{134}
$$

Proposition 9

(α) $F_{z,w}^{\lambda} \partial_z a(z, w) = -\lambda F_{z,w}^{\lambda} a(z, w) = [\partial_w, F_{z,w}^{\lambda}] a(z, w)$,
(β) $F_{z,w}^{\lambda} \partial_w^k \delta(z, w) = \lambda^k$,
(γ) $F_{z,w}^{\lambda} a(w, z) = F_{z,w}^{-\lambda-\partial_w} a(z, w)$ *provided that* $a(z, w)$ *is local,*
(δ) $F_{z,w}^{\lambda} F_{x,w}^{\mu} = F_{x,w}^{\lambda+\mu} F_{z,x}^{\lambda}$.

Proof Since $F_{z,w}^{\lambda} = e^{-\lambda w} F_z^{\lambda}$, (α) and (β) follow from the properties (*a*) and (*b*) in Proposition 8. (δ) holds since

$$
\operatorname{Res}\operatorname{Res} e^{\lambda(z-w)+\mu(z-w)} a(z, w, x)dxdz = \operatorname{Res}\operatorname{Res} e^{\lambda(z-x)} e^{(\lambda+\mu)(x-w)} a(z, w, x)dxdz \tag{135}
$$

Finally, due to the Decomposition theorem, it suffices to check (γ) (interpretation as before) for $a(z, w) = c(w)\partial_w^k \delta(z, w)$:

$$\begin{aligned}\text{LHS} &= \operatorname{Res} e^{\lambda(z-w)} c(z)\partial_z^k \delta(w, z)dz = (-1)^k \operatorname{Res} e^{\lambda(z-w)} c(z)\partial_w \delta(w, z)dz \\ &= (-1)^k e^{-\lambda w}\partial_w^k \operatorname{Res} e^{\lambda z} c(z)\delta(z, w)dz = (-1)^k e^{-\lambda w} \partial_w^k e^{\lambda w} c(w) \\ &= (-\lambda - \partial_w)^k c(w),\end{aligned}$$

using the properties (3) and (5) of the delta function. □

4 Lecture 4 (December 18, 2014)

The Formal Fourier Transform F_z^λ is very important for us, since the λ-bracket (120) is $[a_\lambda b] = F_z^\lambda a(z)b$, i.e., the Fourier transform of the z-product is the λ-bracket. We also note that $: ab : \ (= a_{(-1)}b) = \operatorname{Res} \dfrac{a(z)b}{z} dz$. These observations will be important for studying properties of the normally ordered product : : and the λ-bracket. For simplicity we will further consider vertex algebras V of purely even parity only. The general case follows by the Koszul-Quillen rule.

4.1 *Quasicommutativity, Quasiassociativity and the Noncommutative Wick's Formula*

Lemma 7 (Newton-Leibniz (NL) Lemma) *For any $a(z) \in U[[z]]$, we have*

$$F_z^\lambda \frac{a(z)}{z} = \operatorname{Res} a(z)\frac{dz}{z} + \int_0^\lambda F_z^\mu a(z)d\mu. \tag{136}$$

Proof Both sides are formal power series in λ, they are equal at $\lambda = 0$, and their derivatives by λ are also equal, so they are equal. □

Proposition 10 (Quasicommutativity of : :) *The commutator for the normally ordered product and λ-bracket are related as follows*

$$: ab : - : ba := \int_{-T}^0 [a_\lambda b]d\lambda. \tag{137}$$

Proof Apply F_z^λ to both sides of skewsymmetry, divided by z, and set $\lambda = 0$. We get

$$F_z^\lambda \frac{a(z)b}{z}\bigg|_{\lambda=0} = F_z^\lambda \frac{e^{zT}b(-z)a}{z}\bigg|_{\lambda=0}. \tag{138}$$

By definition

$$LHS = :ab: = a_{(-1)}b. \tag{139}$$

Next, using property (d) of the FFT in Proposition 8, we have

$$\begin{aligned}
RHS &= F_z^{\lambda+T}\frac{b(-z)a}{z}\Big|_{\lambda=0} \\
\text{(by NL Lemma)} \quad &= \mathrm{Res}_z \frac{b(-z)a}{z} + \int_0^{\lambda+T} F_z^{\mu} b(-z)a\, d\mu\Big|_{\lambda=0} \\
\text{(by property (c) of FFT in Prop 8)} \quad &= \ :ba: - \int_0^T F_z^{-\mu} b(z)a\, d\mu \\
&= \ :ba: - \int_0^T [b_{-\mu}a]\, d\mu \\
\text{(by skewsymmetry of the } \lambda\text{-bracket)} \quad &= \ :ba: + \int_0^T [a_{\mu+T}b]\, d\mu \\
&= \ :ba: + \int_{-T}^0 [a_\mu b]\, d\mu.
\end{aligned} \tag{140}$$

□

Next we derive the following important identity.

Proposition 11 *For a, b, c in a vertex algebra V, we have the following identity in $V[[\lambda, w, w^{-1}]]$*

$$[a_\lambda b(w)c] = e^{w\lambda}[a_\lambda b](w)c + b(w)[a_\lambda c]. \tag{141}$$

Proof The following identity in $V[[z^{\pm 1}, w^{\pm 1}]]$ is obvious:

$$a(z)b(w)c = [a(z), b(w)]c + b(w)a(z)c. \tag{142}$$

Applying to both sides $F_z^\lambda = e^{w\lambda}F_{z,w}^\lambda$, we get

$$[a_\lambda b(w)c] = e^{w\lambda}F_{z,w}^\lambda[a(z), b(w)]c + b(w)F_z^\lambda a(z)c = e^{w\lambda}[a_\lambda b](w)c + b(w)[a_\lambda c], \tag{143}$$

where we have used the nth product formula $a(w)_{(n)}b(w) = (a_{(n)}b)(w), n \in \mathbb{Z}_+$.

□

We have the following two important properties of a vertex algebra.

Proposition 12 *Assume a, b, c in a vertex algebra V. Then we have*

(a) Quasiassociativity formula

$$::ab:c:-:a:bc::=:\left(\int_0^T d\lambda a\right)[b_\lambda c]:+:\left(\int_0^T d\lambda b\right)[a_\lambda c]:. \tag{144}$$

(b) Non-commutative Wick's formula

$$[a_\lambda :bc:]=:[a_\lambda b]c:+:b[a_\lambda c]:+\int_0^\lambda [[a_\lambda b]_\mu c]d\mu. \tag{145}$$

Proof

(1) Apply Res $\frac{1}{z}dz$ to the -1st product identity:

$$:ab:(z)c=:a(z)b(z):c=a(z)_+b(z)c+b(z)a(z)_-c, \tag{146}$$

and use that

$$\begin{aligned}\text{Res}\,\frac{1}{z}(:ab:(z)c)dz &= (:ab:)_{(-1)}c=::ab:c:,\\ \text{Res}\,\frac{1}{z}(a(z)_+b(z)c)dz &=:a:bc::+\sum_{j\in\mathbb{Z}_+}a_{(-j-2)}b_{(j)}c\\ &=:a:bc::+:(\int_0^T d\lambda a)[b_\lambda c]:, \qquad (147)\\ \text{Res}\,\frac{1}{z}(b(z)a(z)_-c)dz &= \sum_{j\in\mathbb{Z}_+}b_{(-j-2)}a_{(j)}c\\ &=:(\int_0^T d\lambda b)[a_\lambda c]:\;.\end{aligned}$$

(2) Take Res $\frac{1}{w}dw$ of both sides of formula (141):

$$\text{Res}\,\frac{1}{w}[a_\lambda b(w)c]dw=\text{Res}\,\frac{1}{w}(e^{w\lambda}[a_\lambda b](w)c+b(w)[a_\lambda c])dw. \tag{148}$$

Since Res $\frac{b(w)c}{w}dw=b_{(-1)}c=:bc:$, we have

$$\text{Res}\,\frac{1}{w}[a_\lambda b(w)c]dw=[a_\lambda :bc:]. \tag{149}$$

For the second term of the right-hand side of (148),

$$\operatorname{Res} \frac{1}{w} b(w)[a_\lambda c] dw =: b[a_\lambda c] : . \tag{150}$$

Using the NL Lemma 7 for the first term in the right hand side of (148), we have

$$\begin{aligned} \operatorname{Res} e^{w\lambda} \frac{[a_\lambda b](w)c}{w} dw = F_w^\lambda \frac{[a_\lambda b](w)c}{w} & \\ = \operatorname{Res} \frac{[a_\lambda b](w)c}{w} dw + \int_0^\lambda F_w^\mu [a_\lambda b](w) c d\mu & \qquad (151) \\ =: [a_\lambda b] c : + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu. & \end{aligned}$$

□

Remark 16 The expression $: (\int_0^T d\lambda a)[b_\lambda c] :$ should be understood in the following way. We know that $[b_\lambda c] = \sum_{j \in \mathbb{Z}_+} b_{(j)} c \frac{\lambda^j}{j!}$, so $: a[b_\lambda c] := \sum_{j \in \mathbb{Z}_+} a_{(-1)} b_{(j)} c \frac{\lambda^j}{j!}$. We have $\int_0^T \frac{\lambda^j}{j!} d\lambda = \frac{T^{j+1}}{(j+1)!}$; letting $\frac{T^{j+1}}{(j+1)!}$ act just on a we get $\left(\frac{T^{j+1} a}{(j+1)!} \right)_{(-1)} = a_{(-j-2)}$, so $: (\int_0^T d\lambda a)[b_\lambda c] := \sum_{j \in \mathbb{Z}_+} a_{(-j-2)} b_{(j)} c$.

4.2 *Lie Conformal Algebras vs Vertex Algebras*

Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{g}[[w, w^{-1}]]$ be the space of all $\mathfrak{g}$-valued formal distributions. This space is an $\mathbb{F}[\partial]$-module by defining

$$\partial a(w) := \partial_w a(w). \tag{152}$$

It is closed under the following (formal) λ-bracket: for $a = a(w), b = b(w) \in \mathfrak{g}[[w, w^{-1}]]$. Let

$$[a_\lambda b](w) := F_{z,w}^\lambda [a(z), b(w)]. \tag{153}$$

Indeed, by definition of $F_{z,w}^\lambda$ and its property (β), we have:

$$\begin{aligned} [a_\lambda b](w) &= \operatorname{Res} e^{\lambda(z-w)} [a(z), b(w)] dz \\ &= \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} \operatorname{Res} (z-w)^j [a(z), b(w)] dz \\ &= \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} (a_{(j)} b)(w) \in \mathfrak{g}[[w, w^{-1}]][[\lambda]]. \end{aligned} \tag{154}$$

Thus $[a_\lambda b](w)$ is a generating series for jth products of $a(w)$ and $b(w)$. It is a formal power series in λ in general, but if the pair $(a(w), b(w))$ is local, $[a_\lambda b] \in \mathfrak{g}[[w, w^{-1}]][\lambda]$ is polynomial in λ.

Proposition 13 *Assume $a(w), b(w), c(w) \in \mathfrak{g}[[w, w^{-1}]]$ for some Lie algebra $\mathfrak{g}$ with $\partial = \partial_w$ defined as above. Denote $a = a(w), b = b(w), c = c(w)$. Then the λ-bracket defined as above satisfies the following properties:*

$$\begin{aligned} &\textit{(sesquilinearity)}\ [\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b], \\ &\textit{(skewsymmetry)}\ [b_\lambda a] = -[a_{-\lambda-\partial} b] \quad \textit{if } a, b \textit{ is a local pair}, \\ &\textit{(Jacobi identiy)}\ [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]. \end{aligned} \tag{155}$$

Proof The sesquilinearity comes from (α) and the skewsymmetry comes from (γ) in Proposition 9 about properties of formal Fourier transform in two variables. For the Jacobi identity we have:

$$\begin{aligned} [a_\lambda [b_\mu c]](w) &:= F^\lambda_{z,w}[a(z), F^\mu_{x,w}[b(x), c(w)]] \\ &= F^\lambda_{z,w} F^\mu_{x,w}[a(z), [b(x), c(w)]] \\ &= F^\lambda_{z,w} F^\mu_{x,w}[[a(z), b(x)], c(w)]] + F^\lambda_{z,w} F^\mu_{x,w}[b(x), [a(z), c(w)]]. \end{aligned} \tag{156}$$

The last equality comes from the Jacobi identity in the Lie algebra $\mathfrak{g}$. By property (δ) of the formal Fourier transform in Proposition 9, we have:

$$\begin{aligned} F^\lambda_{z,w} F^\mu_{x,w}[[a(z), b(x)], c(w)]] &= F^{\lambda+\mu}_{x,w} F^\lambda_{z,x}[[a(z), b(x)], c(w)] \\ &= F^{\lambda+\mu}_{x,w}[F^\lambda_{z,x}[[a(z), b(x)], c(w)]] \\ &= [[a_\lambda b]_{\lambda+\mu} c](w), \end{aligned} \tag{157}$$

while $F^\lambda_{z,w} F^\mu_{x,w}[b(x), [a(z), c(w)]] = [b_\mu [a_\lambda c]](w)$ is just by definition. □

Definition 15 A *Lie conformal algebra* (LCA) is an $\mathbb{F}[\partial]$-module R endowed with an $\mathbb{F}$-bilinear λ-bracket $[a_\lambda b] \in R[\lambda]$ for $a, b \in R$, which satisfies the axioms of sesquilinearity, skewsymmetry and the Jacobi identity.

Example 8 The Virasoro formal distribution Lie algebra from Example 5 gives rise, by Proposition 13, to the Virasoro Lie conformal algebra

$$\mathrm{Vir} = \mathbb{F}[\partial]L \oplus \mathbb{F}C \tag{158}$$

with λ-bracket

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}C, \quad [C_\lambda \mathrm{Vir}] = 0\,.$$

Replacing L by $L-\frac{1}{2}\alpha C$, where $\alpha \in \mathbb{F}$, we obtain a λ-bracket with a trivial cocycle added:

$$[L_\lambda L] = (\partial + 2\lambda)L + \alpha\lambda C + \frac{\lambda^3}{12}C, \quad [C_\lambda \mathrm{Vir}] = 0. \tag{159}$$

Example 9 The Kac-Moody formal distribution Lie algebra from Example 6 gives rise to the Kac-Moody Lie conformal algebra

$$\mathrm{Cur}\,\mathfrak{g} = \mathbb{F}[\partial] \otimes \mathfrak{g} + \mathbb{F}K \tag{160}$$

with λ-bracket $(a, b \in \mathfrak{g})$:

$$[a_\lambda b] = [a, b] + \lambda(a|b)K, \quad [K_\lambda \mathrm{Cur}\,\mathfrak{g}] = 0.$$

Fix $s \in \mathfrak{g}$; replacing a by $a-(a|s)K$, we obtain a λ-bracket with a trivial cocycle added:

$$[a_\lambda b] = [a, b] + \lambda(a|b)K + (s|[a, b])K, \quad [K_\lambda \mathrm{Cur}\,\mathfrak{g}] = 0. \tag{161}$$

Of course, adding a trivial cocycle doesn't change the Lie conformal algebra. However this will become crucial in the proof of the integrability of the associated integrable systems.

Due to the nth product identity in a vertex algebra [Corollary 1(c)], we derive from the last proposition the following.

Proposition 14 *A vertex algebra V is a Lie conformal algebra with $\partial = T$, the translation operator, and λ-bracket*

$$[a_\lambda b] = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} a_{(n)} b, \ a, b \in V. \tag{162}$$

Proof The λ-bracket defined by (162) is the formal Fourier transform of the z-product in V. V is obviously an $\mathbb{F}[T]$-module. Moreover, the Fourier coefficients of the formal distributions $\{a(w)|a \in V\} \subset \mathrm{End}\, V[[w, w^{-1}]]$ span a Lie subalgebra of Lie_V of End V (Corollary 2), and they are pairwise local, hence the skewsymmetry is always satisfied. Thus, $(\mathrm{Lie}_V, \{a(w)\}_{a\in V})$ is a formal distribution Lie algebra. Hence, by Proposition 13, the formal distributions $\{a(w)\}_{a\in V}$ satisfy all axioms of a Lie conformal algebra. Due to the nth product identity, Proposition 14 follows. □

We thus obtain the following

Theorem 5 *Let V be a vertex algebra. Then the quintuple $(V, |0\rangle, T, ::, [._\lambda .])$ satisfies the following properties of a "quantum Poisson vertex algebra".*

(a) $(V, T, [\cdot_\lambda \cdot])$ *is a Lie conformal algebra.*

(b) $(V, |0\rangle, T, ::)$ *is a quasicommutative, quasiassociative unital differential algebra.*

(c) The normally order product $::$ *and the* λ*-bracket* $[\cdot_\lambda \cdot]$ *are related by the noncommutative Wick formula* (145).

Remark 17 In fact, properties (a), (b), (c) of Theorem 5 characterize a vertex algebra structure, i.e., a quintuple $(V, |0\rangle, T, ::, [\cdot_\lambda \cdot])$ satisfying the above "quantum Poisson vertex algebra" properties, is a vertex algebra. This is proved in [2].

Example 10 (Computation with the non-commutative Wick's formula) The simplest example is a free boson. Recall Example 3 in Lecture 1. For a free boson field $a(z)$, we have

$$[a(z), a(w)] = \partial_w \delta(z, w). \tag{163}$$

In the language of λ-brackets this means for $a = fs(a(z))$:

$$[a_\lambda a] = \lambda|0\rangle, \tag{164}$$

i.e., $a_{(1)}a = 1$ and $a_{(n)}a = 0$ for $n = 0$ or $n \geqslant 2$.

Now let $L := \frac{1}{2} :aa:$, then

$$[L_\lambda a] = (T + \lambda)a, \qquad [L_\lambda L] = (T + 2\lambda)L + \frac{\lambda^3}{12}|0\rangle. \tag{165}$$

Indeed,

$$2[a_\lambda L] = [a_\lambda :aa:] =: [a_\lambda a]a : + : a[a_\lambda a] : + \int_0^\lambda [[a_\lambda a]_\mu a] d\mu.$$

Using (164), we obtain $[a_\lambda L] = \lambda a$ (since $[|0\rangle_\lambda a] = 0$). By the skewsymmetry of the λ-bracket, the first equation in (165) follows.

Next we have:

$$\begin{aligned}
[L_\lambda L] &= \frac{1}{2}[L_\lambda :aa:] \\
&= \frac{1}{2} : [L_\lambda a]a : + \frac{1}{2} : a[L_\lambda a] : + \int_0^\lambda [[L_\lambda a]_\mu a]\, d\mu \\
&= \frac{1}{2} : ((T + \lambda)a)a : + \frac{1}{2} : a(T + \lambda)a : + \int_0^\lambda [(T + \lambda)a_\mu a]\, d\mu \\
&= \frac{1}{2} T(:aa:) + \lambda :aa: + \int_0^\lambda (\lambda - \mu)\mu\, d\mu\ |0\rangle \\
&= (T + 2\lambda)L + \frac{\lambda^3}{12}\ |0\rangle,
\end{aligned}$$

proving the second equation in (165).

Of course, there is a simpler way of manipulating with free quantum fields, see Theorem 3.3 in [17]. However, exactly the same method as above works well for arbitrary quantum fields (like currents, discussed below).

The following proposition tells us how to prove that a vector L is a conformal vector, hence how to construct a Hamiltonian operator $H = L_0$.

Proposition 15 *Let $(V, |0\rangle, T, \mathcal{F})$ be a pre-vertex algebra and let $L \in V$ be such that for $a(z) \in \mathcal{F}$,*

(i) $[L_\lambda a] = (T + \triangle_a \lambda)a + o(\lambda)$
(ii) *$L(z)$ satisfies the Virasoro relation:* $[L_\lambda L] = (T + 2\lambda)L + \frac{\lambda^3}{12}c|0\rangle$.

Then L is a conformal vector of the corresponding (by the Extension theorem) vertex algebra.

Proof $L(z)$ is already a Virasoro field, so we only need to prove that $L_{-1} = T$ and that L_0 acts diagonalizably on V. By completeness, V is spanned by $a^{j_1}_{(k_1)} \cdots a^{j_s}_{(k_s)}|0\rangle$, where $a^{j_i}(z) \in \mathcal{F}$. Furthermore, property (i) tells us

$$[L_{-1}, a_{(n)}] = -na_{(n-1)} \quad \text{and} \quad [L_0, a_{(n)}] = (\Delta_a - n - 1)a_{(n)}. \tag{166}$$

Moreover, letting $a = |0\rangle$ in (i), we get

$$L_{-1}|0\rangle = 0 \quad \text{and} \quad L_0|0\rangle = 0. \tag{167}$$

Remember that T also satisfies the first equation in (166), so $[L_{-1} - T, a_{(k)}] = 0$ for all $k \in \mathbb{Z}$. Moreover $(L_{-1} - T)|0\rangle = 0$, so $L_{-1} - T$, being a derivation of all nth products, is zero, i.e., $L_{-1} = T$. L_0 is diagonizable by (166). □

It follows from Proposition 15 and (165) that L is a conformal vector for the free boson vertex algebra, the free boson a being primary of conformal weight 1. Exactly the same method works for the affine vertex algebras.

Exercise 12 Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra of level k associated to a simple Lie algebra $\mathfrak{g}$. Let a^i, b^i be dual bases of $\mathfrak{g}$, i.e., $(b^i|a^j) = \delta_{ij}$ with respect to the Killing form. Assume that $k \neq -h^\vee$, where $2h^\vee$ is the eigenvalue of the Casimir element of $U(\mathfrak{g})$ in the adjoint representation ($h^\vee$ is called the dual Coxeter number). Let $L = \frac{1}{2(k+h^\vee)} \sum_i : a^i b^i :$ (the so called Sugawara construction). Show that L is a conformal vector with central charge $c = \frac{k \dim \mathfrak{g}}{2(k+h^\vee)}$, all $a \in \mathfrak{g}$ being primary of conformal weight 1.

4.3 Quasiclassical Limit of Vertex Algebras

Suppose we have a family of vertex algebras, i.e. a vertex algebra $(V_\hbar, T_\hbar, |0\rangle_\hbar, ::_\hbar, [\cdot_\lambda \cdot]_\hbar)$ over $\mathbb{F}[[\hbar]]$, such that

(i) for $v \in V_\hbar$, $\hbar v = 0$ only if $v = 0$ (e.g. if $V_\hbar$ is a free $\mathbb{F}[[\hbar]]$-module),
(ii) $[a_\lambda b]_\hbar \in \hbar V_\hbar$ for $a, b \in V_\hbar$.

Given a vertex algebra $(V_\hbar, T_\hbar, |0\rangle_\hbar, ::_\hbar, [_\lambda]_\hbar)$ over $\mathbb{F}[[\hbar]]$, satisfying the above two conditions, let $\mathcal{V} := V_\hbar/\hbar V_\hbar$. This is a vector space over $\mathbb{F}$. Denote by 1 the image of $|0\rangle_\hbar \in V_\hbar$ in $\mathcal{V}$ and by ∂ the operator on $\mathcal{V}$, induced by $T \in \mathrm{End}\ V_\hbar$ ($\hbar V_\hbar$ is obviously T-invariant). The subspace (over $\mathbb{F}$) $\hbar V_\hbar$ is obviously an ideal for the product $: v :_\hbar$, hence we have the induced product $\cdot$ on $\mathcal{V}$, which is bilinear over $\mathbb{F}$. Finally, define a λ-bracket $\{a_\lambda b\}$ on $\mathcal{V}$ as follows. Let $\tilde{a}$ and $\tilde{b}$ be preimages in $V_\hbar$ of a and b; then we have

$$[\tilde{a}_\lambda \tilde{b}]_\hbar = \hbar[\tilde{a}_\lambda \tilde{b}]'$$

where $[\tilde{a}_\lambda \tilde{b}]'$ is uniquely defined due to (i) and (ii). We let

$$\{a_\lambda b\} = \text{ image of } [\tilde{a}_\lambda \tilde{b}]' \text{ in } \mathcal{V}.$$

Obviously this λ-bracket is independent of the choices of the preimages of a and b.

Definition 16 The *quasiclassical limit* of the family of vertex algebras $V_\hbar$ is the quintuple $(\mathcal{V}, 1, \partial, \cdot, \{\cdot_\lambda \cdot\})$.

Definition 17 A *Poisson vertex algebra* is a quintuple $(\mathcal{V}, |0\rangle, \partial, \cdot, \{\cdot_\lambda \cdot\})$ which satisfies the following axioms,

(A) $(\mathcal{V}, \partial, \{_\lambda\})$ is a Lie conformal algebra,
(B) $(\mathcal{V}, 1, \partial, \cdot)$ is a commutative associative unital differential algebra,
(C) $\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}$ for all $a, b, c \in \mathcal{V}$ (left Leibniz rule).

Theorem 6 *The quasiclassical limit $\mathcal{V}$ of the family of vertex algebras $V_\hbar$ is a Poisson vertex algebra.*

Proof Since $V_\hbar$ is a vertex algebra over $\mathbb{F}[[\hbar]]$, due to Theorem 5 we have the quasicommutativity formula, the quasiassociativity formula and the non-commutative Wick formula for representatives in $V_\hbar$ of elements of $\mathcal{V}$. After taking the images of these formulas in $\mathcal{V}$, the "quantum corrections" disappear, hence $\mathcal{V}$ satisfies properties (B) and (C) of a PVA. Property (C) is satisfied as well since the axioms of a Lie conformal algebra are homogeneous in its elements. □

Exercise 13 Deduce from the left Leibniz rule and the skewcommutativity of the λ-bracket of a Poisson vertex algebra, the right Leibniz rule:

$$\{ab_\lambda c\} = \{b_{\lambda+\partial} c\}_\to a + \{a_{\lambda+\partial} c\}_\to b.$$

Given a Lie algebra $\mathfrak{g}$, we can associate to it two structures: the universal enveloping algebra $U(\mathfrak{g})$ and the Poisson algebra $S(\mathfrak{g})$. The Poisson bracket on $S(\mathfrak{g})$ is the extension of $\{a, b\} = [a, b]$ for all $a, b \in \mathfrak{g}$ by left and right Leibneiz rules. In fact, $S(\mathfrak{g})$ is the quasiclassical limit of $U(\mathfrak{g}_\hbar)$, where $\mathfrak{g}_\hbar$ is the Lie algebra $\mathbb{F}[[\hbar]] \otimes \mathfrak{g}$ over $\mathbb{F}[[\hbar]]$ with bracket $[a, b]_\hbar = \hbar[a, b]$ for $a, b \in \mathfrak{g}$. Indeed it is easy to see that the ordered monomials in a basis of $\mathfrak{g}$ form a basis of $U(\mathfrak{g}_\hbar)$ over $\mathbb{F}[[\hbar]]$. Hence

$U(\mathfrak{g}_\hbar)/\hbar U(\mathfrak{g}_\hbar) = S(\mathfrak{g})$ as associative algebras, and $\{a, b\} = \left.\frac{[\tilde{a}, \tilde{b}]_\hbar}{\hbar}\right|_{\hbar=0} = [a, b]$ for all $a, b \in \mathfrak{g}$ defines the Poisson structure on $S(\mathfrak{g})$.

Similar picture holds if in place of a Lie algebra $\mathfrak{g}$ we take a Lie conformal algebra R, and in place of $U(\mathfrak{g})$ we take $V(R)$, its universal enveloping vertex algebra. Recall its construction. We have the "maximal" formal distribution Lie algebra $(\mathrm{Lie}\, R, R)$, associated to R, which is regular (see [17], Chap. 2). Then $V(R) = V(\mathrm{Lie}\, R, R)$ (for another construction, entirely in terms of R, see [6]).

Consider the vertex algebra $V(R_\hbar)$ over $\mathbb{F}[[\hbar]]$, where $R_\hbar = R[[\hbar]]$ for the Lie conformal algebra R over $\mathbb{F}$, with λ-bracket defined by $[a_\lambda b]_\hbar = \hbar[a_\lambda b]$ for $a, b \in R$. In the same way as in the Lie algebra case, the quasiclassical limit is the Poisson vertex algebra, which, as a differential algebra, is $S(R)$ (the symmetric algebra of the $\mathbb{F}$-vector space R) with ∂, extended as its derivation, endowed with the λ-bracket $\{a_\lambda b\} = [a_\lambda b]$ on R, which is extended to $S(R)$ by the left and right Leibniz rules.

4.4 Representations of Vertex Algebras and Zhu Algebra

We have the following diagram

$$\begin{array}{ccc} PVA & \overset{\text{q.lim}}{\longleftarrow} & VA \\ {\scriptstyle \text{Zhu}}\downarrow & & \downarrow{\scriptstyle \text{Zhu}} \\ PA & \underset{}{\overset{\text{q.lim}}{\longleftarrow}} & AA \end{array}$$

In the diagram, *AA* means associative algebras, *PA* means Poisson algebras, *VA* means vertex algebras and *PVA* means Poisson vertex algebras; q.lim means the quasiclassical limit and Zhu means a functor from vertex algebras to associative algebras (resp. from Poisson vertex algebras to Poisson algebras), explained below.

Let V be a vertex algebra with a Hamiltonian operator H. Throughout this section we will assume (for simplicity) that all eigenvalues of H are integers. Recall the Borcherds identity from Sect. 3.2. For a and $b \in V$ with eigenvalues of H equal Δ_a and Δ_b respectively, we write

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-\Delta_a}, \quad b(z) = \sum_{a\in\mathbb{Z}} b_n z^{-n-\Delta_b} .$$

Then, comparing the coefficients of monomials in z and w in the Borcherds identity we have, for $m, n, k \in \mathbb{Z}$:

$$\sum_{j\geqslant 0} \binom{k}{j} (-1)^j (a_{m+k-j} b_{n+j} - (-1)^n b_{n+k-j} a_{m+j}) = \sum_{j\geqslant 0} \binom{m+\Delta_a-1}{j} (a_{(k+j)} b)_{m+n+k} . \tag{168}$$

Definition 18 A *representation* of the vertex algebra V in a vector space M is a linear map

$$V \longrightarrow (\operatorname{End} M)[[z, z^{-1}]], \quad a \longmapsto a^M(z) = \sum_{n \in \mathbb{Z}} a_n^M z^{-n-\Delta_a}, \tag{169}$$

defined for eigenvectors of H and then extended linearly to V, such that,

(i) $a^M(z)$ is an $\operatorname{End} M$-valued quantum field for all $a \in V$ (i.e., given $m \in M$, $a^M_{(n)} m = 0$ for $n \gg 0$),
(ii) $|0\rangle^M(z) = I_M$,
(iii) Borcherds identity holds, i.e., for $a, b \in V$, $c \in M$, $m, n, k \in \mathbb{Z}$ we have [cf. (168)]:

$$\sum_{j \geqslant 0} \binom{k}{j} (-1)^j (a^M_{m+k-j} b^M_{n+j} c - (-1)^n b^M_{n+k-j} a^M_{m+j} c)$$
$$= \sum_{j \geqslant 0} \binom{m + \Delta_a - 1}{j} (a_{(k+j)} b)^M_{m+n+k} c\,. \tag{170}$$

Remark 18 Note that $(Ta)_n = (-n - \Delta_a) a_n$ and $Ha = \Delta_a a$, hence, $((T+H)a)_0 = 0$.

Now assume that our vertex algebra V contains a conformal vector L of central charge $c \in \mathbb{F}$ (see Definition 11), so that $L_{-1} = T$ and $L_0 = H$ is a Hamiltonian operator. Then we have $L^M(z) = \sum_{n \in \mathbb{Z}} L^M_n z^{-n-2}$, and $[L^M_m, L^M_n] = (m-n) L^M_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} c I_M$.

Definition 19 A *positive energy representation* M of V is a representation with L^M_0 acting diagonalizably on M with spectrum bounded below, i.e., $M = \oplus_{j \geqslant h} M_j$ for some h, where $M_j = \{m \in M | L^M_0 m = jm\}$.

By (126) (which follows from the Borcherds identity) we have

$$a^M_n M_h = 0 \text{ for } n > 0, \; a^M_0 M_h \subset M_h\,. \tag{171}$$

So we have a linear map with $(H + T)V$ contained in the kernel (by Remark 18):

$$\pi_M : V \longrightarrow \operatorname{End} M_h, \; a \longmapsto a^M_0|_{M_h}\,. \tag{172}$$

Taking $m = 1, k = -1, n = 0$ in Borcherds identity (170) for $c \in M_h$, we get, by (171),

$$\pi_M(a) \pi_M(b) c = \pi_M(a * b) c, \; \text{for } a, b \in V,$$

where

$$a * b := \sum_{j \geqslant 0} \binom{\Delta_a}{j} a_{(j-1)} b \,. \tag{173}$$

Thus we get a representation of the algebra $(V, *)$ in the vector space M_h. The multiplication $*$ on V is not associative. However, we have the following remarkable theorem.

Theorem 7 ([21])

(a) $J(V) := ((T+H)V) * V$ *is a two-sided ideal of the algebra* $(V, *)$.
(b) $\mathrm{Zhu}V := (V/J(V), *)$ *is a unital associative algebra with 1 being the image of* $|0\rangle$.
(c) The map $M \to M_h$ *induces a map from the equivalence classes of positive energy* V*-modules to the equivalence classes of* Zhu$V$*-modules, which is bijective on irreducible modules.*

Proof We refer for the proof to the original paper [21] or to [6] for a simpler proof of a similar result without the assumption that the eigenvalues of H are integers. □

Exercise 14 Prove the commutator formula in Zhu algebra:

$$[a, b] := a * b - b * a = \sum\nolimits_{j \geqslant 0} \binom{\Delta_a - 1}{j} a_{(j)} b \tag{174}$$

Exercise 15 Let $\mathcal{V}$ be a Poisson vertex algebra and let H be a diagonalizable operator on $\mathcal{V}$, such that

$$\Delta_{a_{(n)}b} = \Delta_a + \Delta_b - n - 1, \quad \Delta_{\partial a} = \Delta_a + 1, \quad \Delta_{ab} = \Delta_a + \Delta_b,$$

where Δ_a is the eigenvalue of a, and

$$\{a_\lambda b\} = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} a_{(n)} b.$$

Show that $\mathrm{Zhu}\mathcal{V} := \mathcal{V}/((\partial + H)\mathcal{V})\mathcal{V}$ is a unital commutative associative algebra with the well defined Poisson bracket (cf. Exercise 14)

$$\{a, b\} = \sum_{j \geqslant 0} \binom{\Delta_a - 1}{j} a_{(j)} b. \tag{175}$$

Exercise 16 Let V (resp. $\mathcal{V}$) be a vertex algebra (resp. Poisson vertex algebra). Then $J =: (TV)V :$ (resp. $J = (\partial\mathcal{V}) \cdot \mathcal{V}$) is a two-sided ideal of the algebra $(V, ::)$ (resp. $(\mathcal{V}, \cdot)$), and V/J (resp. $\mathcal{V}/J$) is a Poisson algebra with the product, induced by :: (resp $\cdot$), and the well defined bracket, induced by the 0th product of the λ-bracket.

Of course, Zhu's theorem is just the beginning of the representation theory of vertex algebras, which has been a rapidly developing field in the past twenty years. Some of the most remarkable results of this theory are presented in the beautiful lecture course by T. Arakawa in this school.

5 Lecture 5 (January 14, 2015)

Given a vertex algebra V, one can construct its *quasiclassical limit*. As a result we get a Poisson vertex algebra (PVA). This can be done both considering a filtration of the vertex algebra V or by constructing a one parameter family of vertex algebras $V_\hbar$, as previously done in Lecture 4. This construction resembles the way a Poisson algebra arises as a quasiclassical limit of a family of associative algebras, hence the name "Poisson" vertex algebra. The reason we are interested in such structures is that the theory of Poisson vertex algebras has important relation with the theory of integrable systems of PDE's. This relation is parallel to (but a bit different from) the relation of Poisson algebras with the theory of integrable systems of ODE's.

5.1 From Finite-Dimensional to Infinite-Dimensional Poisson Structures

Let us start by recalling the definition of a Poisson vertex algebra:

Definition 20 A PVA is a quintuple $(\mathscr{V},\ \partial,\ 1,\ \cdot, \{\cdot_\lambda\cdot\})$ such that:

1. $(\mathscr{V},\ \partial,\ 1,\ \cdot)$ is a differential algebra;
2. $(\mathscr{V},\ \partial, \{\cdot_\lambda\cdot\})$ is a Lie conformal algebra, whose λ-bracket satisfies the following axioms:
   (i) (sesquilinearity) $\{\partial a_\lambda b\} = -\lambda\{a_\lambda b\}, \qquad \{a_\lambda \partial b\} = (\partial + \lambda)\{a_\lambda b\}$;
   (ii) (skewsymmetry) $\{b_\lambda a\} = -\{a_{-\partial-\lambda} b\}$;
   (iii) (Jacobi identity) $\{a_\lambda\{b_\mu c\}\} - \{b_\mu\{a_\lambda c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\}$;
3. $\{\cdot_\lambda\cdot\}$ and $\cdot$ are related by the following Leibniz rules:
   (i) (left Leibniz rule) $\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}$;
   (ii) (right Leibniz rule) $\{ab_\lambda c\} = \{a_{\lambda+\partial} c\}_\to b + \{b_{\lambda+\partial} c\}_\to a$.

Remark 19 We use the following notation: if $\{a_\lambda b\} = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!} a_{(n)} b$, then when a right arrow appears it means that $\lambda + \partial$ has to be moved to the right: $\{a_{\lambda+\partial} b\}_\to c = \sum_{n\in\mathbb{Z}_+} \frac{a_{(n)}b}{n!}(\lambda + \partial)^n c$. However, if no arrow appears we just have $\{a_{-\partial-\lambda} b\} = \sum_{n\in\mathbb{Z}_+} \frac{(-\lambda-\partial)^n}{n!} a_{(n)} b$.

In the theory of Hamiltonian ODEs the key role is played by the Poisson bracket on the space of smooth functions $\mathcal{F}$ on a manifold. Choosing local coordinates $u_1, \ldots, u_\ell$ on the manifold, we can endow $\mathcal{F}$ with a structure of Poisson algebra, letting

$$\{u_j, u_i\} = H_{ij} \in \mathcal{F}. \tag{176}$$

By the Leibniz rule this extends to polynomials in the variables u_i as follows:

$$\{f, g\} = \frac{\partial g}{\partial u} \cdot H \frac{\partial f}{\partial u}, \tag{177}$$

where $\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f}{\partial u_1} \\ \vdots \\ \frac{\partial f}{\partial u_\ell} \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$, $H = (H_{ij})_{i,j=1}^{\ell}$ is an $\ell \times \ell$ matrix with coefficients in $\mathcal{F}$, and $\cdot$ is the usual dot product of vectors from $\mathcal{F}^\ell$ with values in $\mathcal{F}$. Formula (177) extends to arbitrary functions $f, g \in \mathcal{F}$. This bracket obviously satisfies the Leibniz rule, but it is not necessarily skewsymmetric, neither it satisfies the Jacobi identity. If the matrix H is skewsymmetric (i.e. $H^T = -H$), then the bracket (177) is skewsymmetric. If, in addition, it satisfies the Jacobi identity (which happens iff $[H, H] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket), then the matrix H is called a *Poisson structure* on $\mathcal{F}$.

Definition 21 The *Hamiltonian ODE* associated with this Poisson structure is

$$\frac{du}{dt} = \{h, u\} = H \frac{\partial h}{\partial u}, \tag{178}$$

where the second equality follows from (177). The function $h \in \mathcal{F}$ is called the *Hamiltonian* of this equation.
This is a special case of what is called an *evolution ODE*, that is

$$\frac{du}{dt} = F(u), \text{ for some } F \in \mathcal{F}^\ell.$$

In the theory of Hamiltonian PDEs a similar role is played by PVAs. Let us now see how to construct a similar machinery.

First of all we need to define which kind of differential algebra $\mathcal{V}$ we want for our PVA. The basic example is the *algebra of differential polynomials* in ℓ variables $\mathcal{P}_\ell = \mathbb{F}[u_i^{(n)} \mid i \in I = \{1, \ldots, \ell\}, n \in \mathbb{Z}_+]$, which is a differential algebra with derivation ∂, called the *total derivative*, such that $\partial u_i^{(n)} = u_i^{(n+1)}$.

Definition 22 An *algebra of differential functions* in ℓ variables $\mathcal{V}$ is a differential algebra with a derivation ∂, which is an extension of the algebra of differential polynomials $\mathcal{P}_\ell$, endowed with linear maps $\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \to \mathcal{V}$ for all $i \in I$, $n \in \mathbb{Z}_+$, which are commuting derivations of $\mathscr{V}$, extending the usual partial derivatives in $\mathscr{P}_\ell$, and satisfying the following axioms:

(i) given $f \in \mathscr{V}$, $\frac{\partial f}{\partial u_i^{(n)}} = 0$ for all but finitely many pairs $(i, n) \in I \times \mathbb{Z}_+$;
(ii) $[\frac{\partial}{\partial u_i^{(n)}}, \partial] = \frac{\partial}{\partial u_i^{(n-1)}}$ (where the RHS is considered to be zero if $n = 0$).

Which differential algebras are algebras of differential functions? The algebra of differential polynomials $\mathscr{P}_\ell$ itself clearly satisfies these axioms (it suffices to check (ii) on the generators $u_i^{(n)}$). One can as well consider the corresponding field of fractions $\mathscr{Q}_\ell = \mathbb{F}(u_i^{(n)} \mid i \in I,\ n \in \mathbb{Z}_+)$, or any algebraic extension of $\mathscr{P}_\ell$ or $\mathscr{Q}_\ell$, obtained by adding a solution of a polynomial equation. However, if we want both axioms to hold, we can not add a solution of an arbitrary differential equation: for example, we can add e^u, solution of $f' = fu'$, but we can not add a non-zero solution of $f' = fu$.

Exercise 17 Let $\mathscr{V} = \mathscr{P}_1[v]$ with the derivation ∂, extended from $\mathscr{P}_1$ by $\partial v = vu_1$ or by $\partial v = u_1$. Show that the structure of an algebra of differential functions cannot be extended from $\mathscr{P}_1$ to $\mathscr{V}$.
The reasons why we want both properties (i) and (ii) to hold will soon be clear.

We also want an analogue of the bracket given by (177) and to understand what a Poisson structure is in the infinite-dimensional case. Recall the following (non-rigorous) formula which appears in any textbook on integrable Hamiltonian PDE, cf. [19], but not [12]. It defines the Poisson bracket on generators ($i, j \in I$) as

$$\{u_i(x), u_j(y)\} = H_{ji}(u(y), u'(y), \ldots, u^{(n)}(y); \frac{\partial}{\partial y})\delta(x - y), \tag{179}$$

where $H = (H_{ji})_{i,j=1}^{\ell}$ is an $\ell \times \ell$ matrix differential operator on $\mathscr{V}^\ell$, the u_i's are viewed as functions in x on a one-dimensional manifold, and $\delta(x - y)$ is the usual delta function.

Example 11 The first example is given by the Gardner-Faddeev-Zakharov (GFZ) bracket, for $\mathscr{V} = \mathscr{P}_1$, and it goes back to 1971:

$$\{u(x), u(y)\} = \frac{\partial}{\partial y}\delta(x - y). \tag{180}$$

As in the ODE case, we can extend the bracket defined in (179) by the Leibniz rule. Then, for arbitrary $f, g \in \mathscr{V}$ we have

$$\{f(x), g(y)\} = \sum_{i,j \in I,\, p,q \in \mathbb{Z}_+} \frac{\partial f}{\partial u_i^{(p)}} \frac{\partial g}{\partial u_j^{(q)}} \partial_x^p \partial_y^q \{u_i(x), u_j(y)\}. \tag{181}$$

The basic idea is to introduce the λ-bracket by application of the Fourier transform

$$F(x,y) \mapsto \int e^{\lambda(x-y)} F(x,y)\, dx \tag{182}$$

to both sides of (181):

$$\{f_\lambda g\} := \int e^{\lambda(x-y)} \{f(x), g(y)\} dx. \tag{183}$$

Thus, for arbitrary $f, g \in \mathcal{V}$, we get a rigorous formula, called the *Master Formula*:

$$\{f_\lambda g\} = \sum_{i,j\in I,\, p,q\in\mathbb{Z}_+} \frac{\partial g}{\partial u_j^{(q)}} (\partial+\lambda)^q \{u_{i\,\partial+\lambda} u_j\}_\to (-\partial-\lambda)^p \frac{\partial f}{\partial u_i^{(p)}}. \tag{MF}$$

Here, $\{u_{j\,\partial+\lambda} u_i\} = H_{ij}(\partial+\lambda)$, where $H(\partial) = (H_{ij}(\partial))_{i,j\in I}$ is a matrix differential operator with coefficients in $\mathcal{V}$ for which the λ-bracket is its symbol.

Exercise 18 Derive (MF) from (181).

Note that (MF) is similar to the formula for the Poisson bracket defined by Eq. (177). In fact, to go from the former to the latter we just put λ and ∂ equal to 0.

Theorem 8 ([3]) *Let $\mathcal{V}$ be an algebra of differential functions in the variables $\{u_i\}_{i\in I}$. For each pair $i,j \in I$ choose $\{u_{i\lambda} u_j\} = H_{ji}(\lambda) \in \mathcal{V}[\lambda]$. Then*

1. *The Master Formula (MF) defines a λ-bracket on $\mathcal{V}$ which satisfies sesquilinearity, the left and right Leibniz rules, and extends the given λ-bracket on the variables u_i's. Consequently, any λ-bracket on the algebra of differential polynomials, satisfying these properties, is given by the Master Formula.*
2. *This λ-bracket is skewsymmetric provided skewsymmetry holds for every pair of variables:*

$$\{u_{i\lambda} u_j\} = -\{u_{j_{-\lambda-\partial}} u_i\}, \quad \forall\, i,j \in I. \tag{184}$$

3. *If this λ-bracket is skewsymmetric, then it satisfies the Jacobi identity, provided Jacobi identity holds for every triple of variables:*

$$\{u_{i\lambda}\{u_{j_\mu} u_k\}\} - \{u_{j_\mu}\{u_{i\lambda} u_k\}\} = \{\{u_{i\lambda} u_k\}_{\lambda+\mu} u_j\}, \quad \forall\, i,j,k \in I. \tag{185}$$

It follows from Theorem 8 that, if the corresponding conditions on the variables u_i's hold, the λ-bracket defined by the Master Formula (MF) endows $\mathcal{V}$ with a structure of PVA. As in the finite-dimensional case, this structure is completely defined by $H(\lambda) = (H_{ij}(\lambda)) \in \mathrm{Mat}_{\ell\times\ell}\mathcal{V}[\lambda]$.

Definition 23 We say that the matrix differential operator $H(\partial) \in \mathrm{Mat}_{\ell\times\ell}\mathscr{V}[\partial]$ with the symbol $H(\lambda)$ is a *Poisson structure* if the corresponding λ-bracket defines a PVA structure on $\mathscr{V}$.

Exercise 19 The λ-bracket, given by the Master Formula, is skewsymmetric if and only if the matrix differential operator $H(\partial)$ is skewadjoint.

Example 12 Let $\mathscr{V} = \mathscr{P}_1 = \mathbb{F}[u, u', u'', \ldots]$. From the GFZ bracket defined in Example 11 we get the following λ-bracket: $\{u_\lambda u\} = \lambda$. The skewsymmetry and the Jacobi identity for the λ-bracket, given by the Master Formula, are immediate by Theorem 8. The associated Poisson structure is $H(\partial) = \partial$. This PVA is the quasiclassical limit of the family of free boson vertex algebras $B_\hbar$.

Example 13 Let $\mathscr{V} = \mathscr{P}_1 = \mathbb{F}[u, u', u'', \ldots]$. The Magri-Virasoro PVA with central charge $c \in \mathbb{F}$ is defined by the following λ-bracket:

$$\{u_\lambda u\} = (\partial + 2\lambda)u + c\lambda^3 + \alpha\lambda. \tag{186}$$

Of course, it is straightforward to check that the pair u, u satisfies (184) and the triple u, u, u satisfies (185), hence, by Theorem 8, we get a PVA. It is instructive, however, to give a more conceptual proof. Consider the Lie conformal algebra Vir from Example 8. Then by Theorem 8, $S(\mathrm{Vir})$ is a PVA, hence its quotient $\mathcal{V}^c$ by the ideal, generated by $C - c$, is a PVA, which is obviously isomorphic to the Magri-Virasoro PVA. The corresponding family of Poisson structures is

$$H(\partial) = u' + 2u\partial + c\partial^3 + \alpha\partial. \tag{187}$$

These Poisson structures were discovered by Magri; the name is due to its connection to the Virasoro algebra. Note that $\mathcal{V}^c$ is the quasiclassical limit of the family of universal Virasoro vertex algebras $V_\hbar^{12c}$.
The following exercise shows that the discrete series vertex algebras V_c with c given by (60) is a purely quantum effect.

Exercise 20 Show that the PVA $\mathcal{V}^c$ is simple if $c \neq 0$.

Example 14 Given a vector space U, denote by $\mathcal{P}(U) = S(\mathbb{F}[\partial] \otimes U)$ the algebra of differential polynomials over U. Let $\mathfrak{g}$, $(.\,|\,.)$ be as in Example 6, let $k \in \mathbb{F}$, and fix $s \in \mathfrak{g}$. Then the associated *affine PVA* $\mathcal{V}^k(\mathfrak{g}, s)$ is defined as the algebra of differential polynomials $\mathcal{P}(\mathfrak{g})$, endowed with the λ-brackets ($a, b \in \mathfrak{g}$) :

$$\{a_\lambda b\} = [a, b] + \lambda(a|b)k + (s|[a, b])1. \tag{188}$$

The two proofs from Example 13 apply to show that $\mathcal{V}^k(\mathfrak{g}, s)$ is a PVA. Of course, up to isomorphism, it is independent of s, but the trivial cocycle is important for the associated integrable system, since we get a multiparameter family of Poisson structures. Note that $\mathcal{V}^k(\mathfrak{g}, s)$ is the quasiclassical limit of $V_\hbar^k(\mathfrak{g})$.

Now we recall how one passes from the definition of a Hamiltonian ODE to that of a Hamiltonian PDE. The following idea goes back to the 1970s: in order to get an "honest" Lie algebra bracket, we should not consider the whole algebra of differential functions $\mathcal{V}$, but its quotient $\mathcal{V}/\partial\mathcal{V}$, which is not an algebra anymore, just a vector space. Denote by $\int$ the quotient map $\int : \mathcal{V} \to \mathcal{V}/\partial\mathcal{V}$. The corresponding bracket is defined by

$$\{\textstyle\int f, \int g\} = \displaystyle\int \frac{\delta g}{\delta u} \cdot H(\partial) \frac{\delta f}{\delta u}, \tag{189}$$

where $\frac{\delta f}{\delta u}$ is the vector of *variational derivatives* of f:

$$\frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}.$$

Elements $\int f \in \mathcal{V}/\partial\mathcal{V}$ are called *local functionals*.

Equation (189) is analogous to Eq. (177), with variational derivatives instead of partial derivatives, and a matrix differential operator $H(\partial)$ instead of a matrix of functions. It is rather difficult to prove directly that (189) is a Lie algebra bracket on $\mathcal{V}/\partial\mathcal{V}$. The connection to the PVA theory, explained further on, makes it very easy.

The following exercise shows that (189) is well defined.

Exercise 21 The variational derivative $\frac{\delta f}{\delta u}$ depends only on the image of $f \in \mathcal{V}$ in the quotient space $\mathcal{V}/\partial\mathcal{V}$, since $\frac{\delta}{\delta u} \circ \partial = 0$. Deduce the latter fact from axiom (ii) in the Definition 22 of an algebra of differential functions.

Given a local functional $\int h$, in analogy with (178), one defines the associated *Hamiltonian PDE* as the following evolution PDE:

$$\frac{du}{dt} = H(\partial) \frac{\delta \int h}{\delta u}. \tag{190}$$

The local functional $\int h$ is called the *Hamiltonian* of this equation.

We shall explain further on how these classical definitions fit nicely in the framework of Poisson vertex algebras.

5.2 *Basic Notions of the Theory of Integrable Equations*

An evolution equation in the infinite-dimensional case is quite the same as in the finite-dimensional case, except it is a partial differential equation.

Definition 24 Let $\mathcal{V}$ be an algebra of differential functions in ℓ variables $u_1, \dots, u_\ell$. An *evolution PDE* is

$$\frac{du}{dt} = F(u, u', \dots, u^{(n)}), \tag{191}$$

where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$ and $F = \begin{pmatrix} F_1 \\ \vdots \\ F_\ell \end{pmatrix} \in \mathcal{V}^\ell$. Here, $u_i = u_i(x, t)$ is a function in one independent variable x, and the parameter t is called *time*.

Given an arbitrary differential function $f \in \mathcal{V}$, by the chain rule we have

$$\frac{df}{dt} = \sum_{i \in I,\, n \in \mathbb{Z}_+} \frac{d(u_i^{(n)})}{dt} \frac{\partial f}{\partial u_i^{(n)}}. \tag{192}$$

Since, by (191), we have $\frac{d(u_i^{(n)})}{dt} = \partial^n F_i$, the function f evolves in virtue of Eq. (191) as

$$\frac{df}{dt} = X_F f,$$

where

$$X_F = \sum_{i \in I,\, n \in \mathbb{Z}_+} (\partial^n F_i) \frac{\partial}{\partial u_i^{(n)}} \tag{193}$$

is a derivation of the algebra $\mathcal{V}$, called the *evolutionary vector field* with characteristic $F \in \mathcal{V}^\ell$. It is now clear why Axiom (i) in Definition 22 is important: otherwise, the evolutionary vector field would give a divergent sum when applied to arbitrary functions $f \in \mathcal{V}$.

An important notion in the theory of integrable systems is *compatibility* of evolution equations:

Definition 25 Equation (191) is called *compatible* with the evolution PDE

$$\frac{du}{d\tau} = G(u, u', \dots, u^{(m)}) \in \mathcal{V}^\ell \tag{194}$$

where, as before, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$ and $G = \begin{pmatrix} G_1 \\ \vdots \\ G_\ell \end{pmatrix} \in \mathcal{V}^\ell$, if the corresponding flows commute, that is if $\frac{d}{dt}\frac{d}{d\tau} f = \frac{d}{d\tau}\frac{d}{dt} f$ holds for every function $f \in \mathcal{V}$.

By the above discussion, the compatibility of evolution equations (191) and (194) is equivalent to the property that the corresponding evolutionary vector fields commute: $[X_F, X_G] = 0$, which is a purely Lie algebraic condition. In fact, we can easily see that the commutator of two evolutionary vector fields is again an evolutionary vector field. This follows from the next exercise.

Exercise 22 Prove that $[X_F, X_G] = X_{[F,G]}$, where $[F, G] := X_F G - X_G F$.
Thus, the bracket $[F, G] = X_F G - X_G F$ endows $\mathscr{V}^\ell$ with a Lie algebra structure, called the *Lie algebra of evolutionary vector fields*.

If two evolutionary vector fields commute, then each of them is called a *symmetry* of the other. So if $[X_F, X_G] = 0$, F is a symmetry of G and G is a symmetry of F. Note that every evolutionary vector field commutes with $\partial = X_{u'} = \sum_{i\in I,\, n\in\mathbb{Z}_+} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}$.

Let us now introduce the notion of *integrability* for an evolution equation.

Definition 26 Equation (191) is called *Lie integrable* if X_F is contained in an infinite-dimensional abelian subalgebra of the Lie algebra $\mathscr{V}^\ell$.

Remark 20 Informally, one says that Eq. (191) is Lie integrable if it admits infinitely many commuting symmetries.

Example 15 The linear equations over $\mathscr{P}_1$:

$$u_t = u^{(n)},\ n \in \mathbb{Z}_+,$$

are Lie integrable. Indeed, $X_{u^{(m)}}(u^{(n)}) = u^{(m+n)}$ is symmetric in m and n, hence the corresponding evolutionary vector fields commute.

Example 16 The dispersionless equations over $\mathscr{P}_1$:

$$u_t = f(u)u',\ \ f(u) \in \mathscr{P}_1,$$

are Lie integrable, since

$$X_{f(u)u'}(g(u)u') = \frac{\partial}{\partial u}(f(u)g(u))u'^2 + f(u)g(u)u''$$

is symmetric in f and g, hence the corresponding evolutionary vector fields commute.

The motivation for the definition of Lie integrability of PDE's comes from a theorem of Lie in the theory of ODE's, saying that if the evolution ODE in ℓ variables $\frac{du}{dt} = F(u)$ possesses ℓ commuting symmetries with a non-degenerate Jacobian, then it can be solved in quadratures. Of course, in the PDE case the number of coordinates is infinite, therefore we need to require infinitely many commuting symmetries.

There has been a lot of work trying to establish integrability of various partial differential equations. One well-known method of constructing symmetries of an evolution equation is called *recursion operator*; however, in all examples the recursion operator is actually a pseudodifferential operator (which is an element of $\mathcal{V}((\partial^{-1}))$), hence it can not be applied to functions, as Exercise 17 demonstrates. We will discuss a different approach, the *Hamiltonian* approach, which is completely rigorous.

We shall deduce Lie integrability from the stronger *Liouville integrability* of Hamiltonian PDE, which, analogously to the definition for ODEs, requires the existence of infinitely many integrals of motion in involution.

5.3 Poisson Vertex Algebras and Hamiltonian PDE

In order to translate the traditional language of Hamiltonian PDE's, discussed above, to the language of PVA's, and also, to connect the two notions of integrability, the following simple lemma is crucial.

Lemma 8 (Basic Lemma) *Let $\mathcal{V}$ be a PVA. Let $\bar{\mathcal{V}} := \mathcal{V}/\partial\mathcal{V}$ and let $\int : \mathcal{V} \to \bar{\mathcal{V}}$ be the corresponding quotient map. Then we have the following well-defined brackets:*

(i) $\bar{\mathcal{V}} \times \bar{\mathcal{V}} \longrightarrow \bar{\mathcal{V}}, \qquad \{\int a, \int b\} := \int \{a_\lambda b\}_{\lambda=0},$
(ii) $\bar{\mathcal{V}} \times \mathcal{V} \longrightarrow \mathcal{V}, \qquad \{\int a, b\} := \{a_\lambda b\}_{\lambda=0}.$

Moreover, (i) defines a Lie algebra bracket on $\bar{\mathcal{V}}$, and (ii) defines a representation of the Lie algebra $\bar{\mathcal{V}}$ on $\mathcal{V}$ by derivations of the product and the λ-bracket of $\mathcal{V}$, commuting with ∂.

Proof It all follows directly when we put $\lambda = 0$ in the axioms for the λ-bracket $\{\cdot_\lambda\cdot\}$ of a PVA. First, both brackets are well defined since sesquilinearity holds for $\{\cdot_\lambda\cdot\}$: for every $a, b \in \mathcal{V}$ we have $\{\partial a, b\} = -\lambda\{a_\lambda b\}_{\lambda=0} = 0$ and $\{a, \partial b\} = \{a_\lambda \partial b\}_{\lambda=0} = \partial\{a_\lambda b\} \in \partial\mathcal{V}$.

Let us now verify the Lie algebra axioms for the first bracket: note that $\int\{b_{-\lambda-\partial} a\}_{\lambda=0} = \int\{b_\lambda a\}_{\lambda=0}$ since only the coefficients of the 0th power of $-\lambda-\partial$ and λ respectively survive in $\bar{\mathcal{V}}$, and they obviously coincide. By skewsymmetry of $\{\cdot_\lambda\cdot\}$ we have

$$\{\textstyle\int a, \int b\} = \int \{a_\lambda b\}_{\lambda=0} = -\int \{b_{-\lambda-\partial} a\}_{\lambda=0} = -\int \{b_\lambda a\}_{\lambda=0} = -\{\int b, \int a\}. \tag{195}$$

Hence, skewsymmetry holds for (i). Similarly, the Jacobi identity for $\{\cdot_\lambda\cdot\}$ provides that the Jacobi identity holds for this bracket as well, just putting $\lambda = \mu = 0$ in the corresponding definitions:

$$\{\textstyle\int a, \{\int b, \int c\}\} = \{\int b, \{\int a, \int c\}\} + \{\{\int a, \int b\}, \int c\}. \tag{196}$$

Therefore, $\bar{\mathcal{V}}$ is endowed with a Lie algebra structure with the Lie bracket defined by (i).

Next, we have to check that (ii) is a representation of $\bar{\mathscr{V}}$ on $\mathscr{V}$, i.e., that

$$\{\{\textstyle\int f, \int g\}, a\} = \{\int f, \{\int g, a\}\} - \{\int g, \{\int f, a\}\} \tag{197}$$

holds for all $\int f, \int g \in \bar{\mathscr{V}}$, $a \in \mathscr{V}$. Again, this is due to the Jacobi identity. Then we have to check that $\bar{\mathscr{V}}$ acts on $\mathscr{V}$ as derivations of the product. For $a, b \in \mathscr{V}$ and $\int h \in \bar{\mathscr{V}}$ we have, by the left Leibniz rule:

$$\begin{aligned}\{\textstyle\int h, ab\} &= \{h_\lambda ab\}_{\lambda=0} = (\{h_\lambda a\}b)_{\lambda=0} + (\{h_\lambda b\}a)_{\lambda=0} = \\ &= \{h_\lambda a\}_{\lambda=0}\, b + \{h_\lambda b\}_{\lambda=0}\, a = \{\textstyle\int h, a\}b + \{\int h, b\}a.\end{aligned} \tag{198}$$

Similarly, by the Jacobi identity, we check that it acts by derivations of the λ-bracket. Finally, we have to check that the derivations $\{\int h, \cdot\}$ commute with ∂. For every $a \in \mathscr{V}$ we have

$$\begin{aligned}(\{\textstyle\int h, \cdot\} \circ \partial)a &= \{\textstyle\int h, \partial a\} = \{h_\lambda \partial a\}_{\lambda=0} = ((\lambda + \partial)\{h_\lambda a\})_{\lambda=0} = \partial\{h_\lambda a\}_{\lambda=0} \\ &= (\partial \circ \{\textstyle\int h, \cdot\})a\end{aligned} \tag{199}$$

due to the sesquilinearity of $\{\cdot_\lambda\cdot\}$. □

Definition 27 Given a PVA $\mathscr{V}$ and a local functional $\int h \in \bar{\mathscr{V}}$, the associated *Hamiltonian PDE* is

$$\frac{du}{dt} = \{\textstyle\int h, u\}. \tag{200}$$

The local functional $\int h$ is called the *Hamiltonian* of this equation.

In the case when the PVA $\mathscr{V}$ is an algebra of differential functions in the variables $\{u_i\}_{i\in I}$ and the λ-bracket is given by the Master Formula (MF), we reproduce the traditional definitions:

(i) Hamiltonian PDE: $\frac{du}{dt} = \{\int h, u\} = H\frac{\delta \int h}{\delta u}$;
(ii) Poisson bracket on $\bar{\mathscr{V}}$: $\{\int f, \int g\} = \int \frac{\delta g}{\delta u} \cdot H\frac{\delta f}{\delta u}$.

The first claim is obvious, and the second is obtained by integration by parts.

It follows that in this case $\bar{\mathscr{V}}$ acts on $\mathscr{V}$ by evolutionary vector fields: $\int f \mapsto X_{H\frac{\delta f}{\delta u}}$, and that the following holds.

Corollary 3 *We have a Lie algebra homomorphism* $\bar{\mathscr{V}} \to \mathscr{V}^\ell$, $\int f \mapsto X_{H\frac{\delta f}{\delta u}}$.

Thus, in the case when $\mathscr{V}$ is an algebra of differential functions with the Poisson λ-bracket given by the Master formula, the Hamiltonian equation is a special case of the evolution equation with RHS $H\frac{\delta \int h}{\delta u}$ and the corresponding evolutionary vector field is $X_{H\frac{\delta h}{\delta u}}$.

Definition 28 A local functional $\int f \in \bar{\mathcal{V}}$ is called an *integral of motion* of the evolution equation (191) and f is called a *conserved density*, if $\int \frac{df}{dt} = 0$, or, equivalently, if $\int X_F f = 0$. Integrating by parts, this, in turn, is equivalent to

$$\int \frac{\delta f}{\delta u} \cdot F = 0. \tag{201}$$

Hence, $\int f$ is an integral of motion of the Hamiltonian equation (200) if and only if f and h are in *involution*, that is if $\{\int f, \int h\} = 0$.

So, we have completely translated the language of Hamiltonian PDEs into the language of PVAs.

Definition 29 The Hamiltonian PDE (200) is called *Liouville integrable* if $\int h$ is contained in an infinite-dimensional abelian subalgebra of the Lie algebra $\bar{\mathcal{V}}$. That is, if there exists an infinite sequence of linearly independent local functionals $\int h_n$, such that $\int h_0 = \int h$ and $\{\int h_n, \int h_m\} = 0$ for all $n, m \in \mathbb{Z}_+$.

By Corollary 3, integrals of motion in involution go to commuting evolutionary vector fields $X_{H \frac{\delta \int h}{\delta u}}$. Hence Liouville integrability usually implies Lie integrability (provided we make some weak assumption on $H(\partial)$, such as $H(\partial)$ is non-degenerate). In fact, in order to check that the local functionals are linearly independent, it is usually easier to check that the corresponding evolutionary vector fields are linearly independent.

Exercise 23 Show that the equation $\frac{du}{dt} = u''$ is Lie integrable, but has no non-trivial integrals of motion, hence is not Hamiltonian. On the other hand the equation $\frac{du}{dt} = u'''$ is Hamiltonian with $H = \partial$, $h = -\frac{1}{2}(u')^2$, and it is both Lie and Liouville integrable.

Remark 21 Let $F, G, \ldots$ be a sequence of elements of $\mathcal{V}^\ell$, such that the corresponding evolutionary vector fields commute, i.e. the corresponding evolution equations are compatible. Then we have a *hierarchy* of evolution equations

$$\frac{du}{dt_0} = F, \quad \frac{du}{dt_1} = G, \ \ldots, \tag{202}$$

so that the solution of this hierarchy depends now on x and on infinitely many times: $u = u(x, t_0, t_1, t_2, \ldots)$.

5.4 *The Lenard-Magri Scheme of Integrability*

There is a very simple scheme to prove integrability, called the *Lenard-Magri scheme*. Although it is not a theorem, it always works in practice.

Let $\mathcal{V}$ be an algebra of differential functions in ℓ variables $u_1, \ldots, u_\ell$. First of all, introduce the following symmetric bilinear forms on $\mathcal{V}^\ell$:

$$(\cdot|\cdot) : \mathcal{V}^\ell \times \mathcal{V}^\ell \longrightarrow \bar{\mathcal{V}}, \quad (F|G) = \textstyle\int F \cdot G. \tag{203}$$

Given a matrix differential operator $H(\partial) \in \mathrm{Mat}_{\ell\times\ell}\mathcal{V}[\partial]$

$$\langle\cdot,\cdot\rangle_H : \mathcal{V}^\ell \times \mathcal{V}^\ell \longrightarrow \bar{\mathcal{V}}, \quad \langle F, G\rangle_H = (H(\partial)F|G). \tag{204}$$

Note that $(H(\partial)F|G) = (F|H^*(\partial)G)$, where $H^*(\partial)$ is the adjoint differential operator of $H(\partial)$. Indeed, defining $*$ on $\mathcal{V}[\partial]$ as an anti-involution such that $*(f) = f$ and $*(\partial) = -\partial$, we get $(\partial f|g) = -(f|\partial g)$ because $(\partial f|g) + (f|\partial g) = \int \partial(fg) = 0$ in $\bar{\mathcal{V}}$. Hence, if $H(\partial)$ is skewadjoint, then the bilinear form (204) is skewsymmetric.

Proof of Liouville integrability is based on the following result.

Lemma 9 (Lenard Lemma) *Let $H(\partial)$ and $K(\partial)$ be skewadjoint differential operators on $\mathcal{V}^\ell$. Suppose elements $\xi_0, \ldots, \xi_N \in \mathcal{V}^\ell$ satisfy the following Lenard-Magri relation:*

$$K(\partial)\xi_{n+1} = H(\partial)\xi_n, \quad n = 0, \ldots, N-1. \tag{205}$$

Then, the $\langle\xi_m, \xi_n\rangle = 0$ for all m, $n = 0, \ldots, N$, whenever we consider it with respect to H or K: $\langle\xi_m, \xi_n\rangle_{H,K} = 0$.

Proof Proceed by induction on $i = |m-n|$. If $i = 0$, then $m = n$ and we get $\langle\xi_n, \xi_n\rangle_{H,K} = -\langle\xi_n, \xi_n\rangle_{H,K}$ because the form is skewsymmetric, therefore it is equal to zero. Now let $i > 0$; by skewsymmetry we may assume $m > n$. We have

$$\begin{aligned}\langle\xi_m, \xi_n\rangle_H &= (H(\partial)\xi_m|\xi_n) = -(\xi_m|H(\partial)\xi_n) = -(\xi_m|K(\partial)\xi_{n+1}) = (K(\partial)\xi_m|\xi_{n+1}) \\ &= \langle\xi_m, \xi_{n+1}\rangle_K,\end{aligned} \tag{206}$$

and, by the induction hypothesis, the RHS is zero, since $|m-(n+1)| < |m-n|$. Similarly we have, assuming $n > m$:

$$\begin{aligned}\langle\xi_m, \xi_n\rangle_K &= (K(\partial)\xi_m|\xi_n) = -(\xi_m|K(\partial)\xi_n) = -(\xi_m|H(\partial)\xi_{n-1}) = (H(\partial)\xi_m|\xi_{n-1}) \\ &= \langle\xi_m, \xi_{n-1}\rangle_H\end{aligned} \tag{207}$$

and again, by induction hypothesis the RHS is zero since $|n-1-m| < |n-m|$. □

This lemma is important since, if we can prove that the elements $\xi_m \in \mathcal{V}^\ell$ are variational derivatives, i.e. $\xi_m = \frac{\delta \int h_m}{\delta u}$ for some local functionals $\int h_m$, it guarantees that $\int h_m$ and $\int h_n$ are in involution with respect to both brackets on $\bar{\mathcal{V}}$. Indeed, we know that the bracket on $\bar{\mathcal{V}}$ for the Poisson structure H is given by

$$\{\textstyle\int f, \int g\}_H = \displaystyle\int \frac{\delta g}{\delta u} \cdot H(\partial)\frac{\delta f}{\delta u} = \left(\frac{\delta g}{\delta u}\Big|H(\partial)\frac{\delta f}{\delta u}\right) = \left\langle\frac{\delta f}{\delta u}, \frac{\delta g}{\delta u}\right\rangle_H, \tag{208}$$

therefore, if ξ_n, ξ_m are variational derivatives, then by Lemma 9 we get

$$\{\textstyle\int h_m, \int h_n\}_H = \left\langle \frac{\delta \int h_m}{\delta u}, \frac{\delta \int h_n}{\delta u} \right\rangle_H = \langle \xi_m, \xi_n \rangle_H = 0, \tag{209}$$

and the same holds for K. In other words, we have the following corollary of Lenard's lemma.

Corollary 4 *Let $H(\partial)$ and $K(\partial)$ be skewadjoint differential operators on $\mathcal{V}^\ell$. Suppose that the local functionals $\int h_0, \dots, \int h_N$ satisfy the following relation:*

$$K(\partial)\frac{\delta \int h_{n+1}}{\delta u} = H(\partial)\frac{\delta \int h_n}{\delta u}, \quad n = 0, \dots, N-1. \tag{210}$$

Then all these local functionals are in involution with respect to both brackets $\{.,.\}_H$ and $\{.,.\}_K$ on $\bar{\mathcal{V}}$.
In the case when (210) holds, and K, H are Poisson structures, one says that the evolution equations

$$\frac{du}{dt_n} = K(\partial)\frac{\delta \int h_{n+1}}{\delta u} = H(\partial)\frac{\delta \int h_n}{\delta u}$$

form a hierarchy of *bi-Hamiltonian* equations. Note that if the right-hand sides of these equations span an infinite-dimensional subspace in the space of evolutionary vector fields, then all of these equations are both Lie and Liouville integrable.

We now must address two issues:

1. How can we construct vectors ξ_n's satisfying Eq. (205)?
2. How can we prove that such ξ_n's are variational derivatives?

Although the second issue has been completely solved considering some reduced de Rham complex, called the *variational complex*, discussed in the next lecture, the first and basic issue is far from being resolved, though there are some partial results.

We will now see how to construct a sequence of vectors ξ_n's satisfying the Lenard-Magri relation.

Lemma 10 (Extension Lemma) *[3] Suppose that, in addition to the hypothesis of Lemma 9, we also have the following orthogonality condition: assume to have vectors $\xi_0, \dots, \xi_N \in \mathcal{V}^\ell$, satisfying the Lenard-Magri relation (205), such that*

$$Span\{\xi_0, \dots, \xi_N\}^\perp \subseteq \operatorname{Im} K(\partial),$$

where $Span\{\xi_0, \dots, \xi_N\}^\perp$ is the orthogonal complement with respect to the symmetric bilinear form (203). Then we can extend the given sequence to an infinite sequence of vectors satisfying the Lenard-Magri relation (205) for any $n \in \mathbb{Z}_+$.

Proof It suffices to construct ξ_{N+1} such that Eq. (205) holds for $n = N$. In fact, the orthogonal complement to $\mathrm{Span}\{\xi_0, \dots, \xi_{N+1}\}$ is contained in the orthogonal complement to $\mathrm{Span}\{\xi_0, \dots, \xi_N\}$, hence the orthogonality condition would hold for the extended sequence. By Lemma 9, $H(\partial)\xi_N \perp \xi_n$ for every $n = 0, \dots, N$. Hence, by the orthogonality condition, $H(\partial)\xi_N \subset \mathrm{Im}\, K(\partial)$. Therefore, $H(\partial)\xi_N = K(\partial)\xi_{N+1}$ for some element $\xi_{N+1} \in \mathcal{V}^\ell$. We can now iterate this procedure to construct an infinite sequence of vectors. □

Now, let us address the question why the ξ_n's, satisfying Eq. (205), are variational derivatives. Note that so far we only have used the fact that H and K are skewadjoint, but none of their other properties as Poisson structures. However, we will need these properties in order to prove that the ξ_n's are variational derivatives. Moreover, we will need the notion of *compatibility of Poisson structures*:

Definition 30 (Magri Compatibility) Given two Poisson structures H and K, they (and the corresponding λ-brackets) are *compatible* if any their linear combination $\alpha H + \beta K$ is again a Poisson structure.
Examples 13 and 14 provide multiparameter families of compatible Poisson structures.

The importance of compatibility of Poisson structures is revealed by the following theorem.

Theorem 9 (see [18], Lemma 7.25) *Suppose that $H, K \in Mat_{\ell\times\ell}\mathcal{V}[\partial]$ are compatible Poisson structures, with K non-degenerate (i.e. $KM = 0$ implies $M = 0$ for any differential operator $M \in Mat_{\ell\times\ell}\mathcal{V}[\partial]$). Suppose, moreover, that the Lenard-Magri relation $K(\partial)\xi_{n+1} = H(\partial)\xi_n$ holds for $n = 0, 1$, and that ξ_0, ξ_1 are variational derivatives: $\xi_0 = \frac{\delta \int h_0}{\delta u}$, $\xi_1 = \frac{\delta \int h_1}{\delta u}$ for some $\int h_0, \int h_1 \in \bar{\mathcal{V}}$. Then ξ_2 is closed in the variational complex (discussed in the next lecture).*

Theorem 9 allows us to construct an infinite series of integrals of motion in involution. In fact, if we are given a pair of compatible Poisson structures H, K with K non-degenerate and we know that the first two vectors ξ_0 and ξ_1, satisfying the Lenard-Magri relation, are exact in the variational complex (i.e. they are variational derivatives), it would follow that, whenever we can construct an extending sequence of ξ_n's, then all of them would be closed, and hence exact in some extension $\widetilde{\mathcal{V}}$ of the algebra of differential functions $\mathcal{V}$ (i.e. $\xi_n = \frac{\delta h_n}{\delta u}$ for some $h_n \in \widetilde{\mathcal{V}}$). This is a consequence of the theory of the variational complex, discussed in the next lecture. Note, however, that $\xi_n \in \mathcal{V}^\ell$ for all n.

Remark 22 Let $\xi_{-1} = 0 = \frac{\delta}{\delta u}1$. If $K(\partial)\xi_0 = 0$, then for Theorem 9 to hold it suffices to have only ξ_1 such that $K(\partial)\xi_1 = H(\partial)\xi_0$, since the first step is trivial.

Proposition 16 *Suppose we have two compatible Poisson structures H and K on $\mathcal{V}$, with K non-degenerate, and consider a basis $\xi_0^1, \dots, \xi_0^s$ of* Ker K *(it is finite dimensional since K is non-degenerate). Suppose that each ξ_0^i can be extended to infinity so that Eq. (205) holds for all $n \in \mathbb{Z}_+$, and hence we have ξ_n^i for all $n \in \mathbb{Z}_+$. Assume moreover that all vectors ξ_0^i are exact: $\xi_0^i = \frac{\delta \int h_0^i}{\delta u}$ for some local functional $\int h_0^i$. Then all the ${h_n^i}'$s are in involution. Hence, we have constructed canonically an abelian subalgebra of the Lie algebra $\bar{\mathcal{V}}$, corresponding to the pair of compatible Poisson structures.*

This proposition holds by the following result.

Lemma 11 (Compatibility Lemma) *Let $H(\partial)$ and $K(\partial)$ be skewadjoint differential operators. Suppose we have vectors $\xi_0, \ldots, \xi_N \in \mathcal{V}^\ell$ such that $K(\partial)\xi_0 = 0$ and Eq. (205) holds for $n = 0, \ldots, N$. Suppose moreover to have an infinite sequence of vectors $\xi'_0, \ldots, \xi'_M, \ldots$ satisfying Eq. (205). Then all ξ_i's are in involution with all ξ'_j's with respect to both Poisson structures H and K.*

Proof Proceed by induction on i. The induction basis follows by the fact that for $i = 0$ we have $K(\partial)\xi_0 = 0$:

$$\langle \xi_0, \xi'_j \rangle_K = (K(\partial)\xi_0 | \xi'_j) = (0|\xi'_j) = 0 \tag{211}$$

and

$$\begin{aligned}\langle \xi_0, \xi'_j \rangle_H &= (H(\partial)\xi_0 | \xi'_j) = (\xi_0 | H^*(\partial)\xi'_j) = -(\xi_0 | H(\partial)\xi'_j) \\ &= -(\xi_0 | K(\partial)\xi'_{j+1}) = (K(\partial)\xi_0 | \xi'_{j+1}) = (0|\xi'_{j+1}) = 0.\end{aligned} \tag{212}$$

Now let $N > i > 0$ and suppose $\langle \xi_h, \xi'_j \rangle_{H,K} = 0$ for all $h \leq i$. We want to prove that $\langle \xi_{i+1}, \xi'_j \rangle_{H,K} = 0$. We have

$$\langle \xi_{i+1}, \xi'_j \rangle_K = (K(\partial)\xi_{i+1} | \xi'_j) = (H(\partial)\xi_i | \xi'_j) = \langle \xi_i, \xi'_j \rangle_H = 0 \tag{213}$$

and

$$\begin{aligned}\langle \xi_{i+1}, \xi'_j \rangle_H &= (H(\partial)\xi_{i+1} | \xi'_j) = -(\xi_{i+1} | H(\partial)\xi'_j) = -(\xi_{i+1} | K(\partial)\xi'_{j+1}) \\ &= (K(\partial)\xi_{i+1} | \xi'_{j+1}) = (H(\partial)\xi_i | \xi'_{j+1}) = \langle \xi_i, \xi'_{j+1} \rangle_H = 0,\end{aligned} \tag{214}$$

where in both cases the last equality is given by the induction hypothesis. Hence $\langle \xi_i, \xi'_j \rangle_{H,K} = 0$ for all i, j in question. □

In the next lecture we will demonstrate the Lenard-Magri method on the example of the KdV hierarchy, hence establishing its integrability.

6 Lecture 6 (January 15, 2015)

6.1 An Example: The KdV Hierarchy

We begin this lecture with an example.

Consider the PVA $\mathcal{P}_1 = \mathbb{F}[u, u', u'', \ldots]$ with two compatible λ-brackets: one is the Gardner-Faddeev-Zakharov (GFZ) λ-bracket $\{u_\lambda u\}_K = \lambda$, and the other one is the Magri-Virasoro (MV) λ-bracket $\{u_\lambda u\}_H = (\partial + 2\lambda)u + c\lambda^3$ for some $c \in \mathbb{F}$.

The corresponding compatible Poisson structures are $K(\partial) = \partial$ and $H(\partial) = u' + 2u\partial + c\partial^3$ respectively (see Example 13). Note that $\operatorname{Ker}\partial = \mathbb{F}$.

We shall use the Lenard-Magri scheme discussed in the previous lecture to construct an infinite hierarchy of integrable Hamiltonian equations: we want to construct an infinite sequence of vectors $\xi_n \in \mathscr{P}_1$, such that $K(\partial)\xi_{n+1} = H(\partial)\xi_n$, $n \in \mathbb{Z}_+$, and $\xi_0 \in \operatorname{Ker} K(\partial)$. We also want to compute the conserved densities h_n, such that $\xi_n = \frac{\delta h_n}{\delta u}$.

We can take $\xi_0 = 1$ and, consequently, $h_0 = u$. Taking $\xi_{-1} = 0$, $h_{-1} = 0$, we can apply Theorem 9 to establish by induction on n that all the ξ_n's, satisfying the Lenard-Magri relation, are closed in the variational complex. Since, by Corollary 5 from the next section, every closed 1-form is exact over the algebra of differential polynomials, we conclude that there exist $h_n \in \mathscr{P}_1$, such that $\xi_n = \frac{\delta h_n}{\delta u}$.

The first step of the Lenard-Magri scheme:

$$H(\partial)\xi_0 = K(\partial)\xi_1 \implies u' = \xi_1' \implies \xi_1 = u \implies h_1 = \frac{1}{2}u^2. \tag{215}$$

The second step of the Lenard-Magri scheme:

$$H(\partial)\xi_1 = K(\partial)\xi_2 \implies 3uu' + cu''' = \xi_2' \implies \xi_2 = \frac{3}{2}u^2 + cu'' \implies \xi_2$$
$$= \frac{\delta}{\delta u}h_2,\ h_2 = \frac{1}{2}(u^3 + cuu''). \tag{216}$$

And so on.

Remark 23 All ξ_n's are defined up to adding an element of $\operatorname{Ker} K(\partial)$, hence, in this case, up to adding a constant.

The corresponding KdV hierarchy of Hamiltonian equations is given by $\dfrac{du}{dt_n} = K(\partial)\xi_{n+1} = \partial\xi_{n+1}$, namely:

$$\frac{du}{dt_0} = u', \quad \frac{du}{dt_1} = 3uu' + cu''', \quad \dots. \tag{217}$$

Note that for $n = 1$ we get the classical KdV equation, which is the simplest dispersive equation (cf. Example 16).

The hierarchy can be extended to infinity because the orthogonality condition $(\xi_0)^\perp \subset \operatorname{Im} K(\partial)$ holds (see the Extension Lemma 10): since $\xi_0 = 1$ and $1^\perp = \partial\mathscr{P}_1 = \operatorname{Im} K(\partial)$ (equivalently, if $P \in (\xi_0)^\perp$ then $\int 1 \cdot P = 0 \Leftrightarrow P \in \partial\mathscr{P}_1$, therefore $P \in \operatorname{Im} K(\partial)$). It is easy to show by induction that the differential order of $K(\partial)\xi_n$ is $2n + 1$ (if $c \neq 0$), hence they are linearly independent, and we consequently have Lie integrability. Then automatically all the $\int h_n$'s are linearly independent, and we have Liouville integrability as well. Therefore the KdV equation is integrable, as are all the other equations $\dfrac{du}{dt_n} = K(\partial)\xi_{n+1} = H(\partial)\xi_n$.

Exercise 24 Show that the next equation of the KdV hierarchy is

$$\frac{du}{dt_2} = \partial\xi_3 = \frac{15}{2}u^2u' + 10cu'u'' + 5cuu''' + c^2u^{(5)},$$

and the next conserved density is

$$h_3 = \frac{5}{8}u^4 + \frac{5}{3}cu^2u'' + \frac{5}{6}cuu'^2 + \frac{1}{2}c^2uu^{(4)}.$$

6.2 The Variational Complex

As professor S. S. Chern used to say, "In life both men and women are important; likewise, in geometry both vector fields and differential forms are important". In our theory vector fields are evolutionary vector fields over an algebra of differential functions $\mathcal{V}$:

$$X_P = \sum_{\substack{i=1,\dots,\ell \\ n\in\mathbb{Z}_+}} \partial^n P_i \frac{\partial}{\partial u_i^{(n)}}, \quad P \in \mathcal{V}^\ell, \tag{218}$$

and, as we have already seen in Lecture 5, they commute with $\partial = X_{u'}$. Differential forms in our theory are "variational differential forms" which are obtained by the reduction of the de Rham complex over $\mathcal{V}$ by the image of ∂.

Let $J = \{1,\dots,N\}$, where N can be infinite. Given a unital commutative associative algebra A, containing the algebra of polynomials $\mathbb{F}[x_j | j \in J]$ and endowed with N commuting derivations $\frac{\partial}{\partial x_j}$, extending those on the subalgebra of polynomials, the *de Rham complex* $\widetilde{\Omega}(A)$ over A consists of finite linear combinations of the form

$$\sum_{i_1<\dots<i_k} f_{i_1,\dots,i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \widetilde{\Omega}^k(A), \quad f_{i_1,\dots,i_k} \in A, \tag{219}$$

so that we have the decomposition

$$\widetilde{\Omega}(A) = \bigoplus_{k\in\mathbb{Z}_+} \widetilde{\Omega}^k(A).$$

Moreover, $\widetilde{\Omega}(A)$ is a $\mathbb{Z}_+$-graded associative commutative superalgebra with parity given by $p(A) = \bar{0}$ and $p(dx_i) = \bar{1}$. This is a complex with the usual de Rham differential, namely an odd derivation $d : \widetilde{\Omega}^k(A) \longrightarrow \widetilde{\Omega}^{k+1}(A)$ of $\widetilde{\Omega}(A)$ such that

$$d(dx_i) = 0;\ df = \sum_{j\in J} \frac{\partial f}{\partial x_j} dx_j \text{ for } f \in A. \tag{220}$$

It is easily checked that d is a differential, namely that $d^2 = 0$. We will denote this complex by $(\widetilde{\Omega}(A), d)$.

Let us define now an increasing filtration on A by subalgebras:

$$A_j = \{a \in A \mid \frac{\partial a}{\partial x_i} = 0,\ \forall\ i \geqslant j\}. \tag{221}$$

We call A_1 the subalgebra of *quasiconstants*. If $\dfrac{\partial}{\partial x_j} A_j = A_j$ for all $j \in J$, we call A *normal*. Obviously, the algebra of polynomials in any (including infinite) number of variables is normal.

Lemma 12 (Algebraic Poincaré Lemma) *Let A be a normal commutative associative algebra as above, and let $(\widetilde{\Omega}(A), d)$ be its de Rham complex. Then*

$$H^k(\widetilde{\Omega}(A), d) = 0,\ k > 0;\quad H^0(\widetilde{\Omega}(A), d) = A_0. \tag{222}$$

Proof Extend the filtration of A to $\widetilde{\Omega}(A)$ by letting $\widetilde{\Omega}_j(A)$ be the subalgebra, generated by A_j and $dx_1, \dots, dx_j$. Introduce "local" *homotopy operators* $K_m : \widetilde{\Omega}^k_m(A) \to \widetilde{\Omega}^{k-1}_m(A)$ by

$$K_m(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_s} \wedge dx_m) = \begin{cases} (-1)^s (\int f\, dx_m)\, dx_{i_1} \wedge \cdots \wedge dx_{i_s} \\ 0, \quad \text{if } dx_m \text{ does not occur} \end{cases} \tag{223}$$

where $i_1 < \ldots < i_s < m$. Here the integral $\int f\, dx_m$ is a preimage in A_m of $f \in A_m$ under the map $\frac{\partial}{\partial x_m}$, which exists by normality of A.

Let $\omega \in \widetilde{\Omega}^k_m(A)$. Then it is straightforward to check that

$$K_m d\omega + dK_m\omega - \omega \in \widetilde{\Omega}^k_{m-1}(A), \quad \text{for} \quad m \geqslant 1. \tag{224}$$

Hence, if $\omega \in \widetilde{\Omega}^k_m(A)$ is closed, then

$$d(K_m\omega) - \omega \in \widetilde{\Omega}^k_{m-1}(A). \tag{225}$$

Equivalently $\omega \in \widetilde{\Omega}^k_{m-1}(A) + d\widetilde{\Omega}(A)$, i.e. we may assume that $\omega \in \widetilde{\Omega}^k_{m-1}(A)$ modulo adding an exact tail. Repeating the same argument we proceed downward in the filtration, and after a finite number of steps we get 0, hence $\omega \in d\widetilde{\Omega}(A)$. □

Let $\mathcal{V}$ be an algebra of differential functions. Consider the lexicographic order on pairs $(m, i) \in \mathbb{Z}_+ \times I$, and consider the corresponding filtration by subalgebras as above:

$$\mathscr{V}_{m,i} = \{f \in \mathscr{V} \mid \frac{\partial f}{\partial u_j^{(n)}} = 0,\ \forall\,(n,j) \geqslant (m,i)\}. \tag{226}$$

Hence we can define normality of $\mathscr{V}$ as above.

Example 17 The algebra of differential polynomials in ℓ variables $\mathscr{P}_\ell$ is normal.

The derivation ∂ of $\mathscr{V}$ extends to an even derivation of the superalgebra $\widetilde{\Omega}(\mathscr{V})$ by letting $\partial(du_i^{(n)}) = du_i^{(n+1)}$.

Exercise 25 Show that $d\partial = \partial d$. (Hint: use Axiom (*ii*) in Definition 22.)
Due to this exercise, we may consider the reduced complex

$$(\Omega(\mathscr{V}), d) = (\widetilde{\Omega}(\mathscr{V})/\partial\widetilde{\Omega}(\mathscr{V}), d),$$

called the *variational complex* over the algebra of differential functions $\mathscr{V}$.

Exercise 26 Show that ∂ is injective on $\widetilde{\Omega}^k(\mathscr{V})$ for $k \geqslant 1$.

Theorem 10 ([3]) *Let $\mathscr{V}$ be a normal algebra of differential functions. Then*

$$H^k(\Omega(\mathscr{V}), d) = 0, \quad k > 0; \quad H^0(\Omega(\mathscr{V}), d) = \mathcal{F}/\partial\mathcal{F}, \tag{227}$$

where $\mathcal{F} \subset \mathscr{V}$ is the subalgebra of quasiconstants.

Proof We have a short exact sequence of complexes

$$0 \longrightarrow \partial\widetilde{\Omega}(\mathscr{V}) \longrightarrow \widetilde{\Omega}(\mathscr{V}) \longrightarrow \Omega(\mathscr{V}) \longrightarrow 0, \tag{228}$$

which induces a long exact sequence in cohomology:

$$\begin{aligned} \mathrm{H}^0(\partial\widetilde{\Omega}(\mathscr{V})) \longrightarrow \mathrm{H}^0(\widetilde{\Omega}(\mathscr{V})) \longrightarrow \mathrm{H}^0(\Omega(\mathscr{V})) \longrightarrow \mathrm{H}^1(\partial\widetilde{\Omega}(\mathscr{V})) \longrightarrow \mathrm{H}^1(\widetilde{\Omega}(\mathscr{V})) \\ \longrightarrow \mathrm{H}^1\Omega(\mathscr{V})) \longrightarrow \dots . \end{aligned} \tag{229}$$

Since $\mathscr{V}$ is normal, by Lemma 12 we get

$$\mathrm{H}^k(\widetilde{\Omega}(\mathscr{V})) = 0 \text{ for } k > 0.$$

Now note that $\mathrm{H}^k(\partial\widetilde{\Omega}(\mathscr{V}), d) = 0$ for $k > 0$. Indeed, take $\tilde{\omega} \in \widetilde{\Omega}^k(\mathscr{V})$ with $k \geqslant 0$. If $d(\partial\tilde{\omega}) = 0$, then $\partial(d\tilde{\omega}) = 0$ since d and ∂ commute. So, thanks to Exercise 26, we have $d\tilde{\omega} = 0$. By Lemma 12, since $\tilde{\omega}$ is closed, it is exact: $\tilde{\omega} = d\tilde{\eta}$ for some $\tilde{\eta} \in \widetilde{\Omega}(\mathscr{V})$, hence $\partial\tilde{\omega} = \partial(d\tilde{\eta}) = d(\partial\tilde{\eta})$, and

$$\mathrm{H}^k(\partial\widetilde{\Omega}(\mathscr{V})) = 0 \text{ for } k > 0.$$

Therefore the long cohomology exact sequence (229) becomes

$$\ldots \longrightarrow \mathrm{H}^0(\Omega(\mathcal{V})) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{H}^1(\Omega(\mathcal{V})) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{H}^2(\Omega(\mathcal{V})) \longrightarrow 0 \longrightarrow 0 \ldots, \tag{230}$$

so $H^k(\Omega(\mathcal{V}), d) = 0$ for $k > 0$. When $k = 0$ we obviously get $\mathrm{H}^0(\Omega(\mathcal{V})) \cong \mathrm{H}^0(\widetilde{\Omega}(\mathcal{V}))/\mathrm{H}^0(\partial\widetilde{\Omega}(\mathcal{V})) = \mathcal{F}/\partial\mathcal{F}$. □

Let us study the variational complex more closely. We can write down explicitly the first terms of the complex $\Omega(\mathcal{V})$:

- $\Omega^0(\mathcal{V}) = \mathcal{V}/\partial\mathcal{V}$;
- $\Omega^1(\mathcal{V}) = \mathcal{V}^\ell$;
- $\Omega^2(\mathcal{V}) = \{$skewadjoint $\ell \times \ell$ matrix differential operators over $\mathcal{V}\}$.

The corresponding maps are

$$\Omega^0(\mathcal{V}) \stackrel{d}{\longrightarrow} \Omega^1(\mathcal{V}) \stackrel{d}{\longrightarrow} \Omega^2(\mathcal{V}) \longrightarrow \ldots; \quad \int f \stackrel{d}{\mapsto} \frac{\delta}{\delta u}\int f, F \stackrel{d}{\mapsto} \frac{1}{2}(D_F - D_F^*), \ldots, \tag{231}$$

where $f \in \mathcal{V}, F \in \mathcal{V}^\ell$, and $(D_F)_{ij} = \sum_{n\in\mathbb{Z}_+} \frac{\partial F_j}{\partial u_i^{(n)}}\partial^n, i, j \in I$.

The first identification is clear since $\widetilde{\Omega}(\mathcal{V}) = \mathcal{V}$. Let us explain how to obtain the identification $\Omega^1(\mathcal{V}) = \mathcal{V}^\ell$. We have

$$\begin{aligned}\Omega^1(\mathcal{V}) &= \left\{ \sum_{i\in I, n\in\mathbb{Z}_+} f_{i,n} du_i^{(n)} \right\} / \partial\widetilde{\Omega}^1(\mathcal{V}) = \left\{ \int \sum_{i\in I, n\in\mathbb{Z}_+} f_{i,n} du_i^{(n)} \right\} \\ &= \left\{ \int \sum_{i\in I, n\in\mathbb{Z}_+} f_{i,n}\partial^n du_i \right\},\end{aligned} \tag{232}$$

where last equality is due to the fact that d and ∂ commute. Integrating by parts, we get

$$\int \sum_{i\in I, n\in\mathbb{Z}_+} f_{i,n}\partial^n du_i = \sum_{i=1}^{\ell} \left(\int \sum_{n\in\mathbb{Z}_+} (-\partial)^n f_{i,n} \right) du_i. \tag{233}$$

Thus the identification $\Omega^1(\mathcal{V}) \stackrel{\sim}{\longrightarrow} \mathcal{V}^\ell$ is given by

$$\int \sum_{i\in I, n\in\mathbb{Z}_+} f_{i,n} du_i^{(n)} \mapsto \left(\sum_{n\in\mathbb{Z}_+} (-\partial)^n f_{i,n} \right)_{i\in I} \tag{234}$$

and this is an isomorphism of vector spaces. In particular, we conclude that if we take $df = \sum_{i\in I, n\in\mathbb{Z}_+} \frac{\partial f}{\partial u_i^{(n)}} du_i^{(n)}$ (in this case $f_{i,n} = \frac{\partial f}{\partial u_i^{(n)}}$), then the RHS of (234) becomes exactly the vector of variational derivative of f. It also explains the action of the first differential $d : \Omega^0(\mathcal{V}) \longrightarrow \Omega^1(\mathcal{V})$. Moreover, it is clear that a 1-form $\xi \in \mathcal{V}^\ell$ is exact iff $\xi = \frac{\delta f}{\delta u}$, and it is closed iff D_ξ is self-adjoint.

Exercise 27 Show that the algebra of differential functions $\mathcal{P}_1[u^{-1}, \log u]$ is normal. Show that any algebra of differential functions $\mathcal{V}$ can be included in a normal one.

Since the algebra of differential polynomials $\mathcal{P}_\ell$ is normal, we obtain the following corollary of Theorem 10.

Corollary 5 *Let $\mathcal{V} = \mathcal{P}_\ell$ be an algebra of differential polynomials. Then*

(a) $\mathrm{Ker}\frac{\delta}{\delta u} = \mathbb{F} + \mathrm{Im}\,\partial$.
(b) $\mathrm{Im}\frac{\delta}{\delta u} = \{F \in \mathcal{V}^\ell \mid D_F$ *is selfadjoint*$\}$.
(c) $\omega \in \Omega_k(\mathcal{V}), k \geqslant 1$, *is closed if and only if it is exact.*

Claim (a) is usually attributed to a paper by Gelfand-Manin-Shubin from the 1970s, though it is certainly much older. Claim (b) is called the Helmholtz criterion, and apparently, it was first proved by Volterra in the first half of the twentieth century.

If we know that $\xi \in \mathcal{V}^\ell$ is a variational derivative: $\xi = \frac{\delta h}{\delta u}$ for some $h \in \mathcal{V}$ (which is not unique since we can add to h elements from $\partial\mathcal{V}$), there is a simple formula to find one of such h :

Exercise 28 Let

$$\Delta = \sum_{i\in I, n\in\mathbb{Z}_+} u_i^{(n)} \frac{\partial}{\partial u_i^{(n)}}$$

be the degree evolutionary vector field, and suppose that $\xi \in \mathcal{V}^\ell$ is such that $\Delta(u \cdot \xi) \neq 0$. Let $h \in \Delta^{-1}(u \cdot \xi)$. Show that

$$\frac{\delta h}{\delta u_i} - \xi_i \in \mathrm{Ker}(\Delta + 1) \text{ for all } i \in I.$$

Consequently, if $\mathrm{Ker}(\Delta + 1) = 0$, then $\frac{\delta h}{\delta u} = \xi$.

6.3 Homogeneous Drinfeld-Sokolov Hierarchy and the Classical Affine Hamiltonian Reduction

The method of constructing solutions of the Lenard-Magri relation, described in Sect. 5.4, uses Theorem 9, which assumes that K is non-degenerate. In this section I will describe the direct method of Drinfeld and Sokolov on the example of the so called homogeneous hierarchy, which avoids the use of Theorem 9.

Consider the affine PVA $\mathscr{V} = \mathscr{V}^1(\mathfrak{g}, s)$, where $\mathfrak{g}$ is a reductive Lie algebra with a non-degenerate invariant symmetric bilinear form $(.\,|\,.)$ and s is a semisimple element of $\mathfrak{g}$, with compatible Poisson λ-brackets $(a, b \in \mathfrak{g})$:

$$\{a_\lambda b\}_H = [a, b] + (a|b)\lambda, \quad \{a_\lambda b\}_K = (s|[a, b]), \tag{235}$$

see Example 14. Note that the Poisson structure K is degenerate, as s is a central element of the corresponding λ-bracket.

The Lenard-Magri relation (210) for infinite N can be rewritten as follows:

$$\{\textstyle\int h_n, u\}_H = \{\textstyle\int h_{n+1}, u\}_K,\ n \in \mathbb{Z}_+,\ u \in \mathfrak{g}. \tag{236}$$

The Drinfeld-Sokolov method of constructing solutions to this equation is as follows, see [14] and [8]. Choosing dual bases $\{u_i\}_{i\in I}$ and $\{u^i\}_{i\in I}$ of $\mathfrak{g}$, let

$$L(z) = \partial + \sum_{i\in I} u^i \otimes u_i - z(s \otimes 1) \in \mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathscr{V})[z]\,.$$

The first step consists of finding a solution $F(z) = \sum_{n\geqslant 0} F_n z^{-n} \in (\mathfrak{g} \otimes \mathscr{V})[[z^{-1}]]$ of the following equations in $\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathscr{V})((z^{-1}))$:

$$[L(z), F(z)] = 0,\ [s \otimes 1, F_0] = 0\,. \tag{237}$$

Theorem 11 *Assume that the element s is semisimple, and let $\mathfrak{h}$ be the centralizer of s in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Then*

(a) There exist unique $U(z) \in z^{-1}(\mathfrak{h}^\perp \otimes \mathscr{V})[[z^{-1}]]$ and $f(z) \in (\mathfrak{h} \otimes \mathscr{V})[[z^{-1}]]$, such that

$$e^{ad\, U(z)} L(z) = \partial + f(z) - z(s \otimes 1).$$

The coefficients of $U(z)$ and $f(z)$ can be recursively computed.

(b) Let a be a central element of $\mathfrak{h}$. Then $F^a(z) = e^{-ad\, U(z)}(a \otimes 1)$ satisfies Eq. (237).

Define the variational derivative of $\int f \in \mathscr{V}/\partial\mathscr{V}$ in invariant form:

$$\frac{\delta \int f}{\delta u} = \sum_{i\in I} u^i \otimes \frac{\delta \int f}{\delta u_i}\,.$$

The second step is given by the following.

Theorem 12 *Let $f(z)$, a and $F^a(z)$ be as in Theorem 11. Let $h^a(z) = (a \otimes 1|f(z))$. Then*

(a) $F^a(z) = \frac{\delta \int h^a(z)}{\delta u}$.

(b) The coefficients of $h^a(z) = \sum_{n\geq 0} \int h_n^a z^{-n}$ *satisfy the Lenard-Magri relation (236).*

(c) All the elements $\int h_n^a \in \mathcal{V}/\partial\mathcal{V}$*, where* $n \in \mathbb{Z}_+$ *and* a *is a central element of* $\mathfrak{h}$*, are in involution with respect to both Poisson structures* H *and* K*.*

For proofs of these theorems we refer to [8]. Note that the claim (c) of Theorem 12 follows from claim (b) and Lemma 11.

Example 18 Let s be a regular semisimple element of $\mathfrak{g}$, so that $\mathfrak{h}$ is a Cartan subalgebra, and let $a \in \mathfrak{h}$. Then the above procedure gives the following sequence of densities of local functionals in involution, satisfying the Lenard-Magri relation:

$$h_{-1} = 0,\ h_0 = a,\ h_1 = \frac{1}{2}\sum_{\alpha\in\Delta}\frac{\alpha(a)}{\alpha(s)}e_{-\alpha}e_\alpha,$$

$$h_2 = \frac{1}{2}\sum_{\alpha\in\Delta}\frac{\alpha(a)}{\alpha(s)}e_{-\alpha}e'_\alpha + \frac{1}{2}\sum_{\alpha\in\Delta}\frac{\alpha(a)}{\alpha(s)^2}e_{-\alpha}e_\alpha[e_{-\alpha}, e_\alpha]$$
$$+\frac{1}{3}\sum_{\substack{\alpha,\beta\in\Delta\\ \alpha\neq\beta}}\frac{\alpha(a)}{\alpha(s)\beta(s)}e_{-\beta}e_\alpha[e_{-\alpha}, e_\beta], \ldots,$$

where Δ is the set of roots of $\mathfrak{g}$ and the root vectors e_α are chosen such that $(e_\alpha|e_{-\alpha}) = 1$.

The corresponding Hamiltonian equations are:

$$\frac{db}{dt_n} = 0 \text{ for } b \in \mathfrak{h},\ n \in \mathbb{Z}_+,\ \ \frac{de_\alpha}{dt_0} = \alpha(a)e_\alpha, \tag{238}$$

$$\frac{de_\alpha}{dt_1} = \frac{\alpha(a)}{\alpha(s)}e'_\alpha + \sum_{\beta\in\Delta}\frac{\beta(a)}{\beta(s)}e_{-\beta}[e_\beta, e_\alpha]. \tag{239}$$

The next equation is more complicated, so we give it only for $\mathfrak{g} = s\ell_2$, $a = s$, $\alpha(s) = 1$:

$$\frac{de_\alpha}{dt_2} = e''_\alpha - (2e'_\alpha\alpha + e_\alpha\alpha') - (\alpha|\alpha)e_\alpha^2 e_{-\alpha}\,. \tag{240}$$

Note that the elements of $\mathfrak{h}$ do not evolve since they are central for the Poisson structure K.

In order to construct new PVAs from existing ones, we can use the classical affine Hamiltonian reduction of a PVA $\mathcal{V}$.

The classical affine Hamiltonian reduction of a PVA $\mathcal{V}$, associated to a triple $(\mathcal{V}_0, I_0, \varphi)$, where $\mathcal{V}_0$ is a PVA, $I_0 \subset \mathcal{V}_0$ is a PVA ideal and $\varphi : \mathcal{V}_0 \to \mathcal{V}$ is a PVA homomorphism, is

$$\mathcal{W} = \mathcal{W}(\mathcal{V}, \mathcal{V}_0, I_0, \varphi) = (\mathcal{V}/\mathcal{V}\varphi(I_0))^{\mathrm{ad}_\lambda\varphi(\mathcal{V}_0)}, \tag{241}$$

where $\mathrm{ad}_\lambda\varphi(\mathcal{V}_0)$ means that we are taking the adjoint action of $\mathcal{V}\varphi(I_0)$ on $\mathcal{V}$ with respect to the λ-bracket.

Remark 24 $\mathcal{V}/\mathcal{V}\varphi(I_0)$ is a differential algebra, but the λ-bracket is not well defined on this quotient. However, the λ-bracket is well defined on the subspace of invariants $(\mathcal{V}/\mathcal{V}\varphi(I_0))^{\mathrm{ad}_\lambda\varphi(\mathcal{V}_0)}$.

Theorem 13 *The λ-bracket on $\mathcal{W}$ given by*

$$\{f + \mathcal{V}\varphi(I_0)_\lambda\, g + \mathcal{V}\varphi(I_0)\} = \{f_\lambda g\} + \mathcal{V}\varphi(I_0) \tag{242}$$

is well defined and it endows the differential algebra $\mathcal{W}$ with a structure of a PVA.

Proof Let $\widetilde{\mathcal{W}} = \{f \in \mathcal{V} \mid \{\varphi(\mathcal{V}_0)_\lambda f\} \subset \mathcal{V}[\lambda]\varphi(I_0)\}$, so that $\mathcal{W} = \widetilde{\mathcal{W}}/\mathcal{V}\varphi(I_0)$. It is a subalgebra of the differential algebra $\mathcal{V}$, and $\mathcal{V}\varphi(I_0)$ is its differential ideal.

Check that $\widetilde{\mathcal{W}}$ is closed under the λ-bracket of $\mathcal{V}$ (i.e. $\widetilde{\mathcal{W}}$ is a PVA subalgebra): let $h \in I_0, f, g \in \widetilde{\mathcal{W}}$, then by the Jacobi identity

$$\begin{aligned}\{h_\lambda\{f_\mu g\}\} = \{\{h_\lambda f\}_{\lambda+\mu} g\} + \{f_\mu\{h_\lambda g\}\} \subset \{\mathcal{V}[\lambda]\varphi(I_0)_{\lambda+\mu} g\} + \{f_\mu \mathcal{V}[\lambda]\varphi(I_0)\} \subset \\ \subset \{\mathcal{V}[\lambda]\varphi(\mathcal{V}_0)_{\lambda+\mu} g\} + \{f_\mu \mathcal{V}[\lambda]\varphi(\mathcal{V}_0)\} \subset \mathcal{V}[\lambda,\mu]\varphi(I_0).\end{aligned} \tag{243}$$

Finally, by the right Leibniz rule, $\mathcal{V}\varphi(I_0)$ is a Poisson ideal of $\widetilde{\mathcal{W}}$: for $f \in \widetilde{\mathcal{W}}$ we have

$$\{f_\lambda \mathcal{V}\varphi(I_0)\} \subset \mathcal{V}\{f_\lambda\varphi(I_0)\} + \{f_\lambda \mathcal{V}\}\varphi(I_0) \subset \mathcal{V}\{f_\lambda\varphi(\mathcal{V}_0)\} + \mathcal{V}[\lambda]\varphi(I_0) \subset \mathcal{V}[\lambda]\varphi(I_0). \tag{244}$$

□

The main example of this construction is the classical affine W-algebra.

Example 19 Consider the affine PVA $\mathcal{V} = \mathcal{V}^1(\mathfrak{g}, s)$ with compatible λ-brackets (235). Let f be a nilpotent element of $\mathfrak{g}$ and let $\{f, h, e\}$ be an $s\ell_2$-triple, containing f. Let

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$$

be the $\frac{1}{2} ad\, h$ eigenspace decomposition (so that $f \in \mathfrak{g}_{-1}$). Assume that $s \in \mathfrak{g}_d$, where $d = \max\{j | \mathfrak{g}_j \neq 0\}$. Let $\mathcal{V} = \mathcal{P}(\mathfrak{g}_{>0})$, let $\varphi : \mathcal{V}_0 \to \mathcal{V}$ be the inclusion homomorphism, and let $I_0 \subset \mathcal{V}_0$ be the differential ideal, generated by the set

$$M = \{m - (f|m) \,|\, m \in \mathfrak{g}_{\geq 1}\}.$$

It is easily checked that I_0 is a PVA ideal of $\mathcal{V}_0$ with respect to both λ-brackets (235). Then the *classical affine W-algebra* is the corresponding classical Hamiltonian reduction for both λ-brackets:

$$\mathcal{W}(\mathfrak{g}, f, s) = \mathcal{W}(\mathcal{V}, \mathcal{V}_0, I_0, \varphi).$$

Remark 25 The same construction does not work for vertex algebras because of the presence of quantum corrections. However, for the usual associative algebras it actually works. The *quantum (finite) Hamiltonian reduction* of an associative algebra A is $W = W(A, A_0, I_0, \varphi)$ constructed as above, where A_0 is an associative algebra, $\varphi : A_0 \hookrightarrow A$ is a homomorphism of associative algebras and $I_0 \subset A_0$ is a two-sided ideal. Thus, taking $A = U(\mathfrak{g})$, $A_0 = U(\mathfrak{g}_{>0})$ and I_0 the two-sided ideal generated by the above set M, we get the *quantum finite W-algebra*

$$W(\mathfrak{g}, f) = W(A, A_0, I_0, \varphi). \tag{245}$$

Theorem 14 ([8]) *As a differential algebra, the W-algebra $\mathcal{W}(\mathfrak{g}, f, s)$ is isomorphic to the algebra of differential polynomials on $\mathfrak{g}^f$, the centralizer of f in $\mathfrak{g}$.*
In particular, for f principal nilpotent we get the classical Drinfeld-Sokolov reduction.

The problem is, for which nilpotent elements f can one construct the associated with $\mathcal{W}(\mathfrak{g}, f, s)$ integrable hierarchy of Hamiltonian PDE's? Drinfeld and Sokolov constructed such hierarchy in [14] for the principal nilpotent f. (For $\mathfrak{g} = s\ell_n$ it coincides with the Gelfand-Dickey nth KdV hierarchy, $n = 2$ being the KdV hierarchy.) Their method is similar to that in the homogeneous case. The same method can be extended, but unfortunately not for all nilpotent elements.

Definition 31 A nilpotent element $f \in \mathfrak{g}$ is called of *semisimple type* if $f + s$ is a semisimple element of $\mathfrak{g}$ for some $s \in \mathfrak{g}_d$.
These elements are classified for all simple Lie algebras $\mathfrak{g}$ [15]. For example, principal, subprincipal and minimal nilpotent elements are of semisimple type. In exceptional Lie algebras about one third of the nilpotent elements are of the semisimple type. In $\mathfrak{s}\ell_N$ only those elements corresponding to partitions $(n, \ldots, n, 1, \ldots, 1)$ of N are of semisimple type.

Theorem 15 ([8]) *Let $\mathfrak{g}$ be a simple Lie algebra. If $f \in \mathfrak{g}$ is a nilpotent element, such that $f + s$ is semisimple for $s \in \mathfrak{g}_d$, then there exists a bi-Hamiltonian hierarchy associated to $\mathcal{W}(\mathfrak{g}, f, s)$, which is both Lie and Liouville is integrable.*

Remark 26 In the recent paper [11] for any nilpotent element f of $g\ell_N$ and non-zero $s \in \mathfrak{g}_d$ an integrable hierarchy associated to $W(g\ell_N, f, s)$ is constructed.

6.4 Non-local Poisson Structures and the Dirac Reduction

Unfortunately in many important examples the PVA structure is not enough to deal with integrable systems, as it is in the case of the KdV equation, since in practice most of the Poisson structures are non-local. Thus we need to consider *non-local PVAs*, for which the λ-bracket takes value in $\mathscr{V}((\lambda^{-1}))$. Equivalently, the associated operator $H(\partial) \in \mathrm{Mat}_{\ell\times\ell}\mathscr{V}((\partial^{-1}))$ is now a matrix *pseudodifferential* operator.

Still, we can work with these structures, but we have to check that the axioms for a PVA bracket still make sense when the λ-bracket is a map $\{\cdot_\lambda\cdot\} : \mathscr{V} \otimes \mathscr{V} \to \mathscr{V}((\lambda^{-1}))$. Sesquilinearity and the left and right Leibniz rules are clear. For skewsymmetry we have to make sense of $(\lambda+\partial)^{-1}$: write $(\lambda+\partial)^{-1} = \lambda^{-1}(1+\frac{\partial}{\lambda})^{-1}$ and then expand in the geometric progression, so we get a Laurent series in λ. More generally, for an $n \in \mathbb{Z}$ we let

$$(\lambda + \partial)^n = \sum_{k\in\mathbb{Z}_+} \binom{n}{k} \lambda^{n-k}\partial^k. \tag{246}$$

We only have problems with the Jacobi identity, and in order for it to make sense we need the λ-bracket to satisfy an additional property, called *admissibility*:

$$\{\{a_\lambda b\}_\mu c\} \subset \mathscr{V}[[\lambda^{-1}, \mu^{-1}, (\lambda-\mu)^{-1}]][\lambda, \mu]. \tag{247}$$

The fact is that when we consider a term like $\{a_\lambda\{b_\mu c\}\}$ we have to take Laurent series in λ and then Laurent series in μ and these can not be interchanged, since what we get are completely different spaces. So, two different terms of the Jacobi identity cannot a priori be compared, and we need this admissibility property in order to do so. If $H(\partial) = A(\partial) \circ B(\partial)^{-1}$ is a rational matrix pseudodifferential operator (that is, both $A(\partial)$, $B(\partial)$ are $\ell \times \ell$ matrix differential operators and $B(\partial)$ is non-degenerate), then the λ-bracket defined by the Master Formula (MF) is admissible.

Example 20 For $\mathscr{V} = \mathscr{P}_1 = \mathbb{F}[u, u', u'', \ldots]$ examples of non-local Poisson structures are:

- $H(\partial) = \partial^{-1}$ (Toda)
- $H(\partial) = u'\partial^{-1}u'$ (Sokolov).

More information about non-local PVA can be found in [7]. In particular, it is shown there that the Lenard-Magri scheme can be applied if both $K(\partial)$ and $H(\partial)$ are rational pseudodifferential operators. One of the most important examples is the following pair of compatible non-local Poisson structures on the algebra of differential polynomials in u and v, where $\kappa \in \mathbb{F}$:

$$K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} + 2\kappa \begin{pmatrix} u\partial^{-1}\circ u & -u\partial^{-1}\circ v \\ -v\partial^{-1}\circ u & v\partial^{-1}\circ v \end{pmatrix}, \tag{248}$$

which produces the non-linear Schrödinger (NLS) equation:

$$\begin{aligned} \frac{du}{dt} &= u'' + \kappa u^2 v \\ \frac{dv}{dt} &= -v'' - \kappa u v^2. \end{aligned} \tag{249}$$

An important construction, leading to non-local PVA's, is the Dirac reduction for PVA's, introduced in [9], which generalizes the classical Dirac reduction for Poisson algebras [13].

Theorem 16 ([9]) *Let $(\mathscr{V}, \{._\lambda.\}, \cdot)$ be a (possibly non-local) PVA. Let $\theta_1, \dots, \theta_m \in \mathscr{V}$ be some elements (constraints) such that*

$$C(\partial) = ((\{\theta_\beta\,{}_\partial\,\theta_\alpha\})_{\alpha,\beta=1}^m)_\rightarrow$$

is a non-degenerate matrix pseudodifferential operator. For $f, g \in \mathscr{V}$ let

$$\{f_\lambda g\}^D = \{f_\lambda g\} - \sum_{\alpha,\beta=1}^m \{\theta_\beta\,{}_{\lambda+\partial}\,g\}_\rightarrow (C^{-1})_{\beta\alpha}(\lambda+\partial)\{f_\lambda \theta_\alpha\}\,. \tag{250}$$

Then this modified λ-bracket provides $\mathscr{V}$ with a structure of a non-local PVA, such that all elements θ_α are central. Consequently, the differential ideal of the PVA $\mathscr{V}^D = (\mathscr{V}, \{._\lambda.\}^D, \cdot)$, generated by the θ_α's is a PVA ideal, so that the quotient of $\mathscr{V}^D$ by this ideal is a non-local PVA.

Proof Formula (250) defines the only λ-bracket, which satisfies sesquilinearity and skewsymmetry, and for which all the θ_i are central. The proof of Jacobi identity is a long, but straightforward, calculation. □

Example 21 Consider the affine PVA $\mathscr{V} = \mathscr{V}^1(s\ell_2, s)$ with the two compatible Poisson λ-brackets $\{._\lambda.\}_H$ and $\{._\lambda.\}_K$, given by (235). As in Example 18, choose a basis $e_\alpha, e_{-\alpha}, s$ of $s\ell_2$, such that

$$[e_\alpha, e_{-\alpha}] = s, \quad [s, e_{\pm\alpha}] = \pm e_{\pm\alpha},$$

and the invariant bilinear form, such that $(e_\alpha|e_{-\alpha}) = 1$, $(\alpha|\alpha) = -\kappa$.

Consider the constraint $\theta = s$ (=a multiple of α). This constraint is central with respect to the λ-bracket $\{._\lambda.\}_K$. The quotient of $\mathscr{V}$ by the differential ideal, generated by θ, is the algebra of differential polynomials $\mathscr{P}_2$ in the indeterminates $u = e_\alpha, v = e_{-\alpha}$. The induced on $\mathscr{P}_2$ λ-bracket by $\{._\lambda.\}_K$ is given by the matrix K in (248), and the Dirac reduced λ-bracket $\{._\lambda.\}_H$ on $\mathscr{P}_2$ is given by the matrix H in (248). The reduced by the constraint θ evolution equation (240) is the NLS equation (249).

This approach establishes integrability of the NLS equation, see [10] for details. For other approaches see [19] and [7].

Exercise 29 Dirac reduction of the affine PVA $\mathscr{V}^1(\mathfrak{g}, s)$ by a basis of $\mathfrak{h}$, applied to Eq. (239), gives an integrable Hamiltonian equation on root vectors of the reductive Lie algebra $\mathfrak{g}$:

$$\frac{de_\alpha}{dt} = \frac{\alpha(a)}{\alpha(s)}e'_\alpha + \sum_{\beta\in\Delta, \beta\neq-\alpha} \frac{\beta(a)}{\beta(s)} e_{-\beta}[e_\beta, e_\alpha],$$

where a and s are some fixed elements of $\mathfrak{h}$, s being regular. Find its Poisson structures.

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An Introduction to Algebras of Chiral Differential Operators

Fyodor Malikov

Abstract These notes are an informal introduction to algebras of chiral differential operators. The language used is one of vertex algebras, otherwise the approach chosen is that suggested by Beilinson and Drinfeld. The prerequisites are kept to a minimum, and we even give an informal introduction to the Beilinson-Bernstein localization theory in the example of the projective line.

Keywords Algebra of chiral differential operators • Algebra of differential operators • Lie algebra • Vertex algebra

1 Introduction

These lectures are an informal introduction to algebras of chiral differential operators, the concept that was independently and at about the same time discovered in [25] and, in a significantly greater generality, in [7]. The key to these algebras is the notion of a chiral algebroid, which is a vertex algebra analogue of the notion of a Picard-Lie algebroid. In the context of vertex algebras it was put forward in [17]; in these notes, however, despite the relentless focus on vertex algebras instead of various pseudo-tensor categories, we shall follow a much more natural approach of [7]. Under the assumption that the algebras in question are conformally graded, the results we obtain are the same as in [17].

As a warm-up, we spend considerable time discussing ordinary (and twisted) algebras of differential operators, going so far as to prove parts of the Bernstein-Beilinson localization theory in the case of sl_2. We hope this will create the framework within which things vertex will make more sense.

A little Lie and commutative algebra will suffice to understand much of what follows; dealing with the sheaves will require that the reader is not put off by

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some sheaf-theoretic and algebro-geometric terminology. Formally, no knowledge of vertex algebras is required, and the main definitions are all recorded, but in practice, without some such knowledge some of the sections below will be a tough read. On the other hand, these notes will supply the student studying books such as [11, 19] with a wealth of "real life examples." A number of exercise is intended to enhance the reader's experience.

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2 The Algebra of Differential Operators

For the purposes of these lectures, $\mathbb{C}$ is the ground field, A is a finitely generated commutative $\mathbb{C}$-algebra.

A linear transformation $P \in \mathrm{End}_{\mathbb{C}}(A)$ is called a differential operator of order k if k is the least integer s.t. $[f_{k+1}, [\dots [f_2, [f_1, P]\dots] = 0$ for all $f_1, \dots, f_{k+1} \in A$. Here $f \in A$ means the operator of multiplication by f and $[X, Y] = X \circ Y - Y \circ X$.

Let $D_A^{(k)}$ be the space of all differential operators of order at most k.

Exercise 2.1

(i) The map

$$\mathcal{D}_A^{(0)} \longrightarrow A,\ P \mapsto P(1)$$

is an algebra isomorphism.

(ii) Let $T_A = \{P \in \mathrm{End}_{\mathbb{C}}(A) \text{ s.t. } P(ab) = P(a)b + aP(b)\}$. One has $T_A \subset \mathcal{D}_A^{(1)}$.

(iii) Furthermore, there is a split exact sequence

$$0 \longrightarrow A \longrightarrow \mathcal{D}_A^{(1)} \underset{\longleftarrow}{\overset{\longrightarrow}{}} T_A \longrightarrow 0, \tag{1}$$

with $\mathcal{D}_A^{(1)} \longrightarrow T_A$ defined by $P \mapsto [P, .]$, where $[P, .]$ stands for the map $A \longrightarrow A$, $a \mapsto [P, a]$. Hence, $\mathcal{D}_A^{(1)} = A \oplus T_A$, canonically.

(iv) $\mathcal{D}_A^{(i)} \circ \mathcal{D}_A^{(j)} \subset \mathcal{D}_A^{(i+j)}$.

(v) $[\mathcal{D}_A^{(i)}, \mathcal{D}_A^{(j)}] \subset \mathcal{D}_A^{(i+j-1)}$.

Define $\mathcal{D}_A = \cup_{i\geqslant 0}\mathcal{D}_A^{(i)}$. The assertions of the exercise show that $\mathcal{D}_A$ is an associative (unital) filtered algebra; the corresponding graded object, $\mathrm{Gr}\mathcal{D}_A = \oplus_{i\geqslant 0}\mathcal{D}_A^{(i)}/\mathcal{D}_A^{(i-1)}$, is an associative commutative algebra. (Of course, we let $\mathcal{D}_A^{(-1)} = \{0\}$.) Formula (1) and its various generalization will be the focus of our attention.

Now assume A is "smooth," which we take to mean that the module of Kähler differentials, Ω_A^1, is a finitely generated free A-module.

Lemma 2.1 *There is an algebra isomorphism*

$$Gr\mathcal{D}_A \xrightarrow{\sim} S_A^\bullet T_A,$$

where $S_A^\bullet T_A$ is the symmetric algebra $A \oplus T_A \oplus S_A^2 T_A \oplus \cdots$.

Proof The fact that there is an isomorphism $T_A \xrightarrow{\sim} \mathcal{D}^{(1)}/A$, which was discussed in page 74, furnishes the basis of induction, on the one hand, and gives an algebra map $S_A^\bullet T_A \to \mathrm{Gr}\mathcal{D}_A$, on the other hand. Let us now define the inverse map. There is a map

$$\mathcal{D}^{(i)} \longrightarrow Hom_{\mathbb{C}}(A, \mathcal{D}^{(i-1)}),\ P \mapsto \{a \mapsto [P, a]\}.$$

It clearly descends to a map

$$\mathcal{D}_A^{(i)}/\mathcal{D}_A^{(i-1)} \longrightarrow Hom_{\mathbb{C}}(A, \mathcal{D}_A^{(i-1)}/\mathcal{D}_A^{(i-2)}).$$

Furthermore, its image actually belongs to the space of derivations, Der $(A, \mathcal{D}_A^{(i-1)}/\mathcal{D}_A^{(i-2)})$, which is the same as $\mathrm{Hom}_A(\Omega_A, \mathcal{D}_A^{(i-1)}/\mathcal{D}_A^{(i-2)})$. Thus we obtain a map

$$\mathcal{D}_A^{(i)}/\mathcal{D}_A^{(i-1)} \longrightarrow \mathrm{Hom}_A(\Omega_A, \mathcal{D}_A^{(i-1)}/\mathcal{D}_A^{(i-2)}).$$

All of this is valid for any A, but if Ω_A is free, then we have an identification

$$\mathrm{Hom}_A(\Omega_A, \mathcal{D}_A^{(i-1)}/\mathcal{D}_A^{(i-2)}) \xrightarrow{\sim} T_A \otimes_A \mathcal{D}_A^{(i-1)}/\mathcal{D}_A^{(i-2)}.$$

Using the induction assumption we obtain

$$\mathcal{D}_A^{(i)}/\mathcal{D}_A^{(i-1)} \longrightarrow T_A \otimes_A \mathcal{D}_A^{(i-1)}/\mathcal{D}_A^{(i-2)} \longrightarrow T_A \otimes_A S_A^{i-1} T_A \longrightarrow S_A^i T_A.$$

Exercise 2.2 The two maps constructed above,

$$S_A^\bullet T_A \to \mathrm{Gr}\mathcal{D}_A \text{ and } \mathcal{D}_A^{(i)}/\mathcal{D}_A^{(i-1)} \longrightarrow S_A^i T_A,$$

are each other's inverses.□

In particular, $\mathcal{D}_A$ is generated by vector fields. This is not true in general.

Exercise 2.3 Let $A = \mathbb{C}[x]/(x^n)$. Verify that $\mathcal{D}_A$, which by definition is a subalgebra of $gl(A) = gl_n(\mathbb{C})$, is actually isomorphic to $gl_n(\mathbb{C})$. Describe T_A and show it does not generate $\mathcal{D}_A$.

If A is smooth, then for any "point," i.e. $\mathfrak{m} \in \mathrm{Specm}(A)$, there are elements $x_1, \ldots, x_n \in A$ s.t. the images of their differentials, $dx_1, \ldots, dx_n$, in the fiber $\Omega_A/\mathfrak{m}\Omega_A$ form a basis. Therefore, $\{dx_1, \ldots, dx_n\}$ is a basis of the localization Ω_{A_f}

for some $f \in A$. Let $\{\partial_1, \ldots, \partial_n\}$ be the dual basis of T_{A_f} s.t. $dx_i(\partial_j) = \delta_{ij}$. It follows that $[\partial_i, \partial_j] = 0$. One has $\mathrm{Gr}\mathcal{D}_{A_f} = A_f[\bar{\partial}_1, \ldots, \bar{\partial}_n]$, and so, locally, each differential operator can be written in the form familiar to the calculus student.

A Poisson algebra comprises two structures, one of a commutative associative algebra, another of Lie algebra, that are compatible in the sense that the operator of the Lie bracket with any fixed element, $\{a, .\}$, satisfies the Leibniz identity: $\{a, bc\} = \{a, b\}c + a\{b, c\}$.

If $\mathcal{A} = \cup_i \mathcal{A}_i$ is a filtered associative algebra s.t. the graded object, $\mathrm{Gr}\mathcal{A} = \oplus_i \mathcal{A}_i/\mathcal{A}_{i-1}$, is a commutative algebra, then $\mathrm{Gr}\mathcal{A}$ is naturally a Poisson algebra with the Lie bracket $\{\bar{a}, \bar{b}\} = ab - ba \bmod \mathcal{A}_{i+j-2}$, where $\bar{a}$ ($\bar{b}$ resp.) is a class of $a \in \mathcal{A}_i \setminus \mathcal{A}_{i-1}$ ($b \in \mathcal{A}_j \setminus \mathcal{A}_{j-1}$ resp.) In such a situation it is common to say that $\mathcal{A}$ is a *quantization* of $\mathrm{Gr}\mathcal{A}$.

The commutative associative algebra $S_A^\bullet T_A$ is naturally a Poisson algebra, the bracket being the Lie bracket on T_A extended as a derivation to the whole of $S_A^\bullet T_A$. If A is smooth, then by Lemma 2.1 $S_A^\bullet T_A = \mathrm{Gr}\mathcal{D}_A$ and as such carries another, a priori different, Poisson structure. A moment's thought will show that these two Poisson structure coincide. Hence $\mathcal{D}_A$ is a quantization of $S_A^\bullet T_A$.

Algebras of differential operators localize well.

Lemma 2.2 *If Ω_A is free of finite rank, then $\mathcal{D}_{A_f} \xrightarrow{\sim} A_f \otimes_A \mathcal{D}_A$.*

Proof A differential operator over A defines a differential operator over localization A_f as the following recurrent procedure shows: if $P \in \mathcal{D}_A^{(i)}$, write

$$P(g) = P(f^n \frac{g}{f^n}) = f^n P(\frac{g}{f^n}) + [P, f^n](\frac{g}{f^n}),$$

then solve for $P(\frac{g}{f^n})$, which makes sense, since $[P, f^n] \in \mathcal{D}_A^{(i-1)}$, and so $[P, f^n](\frac{g}{f^n})$ may be assumed to be known.

This gives a map $A_f \otimes_A \mathcal{D}_A \to \mathcal{D}_{A_f}$, which respects the natural filtrations; hence maps of graded objects (Lemma 2.1): $A_f \otimes_A S_A^i T_A \to S_{A_f}^i T_{A_f}$, $i \geqslant 1$. These are isomorphisms as follows from an obvious inductive argument, the basis of induction, $i = 1$, being the standard commutative algebra computation:

$$T_{A_f} \xrightarrow{\sim} \mathrm{Hom}_{A_f}(\Omega_{A_f}, A_f) \xrightarrow{\sim} \mathrm{Hom}_{A_f}(A_f \otimes_A \Omega_A, A_f) \xrightarrow{\sim} A_f \otimes_A \mathrm{Hom}_A(\Omega_A, A) \xrightarrow{\sim} A_f \otimes_A T_A.$$

□

Smoothness is not essential for this result, finite generation is.

Exercise 2.4

(i) Prove $A_f \otimes_A \mathcal{D}_A \xrightarrow{\sim} \mathcal{D}_{A_f}$ for any finitely generated algebra A.
(ii) Find an example of A s.t. T_{A_f} is not isomorphic to $A_f \otimes_A T_A$.

It is now clear that each smooth algebraic variety X carries a sheaf of filtered associative algebras, $\mathcal{D}_X$, s.t. $\mathrm{Gr}\mathcal{D}_X \xrightarrow{\sim} S_{\mathcal{O}_X}^\bullet \mathcal{T}_X$.

A Lie algebra $\mathfrak{g}$ gives rise to the universal enveloping algebra $U(\mathfrak{g})$. A similar construction reproduces $\mathcal{D}_X$. Namely, $\mathcal{D}_X$ is isomorphic to the quotient of $U(\mathcal{O}_X \oplus \mathcal{T}_X)$ modulo the 2-sided ideal J generated by the elements $1_U - 1_{\mathcal{O}}$, $f * g - fg$, $f * \partial - f\partial$; here $1_U \in U(\mathcal{O}_X \oplus \mathcal{T}_X)$ and $1_{\mathcal{O}} \in \mathcal{O}_X$ are the units in the respective algebras, $f, g \in \mathcal{O}_X$, $\partial \in \mathcal{T}_X$, and $*$ denotes the product in $U(\mathcal{O}_X \oplus \mathcal{T}_X)$. Indeed, the universal property of $U(\mathcal{O}_X \oplus \mathcal{T}_X)$ gives a morphism $U(\mathcal{O}_X \oplus \mathcal{T}_X) \to \mathcal{D}_X$ that sends J to 0. Both algebras, $U(\mathcal{O}_X \oplus \mathcal{T}_X)/J$ and $\mathcal{D}_X$ are filtered (use the Poincaré-Birkhof-Witt filtration on the former), and the morphism preserves the filtrations giving us the map $\mathrm{Gr}(U(\mathcal{O}_X \oplus \mathcal{T}_X)/J) \to \mathrm{Gr}\mathcal{D}_X$. Lemma 2.1 shows that this map is an isomorphism.

An obvious analogous construction reproduces $\mathcal{D}_A$ if A is the coordinate ring of a smooth affine variety.

To see what differential operators may be good for, outside PDE, let us consider the simplest case of the Beilinson-Bernstein localization [5, 6]. The Lie algebra of the group

$$\mathrm{SL}_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ ad - bc = 1, a, b, c, d \in \mathbb{C}\}$$

is

$$\mathrm{sl}_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a + d = 0, a, b, c, d \in \mathbb{C}\} \text{ with } [A, B] = AB - BC,$$

the elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

forming its basis.

The group tautologically operates on $\mathbb{C}^2$, hence on $\mathbb{CP}^1$, the set of lines in $\mathbb{C}^2$. Therefore, there arises a Lie algebra morphism

$$\mathrm{sl}_2 \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{T}_{\mathbb{CP}^1}),$$

which induces the associative algebra morphism

$$U(\mathrm{sl}_2) \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1}),$$

Exercise 2.5

(i) Verify that on an appropriate chart $\mathbb{C} \hookrightarrow \mathbb{CP}^1$, this morphism is defined by

$$e \mapsto -\frac{\partial}{\partial x}, \ h \mapsto -2x\frac{\partial}{\partial x}, \ f \mapsto x^2\frac{\partial}{\partial x}.$$

(ii) Prove that the map $U(\mathrm{sl}_2) \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1})$ sends the element $ef + fe + h^2/2$ to 0.

The element $ef + fe + h^2/2$ is, of course, the generator of the center of $U(\mathrm{sl}_2)$ (check at least that it is central!). Denote by $U(\mathrm{sl}_2)_0$ the quotient $U(\mathrm{sl}_2)/(ef + fe + h^2/2)U(\mathrm{sl}_2)$. We obtain the morphism

$$U(\mathrm{sl}_2)_0 \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1}).$$

Lemma 2.3 *This morphism is an isomorphism*

$$U(sl_2)_0 \xrightarrow{\sim} \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1}).$$

Furthermore, $H^i(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1}) = 0$ *if* $i > 0$.

Proof Both algebras at hand are filtered, and the map preserves filtrations; the passage to the graded object gives

$$\mathrm{Gr}U(\mathrm{sl}_2)_0 \longrightarrow \bigoplus_{i=0}^{\infty} \Gamma(\mathbb{CP}^1, \mathcal{D}^{(i)}_{\mathbb{CP}^1})/\Gamma(\mathbb{CP}^1, \mathcal{D}^{(i-1)}_{\mathbb{CP}^1}). \tag{2}$$

We will prove the following two assertions: there are vector space isomorphisms

$$\bigoplus_{i=0}^{\infty} \Gamma(\mathbb{CP}^1, \mathcal{D}^{(i)}_{\mathbb{CP}^1})/\Gamma(\mathbb{CP}^1, \mathcal{D}^{(i-1)}_{\mathbb{CP}^1}) \xrightarrow{\sim} \Gamma(\mathbb{CP}^1, S^\bullet\mathcal{T}_{\mathbb{CP}^1}) \tag{3}$$

and

$$\mathrm{Gr}U(\mathrm{sl}_2)_0 \xrightarrow{\sim} \Gamma(\mathbb{CP}^1, S^\bullet\mathcal{T}_{\mathbb{CP}^1}). \tag{4}$$

These assertions mean that (2) is an isomorphism, and the lemma follows.

Proof of (3) is a simple exercise on locally free sheaves over $\mathbb{CP}^1$. $S^n\mathcal{T}_{\mathbb{CP}^1}$ is the Serre twisting sheaf $\mathcal{O}(2n)$, and so $H^1(\mathbb{CP}^1, S^n\mathcal{T}_{\mathbb{CP}^1}) = 0$. The long exact cohomology sequence attached to the exact sequence

$$0 \longrightarrow \mathcal{D}^{(n-1)}_{\mathbb{CP}^1} \longrightarrow \mathcal{D}^{(n)}_{\mathbb{CP}^1} \longrightarrow S^n\mathcal{T}_{\mathbb{CP}^1} \longrightarrow 0$$

shows, by an obvious induction on n, that $H^1(\mathbb{CP}^1, \mathcal{D}^{(n)}_{\mathbb{CP}^1}) = 0$ and

$$H^0(\mathbb{CP}^1, \mathcal{D}^{(n)}_{\mathbb{CP}^1}) \xrightarrow{\sim} \bigoplus_{i=0}^{n} H^0(\mathbb{CP}^1, \mathcal{O}(n)),$$

as desired.

Proof of (4) is, on the other hand, a pleasing exercise on some classic representation theory. Both sides of (4) are sl_2-modules: the adjoint action of sl_2 on $U(\mathrm{sl}_2)$ clearly descends to an action on the L.H.S; the action on the R.H.S is defined similarly using the morphism $\mathrm{sl}_2 \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{T}_{\mathbb{CP}^1})$ and the Lie bracket. This description shows that map (4) is an sl_2-module morphism. Kostant, [23], has computed the sl_2-module structure of $\mathrm{Gr}U(\mathfrak{g})_0$ for any simple $\mathfrak{g}$. In our case, the result is

$$\mathrm{Gr}U(\mathrm{sl}_2)_0 \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} L_{2n}, \tag{5}$$

which we shall prove a few lines below; here L_{2n} is the unique irreducible $(2n+1)$-dimensional sl_2-module. Furthermore, $L_{2n} \subset \mathrm{Gr}U(\mathrm{sl}_2)_0$ is generated by e^n, the highest weight vector of highest weight $2n$

On the right hand side, some elementary algebraic geometry will show that $\dim\Gamma(\mathbb{CP}^1, S^n\mathcal{T}_{\mathbb{CP}^1}) = 2n+1$ and that $(d/dx)^n \in \Gamma(\mathbb{CP}^1, S^n\mathcal{T}_{\mathbb{CP}^1})$. Since $(d/dx)^n$ is up to sign the image of e^n (under $U(\mathrm{sl}_2) \to \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1})$), this implies the desired isomorphism.

It remains to prove (5). It is clear that $(ef + fe + h^2/2)^m e^n \in S^\bullet\mathrm{sl}_2$ generates an $L_{2n} \subset S^\bullet\mathrm{sl}_2$. Furthermore, the set $\{(ef + fe + h^2/2)^m e^n \in S^\bullet\mathrm{sl}_2,\ m, n \geq 0\}$ is linearly independent. The complete reducibility of sl_2-modules implies that

$$\bigoplus_{N=0}^{\infty}(\mathbb{C}[ef + fe + h^2/2] \otimes L_{2N}) \hookrightarrow S^\bullet\mathrm{sl}_2.$$

Now one can show that both these spaces have the "same size." To any bigraded vectors space, $V = \oplus_{m,n}V_{mn}$, we attach the formal character, $chV = \sum_{mn}\dim V_{mn}x^m t^n$. In our case, the first grading is the canonical grading of the symmetric algebra (s.t. the degree of $x \in \mathrm{sl}_2$ is 1), the second is given by the eigenvalues of $[h, .]$. For example, the reader will readily verify that

$$chS^\bullet\mathrm{sl}_2 = \frac{1}{(1-x^2t)(1-t)(1-x^{-2}t)},\ ch(1 \otimes L_{2n}) = t^n\frac{x^{2n+1} - x^{-2n-1}}{x - x^{-1}}.$$

The following exercise will complete the proof of (5), hence of Lemma 2.3.

Exercise 2.6 Prove that[1]

$$ch(\bigoplus_{N=0}^{\infty}(\mathbb{C}[ef + fe + h^2/2] \otimes L_{2N})) = chS^\bullet\mathrm{sl}_2.$$

The very existence of a morphism $U(\mathrm{sl}_2)_0 \to \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1})$ implies there are two functors

$$\Delta : U(\mathrm{sl}_2)_0\text{-mod} \rightleftarrows \mathcal{D}_{\mathbb{CP}^1}\text{-mod} : \Gamma,\ \Delta(M) = \mathcal{D}_{\mathbb{CP}^1}\otimes_{\mathrm{sl}_2}M, \Gamma(\mathcal{M}) = \Gamma(\mathbb{CP}^1, \mathcal{M}).$$

[1] This short-cut was suggested by V. Kac, who was in class.

The famous theorem of Beilinson-Bernstein [5] asserts that these are quasi-inverses of each other, Lemma 2.3 being an important step in the proof.

Examples-Exercises 2.7

(i) $\Delta(U(sl_2)_0) = \mathcal{D}_{\mathbb{CP}^1}$.

(ii) $\Delta(\mathbb{C}) = \mathcal{O}_{\mathbb{CP}^1}$, *where* $\mathcal{O}_{\mathbb{CP}^1}$ *is the structure sheaf=sheaf of regular functions, a tautological D-module.*

(iii) *Let* τ *be the vector field that is the image of* $e \in sl_2$ *under* $sl_2 \to \Gamma(\mathbb{CP}^1, \mathcal{T}_{\mathbb{CP}^1})$, *and let* $\infty \in \mathbb{CP}^1$ *be the (unique) point where* τ *vanishes. Denote by* $\mathcal{O}_{\mathbb{CP}^1}(\infty)$ *the sheaf of functions that are allowed to have a pole at* ∞, *i.e.,* $\mathcal{O}_{\mathbb{CP}^1}(\infty)(U) = \mathcal{O}_{\mathbb{CP}^1}(U \setminus \infty)$. *Then* $\Gamma(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(\infty)) = M_0^c$, *the* contragradient Verma module *with highest weight 0, that is, an appropriately defined dual of the Verma module* $M_0 = U(sl_2)/U(sl_2)\langle h, e\rangle$.

(iv) *With the notation of (iii), let* $\mathfrak{m}_\infty \subset \mathcal{O}_{\mathbb{CP}^1}$ *be the ideal of* ∞. *Then* $\Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1}/\mathcal{D}_{\mathbb{CP}^1}\mathfrak{m}_\infty)$ *is* $M_{-2} = U(sl_2)/U(sl_2)\langle h+2, e\rangle$, *the Verma module of highest weight -2. Notice that if* y *is a local coordinate s.t.* $\mathfrak{m}_\infty = (y)$, *then* $\Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1}/\mathcal{D}_{\mathbb{CP}^1}\mathfrak{m}_\infty) = \mathbb{C}[\partial/\partial y]$ *and is thought of as the space of distributions supported at* ∞.

(v) *An exact sequence of sheaves*

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1} \longrightarrow \mathcal{O}_{\mathbb{CP}^1}(\infty) \longrightarrow \mathcal{D}_{\mathbb{CP}^1}/\mathcal{D}_{\mathbb{CP}^1}\mathfrak{m}_\infty \longrightarrow 0$$

is transformed by Γ *into the simplest example of the BGG resolution [18, Chap. 6]:*

$$0 \longrightarrow \mathbb{C} \longrightarrow M_0^c \longrightarrow M_{-2} \longrightarrow 0.$$

At this point an obvious question arises: the L.H.S. of the Beilinson-Bernstein localization theorem, $U(\mathrm{sl}_2)_0$, is a member of the family of algebras, $U(\mathrm{sl}_2)_\chi = U(\mathrm{sl}_2)/U(\mathrm{sl}_2)(ef + fe + h^2/2 - \chi)$, $\chi \in \mathbb{C}$.; is there an appropriate $\mathcal{D}^\chi_{\mathbb{CP}^1}$? The answer is, yes, there is.

3 Algebras of Twisted Differential Operators

We shall begin with a class of examples.

Let X be an algebraic variety, $\mathcal{E}$ a $rk = 1$ locally free sheaf of $\mathcal{O}_X$-modules (= the sheaf of sections of a $rk = 1$ complex vector bundle), $\mathcal{E}nd_{\mathbb{C}}(\mathcal{E})$ the sheaf of linear transformations of $\mathcal{E}$. Define (cf. page 74) $\mathcal{D}^{(k)}_{\mathcal{E}} \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{E})$ s.t.

$$\mathcal{D}^{(k)}_{\mathcal{E}}(U) = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{E})(U) : \ [f_{k+1}, [f_k, [\cdots [f_1, P]\cdots] = 0\}$$

and call it the sheaf of differential operators of order at most k operating on $\mathcal{E}$. The reader is invited to verify that our discussion of ordinary differential operators carries over to this case essentially intact as follows:

(i) $\mathcal{D}_{\mathcal{E}}^{(k)} \circ \mathcal{D}_{\mathcal{E}}^{(l)} \subset \mathcal{D}_{\mathcal{E}}^{(k+l)}$; furthermore, $[\mathcal{D}_{\mathcal{E}}^{(k)}, \mathcal{D}_{\mathcal{E}}^{(l)}] \subset \mathcal{D}_{\mathcal{E}}^{(k+l-1)}$

(ii) $\mathcal{D}_{\mathcal{E}}^{(0)} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{E}) = \mathcal{O}_X$.

(iii) to $P \in \mathcal{D}_{\mathcal{E}}^{(1)}$ assign the derivation $\sigma(P) : \mathcal{O}_X \to \mathcal{O}_X, f \mapsto [P,f] \in \mathcal{D}_{\mathcal{E}}^{(0)} = \mathcal{O}_X$. Thus arising map

$$\sigma : \mathcal{D}_{\mathcal{E}}^{(1)} \longrightarrow \mathcal{T}_X,$$

is a surjective Lie algebra morphism. It defines an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}_{\mathcal{E}}^{(1)} \longrightarrow \mathcal{T}_X \longrightarrow 0. \tag{6}$$

It is fundamentally different from (1) in that it does not split; having a splitting $\mathcal{T}_X \to \mathcal{D}_{\mathcal{E}}^{(1)}$ is equivalent to defining a connection on $\mathcal{E}$.

To summarize, $\mathcal{D}_{\mathcal{E}}$ defined to be $\cup_k \mathcal{D}_{\mathcal{E}}^{(k)}$ is a sheaf of filtered algebras, locally isomorphic to $\mathcal{D}_X$; this is because locally $\mathcal{E}$ is indistinguishable from $\mathcal{O}_X$. If X is smooth then $\mathrm{Gr}\mathcal{D}_{\mathcal{E}} \xrightarrow{\sim} S^{\bullet}_{\mathcal{O}_X}\mathcal{T}_X$; therefore $\mathcal{D}_{\mathcal{E}}$ is a quantization of $\mathrm{Gr}\mathcal{D}_{\mathcal{E}} \xrightarrow{\sim} S^{\bullet}_{\mathcal{O}_X}\mathcal{T}_X$ (see page 76), usually not isomorphic to $\mathcal{D}_X$.

Exercise 3.1 If X is smooth, prove $\mathcal{D}_{\mathcal{E}}$ is Nötherian.

Let us now look at an example.

The routine verifications of all the assertions to follow is left to the reader.

$\mathbb{CP}^1$ is covered by an atlas consisting of two charts, both $\mathbb{C}$ with coordinates x and y s.t. over the intersection, $\mathbb{C}^*$, $x = 1/y$. Over each chart Serre's twisting sheaf $\mathcal{O}(n)$, which we have already encountered, is trivialized by sections s and t resp.; over the intersection $s = y^n t$.

The trivializations identify $\mathcal{D}_{\mathcal{O}(n)}$ with $\mathcal{D}_{\mathbb{C}}$ over each chart; we shall write ∇_x for $\partial/\partial x$ over the x-chart and ∇_y for $\partial/\partial y$ over the y-chart. Over the intersection one has

$$\partial/\partial x = -y^2\partial/\partial y \text{ but } \nabla_x = -y^2\nabla_y + ny. \tag{7}$$

(This illustrates how $\mathcal{D}_{\mathcal{O}(n)}$ is different from $\mathcal{D}_{\mathbb{CP}^1}$ and why $\mathrm{Gr}\mathcal{D}_{\mathcal{O}(n)} = \mathrm{Gr}\mathcal{D}_{\mathbb{CP}^1}$.) The assignment, cf. Exercise 2.5,

$$e \mapsto -\frac{\partial}{\partial x},\ h \mapsto -2x\frac{\partial}{\partial x} + n,\ f \mapsto x^2\frac{\partial}{\partial x} - nx$$

extends to morphisms

$$\mathrm{sl}_2 \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathcal{O}(n)}^{(1)}),\ \ U(\mathrm{sl}_2)_{n(n+2)/2} \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathcal{O}(n)}).$$

Lemma 3.1 *The map* $U(sl_2)_{n(n+2)/2} \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{D}_{O(n)})$ *is an isomorphism.*
The enthusiastic reader will discover that the proof of the analogous result Lemma 2.3 carries over to the present case practically word for word for the reason that once the map has been constructed one only has to analyze its effect on the corresponding graded spaces, $\mathrm{Gr}U(sl_2)_{n(n+2)/2}$ and such, where the twisted map is indistinguishable from the one in Lemma 2.3.

The pair of functors

$$\Delta : U(\mathrm{sl}_2)_{n(n+2)/2}\text{-mod} \rightleftarrows \mathcal{D}_{O(n)}\text{-mod} : \Gamma,\ \Delta(M) = \mathcal{D}_{O(n)} \otimes_{\mathrm{sl}_2} M, \Gamma(\mathcal{M}) = \Gamma(\mathbb{CP}^1, \mathcal{M}).$$

is defined as before, but they are each other's inverses only when $n \geqslant 0$. (Q: Why? Hint: consider $H^1(\mathbb{CP}^1, O(n))$.)

The reader is encouraged to find the analogues of the examples 2.7, and especially to define an exact sequence

$$0 \longrightarrow O(n) \longrightarrow O(n)(\infty) \longrightarrow \mathcal{D}_{O(n)}/\mathcal{D}_{O(n)}\mathfrak{m}_\infty \longrightarrow 0$$

and derive from it the BGG resolution

$$0 \longrightarrow L_n \longrightarrow M_n^c \longrightarrow M_{-2-n} \longrightarrow 0.$$

Let X be a smooth algebraic variety. Guided by the discussion at the beginning of the present Section, we shall say (following [5, 6]) that a sheaf of associative algebras $\mathcal{A}$ is an *algebra of twisted differential operators* (TDO for short) if $\mathcal{A}$ carries an increasing filtration $\{\mathcal{A}^{(i)},\ i \geqslant 0\}$ s.t. $\mathrm{Gr}\mathcal{A}$ is commutative and isomorphic to $S^\bullet_{O_X}\mathcal{T}_X$ as a Poisson algebra.

In a word: a TDO is a quantization of $S^\bullet_{O_X}\mathcal{T}_X$.

Of course $\mathcal{D}_\mathcal{E}$ is a TDO, but it is easy to find TDOs that are not $\mathcal{D}_\mathcal{E}$ for any $\mathcal{E}$. For example, although $O(\lambda)$ does not make sense if $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, an explicit construction of $\mathcal{D}_{O(\lambda)}$ as above makes perfect sense for any complex λ. More generally, given a $rk = 1$ locally free sheaf $\mathcal{E}$ the family of TDOs $\mathcal{D}_{\mathcal{E}^{\otimes n}}$, $n \in \mathbb{Z}$, allows "analytic continuation" $\mathcal{D}_{\mathcal{E}^\lambda}$, $\lambda \in \mathbb{C}$.

By definition, $\mathcal{A}^{(0)} = O_X$ and $\mathcal{A}^{(1)}$ fits a familiar by now, cf. (6), exact sequence

$$0 \longrightarrow O_X \longrightarrow \mathcal{A}^{(1)} \longrightarrow \mathcal{T}_X \longrightarrow 0. \tag{8}$$

It is clear that $\mathcal{A}$ is generated as an associative algebra by $\mathcal{A}^{(1)}$ and it should not take much convincing to agree that a classification of TDOs is equivalent to classifications of exact sequences (8), the task that we shall take up next.

The following is a result of abstracting the properties of (8): we shall try to keep $\mathcal{A}^{(1)}$ forgetting about the whole of $\mathcal{A}$.

Let us return to a purely local situation working over a finitely generated $\mathbb{C}$-algebra A. The module of derivations T_A is an A-module and a Lie algebra, but it is not a Lie algebra over A in that the Lie bracket is not linear. Instead, there is a tautological action of T_A on A by derivations, which measures the failure of the bracket to be A-linear:

$$[\xi, a\tau] = a[\xi, \tau] + \xi(a)\tau.$$

This sort of data is called a Lie A-algebroid. More precisely, $\mathcal{L}$ is called a Lie *A-algebroid* if it is a Lie algebra, an A-module, and is equipped with anchor, i.e., a Lie algebra and an A-module map $\sigma : \mathcal{L} \to T_A$. These data are compatible in the sense that the A-module structure map

$$A \otimes \mathcal{L} \longrightarrow \mathcal{L} \tag{9}$$

is an $\mathcal{L}$-module morphisms, where $\mathcal{L}$ is regarded as an adjoint module over itself and A is an $\mathcal{L}$-module via the pull-back w.r.t the anchor $\mathcal{L} \to T_A$. Explicitly,

$$[\xi, a\tau] = \sigma(\xi)(a)\tau + a[\xi, \tau],\ a \in A,\ \xi, \tau \in \mathcal{L}. \tag{10}$$

A Picard-Lie A-algebroid is a Lie A-algebroid $\mathcal{L}$ s.t. the anchor fits in an exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \mathcal{L} \xrightarrow{\sigma} T_A \longrightarrow 0, \tag{11}$$

where the arrows respect all the structures involved; in particular, A is regarded as an A-module and an abelian Lie algebra, and ι makes it an A-submodule and an abelian Lie ideal of $\mathcal{L}$.

Morphisms of Picard-Lie A-algebroids are defined in an obvious way to be morphisms of exact sequences (11) that preserve all the structure involved. More formally, a morphism is a Lie A-algebroid map $f : \mathcal{L}_1 \to \mathcal{L}_2$ s.t. the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota_1} & \mathcal{L}_1 & \xrightarrow{\sigma_1} & T_A & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota_2} & \mathcal{L}_2 & \xrightarrow{\sigma_2} & T_A & \longrightarrow & 0 \end{array} \tag{12}$$

Each such morphism is automatically an isomorphism and we obtain a groupoid $\mathcal{PL}_A$.

Classification of Picard-Lie A-algebroids that split as A-modules is as follows. We have a canonical such algebroid, $A \oplus T_A$ with bracket

$$[a + \xi, b + \tau] = \xi(b) - \tau(a) + [\xi, \tau].$$

By definition, any other bracket must have the form

$$[\xi, \tau]_{new} = [\xi, \tau] + \beta(\xi, \tau),\ \beta(\xi, \tau) \in A.$$

The A-module structure axioms, especially (10), mean that $\beta(.,.)$ is A-bilinear, the anticommutativity of $[.,.]_{new}$ means that β is anticommutative, i.e., $\beta \in \Omega_A^2$.

Exercise 3.2 Verify that the Jacobi identity

$$[\xi, [\eta, \tau]_{new}]_{new} + [\tau, [\xi, \eta]_{new}]_{new} + [\eta, [\tau, \xi]_{new}]_{new} = 0$$

is equivalent to

$$\xi\beta(\tau, \eta) - \tau\beta(\xi, \eta) + \eta\beta(\xi, \tau) - \beta([\xi, \tau], \eta) + \beta([\xi, \eta], \tau) - \beta([\tau, \eta], \xi) = 0.$$

The L.H.S. of the last equation is by definition $d_{DR}\beta(\xi, \tau, \eta)$, d being the De Rham differential. We conclude that $\beta \in \Omega_A^{2,cl}$.

Denote this Picard-Lie algebroid by $T_A(\beta)$. Clearly, any Picard-Lie A-algebroid is isomorphic to $T_A(\beta)$ for some $\beta \in \Omega_A^{2,cl}$.

By definition, a morphism $T_A(\beta) \to T_A(\gamma)$ must have the form $\xi \to \xi + \alpha(\xi)$ for some $\alpha \in \Omega_A^1$. A quick computation (do it!) will show that

$$Hom(T_A(\beta), T_A(\gamma)) = \{\alpha \in \Omega_A^1 \text{ s.t. } d\alpha = \beta - \gamma\}.$$

This can be rephrased as follows—and we will happily omit the technicalities. Let $\Omega_A^{[1,2>}$ be a category with objects $\beta \in \Omega_A^{2,cl}$, morphisms $Hom(\beta, \gamma) = \{\alpha \in \Omega_A^1 \text{ s.t. } d\alpha = \beta - \gamma\}$. It is, in fact, an abelian group in categories meaning that the assignment $\Omega_A^{[1,2>} \times \Omega_A^{[1,2>} \to \Omega_A^{[1,2>}$, $(\beta_1, \beta_2) \mapsto \beta_1 + \beta_2$ is naturally a bifunctor that enjoys a number of properties mimicking the definition of a group.

Next, the assignment $\Omega_A^{[1,2>} \times \mathcal{PL}_A \to \mathcal{PL}_A$, $(\gamma, T_A(\beta)) \mapsto T_A(\beta+\gamma)$ is naturally a bifunctor that enjoys a number of properties that justify calling it a *categorical action of* $\Omega_A^{[1,2>}$ on $\mathcal{PL}_A$. In fact, this action makes $\mathcal{PL}_A$ into an $\Omega_A^{[1,2>}$*-torsor*. What it means is that the assignment $\Omega_A^{[1,2>} \to \mathcal{PL}_A$, $\beta \mapsto T_A(\beta)$ is naturally an equivalence of categories.

We see that the isomorphism classes of Picard-Lie A-algebroids are in 1–1 correspondence with the De Rham cohomology $\Omega_A^{2,cl}/d\Omega_A^1$, and the automorphism group of an object is $\Omega_A^{1,cl}$.

If X is a smooth algebraic variety, then the above considerations give the category of Picard-Lie algebroids over X, $\mathcal{PL}_X$. The meaning of our considerations is that it is a torsor over $\Omega_X^{[1,2>}$ or, if put differently, a gerbe bound by the sheaf complex $\Omega_X^1 \to \Omega_X^{2,cl}$. This gerbe has a global section, the standard $\mathcal{O}_X \oplus \mathcal{T}_X$. The isomorphism of classes of such algebroids are in 1–1 correspondence with the cohomology group $H^1(X, \Omega_X^1, \to \Omega_X^{2,cl})$ (Ω_X^1 being placed in degree 0), and the automorphism group of an object is $H^0(X, \Omega_X^{1,cl})$.

Let us describe the Čech cocycle representing a Picard-Lie algebroid. Consider an affine cover $\{U_i\}$ of X. We obtain a bi-complex with terms $\Omega_X^1(\cap_j U_{i_j})$ and $\Omega_X^{2,cl}(\cap_j U_{i_j})$ and two differentials, De Rham d_{DR} and Čech $d_{\check{C}}$. Now use the classification of Picard-Lie A-algebroids obtained above as follows. The restriction of a Picard-Lie algebroid $\mathcal{L}$ to each U_i is identified with $T_{U_i}(\beta_i)$ for some $\beta_i \in \Omega_X^{2,cl}(U_i)$; on intersections $U_i \cap U_j$ there arise $\alpha_{ij} \in \Omega_X^1(U_i \cap U_j)$ s.t. $(\beta_j - \beta_i)_{U_i \cap U_j} = d_{DR}\alpha_{ij}$, which is interpreted as a patching isomorphism $\phi_{ij} : \mathcal{L}|_{U_i}(U_i \cap U_j) \xrightarrow{\sim} \mathcal{L}|_{U_j}(U_i \cap U_j)$. The transitivity condition, $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$, means the Čech cocycle condition: $d_{\check{C}}(\{\alpha_{ij}\}) = 0$. Therefore, the pair $(\{\alpha_{ij}\}, \{\beta_i\})$ is a 1-cocycle of the total complex.

Replacing $T_{U_i}(\beta_i)$ with an isomorphic $T_{U_i}(\beta_i + d_{DR}\gamma_i)$ results in replacing $(\{\alpha_{ij}\}, \{\beta_i\})$ with a cohomologous cocycle.

If $\mathcal{A}$ is a TDO over X, then $\mathcal{A}^{(1)}$ is a Picard-Lie algebroid over X, by definition, and the assignment $\mathcal{A} \mapsto \mathcal{A}^{(1)}$ gives a functor $\mathcal{TDO}_X \to \mathcal{PL}_X$. This functor has a left adjoint

$$U_{O_X}(.) : \mathcal{PL} \longrightarrow \mathcal{TDO}_X,\ \mathcal{L} \mapsto U_{O_X}(\mathcal{L}).$$

$U_{O_X}(\mathcal{L})$ is called the universal enveloping algebra of $\mathcal{L}$; it is analogous to the concept of the universal enveloping algebra of a Lie algebra and is different insofar as it reflects the partially defined associative product on $\mathcal{L}$: $(f, \tau) \mapsto f \cdot \tau$ for $f \in O_X \subset \mathcal{L}$, $\tau \in \mathcal{L}$. The definition is made in essentially the same way as for the ordinary differential operators, page 77, and we leave working out the details to the interested reader.

The ordinary universal enveloping $U(\mathfrak{g})$ carries a filtration s.t. $\mathrm{Gr}U(\mathfrak{g}) = S^\bullet\mathfrak{g}$, and the same construction applies to $U_{O_X}(\mathcal{L})$.

Exercise 3.3 Find a filtration on $U_{O_X}(\mathcal{L})$ s.t.

(i) $U_{O_X}(\mathcal{L})^{(0)} = O_X$;
(ii) $U_{O_X}(\mathcal{L})^{(1)} = \mathcal{L}$;
(iii) $\mathrm{Gr}U_{O_X}(\mathcal{L}) = S^\bullet_{O_X}\mathcal{T}_X$;
(iv) if $\mathcal{A}$ is a TDO and $\mathcal{A}^{(1)}$ is the corresponding Picard-Lie algebroid, then $U_{O_X}(\mathcal{A}^{(1)}) \xrightarrow{\sim} \mathcal{A}$.

This proves that the two functors

$$U_{O_X}(.) : \mathcal{PL}_X \overset{\leftarrow}{\rightarrow} \mathcal{TDO}\ : (.)^{(1)}$$

are each other's quasi-inverse.

Therefore, the isomorphism classes of TDO's are in bijection with $H^1(X, \Omega_X^1, \to \Omega_X^{2,cl})$.

Example 3.1 If $X = \mathbb{CP}^1$, then the dimensional argument shows that $H^1(X, \Omega_X^1 \to \Omega_X^{2,cl}) = H^1(\mathbb{CP}^1, \Omega_{\mathbb{CP}^1}^1)$.

Exercise 3.4 Prove that $\dim H^1(\mathbb{CP}^1, \Omega_{\mathbb{CP}^1}^1) = 1$ with basis the Čech cocycle dx/x over $\mathbb{C}^*$—we are using the notation introduced in page 81.

This immediately shows that the algebra $\mathcal{D}_{O(n)}$ introduced in *loc. cit.* is exactly the one attached to the indicated cocycle in the classification above: formulas (7) define a Picard-Lie algebroid/TDO whose restriction to each chart is isomorphic to the standard $\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{T}_{\mathbb{CP}^1}$ but the gluing over the intersection $\mathbb{C}^*$ is twisted by an automorphism $\partial/\partial y \mapsto \partial/\partial y - ndy/y(\partial/\partial y)$.

If we replace n with an arbitrary complex number λ or, even better, an indeterminate λ and work over $\mathbb{C}[\lambda]$, then we obtain a universal family of TDOs over $\mathbb{CP}^1$.

4 CDO: An Example

An algebra of chiral differential operators, the subject of these lectures, is a vertex (or chiral) algebra analogue of a TDO. As is our wont, we shall begin with an example.

Let $\mathfrak{a}$ be an infinite dimensional Lie algebra with basis $\{x_n, \partial_n, C; n \in \mathbb{Z}\}$ and the bracket

$$[\partial_i, x_j] = \delta_{i,-j}C,\ [C, \partial_i] = [C, x_j] = [\partial_i, \partial_j] = [x_i, x_j] = 0.$$

Let $\mathfrak{a}_+$ be the subalgebra with basis ∂_i, x_{i+1}, C, $i \geqslant 0$; it is clearly a maximal commutative subalgebra of $\mathfrak{a}$. Let $\mathbb{C}_1$ stand for its 1-dimensional module, where ∂_i and x_{i+1}, $i \geqslant 0$, act trivially, and C acts as multiplication by 1. Let

$$\mathcal{D}^{ch}_{\mathbb{C}[x]} = \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{a}_+}\mathbb{C}_1.$$

Eventually, we shall convince ourselves that this is a reasonable vertex algebra analogue of $\mathcal{D}_{\mathbb{C}[x]}$.

Remark The D-module nature of $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ can be easily seen as follows. The space $\mathbb{C}[[z]] = \{\sum_{n\geqslant 0} x_{-n}z^n\}$ is naturally a scheme, $\mathrm{Spec}\mathbb{C}[x_0, x_{-1}, \ldots]$. The space of Laurent series $\mathbb{C}((z)) = \{\sum_{n\gg -\infty} x_{-n}z^n\}$ can be represented as the union of schemes, $\cup_n z^{-n}\mathbb{C}[[z]]$, and thus given the structure of an ind-scheme. We shall have no use for the algebro-geometric subtleties involved, but there is no doubt that $\mathbb{C}((z))$ has coordinates, $\{x_n,\ n \in \mathbb{Z}\}$, and vector fields, $\{\partial/\partial x_n,\ n \in \mathbb{Z}\}$. With this in mind, the meaning of $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ is clear: it is nothing but an algebraic description of the module of distributions supported on $\mathbb{C}[[z]] \subset \mathbb{C}((z))$; the simplest possible example of such construction was encountered in Examples 7(iv), and the reader is encouraged to compare the two. One cannot multiply distributions, and so this space is not an associative algebra in a natural way, but it is a vertex algebra, and there are geometric reasons for this. Such point of view is developed by Kapranov and Vasserot, [20, 21].

It is perhaps easiest to define a vertex algebra structure on $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ using the *strong reconstruction theorem*, [11, 19], or what V.Kac called an *extension theorem* in his lectures in this volume. Introduce $\mathfrak{a}_-$, the subalgebra "opposite" to $\mathfrak{a}_+$, i.e., the one generated by $\{x_i, \partial_{i-1},\ i\leqslant 0\}$. The PBW theorem identifies $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ with $U(\mathfrak{a}_-) = \mathbb{C}[x_i, \partial_{i-1},\ i\leqslant 0]$, which gives us a basis consisting of monomials and a distinguished vector, 1, to be regarded as a vacuum vector. Next, we introduce the operator $T \in \mathrm{End}_{\mathbb{C}}\mathcal{D}^{ch}_{\mathbb{C}}$ by the formula: $T = -\sum_n nx_n\partial_{-n-1}$. Finally, we have two fields,

$$x(z) = \sum_{n=-\infty}^{+\infty} x_n z^{-n} \text{ and } \partial(z) = \sum_{n=-\infty}^{+\infty} \partial_n z^{-n-1}.$$

Exercise 4.1 Verify the relations

$$[T, x(z)] = x(z)',\ [T, \partial(z)] = \partial(z)',\ x(z)1 = x_0 \bmod z,\ \partial(z)1 = \partial_{-1} \bmod z,$$

$$[x(z), x(w)] = [\partial(z), \partial(w)] = 0,$$

and

$$[\partial(z), x(w)] = \delta(z-w),\ \text{where } \delta(z-w) = \sum_{n=-\infty}^{+\infty} \frac{z^n}{w^{n+1}}. \tag{13}$$

The content of the Reconstruction Theorem in any of its versions is that these relations imply: $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ carries a unique vertex algebra structure s.t. the fields assigned to x_0 and ∂_{-1} are $x(z)$ and $\partial(z)$ resp. More generally, the field assigned to a monomial in $\mathbb{C}[x_i, \partial_{i-1},\ i\leqslant 0]$ is obtained by the operations of normally ordered product and differentiation (w.r.t. z); e.g.,

$$x_{-n}\partial_{-m-1} \mapsto \frac{1}{n!m!} : x(z)^{(n)}\partial(z)^{(m)} : .$$

We would like to think of the just now constructed $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ as a vertex algebra attached to the algebra $\mathbb{C}[x]$. If we are able to suggest a reasonable definition of "localization," $\mathcal{D}^{ch}_{\mathbb{C}[x]_f}$, for any nonzero $f \in \mathbb{C}[x]$, then the assignment $U_f \mapsto \mathcal{D}^{ch}_{\mathbb{C}[x]_f}$ will define a sheaf of vertex algebras, $\mathcal{D}^{ch}_{\mathbb{C}}$ over $\mathbb{C}$.

The polynomial nature of $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ makes it a $\mathbb{C}[x]$-module with x acting as multiplication by x_0. Set $\mathcal{D}^{ch}_{\mathbb{C}[x]_f} = \mathbb{C}[x]_f \otimes_{\mathbb{C}[x]} \mathcal{D}^{ch}_{\mathbb{C}[x]}$. In order to define a vertex algebra structure on this space, one needs a field assigned to $1/f$. So, what is $f(z)^{-1}$? For example, in the case of $\mathcal{D}^{ch}_{\mathbb{C}[x,x^{-1}]}$, what is $x(z)^{-1}$?

Exercise 4.2 (Feigin's Trick) Define using the sum of geometric series formula as motivation

$$x(z)^{-N} =$$

$$\frac{1}{(x_0+\sum_{n\neq 0}x_{-n}z^n)^N} = x_0^{-N}\frac{1}{(1+x_0^{-N}\sum_{n\neq 0}x_{-n}z^n)^N} = x_0^{-N}\sum_{j=0}^{+\infty}\binom{-N}{j}(x_0^{-N}\sum_{n\neq 0}x_{-n}z^n)^j.$$

Verify that this series makes sense as a field, i.e., that $x(z)^{-N}v \in \mathcal{D}^{ch}_{\mathbb{C}[x,x^{-1}]}((z))$ for any $v \in \mathcal{D}^{ch}_{\mathbb{C}[x,x^{-1}]}$, and that

$$[\partial(z), x(w)^{-N}] = -Nx(w)^{-N-1}\delta(z-w). \tag{14}$$

The meaning, hence a generalization, of this construction is obvious: we should think of $\epsilon(z) = \sum_{n\neq 0}x_{-n}z^n$ as a small variation of a constant loop; this of course corresponds with the Kapranov-Vasserot concept of infinitesimal loop, [20]. Therefore, if $g \in \mathbb{C}(x)$ is any rational function (in fact any function holomorphic on an open subset of $\mathbb{C}$ will do), then we define

$$g(x(z)) = \sum_{j=0}^{+\infty}\frac{1}{n!}g^{(n)}(x_0)\epsilon(z)^n.$$

Exercise 4.3 Verify that $g(x(z))$ is a field on $\mathcal{D}^{ch}_{\mathbb{C}[x]_f}$ for any $g \in \mathbb{C}[x]_f$ and check the relations

$$g(x(z))1 = g(x_0) \bmod z,\ \ [T, g(x(z))] = g'(x(z))x(z)' = T(g(x_0))(z),$$

and

$$[\partial(z), g(x(w))] = g'(x(w))\delta(z-w). \tag{15}$$

The Reconstruction Theorem allows us to conclude, as at the beginning of Sect. 5, that $\mathcal{D}^{ch}_{\mathbb{C}[x]_f}$ carries a unique vertex algebra structure s.t. $\partial_{-1} \mapsto \partial(z)$ and $g(x_0) \mapsto g(x(z))$.

It is obvious that the assignment $U_f = \{f \neq 0\} \mapsto \mathcal{D}^{ch}_{\mathbb{C}[x]_f}$ defines a sheaf of vertex algebras on $\mathbb{C}$; in fact, it coincides with the standard algebraic geometry localization of $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ regarded as a $\mathbb{C}[x]$-module. Denote this sheaf by $\mathcal{D}^{ch}_{\mathbb{C}}$.

Now that we have obtained a reasonable sheaf $\mathcal{D}^{ch}_{\mathbb{C}}$ over $\mathbb{C}$, we shall try to glue two such sheaves into a sheaf on $\mathbb{CP}^1$; in other words, we shall play the game similar to the one described on pages 81–82. Thus we have two charts with coordinates to be denoted (this time around) x and $\tilde{x}$. Next, we have a copy of $\mathcal{D}^{ch}_{\mathbb{C}}$ sitting on each of the charts, and two copies of $\mathcal{D}^{ch}_{\mathbb{C}^*}$ on the intersection, $\mathbb{C}^*$, one equal to $\mathbb{C}[x_0^{\pm 1}, x_{-i}, \partial_{-i}, i > 1]$, another to $\mathbb{C}[\tilde{x}_0^{\pm 1}, \tilde{x}_{-i}, \tilde{\partial}_{-i}, i > 1]$. What we want is a way to identify these copies.

A morphism of vertex algebras is a linear map $\pi : V \longrightarrow W$ that preserves the unit, the "product," i.e., s.t. $\pi(a)(z)\pi(b) = \pi(a(z)b)$, and the operator T: $\pi \circ T = T \circ \pi$. We could of course obtain an isomorphism by assigning $\tilde{x}_0$ to x_0 and $\tilde{\partial}_{-1}$ to ∂_{-1}, but this would be unreasonable: the formulas used before, esp. (15) strongly indicate that x_0 has the meaning of the coordinate function and ∂_{-1} has the meaning of the derivative $\partial/\partial x_0$. Therefore, we stipulate (emulating the case of ordinary differential operators, see page 81) that $\pi(x_0) = 1/\tilde{x}_0$ and suggest that $\pi(\partial_{-1}) = -\tilde{x}_0^2\tilde{\partial}_{-1}$. For this assignment to extend to an isomorphism of vertex algebras several identities have to be verified; the mildly interesting one,

$$[\pi(\partial)(z), \pi(x)(w)] = \delta(z-w),$$

is easily checked, but the dull one

$$[\pi(\partial)(z), \pi(\partial)(w)] = 0,$$

fails miserably; in fact, a quick computation using Wick's theorem ([19]) will show

$$[\pi(\partial)(z), \pi(\partial)(w)] = -2\tilde{x}(w)^2\delta(z-w)' - 2\tilde{x}(w)'\tilde{x}(w)^2\delta(z-w).$$

In order to fix this, let us change the transformation law for ∂_{-1} as follows

$$\pi(\partial_{-1}) = -\tilde{x}_0^2\tilde{\partial}_{-1} - \tilde{x}_{-1}, \tag{16}$$

or in terms of fields

$$\pi(\partial_{-1})(z) = -:\tilde{x}(z)^2\tilde{\partial}(z): -\tilde{x}(z)', \tag{17}$$

Exercise 4.4 Use Wick's theorem (having learnt it ([19]) if need be) to verify the relations

$$[\pi(x_0)(z), \pi(x_0)(w)] = [\pi(\partial_{-1})(z), \pi(\partial_{-1})(w)] = 0,$$
$$[\pi(\partial_{-1})(z), \pi(x_0)(w)] = \delta(z-w). \tag{18}$$

Now define a map

$$\mathbb{C}[x_0^{\pm 1}, x_{-i}, \partial_{-i}, i > 1] \longrightarrow \mathbb{C}[\tilde{x}_0^{\pm 1}, \tilde{x}_{-i}, \tilde{\partial}_{-i}, i > 1], \tag{19}$$
$$x_0^{i_0}x_{-1}^{i_1}\cdots\partial_{-1}^{j_1}\partial_{-2}^{j_2}\cdots \mapsto \pi(x_0)_{(-1)}^{i_0}\pi(x_0)_{(-2)}^{i_1}\cdots\pi(\partial_{-1})_{(-1)}^{j_1}\pi(\partial_{-1})_{(-2)}^{j_2}\cdots 1,$$

where an expression such as $a_{(n)}$ means the Fourier coefficient of the field $a(z)$ s.t. $a(z) = \sum_n a_{(n)}z^{-n-1}$.

Lemma 4.1 *The map (19) is a vertex algebra isomorphism.*

Sketch of Proof A little thought shows that this assertion is essentially the Reconstruction Theorem manifesting itself in the case at hand. First of all, relations (18) can be used to define a vertex algebra structure on $\mathbb{C}[\tilde{x}_0^{\pm 1}, \tilde{x}_{-i}, \tilde{\partial}_{-i}, i > 1]$—one only needs to check that the monomials on the right of (19) span $\mathbb{C}[\tilde{x}_0^{\pm 1}, \tilde{x}_{-i}, \tilde{\partial}_{-i}, i > 1]$, and this is easy. Then since relations (18) coincide with those of Exercise 4.2, map (19) is a vertex algebra isomorphism, except that on the right we use the structure that has just been defined. But relations (18) follow from the original relations of the vertex algebra $\mathbb{C}[\tilde{x}_0^{\pm 1}, \tilde{x}_{-i}, \tilde{\partial}_{-i}, i > 1]$, hence the uniqueness assertion of the Reconstruction Theorem will show that the new structure coincides with the old one, which completes the proof. □

Isomorphism (18) is defined on the space of sections over $\mathbb{C}^*$, but it is obviously compatible with localization (defined in pages 87–88) and so defines a sheaf isomorphism $\pi : \mathcal{D}^{ch}_{\mathbb{C}^*} \to \mathcal{D}^{ch}_{\mathbb{C}^*}$. Since we have only two charts, this concludes the definition of the sheaf $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ over $\mathbb{CP}^1$.

Of course, the transition back from $(\tilde{x}, \tilde{\partial})$ to (x, ∂) is defined to be π^{-1}, but it is pleasing to notice that the same formulas will work.

Exercise 4.5 Check that $\tilde{\pi} = \pi^{-1}$ if we define

$$\tilde{\pi}(\tilde{x}_0) = \frac{1}{x_0},\ \tilde{\pi}(\tilde{\partial}_{-1}) = -x_0^2\partial_{-1} - x_{-1}.$$

What sort of a sheaf is $\mathcal{D}^{ch}_{\mathbb{CP}^1}$? Over $\mathbb{C}^*$, $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ is $\mathcal{D}^{ch}_{\mathbb{C}[x,x^{-1}]}$, which is identified with the polynomial ring $\mathbb{C}[x_0^{\pm 1}, x_{-i}, \partial_{-i};\ i > 0]$, and so looks like a $\mathbb{C}[x, x^{-1}]$-module (x operates as multiplication by x_0), and this $\mathbb{C}[x]$-module structure has even been put to use when the localization was defined; but globally, $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ is not an $\mathcal{O}_{\mathbb{CP}^1}$-module in any natural way. The reason for this is as follows: the ordinary multiplication that suggests itself locally, say,

$$(x_0^n, \partial_{-1}) \mapsto x_0^n\partial_{-1},$$

is not given by vertex algebra structure and is not preserved upon gluing. To see the difference, write ab to mean the naive product of $a, b \in \mathbb{C}[x_0^{\pm 1}, x_{-i}, \partial_{-i};\ i > 0]$ and $a_{(n)}b$ for the vertex algebra n-th multiplication, cf. (19).

Exercise 4.6 Verify

$$\text{(i)}\ f(x_0)\partial_{-1} = (\partial_{-1})_{(-1)}f(x_0)\text{but}f(x_0)\partial_{-1} = (f(x_0))_{(-1)}\partial_{-1} - \frac{\partial^2 f}{\partial x^2}(x_0)x_{-1}; \tag{20}$$

$$\text{(ii)}\ \pi(f(x_0)\partial_{-1}) = \pi(f(x_0))\pi(\partial_{-1}) - 2\tilde{x}_0\tilde{x}_{-1}\frac{\partial\pi(f(x_0))}{\partial\tilde{x}}; \tag{21}$$

$$\text{(iii)}\ \pi(x_0) = \tilde{x}_0^{-1},\ \pi(f(x_0)x_{-1}) = -f(\tilde{x}_0^{-1})\tilde{x}_0^{-2}\tilde{x}_{-1}. \tag{22}$$

(Hints: Apart from the definition of $\mathcal{D}^{ch}_{\mathbb{C}[x,x^{-1}]}$ and the patching, use the skew commutativity in vertex algebras: $a_{(-1)}b = b_{(-1)}a - T(b_{(0)}a) + \cdots$, and the Borcherds identity: $(a_{(-1)}b)_{(-1)} = a_{(-1)}b_{(-1)} + a_{(-2)}b_{(0)} + \cdots + b_{(-2)}a_{(0)} + \cdots$. For example, the first part of (i) follows from $[\partial_{-1}, x_0] = 0$, the 2nd follows from the first and skew-commutativity; (ii) follows from (i),the definition of the gluing, and the Borcherds identity.)

The correction terms in these formulas are of the same nature as the "anomalies" we encountered when defining the sheaf, and they teach us a lesson. We see that there is a subsheaf whose restriction to $\mathbb{C}$ is $\mathbb{C}[x_0]$, and this subsheaf is the structure sheaf $\mathcal{O}_{\mathbb{CP}^1}$. Similarly, the subsheaf that restricts to $\mathbb{C}[x_0]x_{-1}$ is isomorphic to the cotangent sheaf $\Omega_{\mathbb{CP}^1}$; in fact, x_{-1} has the meaning of dx. These are the consequences of (22). Thus we obtain a sheaf embedding

$$\mathcal{O}_{\mathbb{CP}^1} \oplus \Omega_{\mathbb{CP}^1} \hookrightarrow \mathcal{D}^{ch}_{\mathbb{CP}^1}. \tag{23}$$

What (21) says is more interesting: $f(x_0)\partial_{-1}$ is not a vector field, but it is modulo 1-forms. More formally, there is a subsheaf $\mathcal{L}^{ch} \subset \mathcal{D}^{ch}_{\mathbb{CP}^1}$ that restricts to $\mathbb{C}[x_0]x_{-1} \oplus \mathbb{C}[x_0]\partial_{-1}$ and fits in the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{CP}^1} \longrightarrow \mathcal{L}^{ch} \longrightarrow \mathcal{T}_{\mathbb{CP}^1} \longrightarrow 0. \tag{24}$$

This extension is destined to be the focus of our attention, and will be understood as a vertex algebra version of the Picard-Lie algebroid, see (11). At the moment, let us point out that although an extension of an $\mathcal{O}_{\mathbb{CP}^1}$-module by another $\mathcal{O}_{\mathbb{CP}^1}$-module, $\mathcal{L}^{ch}$ is not an $\mathcal{O}_{\mathbb{CP}^1}$-module, but merely a sheaf of vector spaces.

Now, a moment's thought will show that this process can be iterated so as to find a filtration on the whole of $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ with quotients that are sheaves of $\mathcal{O}_{\mathbb{CP}^1}$-modules. The simplest (and somewhat crude) such filtration can be defined simply by counting the number of letters ∂_{-n}, $n > 0$. More precisely, define $\mathcal{D}^{ch,\leqslant n}_{\mathbb{CP}^1}$ to be the subsheaf s.t. its restriction to $\mathbb{C} \subset \mathbb{CP}^1$ equals the subspace of $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ that is linearly spanned by polynomials of degree at most n in $\{\partial_{-i},\ i > 0\}$.

Exercise 4.7 Prove that $\oplus_{n\geqslant 0}\mathcal{D}^{ch,\leqslant n}_{\mathbb{CP}^1}/\mathcal{D}^{ch,\leqslant(n-1)}_{\mathbb{CP}^1}$ is naturally a locally free $\mathcal{O}_{\mathbb{CP}^1}$-module. Furthermore, $\mathcal{D}^{ch,\leqslant 0}_{\mathbb{CP}^1}$ is essentially the structure sheaf of the jet scheme $J_\infty\mathbb{CP}^1$. (More precisely, it is the push-forward of $\mathcal{O}_{J_\infty\mathbb{CP}^1}$ w.r.t. the projection $J_\infty\mathbb{CP}^1 \to \mathbb{CP}^1$.)

This filtration is analogous to the one inherent in a TDO (see page 82), and will be essential for us later. It works best if combined with the fact that $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ is graded.

Denote by $\mathcal{D}^{ch}_{\mathbb{CP}^1}[n]$, $n\geqslant 0$, the subsheaf that is locally the $\mathbb{C}$-linear spane of the monomials $f(x_0)\{x_{-i_1}x_{-i_2}\cdots\partial_{-j_1}\partial_{-j_2}\cdots\}$, with $i_\bullet, j_\bullet > 0$ and $\sum_a i_a + \sum_b j_b = n$. It is rather clear from the transformation formulas that then

$$\mathcal{D}^{ch}_{\mathbb{CP}^1} = \bigoplus_{n=0}^{\infty} \mathcal{D}^{ch}_{\mathbb{CP}^1}[n].$$

Furthermore, this grading is compatible with the filtration: if we let $\mathcal{D}^{ch,\leqslant m}_{\mathbb{CP}^1}[n] = \mathcal{D}^{ch,\leqslant m}_{\mathbb{CP}^1} \cap \mathcal{D}^{ch}_{\mathbb{CP}^1}[n]$, then we obtain a finite filtration of each homogeneous piece:

$$\mathcal{D}^{ch,\leqslant 0}_{\mathbb{CP}^1}[n] \subset \mathcal{D}^{ch,\leqslant 1}_{\mathbb{CP}^1}[n] \subset \cdots \subset \mathcal{D}^{ch,\leqslant n}_{\mathbb{CP}^1}[n] = \mathcal{D}^{ch}_{\mathbb{CP}^1}[n].$$

An obvious application of the long cohomology sequence (and the fact that the cohomology of coherent sheaves over $\mathbb{CP}^1$ is finite dimensional) will then show that

$$\dim H^i(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})[n] < \infty. \tag{25}$$

We shall soon be able to compute this dimension.

Let us push these ideas a little further so as to be able to compute the Euler character of $\mathcal{D}^{ch}_{\mathbb{CP}^1}$. Recall that for a sheaf $\mathcal{E}$ over $\mathbb{CP}^1$ with finite dimensional cohomology the *Euler characteristic* is defined by

$$\mathrm{Eu}(\mathcal{E}) = \dim H^0(\mathbb{CP}^1, \mathcal{E}) - \dim H^1(\mathbb{CP}^1, \mathcal{E}).$$

For example, $\mathrm{Eu}(\mathcal{O}(r)) = r + 1$. (Verify this!)

In view of (25) this makes sense for $\mathcal{D}^{ch}_{\mathbb{CP}^1}[n]$, but surely not for the entire $\mathcal{D}^{ch}_{\mathbb{CP}^1}$, where the appropriate notion is one of the *Euler character*, which is nothing but the generating function of the Euler characteristics defined as follows:

$$\mathrm{Eu}_q(\mathcal{D}^{ch}_{\mathbb{CP}^1}) = \sum_{n=0}^{\infty} \mathrm{Eu}(\mathcal{D}^{ch}_{\mathbb{CP}^1}[n])q^n.$$

Lemma 4.2

$$Eu_q(\mathcal{D}^{ch}_{\mathbb{CP}^1}) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}.$$

Proof What makes the computation of the Euler characteristic simpler than that of the cohomology is the fact that the Euler characteristic is additive w.r.t. filtrations: if we have $\mathcal{E}^{-1} = \{0\} \subset \mathcal{E}^1 \subset \cdots \subset \mathcal{E}^N$, then

$$\mathrm{Eu}(\bigcup_{n=0}^{N} \mathcal{E}^n) = \sum_{n=0}^{N} \mathrm{Eu}(\mathcal{E}^n/\mathcal{E}^{n-1}).$$

The key to the proof is a filtration of $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ that is a refinement of the one considered above. Notice that the set of monomials $\{x_{-i_1}x_{-i_2}\cdots\partial_{-j_1}\partial_{-j_2}\cdots,\ i_\bullet, j_\bullet > 0\}$ can be ordered by stipulating that $x_\bullet < \partial_\bullet$, that $a_{-i} < a_{-j}$ if $i < j$, and extending this to the monomials lexicographically. Now define $F_N\mathcal{D}^{ch}_{\mathbb{CP}^1}$ to be subsheaf generated (over functions of x_0) by the least N such monomials.

Exercise 4.8 Verify that in the graded object the class of x_{-i}, $i > 0$, transforms as dx, the class of ∂_{-j} as d/dx, and, more generally, the class of the monomial $x_{-i_1}x_{-i_2}\cdots x_{-i_s}\partial_{-j_1}\partial_{-j_2}\cdots\partial_{-j_t}$ as $(dx)^{\otimes(s-t)}$. (Hint: this follows from the formulas of Exercise 4.6.)

Now recall that $(dx)^{\otimes r}$ is a local section of $\mathcal{O}(-2r)$. Therefore, as follows from the mentioned additivity of the Euler characteristic, the Euler character $\mathrm{Eu}_q(\mathcal{D}^{ch}_{\mathbb{CP}^1})$ is the following sum extended over the indicated monomials:

$$\mathrm{Eu}_q(\mathcal{D}^{ch}_{\mathbb{CP}^1}) = \sum_{\{x_{-i_1}x_{-i_2}\cdots x_{-i_s}\partial_{-j_1}\partial_{-j_2}\cdots\partial_{-j_t}\}} (2(t-s)+1)q^{\sum_a i_a + \sum_b j_b}.$$

The assertion of the lemma easily follows from this equality.

Exercise 4.9 Complete the proof. □

The task of computing the cohomology groups $H^i(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ is harder, and crucial for accomplishing it is the $\widehat{sl}_2$-structure of the sheaf $\mathcal{D}^{ch}_{\mathbb{CP}^1}$

As we have seen in page 77, SL_2 operates on $\mathbb{CP}^1$, and so there is a Lie algebra morphism $sl_2 \to \Gamma(\mathbb{CP}^1, \mathcal{T}_{\mathbb{CP}^1})$ and an associative algebra morphism $U(sl_2)_0 \to \Gamma(\mathbb{CP}^1, \mathcal{D}_{\mathbb{CP}^1})$. If $\mathcal{D}_{\mathbb{CP}^1}$ is to be replaced with $\mathcal{D}^{ch}_{\mathbb{CP}^1}$, then sl_2 must be replaced with the affine $\widehat{\mathrm{sl}}_2$. Recall that the latter is defined to be the central extension$sl_2((t))\oplus\mathbb{C}K$ with bracket defined by

$$[x(.), y(.)] = [x(t), y(t)] + \mathrm{res}_{t=0}\mathrm{Tr} dx(t)y(t)K,\ [K, x(.)] = 0.$$

Here is a train of thought that leads to "chiralization" of the formulas from page 77. A vector field ξ on $\mathbb{C}$ that moves a point x to a nearby point $x + \epsilon f(x)$ defines an infinite family of vector fields ξ_n, $n \in \mathbb{Z}$, on the space of infinitesimal loops $\mathbb{C}((z))$: ξ_n moves the point $x(z) \in \mathbb{C}((z))$ to a nearby point $x(z) + \epsilon z^n f(x(z))$. In terms of coordinates this becomes vaguely familiar:

$$\xi_n = \mathrm{res}_{z=0} f(x(z))\partial(z)dz/z^{-n-1}.$$

In the case of the three vector fields defining an action of sl_2 on $\mathbb{C}$ this gives a Lie algebra morphism $\mathrm{sl}_2((t)) \to \mathcal{T}_{\mathbb{C}((z))}$, slightly informally recorded as follows

$$e(z) \mapsto -\partial(z),\ h(z) \mapsto -2x(z)\partial(z),\ f(z) \mapsto x(z)^2\partial(z),$$

where given $x \in \mathrm{sl}_2$, $x(z) = \sum_{n\in\mathbb{Z}}(x \otimes t^n)z^{-n-1}$ is simply a generating function of the family $\{x \otimes t^n\} \subset \mathrm{sl}_2((t))$.

The term "chiralization" used above means making sense out of such formulas. The problem here is that the indicated vector fields act on functions, not on a vertex algebra $\mathcal{D}^{ch}_{\mathbb{C}[x]}$, which is a space of distributions, *loc. cit.*. A chiralization in the case at hand was accomplished by Wakimoto in the celebrated work [27]. In our terminology his result reads: the assignment

$$e(z) \mapsto -\partial(z),\ h(z) \mapsto -2 : x(z)\partial(z) : -2,\ f(z) \mapsto : x(z)^2\partial(z) : +2x(z)',\ K \mapsto -2 \tag{26}$$

defines an $\widehat{\mathrm{sl}}_2$-module on $\mathcal{D}^{ch}_{\mathbb{C}[x]}$.

Therefore, our $\mathcal{D}^{ch}_{\mathbb{C}[x]}$ is nothing but what is known as the *restricted* Wakimoto module of level -2. The word "restricted" means the following.

Exercise 4.10

(i) Verify that the coefficients of the field $: e(z)f(z)+f(z)e(z)+1/2h(z)^2 :$ commute with $\widehat{\mathrm{sl}}_2$.
(ii) As a field acting on $\mathcal{D}^{ch}_{\mathbb{C}[x]}$, $: e(z)f(z) + f(z)e(z) + 1/2h(z)^2 :$ is 0.

Of course, this is a pleasing chiralization of the formulas from Exercise 2.5. To advance further and chiralize Lemma 2.3 a slight change of tack is needed.

Let $\mathbb{C}_k$ be a 1-dimensional $\mathrm{sl}_2[[t]] \oplus \mathbb{C}K$-module, where $\mathrm{sl}_2[[t]]$ acts trivially and K as multiplication by $k \in \mathbb{C}$. Consider the induced $\widehat{\mathrm{sl}}_2$-module

$$V(\mathrm{sl}_2)_k = \mathrm{Ind}^{\widehat{\mathrm{sl}}_2}_{\mathrm{sl}_2[[t]]\oplus\mathbb{C}K}\mathbb{C}_k.$$

Note that as a vector space $V(\mathrm{sl}_2)_k$ is identified with a polynomial ring $S^\bullet(\mathrm{sl}_2[t^{-1}]t^{-1})$.

The foundational result of Frenkel-Zhu [14] is that $V(\mathrm{sl}_2)_k$ carries a vertex algebra structure determined by the requirements that $1 \in \mathbb{C}_k$ is the vacuum vector and that $(x \otimes t^{-1})(z) = \sum_{n\in\mathbb{Z}}(x \otimes t^n)z^{-n-1}$. Now the vertex algebra content of formula (26) is clear.

Lemma 4.3 *There is a vertex algebra morphism*

$$V(sl_2)_{-2} \longrightarrow \mathcal{D}^{ch}_{\mathbb{C}[x]|}\ s.t. \tag{27}$$
$$e \otimes t^{-1} \mapsto -\partial_{-1},\ h \otimes t^{-1} \mapsto -2x_0\partial_{-1},\ f \otimes t^{-1} \mapsto x_0^2\partial_{-1} + 2x_{-1}.$$

Exercise 4.11 Use the Reconstruction Theorem to prove the Frenkel-Zhu result along with Lemma 4.3.

To return to $\mathbb{CP}^1$. In this context, morphism (27) is interpreted as a vertex algebra morphism

$$V(\mathrm{sl}_2)_{-2} \longrightarrow \mathcal{D}^{ch}_{\mathbb{CP}^1}(\mathbb{C})$$

for $\mathbb{C} \subset \mathbb{CP}^1$ a big cell. Of course, there is the restriction map

$$\Gamma(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) \to \mathcal{D}^{ch}_{\mathbb{CP}^1}(\mathbb{C})$$

and the fact that is crucial for what follows is that the former map factors through the latter. The reason for this is very simple: the correction terms in formulas (16) and (27) coincide s.t. the image of e_{-1} in terms of coordinates on one chart, becomes the image of f_{-1} when written in terms of coordinates on another chart; more formally: $\pi(e_{-1}) = f_{-1}$. A companion equality $\pi(f_{-1}) = e_{-1}$ can be proved by a direct computation, which in fact constitutes the content of Exercise 4.5. This proves

Lemma 4.4 *There is a vertex algebra morphism*

$$V(sl_2)_{-2} \longrightarrow \Gamma(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$$

that is locally defined by (27).

Therefore, the sheaf $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ carries a $V(\mathrm{sl}_2)_{-2}$-module structure.

Denote by $L_{m,k}$ a unique irreducible highest weight $\widehat{sl}_2$-module with highest weight m and level k. Furthermore, $L_{0,k}$ is a quotient of the vertex algebra $V(\widetilde{sl}_2)_k)$, from which it inherits a vertex algebra structure.

The fact that $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ carries a $V(\mathrm{sl}_2)_{-2}$-module structure implies that the cohomology groups $H^i(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$, $i \geqslant 0$, are $V(\mathrm{sl}_2)_{-2}$-modules. On the other hand, $H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ is a vertex algebra, and $H^i(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ is an $H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$.

Theorem 4.1

(i) There is a vertex algebra isomorphism $L_{0,-2} \xrightarrow{\sim} H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ *and an* $L_{0,-2}$*-module isomorphism* $L_{0,-2} \xrightarrow{\sim} H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$. *Furthermore,*
(ii) $H^i(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) = 0$ *if* $i > 1$.

This is an obvious, and perhaps pleasing, analogue of Lemma 2.3, where the associative algebra, $\mathcal{D}_{\mathbb{CP}^1}$, is replaced with a vertex algebra, $\mathcal{D}^{ch}_{\mathbb{CP}^1}$. The two results differ significantly in that the higher cohomology in the latter case does not vanish.

Proof To begin with, item (ii) is nothing but Grothendieck's vanishing theorem that applies on the grounds that $\dim\mathbb{CP}^1 = 1$. Alternatively, one can use the filtration $\mathcal{D}^{ch,\leqslant n}_{\mathbb{CP}^1}$ and the long exact sequence of cohomology groups to reduce to a more elementary result on the cohomology of the Serre twisting sheaves $\mathcal{O}(m)$ over $\mathbb{CP}^1$—a recurrent topic of these notes.

As to (i), notice that the restriction morphism $\Gamma(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) \to \mathcal{D}^{ch}_{\mathbb{CP}^1}(\mathbb{C})$ is an injection (by the definition of the sheaf $\mathcal{D}^{ch}_{\mathbb{CP}^1}$ or because our sheaf is filtered by locally free sheaves of $\mathcal{O}_{\mathbb{CP}^1}$-modules, for which the restriction morphism is always injective); therefore $\Gamma(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ is an $\widehat{sl}_2$-submodule of the Wakimoto module $\mathcal{D}^{ch}_{\mathbb{CP}^1}(\mathbb{C})$. Notice that $\Gamma(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) \subset \mathcal{D}^{ch}_{\mathbb{CP}^1}(\mathbb{C})$ is nontrivial and proper, because $\mathcal{D}^{ch}_{\mathbb{CP}^1}[0] = \mathcal{O}_{\mathbb{CP}^1}$, $\Gamma(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})[0] = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) = \mathbb{C}$, while $(\mathcal{D}^{ch}_{\mathbb{CP}^1}[0])(\mathbb{C}) = \mathcal{O}_{\mathbb{CP}^1})(\mathbb{C}) = \mathbb{C}[x]$. Feigin and Frenkel proved [12] that the Wakimoto module $\mathcal{D}^{ch}_{\mathbb{CP}^1}(\mathbb{C})$ has a unique nontrivial proper submodule, which is isomorphic to $L_{0,-2}$. This proves the isomorphism $L_{0,-2} \xrightarrow{\sim} H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$.

The computation of $H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ is based on the concept of the Euler character, see page 92. We introduce the characters

$$ch_q H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \dim H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}[n]) q^n,$$

$$ch_q H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \dim H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}[n]) q^n,$$

which makes sense, see (25), and so

$$\mathrm{Eu}_q(\mathcal{D}^{ch}_{\mathbb{CP}^1}) = ch_q H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) - ch_q H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}).$$

We have, Lemma 4.2,

$$\mathrm{Eu}_q(\mathcal{D}^{ch}_{\mathbb{CP}^1}) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2},$$

and, [24],

$$ch_q L_{0,-2} = ch_q H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) = \frac{1}{1-q} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}.$$

Solving for $ch_q H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ gives

$$ch_q H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) = \frac{q}{1-q} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}.$$

Therefore,

$$ch_q H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) = q \cdot ch_q H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}),$$

and so the two characters coincide (up to a shift induced by the factor of q.) This is evidence enough to convince the sensible reader that then the modules are also isomorphic. One way to proceed would be to use the Čech resolution to compute $H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1})$ as a quotient of $\mathcal{D}^{ch}_{\mathbb{CP}^1}(\mathbb{C}^*)$ (which is allowed thanks to the filtration by $\mathcal{O}_{\mathbb{CP}^1}$-modules) and then use some results of [24]. Here we shall outline a different approach, more in spirit of these notes.

Exercise 4.12 Do the following:

(i) verify that $H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}[0]) = \{0\}$;

(ii) notice that (24) is equivalent to

$$0 \longrightarrow \Omega_{\mathbb{CP}^1} \longrightarrow \mathcal{D}^{ch}_{\mathbb{CP}^1}[1] \longrightarrow \mathcal{T}_{\mathbb{CP}^1} \longrightarrow 0,$$

and show that in (the segment of) the corresponding long exact sequence of cohomology

$$H^0(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}[1]) \to H^0(\mathbb{CP}^1, \mathcal{T}_{\mathbb{CP}^1}) \to H^1(\mathbb{CP}^1, \Omega_{\mathbb{CP}^1}) \to H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}[1])$$

the leftmost arrow is an isomorphism (this uses (i) of the theorem), and the rightmost arrow is an isomorphism;

(iii) use (ii) and Exercise 3.4 to show that the class of $x_{-1}/x_0 \in (\mathcal{D}^{ch}_{\mathbb{CP}^1}[1])(\mathbb{C}^*)$ defines a basis of $H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}[1])$ and is annihilated by $sl[t]$;

(iv) use (iii) to define a non-trivial $\widehat{sl}_2$-morphism

$$V(\mathrm{sl}_2)_{-2} \longrightarrow H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1});$$

(v) use the above-obtained equality $ch_q H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}) = q \cdot ch_q L_{0,-2}$ to prove that the morphism of (iv) factors through an isomorphism

$$L_{0,-2} \xrightarrow{\sim} H^1(\mathbb{CP}^1, \mathcal{D}^{ch}_{\mathbb{CP}^1}). \square$$

5 CDO: Definition and Classification

The example just now considered may be inspiring enough to conclude that we are onto something. Let us begin abstracting the properties of that example by analyzing a local model.

A higher dimensional generalization of Sect. 4 is immediate and requires nothing but an introduction of an extra index.

In order to define $\mathcal{D}^{ch}_{\mathbb{C}[\vec{x}]}$, where $\mathbb{C}[\vec{x}] = \mathbb{C}[x_1, \ldots, x_N]$, introduce $\mathfrak{a}$, a Lie algebra with generators

$$\{x_{ij}, \partial_{mn}, C;\ 1 \leqslant i, m \leqslant N, n, j \in \mathbb{Z}\}$$

and relations

$$[\partial_{mn}, x_{ij}] = \delta_{mi}\delta_{n,-j}C,\ \ C \text{ being central.}$$

There is a subalgebra, $\mathfrak{a}_+$, defined to be the linear span of x_{ij}, ∂_{mn}, $j > 0$, $n \geqslant 0$, and C. Let $\mathbb{C}_1$ be an $\mathfrak{a}_+$-module, where x_{ij}, ∂_{mn} act trivially, and C as multiplication by 1. The induced representation $\mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{a}_+}\mathbb{C}_1$, which is naturally identified with $\mathbb{C}[x_{ij}, \partial_{mn};\ j \leqslant 0, n < 0]$, is well known to carry a vertex algebra structure; it is often referred to as a "β-γ-system." The shortest way to define this structure is again to notice that the fields

$$x_i(z) = \sum_{n\in\mathbb{Z}} x_{in} z^{-n}, \;\; \partial_j(z) = \sum_{n\in\mathbb{Z}} \partial_{jn} z^{-n-1},$$

the vector space $\mathrm{Ind}_{\mathfrak{a}_+}^{\mathfrak{a}} \mathbb{C}_1$, on which they operate, the choice of a "derivation"

$$T = -\sum_{n\in\mathbb{Z}} n x_{in} \partial_{i,-n-1} : \mathrm{Ind}_{\mathfrak{a}_+}^{\mathfrak{a}} \mathbb{C}_1 \longrightarrow \mathrm{Ind}_{\mathfrak{a}_+}^{\mathfrak{a}} \mathbb{C}_1,$$

and the choice of the vacuum $1 \in \mathbb{C}_1$ satisfy the conditions of the Reconstruction Theorem. Denote

$$\mathcal{D}^{ch}_{\mathbb{C}[\vec{x}]} = \mathrm{Ind}_{\mathfrak{a}_+}^{\mathfrak{a}} \mathbb{C}_1.$$

Over a ring A equipped with an étale morphism $\mathbb{C}[\vec{x}] \to A$, which in practical terms means a choice of a "coordinate system," i.e., a collection of elements $x_i \in A$, $\partial_i \in \mathrm{Der}(A)$, $1\leqslant i\leqslant N = \dim A$, s.t. $\{\partial_i\}$ is an A-basis of $\mathrm{Der}(A)$ and $\partial_i(x_j) = \delta_{ij}$, the construction works along the lines of pages 87–88. We define

$$\mathcal{D}^{ch}_{A,\vec{x}} = A \otimes_{\mathbb{C}[\vec{x}]} \mathrm{Ind}_{\mathfrak{a}_+}^{\mathfrak{a}} \mathbb{C}_1,$$

where $\mathbb{C}[\vec{x}]$ operates on $\mathrm{Ind}_{\mathfrak{a}_+}^{\mathfrak{a}} \mathbb{C}_1$ by $x_i \mapsto x_{i0}$. Of course, as a vector space

$$\mathcal{D}^{ch}_{A,\vec{x}} = A[x_{ij}, \partial_{mn}, C; 1\leqslant i, m\leqslant N, n, j < 0].$$

The space $\mathcal{D}^{ch}_{A,\vec{x}}$ is clearly an $\mathfrak{a}$-module and A-module, with two actions satisfying

$$[\partial_{i,n}, a] = \delta_{n,0} \partial_i(a).$$

In addition to the fields $\partial_j(z)$, T and $1 \in \mathbb{C}_1$, we define a field $a(z)$, for each $a \in A$, as follows

$$a(z) = \sum_{n_1,n_2,\ldots} \frac{\partial_1^{n_1} \partial_2^{n_2} \cdots \partial_N^{n_N} a}{n_1! n_2! \cdots n_N!} \epsilon_1(z)^{n_1} \epsilon_2(z)^{n_2} \cdots \epsilon_N(z)^{n_N},$$

which is an obvious extension of the localization construction of Sect. 5; of course,

$$\epsilon_i(z) \stackrel{\text{def}}{=} \sum_{n\neq 0} x_{in} z^{-n}.$$

The reader will have no trouble formulating and doing an analogue of Exercise 4.3. Therefore, $\mathcal{D}^{ch}_{A,\vec{x}}$ is a vertex algebra. Note the $\vec{x}$ in the notation; the dependence on a choice of a coordinate system is important.

Given an étale morphism $\mathbb{C}[\vec{x}] \to A$, a composition $\mathbb{C}[\vec{x}] \to A \to A_f, f \neq 0$ is also étale. This gives a natural family $\mathcal{D}^{ch}_{A_f,\vec{x}}, f \in A \setminus \{0\}$, hence a sheaf of vertex algebras over Specm(A). This can be rephrased more geometrically as follows:

for any smooth algebraic variety X and any étale morphism $X \to \mathbb{C}^N$, the discussion above defines sheaf of vertex algebras over X, to be denoted $\mathcal{D}^{ch}_{X,\vec{x}}$; $\vec{x}$, the reminder about a fixed morphism, will sometimes be omitted.

What is the meaning of this example? Perhaps the question is: our example is an example of what? Up to this point we have been able to avoid the issue of defining a vertex algebra by making use of the Reconstruction Theorem, but not anymore. We shall nevertheless refrain from making a formal definition, referring the reader to the books such as [11, 19] or V. Kac's lecture notes in this volume. Instead, we shall record the more important structure elements that will be used later.

One of the main features of the "vertex algebra world" is that a vector space is replaced with a vector space with a "derivation." The simplest example is the concept of a unital, commutative, associative algebra with derivation. If we denote by $\mathit{Comm}-\mathcal{D}er$ the category of such algebras and by Comm the category of ordinary unital, commutative, associative algebras, then there is an obvious forgetful functor

$$F: \mathit{Comm} - \mathcal{D}er \longrightarrow \mathit{Comm}.$$

A moment's thought will show that this functor admits a left adjoint

$$J_\infty : \mathit{Comm} \longrightarrow \mathit{Comm} - \mathcal{D}er.$$

Adjoining a universal derivation to an algebra is easy, as the following example illustrates.

Example 5.1

$$J_\infty \mathbb{C}[x_1, \dots, x_N] = \mathbb{C}[x_{in}; 1 \leqslant i \leqslant N, n \leqslant 0]$$

with derivation T defined by the condition $T(x_{i,n+1}) = -n x_{i,n}, n \leqslant -1$.

The adjunction morphism $A \to F \circ J_\infty A = J_\infty A$ makes $J_\infty A$ an A-module. The submodule generated by TA, i.e., $A \cdot TA$ is canonically identified with the module of Kähler differentials, Ω_A. In the above example, we get an identification $\Omega_A = \oplus_i \mathbb{C}[x_1, \dots, x_N] x_{i,-1}$, $x_{i,-1}$ being identified with dx_i.

If X is an affine algebraic variety, then we define the corresponding jet-scheme $J_\infty X$ to be Specm$(J_\infty \mathbb{C}[X])$, where $\mathbb{C}[X]$ means the coordinate ring of X. It is not hard to see that such "local models" can be glued, so as to define, given an algebraic variety X, the corresponding jet scheme $J_\infty X$. Note that the adjunction $\mathbb{C}[X] \to F \circ J_\infty \mathbb{C}[X] = J_\infty \mathbb{C}[X]$ induces the projection $\pi : J_\infty X \to X$.

We have already encountered such objects without using the term "jet." Indeed, part of $\mathcal{D}^{ch}_{A,\vec{x}}$ that does not contain the letters $\partial_\bullet$, that is, $A[x_{in}; 1\leqslant i\leqslant N, n\leqslant 0]$ is naturally $J_\infty A$ s.t. T as defined in page 98 coincides with T mentioned in the example a few lines above.

A similarly defined part of the sheaf $\mathcal{D}^{ch}_{X,\vec{x}}$ is nothing but the push-forward $\pi_*\mathcal{O}_{J_\infty X}$ on X.

We shall soon see that the entire $\mathcal{D}^{ch}_{X,\vec{x}}$ is also related to a jet scheme but in a slightly more complicated manner: it carries a filtration s.t. the corresponding graded object, $\mathrm{Gr}\mathcal{D}^{ch}_{X,\vec{x}}$ is $\pi_*\mathcal{O}_{J_\infty T^*X}$.

A vertex Lie algebra is a vector space V with an endomorphism $T\in\mathrm{End}(V)$ and a family of bilinear products

$$_{(n)}: V\otimes V\longrightarrow V \text{ s.t. } u_{(n)}v=0 \text{ if } n\gg 0$$

for $n=0,1,2,\ldots$. These data must satisfy the following conditions:

$$\text{(i)}\;\; (Tu)_{(n)}v=-nu_{(n-1)}v; \tag{28}$$

$$\text{(ii)}\;\; u_{(n)}v=(-1)^{n+1}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}T^j(v_{(n+j)}u); \tag{29}$$

$$\text{(iii)}\;\; u_{(n)}(v_{(m)}w)-v_{(m)}(u_{(n)}w)=\sum_{j=0}^{\infty}\binom{n}{j}(u_{(j)}v)_{(m+n-j)}w. \tag{30}$$

The structure, if not the details, of this definition is clear: (29), "anticommutativity," and (30), "Jacobi," are analogues of the corresponding ingredients of the definition of an ordinary Lie algebra; (28) is the compatibility condition.

Exercise 5.1 Verify that T is a derivation of all multiplications, i.e., that

$$T(u_{(n)}v)=(Tu)_{(n)}v+u_{(n)}Tv.$$

If $\mathfrak{g}$ is a Lie algebra, then $J_\infty\mathfrak{g}$ defined to be $\mathbb{C}[T]\mathfrak{g}$ carries a vertex Lie algebra structure as follows: let

$$x_{(n)}y=\begin{cases}[x,y] & \text{if } n=0\\ 0 & \text{if } n>0\end{cases}$$

if $x,y\in\mathfrak{g}$ and extend to the whole of $J_\infty\mathfrak{g}$ by setting recurrently $(T^{m+1}x)_{(n)}y=-n(T^mx)_{(n-1)}y$.

Given an invariant inner product $(.,.)$ on $\mathfrak{g}$, $J_\infty\mathfrak{g}$ acquires a central extension. Namely, define $\widehat{J_\infty\mathfrak{g}} = \widehat{J_\infty\mathfrak{g}}_{(.,.)}$ to be $J_\infty\mathfrak{g} \oplus \mathbb{C}K$ with products defined as above except that

$$x_{(n)}y = \begin{cases} [x,y] & \text{if } n=0 \\ (x,y)K & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

plus the requirements $x_{(n)}K = 0$, $T(K) = 0$. We get an exact sequence of vertex Lie algebras

$$0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{J_\infty\mathfrak{g}}_{(.,.)} \longrightarrow J_\infty\mathfrak{g} \longrightarrow 0.$$

If $\mathfrak{g}$ acts on A by derivations, then $J_\infty\mathfrak{g}$ acts on $J_\infty A$ by derivations.

The concept of a vertex Poisson algebra is the simplest way to combine an associative commutative algebra with derivation and a vertex Lie algebra. Namely, a vertex Poisson algebra is a collection $(V, T, 1, _{(-1)}, _{(n)}; n \in \mathbb{Z}_+)$, where the collection $(V, T, 1, _{(-1)})$ is an associative, commutative, unital algebra with derivation (1 is the unit, $_{(-1)}$ is the product), the collection $(V, T, _{(n)}; n \in \mathbb{Z}_+)$ is a vertex Lie algebra, the two structures satisfying the following compatibility condition:

$$u_{(n)}(v_{(-1)}w) = (u_{(n)}(v)_{(-1)}w + v_{(-1)}(u_{(n)}w), \tag{31}$$

i.e., the left multiplication by the n-th product ($n \geqslant 0$) is a derivation of the associative product $_{(-1)}$.

When talking about vertex Poisson algebras, we will usually write uv for $u_{(-1)}v$.

Here is the geometric origin of this notion:

Lemma 5.1 *If A is a Poisson algebra with bracket $\{.,.\}$, then $J_\infty A$ carries a unique vertex Poisson algebra structure s.t for $a, b \in A$*

$$a_{(n)}b = \begin{cases} \{a,b\} & \text{if } n=0 \\ 0 & \text{if } n>0. \end{cases}$$

We leave it for the reader to try to prove this result as an exercise.

Naturally, given a vertex Lie algebra $J_\infty\mathfrak{g}$, the symmetric algebra $S^\bullet J_\infty\mathfrak{g}$ is a vertex Poisson algebra. This is related to Lemma 5.1 as follows: $S^\bullet\mathfrak{g}$ is a Poisson algebra equal to $\mathbb{C}[\mathfrak{g}^*]$ equipped with Kirillov-Kostant-Duflo bracket; $S^\bullet J_\infty\mathfrak{g}$ is precisely $J_\infty\mathbb{C}[\mathfrak{g}^*]$.

Here is an example essential for our purposes. The commutative algebra $S^\bullet_A T_A$ is canonically Poisson, see Sect. 2. By the above, $J_\infty S^\bullet_A T_A$ is vertex Poisson. By analogy with Example 5.1, if $A = \mathbb{C}[x_1, \ldots, x_N]$, then $J_\infty S^\bullet_A T_A$ is a polynomial ring $\mathbb{C}[x_{in}, \partial_{i,n-1}; 1 \leqslant i \leqslant N, n \leqslant 0]$, the adjunction morphisms $\mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C}[x_{in}, \partial_{i,n-1}; 1 \leqslant i \leqslant N, n \leqslant 0]$ being defined by $x_i \mapsto x_{i0}$, $\partial_i \mapsto \partial_{i,-1}$; the latter shift of an index is made merely to conform to some vertex algebra notation. The vertex Poisson bracket is reconstructed uniquely from the axioms and assignment $(\partial_{i,-1})_{(n)}x_j = \delta_{ij}\delta_{n0}$.

For a more general ring with a coordinate system $x_1, \ldots$, one similarly obtains $J_\infty S^\bullet_A T_A = A[x_{i,n-1}, \partial_{i,n-1}; 1\leqslant i\leqslant N, n\leqslant 0]$ and the defining relation $(\partial_{i,-1})_{(n)}a = \partial_i(a)\delta_{n0}$, $a \in A$; this is reminiscent of the formulas at the beginning of Sect. 5—and there is a reason for this similarity.

We shall often replace a slightly awkward $(\partial_{i,-1})_{(n)}$ *with a slightly corrupt* $(\partial_i)_{(n)}$.

It is easiest for us to define a vertex algebra following Borcherds [3] as follows: a vertex algebra is a collection $(V, 1, _{(n)}; n \in \mathbb{Z})$, where $1 \in V$ is a distinguished vector (vacuum), each $_{(n)}$ is a bilinear product s.t. $u_{(n)}v = 0$ if $n \gg 0$; the following three identities must be satisfied:

$$a_{(n)}1 = \begin{cases} a & \text{if } n = -1 \\ 0 & \text{if } n\geqslant 0. \end{cases} \tag{32}$$

$$u_{(n)}(v_{(m)}w) - v_{(m)}(u_{(n)}w) = \sum_{j=0}^{\infty} \binom{n}{j} (u_{(j)}v)_{(m+n-j)}w \tag{33}$$

$$(u_{(-1)}v)_{(n)}w = \sum_{j<0} u_{(j)}(v_{(n-j-1)}w) + \sum_{j\geqslant 0} v_{(n-j-1)}(u_{(j)}w). \tag{34}$$

As we have done previously, one often combines various multiplications in a field:

$$Vu \mapsto u(z) = \sum_{n\in\mathbb{Z}} u_{(n)}z^{-n-1}.$$

Condition (34) is simply the normal ordering formula,

$$(u_{(-1)}v)(z) =: u(z)v(z) :,$$

which we have already used more than once.

Condition (33) is known as the Borcherds commutator formula; it means, in particular, that $\mathrm{Lie}(V)$ defined to be the linear span of $\{u_{(n)},\ u \in V, n \in \mathbb{Z}\} \subset \mathrm{End}_{\mathbb{C}}(V)$ is a Lie algebra—a Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$.

Formulas (33) and (30) coincide, except in the latter the indices are only allowed to be nonnegative. Indeed, the assignment $(V, 1, _{(n)}; n \in \mathbb{Z}) \mapsto (V, T, _{(n)}; n\geqslant 0)$, with $T : V \to V$ defined s.t. $T(a) = a_{(-2)}1$, is a forgetful functor from the category of vertex algebras to the category of vertex Lie algebras: one can verify that thus defined T satisfies what is expected of it, $(Tv)_{(n)} = -nv_{(n-1)} = [T, u_{(n)}]$, and that (29) holds in any vertex algebra, see e.g. [19].

The omission of T from the definition, although legitimate, is misleading. We shall always regard T just defined as part of the data.

Exercise 5.2 Go over the construction of pages 97–98 from the point of view of this definition. In particular, see how T defined there coincides with T introduced here and check that $(x_{i,0}1)_{(n)} = x_{i,n-1}$, $(\partial_{i,-1}1)_{(n)} = \partial_{i,n}$ s.t.

$$(\partial_{i,-1}1)_{(0)}a = \partial_i(a) \text{ if } a \in A. \tag{35}$$

As above, we shall often write simply $(\partial_i)_{(n)}$ meaning $(\partial_{i,-1}1)_{(n)}$ or $\partial_{i,n}$

A vertex algebra is called commutative if $_{(n)} = 0$ for all $n \geqslant 0$. Some motivation for the name lies in (33), because it implies that then $[u_{(n)}, v_{(m)}] = 0$ for all u, v, m, n. More importantly, if V is commutative, then $(V, 1, T, _{(-1)})$ is a commutative, associative, unital algebra. In fact, this assignment sets up an equivalence of the category of commutative vertex algebras and the category of commutative, associative, unital algebras with derivation. (Q: Why is in this case $_{(-1)}$ associative? Hint: (34).)

Exercise 5.3 Prove this equivalence (or read either [11] of [19].)

The definition of a vertex Poisson algebra involves a similar amount of data as that of a vertex algebra; in fact, the latter is to be thought of as a quantization of the former. One way to explain this is to use the concept of a filtered vertex algebra.

We shall call a vertex algebra V filtered if given a sequence of subspaces $\{0\} = V^{-1} \subset V^0 \subset V^1 \subset \cdots \subset V^n \subset \cdots$, $\cup_n V^n = V$ s.t. $1 \in V^0$ and $(V^n)_{(i)}(V^m) \subset V^{m+n}$ for all m, n, i.

If V is filtered then, of course, the graded object $\mathrm{Gr}V = \oplus_n V^n/V^{n-1}$ is naturally a (graded) vertex algebra.

If, in addition, $(V^n)_{(i)}(V^m) \subset V^{m+n-1}$ provided $i \geqslant 0$, then $\mathrm{Gr}V$ is commutative, and so $(\mathrm{Gr}V, T, 1, _{(-1)})$ is an associative, commutative with derivation, but more than that, $\mathrm{Gr}V$ carries traces of products $_{(n)}$ with $n \geqslant 0$. Namely, define for all $m, n, i \geqslant 0$

$$_{(i)} : (V^n/V^{n-1}) \otimes (V^m/V^{m-1}) \longrightarrow V^{n+m-1}/V^{n+m-2} \text{ s.t. } \bar{u}_{(i)}\bar{v} \stackrel{\text{def}}{=} \overline{u_{(i)}v}.$$

A moment's thought will show that this definition makes sense and that thus defined $(\mathrm{Gr}V, 1, T, _{(-1)}, _{(n)}; n \geqslant 0)$ is a vertex Poisson algebra. This prompts the following obvious definition: if P is a vertex Poisson algebra and V is a filtered vertex algebra s.t. $\mathrm{Gr}V$ is commutative, then provided P and $\mathrm{Gr}V$ are isomorphic as vertex Poisson algebras, V is called a *quantization* of P.

The reader has undoubtedly noticed that this idea of "filtration quantization" has been a thread running through these notes starting in Sect. 2.

Our digression on the vertex basics has handed us a key to the understanding of the vertex algebra $\mathcal{D}^{ch}_{A,\vec{x}}$ that appeared at the beginning of Sect. 5. Indeed, define $\mathcal{D}^{ch,\leqslant k}_{A,\vec{x}}$ to be $A[x_{i,n-1}, \partial_{i,n-1}, 1 \leqslant i \leqslant N, n \leqslant 0]^{\leqslant n}$, meaning the subspace of polynomials of degree $\leqslant k$ in variables $\partial_\bullet$; of course, this is precisely the filtration we dealt with in Sect. 4. It is rather clear that this makes $\mathcal{D}^{ch}_{A,\vec{x}}$ filtered s.t. $\mathrm{Gr}\mathcal{D}^{ch}_{A,\vec{x}}$ is commutative—because any nonnegative product will kill at least one $\partial_\bullet$; e.g., by definition $(\partial_{i,-1}1)_{(0)}a = \partial_i(a)$ if $a \in A$ (cf. (35)). A moment's thought shows that in fact

$\mathrm{Gr}\mathcal{D}^{ch}_{A,\vec{x}}$ is naturally identified with $J_\infty S^\bullet_A T_A$. While $\mathcal{D}^{ch}_{A,\vec{x}}$ was constructed by hand and, which is worse, the construction involved a choice of a basis, its graded version is quite canonical, as we have seen, and this is the insight that we needed.

Notice that both vector spaces, $\mathrm{Gr}\mathcal{D}^{ch}_{A,\vec{x}}$ and $J_\infty S^\bullet_A T_A$, are graded; the former by definition, the latter due to the canonical grading of the symmetric algebra $S^\bullet_A T_A$ plus the requirement that the canonical derivation have degree 0; e.g. degree of ∂_i is 1, and so is the degree of ∂_{in} for all $n < 0$. Having made this observation, we conclude that the isomorphism $\mathrm{Gr}\mathcal{D}^{ch}_{A,\vec{x}} \stackrel{\sim}{\longrightarrow} J_\infty S^\bullet_A T_A$ preserves the grading.

Similarly, the sheaf $\mathcal{D}^{ch}_{X,\vec{x}}$ is filtered, and $\mathrm{Gr}\mathcal{D}^{ch}_{A,\vec{x}}$ is isomorphic to $\pi_*\mathcal{O}_{J_\infty T^*X}$ as a graded sheaf, where $\pi : J_\infty T^*X \to X$ is a canonical projection (or rather a composite of two canonical projections $\pi : J_\infty T^*X \to T^*X \to X$.)

Here is then the definition we have been looking for:

Definition 5.1 Let X be a smooth algebraic variety. A sheaf of vertex algebras is called an *algebra of chiral differential operators* (CDO)if it carries a filtration s.t. the corresponding graded object is a vertex Poisson algebra that is isomorphic to $\pi_*\mathcal{O}_{J_\infty T^*X}$ as a graded vertex Poisson algebra.
This has an obvious local counterpart: if A is a ring s.t. $A = \mathbb{C}[X]$ for some smooth affine algebraic variety, then a vertex algebra is called a CDO over A provided it carries a filtration s.t. the corresponding graded object is a vertex Poisson algebra that is isomorphic to $J_\infty S^\bullet_A T_A$.

This definition puts us in a situation analogous to Sect. 3 and we conclude that the notion of CDO is a jet-scheme version of the notion of TDO (just as the notion of a vertex Poisson algebra may be thought of as the jet-scheme version of the notion of Poisson algebra, see page 101.)

It is rather clear what we have to do now: we need to understand what the analogue of (8) is, and then see how a CDO can be obtained as some sort of universal enveloping vertex algebra construction.

Let $\mathcal{A}$ be a CDO over A. By definition, $\mathcal{A}^{\leqslant 0} = J_\infty A$, which is a commutative vertex algebra a.k.a commutative, associative, unital algebra with derivation. The next component, $\mathcal{A}^{\leqslant 1}$, by definition fits into an exact sequence as follows:

$$0 \longrightarrow J_\infty A \longrightarrow \mathcal{A}^{\leqslant 1} \longrightarrow J_\infty T_A \longrightarrow 0. \tag{36}$$

Let us discuss this sequence.

The leftmost nontrivial arrow is but the tautological inclusion of a subset $J_\infty A \subset \mathcal{A}^{\leqslant 1}$, the rightmost nontrivial term is, by definition, the degree 1 component of $J_\infty S^\bullet_A T_A$. This component has an independent description as follows: as we discussed above, $A \mapsto J_\infty A$ is a functor $Comm \to Comm - \mathcal{D}er$ left adjoint to the forgetful functor $\mathit{Comm} - \mathcal{D}\mathit{er} \to \mathit{Comm}$; similarly to this, the pull-back functor $J_\infty A - \mathrm{mod} \to A - \mathrm{mod}$[2] has a left adjoint

$$J_\infty : A - \mathrm{mod} \longrightarrow J_\infty A - \mathrm{mod}.$$

Given an A-module M, the construction of $J_\infty M$ essentially amounts to adjoining a "universal" derivation, much like $J_\infty A$ was defined. We advise the reader to figure out the details having scrutinized our main example.

Example 5.2 In the case of $\mathcal{D}^{ch}_{A,\vec{x}}$ at the beginning of Sect. 5, the above exact sequence takes the form

$$0 \longrightarrow J_\infty A \longrightarrow J_\infty A[\partial_{in}; 1\leqslant i\leqslant N, n < 0]^{\leqslant 1} \longrightarrow \bigoplus_{i=1}^{N} \bigoplus_{n=-1}^{-\infty} J_\infty A\cdot\partial_{in} \longrightarrow 0. \quad (37)$$

In other words, we are dealing with a polynomial ring (over $J_\infty A$) in variables ∂_{in}, $\mathcal{A}^{\leqslant 1}$ is the space of polynomials of degree at most 1, finally $J_\infty T_A$ is the space of polynomials of degree 1. It is a free $J_\infty A$ module, but it also carries a derivation T; this derivation essentially coincides with T that was defined in page 98 by the formula $T = \sum_n -nx_{in}\partial_{i,-n-1}$. We use the term "derivation" because indeed

$$T(a \cdot \xi) = T(a) \cdot \xi + a \cdot T(\xi) \text{ for any } a \in J_\infty A, \xi \in J_\infty T_A.$$

Notice that in this case the exact sequence splits, but this is only because a choice of a basis has been made.

What algebraic structure do the terms of our exact system carry?

By construction, the leftmost term, $J_\infty A$, is closed under all vertex algebra products and the derivation; furthermore, the restriction of $_{(n)}$, $n\geqslant 0$ to $J_\infty A$ is 0, and so $J_\infty A$ is a commutative vertex algebra. In fact, this commutative vertex algebra structure coincides with the one that is induced under the equivalence of *loc. cit.* from the associative, commutative, unital algebra with derivation structure that $J_\infty A$ carries by definition.

Next, $\mathcal{A}^{\leqslant 1}$ carries two structures. Firstly, it is closed (by definition) under all $_{(n)}$ with $n\geqslant 0$, and so is a *vertex Lie algebra.* Furthermore,

$$(\mathcal{A}^{\leqslant 1})_{(n)}(J_\infty A) \subset J_\infty A \text{ for all } n\geqslant 0.$$

Hence, $J_\infty A \subset \mathcal{A}^{\leqslant 1}$ is a vertex Lie algebra *ideal*. Since $J_\infty A$ is commutative, the maps

$$_{(n)} : \mathcal{A}^{\leqslant 1} \otimes J_\infty A \longrightarrow J_\infty A, n\geqslant 0.$$

[2] Pull-back w.r.t. the adjunction $A \to J_\infty A$.

factor through

$$_{(n)}:\ (\mathcal{A}^{\leqslant 1}/J_\infty A)\otimes J_\infty A\longrightarrow J_\infty A,\ n\geqslant 0,$$

making $J_\infty A$ a *vertex Lie algebra module* over either $\mathcal{A}^{\leqslant 1}$ or $\mathcal{A}^{\leqslant 1}/J_\infty A$. Either vertex Lie algebra acts on $J_\infty A$ by *derivations*, meaning that

$$\xi_{(n)}(a_{(-1)}b)=(\xi_{(n)}a)_{(-1)}b+a_{(-1)}(\xi_{(n)}b);\ a,b\in J_\infty A,\xi\in\mathcal{A}^{\leqslant 1}.$$

Exercise 5.4 Use the Borcherds identities to verify this formula.

Secondly, although $\mathcal{A}^{\leqslant 1}$ is not closed under negative numbered multiplications, as in general $\mathcal{A}^{\leqslant 1}_{(n)}\mathcal{A}^{\leqslant 1}\subset\mathcal{A}^{\leqslant 2}$, it is under all multiplications by elements of $J_\infty A$: we have maps

$$_{(n)}:\ J_\infty A\otimes\mathcal{A}^{\leqslant 1}\longrightarrow\mathcal{A}^{\leqslant 1}\text{ for all }n\in\mathbb{Z}.$$

These maps satisfy all the conditions of the Borcherds definition (32)–(34); technically, what it means is that $\mathcal{A}^{\leqslant 1}$ is a *vertex algebra module over* $J_\infty A$.

These two structures are compatible in the following sense:

$$\xi_{(n)}(a_{(m)}\eta)=a_{(m)}(\xi_{(n)}\eta)+\sum_{j=0}^{\infty}\binom{n}{j}(\xi_{(j)}a)_{(n+m-j)}\eta\text{ if }a\in J_\infty A,\xi,\eta\in\mathcal{A}^{\leqslant 1}. \tag{38}$$

Notice that $\xi_{(j)}a\in J_\infty A$, and so this equality (which is nothing but the Borcherds commutator formula (33)) is quite analogous to (10).

Let us finally discuss the rightmost term, $J_\infty T_A$. It carries an amount of structure similar to that of $\mathcal{A}^{\leqslant 1}$, but it is canonical and simpler; furthermore, it is simpler for a specific reason. The fact that $J_\infty A$ is a degree 0 and $J_\infty T_A$ is a degree 1 component of a vertex Poisson algebra implies that (verify this!):

(i) it is a vertex Lie algebra;
(ii) it is a $J_\infty A$-module;
(iii) it acts on $J_\infty A$ by derivations;
(iv) the vertex Lie algebra multiplications on $J_\infty T_A$ are not $J_\infty A$-linear, and the failure to be $J_\infty A$-linear is measured by the action of $J_\infty T_A$ on $J_\infty A$ as follows (cf. (10) and (38)):

$$\xi_{(n)}(a_{(-1)}\eta)=a_{(-1)}(\xi_{(n)}\eta)+(\xi_{(n)}a)_{(-1)}\eta\text{ if }a\in J_\infty A,\xi,\eta\in J_\infty T_A. \tag{39}$$

Define a *vertex Lie A-algebroid* to be a vector space that satisfies conditions (i)–(iv); an analogy with the concept of a Lie A-algebroid reviewed in Sect. 3 will justify the name.

All of this is quite parallel to the discussion of $\mathcal{A}^{\leqslant 1}$, but slightly misleadingly so: $J_\infty T_A$ is a module over $J_\infty A$ as a commutative associative algebra with derivation, which is a more restrictive condition than being a module over $J_\infty A$ as a vertex algebra. For example, the operation $a_{(-1)}$, $a \in J_\infty A$ is associative on $J_\infty T_A$ (i.e., $(a_{(-1)}b)_{(-1)} = a_{(-1)}b_{(-1)}$), but not on $\mathcal{A}^{\leqslant 1}$. (Q: why? Hint: (34); cf. Exercise 20.) Similarly, (38) is different from (39) even if $m = 1$. Both these shortcomings of $\mathcal{A}^{\leqslant 1}$ disappear modulo $J_\infty A$.

Exercise 5.5

(i) Let V be a commutative vertex algebra, M a vertex V-module. Call M *central* if $V_{(n)}M = \{0\}$ for all $n \geqslant 0$. Prove that the category of central vertex M modules is equivalent to the category of modules over V as a commutative associative algebra with derivation.
(ii) Prove that $\mathcal{A}^{\leqslant 1}/J_\infty A$ is a vertex Lie A-algebroid.

We see that the rightmost arrow of sequence (36) gives an isomorphism of vertex Lie algebroids $\mathcal{A}^{\leqslant 1}/J_\infty A \xrightarrow{\sim} J_\infty T_A$; in particular, part of the data defining $\mathcal{A}^{\leqslant 1}$, namely, the action of $\mathcal{A}^{\leqslant 1}$ on $J_\infty A$ by derivations, is a pull-back of the canonical action of $J_\infty T_A$ on $J_\infty A$; the latter assertion is true because $J_\infty A$ is an abelian vertex Lie algebra ideal of $\mathcal{A}^{\leqslant 1}$.

We shall now postulate these properties of $\mathcal{A}^{\leqslant 1}$, thus making our 2nd fundamental definition.

Definition 5.2 A chiral A-algebroid is an exact sequence

$$0 \longrightarrow J_\infty A \stackrel{\iota}{\longrightarrow} \mathcal{L}^{ch} \stackrel{\sigma}{\longrightarrow} J_\infty T_A \longrightarrow 0, \tag{40}$$

where $\mathcal{L}^{ch}$ is a vertex Lie algebra and vertex $J_\infty A$-module s.t. the following conditions hold:

(i) ι is a morphism of vertex modules and vertex Lie algebras ($J_\infty A$ is considered as a vertex module over itself and an abelian vertex Lie algebra):
(ii) σ is also a morphism of vertex $J_\infty A$-modules and vertex Lie algebras;
(iii) according to (ii), $\iota(J_\infty A)$ is a vertex Lie algebra ideal, hence a vertex Lie algebra module over $\mathcal{L}^{ch}$; we require that this module be isomorphic to the pull-back of $J_\infty A$ as a $J_\infty T_A$-module w.r.t. $\sigma : \mathcal{L}^{ch} \to J_\infty T_A$.
(iv) the structure of a vertex $J_\infty A$-module and a vertex Lie algebra on $\mathcal{L}^{ch}$ are compatible in that (cf. (38))

$$\xi_{(n)}(a_{(m)}\eta) - a_{(m)}(\xi_{(n)}\eta) = \sum_{j=0}^{\infty} \binom{n}{j} (\tau(\xi)_{(j)}a)_{(n+m-j)}\eta \text{ if } a \in J_\infty A, \xi, \eta \in \mathcal{L}^{ch}. \tag{41}$$

A morphism of chiral algebroids is a $\mathbb{C}$-linear map $f : \mathcal{L}_1^{ch} \to \mathcal{L}_2^{ch}$ that preserves all operations and makes the following diagram commutative (cf.(12)):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_\infty A & \xrightarrow{\iota_1} & \mathcal{L}_1^{ch} & \xrightarrow{\sigma_1} & J_\infty T_A & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & J_\infty A & \xrightarrow{\iota_2} & \mathcal{L}_2^{ch} & \xrightarrow{\sigma_2} & J_\infty T_A & \longrightarrow & 0 \end{array} \tag{42}$$

Our discussion implies that if $\mathcal{A}$ is a CDO, then $\mathcal{A}^{\leqslant 1}$ is a chiral algebroid; e.g. $\mathcal{D}_{A,\vec{x}}^{ch,\leqslant 1}$ is a chiral algebroid. We shall now explain how given a chiral algebroid to construct a CDO thus establishing, in fact, an equivalence of categories.

The meaning of Definition 5.2 is rather clear: we are given some quasiclassical data, encoded in the direct product $J_\infty A \oplus J_\infty T_A$, and $\mathcal{L}^{ch}$ is its filtration quantization as $\mathrm{Gr}\mathcal{L}^{ch} = J_\infty A \oplus J_\infty T_A$. The clearest manifestation of the difference comes from the comparison of (39), which in effect says that

$$[\xi_{(n)}, a_{(-1)}] = (\xi_{(n)} a)_{(-1)},$$

and a particular case of (41):

$$[\xi_{(n)}(a_{(-1)})] = (\tau(\xi)_{(n)} a)_{(-1)} + \sum_{j=1}^{n-1} \binom{n}{j} (\tau(\xi)_{(j)} a)_{(n-1-j)} \text{ if } a \in J_\infty A, \xi, \eta \in \mathcal{L}^{ch}.$$

The reader is encouraged to figure out why the "quantum correction" terms, $\sum_{j<n} \ldots$, disappear in the quasiclassical limit.

The assignment $\mathcal{A} \mapsto \mathcal{A}^{\leqslant 1}$ is a functor from the category of CDO's to that of chiral algebroids. It has a left adjoint, called a vertex enveloping algebra of a chiral algebroid. There is a closely related (and better known [11, 19]) concept of a vertex enveloping algebra of a vertex Lie algebra. The former is to the latter what the notion of a universal enveloping algebra of a Picard-Lie algebroid (see Sect. 2–3) is to the notion of a universal enveloping algebra of a Lie algebra. Our aim, therefore, is to chiralize the construction in Sect. 2–3.

We have seen at the bottom of page 102, that there is a forgetful functor that makes a vertex algebra into a vertex Lie algebra. This functor admits the left adjoint called the *vertex enveloping* algebra. Let us sketch its construction, cf. [11], 16.1.11.

Given a vertex Lie algebra L, define $\mathrm{Lie}(L)$ to be a linear span of symbols $a_{[n]}$, $a \in L$, $n \in \mathbb{Z}$ modulo the relations

$$(c_1 a_1 + c_2 a_2)_{[n]} = c_1 (a_1)_{[n]} + c_2 (a_2)_{[n]}, \ (Ta)_{[n]} = -n a_{[n-1]}, \ c_1, c_2 \in \mathbb{C}, a, a_1, a_2 \in L.$$

Note that the last one mimics (28). Define the bracket, now mimicking (30),

$$[u_{[n]}, v_{[m]}] = \sum_{j=0}^{\infty} \binom{n}{j} (u_{(j)}v)_{[m+n-j]}. \tag{43}$$

One verifies ([11], 16.1.11 or an exercise) that this makes $\mathrm{Lie}(L)$ a Lie algebra. It follows from the definition that $\mathrm{Lie}(L)_+$ defined to be the linear span of $a_{[n]}$, $a \in L$, $n \geqslant 0$ is a Lie subalgebra. Define $U^{ch}L$ to be $U(\mathrm{Lie}(L))/U(\mathrm{Lie}(L)_+)$. Here $U(.)$ is the ordinary universal enveloping of a Lie algebra.

It is easy to see that the map $L \longrightarrow \mathrm{Lie}(L)$, $a \mapsto a_{[-1]}$, is injective, and so is the composition

$$L \longrightarrow \mathrm{Lie}(L) \hookrightarrow U(\mathrm{Lie}(L)) \twoheadrightarrow U(\mathrm{Lie}(L))/U(\mathrm{Lie}(L)_+) \tag{44}$$

For this reason, we shall usually make no distinction between $a \in L$ and $a_{[-1]}1$.

Given $a \in L$, define a field $a(z) = \sum_n a_{[n]} z^{-n-1}$; these fields clearly "generate" $U^{ch}(L)$. The Reconstruction Theorem, [11], 2.3.11 or [19], 4.5, implies that $U^{ch}(L)$ carries a vertex algebra structure. In terms of $_{(n)}$-products, it is given by a slightly tautological formula

$$(\overline{a_{[-1]}})_{(n)} v = a_{[n]} \cdot v;$$

here $\overline{a_{[-1]}}$ is the image of $a_{[-1]}$ under the above composition, and $\cdot$ on the right means the action of $\mathrm{Lie}(L)$ on $U(\mathrm{Lie}(L))/U(\mathrm{Lie}(L)_+)$.

For example, $U^{ch}(J_\infty \mathfrak{g})$, see page 100, is the vertex algebra attached to the affine Lie algebra at level 0, $\mathfrak{g}((t))$, usually denoted by $V(\mathfrak{g})_0$. To shift the level one has to take $U^{ch}(\widehat{J_\infty \mathfrak{g}_{(.,.)}})$ and then quotient out (the ideal generated by) the element $1_{U^{ch}} - K$; notation: $V(\mathfrak{g})_{(.,.)}$. We encountered one such algebra in Sect. 4, page 94.

The constructed object, $U^{ch}(L)$, is related to a whole menagerie of multiplications, $_{[n]}$ and two copies of $_{(n)}$, one defined on L, another on $U^{ch}(L)$. It is a little relief to know that at least the latter two coincide when both make sense; namely,

$$u_{[n]}v = u_{(n)}v \text{ if } n \geqslant 0, u, v \in L. \tag{45}$$

Indeed, we have due to (43), for $n \geqslant 0$,

$$u_{[n]}v = [u_{[n]}, v_{[-1]}]1 + v_{[-1]}u_{[n]}1 = \sum_{j=0}^{\infty} \binom{n}{j} (u_{(j)}v)_{[-1+n-j]}1 = u_{(n)}v,$$

because by definition $w_{[n]}1 = 0$ if $n \geqslant 0$.

If $\mathcal{L}$ is a chiral A-algebroid, then we can regard it as a vertex Lie algebra and then define $U^{ch}(\mathcal{L})$. This is a vertex algebra, but it is too big to be a CDO s.t. $U^{ch}(\mathcal{L})^{\leqslant 1} \stackrel{\sim}{\longrightarrow} \mathcal{L}$. To see more clearly why, recall that the center of a vertex algebra

V is defined to be $Z(V) = \{v \in V \text{ s.t. } v_{(n)}V = \{0\} \text{ for all } n \geqslant 0\}$. Of course, the Borcherds commutator formula (33) implies

$$v \in Z(V) \Longrightarrow [v_{(n)}, w_{(m)}] = 0 \text{ for all } w \in V, n, m \in \mathbb{Z}.$$

Exercise 5.6 Verify that

(i) $Z(\mathcal{D}^{ch}_{A,\vec{x}}) = \mathbb{C} \cdot 1,\ 1 \in A$;
(ii) the center $U^{ch}(\mathcal{L})$ contains a polynomial ring in one variable; (Hint: consider $1 \in J_\infty A \subset \mathcal{L}$.)

The issue we are dealing with is the same as the one we dealt with in Sect. 3: the vertex enveloping algebra $U^{ch}(\mathcal{L})$ does not "know" about the the "multiplicative" structure that $\mathcal{L}$ carries; "multiplicative" in this context means "negative numbered multiplications." This leads to the existence of a canonical ideal as follows.

Use (44) to identify $\mathcal{L}$ with its image inside $U^{ch}(\mathcal{L})$ and consider the vector subspace $I \subset U^{ch}(\mathcal{L})$ defined as follows

$$I \stackrel{\text{def}}{=} \text{span of } \{1_A - 1_{U^{ch}(\mathcal{L})},\ a_{(-n)}\xi - a_{[-n]}\xi,\ a \in J_\infty A, \xi \in \mathcal{L}, a_{(-n)}\xi \in \mathcal{L})\} \subset U^{ch}(\mathcal{L}).$$

The fact that the brackets (41) and (43) coincide (and (45)) implies that

$$\mathcal{L}_{(n)} I \subset I. \tag{46}$$

Next, set $J = U^{ch}(\mathcal{L})_{(-1)} I$. A repeated application of (46) and the normal ordering axiom (34) gives

$$U^{ch}(\mathcal{L})_{(n)} J \subset J \text{ for all } n \in \mathbb{Z}. \tag{47}$$

A routine verification of these assertions is left to the reader as an exercise.

In other words, J is what is known as a *vertex ideal. Denote* by $U^{ch}_A(\mathcal{L})$ the vertex algebra quotient $U^{ch}(\mathcal{L})/J^{cent}$ and call it the *vertex enveloping algebra of a chiral algebroid.*

Lemma 5.2

(i) If $\mathcal{L}$ is a chiral A-algebroid, then $U^{ch}_A(\mathcal{L})$ is a CDO (over A.)
(ii) The functors

$$U^{ch}_A(.) : \ Chir - \mathcal{A}lg \overset{\longrightarrow}{\longleftarrow} CDO : \ \mathcal{F}$$

are adjoints and inverses of each other.

We shall leave this lemma as an exercise.

5.1 Classification: Objects

Let us classify chiral A-algebroids; the reader is advised to compare what follows with a more familiar material of Sect. 3. Since the situation we have in mind is that of a smooth algebraic variety, we shall always assume that Ω_A is a free A-module, and if need be, the existence of a coordinate system $\{x_i, \partial_i\}$. To make our results more explicit, we shall make one extra assumption as follows:

Recall that a vertex algebra V is called graded by conformal weight if

$$V = \bigoplus_{n\in\mathbb{Z}} V_n \text{ s.t. } (V_n)_{(i)}V_m \subset V_{n+m-i-1}, 1 \in V_0, T(V_n) \subset V_{n+1}.$$

(We shall often omit the descriptor "by conformal weight" if this is deemed unlikely to cause confusion.) The various examples of CDO we have seen, such as $\mathcal{D}^{ch}_{A,\vec{x}}$, are all graded: the degree of x_{ij}, ∂_{ij} is $-j$. Thus $(\mathcal{D}^{ch}_{A,\vec{x}})_0 = A$ and $(\mathcal{D}^{ch}_{A,\vec{x}})_1 = \Omega_A \oplus T_A$, the two components spanned by $x_{i,-1}$ and $\partial_{i,-1}$; more naturally, it fits into an exact sequence

$$0 \longrightarrow \Omega_A \longrightarrow (\mathcal{D}^{ch}_{A,\vec{x}})_1 \longrightarrow T_A \longrightarrow 0. \tag{48}$$

Notice that the reason Ω_A has popped up was explained in page 99: this is the A-submodule of $J_\infty A$ that is spanned by $\{x_{i,-1}\}$.

The reader may be pleased to realize that the 1st time we encountered this exact sequence was (24). The familiar relations $(\partial_{i,-1})_{(0)}x_{i0} = 1$, $(\partial_{i,-1})_{(1)}x_{i,-1} = 1$, etc., are then an illustration of relations $(V_n)_{(i)}V_m \subset V_{n+m-i-1}$ for various n, m, i.

A similar definition applies to a chiral algebroid, and we restrict our task to classifying *graded* chiral algebroids.

If A has a coordinate system, then at least one chiral A-algebroid, $\mathcal{D}^{ch,\leq 1}_{A,\vec{x}}$, exists. How many more are there? Let $\mathcal{L}$ be a chiral algebroid. A choice of a lift of an A-basis of T_A, $\{\partial_i\} \subset \mathcal{L}$, gives a splitting of $\mathcal{L}$: the map defined by

$$J_\infty T_A \longrightarrow \mathcal{L} \text{ s.t. } \sum_i a_{ij}T^j(\partial_i) \mapsto \sum_i (a_{ij})_{(-1)}T^j(\partial_i)$$

gives a splitting of (40) (Q: why?), hence an identification $\mathcal{L}^{ch} = J_\infty A \oplus J_\infty T_A$. With this identification, one observes that the vertex $J_\infty A$-module structure of $\mathcal{L}$ is determined uniquely: it is when restricted to $J_\infty A$ itself, by definition, and on $J_\infty T_A$ one has using (34)

$$a_{(-1)}(b_{(-1)}\partial) = (ab)_{(-1)}\partial - \left(\sum_{n=0}^{\infty}\left(a_{(-n-2)}b_{(n)} + b_{(-n-2)}a_{(n)}\right)\partial\right),$$

where the "correction terms" enclosed in brackets are predetermined: (iii) of Definition 5.2. (We omit extraneous indices and restrict ourselves to product $_{(-1)}$; the reader will see that this can be done without loss of generality.)

Therefore, we have reduced the problem to

$$\mathcal{L} = J_\infty A[\partial_{in}; 1\leqslant i\leqslant N, n < 0]^{\leqslant 1},$$

cf. (37), and observed that potentially only the vertex Lie algebra structure, $\{_{(n)}, n\geqslant 0\}$, can be deformed. To begin with, notice that all the products involving $J_\infty A$, that is,

$$_{(n)} : J_\infty A \otimes (J_\infty A \oplus J_\infty T_A) \longrightarrow (J_\infty A \oplus J_\infty T_A),$$

are predetermined by (iii) of Definition 5.2.

Note that none of this requires either grading or commutativity $[\partial_i, \partial_j] = 0$.

Focus on the subspace $T_A \subset J_\infty T_A$ and now use the grading assumption: when restricted to T_A, of all $\{_{(n)}, n\geqslant 0\}$ only two multiplications can be nonzero:

$$_{(1)} : T_A \otimes T_A \longrightarrow A$$

and

$$_{(0)} : T_A \otimes T_A \longrightarrow T_A \oplus \Omega_A.$$

As to the former, there is no room for maneuver at all: if we replace

$$\partial_i \text{ with } \partial_i - 1/2 \sum_j \left((\partial_i)_{(1)}\partial_j\right) \omega_j,$$

where $\{\omega_j\} \subset \Omega_A$ is the basis dual to $\{\partial_i\} \subset T_A$, then we obtain (check this!)

$$(\partial_i)_{(1)}\partial_j = 0 \text{ for all } i, j,$$

a relation we will assume throughout. If so, then an application of (34) will allow unambiguously to compute $(a_{(-1)}\partial_i)_{(1)}(b_{(-1)}\partial_j)$.

As to the latter, modulo Ω_A, the indicated product is nothing but the Lie bracket of vector fields, as follows from (i) of Definition 5.2. We are left, therefore, with the task of analyzing a map

$$T_A \otimes T_A \longrightarrow \Omega_A,$$

the composition of $_{(0)}$ with projection on Ω_A.

Let us put it this way: given $_{(0)} : T_A \otimes T_A \longrightarrow T_A \oplus \Omega_A$ define

$$_{(0)^{new}} : T_A \otimes T_A \longrightarrow T_A \oplus \Omega_A.$$

by the formula

$$\xi_{(0)^{new}}\eta = \xi_{(0)}\eta + \alpha(\xi,\eta),$$

where

$$\alpha : T_A \times T_A \longrightarrow \Omega_A$$

is a function of two variables. What kind of a function is it?

Exercise 5.7

(i) Use (34) to prove that

$$(a_{(-1)}\xi)_{(0)}\eta = a_{(-1)}(\xi_{(0)}\eta) \text{ modulo predetermined terms;}$$

(ii) Use skew-commutativity (29) to prove that

$$\xi_{(0)}\eta = -\eta_{(0)}\xi \text{ modulo predetermined terms;}$$

(iii) Derive that α is A-bilinear and anti-symmetric.

In fact, more is true. Dualizing, one obtains that α can be thought of as a function

$$\alpha :\ T_A \times T_A \times T_A \longrightarrow A,$$

which is anti-symmetric and A-linear in the first two arguments. Notice that as a function of three variables, ξ, η, ϵ, it equals $\epsilon_{(1)}(\xi_{(0)}\eta)$.

Exercise 5.8

(i) Use the Borcherds commutator formula (33) to verify that

$$\epsilon_{(1)}(\xi_{(0)}\eta) = \eta_{(1}(\epsilon_{(0)}\xi) \text{ modulo predetermined terms.}$$

(ii) Derive that α is totally antisymmetric and A-trilinear.

Therefore, $\alpha \in \Omega^3_A$.

Exercise 5.9

(i) Derive from the definition of a graded chiral algebroid that part of what we called "predetermined operations," namely

$$_{(0)} : T_A \otimes \Omega_A \longrightarrow \Omega_A \text{ and}$$

$$_{(1)} : T_A \otimes \Omega_A \longrightarrow A,$$

are the classic Lie derivative and contraction (resp.).

(ii) Use the $n = m = 0$ case of ((33) , which reads

$$\xi_{(0)}(\eta_{(0)}\epsilon) - \eta_{(0)}(\xi_{(0)}\epsilon) = (\xi_{(0)}\eta)_{(0)}\epsilon,$$

to prove that α must be closed: $d_{DR}\alpha = 0$. (Hint: this is parallel but more computationally involved than Exercise 3.2; repeatedly use (i) and the Borcherds commutator formula.)

Therefore, $\alpha \in \Omega_A^{3,cl}$. In fact, any closed 3-form α defines a chiral algebroid.

Lemma 5.3 *Let A carry a coordinate system $\{x_i, \partial_i\}$ and $\alpha \in \Omega_A^{3,cl}$. Then the assignment*

$$(\partial_i)_{(n)}\partial_j = \begin{cases} \alpha(\partial_i, \partial_j, .) & \textit{if } n = 0 \\ 0 & \textit{if } n > 0 \end{cases}$$

defines a conformally graded chiral algebroid structure on $J_\infty A[\partial_{in}; 1 \leqslant i \leqslant N, n < 0]^{\leqslant 1}$. Any conformally graded chiral algebroid is isomorphic to an algebroid of this kind.

Denote the constructed chiral algebroid $\mathcal{L}(\alpha)$.

Proof If $\alpha = 0$, then we have our original chiral algebroid $\mathcal{D}^{ch,\leqslant 1}_{A,\vec{x}}$. It is a vertex Lie algebra, and it has a vertex Lie subalgebra $J_\infty A \oplus (\oplus_{in}\mathbb{C}\partial_{i,n})$. It is an extension of an abelian vertex Lie algebra $\oplus_{in}\mathbb{C}\partial_{i,n}$ by an ideal $J_\infty A$. The point we are trying to make is that the prescription of the lemma allows us to deform this vertex Lie algebra structure. First of all, the prescription of the lemma defines a *truncated* such structure on the space $J_\infty A \oplus (\oplus_i \mathbb{C}\partial_{i,-1})$. What I mean by this is that we have defined all products on this space s.t. the Borcherds commutator formula (33) holds true, but the space has been truncated and so it no longer carries the derivation T. (Indeed, expressions such as $(\partial_i)_{(n)}a$, $a_{(n)}b$, $a, b \in J_\infty A$ are given to us with $a_{(n)}b = 0$, and (33) must be verified only for u, v, w being various ∂'s. In this case, the only nontrivial relation is

$$\xi_{(0)}(\eta_{(0)}\epsilon) - \eta_{(0)}(\xi_{(0)}\epsilon) = (\xi_{(0)}\eta)_{(0)}\epsilon,$$

and its validity is the content of Exercise 5.8.

This truncated vertex Lie algebra structure uniquely extends to the whole of $J_\infty A \oplus (\oplus_{in}\mathbb{C}\partial_{i,n})$: we define T by the old formula $T = \sum_{i,n} -n x_{in}\partial_{i,-n-1}$, and then use axiom (28) as motivation to define

$$(T^n\partial_i)_{(m)} = (-1)^m m(m-1)\cdots(m-n+1)(\partial_i)_{(m-n)}.$$

Clearly,

$$(T^n\partial_i)_{(m)} = \begin{cases} (-1)^n n!(\partial_i)_{(0)} & \text{if } n = m \\ 0 & \text{if } n \neq 0 \end{cases},$$

and it follows easily that with this definition the axioms of vertex Lie algebra hold true. (Think about the details!)

Denote the constructed vertex Lie algebra by $\mathcal{L}^{aux}(\alpha)$.

What remains to be done is to extend the structure obtained to the "multiplicatively closed" $J_\infty A[\partial_{in}; 1\leqslant i\leqslant N, n < 0]^{\leqslant 1}$. To do so, one either defines $a_{(-1)}\partial_{i,n}$ to be $a\partial_{i,n} \in J_\infty A[\partial_{in}; 1\leqslant i\leqslant N, n < 0]^1$, and then uses Definition 5.2(iv) to compute the operations, which is computationally laborious, or largely bypasses the computational hurdles by adjusting the discussion on the vertex enveloping algebra of a chiral algebroid to the present situation as follows.

First, consider the vertex enveloping algebra $U^{ch}(\mathcal{L}^{aux}(\alpha))$. Next, introduce the vector subspace

$$I = \text{ span of } \{1_A - 1_{U^{ch}(\mathcal{L}^{aux}(\alpha))} \text{ and } a_{[-1]}b - a_{(-1)}b,\ a, b \in J_\infty A\},$$

where $a_{(-1)}b$ is regarded as $ab \in J_\infty A$. We assert that

$$U^{ch}(\mathcal{L}^{aux}(\alpha))_{(n)} I \subset I \text{ if } n\geqslant 0.$$

Let us prove this focusing on elements of the type $a_{[-1]}b - a_{(-1)}b$. We will be repeatedly using the identification of $a \in \mathcal{L}^{aux}(\alpha)$ with $a_{[-1]}1_{U^{ch}(\mathcal{L}^{aux}(\alpha))}$. We have

$$\xi_{[n]}(a_{(-1)}b) = \xi_{(n)}(a_{(-1)}b) = (\xi_{(n)}a)_{(-1)}b + a_{(-1)}(\xi_{(n)}b),$$

thanks to (31). Similarly,

$$\xi_{[n]}(a_{[-1]}b) = ([\xi_{[n]}, a_{[-1]}])b + a_{[-1]}\xi_{[n]}b = \sum_{j=0}^{\infty}\binom{n}{j}(\xi_{(j)}a)_{[-1+n-j]}b + a_{(-1)}(\xi_{(n)}b) =$$

$$\sum_{j=0}^{n-1}\binom{n}{j}(\xi_{(j)}a)_{(-1+n-j)}b + (\xi_{(n)}a)_{[-1]}b + a_{(-1)}(\xi_{(n)}b) = (\xi_{(n)}a)_{[-1])}b + a_{(-1)}(\xi_{(n)}b),$$

the terms $(\xi_{(j)}a)_{(-1+n-j)}b$, $0\leqslant j\leqslant n-1$ vanishing, because $\xi_{(j)}a \in J_\infty A$ and $(J_\infty A)_{(i)}J_\infty A = \{0\}$ if $i\geqslant 0$. By definition

$$\xi_{[n]}(a_{(-1)}b) - \xi_{[n]}(a_{[-1]}b) \in I,$$

as desired. As in (47), $U^{ch}(\mathcal{L}^{aux}(\alpha))_{(-1)}I \subset U^{ch}(\mathcal{L}^{aux}(\alpha))$ is a vertex ideal. It is easy to understand (do this!) that the quotient $U^{ch}(\mathcal{L}^{aux}(\alpha))/U^{ch}(\mathcal{L}^{aux}(\alpha))_{(-1)}I$ is a CDO. The component $(U^{ch}(\mathcal{L}^{aux}(\alpha))/U^{ch}(\mathcal{L}^{aux}(\alpha))_{(-1)}I)^{\leqslant 1}$ is the desired chiral algebroid. □

5.2 Classification: Morphisms

Since by construction each $\mathcal{L}(\alpha)$ (i.e., the algebroid constructed in Lemma 5.3) comes equipped with a splitting, $\mathcal{L}(\alpha) = J_\infty A \oplus J_\infty T_A$, definition (42) implies that a *graded* morphism

$$f : \mathcal{L}(\alpha_1) \longrightarrow \mathcal{L}(\alpha_2)$$

is determined by a map

$$\beta : T_A \longrightarrow \Omega_A$$

s.t. $f(\xi) = \xi + \beta(\xi)$. As before, it is convenient to dualize and introduce

$$\beta : T_A \otimes T_A \longrightarrow A$$

s.t. $f(\xi) = \xi + \beta(\xi, .)$. The interested reader may be anticipating what is to follow.

Exercise 5.10 Verify that β must be A-linear and antisymmetric. (Remark: we have used normalization $(\partial_i)_{(1)}\partial_j = 0$; this is the reason why β must be antisymmetric.) Therefore, $\beta \in \Omega_A^2$.

Lemma 5.4

$$Hom(\mathcal{L}(\alpha_1, \alpha_2) = \{\beta \in \Omega_A^2 \ s.t. \ d_{DR}\beta = \alpha_1 - \alpha_2\},$$

where the morphism attached to β *is defined by*

$$T_A\xi \mapsto \xi + \beta(\xi, .).$$

Proof This is in the spirit of Sect. 3 and Exercise 5.9, but less laborious, and we shall go over some details. We need to compare $f(\xi_{(0)}\eta)$ and $f(\xi)_{(0)}f(\eta)$, $\xi, \eta \in T_A$. We have

$$f(\xi_{(0)}\eta) = \cdots \alpha_1(\xi, \eta, .) + \beta([\xi, \eta], .),$$

$$\begin{aligned} f(\xi)_{(0)}f(\eta) &= (\xi + \beta(\xi, .))_{(0)}(\eta + \beta(\eta, .)) = \cdots + \alpha_2(\xi, \eta, .) + \xi_{(0)}\beta(\eta, .) + \beta(\xi, .)_{(0)}\eta \\ &= \cdots + \alpha_2(\xi, \eta, .) + \xi_{(0)}\beta(\eta, .) - \eta_{(0)}\beta(\xi, .) + d_{DR}\beta(\xi, \eta), \end{aligned}$$

where $\cdots$ stands for the terms uniquely determined by the axioms. (Note that the last equality uses the skew-symmetry (29) as follows:

$$\beta(\xi, .)_{(0)}\eta = -\eta_{(0)}\beta(\xi, .) + T(\eta_{(1)}\beta(\xi, .)).)$$

To compare the two 1-forms we have to evaluate them on an arbitrary $\zeta \in T_A$ and subtract from one another; in vertex algebra terms to evaluate means to apply $\zeta_{(1)}$. We have

$$\zeta_{(1)}(f(\xi)_{(0)}f(\eta)) - \zeta_{(1)}f(\xi_{(0)}\eta) = \alpha_2(\xi,\eta,\zeta) - \alpha_1(\xi,\eta,\zeta) +$$
$$\zeta_{(1)}\xi_{(0)}\beta(\eta,.) - \zeta_{(1)}\eta_{(0)}\beta(\xi,.) + \zeta_{(1)}d_{DR}\beta(\xi,\eta) - \zeta_{(1)}\beta([\xi,\eta],.)$$

The R.H.S. must be zero. To evaluate the R.H.S., use the Borcherds commutator formula, e.g.,

$$\zeta_{(1)}\xi_{(0)} = \xi_{(0)}\zeta_{(1)} + (\zeta_{(0)}\xi)_{(1)} + (\zeta_{(1)}\xi)_{(0)},$$

and then the fact (Exercise 5.9(i)) that $\xi_{(0)}$ is the Lie derivative along ξ and $\xi_{(1)}$ is the contraction with ξ. We obtain

$$\alpha_1(\xi,\eta,\zeta) - \alpha_2(\xi,\eta,\zeta) =$$
$$\xi\beta(\eta,\zeta) - \eta\beta(\xi,\zeta) + \zeta\beta(\xi,\eta) - \beta([\xi,\eta],\zeta) + \beta([\xi,\zeta],\eta) - \beta([\eta,\zeta],\xi),$$

which by definition is the desired

$$\alpha_1(\xi,\eta,\zeta) - \alpha_2(\xi,\eta,\zeta) = d_{DR}\beta(\xi,\eta,\zeta).\square$$

5.3 Classification: Synthesis

All of this is delightfully analogous to Sect. 3—analogous in a nontrivial manner, as the dimension has gone up by 1. We leave it to the reader to push the analogy a little further and to define the category $\Omega_A^{[2,3>}$ with objects $\{\alpha \in \Omega_A^{3,cl}\}$ and morphisms $\mathrm{Hom}(\alpha_1,\alpha_2) = \{\beta \in \Omega_A^2 \text{ s.t. } \alpha_1 - \alpha_2 = d_{DR}\beta\}$; then verify that $\Omega_A^{[2,3>}$ is an abelian group in categories, and that the category of graded chiral A-algebroids, hence of CDOs over A, is an $\Omega_A^{[2,3>}$-torsor.

What this means geometrically is that on any smooth algebraic variety X there is a sheaf of groupoids *CDO* bound by the complex $\Omega_X^2 \to \Omega_X^{3,cl}$. In particular, given a CDO $\mathcal{D}_U^{ch}$ defined over an open $U \subset X$ and a closed 3-form $\alpha \in \Omega_X^{3,cl}(U)$ we have a naturally defined CDO $\mathcal{D}_U^{ch}(\alpha)$ over U, see the line that follows Lemma. 5.3.

We shift the cohomological degree so as to place Ω_X^2 in degree 0 and consider the (hyper)cohomology $H^\bullet(X, \Omega_X^2 \to \Omega_X^{3,cl})$. The cohomology groups have usual interpretations: if the category of globally defined CDOs is nonempty, then the set of isomorphism classes of such is $H^1(X, \Omega_X^2 \to \Omega_X^{3,cl})$; $H^0(X, \Omega_X^2 \to \Omega_X^{3,cl})$ is the group of automorphisms of any CDO if, again, one exists. All of this is not really different from Sect. 3, but here is the point: *CDO* is characterized by, well, its characteristic class $\chi(CDO)$, which is an element of $H^2(X, \Omega_X^2 \to \Omega_X^{3,cl})$. This class is an obstruction to the existence of a globally defined CDO: if $\chi(CDO) \neq 0$, then no such CDO exists. The class was computed in [17] to the effect that it equals the 2nd component of the Chern character:

$$\chi(CDO) = \frac{1}{2} ch_2(\mathcal{T}_X).$$

A more general result can be found in [7]. We shall leave this computation out, referring the reader to *loc. cit.*, but give a bit of an insight here.

First of all, the definition of the characteristic class. Cover X by affine subset $\{U_a\}$ along with a choice of coordinate system $\{x_i^a, \partial_i^a\}$. We have a description of the category of CDOs over each U_a, and we can make a choice of an object; say, $\mathcal{D}^{ch}_{U_a,\vec{x}^a}$, as in Sect. 4. Over an intersection $U_a \cap U_b$ we have two sheaves, $\mathcal{D}^{ch}_{U_a,\vec{x}^a}|_{U_a\cap U_b}$ and $\mathcal{D}^{ch}_{U_b,\vec{x}^b}|_{U_a\cap U_b}$ and we attempt to find an isomorphism

$$f_{ab} : \mathcal{D}^{ch}_{U_a,\vec{x}^a}|_{U_a\cap U_b} \longrightarrow \mathcal{D}^{ch}_{U_b,\vec{x}^b}|_{U_a\cap U_b}.$$

This isomorphism is not only a vertex algebra morphism, but it must satisfy the definition of the category of chiral algebroids, (42); namely, the induced maps

$$\mathcal{D}^{ch,\leqslant 0}_{U_a,\vec{x}^a}|_{U_a\cap U_b} \longrightarrow \mathcal{D}^{ch,\leqslant 0}_{U_b,\vec{x}^b}|_{U_a\cap U_b},$$

$$\mathcal{D}^{ch,\leqslant 1}_{U_a,\vec{x}^a}|_{U_a\cap U_b}/\mathcal{D}^{ch,\leqslant 0}_{U_a,\vec{x}^a}|_{U_a\cap U_b} \longrightarrow \mathcal{D}^{ch,\leqslant 1}_{U_b,\vec{x}^b}|_{U_a\cap U_b}/\mathcal{D}^{ch,\leqslant 0}_{U_b,\vec{x}^b}|_{U_a\cap U_b},$$

must be identities, as both of the former spaces are equal to $\mathcal{O}_X|_{U_a\cap U_b}$ and of the latter to $\mathcal{T}_X|_{U_a\cap U_b}$. The result of implementing this is not quite what we wanted, but an isomorphism

$$f_{ab} : \mathcal{D}^{ch}_{U_a,\vec{x}^a}|_{U_a\cap U_b} \longrightarrow \mathcal{D}^{ch}_{U_b,\vec{x}^b}|_{U_a\cap U_b}(\alpha_{ab}),$$

for some $\alpha_{ab} \in \Omega^{3,cl}_X(U_a \cap U_b)$. We shall say a few words on how these α's are computed below.

On a triple intersection, $U_a \cap U_b \cap U_c$, an appropriate composition of $f_\bullet$'s gives an isomorphism

$$\mathcal{D}^{ch}_{U_a,\vec{x}^a}|_{U_a\cap U_b\cap U_c} \longrightarrow \mathcal{D}^{ch}_{U_a,\vec{x}^a}|_{U_a\cap U_b\cap U_c}(\alpha_{bc} - \alpha_{ac} + \alpha_{ab}),$$

hence, see Lemma 5.4, a form $\beta_{abc} \in \Omega^2_X(U_a \cap U_b \cap U_c)$ s.t. $d_{DR}\beta_{abc} = \alpha_{bc} - \alpha_{ac} + \alpha_{ab}$.

We have thus obtained two sets $\{\alpha_{ab}\}$ and $\{\beta_{abc}\}$ s.t.

$$d_{DR}\{\beta_{abc}\} = d_{\check{C}}\{\alpha_{ab}\}.$$

It is not hard to see that $d_{\check{C}}\{\beta_{abc}\} = 0$—think about it(!) and then recall that $d_{DR}\{\alpha_{ab}\} = 0$. This means that the pair $(\{\alpha_{ab}\}, \{\beta_{abc}\})$ is a cocycle of an obviously defined Čech-De Rham bi-complex $C^\bullet(X, \Omega_X^2 \to \Omega_X^{3,cl})$. Computing we have made various choices; other choices will replace the cocycle with a cohomologous one, hence a well-defined element of $H^2(X, \Omega_X^2 \to \Omega_X^{3,cl})$.

To conclude, a few promised words on the computation of the α's. We have two copies of the same vertex algebra attached to various coordinate systems, $\mathcal{D}^{ch}_{U_a,\vec{x}^a}|_{U_a\cap U_b}$ and $\mathcal{D}^{ch}_{U_b,\vec{x}^b}|_{U_a\cap U_b}$. The desired map must send[3]

$$(\partial_i^a)_{(-1)}1 \mapsto (\frac{\partial x_s^b}{\partial x_i^a})_{(-1)}(\partial_s^b)_{(-1)}1 + \cdots ,$$

$\cdots$ meaning the terms we have control over. If we let $\cdots = 0$, shall we obtain a morphism? This situation is familiar from as early as Sect. 4: compute

$$\left((\frac{\partial x_s^b}{\partial x_i^a})_{(-1)}(\partial_s^b)_{(-1)}1\right)_{(1)}\left((\frac{\partial x_s^b}{\partial x_j^a})_{(-1)}(\partial_s^b)_{(-1)}1\right)$$

and if it is nonzero add to the morphism an appropriate 1-form, γ_i, to ensure that this product vanishes. Then compute

$$\left(\left((\frac{\partial x_s^b}{\partial x_i^a})_{(-1)}(\partial_s^b)_{(-1)} + \gamma_i\right)1\right)_{(0)}\left(\left((\frac{\partial x_s^b}{\partial x_j^a})_{(-1)}(\partial_s^b)_{(-1)} + \gamma_j\right)1\right).$$

The result will automatically be $\alpha_{ab}(\partial_i^a, \partial_j^b, .)$ for some α_{ab}, Sect. 5.1 , as desired. We hope this may serve as a useful guide to [17].

6 Further Examples

This section contains some material that I really did not have time for in class. It is included for the sake of completeness and regarded as a review, except perhaps for Sect. 6.1.1, which I hope is a useful illustration of the ideas that appeared above.

6.1 Homogeneous Spaces

6.1.1 CDO on *G*

This is based on [15]. Let $\mathfrak{g}$ be a finite dimensional, simple Lie algebra, G the corresponding algebraic group. G operates on itself by left multiplication, which gives a Lie algebra morphism

$$j_l : \mathfrak{g} \longrightarrow \Gamma(G, \mathcal{T}_G).$$

[3]In the formulas to follow, the summation w.r.t. the repeated indices is assumed.

The tangent bundle TG is trivial, and so $ch_2(\mathcal{T}_G) = 0$ and the category of CDOs over G is nonempty. Next, G being affine,

$$H^1(G, \Omega^2_G \to \Omega^{3,cl}_G) = \Gamma(G, \Omega^{3,cl})/d_{DR}\Gamma(G, \Omega^2) \xrightarrow{\sim} \mathbb{C},$$

with basis a G-invariant form ω s.t. $\omega(x, y, z) = (x, [y, z])$, where $x, y, z \in j_l(\mathfrak{g})$, $(., .)$ a choice of an invariant inner product on $\mathfrak{g}$, $[., .]$ the Lie bracket on $\mathfrak{g}$.

According to our classification, Sect. 5.3, the isomorphism classes of CDOs over G form a $\mathbb{C}$-torsor. In fact, it is easy and instructive to construct a universal 1-dimensional family of these CDOs.

Lemma 5.3 asserts that such quantization is possible if T_A has an abelian basis. Some further thought shows that the abelian condition is inessential and can be weakened. To begin with, assume given a Lie algebra morphism

$$\mathfrak{a} \longrightarrow T_A.$$

This lets $\mathfrak{a}$ act on A by derivations and we obtain $A \rtimes \mathfrak{a}$, an extension of $\mathfrak{a}$ by an abelian ideal A

This can be chiralized: let $J_\infty\mathfrak{a} = \mathbb{C}[T]\mathfrak{a}$, which is a vertex Lie algebra, see page 100, for the definition, and then $J_\infty A$ is a $J_\infty\mathfrak{a}$-module. We thus obtain a vertex Lie algebra $J_\infty A \rtimes J_\infty\mathfrak{a}$, an extension of $J_\infty\mathfrak{a}$ by $J_\infty A$. This gives us a vertex Poisson algebra $J_\infty A \otimes S^\bullet J_\infty\mathfrak{a}$ (the vertex Poisson algebra $S^\bullet J_\infty\mathfrak{a}$ first appeared in page 101) along with a vertex Poisson algebra morphism:

$$J_\infty A \otimes S^\bullet J_\infty\mathfrak{a} \longrightarrow S^\bullet J_\infty T_A.$$

The domain of this map can be quantized, and this is the point. Namely, consider $U^{ch}(J_\infty A \rtimes J_\infty\mathfrak{a})$, and then quotient out by the vertex ideal generated by the elements $1_A - 1_{U^{ch}}, a_{[-1]}b - a_{(-1)}b, a, b \in J_\infty A$—exactly as in the proof of Lemma 5.3. Denote the vertex algebra thus obtained by $\mathcal{D}^{ch}_{A,\mathfrak{a}}$.

$\mathcal{D}^{ch}_{A,\mathfrak{a}}$ is filtered, and its graded object is exactly $J_\infty A \otimes S^\bullet J_\infty\mathfrak{a}$.

Now assume that the map $A \otimes \mathfrak{a} \to T_A$, which is induced by the above $\mathfrak{a} \longrightarrow T_A$, is an isomorphism. It follows that the map $J_\infty A \otimes S^\bullet J_\infty\mathfrak{a} \longrightarrow S^\bullet J_\infty T_A$ is also an isomorphism, hence $\mathcal{D}^{ch}_{A,\mathfrak{a}}$ is a quantization of $S^\bullet J_\infty T_A$.

This construction can be deformed. Namely, consider a central extension of vertex Lie algebras,

$$0 \longrightarrow \mathbb{C} \cdot 1_L \longrightarrow L \longrightarrow J_\infty\mathfrak{a} \longrightarrow 0.$$

This also gives us an extension $J_\infty A \rtimes L$ and a vertex Poisson algebra morphism

$$J_\infty A \otimes S^\bullet L \longrightarrow S^\bullet J_\infty T_A.$$

Defining as above $\mathcal{D}^{ch}_{A,L}$ to be $U^{ch}(J_\infty A \otimes S^\bullet L)$ modulo the ideal generated by $1_A - 1_{U^{ch}}$, $a_{[-1]}b - a_{(-1)}b$ AND $1_A - 1_L$ gives us a quantization of $S^\bullet J_\infty T_A$—again if $A \otimes \mathfrak{a} \to T_A$ is an isomorphism.

This is exactly the set-up of the group G, where $\mathfrak{a}$ is replaced with $\mathfrak{g}$, $\mathfrak{a} \to T_A$ with $j_l : \mathfrak{g} \to \Gamma(G, \mathcal{T}_G)$ and L with the central extension $\widehat{J_\infty \mathfrak{g}}_{(.,.)}$. The result is a family of CDOs, $\mathcal{D}^{ch}_{G,(.,.)}$, along with the tautological embedding

$$j_l^{ch} : V(\mathfrak{g})_{(.,.)} \longrightarrow \mathcal{D}^{ch}_{G,(.,.)};$$

the vertex algebra $V(\mathfrak{g})_{(.,.)}$ was defined in Sect. 5.

One remarkable fact about $\mathcal{D}^{ch}_{G,(.,.)}$ is that the action by right translations (discovered in [2])

$$j_r : \mathfrak{g} \longrightarrow \Gamma(G, \mathcal{T}_G)$$

also chiralizes: there is a diagram of vertex algebra embeddings

$$j_l^{ch} : V(\mathfrak{g})_{(.,.)} \longrightarrow \mathcal{D}^{ch}_{G,(.,.)} \longleftarrow V(\mathfrak{g})_{(.,.)^\vee} \text{ s.t. } (j_l^{ch} V(\mathfrak{g})_{(.,.)})_{(n)}(j_r^{ch} V(\mathfrak{g})_{(.,.)^\vee}) \text{ if } n \geqslant 0.$$

Note that the "right" action requires a change of level from $(.,.)$ to the dual $(.,.)^\vee$.

The CDO $\mathcal{D}^{ch}_{G,k}$ has found nontrivial applications to geometry and representation theory, [2, 13].

6.1.2 Flag Manifolds and Base Affine Spaces

The CDO on $\mathbb{CP}^1$ constructed by hand in Sect. 4 owes its existence to the fact that $\dim \mathbb{CP}^1 = 1$ and so the obstruction, Sect. 5.3, vanishes for the trivial dimensional reason. Likewise, $H^1(\mathbb{CP}^1, \Omega^2_v \to \Omega^{3,cl}_{\mathbb{CP}^1}) = 0$, and so that sheaf is unique up to isomorphism.

The dimensional argument does not work for $\mathbb{CP}^2$, and indeed there are no CDOs on $\mathbb{CP}^n$ for any $n > 1$. The appropriate generalization of $\mathbb{CP}^1$ for a simple G is the flag manifold, where there is a unique up to isomorphism CDO $\mathcal{D}^{ch}_{G/B}$, [15]. A simple way to construct it is to use the ideas of Hamiltonian reduction.

Consider a chain $N \subset B \subset G$, where N is the maximal unipotent and B the Borel subgroups.

For any smooth X, T^*X is simplectic.

The action of G (by left translations) on T^*G is simplectic; the same applies to any subgroup of G.

The simplectic manifolds $T^*(G/N)$ and $T^*(G/B)$ are, in fact, Hamiltonian reductions of T^*G w.r.t. N (B resp.)

This has a chiral analogue due to the pioneering work of Feigin [10]. Consider $\mathcal{D}^{ch}_{G,(.,.)}$, Sect. 6.1.1. It carries two actions of the affine Lie algebra $\widehat{\mathfrak{g}}$. Pick the one coming from j_r^{ch} and pull it back w.r.t. $\mathfrak{n}((t)) \hookrightarrow \widehat{\mathfrak{g}}$. A sheaf version of Feigin's semi-infinite cohomology [10][4] will give a sheaf $\mathcal{H}^{0+\infty}(\mathfrak{n}((t)), \mathcal{D}^{ch}_{G,(.,.)})$ on G and then the push-forward $\pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{n}((t)), \mathcal{D}^{ch}_{G,(.,.)})$, where $\pi : G \to G/N$. It is not hard to see that $\{\pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{n}((t)), \mathcal{D}^{ch}_{G,(.,.)}), k \in \mathbb{C}\}$ is a family of CDOs on the *base affine space* G/N, [15]. The passage to the cohomology destroys j_r^{ch}, but j_l^{ch} survives and we obtain a $\widehat{\mathfrak{g}}$-structure

$$V(\mathfrak{g})_{(.,.)} \longrightarrow \pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{n}((t)), \mathcal{D}^{ch}_{G,(.,.)}).$$

The flag manifold G/B is dealt with similarly, except that in this case the relevant subgroup B is not contractible, $\mathfrak{b}((t))$ undergoes a central extension, and the cohomology $\pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{b}((t)), \mathcal{D}^{ch}_{G,(.,.)})$ does not quite make sense. To straighten things out, one has to replace the absolute semi-infinite cohomology with the relative, and specialize the central charge to the critical one, [15]. The result is a unique CDO on G/B, $\pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{b}((t)), \mathfrak{b}; \mathcal{D}^{ch}_{G,(.,.)^{crit}})$, along with a morphism

$$V(\mathfrak{g})_{(.,.)^{crit}} \longrightarrow \pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{b}((t)), \mathfrak{b}; \mathcal{D}^{ch}_{G,(.,.)^{crit}}).$$

This is a proper generalization of Sect. 4. The generalization of Theorem 4.1 is the following result proved in [1]:

$$H^i(G/B, \pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{b}((t)), \mathfrak{b}; \mathcal{D}^{ch}_{G,(.,.)^{crit}})) = \bigoplus_{w\in W^{(i)}} L_{0,(.,.)^{crit}},$$

where $W^{(i)}$ is the set of length i elements of the Weyl group W.

Note that if we let $U \subset G/B$ be the big cell, then $\pi_\bullet\mathcal{H}^{0+\infty}(\mathfrak{b}((t)), \mathfrak{b}; \mathcal{D}^{ch}_{G,(.,.)^{crit}})(U)$ is, by definition, the Wakimoto module, [12]. In fact, the approach sketched here is a way to introduce the Wakimoto module independent of [12].

6.2 Chiral De Rham and String Theory

The story told above has a straightforward super-analogue: one should simply deal with a sheaf of vertex superalgebras over a supervariety. Unfortunately, the only case that was treated in some detail is that of the supervariety $\Pi\mathcal{M}X$, where $\mathcal{M} \to X$ is a vector bundle. What this really means is that the structure sheaf in question is the supercommutative algebra $\Lambda^\bullet_{\mathcal{O}_X}\mathcal{M}^*$. In this case, [16], the characteristic class of the arising groupoid of categories over X is $ch_2(\mathcal{T}_X) - ch_2(\mathcal{M})$. In particular, if $\mathcal{M} = \mathcal{T}_X$, then the characteristic classes vanishes and there arises a category of superCDOs over any smooth X. What is striking, however, is that amongst those various CDOs

[4]The reader will find more information on chiral Hamiltonian reduction in lectures by T. Arakawa in this volume.

over various X there is a universal one. More precisely, for each smooth(purely even) X, there is a doubly graded sheaf of vertex superalgebras over X, to be denoted $\Omega^{ch,\bullet}_{X,\bullet}$. It is universal in that for each étale $f : X \to Y$, there is a vertex superalgebra morphism $f^{-1}\Omega^{ch,\bullet}_{X,\bullet} \to \Omega^{ch,\bullet}_{Y,\bullet}$ that satisfies an easy to work out cocycle condition. This sheaf satisfies various other favorable properties; e.g. its degree 0 component is the usual De Rham $\Omega^{\bullet}_{X}$; it carries a differential—so to say, a chiral De Rham differential—s.t. the embedding $\Omega^{\bullet}_{X} \to \Omega^{ch,\bullet}_{X,\bullet}$ is a quasiisomorphism. This sheaf was constructed by hands in [25] and called the *chiral De Rham complex*; this is where the CDO story began.

L. Borisov wrote a series of papers, starting with the strikingly original [4], linking and applying $\Omega^{ch,\bullet}_{X,\bullet}$ to mirror symmetry on toric varieties. A relation of $\Omega^{ch,\bullet}_{X,\bullet}$ to the concept of elliptic genus is discussed in [8]. Incidentally, an analogue of this concept to the purely even $\mathcal{D}^{ch}_{X}$, the Euler character $\mathrm{Eu}(\mathcal{D}^{ch}_{X})$, which appeared in Sect. 4 in an example, is essentially the Witten genus [9].

We will conclude by mentioning that this theory has been analyzed from various physics viewpoints by A. Kapustin, N. Nekrasov, and E. Witten in [22, 26, 28].

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Representations of Lie Superalgebras

Vera Serganova

Abstract Abstract In these notes we give an introduction to representation theory of simple finite-dimensional Lie superalgebras. We concentrate on so called basic superalgebras. Those are superalgebras which have even reductive part and admit an invariant form. We start with structure theory of basic superalgebras emphasizing abstract properties of roots and then proceed to representations, trying to demonstrate the variety of methods: Harish-Chandra homomorphism, support variety, translation functors, Borel-Weil-Bott theory and localization.

Keywords Atypicality • Blocks • BorelŰWeilŰBott theorem • Harish-Chandra homomorphism • Lie superalgebras • Supermanifold • Translation functors

1 Introduction

In these notes we give an introduction to representation theory of simple finite-dimensional Lie superalgebras. We concentrate on so called basic superalgebras. Those are superalgebras which have even reductive part and admit an invariant form. Representation theory of these superalgebras was initiated in 1978 by V. Kac, see [23]. It turned out that finite-dimensional representations of basic superalgebras are not easy to describe completely and questions which arise in this theory are analogous to similar questions in positive characteristic.

We start with structure theory of basic superalgebras emphasizing abstract properties of roots and then proceed to representations, trying to demonstrate the variety of methods: Harish-Chandra homomorphism, support variety, translation functors, Borel-Weil-Bott theory and localization.

We assume from the reader the thorough knowledge of representation theory of reductive Lie algebras (in characteristic zero) and rudimentary knowledge of algebraic geometry.

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Let me mention several monographs related to the topic of these lectures: [32] and [4] on Lie superalgebras and [3] on supermanifolds. The reader can find some details in these books.

2 Preliminaries

2.1 Superalgebras in General

In supermathematics we study $\mathbf{Z}_2$-graded objects. The word super means simply "$\mathbf{Z}_2$-graded", whenever it is used (superalgebra, superspace etc.).

We denote by k the ground field and assume that $char(k) \neq 2$.

Definition 1 An *associative superalgebra* is a $\mathbf{Z}_2$ graded algebra $A = A_0 \oplus A_1$. If $a \in A_i$ is a homogeneous element, then $\bar{a}$ will denote the parity of a, that is $\bar{a} = 0$ if $a \in A_0$ or $\bar{a} = 1$ if $a \in A_1$.

All modules over an associative superalgebra A are also supposed to be $\mathbf{Z}_2$-graded. Thus, an A-module M has a grading $M = M_0 \oplus M_1$ such that $A_i M_j \subset M_{i+j}$.

In particular, a vector superspace is a $\mathbf{Z}_2$-graded vector space. The associative algebra $\mathrm{End}_k(V)$ of all k-linear transformation of a vector superspace V has a natural structure of a superalgebra with the $\mathbf{Z}_2$-grading given by:

$$\mathrm{End}_k(V)_0 = \{\phi \mid \phi(V_i) \subset V_i\}, \quad \mathrm{End}_k(V)_1 = \{\phi \mid \phi(V_i) \subset V_{i+1}\},$$

If $e_1, \dots, e_m$ is a basis of V_0 and $e_{m+1}, \dots, e_{m+n}$ is a basis of V_1, then we can identify $\mathrm{End}_k(V)$ with block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and

$$\mathrm{End}_k(V)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad \mathrm{End}_k(V)_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

All formulas are written for homogeneous elements only and then extended to all objects by linearity. Every term has a sign coefficient, which is determined by following the *sign rule*:

If one term is obtained from another by swapping adjacent symbols x and y we put the coefficient $(-1)^{\bar{x}\bar{y}}$.

Example 1 Consider the commutator $[x, y]$. In the classical world it is defined by $[x, y] = xy - yx$. In superworld we write instead:

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx.$$

The sign rule has its roots in the tensor category theory. More precisely, the category *SVect* of supervector spaces is an abelian rigid symmetric tensor category with brading $s : V \otimes W \to W \otimes V$ given by the sign rule

$$s(v \otimes w) = (-1)^{\bar{v}\bar{w}} w \otimes v.$$

All objects, which can be defined in context of tensor category: affine schemes, algebraic groups etc. can be generalized to superschemes, supergroups etc. if we work in the category *SVect* instead of the category *Vect* of vector spaces. We refer the reader to [9] for details in this approach. We will follow the sign rule naively and see that it always gives the correct answer.

Definition 2 We say that a superalgebra A is *supercommutative* if

$$xy = (-1)^{\bar{x}\bar{y}} yx$$

for all homogeneous $x, y \in A$.

Exercise Show that a free supercommutative algebra with odd generators $\xi_1, \dots, \xi_n$ is the exterior (Grassmann) algebra $\Lambda(\xi_1, \dots \xi_n)$.

All the morphisms between superalgebras, modules etc. have to preserve parity. In this way if A is a superalgebra then the category of A-modules is an abelian category. This category is equipped with the *parity change functor* Π. If $M = M_0 \oplus M_1$ is an A-module we set $\Pi M := M$ with new grading $(\Pi M)_0 = M_1, (\Pi M)_1 = M_0$ and the obviuos A-action. It is clear that Π is an autoequivalence of the abelian category of A-modules.

Exercise Let V be a finite dimensional vector superspace and V^* be the dual vector space with $\mathbf{Z}_2$-grading defined in the obvious way. Consider a linear operator $X : V \longrightarrow V$. We would like to define the adjoint operator $X^* : V^* \longrightarrow V^*$ following the sign rule. For $\phi \in V^*$ and $v \in V$ we set

$$< X^*\phi, v >=< \phi, (-1)^{\bar{X}\bar{\phi}} Xv >,$$

where $< \cdot, \cdot >: V^* \otimes V \to k$ is the natural pairing. Let $\{e_i\}$, $i = 1, \dots, m+n$ be a homogeneous basis of V as above and $\{f_i\}$ be the dual basis of V^* in the sense that $< f_j, e_i >= \delta_{i,j}$. Show that if the matrix of X in the basis $\{e_i\}$ is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then the matrix of X^* in the basis $\{f_i\}$ equals $X^{st} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}$. The operation $X \mapsto X^{st}$ is called the *supertransposition* and it satisfies the identity

$$(XY)^{st} = (-1)^{\bar{X}\bar{Y}} Y^{st} X^{st}.$$

Our next example is the *supertrace*. To define it we use the canonical identification $V \otimes V^* \cong \mathrm{End}_k(V)$ given by

$$v \otimes \phi(w) =< \phi, w > v \text{ for all } v, w \in V, \phi \in V^*.$$

The supertrace str : $\mathrm{End}_k(V) \to k$ is the composition

$$\mathrm{str} : V \otimes V^* \xrightarrow{s} V^* \otimes V \xrightarrow{<\cdot,\cdot>} k.$$

Exercise Prove that if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then

(a) $\mathrm{str}(X) = \mathrm{tr}(A) - \mathrm{tr} D$,
(b) $\mathrm{str}([X, Y]) = 0$.

The *superdimension* sdim V of a superspace V is by definition the supertrace of the identity operator in V. It follows from the above exercise that $\mathrm{sdim}\, V = \dim V_0 - \dim V_1$. It is important sometimes to remember both even and odd dimension of V. So we set $\dim V = (\dim V_0 | \dim V_1) = (m|n)$ be an element $m + n\varepsilon$ in the ring $\mathbf{Z}(\varepsilon)/(\varepsilon^2 - 1)$.

Exercise Show that

(a) $\mathrm{sdim}(V \oplus W) = \mathrm{sdim}\, V + \mathrm{sdim}\, W$ and $\dim(V \oplus W) = \dim V + \dim W$,
(b) $\mathrm{sdim}(V \otimes W) = \mathrm{sdim}\, V\, \mathrm{sdim}\, W$ and $\dim(V \otimes W) = \dim V \dim W$,
(c) $\mathrm{sdim}(\Pi V) = -\,\mathrm{sdim}\, V$ and $\dim(\Pi V) = \varepsilon \dim V$.

2.2 Lie Superalgebras

Definition 3 A *Lie superalgebra* $\mathfrak{g}$ is a vector superspace with a bilinear even map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that:

1. $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$,
2. $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$.

Example 2 If A is an associative superalgebra, one can make it into a Lie superalgebra Lie(A) by defining the bracket:

$$[a, b] = ab - (-1)^{\bar{a}\bar{b}} ba.$$

For example if $A = \mathrm{End}(V)$, $\dim(V) = (m|n)$, then *Lie*(A) is the Lie superalgebra which we denote by $\mathfrak{gl}(m|n)$.

Definition 4 If A is an associative superalgebra, $d : A \longrightarrow A$ is a derivation of A if:

$$d(ab) = d(a)b + (-1)^{\bar{d}\bar{a}} a d(b).$$

Exercise

(a) Check that the space $Der(A)$ of all derivations of A with bracket given by the supercommutator is a Lie superalgebra.
(b) Consider $A = \Lambda(\xi_1, \dots \xi_n)$. Then $Der(A)$ is a finite-dimensional superalgebra denoted by $W(0|n)$. Show that its dimension is $(2^{n-1}n|2^{n-1}n)$.

Exercise Show that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with bracket $[\cdot, \cdot]$ is a Lie superalgebra if and only if

1. $\mathfrak{g}_0$ is a Lie algebra;
2. $[\cdot, \cdot] : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$ equips $\mathfrak{g}_1$ with the structure of a $\mathfrak{g}_0$-module;
3. $[,] : S^2\mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is a homomorphism of $\mathfrak{g}_0$-modules;
4. for all $x \in \mathfrak{g}_1$, $[x, [x, x]] = 0$.

Example 3 Let us introduce the "smallest" simple Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2)$ of dimension $(3|2)$. Take $\mathfrak{g}_0 = \mathfrak{sl}(2)$ and $\mathfrak{g}_1 = V$, where V is the two dimensional irreducible representation of $\mathfrak{sl}(2)$. The isomorphims $S^2V \simeq \mathfrak{sl}(2)$ of $\mathfrak{sl}(2)$-modules defines the bracket $S^2\mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$. One can easily check that $[x, [x, x]] = 0$ for all $x \in \mathfrak{g}_1$ and hence by the previous exercise these data define a Lie superalgebra structure.

Example 4 (Bernstein) Consider a symplectic manifold M, with symplectic form $\omega \in \Omega^2 M$. Consider the following operators acting on the de Rham complex $\Omega(M)$:

- $\omega : \Omega^i(M) \longrightarrow \Omega^{i+2}(M)$, given by $\wedge\omega$,
- $i_\omega : \Omega^i(M) \longrightarrow \Omega^{i-2}(M)$, given by contraction with bivector ω^*,
- grading operator $h : \Omega^i(M) \longrightarrow \Omega^i(M)$.

It is a well known fact that ω, h, i_ω form an sl(2)-triple.

Assume now that $\mathcal{L}$ is a line bundle on M with a connection ∇. Assume further that the curvature of ∇ equals $t\omega$ for some non-zero t. Recall that ∇ is an operator of degree 1 on the sheaf $\mathcal{L} \otimes \Omega(M)$ of differential forms with coefficients in $\mathcal{L}$

$$\nabla : \mathcal{L} \otimes \Omega^i \longrightarrow \mathcal{L} \otimes \Omega^{i+1}.$$

On the other hand, ω, h, i_ω act on $\mathcal{L} \otimes \Omega$ in the same manner as before. Set $\nabla^* := [\nabla, i_\omega]$. One can check that ∇, ∇^*, together with ω, h, i_ω span the superalgebra isomorphic to $\mathfrak{osp}(1|2)$.

The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is the associative superalgebra which satisfies the natural universality property in the category of superalgebras. It can be defined as the quotient of the tensor superalgebra $T(\mathfrak{g})$ by the ideal generated by $XY - (-1)^{\bar{X}\bar{Y}}YX - [X, Y]$ for all homogeneous $X, Y \in \mathfrak{g}$. The PBW theorem holds in the supercase, i.e. $Gr\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$. However, $S(\mathfrak{g})$ is a free commutative superalgebra. From the point of view of the usual tensor algebra we have an isomorphism $S(\mathfrak{g}) \simeq S(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$.

3 Basic Lie Superalgebras

3.1 Simple Lie Superalgebras

A Lie superalgebra is *simple* if it does not have proper non-trivial ideals (ideals are of course $\mathbf{Z}_2$-graded).

Exercise Prove that if a Lie superalgebra $\mathfrak{g}$ is simple, then $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$ and $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$.

In 1977 Kac classified simple Lie superalgebras over an algebraically closed field k of characteristic zero, [22]. He divided simple Lie superalgebras into three groups:

- *basic*: classical and exceptional,
- *strange*: $P(n)$, $Q(n)$,
- *Cartan type*: $W(0|n) = Der\Lambda(\xi_1, \dots \xi_n)$ and some subalgebras of it.

Basic and strange Lie superalgebras have a reductive even part. Cartan type superalgebras have a non-reductive $\mathfrak{g}_0$.

Definition 5 A simple Lie superalgebra $\mathfrak{g}$ is *basic* if $\mathfrak{g}_0$ is reductive and if $\mathfrak{g}$ admits a non-zero invariant even symmetric form $(\cdot,\cdot)$, i. e. the form satisfying the condition

$$([x,y],z) + (-1)^{\bar{x}\bar{y}}(y,[x,z]) = 0, \quad \text{for all} \quad x,y,z \in \mathfrak{g},$$

or, equivalently,

$$([x,y],z) = (x,[y,z]).$$

and $(x,y) \neq 0$ implies $\bar{x} = \bar{y}$.

Exercise 1 Let V be a finite-dimensional $\mathfrak{g}$-module. Then the form

$$(x,y) := \mathrm{str}_V(yx)$$

is an invariant even symmetric form.

In this section we describe the basic Lie superalgebras. We start with classical Lie superalgebras. The invariant symmetric form is given by the supertrace in the natural module V.

Special linear Lie Superalgebra $\mathfrak{sl}(m|n)$ is the subalgebra of $\mathfrak{gl}(m|n)$ of matrices with supertrace zero. It is not hard to verify that $\mathfrak{sl}(m|n)$ is simple if $m \neq n$ and $m + n \geqslant 2$. What happens when $m = n$? In this case the supertrace of the identity matrix is zero and therefore $\mathfrak{sl}(n|n)$ has a one-dimensional center $\mathfrak{z}$ consisting of all scalar matrices. We define $\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\mathfrak{z}$.

Exercise Check that $\mathfrak{psl}(n|n)$ is simple if $n \geqslant 2$.

Look at the case $n = 1$. Then $\mathfrak{sl}(1|1) =< x, y, z >$, where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the commutators are:

$$[x, z] = [y, z] = 0, \quad [x, y] = z,$$

and we see that $\mathfrak{sl}(1|1)$ is a nilpotent $(1|2)$-dimensional Lie superalgebra, which is the superanalogue of the Heisenberg algebra. Furthermore, $\mathfrak{psl}(1|1)$ is an abelian $(0|2)$-dimensional superalgebra.

We have $\mathfrak{sl}(m|n)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid \mathrm{tr}\,(A) = \mathrm{tr}\,(D) \right\}$. Hence

$$\mathfrak{sl}(m|n)_0 \cong \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus k.$$

Note also that $\mathfrak{g} = \mathfrak{sl}(m|n)$ has a compatible $\mathbf{Z}$-grading[1]:

$$\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$$

with $\mathfrak{g}_0 = \mathfrak{g}(0)$ and

$$\mathfrak{g}(1) = V_0 \otimes V_1^*, \qquad \mathfrak{g}(-1) = V_0^* \otimes V_1.$$

The Orthosymplectic Lie Superalgebra $\mathfrak{osp}(m|n)$ is also a subalgebra of $\mathfrak{gl}(m|n)$. Let V be a vector superspace of dimension $(m|n)$ equipped with an even non-degenerate bilinear symmetric form $(\cdot,\cdot)$, i.e. for all homogeneous $v, w \in V$ we have

$$(v, w) = (-1)^{\bar{v}\bar{w}}(w, v), \quad (v, w) \neq 0 \implies \bar{v} = \bar{w}.$$

Note that $(\cdot,\cdot)$ is symmetric on V_0 and symplectic on V_1. Hence the dimension of V_1 must be even, $n = 2l$. We define:

$$\mathfrak{osp}(m|n) := \{X \in \mathfrak{gl}(m|n) \mid (Xv, w) + (-1)^{\bar{X}\bar{v}}(v, Xw) = 0\}.$$

It is easy to see that $\mathfrak{g}_0 = \mathfrak{so}(m) \oplus \mathfrak{sp}(2l)$. So the two classical series, orthogonal and symplectic, come together in the superalgebra theory. One can see also that $\mathfrak{g}_1$ is isomorphic to $V_0 \otimes V_1$ as a $\mathfrak{g}_0$-module. Furthermore it is easy to check that $\mathfrak{osp}(m|2l)$ is simple for all $m, l > 0$.

[1] A grading $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is compatible if $\mathfrak{g}(2j) \subset \mathfrak{g}_0$ and $\mathfrak{g}(2j+1) \subset \mathfrak{g}_1$.

Lemma 1 *Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra over an algebraically closed field k. Then the center of $\mathfrak{g}_0$ is at most one dimensional.*

Proof Assume the opposite. Let z_1, z_2 be two linearly independent elements in the center of $\mathfrak{g}_0$. For all $a, b \in k$ set

$$\mathfrak{g}(a,b) = \{x \in \mathfrak{g}_1 \mid (\mathrm{ad}_{z_1} - a)^{\dim \mathfrak{g}_1} x = 0,\ (\mathrm{ad}_{z_2} - b)^{\dim \mathfrak{g}_1} x = 0\}.$$

Then we have

1. $\mathfrak{g}_1 = \bigoplus \mathfrak{g}(a,b)$;
2. $[\mathfrak{g}_0, \mathfrak{g}(a,b)] \subset \mathfrak{g}(a,b)$;
3. $[\mathfrak{g}(a,b), \mathfrak{g}(c,d)] \neq 0$ implies $a = -c, b = -d$.

These conditions imply that $[\mathfrak{g}(a,b), \mathfrak{g}(-a,-b)] + \mathfrak{g}(a,b) + \mathfrak{g}(-a,-b)$ is an ideal in $\mathfrak{g}$. Therefore by simplicity of $\mathfrak{g}$ we obtain that for some $a, b \in k$, $\mathfrak{g} = [\mathfrak{g}(a,b), \mathfrak{g}(-a,-b)] + \mathfrak{g}(a,b) + \mathfrak{g}(-a,-b)$. Set $z = bz_1 - az_2$ if $a \neq 0$ and $z = z_1$ if $a = 0$. Then ad_z acts nilpotently on $\mathfrak{g}_1$. But $\mathfrak{g}_0 \oplus [z, \mathfrak{g}_1]$ is an ideal in $\mathfrak{g}$. Hence $z = 0$ and we obtain a contradiction.

Lemma 2 *Let $\mathfrak{g}$ be a basic Lie superalgebra and $\mathfrak{g}_1 \neq 0$. Then one of the following holds.*

1. *There is a $\mathbf{Z}$-grading $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$, such that $\mathfrak{g}(0) = \mathfrak{g}_0$ and $\mathfrak{g}(\pm 1)$ are irreducible $\mathfrak{g}_0$-modules.*
2. *The even part $\mathfrak{g}_0$ is semisimple and $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module.*

Proof Consider the restriction of the invariant form $(\cdot,\cdot)$ on $\mathfrak{g}_1$. Let $M, N \subset \mathfrak{g}_1$ be two $\mathfrak{g}_0$ submodules such that $(M, N) = 0$. Then by invariance of the form we have $([M,N], \mathfrak{g}_0) = (M, [\mathfrak{g}_0, N]) = 0$. Hence $[M, N] = 0$. In particular, let $M \subset \mathfrak{g}_1$ be an irreducible $\mathfrak{g}_0$ submodule. Then the restriction of $(\cdot,\cdot)$ on M is either non-degenerate or zero.

In the first case, let $N = M^{\perp}$ and $I = M \oplus [M, M]$. Then $[N, I] = 0$ and $[\mathfrak{g}_0, I] \subset I$. Hence I is an ideal of $\mathfrak{g}$, which implies $N = 0$, $M = \mathfrak{g}_1$ and $\mathfrak{g}$ satisfies 2. It follows from the proof of Lemma 1 that $\mathfrak{g}_0$ has trivial center.

In the second case there exists an irreducible isotropic submodule $M' \subset \mathfrak{g}_1$ such that $(\cdot,\cdot)$ defines a $\mathfrak{g}_0$-invariant non-degenerate pairing $M \times M' \to k$. By the same argument as in the previous case we have $\mathfrak{g}_1 = M \oplus M'$, $[M, M] = [M', M'] = 0$. Thus, we can set

$$\mathfrak{g}(1) = M, \mathfrak{g}(-1) = M', \mathfrak{g}(0) = \mathfrak{g}_0.$$

Hence $\mathfrak{g}$ satisfies 1.

We say that a basic $\mathfrak{g}$ is of *type 1* (resp. of *type 2*) if it satisfies 1 (resp. 2). Note that if $\mathfrak{g}$ is of type 1, then $\mathfrak{g}(1)$ and $\mathfrak{g}(-1)$ are dual $\mathfrak{g}_0$-modules.

Exercise Check that $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(m|m)$ and $\mathfrak{osp}(2|2n)$ are of type 1, and $\mathfrak{osp}(m|2n)$ is of type 2 if $m \neq 2$.

In contrast with simple Lie algebras, simple Lie superalgebras can have non-trivial central extensions, derivations and deformations. Besides, finite-dimensional representations of simple Lie superalgebras are not completely reducible.

Example 5 Consider the short exact sequence of Lie superalgebras:

$$0 \longrightarrow k \longrightarrow \mathfrak{sl}(2|2) \longrightarrow \mathfrak{psl}(2|2) \longrightarrow 0.$$

One can see that this sequence does not split. In other words, a simple Lie superalgebra $\mathfrak{psl}(2|2)$ has a non-trivial central extension. The dual of this example implies that a finite-dimensional representation of a simple Lie algebra may be not completely reducible, just look at the representation of $\mathfrak{psl}(2|2)$ in $\mathfrak{pgl}(2|2)$ and the exact sequence

$$0 \longrightarrow \mathfrak{psl}(2|2) \longrightarrow \mathfrak{pgl}(2|2) \longrightarrow k \longrightarrow 0.$$

The next example will show that sometimes simple Lie superalgebras have non-trivial deformations.

Example 6 Let $\mathfrak{g} = \mathfrak{osp}(4|2)$. We have

$$\mathfrak{g}_0 = \mathfrak{so}(4) \oplus \mathfrak{sl}(2) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2).$$

In fact, this is the only example of a classical Lie superalgebra whose even part has more then two simple ideals. If V denotes the irreducible 2-dimensional representation of $\mathfrak{sl}(2)$, then $\mathfrak{g}_1$ is isomorphic to $V \boxtimes V \boxtimes V$ as a $\mathfrak{g}_0$-module.

We will construct a one parameter deformation of this superalgebra by deforming the bracket $S^2\mathfrak{g}_1 \to \mathfrak{g}_0$. Let $\psi : S^2V \to \mathfrak{sl}(2)$ and $\omega : \Lambda^2 V \to \mathfrak{sl}(2)$ be non-trivial $\mathfrak{sl}(2)$-equivariant maps. Define the bracket between two odd elements by the formula

$$\begin{aligned}&[v_1 \otimes v_2 \otimes v_3, w_1 \otimes w_2 \otimes w_3]\\ &\quad = (t_1\omega(v_2, w_2)\omega(v_3, w_3)\psi(v_1, w_1), t_2\omega(v_1, w_1)\omega(v_3, w_3)\psi(v_2, w_2),\\ &\quad t_3\omega(v_1, w_1)\omega(v_2, w_2)\psi(v_3, w_3))\end{aligned}$$

for some $t_1, t_2, t_3 \in k$.

Exercise The Jacobi identity holds if and only if $t_1 + t_2 + t_3 = 0$.

When $t_1+t_2+t_3 = 0$ we obtain a new Lie superalgebra structure on $\mathfrak{g}$: we denote the corresponding Lie superalgebra by $D(2,1|t_1,t_2,t_3)$. We see immediately that

$$D(2,1|t_1,t_2,t_3) \cong D(2,1|t_{s(1)},t_{s(2)},t_{s(3)}) \cong D(2,1|ct_1,ct_2,ct_3)$$

for all $c \in k^*$ and $s \in S_3$. One can check that $D(2,1|1,1,-2) \cong \mathfrak{osp}(4|2)$ and that $D(2,1|t_1,t_2,t_3)$ is simple whenever $t_1t_2t_3 \neq 0$. By setting $a = \frac{t_2}{t_1}$ one obtains a one-parameter family $D(2,1,a)$ of Lie superalgebras. One can consider a as a local coordinate in $\mathbf{P}^1 \setminus \{0,-1,\infty\}$.

Exercise Prove that, if $a = 0$, then $D(2,1,a)$ has the ideal J isomorphic to $\mathfrak{psl}(2|2)$ with the quotient $D(2,1,a)/J$ isomorphic to $\mathfrak{sl}(2)$. Use this to prove that the superalgebra of derivations of $\mathfrak{psl}(2|2)$ is isomorphic to $D(2,1,0)$.

Consider now the following general construction of a basic Lie superalgebra of type 2. Let

$$\mathfrak{g}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2, \quad \mathfrak{g}_1 = M \otimes N$$

where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are simple Lie algebras, M is a simple $\mathfrak{l}_1$-module and N a simple $\mathfrak{l}_2$-module. Assume in addition that M has an $\mathfrak{l}_1$-invariant skewsymmetric form ω, while N has an $\mathfrak{l}_2$-invariant symmetric form γ. Then we have isomorphisms $S^2M \simeq \mathfrak{sp}(M)$ and $\Lambda^2 N \simeq \mathfrak{so}(N)$. Hence $\mathfrak{l}_1$ is a submodule in S^2M and $\mathfrak{l}_2$ is a submodule in $\Lambda^2 N$. Let $\phi : S^2M \longrightarrow \mathfrak{l}_1$, $\psi : \wedge^2 N \longrightarrow \mathfrak{l}_2$ denote the projections on the corresponding submodules. For some $t \in k$ and all $x, x' \in M$, $y, y' \in N$ we set

$$[x \otimes y, x' \otimes y'] := \omega(x,x')\psi(y \wedge y') + t\gamma(y,y')\phi(x \cdot x')$$

If for a suitable $t \in k$ we have $[X,[X,X]] = 0$ for all $X \in \mathfrak{g}_1$, then $\mathfrak{g}$ is a Lie superalgebra. For instance, this construction works for $\mathfrak{osp}(m|2n)$ with $\mathfrak{l}_1 = \mathfrak{sp}(2n)$, $\mathfrak{l}_2 = \mathfrak{so}(m)$ and M, N being the standard modules.

This construction also works for exceptional Lie superalgebras: G_3 and F_4 (in Kac's notation). We prefer to use the notation AG_2 and AB_3 to avoid confusion with Lie algebras.

- $\mathfrak{g} = AG_2$ with $\mathfrak{l}_1 = \mathfrak{sl}(2)$, $\mathfrak{l}_2 = G_2$, M is the 2-dimensional irreducible $\mathfrak{sl}(2)$-module and N is the smallest irreducible G_2-module of dimension 7. One can easily see that $\dim AG_2 = (17|14)$.
- $\mathfrak{g} = AB_3$ with $\mathfrak{l}_1 = \mathfrak{sl}(2)$, $\mathfrak{l}_2 = \mathfrak{so}(7)$, M is again the 2-dimensional irreducible $\mathfrak{sl}(2)$-module, N is the spinor representation of $\mathfrak{so}(7)$. Clearly, $\dim AB_3 = (24|16)$.

Theorem 1 (Kac, [22]) *Let k be an algebraically closed field of characteristic zero and $\mathfrak{g}$ be a basic Lie superalgebra over k with nontrivial $\mathfrak{g}_1$. Then $\mathfrak{g}$ is isomorphic to one of the following superalgebras:*

- $\mathfrak{sl}(m|n)$, $1 \leqslant m < n$;
- $\mathfrak{psl}(n|n)$, $n \geqslant 2$;
- $\mathfrak{osp}(m|2n)$, $m, n \geqslant 1$, $(m,n) \neq (2,1), (4,1)$;
- $D(2,1,a)$, $a \in (\mathbf{P}^1 \setminus \{0,-1,\infty\})/S_3$;
- AB_3;
- AG_2.

For the proof of Theorem 1 we refer the reader to the original paper of Kac. Some hints can be also found in the next Section.

Exercise Show that $\mathfrak{sl}(1|2)$ and $\mathfrak{osp}(2|2)$ are isomorphic Lie superalgebras. Check that the list in Theorem 1 does not contain isomorphic superalgebras.

3.2 Roots Decompositions of Basic Lie Superalgebras

From now on we will always assume that k is algebraically closed of characteristic zero.

Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and W denote the Weyl group of $\mathfrak{g}_0$. If $\mathfrak{g}$ is of type 1 but $\mathfrak{g}_0$ is semisimple it will be convenient to consider a bigger superalgebra $\tilde{\mathfrak{g}}$ by adding to $\mathfrak{g}$ the grading element z (if $\mathfrak{g} = \mathfrak{psl}(n|n)$, then $\tilde{\mathfrak{g}} = \mathfrak{pgl}(n|n)$). In this case we set $\tilde{\mathfrak{h}}_0 := \mathfrak{h}_0 + kz$, otherwise $\tilde{\mathfrak{h}}_0 := \mathfrak{h}_0$. Let $\mathfrak{h}$ be the centralizer of $\tilde{\mathfrak{h}}_0$ in $\mathfrak{g}$.

Lemma 3 *We have* $\mathfrak{h} = \mathfrak{h}_0$*and* $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_0$.

Proof If $\mathfrak{g}$ is of type 1, the statement is trivial. If $\mathfrak{g}$ is of type 2, then $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module which admits invariant symplectic form. Then such representation does not have zero weight, see [34, Chap. 4.3, Exercise 13].

Lemma 3 implies that $\tilde{\mathfrak{h}}$ acts semisimply on $\mathfrak{g}$. Hence we have a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \text{for all}\, h \in \tilde{\mathfrak{h}}\}.$$

Here Δ is a finite subset of non-zero vectors in $\tilde{\mathfrak{h}}^*$, whose elements are called *roots*. The subalgebra $\mathfrak{h}$ is called a *Cartan subalgebra* of $\mathfrak{g}$.

The following conditions are straightforward

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \neq 0$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.
- The invariant form $(\cdot, \cdot)$ defines a non-degenerate pairings $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to k$ for all $\alpha \in \Delta$ and $\mathfrak{h} \times \mathfrak{h} \to k$.
- $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a one-dimensional subspace in $\mathfrak{h}$. That follows from the first two properties and the identity $([x, y], h) = \alpha(h)(x, y)$ for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}, h \in \mathfrak{h}_0$.

We can define the non-degenerate symmetric form on $(\cdot|\cdot)$ on $\tilde{\mathfrak{h}}^*$ as the pull back of $(\cdot, \cdot)$ with respect to $\tilde{\mathfrak{h}}^* \xrightarrow{p} \mathfrak{h}^* \xrightarrow{s} \mathfrak{h}$, where p is the canonical projection and $s : \mathfrak{h}^* \to \mathfrak{h}$ is an isomorphism induced by $(\cdot, \cdot)$. For any two roots $\alpha, \beta \in \Delta$

$$\beta(\mathfrak{h}_\alpha) = 0 \quad \text{if and only if} \quad (\alpha, \beta) = 0. \tag{1}$$

Lemma 4 *Let* $\alpha \in \Delta$ *be a root.*

1. $\dim(\mathfrak{g}_\alpha)_0 \leqslant 1$*;*
2. If $(\mathfrak{g}_\alpha)_0 \neq 0$*, then* $(\mathfrak{g}_\alpha)_1 = 0$.

Proof Since $\mathfrak{g}_0$ is reductive 1 is trivial. To prove 2 consider the root $\mathfrak{sl}(2)$-subalgebra $\{x_\alpha, h_\alpha, y_\alpha\} \subset \mathfrak{g}_0$. Let $x \in (\mathfrak{g}_\alpha)_1$ and $x \neq 0$. Then from representation theory of $\mathfrak{sl}(2)$ we know that $[y_\alpha, x] \neq 0$. But $[y_\alpha, x] \in \mathfrak{h}_1 = 0$. Contradiction.

We call $\alpha \in \Delta$ even (resp. odd) if $(\mathfrak{g}_\alpha)_1 = 0$, (resp. $(\mathfrak{g}_\alpha)_0 = 0$). We denote by Δ_0 (resp. Δ_1) the set of even (resp. odd roots). The preceding lemma implies that Δ is the disjoint union of Δ_0 and Δ_1.

Lemma 5

1. *If* $\alpha \in \Delta_0$, *then* $(\alpha|\alpha) \neq 0$.
2. *If* $\alpha \in \Delta_1$ *and* $(\alpha|\alpha) \neq 0$, *then for any non-zero* $x \in \mathfrak{g}_\alpha$, $[x,x] \neq 0$. *Hence* $2\alpha \in \Delta_0$.
3. *If* $\alpha \in \Delta_1$ *and* $(\alpha|\alpha) \neq 0$, *then* $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \{-1, 0, 1\}$ *for any* $\beta \in \Delta_0$.
4. *If* $\alpha \in \Delta_1$ *and* $(\alpha|\alpha) = 0$, *then* $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \{-2, -1, 0, 1, 2\}$ *for any* $\beta \in \Delta_0$.

Proof 1 is the property of root decomposition of reductive Lie algebras. To show 2 let $y \in \mathfrak{g}_{-\alpha}$ be such that $(x, y) \neq 0$. Then $h = [y, x] \neq 0$ and by (1) we obtain

$$[y, [x, x]] = 2[h, x] = 2\alpha(h)x \neq 0.$$

To prove the last two statements we consider the root $\mathfrak{sl}(2)$-triple $\{x_\beta, h_\beta, y_\beta\}$. Then from the representation theory of $\mathfrak{sl}(2)$ we obtain that $\frac{2(\alpha|\beta)}{(\beta|\beta)} = \alpha(h_\beta)$ must be an integer.

To show 3 we use the fact that 2α is an even root. We know from the structure theory of reductive Lie algebras that

$$\frac{2(2\alpha|\beta)}{(\beta,\beta)} \in \{-3, -2-1, 0, 1, 2, 3\}.$$

Taking into account that $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \mathbf{Z}$, we obtain the assertion.

Finally, let us prove 4. Without loss of generality we may assume that $k = \alpha(h_\beta) > 1$. Then we claim that $y_\beta(\mathfrak{g}_\alpha) \neq 0$, hence $\alpha - \beta$ is a root. Moreover

$$(\alpha - \beta|\alpha - \beta) = (\beta|\beta)(1 - k) \neq 0.$$

Therefore $\gamma := 2(\alpha - \beta)$ is an even root and we have

$$\frac{2(\beta|\gamma)}{(\gamma|\gamma)} = \frac{k/2 - 1}{1 - k} \in \mathbf{Z},$$

which implies $k = 2$.

Exercise An odd root α is called *isotropic* if $(\alpha|\alpha) = 0$. Show that if $\mathfrak{g}$ is of type 1, then all odd roots are isotropic.

It is clear that W acts on Δ and preserves the parity and the scalar products between roots.

Lemma 6

(a) If $\mathfrak{g}$ *is of type* 1 *then* W *has two orbits in* Δ_1*, the roots of* $\mathfrak{g}(1)$ *and the roots of* $\mathfrak{g}(-1)$.

(b) If $\mathfrak{g}$ *is of type* 2*, then* W *acts transitively on the set of isotropic and the set of non-isotropic odd roots.*

Proof If all roots of $\mathfrak{g}$ are isotropic, then it follows from the proof of Lemma 5 (4) that $\alpha(h_\beta) = \pm 1$ or 0 for any odd root α and even root β. In particular, if we fix positive roots in Δ_0 and consider a highest weight α in $\mathfrak{g}_1$ (or $\mathfrak{g}(\pm 1)$ in type 1 case), the above condition implies that $\mathfrak{g}_1$ (resp. $\mathfrak{g}(\pm 1)$) is a minuscule representation of $\mathfrak{g}_0$.

If $\mathfrak{g}$ is of type 2 and the highest weight α is isotropic, then we have $\alpha(h_\beta) = \pm 1, \pm 2$ or 0 for any positive β. That implies the existence of two orbits. Finally if α is not isotropic, then $\mathfrak{g}_1$ is minuscule, hence there is one W-orbit consisting of non-isotropic roots.

Corollary 1 *For any root* $\alpha \in \Delta$ *the root space* $\mathfrak{g}_\alpha$ *has dimension* $(1|0)$ *or* $(0|1)$.

Proof We need to prove the statement only for odd α. If $\mathfrak{g}$ is of type 1 or of type 2 with only isotropic or only non-isotropic odd roots, then the statement follows from Lemma 6 since the multiplicity of the highest weight is 1. If $\mathfrak{g}$ contains both isotropic and non-isotropic roots, we have to show only that $\dim\mathfrak{g}_\alpha = (0|1)$ for a non-isotropic odd root α, which easily follows from Lemma 5 (2).

Remark 1 Note that if we do not extend $\mathfrak{psl}(2|2)$ to $\mathfrak{pgl}(2|2)$, then Corollary 1 does not hold since the dimension of $\mathfrak{g}_\alpha$ equals $(0|2)$ for any odd α.

Example 7 Let $\mathfrak{g} = \mathfrak{sl}(m|n)$. We take as our Cartan subalgebra $\mathfrak{h}$ the subalgebra of diagonal matrices. Let us denote by $\epsilon_1, \dots \epsilon_m, \delta_1, \dots \delta_n$ the roots in $\mathfrak{h}^*$ ($\epsilon_i(\mathrm{diag}(a_1, \dots a_m)) = a_i$ and similarly for δ_j). We have:

$$\Delta_0 = \{\epsilon_i - \epsilon_j,\ 1 \leqslant i \neq j \leqslant m\} \cup \{\delta_i - \delta_j,\ 1 \leqslant i \neq j \leqslant n\}, \qquad \Delta_1 = \{\pm(\epsilon_i - \delta_j)\}.$$

The invariant form is:

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\epsilon_i, \delta_j) = 0, \quad (\delta_i, \delta_j) = -\delta_{ij},$$

All odd roots are isotropic.

Example 8 Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. $\mathfrak{g}_0 = \mathfrak{sp}(2n)$.

$$\Delta_0 = \{\pm(\epsilon_i \pm \epsilon_j),\ \pm 2\epsilon_i \mid i,j = 1 \dots n,\ i \neq j\}, \qquad \Delta_1 = \{\pm\epsilon_i \mid i = 1 \dots n\}.$$

This is the only example of a basic superalgebra such that all odd roots are non-isotropic.

The above implies that we have in general three types of roots:

1. $\alpha \in \Delta_0$. In this case the root spaces $\mathfrak{g}_{\pm\alpha}$ generate a $\mathfrak{sl}(2)$ subalgebra (white node in a Dynkin diagram).

2. $\alpha \in \Delta_1$, $(\alpha, \alpha) \neq 0$. Then the root spaces $\mathfrak{g}_{\pm\alpha}$ generate a subalgebra isomorphic to $\mathfrak{osp}(1|2)$ (black node in a Dynkin diagram).
3. $\alpha \in \Delta_1$, $(\alpha, \alpha) = 0$. The roots spaces $\mathfrak{g}_{\pm\alpha}$ generate a subalgebra isomorphic to $\mathfrak{sl}(1|1)$ (grey node in a Dynkin diagram).

Definition 6 Let E be a vector space (over k) equipped with non-degenerate scalar product $(\cdot|\cdot)$. A finite subset $\Delta \subset E \setminus \{0\}$ is called a *generalized root system* if the following conditions hold:

- if $\alpha \in \Delta$, then $-\alpha \in \Delta$;
- if $\alpha, \beta \in \Delta$ and $(\alpha|\alpha) \neq 0$, then $k_{\alpha,\beta} = \frac{2(\alpha|\beta)}{(\alpha|\alpha)}$ is an integer and $\beta - k_{\alpha,\beta}\alpha \in \Delta$;
- if $\alpha \in \Delta$ and $(\alpha|\alpha) = 0$, then there exists an invertible map $r_\alpha : \Delta \to \Delta$ such that

$$r_\alpha(\beta) = \begin{cases} \beta \text{ if } (\alpha|\beta) = 0 \\ \beta \pm \alpha \text{ if } (\alpha|\beta) \neq 0 \end{cases} .$$

Exercise Check that if $\mathfrak{g}$ is a basic Lie superalgebra, then the set of roots Δ is a generalized root system.

Indecomposable generalized root systems are classified in [39]. In fact, they coincide with root systems of basic Lie superalgebras. That gives an approach to the proof of Theorem 1.

Exercise Let Q_0 be the lattice generated by Δ_0 and Q be the lattice generated by Q. Check that

- If $\mathfrak{g}$ is of type 1, then Q_0 is a sublattice of corank 1 in Q.
- If $\mathfrak{g}$ is of type 2, then Q_0 is a finite index subgroup in Q.

3.3 Bases and Odd Reflections

As in the case of simple Lie algebras we can represent Δ as a disjoint union $\Delta^+ \coprod \Delta^-$ of positive and negative roots (by dividing $\tilde{\mathfrak{h}}^*$ in two half-spaces).

We are going to use the *triangular decomposition*:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \text{where} \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha,$$

The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called a *Borel subalgebra* of $\mathfrak{g}$.

We call $\alpha \in \Delta^+$ indecomposable if it is not a sum of two positive roots. We call the set of indecomposable roots $\alpha_1, \ldots \alpha_n \in \Delta^+$ *simple roots* or a *base* as in the Lie algebra case. Clearly, W action on Δ permutes bases. However, not all bases can be obtained from one by the action of W.

Example 9 The Weyl group of $\mathfrak{gl}(2|2)$ is isomorphic to $S_2 \times S_2$. One can see that the following two bases are not conjugate by the action of W: $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \delta_1, \delta_1 - \delta_2\}$, $\Pi' = \{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \epsilon_2 - \delta_2\}$.

Since W does not act transitively on the set of bases, more than one Dynkin diagram may be associated to the same Lie superalgebra. The existence of several Dynkin diagrams implies existence of several non conjugate Borel subalgebras, which in turn implies that there are several non isomorphic flag supervarieties.

To every base Π we associate the *Cartan matrix* in the following way. Take $X_i \in \mathfrak{g}_{\alpha_i}$, $Y_i \in \mathfrak{g}_{-\alpha_i}$, and set $H_i := [X_i, Y_i]$ and $a_{ij} := \alpha_j(H_i)$. In the classical theory of Kac-Moody algebras Cartan matrices are normalized so that the diagonal entries are equal to 2. In the supercase we can do the same for non-isotropic simple roots. It is not difficult to see that H_i, X_i, Y_i for $i = 1, \dots, n$ generate $\mathfrak{g}$ and satisfy the relations

$$[H_i, X_j] = a_{ij}X_j, \qquad [H_i, Y_j] = -a_{ij}Y_j, \qquad [X_i, Y_j] = \delta_{ij}H_i, \qquad [H_i, H_j] = 0.$$

Let $\bar{\mathfrak{g}}$ be the free Lie superalgebra with above generators and relations. We define the Kac-Moody superalgebra $\mathfrak{g}(A)$ as the quotient of $\bar{\mathfrak{g}}$ by the maximal ideal which intersects trivially the Cartan subalgebra. In this way we recover basic finite dimensional Lie superalgebras. In contrast with Lie algebra case we may get a finite-dimensional Kac-Moody superalgebra even if $det(A) = 0$, for example, $\mathfrak{g}(A) = \mathfrak{gl}(n|n)$. Note that in this case $\mathfrak{g}(A)$ is not simple but a non-trivial central extension of the corresponding simple superalgebra. In many applications, it is better to consider $\mathfrak{g}(A)$ instead of the corresponding quotient, which essentially means that in what follows we rather discuss representations and structure theory of $\mathfrak{gl}(n|n)$ instead of $\mathfrak{psl}(n|n)$.

Definition 7 Let Π be a base (set of simple roots) and let $\alpha \in \Pi$ be an isotropic odd root. We define an *odd reflection* $r_\alpha : \Pi \to \Pi'$ by

$$r_\alpha(\beta) = \begin{cases} \beta + \alpha \text{ if } (\alpha|\beta) \neq 0 \\ \beta \text{ if } (\alpha|\beta) = 0,\ \beta \neq \alpha \\ -\alpha \text{ if } \beta = \alpha \end{cases}$$

Exercise Check that $\Pi' = r_\alpha(\Pi)$ is a base.

Notice that if $(\alpha|\alpha) \neq 0$ we can define the usual reflection $r_\alpha(x) := x - \frac{2(x|\alpha)}{(\alpha|\alpha)}\alpha$, which is an orthogonal linear transformation of $\mathfrak{h}^*$. In fact, since $r_\alpha = r_{2\alpha}$, one can see that these reflections generate W. Though the odd reflections are defined on simple roots only, one can show that they may be extended (uniquely) to permutations of all roots. However, in most cases such extension can not be further extended to a linear map of the root lattice.

Proposition 1 *Let $\mathfrak{g}$ be a basic Lie superalgebra.*

1. *If Π and Π' are two bases, then Π' can be obtained from Π by application of odd and even reflections.*
2. *If Π and Π' are bases such that $\Delta^+ \cap \Delta_0 = (\Delta')^+ \cap \Delta_0$, then Π' can be obtained from Π by application of odd reflections.*

Go back to the example of $\mathfrak{gl}(2|2)$. The Cartan matrix associated with Π is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The odd reflection r_α associated with the root $\alpha = \epsilon_2 - \delta_1 \in \Pi$ maps Π to Π'. Indeed, we have:

$$\begin{aligned} r_\alpha(\epsilon_1 - \epsilon_2) &= \epsilon_1 - \delta_1 = (\epsilon_1 - \epsilon_2 + \epsilon_2 - \delta_1) \\ r_\alpha(\epsilon_2 - \epsilon_1) &= \delta_1 - \epsilon_2 \\ r_\alpha(\delta_1 - \delta_2) &= \epsilon_2 - \delta_2 = (\epsilon_2 - \delta_1 + \delta_1 - \delta_2). \end{aligned}$$

The Cartan matrix associated with Π' is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Exercise Use odd reflections to get all bases of AG_2.

Remark 2 Let $\mathfrak{g}$ be of type 1 and let us fix a Borel subalgebra $\mathfrak{b}_0 \subset \mathfrak{g}_0$. We have two especially convenient Borel subalgebras:

$$\mathfrak{b}_d = \mathfrak{b}_0 \oplus \mathfrak{g}(1), \qquad \mathfrak{b}_{ad} = \mathfrak{b}_0 \oplus \mathfrak{g}(-1).$$

We call them *distinguished* and *antidistinguished*, respectively.

4 Representations of Basic Superalgebras

4.1 Highest Weight Theory

We assume in this section that $\mathfrak{g}$ is a basic superalgebra or its Kac Moody extension (in the case of $\mathfrak{gl}(n|n)$). Let us fix a triangular decomposition: $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Define the *Verma module*:

$$M_{\mathfrak{b}}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} C_\lambda,$$

where C_λ is the one-dimensional $\mathfrak{b}$-module with trivial action of $\mathfrak{n}^+$ and weight λ. One can prove exactly as in the Lie algebra case that $M_{\mathfrak{b}}(\lambda)$ has a unique simple quotient which we denote by $L_{\mathfrak{b}}(\lambda)$.

We say that λ is *integral dominant* if $L_{\mathfrak{b}}(\lambda)$ is finite dimensional.

Exercise Prove that if λ is integral dominant, then $M_{\mathfrak{b}}(\lambda)$ has the unique maximal finite dimensional quotient $K_{\mathfrak{b}}(\lambda)$. If $\mathfrak{g}$ is of type 1 and $\mathfrak{b}$ is distinguished, then $K_{\mathfrak{b}}(\lambda)$ is isomorphic to the induced module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}(1))} L_{\mathfrak{b}_0}(\lambda)$, where $L_{\mathfrak{b}_0}(\lambda)$ is the simple $\mathfrak{g}_0$-module with trivial action of $\mathfrak{g}(1)$. In this case it is called a *Kac module*.

Proposition 2 *Any finite-dimensional simple* $\mathfrak{g}$*-module is isomorphic to* $L_{\mathfrak{b}}(\lambda)$ *for some integral dominant* λ.

Proof Any finite dimensional simple module M is semisimple over $\mathfrak{h}$ and hence has a finite number of weights. Let λ be a weight such that $\lambda + \alpha$ is not a weight for all positive roots α. Then, by Frobenius reciprocity, M is a quotient of $M_{\mathfrak{b}}(\lambda)$.

Remark 3 Let O be the category of finitely generated $\mathfrak{h}$-semisimple $\mathfrak{g}$-modules with locally nilpotent action of $\mathfrak{n}^+$. Note that this definition depends on the choice of a Borel subalgebra $\mathfrak{b}$. In fact, it depends only on the choice of $\mathfrak{b}_0$, since the local nilpotency of $\mathfrak{n}_0^+$ implies the local nilpotency of $\mathfrak{n}^+$.

How do we check whether λ is dominant integral with respect to a particular Borel subalgebra $\mathfrak{b}$? If $\mathfrak{g}$ is of type 1 and $\mathfrak{b}$ is distinguished or antidistinguished, it is sufficient to check that λ is integral dominant with respect to $\mathfrak{b}_0$, i.e. $\lambda(h_\alpha) \in \mathbb{N}$ for all simple even roots α. In general, the condition of dominance is more complicated.

Exercise

(a) If $\mathfrak{b}$ and $\mathfrak{b}'$ are two Borel subalgebras of $\mathfrak{g}$ with the same even part, then we must have an isomorphism $L_{\mathfrak{b}}(\lambda) \simeq L_{\mathfrak{b}'}(\lambda')$ for some weights λ and λ'. Let $\mathfrak{b}'$ be obtained from $\mathfrak{b}$ by an odd reflection r_α. Check that

$$\lambda' = \begin{cases} \lambda - \alpha \text{ if } (\lambda, \alpha) \neq 0 \\ \lambda \text{ if } (\lambda, \alpha) = 0. \end{cases} \tag{2}$$

(b) Fix a base Π and the corresponding Borel subalgebra $\mathfrak{b}$. Let Π_0 denote the base of Δ_0^+. Prove that $L_{\mathfrak{b}}(\lambda)$ is finite-dimensional if and only if for any $\beta \in \Pi_0$ and a base Π' obtained from Π by odd reflections such that $\beta \in \Pi'$ or $\frac{\beta}{2} \in \Pi'$ we have $\frac{2(\lambda|\beta)}{(\beta|\beta)} \in \mathbb{N}$. (*Hint*: you just have to check that $y_\beta \in \mathfrak{g}_{-\beta}$ acts locally nilpotently on $L_{\mathfrak{b}}(\lambda)$.)

4.2 Typicality

We define the *Weyl* vector $\rho_{\mathfrak{b}} \in \mathfrak{h}^*$ by:

$$\rho_{\mathfrak{b}} := \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha.$$

If $\mathfrak{b}$ is fixed and clear we simplify notation by setting $\rho = \rho_{\mathfrak{b}}$.

Exercise Let Π be the base corresponding to $\mathfrak{b}$. Show that

$$(\rho|\alpha) = \begin{cases} \frac{1}{2}(\alpha|\alpha) \text{ if } \alpha \in \Pi \cap \Delta_0 \\ (\alpha|\alpha) \text{ if } \alpha \in \Pi \cap \Delta_1 \end{cases} .$$

Definition 8 A weight λ is called *typical* if $(\lambda + \rho, \alpha) \neq 0$ for all isotropic roots $\alpha \in \Delta$.

Exercise Check that the definition of typicality does not depend on the choice of $\mathfrak{b}$. To show this assume that $\mathfrak{b}'$ is obtained from $\mathfrak{b}$ by an odd reflection r_α and λ is typical. Then $\rho'_{\mathfrak{b}} = \rho_{\mathfrak{b}} + \alpha$ and $L_{\mathfrak{b}}(\lambda) = L'_{\mathfrak{b}}(\lambda')$, where $\lambda + \rho_{\mathfrak{b}} = \lambda' + \rho_{\mathfrak{b}'}$.

4.3 Characters of Simple Finite-Dimensional Modules

If M is in the category $\mathcal{O}$, then, by definition, M is $\mathfrak{h}$-semisimple, and therefore has weight decomposition $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$. The character $\operatorname{ch} M$ is the generating function

$$\operatorname{ch} M := \sum \operatorname{sdim}(M_\mu) e^\mu.$$

Exercise Show, using Corollary 1, that if M is generated by one weight vector, in particular, if M is simple then every weight space M_μ is either purely even or purely odd.

Theorem 2 ([23]) *If λ is a typical integral dominant weight then*

$$\operatorname{ch} L_{\mathfrak{b}}(\lambda) = \frac{D_1}{D_0} \sum_{w \in W} sgn(w) e^{w(\lambda + \rho_{\mathfrak{b}})}, \tag{3}$$

where W is the Weyl group of the even part $\mathfrak{g}_0$ and

$$D_1 = \prod_{\alpha \in \Delta_1^+} (e^{\alpha/2} - e^{-\alpha/2}), \qquad D_0 = \prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

Exercise Using the isomorphism of $\mathfrak{h}$-modules $\mathcal{U}(\mathfrak{n}^-) \simeq S(\mathfrak{n}^-)$ show that

$$\operatorname{ch}\mathcal{U}(\mathfrak{n}^-) = \prod_{\alpha\in\Delta_1}(1-e^{-\alpha})/\prod_{\alpha\in\Delta_0}(1-e^{-\alpha}),$$

and

$$\operatorname{ch} M_{\mathfrak{b}}(\lambda)) = e^{\lambda+\rho}\frac{D_1}{D_0}.$$

Remark 4

- If $\mathfrak{g} = \mathfrak{g}_0$ then we get the usual Weyl character formula.
- The formula (3) is invariant with respect to the change of Borel subalgebra.
- The formula (3) can be rewritten in the form

  $$\operatorname{ch} L_{\mathfrak{b}}(\lambda) = \sum_{w\in W} sgn(w)\operatorname{ch} M_{\mathfrak{b}}(w\cdot\lambda),$$

  where $w\cdot\lambda := w(\lambda+\rho)-\rho$ is the shifted action.

Proof of Theorem 2 We will give the proof for type 1 superalgebras, i.e. assuming a compatible grading $\mathfrak{g} = \mathfrak{g}(-1) + \mathfrak{g}(0) + \mathfrak{g}(1)$. By Remark 4 it suffices to prove the formula for the distinguished $\mathfrak{b} = \mathfrak{b}_d$.

Note that the Kac module $K_{\mathfrak{b}}(\lambda)$ is isomorphic to

$$\mathcal{U}(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda) = \Lambda(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda)$$

as a $\mathfrak{g}_0 + \mathfrak{g}(-1)$-module. Therefore

$$\operatorname{ch} K_{\mathfrak{b}}(\lambda) = \operatorname{ch}\Lambda(\mathfrak{g}(-1))\operatorname{ch} L_{\mathfrak{b}_0}(\lambda) = \prod_{\alpha\in\Delta_1^+}(1-e^{-\alpha})\operatorname{ch} L_{\mathfrak{b}_0}(\lambda).$$

Furthermore, if $\rho_i = \frac{1}{2}\sum_{\alpha\in\Delta_i}\alpha$, for $i = 0, 1$, then

$$\prod_{\alpha\in\Delta_1^+}(1-e^{-\alpha}) = e^{\rho_1}D_1, \quad \operatorname{ch} L_{\mathfrak{b}_0}(\lambda) = \frac{1}{D_0}\sum_{w\in W} sgn(w)e^{w(\lambda+\rho_0)}.$$

Note also that $w(\rho_1) = \rho_1$ for all $w \in W$. Therefore $\operatorname{ch} K_{\mathfrak{b}}(\lambda)$ is given by (3). Thus, it remains to show that $K_{\mathfrak{b}}(\lambda) = L_{\mathfrak{b}}(\lambda)$.

One can see easily that any submodule of $K_{\mathfrak{b}}(\lambda)$ contains a simple $\mathfrak{g}_0$-submodule

$$\Lambda^{top}(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda).$$

Hence $K_{\mathfrak{b}}(\lambda)$ has a unique simple submodule isomorphic to $L_{\mathfrak{b}}(\mu)$ for some μ.

Next we observe that

$$\lambda' := \lambda - \sum_{\alpha \in \Delta_1^+} \alpha$$

is the highest weight of $L_{\mathfrak{b}}(\mu)$ with respect to the anti-distinguished Borel $\mathfrak{b}_{ad}$, since λ' is the $\mathfrak{b}_0$-highest weight in $\Lambda^{top}(\mathfrak{g}(-1)) \otimes L_{\mathfrak{b}_0}(\lambda)$ and

$$\mathfrak{g}(-1)\Lambda^{top}(\mathfrak{g}(-1)) = 0.$$

Therefore we have

$$L_{\mathfrak{b}}(\mu) = L_{\mathfrak{b}_{ad}}(\lambda').$$

Applying (2) several times to move from $\mathfrak{b}$ to $\mathfrak{b}_{ad}$ and using the typicality of λ we obtain $\lambda = \mu$. Hence $K_{\mathfrak{b}}(\lambda) = L_{\mathfrak{b}}(\lambda)$.

4.4 The Center of $\mathcal{U}(\mathfrak{g})$

Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In the supersetting the Duflo theorem states that there exists an isomorphism of supercommutative rings

$$S(\mathfrak{g})^{\mathfrak{g}} \simeq \mathcal{Z}(\mathfrak{g}).$$

For the proof in the supercase see [26].

Recall that if $\mathfrak{g}$ is a reductive Lie algebra then $\mathcal{Z}(\mathfrak{g})$ is a polynomial ring, see, for example, [10]. This fact follows from so called Harish-Chandra homomorphism. One can generalize the Harish-Chandra homomorphism for basic superalgebras, however, as we will see, $\mathcal{Z}(\mathfrak{g})$ is not Noetherian.

Choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, then by PBW theorem we have the decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^+).$$

The Harish-Chandra map

$$HC : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) = k[\mathfrak{h}^*]$$

is the projection with kernel $\mathfrak{n}^-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$. The restriction

$$HC : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h}) = k[\mathfrak{h}^*]$$

is a homomorphism of rings.

For any $w \in W$ and $\lambda \in \mathfrak{h}^*$ we set $w \cdot \lambda := w(\lambda + \rho) - \rho$.

Theorem 3 *The homomorphism $HC : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ is injective and $f \in k[\mathfrak{h}^*]$ belongs to $HC(\mathcal{Z}(\mathfrak{g}))$ if and only if*

- *$f(w \cdot \lambda) = f(\lambda)$, for any $\lambda \in \mathfrak{h}^*, w \in W$;*
- *if $(\lambda + \rho|\alpha) = 0$ for some isotropic root α then $f(\lambda + t\alpha) = f(\lambda)$ for all $t \in k$.*

The proof of this Theorem can be found in [24, 45] or [16]. One of the consequences of the above theorem is that the supercommutative ring $\mathcal{Z}(\mathfrak{g})$ has trivial odd part and hence is in fact a usual commutative ring.

The proof in [45] makes use of the superanalogue of the Chevalley restriction theorem. Since $\mathfrak{g}$ is basic, then the adjoint representation is self-dual. Thus, we can identify the invariant polynomials on $\mathfrak{g}$ and $\mathfrak{g}^*$:

$$k[\mathfrak{g}]^{\mathfrak{g}} \simeq k[\mathfrak{g}^*]^{\mathfrak{g}}.$$

If $F : k[\mathfrak{g}]^{\mathfrak{g}} \to k[\mathfrak{h}]$ denotes the restriction map induced by the embedding $\mathfrak{h} \subset \mathfrak{g}$, then the image of F consists of W-invariant polynomials on $\mathfrak{h}$ satisfying the additional condition:

if $(\lambda|\alpha) = 0$ for some isotropic root α then $f(\lambda + t\alpha) = f(\lambda)$ for all $t \in k$.

Example 10 Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. The ring $S(\mathfrak{g}^*)^{\mathfrak{g}}$ is generated by $str(X^s)$ $s = 1, 2, 3 \ldots$. After restriction to the diagonal subalgebra they become polynomials in $P_1, P_2, \dots \in k[x_1, \dots, x_m, y_1, \dots, y_n]$ given by the formula Set

$$P_s := x_1^s + \dots x_m^s - y_1^s - \dots - y_n^s.$$

One can see that the subring in $k[x_1, \dots, x_m, y_1, \dots, y_n]$ generated by P_s is not a Noetherian ring.

If *Specm* stands for the spectrum of maximal ideals, then HC induces the map $\theta : Specm(k[\mathfrak{h}^*]) = \mathfrak{h}^* \longrightarrow Specm(\mathcal{Z}(\mathfrak{g}))$. In other words we associate with every weight $\lambda \in \mathfrak{h}^*$ the central character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to k$ by setting $\chi_\lambda(z) := HC(z)(\lambda)$. We would like to describe the fibers of θ. The following corollary implies that every fiber is a union of finitely many affine subspaces of the same dimension.

Corollary 2 *Let $\lambda \in \mathfrak{h}^*$ and let $\{\alpha_1, \dots, \alpha_k\}$ be a maximal set of mutually orthogonal linearly independent isotropic roots such that $(\lambda + \rho|\alpha_i) = 0$. If $\chi = \chi_\lambda$, then*

$$\theta^{-1}(\chi) = \bigcup_{w \in W} w \cdot (\lambda + \sum_{i=1}^{k} k\alpha_i).$$

Example 11 If $\mathfrak{g} = \mathfrak{sl}(1|2)$, then $\dim \mathfrak{h} = 2$ and the image of the Harish Chandra homomorphism in $k[x, y]$ consists of polynomials $k[x, y^2]$ which are constant on the cross $y = \pm x$.

Corollary 3 *If λ is typical then $(\theta)^{-1}(\chi_\lambda) = W \cdot \lambda$.*

Corollary 4 *If λ is dominant integral and typical, then* $\mathrm{Ext}^1(L_{\mathfrak{b}}(\lambda), L_{\mathfrak{b}}(\mu)) = 0$ *for any integral dominant $\mu \neq \lambda$. Hence $L_{\mathfrak{b}}(\lambda)$ is projective in the category $\mathcal{F}$ of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$.*

Proof If λ is dominant integral and typical, then $W \cdot \lambda$ does not contain any other integral dominant weight. Therefore $L_{\mathfrak{b}}(\lambda)$ and $L_{\mathfrak{b}}(\mu)$ admit different central characters. Hence $\mathrm{Ext}^1(L_{\mathfrak{b}}(\lambda), L_{\mathfrak{b}}(\mu)) = 0$. Semisimplicity over $\mathfrak{g}_0$ ensures that $\mathrm{Ext}^1_{\mathcal{F}}(L_{\mathfrak{b}}(\lambda), L_{\mathfrak{b}}(\lambda)) = 0$.

Remark 5 If $\mathfrak{g}$ is of type 2, then any finite-dimensional $\mathfrak{g}$-module is semisimple over $\mathfrak{g}_0$. In type 1 case, $L_{\mathfrak{b}}(\lambda)$ is not projective in the category of *all* finite-dimensional $\mathfrak{g}$-modules since it has non-trivial self-extension.

Definition 9 (Kac–Wakimoto) The dimension of $\theta^{-1}(\chi)$ is called the *atypicality degree* of χ. We will denote it by $\mathrm{at}(\chi)$. It follows from Corollary 2 that if $\chi_\lambda = \chi$, then $\mathrm{at}(\chi)$ is the maximal number of mutually orthogonal linearly independent isotropic roots α such that $(\lambda + \rho|\alpha) = 0$. We also use the notation $\mathrm{at}(\lambda) = \mathrm{at}(\chi_\lambda)$. The central character χ is typical (resp. atypical) if $\mathrm{at}(\chi) = 0$ (resp. $f(\chi) > 0$).

The *defect* $\mathrm{def}\,\mathfrak{g}$ of $\mathfrak{g}$ is the maximal number of mutually orthogonal linearly independent isotropic roots, i.e. the maximal dimension of the fiber of θ.

Exercise Show that

$$\mathrm{def}\,\mathfrak{gl}(m|n) = \mathrm{def}\,\mathfrak{osp}(2m|2n) = \mathrm{def}\,\mathfrak{osp}(2m+1|2n) = \min(m, n).$$

Check that the defect of the exceptional superalgebras AG_2, AB_3 and $D(1, 2; a)$ is 1.

Note that $\mathfrak{osp}(1|2n)$ is the only basic superalgebra with defect zero. Hence we have the following proposition.

Proposition 3 *All finite-dimensional representations of $\mathfrak{osp}(1|2n)$ are completely reducible and the character of any irreducible finite-dimensional representation of $\mathfrak{osp}(1|2n)$ is given by (3).*

Finally, let us formulate without proof the following general result which allows to reduce many questions about typical representations (finite or infinite-dimensional) to the same questions for the even part $\mathfrak{g}_0$.

Theorem 4 ([15, 36]) *Suppose that $\chi = \chi_\lambda$ is a typical central character such that $(\lambda + \rho|\beta) \neq 0$ for any non-isotropic root β. Let $\mathcal{U}_\chi(\mathfrak{g}) := \mathcal{U}(\mathfrak{g})/(Ann(\chi))$. Then there exists a central character χ_0 of $\mathcal{Z}(\mathfrak{g}_0)$ such that $\mathcal{U}_\chi(\mathfrak{g})$ is Morita equivalent to $\mathcal{U}_{\chi_0}(\mathfrak{g}_0) := \mathcal{U}(\mathfrak{g}_0)/(Ann(\chi_0))$.*

Remark 6 If $\mathfrak{g}$ is of type 1, then $\mathcal{U}_\chi(\mathfrak{g})$ is isomorphic to the matrix algebra over $\mathcal{U}_{\chi_0}(\mathfrak{g}_0)$.

5 Associated Variety

5.1 Self-Commuting Cone

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra. The self-commuting cone X is the subvariety of $\mathfrak{g}_1$ defined by

$$X = \{x \in \mathfrak{g}_1 \mid [x, x] = 0\}$$

This cone was studied first in [17] for applications to Lie superalgebras cohomology.

Example 12 Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Then

$$X = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid AB = 0 = BA \right\}.$$

We discuss geometry of X for basic classical $\mathfrak{g}$. Let G_0 be a connected, reductive algebraic group such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$ and let B_0 be a Borel subgroup of G_0. It is clear that X is G_0-stable with respect to the adjoint action of G_0 on $\mathfrak{g}_1$. Denote by X/B_0 (resp. X/G_0) the set of B_0 (resp. G_0)-orbits in X. We will see that both sets are finite.

Denote by S_p the set of all p-tuples of linearly independent and mutually orthogonal isotropic roots and set

$$S := \coprod_{p=0}^{\mathrm{def}\,\mathfrak{g}} S_p, \quad \text{where} \quad S_0 = \{\emptyset\}.$$

Let $u = \{\alpha_1, \ldots, \alpha_p\} \in S_p$, choose non-zero $x_i \in \mathfrak{g}_{\alpha_i}$ and set

$$x_u := x_1 + \cdots + x_p.$$

Then $x_u \in X$ and it is not hard to see that a different choice of the x_i-s produces an element in the same H-orbit, where H is the maximal torus in G_0 with Lie algebra $\mathfrak{h}$. Therefore we have a well-defined map

$$\Phi : S \to X/B_0.$$

Furthermore, the Weyl group W acts on S and clearly $x_{w(u)}$ and x_u belong to the same G_0-orbit. Therefore we also have a map

$$\Psi : S/W \to X/G_0.$$

Theorem 5 *Both maps Φ and Ψ are bijections.*

The proof that Ψ is a bijection can be found in [11] and it is done by case by case inspection. It would be interesting to find a conceptual proof, using for example only properties of the root decomposition. For the proof that Φ is a bijection we refer the reader to [7]. It uses the result about Ψ and the Bruhat decomposition of G_0. It is possible that a conceptual proof of Theorem 5 is related to the following analogue of the Jacobson–Morozov theorem.

Theorem 6 *Let $\mathfrak{g}$ be a basic classical Lie superalgebra and $x \in \mathfrak{g}_1$ be an odd element such that $[x, x]$ is nilpotent. Then*

1. *If $[x, x] = 0$, then x can be embedded into an $\mathfrak{sl}(1|1)$-subalgebra of $\mathfrak{g}$.*
2. *If $[x, x] \neq 0$ then x can be embedded into an $\mathfrak{osp}(1|2)$-subalgebra of $\mathfrak{g}$.*

As a consequence of Theorem 5 we know that every $x \in X$ is G_0-conjugate to x_u for $u \in S_p$. We call the number p the rank of x. If $\mathfrak{g} = \mathfrak{gl}(m|n)$, then the rank coincides with the usual rank of the matrix. We denote by X_p the set of all elements in X of rank p. In this way we define the stratification

$$X = \coprod_{p=0}^{\operatorname{def}\mathfrak{g}} X_p,$$

where $X_0 = \{0\}$. Clearly, the Zariski closure of X_p is the disjoint union of X_q for all $q \leqslant p$.

Proposition 4 *The closure of every stratum X_p is an equidimensional variety or, equivalently, if $x, y \in X$ have the same rank, then $\dim G_0 x = \dim G_0 y$. Furthermore, if $u = \{\alpha_1, \ldots, \alpha_p\} \in S_p$ and*

$$u^{\perp} := \{\beta \in \Delta_1 \mid (\beta|\alpha_i) = 0,\ i = 1, \ldots, p\},$$

then

$$\dim G_0 x_u = \frac{1}{2}|\Delta_1 \setminus u^{\perp}| + p.$$

Proof We start with proving the second assertion. For any $x \in \mathfrak{g}_1$ consider the odd analogue of the Kostant-Kirillov form:

$$\omega(y, z) = (x, [y, z]).$$

This is an odd skew-symmetric form. It is easy to see that $\ker(\omega) = \ker(ad_x)$. Using the isomorphism $[x, \mathfrak{g}] \simeq \mathfrak{g}/\ker(ad_x)$ we can push forward ω to $[x, \mathfrak{g}]$, where it becomes non-degenerate. Since ω is odd, we obtain

$$\dim G_0 x = \dim [x, \mathfrak{g}_0] = \dim [x, \mathfrak{g}_1] = \frac{1}{2}\dim [x, \mathfrak{g}].$$

We compute $\dim[x,\mathfrak{g}]$. Let $x = x_u = x_1 + \cdots + x_p$. Fix some $y_i \in \mathfrak{g}_{-\alpha}$ and let $h_i := [x, y_i] \in \mathfrak{h}_{\alpha_i}$. Consider a generic linear combination $y = c_1 y_1 + \cdots + c_p y_p$ and set $h = [x, y]$. Then x, h, y span an $\mathfrak{sl}(1|1)$-subalgebra $\mathfrak{l}$. Let $\mathfrak{g}'$ be the direct sum of all eigenspaces of ad_h with non-zero eigenvalue and $\mathfrak{g}^h$ denote the centralizer of h. Clearly, $\mathfrak{g}'$ and $\mathfrak{g}^h$ are $\mathfrak{l}$-stable. Furthermore, it is easy to see that

$$\operatorname{sdim}\mathfrak{g}' = 0, \quad [x,\mathfrak{g}'] = \mathfrak{g}' \cap \ker \mathrm{ad}_x \quad \text{hence} \quad \dim[x,\mathfrak{g}'] = \frac{1}{2}\dim\mathfrak{g}' = \dim\mathfrak{g}'_1.$$

For generic $c_1, \dots, c_p$ we have

$$\mathfrak{g}'_1 = \bigoplus_{\beta\in\Delta_1\setminus u^\perp} \mathfrak{g}_\beta.$$

Therefore we obtain

$$\dim[x,\mathfrak{g}'] = |\Delta_1 \setminus u^\perp|.$$

On the other hand, a simple calculation shows that

$$[\mathfrak{g}^h, x] = [\mathfrak{l}, x] \oplus [\mathfrak{h}, x] = \bigoplus_{i\leqslant p}(kx_i \oplus kh_i).$$

Therefore $\dim[\mathfrak{g}^h, x] = 2p$.

$$\dim G_0 x = \frac{1}{2}(\dim[x,\mathfrak{g}'] + \dim[x,\mathfrak{g}^h]) = \frac{1}{2}|\Delta_1 \setminus u^\perp| + p.$$

The first assertion follows from the fact that for any two $u, u' \in S_p$ there exists $w \in W$ such that $wu' \subset u \cup -u$. This fact is established by case by case inspection.

Corollary 5 *X is an equidimensional variety.*

5.2 Functor F_x

Let $\mathfrak{g}$ be an arbitrary superalgebra and $x \in \mathfrak{g}_1$ satisfy $[x,x] = 2x^2 = 0$. For any $\mathfrak{g}$-module M we have $x^2M = 0$ and therefore can define the cohomology

$$M_x := \ker x / xM.$$

Lemma 7

1. $(M \oplus N)_x = M_x \oplus N_x$.
2. $\operatorname{sdim}(M_x) = \operatorname{sdim}(M)$ *(superdimension).*
3. $M_x^* \simeq (M_x)^*$.
4. *We have a canonical isomorphism* $(M \otimes N)_x \simeq M_x \otimes N_x$.

Proof 1, 2 and 3 are straightforward. To prove 4 consider M as a $k[x]/(x^2)$-module. We have the obvious map $M_x \otimes N_x \to (M \otimes N)_x$. On the other hand, we have decompositions $M = M_x \oplus F$ and $N = N_x \oplus F'$, where F and F' are free $k[x]/(x^2)$-modules.

$$M \times N \simeq M_x \otimes N_x \oplus (F \otimes N \oplus M \otimes F').$$

Since a tensor product of any $k[x]/(x^2)$-module with a free $k[x]/(x^2)$-module is free we obtain the isomorphism $(M \otimes N)_x \simeq M_x \otimes N_x$.

Applying the above construction to the adjoint representations we get

$$\mathfrak{g}_x = \ker(ad_x)/[x,\mathfrak{g}] = \mathfrak{g}^x/[x,\mathfrak{g}].$$

Exercise Check that $[x,\mathfrak{g}]$ is an ideal in $\mathfrak{g}^x$. Hence $\mathfrak{g}_x$ is a Lie superalgebra.

Let M be a $\mathfrak{g}$-module. Then we have a canonical $\mathfrak{g}_x$-module structure on M_x. Indeed, it is easy to check that both $\ker x$ and xM are $\mathfrak{g}^x$-stable, For any $y \in \mathfrak{g}$ we have $[x,y]m = xym \in [\mathfrak{g},x]m$. Therefore $[\mathfrak{g},x]\ker x \subset xM$ and the induced action of $[\mathfrak{g},x]$ on M_x is trivial. Thus, we obtain the following proposition.

Proposition 5 *Let $\mathfrak{g}$ be a superalgebra and x be an odd self-commuting element. The assignment $M \to M_x$ induces a tensor functor F_x from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules.*

Remark 7 F_x is neither left nor right exact.
Note that if x,y lie in the same orbit of G_0 then $\mathfrak{g}_x$ and $\mathfrak{g}_y$ are isomorphic Lie superalgebras. Moreover, if $\mathfrak{g}$ is basic, then $\mathfrak{g}_x$ is constant on each stratum $X_p \subset X$.

Lemma 8 *Let $\mathfrak{g}$ be a basic Lie superalgebra, then $\mathfrak{g}_x \simeq \mathfrak{g}_y$ if $x,y \in X_p$.*

Proof Let $x = x_u = x_1 + \cdots + x_p$, y_i and h_i be as in the proof of Proposition 4. Let $\mathfrak{k}$ be the subalgebra generated by x_i, y_i, h_i for all $i \leqslant p$. Then it follows from the proof of Proposition 4 that $\mathfrak{g}_x$ is the quotient of the centralizer of $\mathfrak{k}$ by the center of $\mathfrak{k}$. Note that by the last remark in the same proof we know that y is G_0-conjugate to x_v for some $v \in u \cup -u$. It follows that $\mathfrak{g}_{x_u} = \mathfrak{g}_{x_v}$. Hence the statement.

Exercise Let $\mathfrak{g}$ be one of the basic superalgebras and $x \in X_p$, check that $\mathfrak{g}_x$ is the following:

- $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{g}_x = \mathfrak{gl}(m-p|n-p)$;
- $\mathfrak{g} = \mathfrak{osp}(m|2n)$, $\mathfrak{g}_x = \mathfrak{osp}(m-2p|2n-2p)$;
- $\mathfrak{g} = AG_2, p = 1$, $\mathfrak{g}_x = \mathfrak{sl}_2$;
- $\mathfrak{g} = AB_3, p = 1$, $\mathfrak{g}_x = \mathfrak{sl}_3$;
- $\mathfrak{g} = D(2,1;a), p = 1$, $\mathfrak{g}_x = \mathfrak{sl}_2$.

Consider $\mathcal{U}(\mathfrak{g})$ as the adjoint $\mathfrak{g}$-module. Then it is not difficult to see that $(\mathcal{U}(\mathfrak{g}))_x \simeq \mathcal{U}(\mathfrak{g}_x)$, hence we have a projection $f_x : \mathcal{U}(\mathfrak{g})^{ad(x)} \longrightarrow \mathcal{U}(\mathfrak{g}_x)$. Note that $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})^{ad(x)}$ and the restriction of f_x to $\mathcal{Z}(\mathfrak{g})$ defines a homomorphism $\phi_x : \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{g}_x)$.

We are interested in the dual map.

$$\check{\phi}_x : \mathrm{Hom}(\mathcal{Z}(\mathfrak{g}_x), k) \longrightarrow \mathrm{Hom}(\mathcal{Z}(\mathfrak{g}), k).$$

Theorem 7 *Let $\psi \in \mathrm{Hom}(\mathcal{Z}(\mathfrak{g}_x), k)$, $x \in X_p$, then*

1. $\mathrm{at}(\check{\phi}_x(\psi)) = p + \mathrm{at}(\psi)$.
2. *The image of $\check{\phi}_x$ consists of all central characters of atypicality degree greater or equal than p.*
3. *If $\mathrm{at}(\chi) \geqslant p$, then the fiber $\check{\phi}_x^{-1}(\chi)$ consists of one or two points.*

Proof Let $x = x_u$ where $u = \{\alpha_1, \ldots, \alpha_p\}$. It is always possible to find a triangular decomposition such that $\alpha_1, \ldots, \alpha_p$ are simple roots. We consider the Harish-Chandra map $HC : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ related to this particular triangular decomposition and the analogous map $HC_x : \mathcal{Z}(\mathfrak{g}_x) \longrightarrow S(\mathfrak{h}_x)$ with dual map denoted by θ_x. Let

$$\mathfrak{h}_u := \bigcap_{i=1}^{p} \ker \alpha_i,$$

from the proof of Lemma 8 we have

$$\mathfrak{h}_x = \mathfrak{h}_u / \operatorname{span}\{h_1, \ldots, h_p\}.$$

Let $i_x : \mathfrak{h}_x^* \to \mathfrak{h}_u^*$ be the map dual to the natural projection. We claim the existence of the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_x^* & \xrightarrow{\theta_x} & Specm\mathcal{Z}(\mathfrak{g}_x) \\ \downarrow{\scriptstyle 1_x} & & \downarrow{\scriptstyle \check{\phi}_x} \\ \mathfrak{h}_u^* & \xrightarrow{\theta} & Specm\mathcal{Z}(\mathfrak{g}_x) \end{array}$$

Indeed, for any $\mu \in \mathfrak{h}_x^*$ let $\lambda = i_x(\mu)$ and $M = L_{\mathfrak{b}}(\lambda)$ be the irreducible module with highest weight λ (may be infinite-dimensional). The highest weight vector of this module belongs to M_x and therefore M_x contains a $\mathfrak{g}_x$-submodule which admits central character χ_μ while M admit central character χ_λ. That implies $\check{\phi}_x(\chi_\mu) = \chi_\lambda$.

2 is a direct consequence of 1 and 3 is obtained by case by case inspection using Corollary 2.

Exercise If a $\mathfrak{g}$-module M admits central character χ, then M_x is a sum of modules which admit central characters in $\check{\phi}_x^{-1}(\chi)$.

Corollary 6 *Assume that M admits central character χ with atypicality degree p.*

(a) *$F_x(M) = 0$ for any $x \in X_q$ such that $q > p$. In particular, if χ is typical, then $F_x(M) = 0$ for any $x \neq 0$.*

(b) *If* $x \in X_p$, *then* $F_x(M)$ *is a direct sum of* $\mathfrak{g}_x$*-modules with typical central character.*

Conjecture 1 Let $\mathfrak{g}$ be a basic Lie superalgebra. If M is a finite dimensional simple $\mathfrak{g}$-module, then M_x is a semisimple $\mathfrak{g}_x$-module.
By Corollary 6 Conjecture 1 is true when the rank of x equals the atypicality degree of M. In particular, it holds if the rank of x equals the defect of $\mathfrak{g}$. In this case $\mathfrak{g}_x$ is either a Lie algebra or $\mathfrak{osp}(1|2k)$. For general x the conjecture is proven for $\mathfrak{g} = \mathfrak{gl}(m|n)$ in [21].

5.3 Associated Variety

Definition 10 Let $\mathfrak{g}$ be a Lie superalgebra, X self-commuting cone and M a $\mathfrak{g}$-module. The *associated variety* of M is

$$X_M = \{x \in X \mid M_x \neq 0\}.$$

Exercise In general X_M may be not closed, see [7]. Prove that if M is finite dimensional then X_M is a closed G_0 invariant subvariety of X. If M is an object of the category $\mathcal{O}$, then X_M is B_0-invariant.

The following properties of X_M follow immediately from the corresponding properties of F_x

1. $X_{M\oplus N} = X_M \cup X_N$.
2. $X_{M\otimes N} = X_M \cap X_N$.
3. $X_{M^*} = X_M$.

Note also that Corollary 6 implies the following:

Proposition 6 *Let* $\mathfrak{g}$ *be a basic superalgebra. If* M *admits a central character* χ *of atypicality degree* p*, then* X_M *belongs to the Zariski closure of* X_p*.*

The following result has a rather complicated proof which can be found in [42] for classical superalgebras and in [14, 29] for exceptional.

Theorem 8 *Let* $\mathfrak{g}$ *be a classical Lie superalgebra and* L *be a finite dimensional simple* $\mathfrak{g}$*-module of atypicality degree* p*. Then the associated variety* X_L *coincides with the Zariski closure of* X_p*.*

Finally, let us mention that to every $\mathfrak{g}$-module M integrable over G_0 we can associate a G_0-equivariant coherent sheaf $\mathcal{M}$ on X in the following way. Let $k[X]$ denote the ring of regular functions on X and $k[X] \otimes M$ be a free $k[X]$-module. Define $\partial : k[X] \otimes M \to k[X] \otimes M$ by setting

$$\partial f(x) = xf(x) \quad \text{for every} \quad x \in X.$$

Then $\partial^2 = 0$ and the cohomology of ∂ is a $k[X]$-module $\mathcal{M}$. It is clear that supp$\mathcal{M} \subset X_M$ and it is proven in [11] that supp$\mathcal{M} = X$ if $X_M = X$.

Conjecture 2 supp$\mathcal{M} = X$.

5.4 Some Applications

Conjecture 3 (Kac–Wakimoto,[25]) Let $\mathfrak{g}$ be a basic Lie superalgebra and L be a simple finite-dimensional $\mathfrak{g}$-module. Then $\operatorname{sdim} L \neq 0$ if and only the degree of atypicality of L equals the defect of $\mathfrak{g}$.

Kac-Wakimoto conjecture was verified for classical superalgebras in [42] and for exceptional in [29]. Here we can give a simple proof in one direction. Since F_x preserves superdimension, Corollary 6 (a) implies the following statement.

Corollary 7 *Let M be a finite-dimensional $\mathfrak{g}$-module which admits central character χ. If* $\operatorname{at}(\chi) < \operatorname{def}\mathfrak{g}$ *then* $\operatorname{sdim} M = 0$.

Let $k = \mathbf{C}$, M be a finite dimensional $\mathfrak{g}$-module, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Define a function p_M on $\mathfrak{h}$ by setting

$$p_M(h) = str_M(e^h).$$

It is clear that p_M is analytic. Consider the Taylor series for p_M at $h = 0$

$$p_M(h) = \sum_{i=0}^{\infty} p_i(h),$$

where p_i is a homogeneous polynomial of degree i. The order of zero is the minimal i such that $p_i \neq 0$.

The following result can be considered as a generalization of the Kac-Wakimoto conjecture.

Theorem 9 ([11]) *Assume that $\mathfrak{g}$ does not have non-isotropic odd roots and let M be simple. Then the order of $p_M(h)$ equals the codimension of X_M in X.*

6 Classification of Blocks

6.1 General Results

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. Recall that we denote by $\mathcal{F}$ the category of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$.

Lemma 9 *Let $\mathfrak{g}_0$ be reductive and $\mathfrak{g}_1$ be a semisimple $\mathfrak{g}$-module. Then the category $\mathcal{F}$ has enough projective and injective objects. Moreover, $\mathcal{F}$ is a Frobenius category, i.e. every projective module is injective and vice versa.*

Proof To prove the first assertion note that if M is a simple $\mathfrak{g}_0$-module, then by Frobenius reciprocity the induced module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M$ is projective in $\mathcal{F}$ and the coinduced module $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), M)$ is injective. For the second assertion use the following.

Exercise Show the isomorphism of $\mathfrak{g}$-modules

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M \simeq \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), M \otimes \Lambda^{top}\mathfrak{g}_1).$$

From now on we assume that $\mathfrak{g}$ is basic. For a central character $\chi : \mathcal{Z}(\mathfrak{g}) \to k$ let $\mathcal{F}_\chi$ be the subcategory of $\mathcal{F}$ consisting of modules which admit generalized central character χ.

Lemma 10

(a) We have a decomposition of $\mathcal{F}$ into a direct sum of subcategories

$$\mathcal{F} = \bigoplus_\chi \mathcal{F}_\chi.$$

(b) For every χ with non-empty $\mathcal{F}_\chi$ we have a decomposition

$$\mathcal{F}_\chi = \mathcal{F}_\chi^+ \oplus \mathcal{F}_\chi^-$$

such that $\mathcal{F}_\chi^- = \Pi\mathcal{F}_\chi^+$. (Recall that Π is the change of parity functor.)

Proof

(a) If M is finite-dimensional, then $\mathcal{Z}(\mathfrak{g})$ acts locally finitely on M, so M decomposes into the direct sum of generalized weight spaces of $\mathcal{Z}(\mathfrak{g})$.
(b) Every module $M \in \mathcal{F}$ is $\mathfrak{h}$-semisimple. Thus, M has a weight decomposition $M = \oplus M_\mu$. One can define a function $p : \mathfrak{h}^* \to \mathbf{Z}_2$ such that $p(\lambda + \alpha) = p(\lambda)$ for any even root α and $p(\lambda + \alpha) = p(\lambda) + 1$ for any odd root α. Set

$$M_\mu^+ := \begin{cases} (M_\mu)_0 \text{ if } p(\mu) = 0 \\ (M_\mu)_1 \text{ if } p(\mu) = 1 \end{cases}, \quad M_\mu^- := \begin{cases} (M_\mu)_1 \text{ if } p(\mu) = 1 \\ (M_\mu)_1 \text{ if } p(\mu) = 0 \end{cases},$$

and let $M^\pm := \oplus M_\mu^\pm$. Then $M^\pm$ are submodules of M and M is the direct sum $M^+ \oplus M^-$. Therefore we can define $\mathcal{F}_\chi^\pm$ as the full subcategory of $\mathcal{F}_\chi$ consisting of modules M such that $M^\mp = 0$.

We call *principal block* the subcategory $\mathcal{F}_{\chi_0}^+$ which contains the trivial module.

Theorem 10

1. *The subcategories* $\mathcal{F}_\chi^\pm$ *are indecomposable.*
2. *If* $\mathfrak{g} = \mathfrak{gl}(m|n)$ *(resp.* $\mathfrak{osp}(2m+1|2n)$*), and* $p = \mathrm{at}\chi$*, then* $\mathcal{F}_\chi^\pm$ *is equivalent to the principal block of* $\mathfrak{gl}(p|p)$ *(resp.* $\mathfrak{osp}(2p+1|2p)$*).*
3. *If* $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ *then* $\mathcal{F}_\chi^\pm$ *is equivalent to the principal block of* $\mathfrak{osp}(2p|2p)$ *or* $\mathfrak{osp}(2p+2|2p)$*.*
4. *For exceptional superalgebras* $D(2,1,a)$ AG_2 *or* AB_3 $\mathcal{F}_\chi^\pm$ *with atypical* χ *is equivalent to the principal block of* $\mathfrak{gl}(1|1)$ *or* $\mathfrak{osp}(3|2)$*.*

In these notes we give the proof for $\mathfrak{g} = \mathfrak{gl}(m|n)$. One can find the proof for all classical superalgebras in [19] and for exceptional in [14] and [29].

Remark 8 If χ is typical, then $\mathcal{F}_\chi^\pm$ is semisimple and has one up to isomorphism simple object.

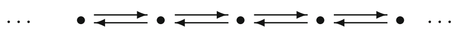
Remark 9 The problem of classifying blocks in the category $\mathcal{O}$ is still open. In contrast with $\mathcal{F}$, there are infinitely many non-equivalent blocks of given atypicality degree, [7].

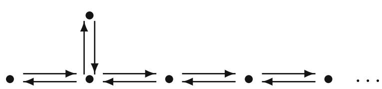
6.2 Tame Blocks

Using general approach, see [12], every block is equivalent to the category of finite-dimensional representations of a certain quiver with relations. This approach for Lie superalgebras was initiated by J. Germoni, [13]. In this method an important role is played by the dichotomy: wild vs tame categories. Roughly speaking, in tame categories, we can describe indecomposable modules by a finite number of parameters, while in wild categories it is impossible.

The following statement was originally conjectured by Germoni and now follows from Theorem 10 and results in [14, 18, 29] and [33].

Proposition 7 *A block* $\mathcal{F}_\chi^\pm$ *is tame if and only if* $\mathrm{at}(\chi) \leqslant 1$*. An atypical tame block is equivalent to the category of finite-dimensional representations of one of the following two quivers:*

$A(\infty)$:


$D(\infty)$:


with relations $ba = cd$, $ac = 0 = db$ for any subquiver isomorphic to:

$$\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet \underset{d}{\overset{c}{\rightleftarrows}} \bullet$$

Remark 10 It follows from Corollary 6 that for any $x \in X$ the functor F_x maps a block $\mathcal{F}_\chi^\pm$ to

$$\bigoplus_{\tau \in \check{\phi}_x^{-1}} \mathcal{F}_\tau.$$

There is some evidence that a more subtle relation is true, namely

$$F_x(\mathcal{F}_\chi^\pm) = \bigoplus_{\tau \in \check{\phi}_x^{-1}} \mathcal{F}_\tau^\pm.$$

In the case of the most atypical block it is possible to show that the superdimension is constant on a Zariski open subset of simple modules in the block.

6.3 *Proof of Theorem 10 for* $\mathfrak{gl}(m|n)$

In this subsection $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{b} = \mathfrak{b}_d$ is the distinguished Borel, and we skip the low index in the notation for simple, Kac and projective modules. For instance $L(\lambda) := L_\mathfrak{b}(\lambda)$. The weight

$$\lambda = c_1\epsilon_1 + \cdots + c_m\epsilon_m + d_1\delta_1 + \cdots + d_n\delta_n = (c_1, \ldots, c_m \,|\, d_1, \ldots, d_n)$$

is integral dominant if and only if $c_i - c_{i+1} \in \mathbf{Z}_+$, $d_j - d_{j+1} \in \mathbf{Z}_+$ for all $i \leqslant m-1$, $j \leqslant n-1$. We assume in addition that $c_i, d_j \in \mathbf{Z}$.[2]

For the Weyl vector we use

$$\rho = (m-1, \ldots, 1, 0 \,|\, 0, -1, \ldots, -n).$$

In [2] Brundan and Stroppel introduced an extremely useful way to represent weights by the so called *weight diagrams*.

Let λ be a dominant integral weight, and

$$\lambda + \rho = (a_1, \ldots, a_m \,|\, b_1, \ldots, b_n), \quad a_i > a_{i+1},\ b_j > b_{j+1}.$$

[2]This assumption is not essential and can be dropped. It is here only for convenience of notations.

The weight diagram f_λ is the map $\mathbf{Z} \to \{\circ, >, <, \times\}$ defined as follows

$$f_\lambda(t) = \begin{cases} \circ \text{ if } \ a_i \neq t, b_j \neq -t \ \text{ for all } \ i = 1, \dots, m, j = 1, \dots n; \\ > \ \text{ if } \ a_i = t \ \text{ for some } \ i, \quad b_j \neq -t \ \text{ for all } j = 1, \dots, n; \\ < \ \text{ if } \ b_i = -t \text{ for some } i, \quad a_j \neq t \ \text{ for all } j = 1, \dots, m; \\ \times \text{ if } \ a_i = t, b_j = -t \ \text{ for some } \ i, j. \end{cases}$$

We represent f_λ by a picture on the number line with position $t \in \{0, \pm 1, \pm 2, \pm 3, \dots\}$ filled with $f_\lambda(t)$. We consider $\circ$ as a placeholder for an empty position. The *core* diagram $\bar{f}_\lambda$ is obtained from f_λ by removing all $\times$. We call $>$ and $<$ core symbols.

Example 13 Take the adjoint representation of $\mathfrak{gl}(2|3)$. Then

$$\lambda = (1, 0 \,|\, 0, 0, -1), \quad \lambda + \rho = (2, 0 \,|\, 0, -1, -3)$$

and f_λ can be represented by the picture $\circ \; \times \; < \; > \; < \; \circ$
where all negative positions and all positions $t > 3$ are empty. The core diagram is

$\circ \; \circ \; < \; > \; < \; \circ$

Exercise Check that

- The degree of atypicality of λ equals the number of $\times$-s in the weight diagram f_λ.
- Core diagrams parametrize blocks, namely, $\chi_\lambda = \chi_\mu$ if and only if $\bar{f}_\lambda = \bar{f}_\mu$.

The above exercise implies that blocks $\mathcal{F}^+_\chi$ can be parametrized by weight diagrams without $\times$-s. We use the notation $f_\chi := \bar{f}_\lambda$ for any λ such that $\chi = \chi_\lambda$.

Definition 11 We define the following operations on a weight diagram:

- *Left simple move*: Move $>$ one position to the right or move $<$ one position to the left.
- *Right simple move*: Move $>$ one position to the left or move $<$ one position to the right.

In this definition we assume that $\times$ is the union $><$, and we can split it or join $><$ into $\times$.

Example 14 Let f be as in the previous example: $\circ \; \times \; < \; > \; < \; \circ$
Then the following are possible right simple moves

1. Moving the rightmost $<$ one position right: $\circ \; \times \; < \; > \; \circ \; <$
2. Moving the leftmost $<$ one position right (new $\times$ in position 2 appears):

   $\circ \; \times \; \circ \; \times \; \circ \; <$

3. Moving $>$ one position left (new $\times$ in position 1 appears): $\circ \; \times \; \times \; \circ \; \circ \; <$
4. Splitting $\times$. Here we can not move $<$ to the right since it does not produce a valid diagram. But we can move $>$ to the left. $> \; < \; < \; > \; < \; \circ$

Let V and V^* denote the natural and conatural representations respectively.

Lemma 11 *If $K(\lambda)$ is the Kac module with highest weight λ, then $K(\lambda) \otimes V$ (resp. $K(\lambda) \otimes V^*$) has a filtration by Kac modules $K(\mu)$ for all f_μ obtained from f_λ by a left (resp. right) simple move.*

Proof Recall that $K(\lambda) = U(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}(1))} L_{\mathfrak{b}_0}(\lambda)$. Hence

$$K(\lambda) \otimes V \simeq U(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}(1))} (L_{\mathfrak{b}_0}(\lambda) \otimes V).$$

Since the weights of V are $\{\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n\}$, then $K(\lambda) \otimes V$ has a filtration by $K(\mu)$ for all dominant μ in $\{\lambda + \epsilon_1, \dots, \lambda + \epsilon_m, \lambda + \delta_1, \dots, \lambda + \delta_n\}$. The corresponding weight diagrams are exactly those obtained from f_λ by a right simple move. The case of $K(\lambda) \otimes V^*$ is similar.

Next step is to define translation functors inspired by translation functors in classical category $\mathcal{O}$. For every M in $\mathcal{F}$ we denote by $(M)_\tau$ the projection on the block $\mathcal{F}_\tau^+$. Then the translation functors between $\mathcal{F}_\chi^+$ and $\mathcal{F}_\tau^+$ are defined by

$$T_{\chi,\tau} : \mathcal{F}_\chi^+ \longrightarrow \mathcal{F}_\tau^+, \qquad M \mapsto (M \otimes V)_\tau$$

$$T^*_{\tau,\chi} : \mathcal{F}_\tau^+ \longrightarrow \mathcal{F}_\chi^+, \qquad M \mapsto (M \otimes V^*)_\chi$$

Exercise Show that:

1. The functors $T_{\chi,\tau}$, $T^*_{\tau,\chi}$ are exact.
2. $T^*_{\tau,\chi}$ is left and right adjoint to $T_{\chi,\tau}$.
3. $T_{\chi,\tau}$, $T^*_{\tau,\chi}$ map projective modules to projective modules.
4. Assume that $T_{\chi,\tau}$ and $T^*_{\tau,\chi}$ establish a bijection between simple modules in both blocks, then they establish an equivalence $\mathcal{F}_\chi^+ \cong \mathcal{F}_\tau^+$ of abelian categories.
5. If $T_{\chi,\tau}$ and $T^*_{\tau,\chi}$ establish a bijection between Kac modules in both blocks, they also establish a bijection between simple modules.

Proposition 8 *Assume that* $\mathrm{at}(\chi) = \mathrm{at}(\tau)$ *and f_τ is obtained from f_χ by a left (resp. right) simple move, then $T_{\chi,\tau} : \mathcal{F}_\chi \to \mathcal{F}_\tau$ (resp. $T^*_{\chi,\tau} : \mathcal{F}_\chi \to \mathcal{F}_\tau$) is an equivalence of abelian categories.*

Proof Without loss of generality we do the proof in the case of a left move. Using Lemma 11 one can easily check that $T_{\chi,\tau}$ and $T^*_{\tau,\chi}$ provide a bijection between Kac modules in both blocks. Hence the statement follows from the preceding exercise.

Definition 12 A weight λ is *stable* if all $\times$-s in the weight diagram f_λ stay to the left of $<$ and $>$.

Introduce an order on the set of weights in the same block by setting $\nu \leqslant \mu$ if $\mu - \nu$ is a sum of positive roots. One can easily see that $\nu < \mu$ if ν is obtained from μ by moving some $\times$ to the left. Therefore if μ is stable and $\nu < \mu$, then ν is also stable. We denote by $\mathcal{F}_\chi^\mu$ the full subcategory of $\mathcal{F}_\chi^+$ whose simple constituents $L(\lambda)$ satisfy $\lambda \leqslant \mu$. We call $\mathcal{F}_\chi^\mu$ a *truncated* block.

Proposition 9 *Let μ be a stable weight of atypicality degree p, $\chi = \chi_\mu$. Let $s \in \mathbf{Z}$ be minimal such that $f_\chi(s) \neq \circ$. Let ν be the weight of the principal block of $\mathfrak{gl}(p|p)$ with weight diagram*

$$f_\nu = \begin{cases} \times \text{ if } s-p \leqslant t \leqslant s-1 \\ \circ \text{ otherwise} \end{cases} .$$

Then $\mathcal{F}_\chi^\mu$ is equivalent to the truncation $\mathcal{F}_0^\nu$ of the principal block of $\mathfrak{gl}(p|p)$.

Proof (Sketch) We just explain how to define the functors establishing the equivalence. Let $\mu = (c_1, \ldots, c_m \mid d_1, \ldots, d_n)$. Start with defining the parabolic subalgebra

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta'} \mathfrak{g}_\alpha,$$

where

$$\Delta' = \Delta^+ \cup \{\epsilon_i - \epsilon_j \mid m-p < j < i \leqslant m\} \cup \{\delta_i - \delta_j \mid 1 \leqslant j < i \leqslant p\}$$
$$\cup \{\delta_i - \epsilon_j \mid 1 \leqslant i \leqslant p,\ m-p < j \leqslant m\}.$$

in other words $\mathfrak{p}$ consists of block matrices of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

where the middle square block has size $p|p$. Set

$$\mathfrak{l} := \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\}, \quad \mathfrak{m} := \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Clearly, $\mathfrak{p}$ is a semi-direct product of the subalgebra $\mathfrak{l} \simeq \mathfrak{gl}(p|p) \oplus k^{m+n-2p}$ and the nilpotent ideal $\mathfrak{m}$. Consider the functor $R : \mathcal{F}_\chi^\mu \to \mathcal{F}_0^\nu$ defined by $R(M) = M^{\mathfrak{m}}$. Then its left adjoint $I : \mathcal{F}_0^\nu \to \mathcal{F}_\chi^\mu$ maps a $\mathfrak{gl}(p|p)$-module N to the maximal finite-dimensional quotient of the parabolically induced module

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (N \boxtimes C_\mu),$$

where C_μ is the one-dimensional representation of k^{m+n-2p} with weight

$$\mu := (c_1, \ldots, c_{m-p} \mid d_{p+1}, \ldots, d_n).$$

It suffices to show that R and I are exact and establish the bijection between simple modules. Indeed, the exactness of R can be proven by noticing that R picks up the eigenspace of k^{m+n-2p} with weight μ. Furthermore, if $L(\lambda)$ is a simple module in $\mathcal{F}_\chi^\mu$, then

$$\lambda = (c_1, \dots, c_{m-p}, t_1, \dots, t_p \mid -t_p, \dots, -t_1, d_{p+1}, \dots, d_n)$$

for some $t_1, \dots, t_p$. It is easy to see that $R(L(\lambda)) = L(\lambda')$, where $\lambda' = (t_1, \dots, t_p \mid -t_p, \dots, -t_1)$ and that $I(L(\lambda')) = L(\lambda)$. The exactness of I can be now proven by induction on the length of a module.

The following combinatorial lemma is straightforward.

Lemma 12 *For any weight diagram f_μ there exists a stable weight diagram $f_{\mu'}$ obtained from f_μ by a sequence of simple moves which do not change the degree of atypicality.*

Now we are ready to prove Theorem 10. Indeed, let $\mathcal{F}_\chi^+$ be a block with atypicality degree p. Lemma 12 and Proposition 8 imply that any truncated block $\mathcal{F}_\chi^\mu$ is equivalent to a stable truncated block of the same atypicality. Hence by Proposition 9 $\mathcal{F}_\chi^\mu$ is equivalent to some truncation of a principal block of $\mathfrak{gl}(p|p)$. Taking the direct limit of $\mathcal{F}_\chi^\mu$ we obtain equivalence between $\mathcal{F}_\chi^+$ and the principal block of $\mathfrak{gl}(p|p)$.

It remains to prove the indecomposability of the principal block of $\mathfrak{gl}(p|p)$. Note that $f_{\nu'}$ is obtained from f_ν by moving a $\times$ one position left, then $[K(\nu) : L(\nu')] = 1$. Since $K(\nu)$ is indecomposable, $L(\nu)$ lies in the indecomposable block containing $L(\nu')$. Since any diagram in the principal block can be obtained from the fixed one by repeatedly moving $\times$-s one position left or right, the statement follows.

6.4 *Calculating the Kazhdan-Lusztig Multiplicities*

We would like to mention without proof other applications of weight diagrams and translation functors. We still assume that $\mathfrak{g} = \mathfrak{gl}(m|n)$. Then the category $\mathcal{F}$ is a highest weight category, [47], where standard objects are Kac modules. In particular, we have BGG reciprocity for the multiplicities:

$$[K(\lambda) : L(\mu)] = [P(\mu) : K(\lambda)],$$

where $P(\mu)$ denotes the projective cover of $L(\mu)$. It is useful to compute these multiplicities. It was done in [40] and in [1] by different methods. The answer is very easy to formulate in terms of weight diagrams.

Let f be some weight diagram. We decorate it with caps by the following rule:

- Every cap has left end at $\times$ and right end at $\circ$.
- Every $\times$ is engaged in some cap, so the number of caps equals the number of crosses.

- There are no $\circ$ under a cap.
- Caps do not cross.

We say that f' is *adjacent* to f if f' is obtained from f by moving one $\times$ from the left end of its cap to the right end. We say that f' is *adjoint* to f if f' is obtained from f by moving several $\times$ from the left end of its cap to the right end. We assume that f is adjoint to itself. If f has p $\times$-s, then it has exactly p adjacent diagrams and 2^p adjoint diagrams

Theorem 11 **([1, 33])**

$$Ext^1_{\mathcal{F}}(L(\lambda), L(\mu)) = \begin{cases} k \text{ if } f_\lambda \text{ is adjacent to } f_\mu \text{ or } f_\mu \text{ is adjacent to } f_\lambda \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 12 **([1])**

$$[P(\lambda) : K(\mu)] = \begin{cases} 1 \text{ if } f_\mu \text{ is adjoint to } f_\lambda \\ 0 \text{ otherwise.} \end{cases}$$

7 Supergeometry and Borel–Weil–Bott Theorem

7.1 Supermanifolds

The notion of supermanifold exists in three flavors: smooth, analytic and algebraic. We concentrate here on the algebraic version. The main idea is the same: we define first superdomains and then glue them together.

By a *superdomain* we understand a pair $(U_0, \mathcal{O}_U)$, where U_0 is an affine manifold and $\mathcal{O}_U$ is the sheaf of superalgebras isomorphic to

$$\Lambda(\xi_1, \ldots, \xi_n) \otimes \mathcal{O}_{U_0},$$

$\mathcal{O}_{U_0}$ denotes the structure sheaf on U_0. The dimension of U is $(m|n)$ where $m = \dim U_0$.

For example, the affine superspace $\mathbb{A}^{m|n}$ is a pair $(\mathbb{A}^m, \mathcal{O}_{\mathbb{A}^{m|n}})$. The ring of global sections of $\mathcal{O}(\mathbb{A}^{m|n})$ is a free supercommutative ring $k[x_1 \ldots x_m, \xi_1, \ldots \xi_n]$. If we work in local coordinates, then we use roman letters for even variables, greek letters for odd ones.

Definition 13 A supermanifold is a pair $(X_0, \mathcal{O}_X)$ where X_0 is a manifold and $\mathcal{O}_X$ is a sheaf locally isomorphic to $(U_0, \mathcal{O}_U)$ for a superdomain U. The manifold X_0 is called the *underlying manifold* of X and $\mathcal{O}_X$ is called the *structure sheaf*.

One way to define a supermanifold is by introducing local charts U_i and then gluing them together.

Example 15 Consider two copies of $\mathbb{A}^{1|2}$ with coordinates (x, ξ_1, ξ_2) and (y, η_1, η_2). We give the gluing by setting:

$$y = x^{-1} + \xi_1\xi_2, \quad \eta_1 = x^{-1}\xi_1, \quad \eta_2 = \xi_2.$$

Example 16 Let X_0 be a manifold, $\mathcal{V}$ be a vector bundle on X_0 and $\mathcal{O}_X$ is the sheaf of sections of the exterior algebra bundle $\Lambda(\mathcal{V})$. In particular, X_0 with the sheaf of forms Ω_{X_0} is a supermanifold.

Given the supermanifold X, we have the canonical embedding $X_0 \to X$ and the corresponding morphism of structure sheaves $\mathcal{O}_X \to \mathcal{O}_{X_0}$. Denote by I_{X_0} the kernel of this map. It is not difficult to see that I_{X_0} is the nilpotent ideal generated by all odd sections of $\mathcal{O}_X$. Consider the filtration

$$\mathcal{O}_X \supset I_{X_0} \supset I_{X_0}^2 \supset \dots.$$

Then $Gr(X) := (X_0, Gr\mathcal{O}_X)$ is again a supermanifold. One can identify $Gr(X)$ with $(X_0, \Gamma(\Lambda(N^*_{X_0}X))$, where $N^*_{X_0}X$ denotes the conormal bundle for $X_0 \subset X$.

A supermanifold X is called *split* if it is isomorphic to $Gr(X)$. In the category of smooth supermanifolds all supermanifolds are split but this is not true for algebraic supermanifolds.

Exercise Show that any supermanifold of dimension $(m|1)$ is split. Is the supermanifold defined in Example 15 split?

Another way to define a supermanifold is to use the functor of points, which is a functor from the category (Salg) of commutative superalgebras to the category (Sets). For general definitions see [3]. Let us illustrate this approach with the following example.

Example 17 We define the projective superspace $X = \mathbf{P}^{1|1}$ as follows. For a commutative superalgebra $\mathcal{A}$ the set of $\mathcal{A}$-points is the set of all submodules $\mathcal{A}^{1|0} \subset \mathcal{A}^{2|1}$. This is the set of all triples (z_1, z_2, ζ) with $z_1, z_2 \in \mathcal{A}_0$ and $\zeta \in \mathcal{A}_1$, such that at least one of z_1, z_2 is invertible, modulo rescaling by an invertible element of $\mathcal{A}_0$. This supermanifold has two affine charts $\{(1, x, \xi)\}$ and $\{(y, 1, \eta)\}$ with gluing functions $\xi = x^{-1}\eta$, $y = x^{-1}$.

Exercise Check that in Example 17 $X_0 = \mathbf{P}^1$ and $\mathcal{O}_X \simeq \mathcal{O} \oplus \Pi\mathcal{O}(-1)$.

8 Algebraic Supergroups

An affine supermanifold G equipped with morphisms $m : G \times G \to G$, $i : D \to G$ and $e : \{point\} \to G$ satisfying usual group axioms is called an affine algebraic supergroup. We skip the word “affine” in what follows.

The ring $\mathcal{O}(G)$ of global sections of $\mathcal{O}_G$ has a structure of Hopf superalgebra. In fact, one can start with a Hopf superalgebra $\mathcal{O}(G)$ and define a supergroup as a functor:

$$G : (\text{Salg}) \longrightarrow \{\text{Groups}\}, \quad G(\mathcal{A}) = \text{Hom}(\mathcal{O}(G), \mathcal{A}).$$

Properties of Hopf algebras allow one to define the group structure on $G(\mathcal{A})$.

The ideal I generated by the odd elements in $\mathcal{O}(G)$ is an Hopf ideal. The quotient Hopf algebra $\mathcal{O}(G)/I$ is the Hopf algebra of regular functions on the underlying algebraic group G_0.

Exercise $\text{GL}(m|n)$.

$$\text{GL}(m|n)(\mathcal{A}) = \left\{ Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$$

satisfying the following conditions

- the entries on A and D are even elements in $\mathcal{A}$, while the the entries of B and C are odd;
- Y is invertible.

Show that $\text{GL}(m|n)$ is representable and construct the corresponding Hopf superalgebra.

Example 18 (Exercise) Consider the functor

$$\text{Ber} : \text{GL}(m|n) \longrightarrow \text{GL}(1), \qquad \text{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BDC)/\det(D).$$

Check that Ber is a homomorphism. Hint: Write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}.$$

We define $\text{SL}(m|n)$ by imposing the condition $\text{Ber} = 1$.

Show that $\text{GL}(m|n)_0 = \text{GL}(m) \times \text{GL}(n)$ and

$$\text{SL}(m|n)_0 = \{(A, D) \in \text{GL}(m) \times \text{GL}(n) \mid \det A = \det D\}.$$

Definition 14 $\text{Lie}(G)$ is the Lie superalgebra of left invariant derivations of $\mathcal{O}(G)$ and can be identified with $T_e(G)$.

Exercise $\text{Lie}(\text{GL}(m|n)) = \mathfrak{gl}(m|n)$, $\text{Lie}(\text{SL}(m|n)) = \mathfrak{sl}(m|n)$.

A useful approach to algebraic supergroups is via the so called Harish-Chandra pairs. In the case of Lie groups it is due to Koszul and Kostant, [27, 28], for complex analytic category it is done in [46], for algebraic groups see [31].

We call an HC pair the following data

- a finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$;
- an algebraic group G_0 such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$;
- a G_0-module structure on $\mathfrak{g}_1$ with differential equal to the superbracket $\mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$.

Theorem 13 *The category of HC pairs is equivalent to the category of algebraic supergroups.*

Let us comment on the proof. It is clear that every supergroup G defines uniquely a HC pair $(\mathfrak{g}, G_0)$. The difficult part is to go back: given an HC pair $(\mathfrak{g}, G_0)$, define a Hopf superalgebra $\mathcal{O}(G)$. One way to approach this problem is to set

$$R = \mathcal{O}(G) := \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0)).$$

Define a multiplication map $m : R \otimes R \to R$ by

$$m(f_1, f_2)(X) := m_0((f_1 \otimes f_2)(\Delta_U(X))),$$

where m_0 is the multiplication in $\mathcal{O}(G_0)$ and Δ_U is the comultiplication in $\mathcal{U}(\mathfrak{g})$:

$$\Delta_u(x) = x \otimes 1 + 1 \otimes x, \quad x \in \mathfrak{g}.$$

It is easy to see that R is a commutative superalgebra isomorphic to $S(\mathfrak{g}_1^*) \otimes \mathcal{O}(G_0)$, [28]. In particular, this implies that an algebraic group is a split supermanifold.

Next define the comultiplication $\Delta : R \to R \otimes R$. For $g, h \in G_0$ and $x, y \in \mathcal{U}(\mathfrak{g})$ we set

$$\Delta f(x, y)_{g,h} = f(\mathrm{Ad}(h^{-1})(x)y)_{gh}.$$

The counit map $\epsilon : R \to k$ is defined by

$$\epsilon f := \epsilon_0 \circ f(1),$$

where ϵ_0 is the counit in $\mathcal{O}(G_0)$. Finally, define the antipode $s : R \to R$ by setting for all $g \in G_0$, $x \in \mathcal{U}(\mathfrak{g})$

$$sf(X)_g = f(\mathrm{Ad}(g)s_U(X))_{g^{-1}},$$

where s_U is the antipode in $\mathcal{U}(\mathfrak{g})$.

Theorem 14 ([31]) *The category of representations of G is equivalent to the category of $(\mathfrak{g}, G_0)$-modules.*

We now concentrate on the case of reductive G_0. By the above Theorem the category $Rep(G)$ of finite-dimensional representations of G is a full subcategory of $\mathcal{F}$. Therefore we immediately obtain the following.

Corollary 8 *Let G_0 be reductive.*

- *Then $Rep(G)$ has enough projective and injective objects.*
- *Every injective G-module is projective.*

Exercise Assume that G_0 is reductive. Check that

$$\mathcal{O}(G) \simeq \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0))$$

is an isomorphism of $(\mathfrak{g}, G_0)$-modules and use it prove that

$$\mathcal{O}(G) = \oplus P(L)^{\dim(L_0)}$$

where L runs the set of irreducible representations of G and $P(L)$ is the projective cover of L. *Hint:* Use Frobenius reciprocity and the structure of $\mathcal{O}(G_0)$ as a G_0-module.

9 Geometric Induction

9.1 General Construction

Let $H \subset G$ be a subsupergroup. It is possible to show that G/H is a supermanifold, see [30]. The space of global sections of the structure sheaf is given by

$$\mathcal{O}(G/H) := \mathcal{O}(G)^H,$$

where H-invariants are defined with respect to the right action of H on G. Furthermore, if M is a representation of H, then $G \times_H M$ is a G-equivariant vector bundle on G/H. We define:

$$\mathcal{O}(G/H, M) = (\mathcal{O}(G) \otimes M)^H = \{f : G \to M \,|\, f(gh) = h^{-1}f(g),\ h \in H\}.$$

Thus, we associated in functorial way to every representation of H a representation of G, namely, the space of global sections of $G \times_H M$. The corresponding functor $\Gamma : Rep(H) \longrightarrow Rep(G)$ is left exact. The right derived functor is given by the cohomology

$$R^i\Gamma(M) = H^i(G/H, G \times_H M).$$

It is a little bit more convenient to us to work with dual functors $\Gamma_i(G/H, \cdot)$ defined by

$$\Gamma_i(G/H, M) := H^i(G/H, G \times_H M^*)^*.$$

The following statement is the Frobenius reciprocity for geometric induction and the proof is the same as for algebraic groups.

Proposition 10 *For any H-module M and G-module N we have a canonical isomorphism*

$$\mathrm{Hom}_G(\Gamma_0(G/H, M), N) \simeq \mathrm{Hom}_H(M, N).$$

Exercise If $G = G_0$, then $\Gamma_i(M) = 0$ for $i > 0$ and $\Gamma_0(M) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M$.

9.2 The Borel-Weil-Bott Theorem

Let G be an algebraic supergroup with basic Lie superalgebra $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ and denote by $B \subset G$ and $H \subset B$ the corresponding subgroups. The supermanifold G/B is called a *flag supermanifold*. Its underlying manifold G_0/B_0 is a classical flag manifold.

Recall that in the Lie algebra case flag manifolds play a crucial role in the representation theory of $\mathfrak{g}$. In particular, all the irreducible representations of a reductive algebraic group can be realized as global sections of line bundles on the flag variety by the Borel–Weil–Bott theorem. Let us see what happens in the supercase.

Consider the H-weight lattice Λ in $\mathfrak{h}^*$. Every $\lambda \in \Lambda$ defines a unique one-dimensional representation of B which we denote by c_λ. We are interested in computing $\Gamma_i(G/B, c_\lambda) = 0$. The Frobenius reciprocity (Proposition 10) implies the following

Corollary 9 *$\Gamma_i(G/B, c_\lambda)$ is isomorphic to the maximal finite-dimensional quotient $K_\mathfrak{b}(\lambda)$ of the Verma module $M_\mathfrak{b}(\lambda)$.*

Lemma 13 *Assume that the defect of $\mathfrak{g}$ is positive. Then the flag supervariety G/B is split if and only if $\mathfrak{g}$ is type 1 and $\mathfrak{b}$ is distinguished or antidistinguished.*

Proof First, let us assume that G/B is split. Then we have a projection $\pi : G/B \to G_0/B_0$ and the pull back map

$$\pi^* : G_0 \times_{B_0} c_{-\lambda} \to G \times_B c_{-\lambda}$$

which induces the embedding

$$H^0(G_0/B_0, G_0 \times_{B_0} c_{-\lambda}) \to H^0(G/B, G \times_B c_{-\lambda}).$$

After dualizing we obtain a surjection

$$\Gamma_0(G/B, c_\lambda) \to \Gamma_0(G_0/B_0, c_\lambda).$$

If λ is a G_0-dominant weight, then $\Gamma_0(G_0/B_0, c_\lambda) = L_{\mathfrak{b}_0}(\lambda) \neq 0$. By Corollary 9 $K_{\mathfrak{b}}(\lambda) \neq 0$. Hence λ is G-dominant. Thus, every dominant G_0-weight is G dominant and this is possible only for distinguished Borel or for $\mathfrak{osp}(1|2n)$.

Now let $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{g}(\pm 1)$ be a distinguished or antidistinguished Borel subalgebra. Then it is easy to see that

$$\mathcal{O}_{G/B} = \mathcal{O}_{G_0/B_0} \otimes \Lambda(\mathfrak{g}(\pm 1)^*).$$

The following result is a generalization of Borel–Weil–Bott theorem in the case of typical λ. We call a weight μ regular (resp. singular) if it has trivial (resp. non-trivial) stabilizer in W. We denote by $\Lambda^+ \subset \Lambda$ the set of all $\mu \in \Lambda$ such that $\frac{2(\mu|\alpha)}{(\alpha|\alpha)} \in \mathbf{Z}_+$ for all even positive roots α. It follows from Sect. 4.1 that a typical λ is dominant if and only if $\lambda + \rho \in \Lambda^+$.

Theorem 15 ([35]) *Let $\lambda \in \Lambda$ be typical.*

1. *If $\lambda + \rho$ is singular then $\Gamma_i(G/B, c_\lambda) = 0$ for all i.*
2. *If $\lambda + \rho$ is regular there exists a unique $w \in W$ such that $w(\lambda + \rho) \in \Lambda^+$. Let l be the length of w. Then*

$$\Gamma_i(G/B, c_\lambda) = \begin{cases} 0 & \text{if } i \neq l, \\ L(w \cdot \lambda), & \text{if } i = l. \end{cases}$$

Proof We give here just the outline, see details in [35]. First, if λ is dominant then by Corollary 9 $\Gamma_0(G/B, c_\lambda) = K_{\mathfrak{b}}(\lambda)$ and by typicality of λ we have $K_{\mathfrak{b}}(\lambda) = L_{\mathfrak{b}}(\lambda)$.

If α or $\frac{1}{2}\alpha$ is a simple root of B, then one can show using the original Demazure argument, that

$$\Gamma_i(G/B, c_\mu) \simeq \Gamma_{i+1}(G/B, c_{r_\alpha \cdot \mu}), \tag{4}$$

if $\frac{2(\mu|\alpha)}{(\alpha|\alpha)} > 0$. Furthermore, if $\frac{2(\mu|\alpha)}{(\alpha|\alpha)} = 0$, then

$$\Gamma_i(G/B, c_\mu) = 0 \tag{5}$$

for all i.

However, not every simple root of $\mathfrak{b}_0$ is a simple root of $\mathfrak{b}$ and therefore we need to involve odd reflections and change of Borel subalgebras.

Let α be an isotropic simple root and $\mathfrak{b}'$ be obtained from $\mathfrak{b}$ by the odd reflection r_α. Then we claim that

$$\Gamma_i(G/B, c_\lambda) \simeq \Gamma_i(G/B', c_{\lambda'}), \tag{6}$$

where $\lambda+\rho = \lambda'+\rho'$. To show this we consider the parabolic subalgebra $\mathfrak{p} = \mathfrak{b}+\mathfrak{b}'$. Then we have two projections

$$p : G/B \to G/P, \quad p' : G/B' \to G/P,$$

the fiber of both projections is a $(0|1)$-dimensional affine space and we have

$$p_*(G \times_B c_{-\lambda}) = p'_*(G \times_{B'} c_{-\lambda'}) = G \times_P V_\lambda,$$

where V_λ is the two-dimensional simple P-module with weights $-\lambda$ and $-\lambda'$. Note that here we use that $(\lambda + \rho, \alpha) \neq 0$ by the typicality of λ. This implies

$$H^i(G/B, G \times_B c_{-\lambda}) \simeq H^i(G/P, G \times_P V_\lambda) \simeq H^i(G/B', G \times_{B'} c'_{-\lambda}).$$

After dualization we obtain (6).

Let us assume again that λ is dominant and consider the Borel subalgebra $\mathfrak{b}'$ opposite to $\mathfrak{b}$. Combining (4) and (6) we obtain

$$\Gamma_i(G/B, c_\lambda) = \Gamma_{i+d}(G/B', c_{w_0 \cdot \lambda}),$$

where w_0 is the longest element of W and its length d equals $\dim G_0/B_0$. That implies the second statement of the theorem for dominant λ. Using (4) and (6) we can reduce the case of arbitrary regular $\lambda + \rho$ to the dominant case.

If $\lambda + \rho$ is singular, then there is a simple root α of $\mathfrak{b}_0$ such that $(\lambda + \rho, \alpha) = 0$. Using odd reflections and (6) we can change the Borel subgroup B to B' and λ so that α or $\frac{1}{2}\alpha$ is a simple root of B'. Then the vanishing of cohomology follows from (5).

Computing $\Gamma_i(G/B, c_\lambda)$ for atypical λ is an open question. The main reason why the proof in this case does not work is the absence of (6). It is known from examples that $\Gamma_i(G/B, c_\lambda)$ may not vanish for several i.

Finally let us formulate the following analogue of Bott's reciprocity relating Γ_i with Lie superalgebra cohomology. The proof is straightforward using the definition of the derived functor (see [20]).

Proposition 11 *For any finite-dimensional B-module M and any dominant weight λ, we have*

$$\left[H^i(G/B, G \times_B M) : L_{\mathfrak{b}}(\lambda)\right] = \dim \operatorname{Ext}^i_B(P_{\mathfrak{b}}(\lambda), M) = \dim H^i(\mathfrak{n}^+, P^*_{\mathfrak{b}}(\lambda) \otimes M)^{\mathfrak{h}},$$

where $P_{\mathfrak{b}}(\lambda)$ denotes the projective cover of $L_{\mathfrak{b}}(\lambda)$.

After dualizing and setting $M = c_{-\nu}$ we obtain the following

Corollary 10

$$[\Gamma_i(G/B, c_\nu) : L_{\mathfrak{b}}(\lambda)] = \dim \operatorname{Hom}_{\mathfrak{h}}\left(c_\nu, H^i(\mathfrak{n}^+, P_{\mathfrak{b}}(\lambda))\right).$$

9.3 Application to Characters

Although we do not know $\Gamma_i(G/B, c_\lambda)$ for atypical λ, we can calculate the character of the Euler characteristic.

Theorem 16 *The character of the Euler characteristic is given by the typical character formula, i.e.*

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch}\Gamma_i(G/B, c_\lambda) = \frac{D_1}{D_0} \sum_{w\in W} sgn(w) e^{w(\lambda+\rho)}.$$

Proof Consider the associated split manifold $Gr(G/B)$ and the associated graded $\mathcal{L} = Gr(\Gamma)$ of the sheaf Γ of sections of $G \times_B c_{-\lambda}$. Since Euler characteristic is preserved after going to the associated graded sheaf we have

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} H^i(G/B, G\times_B c_{-\lambda}) = \sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} H^i(Gr(G/B), \mathcal{L}).$$

Note that $\mathcal{L}$ is a G_0-equivariant vector bundle on G_0/B_0, and the classical Borel–Weil–Bott theorem allows us to calculate the right hand side of the above equality. Indeed, if $\mathcal{N}$ denotes the conormal bundle to G_0/B_0, then

$$\mathcal{L} \simeq \Lambda(\mathcal{N}) \otimes (G_0 \times_{B_0} c_{-\lambda}) = G_0 \times_{B_0} \left(c_{-\lambda} \otimes \Lambda^*(\mathfrak{g}_1/\mathfrak{b}_1)\right),$$

and

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} H^i(G/B, \mathcal{L}) = \frac{1}{D_0} \sum_{w\in W} sgn(w) w(e^{\lambda+\rho_0} \prod_{\alpha\in\Delta_1^+} (1-e^{-\alpha})),$$

which is equivalent to the typical character formula.

Note that Theorems 16 and 15 imply Theorem 2.

Definition 15 Let λ be a weight of atypicality degree p. It is called *tame* with respect to the Borel subalgebra $\mathfrak{b}$ if there exists isotropic mutually orthogonal *simple* roots $\alpha_1, \dots, \alpha_p$ such that

$$(\lambda + \rho|\alpha_1) = \cdots = (\lambda + \rho|\alpha_p) = 0.$$

Conjecture 4 (Kac–Wakimoto, [25]) If λ is dominant and tame with respect to $\mathfrak{b}$, then

$$\operatorname{ch} L_{\mathfrak{b}}(\lambda) = \frac{D_1}{D_0} \sum_{w\in W} sgn(w) w \left(\frac{e^{\lambda+\rho}}{\prod_{i=1}^{p}(1-e^{-\alpha_i})}\right). \tag{7}$$

The right hand side of formula (7) is the character of the Euler characteristic

$$\sum_{i=1}^{\dim(G_0/B_0)} (-1)^i \operatorname{ch} \Gamma_i(G/Q, c_\lambda),$$

where Q is the parabolic subgroup with Lie superalgebra

$$\mathfrak{q} := \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{-\alpha_p}.$$

Hence one way to prove Conjecture 4 is to prove the following

Conjecture 5 If λ is tame with respect to $\mathfrak{b}$, then $\Gamma_i(G/Q, c_\lambda) = 0$ if $i > 0$ and $\Gamma_0(G/Q, c_\lambda) = L_\mathfrak{b}(\lambda)$.
For classical Lie superalgebras Conjecture 5 is proven in [5].

9.4 Weak BGG Reciprocity

Let $\mathcal{K}(G)$ denote the Grothendieck group of the category $Rep(G)$ and $[M]$ denote the class of a G-module M. Clearly $[L_\mathfrak{b}(\lambda)]$, for all dominant $\lambda \in \Lambda$, is a basis of $\mathcal{K}(G)$. Set

$$[\mathcal{E}_\mathfrak{b}(\lambda)] = \sum_i (-1)^i [\Gamma_i(G/B, c_\lambda)].$$

As we already mentioned in Sect. 6.4, if $\mathfrak{g}$ is of type 1 then $Rep(G)$ is a highest weight category. For type 2 superalgebras this is not true. Nevertheless one can use virtual modules $\mathcal{E}_\mathfrak{b}(\lambda)$ instead of $K_\mathfrak{b}(\lambda)$ and obtain the following weak BGG reciprocity.

Theorem 17 ([20]) *Let $\lambda \in \Lambda$ be dominant and $\mu \in \Lambda$ be such that $\mu + \rho \in \Lambda^+$. There exists unique $a_{\lambda,\mu} \in \mathbf{Z}$ such that*

$$[\mathcal{E}_\mathfrak{b}(\mu)] = \sum a_{\lambda,\mu}[L_\mathfrak{b}(\lambda)]$$

and

$$[P_\mathfrak{b}(\lambda)] = \sum a_{\lambda,\mu}[\mathcal{E}_\mathfrak{b}(\mu)].$$

9.5 $\mathcal{D}$-Modules

In this subsection we discuss briefly possible generalizations of the Beilinson-Bernstein localization theorem for basic classical Lie superalgebras. The basics on

$\mathcal{D}$-modules on supermanifold can be found in [36]. The main result there is that if X is a supermanifold with underlying manifold X_0 then Kashiwara extension functor provides the equivalence between categories of $\mathcal{D}_{X_0}$-modules and $\mathcal{D}_X$-modules.

This fact is easy to explain in the case when X is a superdomain. Indeed, in this case

$$\mathcal{O}(X) = \mathcal{O}(X_0) \otimes \Lambda(\xi_1, \dots, \xi_n),$$

and this implies an isomorphism

$$\mathcal{D}(X) = \mathcal{D}(X_0) \otimes D(\Lambda(\xi_1, \dots, \xi_n)),$$

where $D(\Lambda(\xi_1, \dots, \xi_n)$ is the superalgebra of the differential operators on $(0|n)$-dimensional supermanifold $\mathbb{A}^{(0|n)}$. Since $\Lambda(\xi_1, \dots, \xi_n)$ is finite-dimensional, the superalgebra $D(\Lambda(\xi_1, \dots, \xi_n))$ coincides with the superalgebra $\mathrm{End}_k(\Lambda(\xi_1, \dots, \xi_n))$. This immediately implies the Morita equivalence of $\mathcal{D}(X)$ and $\mathcal{D}(X_0)$.

Let λ be a weight of $\mathfrak{g}$ and $X = G/B$ be a flag supermanifold. As in the usual case one can define the sheaf of twisted differential operators D_X^λ. Let $\mathcal{U}^\lambda(\mathfrak{g})$ denote the quotient of $\mathcal{U}(\mathfrak{g})$ by the ideal generated by the kernel of the central character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to k$. The embedding of the Lie superalgebra $\mathfrak{g}$ to the Lie superalgebra of vector fields on X induces the homomorphism of superalgebras

$$p_\lambda : \mathcal{U}^\lambda(\mathfrak{g}) \to D^\lambda(X).$$

Recall that it is an isomorphism if $\mathfrak{g}$ is a reductive Lie algebra. Moreover, for dominant λ the localization functor provides equivalence of categories of $\mathcal{U}^\lambda(\mathfrak{g})$-modules and $\mathcal{D}_X^\lambda$-modules. In the supercase, the similar result is true for generic typical λ, see [36].

Theorem 18 *Let λ be a generic typical weight such that $\frac{2(\lambda|\alpha)}{(\alpha|\alpha)} \notin \mathbf{Z}_{<0}$ for all even positive roots α. Then the functors of localization and global sections establish equivalence of categories of $\mathcal{U}^\lambda(\mathfrak{g})$-modules and $\mathcal{D}_X^\lambda$-modules.*

Note that essentially this theorem is equivalent to Theorem 4. In fact Theorem 18 was used by Penkov for the proof of Theorem 4. If λ is not typical, then the homomorphism p_λ is neither surjective nor injective. On the other hand, it is not difficult to see that for atypical λ the superalgebra $\mathcal{U}^\lambda(\mathfrak{g})$ has a non-trivial Jacobson radical, see [41]. There is an evidence that the following conjecture may hold.

Conjecture 6 Let λ be a regular weight, tame with respect to $\mathfrak{b}$, and let $\bar{\mathcal{U}}^\lambda(\mathfrak{g})$ denote the quotient of $\mathcal{U}^\lambda(\mathfrak{g})$ by the Jacobson radical. Let $\mathcal{Z}$ denote the center of $\bar{\mathcal{U}}^\lambda(\mathfrak{g})$. Let $Q \supset B$ be the maximal parabolic subgroup of G such that its Lie superalgebra $\mathfrak{q}$ admits one-dimensional representation c_λ. Finally let $Y := G/Q$.

If $\tau : \mathcal{Z} \to k$ is a generic central character and $\bar{\mathcal{U}}_\tau^\lambda(\mathfrak{g})$ is the quotient of $\bar{\mathcal{U}}^\lambda(\mathfrak{g})$ by the ideal $(\ker \tau)$, then the categories of $\bar{\mathcal{U}}_\tau^\lambda(\mathfrak{g})$-modules and $\mathcal{D}_Y^\lambda$-modules are equivalent.

10 Direct Limits of Lie Algebras and Superalgebras

The goal of this section is to say few words about representations of direct limits of classical Lie superalgebras. We will discuss here only the case of $\mathfrak{gl}(\infty|\infty)$ and refer to [43] for the case of $\mathfrak{osp}(\infty|\infty)$. Surprisingly, for some class of representations the difference between the Lie superalgebra $\mathfrak{gl}(\infty|\infty)$ and the Lie algebra $\mathfrak{gl}(\infty)$ disappears.

10.1 Category of Tensor Modules

Let V, W be countable-dimensional vector spaces (resp. superspaces) with non-degenerate even pairing $\langle\cdot,\cdot\rangle : W \times V \to k$. It is known that one can choose a pair of dual bases in V and W. The tensor product $V \otimes W$ is a Lie algebra (resp. superalgebra) $\mathfrak{g}$ with the following bracket:

$$[v_1 \otimes w_1, v_2 \otimes w_2] = \langle w_1, v_2\rangle v_1 \otimes w_2 - (-1)^{(\bar{v}_1+\bar{w}_1)(\bar{v}_2+\bar{w}_2)}\langle w_2, v_1\rangle v_2 \otimes w_1.$$

We denote this (super)algebra $\mathfrak{gl}(\infty)$ in the even case and $\mathfrak{gl}(\infty|\infty)$ in the supercase. Note that both V and W are $\mathfrak{g}$-modules and $\mathfrak{g}$ acts on V and W by linear operators of finite rank. It is not difficult to see that $\mathfrak{g}$ can be identified with infinite matrices with finitely many non-zero entries and hence

$$\mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n), \quad \mathfrak{gl}(\infty|\infty) = \varinjlim \mathfrak{gl}(m|n).$$

Let $T^{p,q} = V^{\otimes p} \otimes W^{\otimes q}$. We would like to understand the structure of $\mathfrak{g}$-module on $T^{p,q}$. It is clear that the product of symmetric groups $S_p \times S_q$ acts on $T^{p,q}$ and this action commutes with the action of $\mathfrak{g}$. Irreducible representations of $S_p \times S_q$ are parametrized by bipartitions (λ, μ) such that $|\lambda| = p$, $|\mu| = q$. The following result is a classical Schur–Weyl duality. In the supercase its proof is due to Sergeev, [44].

Theorem 19 *Let $\mathfrak{g} = \mathfrak{gl}(\infty)$ or $\mathfrak{gl}(\infty|\infty)$. Then we have the following decomposition*

$$T^{p,q} = \bigoplus_{|\lambda|=p,|\mu|=q} S_\lambda(V) \otimes S_\mu(W) \otimes Y_{\lambda,\mu},$$

where $S_\lambda(V)$ and $S_\mu(W)$ are simple $\mathfrak{g}$-modules and $Y_{\lambda,\mu}$ is the irreducible representation of $S_p \times S_q$ associated with a bipartition (λ, μ).

Let $\mathfrak{g} = \mathfrak{gl}(\infty)$. It is proven in [37] that $S_\lambda(V) \otimes S_\mu(W)$ is an indecomposable $\mathfrak{g}$-module of finite length with simple socle $V(\lambda, \mu)$. Denote by $\mathrm{Trep}\,\mathfrak{g}$ the abelian category of $\mathfrak{g}$-modules generated by finite direct sums of $T^{p,q}$ and all their subquotients. This is a symmetric monoidal category which in the case of $\mathfrak{g} = \mathfrak{gl}(\infty)$ was studied in [8] and [38].

Theorem 20 ([8]) *Let* $\mathfrak{g} = \mathfrak{gl}(\infty)$. *Any simple object of* $\mathrm{Trep}\,\mathfrak{g}$ *is isomorphic to* $V(\lambda, \mu)$ *for some bipartition* (λ, μ) *and* $S_\lambda(V) \otimes S_\mu(W)$ *is the injective hull of* $V(\lambda, \mu)$. *In particular, the category* $\mathrm{Trep}\,\mathfrak{g}$ *has enough injective objects. Moreover, any object in* $\mathrm{Trep}\,\mathfrak{g}$ *has a finite injective resolution.*

It is also proven in [8] that $\mathrm{Trep}\,\mathfrak{g}$ is a Koszul self-dual category.

Let us consider the case $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$. We start by constructing two functors F_l and F_r from the category $\mathrm{Trep}\,\mathfrak{g}$ to the category $\mathrm{Trep}\,\mathfrak{gl}(\infty)$. Observe that the even part $\mathfrak{gl}(\infty|\infty)_0$ is a direct sum $\mathfrak{g}_l \oplus \mathfrak{g}_r$ with both $\mathfrak{g}_l = V_0 \otimes W_0$ and $\mathfrak{g}_r = V_1 \otimes W_1$ isomorphic to $\mathfrak{gl}(\infty)$. For any $M \in \mathrm{Trep}\,\mathfrak{g}$ we set

$$F_l(M) := M^{\mathfrak{g}_r}, \quad F_r(M) := M^{\mathfrak{g}_l}.$$

Theorem 21 ([43]) *Let* $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$.

(a) F_l *and* F_r *are exact tensor functors, i.e.* $F_l(M \otimes N) = F_l(M) \otimes F_l(N)$ *and the same for* F_r.
(b) F_l *and* F_r *have left adjoint functors which we denote by* R_l *and* R_r *respectively.*
(c) F_l *and* R_l *(resp.* F_r *and* R_r*) are mutually inverse equivalences of tensor categories* $\mathrm{Trep}\,\mathfrak{g}$ *and* $\mathrm{Trep}\,\mathfrak{g}_l$ *(resp.* $\mathrm{Trep}\,\mathfrak{g}_r$*).*

Remark 11 The compositions $F_r \circ R_l$ and $F_l \circ R_r$ provide an autoequivalence of $\mathrm{Trep}\,\mathfrak{gl}(\infty)$ which sends a simple module $V(\lambda, \mu)$ to the simple module $V(\lambda', \mu')$, where ν' stands for the partition conjugate to ν.

Remark 12 The corresponding construction works as well for the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(\infty|\infty)$. Here $\mathfrak{g}_l = \mathfrak{so}(\infty)$ and $\mathfrak{g}_r = \mathfrak{sp}(\infty)$. In particular, we establish equivalence of tensor categories $\mathrm{Trep}\,\mathfrak{so}(\infty)$ and $\mathrm{Trep}\,\mathfrak{sp}(\infty)$.

Remark 13 The category $\mathrm{Trep}\,\mathfrak{g}$ contains a semisimple subcategory $\mathrm{Trep}^+\mathfrak{g}$ consisting of modules appearing in $T^{p,0}$, $p \in \mathbb{N}$.

10.2 Equivalences for Parabolic Category $\mathcal{O}$

In this subsection we will show how functors F_r and F_l help to prove equivalence of certain parabolic category $\mathcal{O}$ for $\mathfrak{gl}(\mathfrak{m}|\infty)$ and $\mathfrak{gl}(\infty)$. This result is originally proven in [6] by using infinite chain of odd reflections.

Let $\mathfrak{g}' = \mathfrak{gl}(\infty)$, $\mathfrak{g}'' = \mathfrak{gl}(m|\infty)$ and $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$. We fix the embeddings $\mathfrak{g}'$ and $\mathfrak{g}''$ into $\mathfrak{g}$ in the following way. Realize $\mathfrak{g}$ as matrices with finitely many non-zero entries written in the block form

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix},$$

where $A_{1,1}$ has size $m \times m$, $A_{1,2}$ and $A_{1,3}$ have size $m \times \infty$, $A_{2,1}$ and $A_{3,1}$ have size $\infty \times m$ and $A_{2,2}$ and $A_{3,3}$ have size $\infty \times \infty$. The even part $\mathfrak{g}_0$ consists of matrices of the form

$$\begin{pmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} & 0 \\ 0 & 0 & A_{3,3} \end{pmatrix},$$

and the odd part $\mathfrak{g}_1$ of matrices of the form

$$\begin{pmatrix} 0 & 0 & A_{1,3} \\ 0 & 0 & A_{2,3} \\ A_{3,1} & A_{3,2} & 0 \end{pmatrix}.$$

Then $\mathfrak{g}'$ consists of matrices

$$\begin{pmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\mathfrak{g}''$ of matrices

$$\begin{pmatrix} A_{1,1} & 0 & A_{1,3} \\ 0 & 0 & 0 \\ A_{3,1} & 0 & A_{3,3} \end{pmatrix}.$$

Let $\mathfrak{k}'$ and $\mathfrak{k}''$ be subalgebras of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{3,3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Then it is not hard to see that $\mathfrak{g}'$ is the centralizer of $\mathfrak{k}'$ and $\mathfrak{g}''$ is the centralizer of $\mathfrak{g}''$.

Next we consider the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ consisting of matrices

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ 0 & A_{2,2} & A_{2,3} \\ 0 & A_{3,2} & A_{3,3} \end{pmatrix},$$

with abelian ideal $\mathfrak{m}$

$$\begin{pmatrix} 0 & A_{1,2} & A_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the Levi subalgebra $\mathfrak{l}$

$$\begin{pmatrix} A_{1,1} & 0 & 0 \\ 0 & A_{2,2} & A_{2,3} \\ 0 & A_{3,2} & A_{3,3} \end{pmatrix},$$

isomorphic to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(\infty|\infty)$.

Finally we set $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ and $\mathfrak{p}'' := \mathfrak{p} \cap \mathfrak{g}''$. Note that $\mathfrak{p}' \subset \mathfrak{g}'$ and $\mathfrak{p}'' \subset \mathfrak{g}''$ are parabolic subalgebras. Now we consider the category $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ consisting of all $\mathfrak{g}$-modules M satisfying the following conditions

- M is finitely generated;
- M is semisimple over the diagonal subalgebra of $\mathfrak{g}$ with integral weights;
- M is an integrable $\mathfrak{p}$-module and the restriction to the subalgebra $\mathfrak{gl}(\infty|\infty) \subset \mathfrak{p}$ belongs to the inductive completion of $\mathrm{Trep}^+\mathfrak{gl}(\infty|\infty)$.

In a similar way we define the categories $\mathcal{O}(\mathfrak{g}', \mathfrak{p}')$ and $\mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$ for algebras $\mathfrak{g}'$ and $\mathfrak{g}''$ respectively. As in the previous subsection we define the functors

$$F' : \mathcal{O}(\mathfrak{g}, \mathfrak{p}) \to \mathcal{O}(\mathfrak{g}', \mathfrak{p}'), \quad F'' : \mathcal{O}(\mathfrak{g}, \mathfrak{p}) \to \mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$$

by setting

$$F'(M) = M^{\mathfrak{k}'}, \quad F''(M) = M^{\mathfrak{k}''}.$$

Then we have the following analogue of Theorem 21.

Theorem 22

(a) F' and F'' have left adjoint functors which we denote by R' and R'' respectively.
(b) F' and R' (resp. F'' and R'') are mutually inverse equivalences of abelian categories $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ and $\mathcal{O}(\mathfrak{g}', \mathfrak{p}')$ (resp. $\mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$).
(c) The composite functors $F'' \circ R'$ and $F' \circ R''$ are mutually inverse equivalences of abelian categories $\mathcal{O}(\mathfrak{g}', \mathfrak{p}')$ and $\mathcal{O}(\mathfrak{g}'', \mathfrak{p}'')$.

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Introduction to W-Algebras and Their Representation Theory

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Abstract These are lecture notes from author's mini-course on W-algebras during Session 1: "Vertex algebras, W-algebras, and application" of INdAM Intensive research period "Perspectives in Lie Theory", at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy. December 9, 2014–February 28, 2015.

Keywords Vertex algebras • W-algebras

1 Introduction

This note is based on lectures given at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, in Winter of 2014–2015. They are aimed as an introduction to W-algebras and their representation theory. Since W-algebras appear in many areas of mathematics and physics there are certainly many other important topics untouched in the note, partly due to the limitation of the space and partly due to the author's incapability.

The W-algebras can be regarded as generalizations of affine Kac-Moody algebras and the Virasoro algebra. They appeared [34, 70, 78] in the study of the classification of two-dimensional rational conformal field theories. There are several ways to define W-algebras, but it was Feigin and Frenkel [36] who found the most conceptual definition of principal W-algebras that uses the *quantized Drinfeld-Sokolov reduction*, which is a version of Hamiltonian reduction. There are a lot of works on W-algebras (see [26] and references therein) mostly by physicists in 1980s and 1990s, but they were mostly on principal W-algebras, that is, the W-algebras associated with principal nilpotent elements. It was quite recent that Kac et al. [60] defined the W-algebra $\mathscr{W}^k(\mathfrak{g}, f)$ associated with a simple Lie algebra and its arbitrary nilpotent element f by generalizing the method of quantized Drinfeld-Sokolov reduction.

The advantage of the method of quantized Drinfeld-Sokolov reduction is its functoriality, in the sense that it gives rise to a functor from the category of

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representations of affine Kac-Moody algebras and to the category of representations of W-algebras. Since it is difficult to study W-algebras directly (as no presentation by generators and relations (OPE's) is known for a general W-algebra), in this note we spend the most of our efforts in understanding this functor.

Although our methods apply to much more general settings [4, 6, 9, 10, 12] we focus on the W-algebras associated with Lie algebras $\mathfrak{g}$ of type A and its principal nilpotent element that were originally defined by Fateev and Lykyanov [34]. They can be regarded as affinization of the center of the universal enveloping algebra of $\mathfrak{g}$ via Konstant's Whittaker model [65] and Kostant-Sternberg's description [66] of Hamiltonian reduction via BRST cohomolgy, as explained in [36]. For this reason we start with a review of Kostant's results and proceed to the construction of BRST complex in the finite-dimensional setting in Sect. 2. W-algebras are *not* Lie algebras, not even associated algebras in general, but *vertex algebras*. In many cases a vertex algebra can be considered as a quantization of arc spaces of an affine Poisson scheme. In Sect. 3 we study this view point that is useful in understanding W-algebras and their representation theory. In Sect. 4 we study Zhu's algebras of vertex algebras that connects W-algebras with *finite W-algebras* [27, 75]. In Sect. 5 we introduce W-algebras and study their basic properties. In Sect. 6 we start studying representation theory of W-algebras. In Sect. 7 we quickly review some fundamental results on irreducible representations of W-algebras obtained in [5]. One of the fundamental problems (at least mathematically) on W-algebras was the conjecture of Frenkel et al. [44] on the existence and construction of so called the *minimal models* of W-algebras, which give rive to rational conformal field theories as in the case of the integrable representations of affine Kac-Moody algebras and the minimal models of the Virasoro algebra. In Sect. 8 we give an outline of the proof [10] of this conjecture.

2 Review of Kostant's Results

2.1 Companion Matrices and Invariant Polynomials

Let $G = GL_n(\mathbb{C})$ be the general linear group, and let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the general linear Lie algebra consisting of $n \times n$ matrices. The group G acts on $\mathfrak{g}$ by the adjoint action: $x \mapsto \mathrm{Ad}(g)x = gxg^{-1}$, $g \in G$. Let $\mathbb{C}[\mathfrak{g}]^G$ be the subring of the ring $\mathbb{C}[\mathfrak{g}]$ of polynomial functions on $\mathfrak{g}$ consisting of G-invariant polynomials.

Recall that a matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdot & 0 & -a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & -a_n \end{pmatrix} \tag{1}$$

is called the *companion matrix* of the polynomial $a_1 + a_2 t + a_3 t^2 + \cdots + a_n t^{n-1} + t^n \in \mathbb{C}[t]$ since

$$\det(tI - A) = a_1 + a_2 t + a_3 t^2 + \cdots + a_n t^{n-1} + t^n. \tag{2}$$

Let $\mathcal{S}$ be the affine subspace of $\mathfrak{g}$ consisting of companion matrices of the form (1).

Lemma 1 *For $A \in \mathfrak{g}$ the following conditions are equivalent.*

1. *$A \in G \cdot \mathcal{S}$.*
2. *There exists a vector $v \in \mathbb{C}^n$ such that $v, Av, A^2 v, \dots, A^{n-1} v$ are linearly independent.*

Theorem 1 *The restriction map gives the isomorphism*

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathcal{S}].$$

Proof Let $f \in \mathbb{C}[\mathfrak{g}]$ be a G-invariant polynomial such that $f|_{\mathcal{S}} = 0$. Then clearly $f|_{G.\mathcal{S}} = 0$. On the other hand it follows from Lemma 1 that $G.\mathcal{S}$ is a Zariski open subset in $\mathfrak{g}$. Therefore $f = 0$. To see the surjectiveness define $p_1, \dots, p_n \in \mathbb{C}[\mathfrak{g}]^G$ by

$$\det(tI - A) = t^n + p_1(A) t^{n-1} - \cdots + p_n(A), \quad A \in \mathfrak{g}.$$

By (2), we have $\mathbb{C}[\mathcal{S}] = \mathbb{C}[p_1|_{\mathcal{S}}, \dots, p_n|_{\mathcal{S}}]$. This completes the proof.

Put

$$f := \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathcal{S}. \tag{3}$$

Note that f is a *nilpotent element* of $\mathfrak{g}$, that is, $(\mathrm{ad} f)^r = 0$ for a sufficiently large r. We have

$$\mathcal{S} = f + \mathfrak{a},$$

where

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \cdots & 0 & * \\ \vdots & & \vdots & * \\ \vdots & & \vdots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \right\}.$$

Let $\mathfrak{b}$, $\mathfrak{n}$ be the subalgebras of $\mathfrak{g}$ defined by

$$\mathfrak{b} = \left\{ \begin{pmatrix} * & & & \\ & \ddots & * & \\ & 0 & \ddots & \\ & & & * \end{pmatrix} \right\}, \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & & & \\ & \ddots & * & \\ & 0 & \ddots & \\ & & & 0 \end{pmatrix} \right\} \subset \mathfrak{b},$$

and let N be the unipotent subgroup of G corresponding to $\mathfrak{n}$, i.e.,

$$N = \left\{ \begin{pmatrix} 1 & & & \\ & \ddots & * & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix} \right\}. \tag{4}$$

Let $(\ |\ )$ be the invariant inner product of $\mathfrak{g}$ defined by $(x|y) = \mathrm{tr}(xy)$. This gives a G-equivariant isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

Define $\chi \in \mathfrak{n}^*$ by

$$\chi(x) = (f|x) \quad \text{for } x \in \mathfrak{n}.$$

Note that χ is a character of $\mathfrak{n}$, that is, $\chi([\mathfrak{n}, \mathfrak{n}]) = 0$. Hence χ defines a one-dimensional representation of N.

Consider the restriction map

$$\mu : \mathfrak{g}^* \to \mathfrak{n}^*.$$

Then

$$\mu^{-1}(\chi) = \chi + \mathfrak{n}^{\perp} \cong f + \mathfrak{b}.$$

Here $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ via $(\ |\ )$. Since μ is N-equivariant and χ is a one-point orbit of N, it follows that $f + \mathfrak{b}$ is stable under the action of N.

Theorem 2 (Kostant [65]) *The adjoint action gives the isomorphism*

$$N \times \mathcal{S} \xrightarrow{\sim} f + \mathfrak{b}, \quad (g, x) \mapsto \mathrm{Ad}(g)x$$

of affine varieties.

Proof It is not difficult to see that the adjoint action gives the bijection $N \times \mathcal{S} \xrightarrow{\sim} f + \mathfrak{b}$. Since it is a morphism of irreducible varieties and $f + \mathfrak{b}$ is normal, the assertion follows from Zariski's Main Theorem (see e.g., [76, Corollary 17.4.8]).

Corollary 1 *The restriction map gives the isomorphisms*

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[f+\mathfrak{b}]^N \xrightarrow{\sim} \mathbb{C}[\mathcal{S}].$$

Proof By Theorem 2, we have

$$\mathbb{C}[f+\mathfrak{b}]^N \cong \mathbb{C}[N]^N \otimes \mathbb{C}[\mathcal{S}] \cong \mathbb{C}[\mathcal{S}].$$

Hence the assertion follows from Theorem 1.

2.2 Transversality of $\mathcal{S}$ to G-Orbits

Lemma 2 *The affine spaces $\mathcal{S}$ and $f+\mathfrak{b}$ intersect transversely at f to $\operatorname{Ad}G\cdot f$.*

Proof We need to show that

$$T_f\mathfrak{g} = T_f\mathcal{S} + T_f(\operatorname{Ad}G\cdot f) \tag{5}$$

But $T_f\mathfrak{g} \cong \mathfrak{g}$, $T_f\mathcal{S} \cong \mathfrak{a}$, $T_f(\operatorname{Ad}G\cdot f) \cong [\mathfrak{g},f]$. The assertion follows since $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g},f]$.

Using the Jacobson-Morozov theorem, we can embed f into an $\mathfrak{sl}_2$-triple $\{e,f,h\}$ in $\mathfrak{g}$. Explicitly, we can choose the following elements for e and h:

$$e = \sum_{i=1}^{n-1} i(n-i)e_{i,i+1}, \quad h = \sum_{i=1}^{n}(n+1-2i)e_{i,i}, \tag{6}$$

where $e_{i,j}$ denotes the standard basis element of $\mathfrak{g} = \operatorname{Mat}_n(\mathbb{C})$.

The embedding $\mathfrak{sl}_2 = \operatorname{span}_{\mathbb{C}}\{e,h,f\} \to \mathfrak{g}$ exponents to a homomorphism $SL_2 \to G$. Restricting it to the torus $\mathbb{C}^*$ consisting of diagonal matrices we obtain a one-parameter subgroup $\gamma : \mathbb{C}^* \to G$. Set

$$\rho : \mathbb{C}^* \ni t \longmapsto t^2 \operatorname{Ad}\gamma(t) \in GL(\mathfrak{g}). \tag{7}$$

Then

$$\rho(t)(f + \sum_{i\leqslant j} c_{ij}e_{i,j}) = f + \sum_{i\leqslant j} t^{2(i-j+1)}c_{ij}e_{i,j}.$$

Thus it define a $\mathbb{C}^*$-action on $\mathfrak{g}$ that preserves $f+\mathfrak{b}$ and $\mathcal{S}$. This action on $f+\mathfrak{b}$ and $\mathcal{S}$ contracts to f, that is, $\rho(t)x \to f$ when $t \to 0$.

Proposition 1 *The affine space $f + \mathfrak{b}$ (resp. $\mathcal{S}$) intersects* $\mathrm{Ad}\,G \cdot x$ *transversely at any point $x \in f + \mathfrak{b}$ (resp. $x \in \mathcal{S}$).*

Proof By Lemma 2 the intersection of $f + \mathfrak{b}$ with $\mathrm{Ad}\,G$-orbits is transversal at each point in some open neighborhood of f in $f + \mathfrak{b}$. By the contracting $\mathbb{C}^*$-action ρ, it follows that the same is true for all points of $f + \mathfrak{b}$.

2.3 The Transversal Slice $\mathcal{S}$ as a Reduced Poisson Variety

The affine variety $\mathfrak{g}^*$ is equipped with the Kirillov-Kostant Poisson structure: the Poisson algebra structure of $\mathbb{C}[\mathfrak{g}^*]$ is given by

$$\{x, y\} = [x, y] \quad \text{for } x, y \in \mathfrak{g} \subset \mathbb{C}[\mathfrak{g}^*].$$

Consider the restriction map $\mu : \mathfrak{g}^* \to \mathfrak{n}^*$, which is a *moment map* for the N-action on $\mathfrak{g}^*$. That is, μ is a regular N-equivariant morphism that gives the following commutative diagram of Lie algebras:

$$\begin{array}{ccc} & & \mathfrak{n} \\ & \overset{\mu^*}{\swarrow} & \downarrow \\ \mathbb{C}[\mathfrak{g}^*] & \longrightarrow & \mathrm{Der}\,\mathbb{C}[\mathfrak{g}^*] \end{array}$$

Here $\mu^* : \mathfrak{n} \to \mathfrak{g} \subset \mathbb{C}[\mathfrak{g}^*]$ is the pullback map, the map $\mathbb{C}[\mathfrak{g}^*] \to \mathrm{Der}\,\mathbb{C}[\mathfrak{g}^*]$ is given by $\phi \mapsto \{\phi, ?\}$, and $\mathfrak{n} \to \mathrm{Der}\,\mathbb{C}[\mathfrak{g}^*]$ is the Lie algebra homomorphism induced by the coadjoint action of G on $\mathfrak{g}^*$.

The transversality statement of Proposition 1 for $f + \mathfrak{b}$ is equivalent to that χ is a regular value of μ. By Theorem 2, the action of N on $\mu^{-1}(\chi) = \chi + \mathfrak{n}^\perp$ is free and

$$\mathcal{S} \cong \mu^{-1}(\chi)/N.$$

Therefore $\mathcal{S}$ has the structure of the *reduced Poisson variety*, obtained from $\mathfrak{g}^*$ by the Hamiltonian reduction.

The Poisson structure of $\mathcal{S}$ is described as follows. Let

$$I_\chi = \mathbb{C}[\mathfrak{g}^*] \sum_{x \in \mathfrak{n}} (x - \chi(x)),$$

so that

$$\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathfrak{g}^*]/I_\chi.$$

Then $\mathbb{C}[\mathcal{S}]$ can be identified as the subspace of $\mathbb{C}[\mathfrak{g}^*]/I_\chi$ consisting of all cosets $\phi + \mathbb{C}[\mathfrak{g}^*]I_\chi$ such that $\{x, \phi\} \in \mathbb{C}[\mathfrak{g}^*]I_\chi$ for all $x \in \mathfrak{n}$. In this realization, the Poisson structure on $\mathbb{C}[\mathcal{S}]$ is defined by the formula

$$\{\phi + \mathbb{C}[\mathfrak{g}^*]I_\chi, \phi' + \mathbb{C}[\mathfrak{g}^*]I_\chi\} = \{\phi, \phi'\} + \mathbb{C}[\mathfrak{g}^*]I_\chi$$

for ϕ, ϕ' such that $\{x, \phi\}, \{x, \phi'\} \in \mathbb{C}[\mathfrak{g}^*]I_\chi$ for all $x \in \mathfrak{n}$.

Proposition 2 *We have the isomorphism* $\mathbb{C}[\mathfrak{g}^*]^G \xrightarrow{\sim} \mathbb{C}[\mathcal{S}]$ *as Poisson algebras. In particular the Poisson structure of* $\mathcal{S}$ *is trivial.*

Proof The restriction map $\mathbb{C}[\mathfrak{g}^*]^G \xrightarrow{\sim} \mathbb{C}[\mathcal{S}]$ (see Corollary 1) is obviously a homomorphism of Poisson algebras.

In the next subsection we shall describe the above Hamiltonian reduction in more factorial way, in terms of the *BRST cohomology* (where BRST refers to the physicists Becchi, Rouet, Stora and Tyutin) for later purpose.

2.4 BRST Reduction

Let *Cl* be the *Clifford algebra* associated with the vector space $\mathfrak{n} \oplus \mathfrak{n}^*$ and its non-degenerate bilinear form $(\cdot|\cdot)$ defined by $(f + x|g + y) = f(y) + g(x)$ for $f, g \in \mathfrak{n}^*, x, y \in \mathfrak{n}$. Namely, *Cl* is the unital $\mathbb{C}$-superalgebra that is isomorphic to $\Lambda(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*)$ as $\mathbb{C}$-vector spaces, the natural embeddings $\Lambda(\mathfrak{n}) \hookrightarrow Cl$, $\Lambda(\mathfrak{n}^*) \hookrightarrow Cl$ are homogeneous homomorphism of superalgebras, and

$$[x,f] = f(x) \quad x \in \mathfrak{n} \subset \Lambda(\mathfrak{n}),\, f \in \mathfrak{n}^* \subset \Lambda(\mathfrak{n}^*).$$

(Note that $[x,f] = xf + fx$ since x,f are odd.)

Let $\{x_\alpha\}_{\alpha\in\Delta_+}$ be a basis of $\mathfrak{n}$, $\{x_\alpha^*\}_{\alpha\in\Delta_+}$ the dual basis of $\mathfrak{n}^*$, and $c_{\alpha,\beta}^\gamma$ the structure constants of $\mathfrak{n}$, that is, $[x_\alpha, x_\beta] = \sum_{\alpha,\beta}^\gamma c_{\alpha\beta}^\gamma x_\gamma$.

Lemma 3 *The following map gives a Lie algebra homomorphism.*

$$\rho : \mathfrak{n} \longrightarrow Cl$$
$$x_\alpha \longmapsto \sum_{\beta,\gamma\in\Delta_+} c_{\alpha,\beta}^\gamma x_\gamma x_\beta^*.$$

We have

$$[\rho(x), y] = [x, y] \in \mathfrak{n} \subset Cl \qquad \textit{for } x, y \in \mathfrak{n}.$$

Define an increasing filtration on Cl by setting $Cl_p := \Lambda^{\leqslant p}(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*)$. We have

$$0 = Cl_{-1} \subset Cl_0 \subset Cl_1 \cdots \subset Cl_N = Cl,$$

where $N = \dim \mathfrak{n} = \frac{n(n-1)}{2}$, and

$$Cl_p \cdot Cl_q \subset Cl_{p+q}, \quad [Cl_p, Cl_q] \subset Cl_{p+q-1}. \tag{8}$$

Let $\overline{Cl}$ be its associated graded algebra:

$$\overline{Cl} := \operatorname{gr} Cl = \bigoplus_{p \geqslant 0} \frac{Cl_p}{Cl_{p-1}}.$$

By (8), $\overline{Cl}$ is naturally a graded Poisson superalgebra, called the *classical Clifford algebra*.

We have $\overline{Cl} = \Lambda(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*)$ as a commutative superalgebra. Its Poisson (super)bracket is given by

$$\{x, f\} = f(x), \quad x \in \mathfrak{n} \subset \Lambda(\mathfrak{n}), f \in \mathfrak{n}^* \subset \Lambda(\mathfrak{n}^*),$$
$$\{x, y\} = 0, \quad x, y \in \mathfrak{n} \subset \Lambda(\mathfrak{n}), \quad \{f, g\} = 0, f, g \in \mathfrak{n}^* \subset \Lambda(\mathfrak{n}^*).$$

Lemma 4 *We have* $\overline{Cl}^{\mathfrak{n}} = \Lambda(\mathfrak{n})$, *where* $\overline{Cl}^{\mathfrak{n}} := \{w \in \overline{Cl} \mid \{x, w\} = 0, \forall x \in \mathfrak{n}\}$.

The Lie algebra homomorphism $\rho : \mathfrak{n} \longrightarrow Cl_1 \subset Cl$ induces a Lie algebra homomorphism

$$\bar{\rho} := \sigma_1 \circ \rho : \mathfrak{n} \longrightarrow \overline{Cl}, \tag{9}$$

where σ_1 is the projection $Cl_1 \to Cl_1/Cl_0 \subset \operatorname{gr} Cl$. We have

$$\{\bar{\rho}(x), y\} = [x, y] \quad \text{for } x, y \in \mathfrak{n}.$$

Set

$$\bar{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}.$$

Since it is a tensor product of Poisson superalgebras, $\bar{C}(\mathfrak{g})$ is naturally a Poisson superalgebra.

Lemma 5 *The following map gives a Lie algebra homomorphism:*

$$\bar{\theta}_\chi : \mathfrak{n} \longrightarrow \bar{C}(\mathfrak{g})$$
$$x \longmapsto (\mu^*(x) - \chi(x)) \otimes 1 + 1 \otimes \bar{\rho}(x),$$

that is, $\{\bar{\theta}_\chi(x), \bar{\theta}_\chi(y)\} = \bar{\theta}_\chi([x, y])$ *for* $x, y \in \mathfrak{n}$.

Let $\bar{C}(\mathfrak{g}) = \bigoplus_{n\in\mathbb{Z}} \bar{C}^n(\mathfrak{g})$ be the $\mathbb{Z}$-grading defined by $\deg \phi \otimes 1 = 0$ ($\phi \in \mathbb{C}[\mathfrak{g}^*]$), $\deg 1 \otimes f = 1$ ($f \in \mathfrak{n}^*$), $\deg 1 \otimes x = -1$ ($x \in \mathfrak{n}$). We have

$$\bar{C}^n(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes (\bigoplus_{j-i=n} \Lambda^i(\mathfrak{n}) \otimes \Lambda^j(\mathfrak{n}^*)).$$

Lemma 6 ([20, Lemma 7.13.3]) *There exists a unique element* $\bar{Q} \in \bar{C}^1(\mathfrak{g})$ *such that*

$$\{\bar{Q}, 1 \otimes x\} = \bar{\theta}_\chi(x) \quad \text{for } x \in \mathfrak{n}.$$

We have $\{\bar{Q}, \bar{Q}\} = 0$.

Proof Existence. It is straightforward to see that the element

$$\bar{Q} = \sum_\alpha (x_\alpha - \chi(x_\alpha)) \otimes x_\alpha^* - 1 \otimes \frac{1}{2} \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta}^\gamma x_\alpha^* x_\beta^* x_\gamma$$

satisfies the condition.

Uniqueness. Suppose that $\bar{Q}_1, \bar{Q}_2 \in \bar{C}^1(\mathfrak{g})$ satisfy the condition. Set $R = Q_1 - Q_2 \in \bar{C}^1(\mathfrak{g})$. Then $\{R, 1 \otimes x\} = 0$, and so, $R \in \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}^{\mathfrak{n}}$. But by Lemma 4, $\overline{Cl}^{\mathfrak{n}} \cap \overline{Cl}^1 = 0$. Thus $R = 0$ as required.

To show that $\{\bar{Q}, \bar{Q}\} = 0$, observe that

$$\{1 \otimes x, \{1 \otimes y, \{\bar{Q}, \bar{Q}\}\}\} = 0, \quad \forall x, y \in \mathfrak{n}$$

(note that $\bar{Q}$ is odd). Applying Lemma 4 twice, we get that $\{\bar{Q}, \bar{Q}\} = 0$.

Since $\bar{Q}$ is odd, Lemma 6 implies that

$$\{\bar{Q}, \{\bar{Q}, a\}\} = \frac{1}{2}\{\{\bar{Q}, \bar{Q}\}, a\} = 0$$

for any $a \in \bar{C}(\mathfrak{g})$. That is, $\operatorname{ad}\bar{Q} := \{\bar{Q}, \cdot\}$ satisfies that

$$(\operatorname{ad}\bar{Q})^2 = 0.$$

Thus, $(\bar{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})$ is a *differential graded Poisson superalgebra*. Its cohomology $H^\bullet(\bar{C}(\mathfrak{g}), \operatorname{ad}\bar{Q}) = \bigoplus_{i\in\mathbb{Z}} H^i(\bar{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})$ inherits a graded Poisson superalgebra structure from $\bar{C}(\mathfrak{g})$.

According to Kostant and Sternberg [66] the Poisson structure of $\mathbb{C}[\mathcal{S}]$ may be described through the following isomorphism:

Theorem 3 ([66]) *We have* $H^i(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q}) = 0$ *for* $i \neq 0$ *and*

$$H^0(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q}) \cong \mathbb{C}[\mathcal{S}]$$

as Poisson algebras.

Proof Give a bigrading on $\bar{C} := \bar{C}(\mathfrak{g})$ by setting

$$\bar{C}^{i,j} = \mathbb{C}[\mathfrak{g}^*] \otimes \Lambda^i(\mathfrak{n}^*) \otimes \Lambda^{-j}(\mathfrak{n}),$$

so that $\bar{C} = \bigoplus_{i \geqslant 0, j \leqslant 0} \bar{C}^{i,j}$.

Observe that $\mathrm{ad}\,\bar{Q}$ decomposes as $\mathrm{ad}\,\bar{Q} = d_+ + d_-$ such that

$$d_-(\bar{C}^{i,j}) \subset \bar{C}^{i,j+1}, \quad d_+(\bar{C}^{i,j}) \subset \bar{C}^{i+1,j}. \tag{10}$$

Explicitly, we have

$$d_- = \sum_i (x_i - \chi(x_i)) \otimes \mathrm{ad}\,x_i^*,$$

$$d_+ = \sum_i \mathrm{ad}\,x_i \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k x_i^* x_j^* \mathrm{ad}\,x_k + \sum_i 1 \otimes \bar{\rho}(x_i) \mathrm{ad}\,x_i^*.$$

Since $\mathrm{ad}\,\bar{Q}^2 = 0$, (10) implies that

$$d_-^2 = d_+^2 = [d_-, d_+] = 0.$$

It follows that there exists a spectral sequence

$$E_r \Longrightarrow H^\bullet(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})$$

such that

$$E_1^{\bullet,q} = H^q(\bar{C}(\mathfrak{g}), d_-) = H^q(\mathbb{C}[\mathfrak{g}^*] \otimes \Lambda(\mathfrak{n}), d_-) \otimes \Lambda^\bullet(\mathfrak{n}^*),$$

$$E_2^{p,q} = H^p(H^q(\bar{C}(\mathfrak{g}), d_-), d_+).$$

Observe that $(\bar{C}(\mathfrak{g}), d_-)$ is identical to the Koszul complex $\mathbb{C}[\mathfrak{g}^*]$ associated with the sequence $x_1 - \chi(x_1), x_2 - \chi(x_2.) \ldots, x_N - \chi(x_N)$ tensorized with $\Lambda(\mathfrak{n}^*)$. Since $\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathfrak{g}^*]/\sum_i \mathbb{C}[\mathfrak{g}^*](x_i - \chi(x_i))$, we get that

$$H^i(\bar{C}(\mathfrak{g}), d_-) = \begin{cases} \mathbb{C}[\mu^{-1}(\chi)] \otimes \Lambda(\mathfrak{n}^*), & \text{if} \quad i = 0 \\ 0, & \text{if} \quad i \neq 0. \end{cases}$$

Next, notice that $(H^0(C(\mathfrak{g}), d_-), d_+)$ is identical to the Chevalley complex for the Lie algebra cohomology $H^\bullet(\mathfrak{n}, \mathbb{C}[\mu^{-1}(\chi)])$. Therefore Theorem 2 gives that

$$H^i(H^\bullet(C(\mathfrak{g}), d_-), d_+) = \begin{cases} \mathbb{C}[\mathcal{S}], & i = 0 \\ 0, & i \neq 0. \end{cases}$$

Hence the spectral sequence collapses at $E_2 = E_\infty$ and we get that $H^i(\bar{C}(\mathfrak{g}), \operatorname{ad}\bar{Q}) = 0$ for $i \neq 0$. Moreover, there is an isomorphism

$$H^0(\bar{C}(\mathfrak{g}), \operatorname{ad}\bar{Q}) \xrightarrow{\sim} H^0(H^0(\bar{C}(\mathfrak{g}), d_-), d_+) = \mathbb{C}[\mathcal{S}], \quad [c] \mapsto [c].$$

This completes the proof.

Theorem 4 *The natural map* $\mathbb{C}[\mathfrak{g}^*]^G \longrightarrow H^0(\bar{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})$ *defined by sending* p *to* $p \otimes 1$ *is an isomorphism of Poisson algebras.*

Proof It is clear that the map is a well-defined homomorphism of Poisson algebras since $\mathbb{C}[\mathfrak{g}^*]^G$ is the Poisson center of $\mathbb{C}[\mathfrak{g}^*]$. The assertion follows from the commutativity of the following diagram.

$$\begin{array}{ccc} & & \mathbb{C}[\mathfrak{g}^*]^G \\ & \swarrow{\scriptstyle\cong} & \downarrow \\ \mathbb{C}[\mathcal{S}] & \overset{\cong}{\longleftarrow} & H^0(\bar{C}(\mathfrak{g}), \operatorname{ad}\bar{Q}). \end{array}$$

2.5 Quantized Hamiltonian Reduction

We shall now quantize the above construction following [66].

Let $\{U_i(\mathfrak{g})\}$ be the PBW filtration of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, that is, $U_i(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ spanned by the products of at most i elements of $\mathfrak{g}$. Then

$$0 = U_{-1}(\mathfrak{g}) \subset U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \dots, \quad U(\mathfrak{g}) = \bigcup_i U_i(\mathfrak{g}),$$

$$U_i(\mathfrak{g}) \cdot U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g}), \quad [U_i(\mathfrak{g}), U_j(\mathfrak{g})] \subset U_{i+j-1}(\mathfrak{g}).$$

The associated graded space $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i \geqslant 0} U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g})$ is naturally a Poisson algebra, and the PBW Theorem states that

$$\operatorname{gr} U(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$$

as Poisson algebras. Thus, $U(\mathfrak{g})$ is a quantization of $\mathbb{C}[\mathfrak{g}^*]$.

Define

$$C(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cl.$$

It is naturally a $\mathbb{C}$-superalgebra, where $U(\mathfrak{g})$ is considered as a purely even subsuperalgebra. The filtration of $U(\mathfrak{g})$ and Cl induces the filtration of $C(\mathfrak{g})$: $C_p(\mathfrak{g}) = \sum_{i+j\leqslant p} U_i(\mathfrak{g}) \otimes Cl_j$, and we have

$$\operatorname{gr} C(\mathfrak{g}) \cong \bar{C}(\mathfrak{g})$$

as Poisson superalgebras. Therefore, $C(\mathfrak{g})$ is a quantization of $\bar{C}(\mathfrak{g})$.

Define the $\mathbb{Z}$-grading $C(\mathfrak{g}) = \bigoplus_{n\in\mathbb{Z}} C^n(\mathfrak{g})$ by setting $\deg(u \otimes 1) = 0$ ($u \in U(\mathfrak{g})$), $\deg(1 \otimes f) = 1$ ($f \in \mathfrak{n}^*$), $\deg(1 \otimes x) = -1$ ($x \in \mathfrak{n}$). Then

$$C^n(\mathfrak{g}) = U(\mathfrak{g}) \otimes (\bigoplus_{j-i=n} \Lambda^i(\mathfrak{n}) \otimes \Lambda^j(\mathfrak{n}^*)).$$

Lemma 7 *The following map defines a Lie algebra homomorphism.*

$$\begin{aligned} \theta_\chi : \mathfrak{n} &\longrightarrow C(\mathfrak{g}) \\ x &\longmapsto (x - \chi(x)) \otimes 1 + 1 \otimes \rho(x) \end{aligned}$$

Lemma 8 ([20, Lemma 7.13.7]) *There exists a unique element $Q \in C^1(\mathfrak{g})$ such that*

$$[Q, 1 \otimes x] = \theta_\chi(x), \quad \forall x \in \mathfrak{n}.$$

We have $Q^2 = 0$.

Proof The proof is similar to that of Lemma 6. In fact the element Q is explicitly given by the same formula as $\bar{Q}$:

$$Q = \sum_\alpha (x_\alpha - \chi(x_\alpha)) \otimes x_\alpha^* - 1 \otimes \frac{1}{2} \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta}^\gamma x_\alpha^* x_\beta^* x_\gamma.$$

Since Q is odd, Lemma 8 implies that

$$(\operatorname{ad} Q)^2 = 0.$$

Thus, $(C(\mathfrak{g}), \operatorname{ad} Q)$ is a *differential graded algebra*, and its cohomology $H^\bullet(C(\mathfrak{g}), \operatorname{ad} Q)$ is a graded superalgebra.

However the operator on $\operatorname{gr} C(\mathfrak{g}) = \bar{C}(\mathfrak{g})$ induced by $\operatorname{ad} Q$ does not coincide with $\operatorname{ad} \bar{Q}$. To remedy this, we introduce the *Kazhdan filtration* $K_\bullet C(\mathfrak{g})$ of $C(\mathfrak{g})$ as follows: Defined a $\mathbb{Z}$-grading on $\mathfrak{g}$ by

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g} : [h, x] = 2jx\}$$

where h is defined in (6). Then $\mathfrak{n} = \bigoplus_{j>0} \mathfrak{g}_j \subset \mathfrak{b} = \bigoplus_{j\geqslant 0} \mathfrak{g}_j$, and

$$\mathfrak{h} := \mathfrak{g}_0$$

is the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices. Extend the basis $\{x_\alpha\}_{\alpha\in\Delta_+}$ of $\mathfrak{n}$ to the basis $\{x_a\}_{a\in\Delta_+\sqcup I}$ of $\mathfrak{b}$ by adding a basis $\{x_i\}_{i\in I}$ of $\mathfrak{h}$. Let $c_{a,b}^d$ denote the structure constant of $\mathfrak{b}$ with respect to this basis.

Lemma 9 *The map $\rho : \mathfrak{n} \to Cl$ extends to the Lie algebra homomorphism*

$$\rho : \mathfrak{b} \to Cl, \quad x_a \mapsto \sum_{\beta,\gamma\in\Delta_+} c_{a,\beta}^{\gamma} x_\gamma x_\beta^*.$$

Define the Lie algebra homomorphism

$$\theta_0 : \mathfrak{b} \to C(\mathfrak{g}), \quad x_i \mapsto x_i \otimes 1 + 1 \otimes \rho(x_i),$$

and define a $\mathbb{Z}$-grading on $C(\mathfrak{g})$ by

$$C(\mathfrak{g}) = \bigoplus_{j\in\mathbb{Z}} C(\mathfrak{g})[j], \quad C(\mathfrak{g})[j] = \{c \in C(\mathfrak{g}) \mid [\theta_0(h), c] = 2jx\}.$$

Set

$$K_p C(\mathfrak{g}) = \sum_{i-j\leqslant p} C_i(\mathfrak{g})[j], \quad \text{where } C_i(\mathfrak{g})[j] = C_i(\mathfrak{g}) \cap C(\mathfrak{g})[j].$$

Then $K_\bullet C(\mathfrak{g})$ defines an increasing, exhaustive, separated filtration of $C(\mathfrak{g})$ such that $K_p C(\mathfrak{g}) \cdot K_q C(\mathfrak{g}) \subset K_{p+q} C(\mathfrak{g})$, $[K_p C(\mathfrak{g}), K_q C(\mathfrak{g})] \subset K_{p+q-1} C(\mathfrak{g})$, and $\operatorname{gr}_K C(\mathfrak{g}) = \bigoplus_p K_p C(\mathfrak{g})/K_{p-1} C(\mathfrak{g})$ is isomorphic to $\bar{C}(\mathfrak{g})$ as Poisson superalgebras. Moreover, the complex $(\operatorname{gr}_K C(\mathfrak{g}), \operatorname{ad} Q)$ is identical to $(\bar{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$.

Let $\mathcal{Z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$.

Theorem 5 ([65]) *We have $H^i(C(\mathfrak{g}), \operatorname{ad} Q) = 0$ for $i \neq 0$ and the map $\mathcal{Z}(\mathfrak{g}) \longrightarrow H^0(C(\mathfrak{g}), \operatorname{ad} Q)$ defined by sending z to $[z \otimes 1]$ is an isomorphism of algebras. Here $[z \otimes 1]$ denotes the cohomology class of $z \otimes 1$.*

Proof We have the spectral sequence

$$E_r \Longrightarrow H^\bullet(C(\mathfrak{g}), \operatorname{ad} Q)$$

such that

$$E_1^{\bullet,i} = H^i(\mathrm{gr}_K C(\mathfrak{g}), \mathrm{ad}\,\bar{Q}) \cong \begin{cases} \mathbb{C}[\mathfrak{g}^*]^G, & \text{if} \quad i = 0 \\ 0, & \text{if} \quad i \neq 0. \end{cases}$$

Therefore the spectral sequence collapses at $E_1 = E_\infty$, so we get

$$\mathrm{gr}\, H^0(C(\mathfrak{g}), \mathrm{ad}\, Q) \cong \mathbb{C}[\mathfrak{g}^*]^G.$$

Since the homomorphism $\mathcal{Z}(\mathfrak{g}) \to H^0(C(\mathfrak{g}), \mathrm{ad}\, Q)$, $z \mapsto [z \otimes 1]$, respects the filtration $\mathcal{Z}_\bullet(\mathfrak{g})$ and $K_\bullet H^0(C(\mathfrak{g}), \mathrm{ad}\, Q)$, where $\mathcal{Z}_p(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) \cap U_p(\mathfrak{g})$, $K_p H^\bullet(C(\mathfrak{g}), \mathrm{ad}\, Q) = \mathrm{Im}(H^0(K_p C(\mathfrak{g}), \mathrm{ad}\, Q) \to H^0(C(\mathfrak{g}), \mathrm{ad}\, Q))$, we get the desired isomorphism.

Remark 1 (See [5, § 2] for the Details) As in the case of $\bar{C}(\mathfrak{g})$, $C(\mathfrak{g})$ is also bigraded, we can also write $\mathrm{ad}\, Q = d_+ + d_-$ such that $d_+(C^{i,j}) \subset C^{i+1,j}$, $d_-(C^{i,j}) \subset C^{i,j+1}$ and get a spectral sequence

$$E_r \Longrightarrow H^\bullet(C(\mathfrak{g}), \mathrm{ad}\, Q)$$

such that

$$E_2^{p,q} = H^p(H^q(C(\mathfrak{g}), d_-), d_+) \cong \delta_{q,0} H^p(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi)$$
$$\cong \delta_{p,0}\delta_{q,0} H^0(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi) \cong \mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi)^{op}.$$

Where $\mathbb{C}_\chi$ is the one-dimensional representation of $\mathfrak{n}$ defined by the character χ. Thus we get the Whittaker model isomorphism [65]

$$\mathcal{Z}(\mathfrak{g}) \cong H^0(C(\mathfrak{g}), \mathrm{ad}\, Q) \cong \mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi)^{op}.$$

2.6 Classical Miura Map

Let $\mathfrak{n}_- = \bigoplus_{j<0} \mathfrak{g}_j$ be the subalgebra of $\mathfrak{g}$ consisting of lower triangular matrices, and set $\mathfrak{b}_- = \bigoplus_{j\leqslant 0} \mathfrak{g}_j = \mathfrak{n}_- \oplus \mathfrak{h}$. We have

$$\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{n}_+. \tag{11}$$

Extend the basis $\{x_a\}_{a\in\Delta_+\sqcup I}$ to the basis $\{x_a\}_{a\in\Delta_+\sqcup I\sqcup\Delta_-}$ by adding a basis $\{x_\alpha\}_{\alpha\in\Delta_-}$ of $\mathfrak{n}_-$. Let $c_{a,b}^d$ be the structure constant with respect to this basis. Extend

$\theta_0 : \mathfrak{b} \to C(\mathfrak{g})$ to the linear map $\theta_0 : \mathfrak{g} \to C(\mathfrak{g})$ by setting

$$\theta_0(x_a) = x_a \otimes 1 + 1 \otimes \sum_{\beta,\gamma \in \Delta_+} c_{a,\beta}^{\gamma} x_\gamma x_\beta^*.$$

We already know that the restriction of θ_0 to $\mathfrak{n}$ is a Lie algebra homomorphism and

$$[\theta_0(x), 1 \otimes y] = 1 \otimes [x, y] \quad \text{for } x, y \in \mathfrak{n}.$$

Although θ_0 is not a Lie algebra homomorphism, we have the following.

Lemma 10 *The restriction of θ_0 to $\mathfrak{b}_-$ is a Lie algebra homomorphism. We have $[\theta_0(x), 1 \otimes y] = 1 \otimes \mathrm{ad}^*(x)(y)$ for $x \in \mathfrak{b}_-$, $y \in \mathfrak{n}^*$, where ad^* denote the coadjoint action and $\mathfrak{n}^*$ is identified with $(\mathfrak{g}/\mathfrak{b}_-)^*$.*

Let $C(\mathfrak{g})_+$ denote the subalgebra of $C(\mathfrak{g})$ generated by $\theta_0(\mathfrak{n})$ and $\Lambda(\mathfrak{n}) \subset Cl$, and let $C(\mathfrak{g})_-$ denote the subalgebra generated by $\theta_0(\mathfrak{b}_-)$ and $\Lambda(\mathfrak{n}^*) \subset Cl$.

Lemma 11 *The multiplication map gives a linear isomorphism*

$$C(\mathfrak{g})_- \otimes C(\mathfrak{g})_+ \xrightarrow{\sim} C(\mathfrak{g}).$$

Lemma 12 *The subspaces $C(\mathfrak{g})_-$ and $C(\mathfrak{g})_+$ are subcomplexes of $(C(\mathfrak{g}), \mathrm{ad}\, Q)$. Hence $C(\mathfrak{g}) \cong C(\mathfrak{g})_- \otimes C(\mathfrak{g})_+$ as complexes.*

Proof The fact that $C(\mathfrak{g})_+$ is subcomplex is obvious (see Lemma 8). The fact that $C(\mathfrak{g})_-$ is a subcomplex follows from the following formula.

$$[Q, \theta_0(x_a)] = \sum_{b \in \Delta_- \sqcup I, \alpha \in \Delta_+} c_{\alpha,a}^{b} \theta_0(x_b)(1 \otimes x_\alpha^*) - 1 \otimes \sum_{\beta,\gamma \in \Delta_+} c_{a,\beta}^{\gamma} \chi(x_\gamma) x_\beta^*$$

$$[Q, 1 \otimes x_\alpha^*] = -1 \otimes \frac{1}{2} \sum_{\beta,\gamma \in \Delta_+} c_{\beta,\gamma}^{\alpha} x_\beta^* x_\gamma^*$$

$(a \in \Delta_- \sqcup I, \alpha \in \Delta_+)$.

Proposition 3 $H^\bullet(C(\mathfrak{g})_-, \mathrm{ad}\, Q) \cong H^\bullet(C(\mathfrak{g}), \mathrm{ad}\, Q)$.

Proof By Lemma 12 and Kunneth's Theorem,

$$H^p(C(\mathfrak{g}), \mathrm{ad}\, Q) \cong \bigoplus_{i+j=p} H^i(C(\mathfrak{g})_-, \mathrm{ad}\, Q) \otimes H^j(C(\mathfrak{g})_+, \mathrm{ad}\, Q).$$

On the other hand we have $\mathrm{ad}\,(Q)(1 \otimes x_\alpha) = \theta_\chi(x_\alpha) = \theta_0(x_\alpha) - \chi(x_\alpha)$ for $\alpha \in \Delta_-$. Hence $C(\mathfrak{g})_-$ is isomorphic to the tensor product of complexes of the form $\mathbb{C}[\theta_\chi(x_\alpha)] \otimes \Lambda(x_\alpha)$ with the differential $\theta_\chi(x_\alpha) \otimes x_\alpha^*$, where x_α^* denotes the odd derivation of the exterior algebra $\Lambda(x_\alpha)$ with one variable x_α such that $x_\alpha^*(x_\alpha) = 1$.

Each of these complexes has one-dimensional zeroth cohomology and zero first cohomology. Therefore $H^i(C(\mathfrak{g})_+, \operatorname{ad} Q) = \delta_{i,0}\mathbb{C}$. This completes the proof.

Note that the cohomological gradation takes only non-negative values on $C(\mathfrak{g})_-$. Hence by Proposition 3 we may identify $\mathcal{Z}(\mathfrak{g}) = H^0(C(\mathfrak{g}), \operatorname{ad} Q)$ with the subalgebra $H^0(C(\mathfrak{g})_-, \operatorname{ad} Q) = \{c \in C(\mathfrak{g})_-^0 \mid \operatorname{ad} Q(c) = 0\}$ of $C(\mathfrak{g})_-$.

Consider the decomposition

$$C(\mathfrak{g})_-^0 = \bigoplus_{j \leqslant 0} C(\mathfrak{g})_{-,j}^0, \quad C(\mathfrak{g})_{-,j}^0 = \{c \in C(\mathfrak{g})_-^0 \mid [\theta_0(h), c] = 2jc\}.$$

Note that $C(\mathfrak{g})_{-,0}^0$ is generated by $\theta_0(\mathfrak{h})$ and is isomorphic to $U(\mathfrak{h})$. The projection

$$C(\mathfrak{g})_-^0 \to C(\mathfrak{g})_{-,0}^0 \cong U(\mathfrak{h})$$

is an algebra homomorphism, and hence, its restriction

$$\Upsilon : \mathcal{Z}(\mathfrak{g}) = H^0(C(\mathfrak{g})_-, \operatorname{ad} Q) \to U(\mathfrak{h})$$

is also an algebra homomorphism.

Proposition 4 *The map Υ is an embedding.*

Let $K_\bullet C(\mathfrak{g})_\pm$ be the filtration of $C(\mathfrak{g})_\pm$ induced by the Kazhdan filtration of $C(\mathfrak{g})$. We have the isomorphism

$$\bar{C}(\mathfrak{g}) = \operatorname{gr}_K C(\mathfrak{g}) \cong \operatorname{gr}_K C(\mathfrak{g})_- \otimes \operatorname{gr}_K C(\mathfrak{g})_+$$

as complexes. Similarly as above, we have $H^i(\operatorname{gr}_K C(\mathfrak{g})_+, \operatorname{ad} \bar{Q}) = \delta_{i,0}\mathbb{C}$, and

$$H^0(\bar{C}(\mathfrak{g}), \operatorname{ad} \bar{Q}) \cong H^0(\operatorname{gr}_K C(\mathfrak{g})_-, \operatorname{ad} \bar{Q}). \tag{12}$$

Proof (Proof of Proposition 4) The filtration $K_\bullet U(\mathfrak{h})$ of $U(\mathfrak{h}) \cong C(\mathfrak{g})_{-,0}^0$ induced by the Kazhdan filtration coincides with the usual PBW filtration. By (12) and Theorem 3, the induced map

$$H^0(\operatorname{gr}_K C(\mathfrak{g})_-, \operatorname{ad} Q) \to \operatorname{gr}_K U(\mathfrak{h})$$

can be identified with the restriction map

$$\bar{\Upsilon} : \mathbb{C}[\mathcal{S}] = \mathbb{C}[f + \mathfrak{b}]^N \to \mathbb{C}[f + \mathfrak{h}]. \tag{13}$$

It is sufficient to show that $\bar{\Upsilon}$ is injective.

If $\varphi \in \mathbb{C}[f+\mathfrak{b}]^N$ is in the kernel, $\varphi(g.x) = 0$ for all $g \in N$ and $x \in f+\mathfrak{h}$. Hence it is enough to show that the image of the the action map

$$N \times (f+\mathfrak{h}) \to f+\mathfrak{b}, \quad (g,x) \mapsto \mathrm{Ad}(g)x, \tag{14}$$

is Zariski dense in $f+\mathfrak{b}$.

The differential of this morphism at $(1,x) \in N \times (f+\mathfrak{h})$ is given by

$$\mathfrak{n} \times \mathfrak{h} \to \mathfrak{b}, \quad (y,z) \mapsto [y,x]+z.$$

This is an isomorphism if $x \in f+\mathfrak{h}_{\mathrm{reg}}$, where $\mathfrak{h}_{\mathrm{reg}} = \{x \in \mathfrak{h} \mid \mathfrak{n}^x = 0\}$. Hence (14) is a dominant morphism as required, see e.g. [76, Theorem 16.5.7].

Remark 2 The fact that $\bar{\Upsilon}$ is injective is in fact well-known. Indeed, under the identifications $\mathbb{C}[\mathcal{S}] \cong \mathbb{C}[\mathfrak{g}]^G$, $\mathbb{C}[f+\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]$, $\bar{\Upsilon}$ is identified with the Chevalley restriction map $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$, where $W = \mathfrak{S}_n$.

The advantage of the above proof is that it applies to a general finite W-algebra [71], and also, it generalizes to the affine setting, see Sect. 5.9.

The map Υ is called the classical *Miura map*.

2.7 Generalization to an Arbitrary Simple Lie Algebra

It is clear that the above argument works if we replace $\mathfrak{gl}_n$ by $\mathfrak{sl}_n$, and $\mathfrak{a}$ by $\mathfrak{a} \cap \mathfrak{sl}_n$.

More generally, let $\mathfrak{g}$ be an arbitrary simple Lie algebra. Let f be a *principal* (regular) nilpotent element of $\mathfrak{g}$, $\{e,f,h\}$ an associated $\mathfrak{sl}_2$-triple. One may assume that

$$f = \sum_{i \in I} f_i,$$

where f_i is a root vector of roots α_i and $\{\alpha_i\}_{i \in I}$ is the set of simple roots of $\mathfrak{g}$. Define the *Kostant slice* $\mathcal{S}$ by

$$\mathcal{S} := f + \mathfrak{g}^e \subset \mathfrak{g} = \mathfrak{g}^*,$$

where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$.

Then all the statements in previous subsections that make sense hold by replacing the set of companion matrices by the Kostant slice [65].

2.8 Generalization to Finite W-Algebras

In fact, the above argument works in more general setting of Hamiltonian reduction. In particular for *Slodowy slices*. Namely, for a non-zero nilpotent element f of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, we can use Jacobson-Morozov's theorem to embed f into an sl_2-triple $\{e, f, h\}$. The Slodowy slice at f is defined to be the affine subslace

$$\mathcal{S}_f = f + \mathfrak{g}^e$$

of $\mathfrak{g}$.

The Slodowy slice $\mathcal{S}_f$ has the following properties.

- $\mathcal{S}_f$ intersects the G-orbits at any point of $\mathcal{S}_f$, where G is the adjoint group of $\mathfrak{g}$.
- $\mathcal{S}_f$ admits a $\mathbb{C}^*$-action which is contracting to f.

As in the case of the set of companion matrices $\mathcal{S}_f$ can be realized by Hamiltonian reduction. Let $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h, x] = 2jx\}$, so that

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j.$$

Then the subspace $\mathfrak{g}_{1/2}$ admits a symplectic form defined by $\langle x|y\rangle = (f|[x, y])$. Choose a Lagrangian subspace l of $\mathfrak{g}_{1/2}$ with respect to this form, and set $\mathfrak{m} = l + \sum_{j \geqslant 1} \mathfrak{g}_j$. Then $\mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\chi : \mathfrak{m} \to \mathbb{C}$, $x \mapsto (f|x)$, defines a character. Let M be the unipotent subgroup of G corresponding to $\mathfrak{m}$, that is, $\operatorname{Lie} M = \mathfrak{m}$. The adjoint action of M on $\mathfrak{g}$ is Hamiltonian, so we can consider the moment map of this action

$$\mu : \mathfrak{g}^* \longrightarrow \mathfrak{m}^*,$$

which is just a restriction map. Then we have the following realization of the Slodowy slice.

$$\mathcal{S}_f \cong \frac{\mu^{-1}(\chi)}{M}$$

To obtain the BRST realization of this Hamiltonian reduction we simply replace the Clifford algebra Cl by $Cl_{\mathfrak{m}}$, i.e., the Clifford algebra associated to $\mathfrak{m} \oplus \mathfrak{m}^*$. Then we can define the operator $\operatorname{ad} \bar{Q}$ similarly and get a differential cochain complex $(\mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}_{\mathfrak{m}}, \operatorname{ad} \bar{Q})$. We have

$$\mathbb{C}[\mathcal{S}_f] \cong H^0(\mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}_{\mathfrak{m}}, \operatorname{ad} \bar{Q})$$

as Poisson algebras.

As above, this construction has a natural quantization and the quantization $U(\mathfrak{g},f)$ of $\mathcal{S}_f$ thus defined is called the *finite W-algebra associated to the pair* $(\mathfrak{g},f)$ [75]:

$$U(\mathfrak{g},f) := H^0(U(\mathfrak{g}) \otimes Cl_{\mathfrak{m}}, \operatorname{ad}\bar{Q}_+) \cong \operatorname{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi)^{op},$$

where $\mathbb{C}_\chi$ is the one-dimensional representation of $\mathfrak{m}$ defined by χ (cf. [5, 29]).

3 Arc Spaces, Poisson Vertex Algebras, and Associated Varieties of Vertex Algebras

3.1 Vertex Algebras

A *vertex algebra* is a vector space V equipped with $|0\rangle \in V$ (the vacuum vector), $T \in \operatorname{End} V$ (the translation operator), and a bilinear product

$$V \times V \to V((z)), \quad (a,b) \mapsto a(z)b,$$

where $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, $a_{(n)} \in \operatorname{End} V$, such that

1. $(|0\rangle)(z) = \mathrm{id}_V$,
2. $a(z)|0\rangle \in V[[z]]$ and $\lim_{z\to 0} a(z)|0\rangle = a$ for all $a \in V$,
3. $(Ta)(z) = \partial_z a(z)$ for all $a \in V$, where $\partial_z = d/dz$,
4. for any $a,b \in V$, $(z-w)^{N_{a,b}}[a(z),b(w)] = 0$ for some $N_{a,b} \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$.

The last condition is called the *locality*, which is equivalent to the fact that

$$[a(z),b(w)] = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w)\frac{1}{n!}\partial_w^n \delta(z-w), \tag{15}$$

where $\delta(z-w) = \sum_{n\in\mathbb{Z}} w^n z^{-n-1} \in \mathbb{C}[[z,w,z^{-1},w^{-1}]]$.

A consequence of the definition is the following *Borcherds identities*:

$$[a_{(m)}, b_{(n)}] = \sum_{i\geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}, \tag{16}$$

$$(a_{(m)}b)_{(n)} = \sum_{j\geqslant 0} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}). \tag{17}$$

We write (15) as

$$[a_\lambda b] = \sum_{n\geqslant 0} \frac{\lambda^n}{n!} a_{(n)} b \in V[\lambda],$$

and call it the *λ-bracket* of a and b. (We have $a_{(n)}b = 0$ if $(z-w)^n[a(z), b(w)] = 0$.)

Here are some properties of λ-brackets.

$$[(Ta)_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda(Tb)] = (\lambda + T)[a_\lambda b], \tag{18}$$

$$[b_\lambda a] = -[a_{-\lambda-T} b], \tag{19}$$

$$[a_\lambda[b_\mu c]] - [b_\mu[a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]. \tag{20}$$

The normally ordered product on V is defied as $:ab: := a_{(-1)}b$. We also write $:ab:(z) =: a(z)b(z):$. We have

$$:a(z)b(z): = a(z)_+ b(z) + b(w)a(z)_-,$$

where $a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}$, $a(z)_- = \sum_{n\geqslant 0} a_{(n)} z^{-n-1}$. We have the following *non-commutative Wick formula*.

$$[a_\lambda :bc:] =: [a_\lambda b]c: + :[a_\lambda c]b: + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu, \tag{21}$$

$$[:ab:_\lambda c] =: (e^{T\partial_\lambda} a)[b_\lambda c]: + :(e^{T\partial_\lambda} b)[a_\lambda c]: + \int_0^\lambda [b_\mu[a_{\lambda-\mu} c]] d\mu. \tag{22}$$

3.2 Commutative Vertex Algebras and Differential Algebras

A vertex algebra V is called *commutative* if

$$[a_\lambda b] = 0, \quad \forall a, b \in V,$$

or equivalently, $a_{(n)} = 0$ for $n \geqslant 0$ in $\operatorname{End} V$ for all $a \in V$. This condition is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0 \quad \forall a, b \in \mathbb{Z},\ m, n \in \mathbb{Z}$$

by (16).

A commutative vertex algebra has the structure of a unital commutative algebra by the product

$$a \cdot b =: ab := a_{(-1)} b,$$

where the unite is given by the vacuum vector $|0\rangle$. The translation operator T of V acts on V as a derivation with respect to this product:

$$T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb).$$

Therefore a commutative vertex algebra has the structure of a *differential algebra*, that is, a unital commutative algebra equipped with a derivation. Conversely, there is a unique vertex algebra structure on a differential algebra R with a derivation T such that

$$Y(a, z) = e^{zT} a$$

for $a \in R$. This correspondence gives the following.

Theorem 6 ([24]) *The category of commutative vertex algebras is the same as that of differential algebras.*

3.3 Arc Spaces

Define the (formal) disc as

$$D = \mathrm{Spec}(\mathbb{C}[[t]]).$$

For a scheme X, a homomorphism $\alpha : D \to X$ is called an *arc* of X.

Theorem 7 ([25, 33, 52]) *Let X be a scheme of finite type over $\mathbb{C}$, Sch the category of schemes of finite type over $\mathbb{C}$, Set the category of sets. The contravariant functor*

$$Sch \to Set, \quad Y \mapsto \mathrm{Hom}_{Sch}(Y \widehat{\times} D, X),$$

is represented by a scheme JX, that is,

$$\mathrm{Hom}_{Sch}(Y, JX) \cong \mathrm{Hom}_{Sch}(Y \widehat{\times} D, X).$$

for any $Y \in Sch$. Here $Y \widehat{\times} D$ is the completion of $Y \times D$ with respect to the subscheme $Y \widehat{\times} \{0\}$.

By definition, the $\mathbb{C}$-points of JX are

$$\mathrm{Hom}_{Sch}(\mathrm{Spec}\,\mathbb{C}, JX) = \mathrm{Hom}_{Sch}(D, X),$$

that is, the set of arcs of X. The reason we need the completion $Y \widehat{\times} D$ in the definition is that $A \otimes \mathbb{C}[[t]] \subsetneq A[[t]] = A \widehat{\otimes} \mathbb{C}[[t]]$ in general.

The scheme JX is called the *arc space*, or the *infinite jet scheme*, of X.

It is easy to describe JX when X is affine:

First, consider the case $X = \mathbb{C}^N = \operatorname{Spec}\mathbb{C}[x_1, x_2, \cdots, x_N]$. The $\mathbb{C}$-points of JX are the arcs $\operatorname{Hom}_{Sch}(D, JX)$, that is, the ring homomorphisms

$$\gamma : \mathbb{C}[x_1, x_2, \cdots, x_N] \to \mathbb{C}[[t]].$$

Such a map is determined by the image

$$\gamma(x_i) = \sum_{n \geqslant 0} \gamma_{i,(-n-1)} t^n \tag{23}$$

of each x_i, and conversely, the coefficients $\{\gamma_{i,(-n-1)}\}$ determines a $\mathbb{C}$-point of JX. If we choose coordinates $x_{i,(-n-1)}$ of JX as $x_{i,(-n-1)}(\gamma) = \gamma_{i,(-n-1)}$, we have

$$J\mathbb{C}^N = \operatorname{Spec}\mathbb{C}[x_{i,(n)} | i = 1, 2, \cdots, N, n = -1, -2, \cdots].$$

Next, let $X = \operatorname{Spec} R$, with $R = \mathbb{C}[x_1, x_2, \cdots, x_N]/\langle f_1, f_2, \cdots, f_r\rangle$. The arcs of X are

$$\operatorname{Hom}_{\text{ring}}\Big(\frac{\mathbb{C}[x_1, x_2, \cdots, x_n]}{< f_1, f_2, \cdots, f_r >}, \mathbb{C}[[t]]\Big) \subset \operatorname{Hom}_{\text{ring}}(\mathbb{C}[x_1, x_2, \cdots, x_n], \mathbb{C}[[t]]).$$

An element $\gamma \in \operatorname{Hom}_{\text{ring}}(\mathbb{C}[x_1, x_2, \cdots, x_n], \mathbb{C}[[t]])$ is an element of this subset if and only if $\gamma(f_i) = 0$ for $i = 1, 2, \cdots, r$. By writing

$$f_i(x_1(t), x_2(t), \ldots, x_N(t)) = \sum_{m \geqslant 0} \frac{f_{i,m}}{m!} t^m$$

with $f_{i,m} \in \mathbb{C}[x_{i,(-n-1)}]$, where $x_i(t) := \sum_{m \geqslant 0} x_{i,(-m-1)} t^m$, we get that

$$JX = \operatorname{Spec} \frac{\mathbb{C}[x_{i(n)} | i = 1, 2, \cdots, N; n = -1, -2, \cdots]}{\langle f_{i,m}(x_{i(n)}), i = 1, 2, \cdots, r; m \geqslant 0\rangle}.$$

Lemma 13 *Define the derivation T of $\mathbb{C}[x_{i(n)} | i = 1, 2, \cdots, N; n = -1, -2, \cdots]$ by*

$$Tx_{i(n)} = -n x_{i(n-1)}.$$

Then $f_{i,m} = T^n f_i$ for $n \geqslant 0$. Here we identify x_i with $x_{i(-1)}$.

With the above lemma, we conclude that for the affine scheme $X = \operatorname{Spec} R$, $R = \mathbb{C}[x_1, x_2, \cdots, x_n]/\langle f_1, f_2, \cdots, f_r\rangle$, its arc space JX is the affine scheme $\operatorname{Spec}(JR)$, where

$$JR := \frac{\mathbb{C}[x_{i(n)} | i = 1, 2, \cdots, N; n = -1, -2, \cdots]}{\langle T^n f_i, i = 1, 2, \cdots, r; n \geqslant 0\rangle}$$

and T is as defined in the lemma.

The derivation T acts on the above quotient ring JR. Hence for an affine scheme $X = \operatorname{Spec} R$, the coordinate ring $JR = \mathbb{C}[JX]$ of its arc space JX is a differential algebra, hence is a commutative vertex algebra.

Remark 3 The differential algebra JR has the universal property that

$$\mathrm{Hom}_{\text{dif.alg.}}(JR, A) \cong \mathrm{Hom}_{\text{ring}}(R, A)$$

for any differential algebra A, where $\mathrm{Hom}_{\text{diff. alg.}}(JR, A)$ is the set of homomorphisms $JR \to A$ of differential algebras.

For a general scheme Y of finite type with an affine open covering $\{U_i\}_{i\in I}$, its arc space JY is obtained by glueing JU_i (see [33, 52]). In particular, the structure sheaf $\mathcal{O}_{JY}$ is a sheaf of commutative vertex algebras.

There is a natural projection $\pi_\infty : JX \to X$ that corresponds to the embedding $R \hookrightarrow JR$, $x_i \to x_{i,(-1)}$, in the case X is affine. In terms of arcs, $\pi_\infty(\alpha) = \alpha(0)$ for $\alpha \in \mathrm{Hom}_{Sch}(D, X)$, where 0 is the unique closed point of the disc D.

The map from a scheme to its arc space is functorial. i.e., a scheme homomorphism $f : X \to Y$ induces a scheme homomorphism $Jf : JX \to JY$ that makes the following diagram commutative:

$$\begin{array}{ccc} JX & \xrightarrow{Jf} & JY \\ \downarrow{\scriptstyle \pi_\infty} & & \downarrow{\scriptstyle \pi_\infty} \\ X & \xrightarrow{f} & Y. \end{array}$$

In terms of arcs, $Jf(\alpha) = f \circ \alpha$ for $\alpha \in \mathrm{Hom}_{Sch}(D, X)$.

We also have

$$J(X \times Y) \cong JX \times JY. \tag{24}$$

Indeed, for any scheme Z,

$$\begin{aligned} \mathrm{Hom}(Z, J(X \times Y)) &= \mathrm{Hom}(Z \widehat{\times} D, X \times Y) \\ &\cong \mathrm{Hom}(Z \widehat{\times} D, X) \times \mathrm{Hom}(Y \widehat{\times} D, Y) \\ &= \mathrm{Hom}(Z, JX) \times \mathrm{Hom}(Y, JY) \\ &\cong \mathrm{Hom}(Z, JX \times JY). \end{aligned}$$

Lemma 14 *The natural morphism $X_{\text{red}} \to X$ induces an isomorphism $JX_{\text{red}} \to JX$ of topological spaces, where X_{red} denotes the reduced scheme of X.*

Proof We may assume that $X = \operatorname{Spec} R$. An arc α of X corresponds to a ring homomorphism $\alpha^* : R \to \mathbb{C}[[t]]$. Since $\mathbb{C}[[t]]$ is an integral domain it decomposes as $\alpha^* : R \to R/\sqrt{0} \to \mathbb{C}[[t]]$. Thus, α is an arc of X_{red}.

If X is a point, then JX is also a point, since $\mathrm{Hom}(D, X) = \mathrm{Hom}(\mathbb{C}, \mathbb{C}[[t]])$ consists of only one element. Thus, Lemma 14 implies the following.

Corollary 2 *If X is zero-dimensional then JX is also zero-dimensional.*

Theorem 8 ([64]) *JX is irreducible if X is irreducible.*

Lemma 15 *Let Y be irreducible, and let $f : X \to Y$ be a morphism that restricts to a bijection between some open subsets $U \subset X$ and $V \subset Y$. Then $Jf : JX \to JY$ is dominant.*

Proof Jf restricts to the isomorphism $JU \xrightarrow{\sim} JV$, and the open subset JV is dense in JY since JY is irreducible.

3.4 Arc Space of Poisson Varieties and Poisson Vertex Algebras

Let V be a commutative vertex algebra, or equivalently, a differential algebra. V is called a *Poisson vertex algebras* if it is equipped with a bilinear map

$$V \times V \to V[\lambda], \quad (a, b) \mapsto \{a_\lambda b\} = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} a_{(n)} b, \quad a_{(n)} \in \mathrm{End}\, V,$$

also called the *λ-bracket*, satisfying the following axioms:

$$\{(Ta)_\lambda b\} = -\lambda \{a_\lambda b\}, \quad \{a_\lambda (Tb)\} = (\lambda + T)\{a_\lambda b\}, \tag{25}$$

$$\{b_\lambda a\} = -\{a_{-\lambda - T} b\}, \tag{26}$$

$$\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\}, \tag{27}$$

$$\{a_\lambda (bc)\} = \{a_\lambda b\} c + \{a_\lambda c\} b, \quad \{(ab)_\lambda c\} = \{a_{\lambda+T} c\}_\to b + \{b_{\lambda+T} c\}_\to a, \tag{28}$$

where the arrow means that $\lambda + T$ should be moved to the right, that is, $\{a_{\lambda+T} c\}_\to b = \sum_{n \geqslant 0} (a_{(n)} c) \frac{(\lambda+T)^n}{n!} b$.

The first equation in (28) says that $a_{(n)}$, $n \geqslant 0$, is a derivation of the ring V. (Do not confuse $a_{(n)} \in \mathrm{Der}(V)$, $n \geqslant 0$, with the multiplication $a_{(n)}$ as a vertex algebra, which should be zero for a commutative vertex algebra.)

Note that (25), (26), (27) are the same as (18), (19), (20), and (28) is the same with (21) and (22) without the third terms. In particular, by (27), we have

$$[a_{(m)}, b_{(n)}] = \sum_{i \geqslant 0} \binom{m}{i} (a_{(i)} b)_{(m+n-i)}, \quad m, n \in \mathbb{Z}_+. \tag{29}$$

Theorem 9 ([7, Proposition 2.3.1]) *Let X be an affine Poisson scheme, that is, $X = \mathrm{Spec}\, R$ for some Poisson algebra R. Then there is a unique Poisson vertex* *algebra structure on* $JR = \mathbb{C}[JX]$ *such that*

$$\{a_\lambda b\} = \{a, b\} \quad \text{for } a, b \in R \subset JR,$$

where $\{a, b\}$ *is the Poisson bracket in* R*.*

Proof The uniqueness is clear by (18) since JR is generated by R as a differential algebra. We leave it to the reader to check the well-definedness.

Remark 4 More generally, let X be a Poisson scheme which is not necessarily affine. Then the structure sheaf $\mathcal{O}_{JX}$ carries a unique vertex Poisson algebra structure such that $\{f_\lambda g\} = \{f, g\}$ for $f, g \in \mathcal{O}_X \subset \mathcal{O}_{JX}$, see [16, Lemma 2.1.3.1].

Example 1 Let G be an affine algebraic group, $\mathfrak{g} = \operatorname{Lie} G$. The arc space JG is naturally a proalgebraic group. Regarding JG as the $\mathbb{C}[[t]]$-points of G, we have $JG = G[[t]]$. Similarly, $J\mathfrak{g} = \mathfrak{g}[[t]] = \operatorname{Lie}(JG)$.

The affine space $\mathfrak{g}^*$ is a Poisson variety by the Kirillov-Kostant Poisson structure, see Sect. 2.3. If $\{x_i\}$is a basis of $\mathfrak{g}$, then

$$\mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[x_1, x_2, \cdots, x_n].$$

Thus

$$J\mathfrak{g}^* = \operatorname{Spec} \mathbb{C}[x_{i(-n)} | i = 1, 2, \cdots, l; n \geqslant 1]. \tag{30}$$

So we may identify $\mathbb{C}[J\mathfrak{g}^*]$ with the symmetric algebra $S(\mathfrak{g}[t^{-1}]t^{-1})$.

Let $x = x_{(-1)}|0\rangle = (xt^{-1})|0\rangle$, where we denote by $|0\rangle$ the unite element in $S(\mathfrak{g}[t^{-1}]t^{-1})$. Then (29) gives that

$$[x_{(m)}, y_{(n)}] = [x, y]_{(m+n)}, \quad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z}_{\geqslant 0}. \tag{31}$$

So the Lie algebra $J\mathfrak{g} = \mathfrak{g}[[t]]$ acts on $\mathbb{C}[J\mathfrak{g}^*]$. This action coincides with that obtained by differentiating the action of $JG = G[[t]]$ on $J\mathfrak{g}^*$ induced by the coadjoint action of G. In other words, the vertex Poisson algebra structure of $\mathbb{C}[J\mathfrak{g}^*]$ comes from the JG-action on $J\mathfrak{g}^*$.

3.5 Canonical Filtration of Vertex Algebras

Haisheng Li [68] has shown that *every* vertex algebra is canonically filtered: For a vertex algebra V, let $F^p V$ be the subspace of V spanned by the elements

$$a^1_{(-n_1-1)} a^2_{(-n_2-1)} \cdots a^r_{(-n_r-1)} |0\rangle$$

with $a^1, a^2, \cdots, a^r \in V, n_i \geqslant 0, n_1 + n_2 + \cdots + n_r \geqslant p$. Then

$$V = F^0 V \supset F^1 V \supset \dots.$$

It is clear that $TF^p V \subset F^{p+1} V$.

Set $(F^p V)_{(n)} F^q V := \mathrm{span}_{\mathbb{C}}\{a_{(n)} b | a \in F^p V, b \in F^q V\}$.

Lemma 16 *We have*

$$F^p V = \sum_{j \geqslant 0} (F^0 V)_{(-j-1)} F^{p-j} V.$$

Proposition 5

(1) $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n-1} V$. *Moreover, if* $n \geqslant 0$, *we have* $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n} V$.
(2) The filtration $F^\bullet V$ *is separated, that is,* $\bigcap_{p \geqslant 0} F^p V = \{0\}$, *if* V *is a positive energy representation over itself.*

Proof It is straightforward to check. [(2) also follows from Lemma 17 below.]

In this note we assume that the filtration $F^\bullet V$ is separated.

Set

$$\mathrm{gr}\, V = \bigoplus_{p \geqslant 0} F^p V / F^{p+1} V.$$

We denote by $\sigma_p : F^p V \mapsto F^p V / F^{p+1} V$ for $p \geqslant 0$, the canonical quotient map.

Proposition 5 gives the following.

Proposition 6 ([68]) *The space* $\mathrm{gr}\, V$ *is a Poisson vertex algebra by*

$$\sigma_p(a) \cdot \sigma_q(b) := \sigma_{p+q}(a_{(-1)} b), \quad \sigma_p(a)_{(n)} \sigma_q(b) := \sigma_{p+q-n}(a_{(n)} b)$$

for $a \in F^p V$, $b \in F^q V$, $n \geqslant 0$.

Set

$$R_V := F^0 V / F^1 V \subset \mathrm{gr}\, V.$$

Note that $F^1 V = \mathrm{span}_{\mathbb{C}}\{a_{(-2)} b \mid a, b \in V\}$.

Proposition 7 ([68, 79]) *The restriction of the Poisson structure gives* R_V *a Poisson algebra structure, that is,* R_V *is a Poisson algebra by*

$$\bar{a} \cdot \bar{b} := \overline{a_{(-1)} b}, \quad \{\bar{a}, \bar{b}\} = \overline{a_{(0)} b},$$

where $\bar{a} = \sigma_0(a)$.

Proof It is straightforward from Proposition 6.

In the literature F^1V is often denoted by $C_2(V)$ and the Poisson algebra R_V is called *Zhu's* C_2*-algebra*.

A vertex algebra V is called *finitely strongly generated* if R_V is finitely generated as a ring. If the images of vectors $a_1, \dots, a_N \in V$ generate R_V, we say that V is strongly generated by $a_1, \dots, a_N$.

Below we always assume that a vertex algebra V is finitely strongly generated.

Note that if $\phi : V \to W$ is a homomorphism of vertex algebras, ϕ respects the canonical filtration, that is, $\phi(F^pV) \subset F^pW$. Hence it induces the homomorphism $\operatorname{gr} V \to \operatorname{gr} W$ of Poisson vertex algebra homomorphism which we denote by $\operatorname{gr}\phi$.

3.6 Associated Variety and Singular Support of Vertex Algebras

Definition 1 Define the *associated scheme* $\tilde{X}_V$ and the *associated variety* X_V of a vertex algebra V as

$$\tilde{X}_V := \operatorname{Spec} R_V, \quad X_V := \operatorname{Specm} R_V = (\tilde{X}_V)_{\mathrm{red}}.$$

It was shown in [68, Lemma 4.2] that $\operatorname{gr} V$ is generated by the subring R_V as a differential algebra. Thus, we have a surjection $JR_V \to \operatorname{gr} V$ of differential algebras by Remark 3. This is in fact a homomorphism of Poisson vertex algebras:

Theorem 10 ([68, Lemma 4.2], [7, Proposition 2.5.1]) *The identity map* $R_V \to R_V$ *induces a surjective Poisson vertex algebra homomorphism*

$$JR_V = \mathbb{C}[J\tilde{X}_V] \twoheadrightarrow \operatorname{gr} V.$$

Let $a^1, \dots, a^n$ be a set of strong generators of V. Since $\operatorname{gr} V \cong V$ as $\mathbb{C}$-vector spaces by the assumption that $F^\bullet V$ is separated, it follows from Theorem 10 that V is spanned by elements

$$a^{i_1}_{(-n_1)} \dots a^{i_r}_{(-n_r)}|0\rangle \quad \text{with } r \geqslant 0,\ n_i \geqslant 1.$$

Definition 2 Define the *singular support* of a vertex algebra V as

$$SS(V) := \operatorname{Spec}(\operatorname{gr} V) \subset J\tilde{X}_V.$$

Theorem 11 *We have* $\dim SS(V) = 0$ *if and only if* $\dim X_V = 0$.

Proof The "only if" part is obvious sine $\pi_\infty(SS(V)) = \tilde{X}_V$, where $\pi_\infty : J\tilde{X}_V \to \tilde{X}_V$ is the projection. The "if" part follows from Corollary 2.

Definition 3 We call V *lisse* (or C_2*-cofinite*) if $\dim X_V = 0$.

Remark 5 Suppose that V is $\mathbb{Z}_+$-graded, so that $V = \bigoplus_{i \geqslant 0} V_i$, and that $V_0 = \mathbb{C}|0\rangle$. Then $\operatorname{gr} V$ and R_V are equipped with the induced grading:

$$\operatorname{gr} V = \bigoplus_{i \geqslant 0} (\operatorname{gr} V)_i, \quad (\operatorname{gr} V)_0 = \mathbb{C},$$

$$R_V = \bigoplus_{i \geqslant 0} (R_V)_i, \quad (R_V)_0 = \mathbb{C}.$$

So the following conditions are equivalent:

1. V is lisse.
2. $X_V = \{0\}$.
3. The image of any vector $a \in V_i$ for $i \geqslant 1$ in $\operatorname{gr} V$ is nilpotent.
4. The image of any vector $a \in V_i$ for $i \geqslant 1$ in R_V is nilpotent.

Thus, lisse vertex algebras can be regarded as a generalization of finite-dimensional algebras.

Remark 6 Suppose that the Poisson structure of R_V is trivial. Then the Poisson vertex algebra structure of JR_V is trivial, and so is that of $\operatorname{gr} V$ by Theorem 10. This happens if and only if

$$(F^p V)_{(n)}(F^q V) \subset F^{p+q-n+1} V \quad \text{for all } n \geqslant 0.$$

If this is the case, one can give $\operatorname{gr} V$ yet another Poisson vertex algebra structure by setting

$$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n+1}(a_{(n)}b) \quad \text{for } n \geqslant 0. \tag{32}$$

(We can repeat this procedure if this Poisson vertex algebra structure is again trivial.)

3.7 Comparison with Weight-Depending Filtration

Let V be a vertex algebra that is $\mathbb{Z}$-graded by some Hamiltonian H:

$$V = \bigoplus_{\Delta \in \mathbb{Z}} V_\Delta \quad \text{where} \quad V_\Delta := \{v \in V | Hv = \Delta v\}.$$

Then there is [67] another natural filtration of V defined as follows.

For a homogeneous vector $a \in V_\Delta$, Δ is called the *conformal weight* of a and is denote by Δ_a. Let $G_p V$ be the subspace of V spanned by the vectors

$$a^1_{(-n_1-1)} a^2_{(-n_2-1)} \cdots a^r_{(-n_r-1)} |0\rangle$$

with $\Delta_{a^1} + \cdots + \Delta_{a^r} \leqslant p$. Then $G_\bullet V$ defines an increasing filtration of V:

$$0 = G_{-1}V \subset G_0 V \subset \ldots G_1 V \subset \ldots, \quad V = \bigcup_p G_p V.$$

Moreover we have

$$\begin{aligned} &TG_pV \subset G_pV, \\ &(G_p)_{(n)}G_qV \subset G_{p+q}V \quad \text{for } n \in \mathbb{Z}, \\ &(G_p)_{(n)}G_qV \subset G_{p+q-1}V \quad \text{for } n \in \mathbb{Z}_+, \end{aligned}$$

It follows that $\operatorname{gr}_G V = \bigoplus G_pV/G_{p-1}V$ is naturally a Poisson vertex algebra.

It is not too difficult to see the following.

Lemma 17 ([7, Proposition 2.6.1]) *We have*

$$F^pV_\Delta = G_{\Delta-p}V_\Delta,$$

where $F^pV_\Delta = V_\Delta \cap F^pV$, $G_pV_\Delta = V_\Delta \cap G_pV$. *Therefore*

$$\operatorname{gr} V \cong \operatorname{gr}_G V$$

as Poisson vertex algebras.

3.8 Example: Universal Affine Vertex Algebras

Let $\mathfrak{a}$ be a Lie algebra with a symmetric invariant bilinear form κ. Let

$$\widehat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$$

be the Kac-Moody affinization of $\mathfrak{a}$. It is a Lie algebra with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}\kappa(x, y)\mathbf{1}, \quad x, y \in \mathfrak{a},\ m, n \in \mathbb{Z}, \qquad [\mathbf{1}, \widehat{\mathfrak{a}}] = 0.$$

Let

$$V^\kappa(\mathfrak{a}) = U(\hat{\mathfrak{a}}) \otimes_{U(\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C},$$

where $\mathbb{C}$ is one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{a}[t]$ acts trivially and $\mathbf{1}$ acts as the identity. The space $V^\kappa(\mathfrak{a})$ is naturally graded: $V^\kappa(\mathfrak{a}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} V^\kappa(\mathfrak{a})_\Delta$, where the grading is defined by setting $\deg xt^n = -n$, $\deg |0\rangle = 0$.

Here $|0\rangle = 1 \otimes 1$. We have $V^{\kappa}(\mathfrak{a})_0 = \mathbb{C}|0\rangle$. We identify $\mathfrak{a}$ with $V^{\kappa}(\mathfrak{a})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

There is a unique vertex algebra structure on $V^{\kappa}(\mathfrak{a})$ such that $|0\rangle$ is the vacuum vector and

$$Y(x,z) = x(z) := \sum_{n\in\mathbb{Z}} (xt^n)z^{-n-1}, \quad x \in \mathfrak{a}.$$

(So $x_{(n)} = xt^n$ for $x \in \mathfrak{a} = V^{\kappa}(\mathfrak{a})_1$, $n \in \mathbb{Z}$).

The vertex algebra $V^{\kappa}(\mathfrak{a})$ is called the *universal affine vertex algebra associated with* $(\mathfrak{a}, \kappa)$.

We have $F^1 V^{\kappa}(\mathfrak{a}) = \mathfrak{a}[t^{-1}]t^{-2}V^{\kappa}(\mathfrak{a})$, and the Poisson algebra isomorphism

$$\begin{aligned} \mathbb{C}[\mathfrak{a}^*] &\xrightarrow{\sim} R_{V^{\kappa}(\mathfrak{a})} = V^k(\mathfrak{a})/\mathfrak{g}[t^{-1}]t^{-2}V^{\kappa}(\mathfrak{a}) \\ x_1 \dots x_r &\longmapsto \overline{(x_1t^{-1})\dots(x_rt^{-1})|0\rangle} \quad (x_i \in \mathfrak{a}). \end{aligned} \tag{33}$$

Thus

$$X_{V^{\kappa}(\mathfrak{a})} = \mathfrak{a}^*.$$

We have the isomorphism

$$\mathbb{C}[J\mathfrak{a}^*] \simeq \operatorname{gr} V^{\kappa}(\mathfrak{a}) \tag{34}$$

because the graded dimensions of both sides coincide. Therefore

$$SS(V^{\kappa}(\mathfrak{a})) = J\mathfrak{a}^*.$$

The isomorphism (34) follows also from the fact that

$$G_p V^{\kappa}(\mathfrak{a}) = U_p(\mathfrak{a}[t^{-1}]t^{-1})|0\rangle,$$

where $\{U_p(\mathfrak{a}[t^{-1}]t^{-1})\}$ is the PBW filtration of $U(\mathfrak{a}[t^{-1}]t^{-1})$.

3.9 Example: Simple Affine Vertex Algebras

For a finite-dimensional simple Lie algebra $\mathfrak{g}$ and $k \in \mathbb{C}$, we denote by $V^k(\mathfrak{g})$ the universal affine vertex algebra $V^{k\kappa_0}(\mathfrak{g})$, where κ_0 is the normalized invariant inner product of $\mathfrak{g}$, that is,

$$\kappa_0(\theta, \theta) = 2,$$

where θ is the highest root of $\mathfrak{g}$. Denote by $V_k(\mathfrak{g})$ the unique simple graded quotient of $V^k(\mathfrak{g})$. As a $\hat{\mathfrak{g}}$-module, $V_k(\mathfrak{g})$ is isomorphic to the irreducible highest weight representation $L(k\Lambda_0)$ of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, where Λ_0 is the weight of the basic representation of $\widehat{\mathfrak{g}}$.

Theorem 12 *The vertex algebra $V_k(\mathfrak{g})$ is lisse if and only if $V_k(\mathfrak{g})$ is integrable as a $\hat{\mathfrak{g}}$-module, which is true if and only if $k \in \mathbb{Z}_+$.*

Lemma 18 *Let (R, ∂) be a differential algebra over $\mathbb{Q}$, I a differential ideal of R, i.e., I is an ideal of R such that $\partial I \subset I$. Then $\partial\sqrt{I} \subset \sqrt{I}$.*

Proof Let $a \in \sqrt{I}$, so that $a^m \in I$ for some $m \in \mathbb{N} = \{1, 2, \ldots\}$. Since I is ∂-invariant, we have $\partial^m a^m \in I$. But

$$\partial^m a^m = \sum_{0 \leqslant i \leqslant m} \binom{m}{i} a^{m-i} (\partial a)^i \equiv m!(\partial a)^m \pmod{\sqrt{I}}.$$

Hence $(\partial a)^m \in \sqrt{I}$, and therefore, $\partial a \in \sqrt{I}$.

Proof (Proof of the "if" Part of Theorem 12) Suppose that $V_k(\mathfrak{g})$ is integrable. This condition is equivalent to that $k \in \mathbb{Z}_+$ and the maximal submodule N_k of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$ [54]. The exact sequence $0 \to N_k \to V^k(\mathfrak{g}) \to V_k(\mathfrak{g}) \to 0$ induces the exact sequence

$$0 \to I_k \to R_{V^k(\mathfrak{g})} \to R_{V_k(\mathfrak{g})} \to 0,$$

where I_k is the image of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$, and so, $R_{V_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_k$. The image of the singular vector in I_k is given by e_θ^{k+1}. Therefore, $e_\theta \in \sqrt{I}$. On the other hand, by Lemma 18, $\sqrt{I_k}$ is preserved by the adjoint action of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{g} \subset \sqrt{I}$. This proves that $X_{V_k(\mathfrak{g})} = \{0\}$ as required.
The proof of "only if" part follows from [30]. We will give a different proof using W-algebras in Remark 13.

In view of Theorem 12, one may regard the lisse condition as a generalization of the integrability condition to an arbitrary vertex algebra.

4 Zhu's Algebra

In this section we will introduce and study the Zhu's algebra of a vertex algebra, which plays an important role in the representation theory.

See [55] in this volume for the definition of modules over vertex algebras.

4.1 Zhu's C_2-Algebra and Zhu's Algebra of a Vertex Algebra

Let V be a $\mathbb{Z}$-graded vertex algebra. *Zhu's algebra* Zhu$V$ [43, 79] is defined as

$$\mathrm{Zhu}(V) := V/V \circ V$$

where $V \circ V := \mathrm{span}\{a \circ b | a, b \in V\}$ and

$$a \circ b := \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} b$$

for homogeneous elements a, b and extended linearly. It is an associative algebra with multiplication defined as

$$a * b := \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-1)} b$$

for homogeneous elements $a, b \in V$.

For a simple positive energy representation $M = \bigoplus_{n \in \mathbb{Z}_+} M_{\lambda+n}$, $M_\lambda \neq 0$, $\lambda \in \mathbb{C}$, of V, let M_{top} be the top degree component M_λ of M. Also, for a homogeneous vector $a \in V$, let $o(a) = a_{(\Delta_a - 1)}$, so that $o(a)$ preserves the homogeneous components of any graded representation of V.

The importance of Zhu's algebra in vertex algebra theory is the following fact that was established by Yonchang Zhu.

Theorem 13 ([79]) *For any positive energy representation M of V, $\bar{a} \mapsto o(a)$ defines a well-defined representation of* Zhu(V) *on M_{top}. Moreover, the correspondence $M \mapsto M_{top}$ gives a bijection between the set of isomorphism classes of irreducible positive energy representations of V and that of simple* Zhu(V)*-modules.*
A vertex algebra V is called a *chiralization* of an algebra A if $\mathrm{Zhu}(V) \cong A$.

Now we define an increasing filtration of Zhu's algebra. For this, we assume that V is $\mathbb{Z}_+$-graded: $V = \bigoplus_{\Delta \geqslant 0} V_\Delta$. Then $V_{\leqslant p} = \bigoplus_{\Delta=0}^{p} V_\Delta$ gives an increasing filtration of V. Define

$$\mathrm{Zhu}_p(V) := \mathrm{Im}(V_{\leqslant p} \to \mathrm{Zhu}(V)).$$

Obviously, we have

$$0 = \mathrm{Zhu}_{-1}(V) \subset \mathrm{Zhu}_0(V) \subset \mathrm{Zhu}_1(V) \subset \cdots, \quad \text{and} \quad \mathrm{Zhu}(V) = \bigcup_{p \geqslant -1} \mathrm{Zhu}_p(V).$$

Also, since $a_{(n)} b \in V_{\Delta_a + \Delta_b - n - 1}$ for $a \in V_{\Delta_a}$, $b \in V_{\Delta_b}$, we have

$$\mathrm{Zhu}_p(V) * \mathrm{Zhu}_q(V) \subset \mathrm{Zhu}_{p+q}(V). \tag{35}$$

The following assertion follows from the skew symmetry.

Lemma 19 *We have*

$$b * a \equiv \sum_{i\geqslant 0} \binom{\Delta_a - 1}{i} a_{(i-1)}b \quad (\mathrm{mod}\ V \circ V),$$

and hence,

$$a * b - b * a \equiv \sum_{i\geqslant 0} \binom{\Delta_a - 1}{i} a_{(i)}b \quad (\mathrm{mod}\ V \circ V).$$

By Lemma 19, we have

$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(V)] \subset \mathrm{Zhu}_{p+q-1}(V). \tag{36}$$

By (35) and (36), the associated graded $\mathrm{gr}\,\mathrm{Zhu}(V) = \bigoplus_p \mathrm{Zhu}_p(V)/\mathrm{Zhu}_{p-2}(V)$ is naturally a graded Poisson algebra.

Note that $a \circ b \equiv a_{(-2)}b$ $(\mathrm{mod} \bigoplus_{\Delta \leqslant \Delta_a + \Delta_b} V_\Delta)$ for homogeneous elements $a, b \in V$.

Lemma 20 (Zhu, See [29, Proposition 2.17(c)], [17, Proposition 3.3]) *The following map defines a well-defined surjective homomorphism of Poisson algebras.*

$$\begin{aligned} \eta_V : R_V &\longrightarrow \mathrm{gr}\,\mathrm{Zhu}(V) \\ \bar{a} &\longmapsto a \ \ (\mathrm{mod}\ V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta). \end{aligned}$$

Remark 7 The map η_V is not an isomorphism in general. For an example, let $\mathfrak{g}$ be the simple Lie algebra of type E_8 and $V = V_1(\mathfrak{g})$. Then $\dim R_V > \dim \mathrm{Zhu}(V) = 1$.

Corollary 3 *If V is lisse then* Zhu(V) *is finite dimensional. Hence the number of isomorphic classes of simple positive energy representations of V is finite.*

In fact the following stronger facts are known.

Theorem 14 ([1]) *Let V be lisse. Then any simple V-module is a positive energy representation. Therefore the number of isomorphic classes of simple V-modules is finite.*

Theorem 15 ([31, 73]) *Le V be lisse. Then the abelian category of V-modules is equivalent to the module category of a finite-dimensional associative algebra.*

4.2 Computation of Zhu's Algebras

We say that a vertex algebra V *admits a PBW basis* if R_V is a polynomial algebra and the map $\mathbb{C}[JX_V] \twoheadrightarrow \mathrm{gr}\,V$ is an isomorphism.

Theorem 16 *If V admits a PBW basis, then $\eta_V : R_V \twoheadrightarrow \operatorname{gr}\operatorname{Zhu}V$ is an isomorphism.*

Proof We have $\operatorname{gr}\operatorname{Zhu}(V) = V/\operatorname{gr}(V \circ V)$, where $\operatorname{gr}(V \circ V)$ is the associated graded space of $V \circ V$ with respect to the filtration induced by the filtration $V_{\leqslant p}$. We wish to show that $\operatorname{gr}(V \circ V) = F^1 V$. Since $a \circ b \equiv a_{(-2)}b \pmod{F_{\leqslant \Delta_a + \Delta_b} V}$, it is sufficient to show that $a \circ b \neq 0$ implies that $a_{(-2)}b \neq 0$.

Suppose that $a_{(-2)}b = (Ta)_{(-1)}b = 0$. Since V admits a PBW basis, $\operatorname{gr} V$ has no zero divisors. That fact that V admits a PBW basis also shows that $Ta = 0$ implies that $a = c|0\rangle$ for some constant $c \in \mathbb{C}$. Thus, a is a constant multiple of $|0\rangle$, in which case $a \circ b = 0$.

Example 2 (Universal Affine Vertex Algebras) The universal affine vertex algebra $V^\kappa(\mathfrak{a})$ (see Sect. 3.8) admits a PBW basis. Therefore

$$\eta_{V^\kappa(\mathfrak{a})} : R_{V^\kappa(\mathfrak{a})} = \mathbb{C}[\mathfrak{a}^*] \xrightarrow{\sim} \operatorname{gr}\operatorname{Zhu}V^\kappa(\mathfrak{a}).$$

On the other hand, from Lemma 19 one finds that

$$\begin{aligned} U(\mathfrak{a}) &\longrightarrow \operatorname{Zhu}(V^\kappa(\mathfrak{a})) \\ \mathfrak{a}x &\longmapsto \bar{x} = \overline{x_{(-1)}|0\rangle} \end{aligned} \tag{37}$$

gives a well-defined algebra homomorphism. This map respects the filtration on both sides, where the filtration in the left-hand-side is the PBW filtration. Hence it induces a map between their associated graded algebras, which is identical to $\eta_{V^\kappa(\mathfrak{a})}$. Therefore (37) is an isomorphism, that is to say, $V^\kappa(\mathfrak{a})$ is a chiralization of $U(\mathfrak{a})$.

Exercise 30 Extend Theorem 16 to the case that $\mathfrak{a}$ is a Lie superalgebra.

Example 3 (Free Fermions) Let $\mathfrak{n}$ be a finite-dimensional vector space. The *Clifford affinization* $\hat{Cl}$ of $\mathfrak{n}$ is the Clifford algebra associated with $\mathfrak{n}[t,t^{-1}] \oplus \mathfrak{n}^*[t,t^{-1}]$ and its symmetric bilinear form defined by

$$(xt^m|ft^n) = \delta_{m+n,0}f(x), \ (xt^m|yt^n) = 0 = (ft^m|gt^n)$$

for $x, y \in \mathfrak{n}, f, g \in \mathfrak{n}^*, m, n \in \mathbb{Z}$.

Let $\{x_\alpha\}_{\alpha \in \Delta_+}$ be a basis of $\mathfrak{n}$, $\{x_\alpha^*\}$ its dual basis. We write $\psi_{\alpha,m}$ for $x_\alpha t^m \in \hat{Cl}$ and $\psi^*_{\alpha,m}$ for $x^*_\alpha t^m \in \hat{Cl}$, so that $\hat{Cl}$ is the associative superalgebra with

- odd generators: $\psi_{\alpha,m}, \psi^*_{\alpha,m}, m \in \mathbb{Z}, \alpha \in \Delta_+$.
- relations: $[\psi_{\alpha,m}, \psi_{\beta,n}] = [\psi^*_{\alpha,m}, \psi^*_{\beta,n}] = 0, [\psi_{\alpha,m}, \psi^*_{\beta,n}] = \delta_{\alpha,\beta}\delta_{m+n,0}$.

Define the *charged fermion Fock space* associated with $\mathfrak{n}$ as

$$\mathcal{F}_\mathfrak{n} := \hat{Cl}/(\sum_{\substack{m \geqslant 0 \\ \alpha \in \Delta_+}} \hat{Cl}\psi_{\alpha,m} + \sum_{\substack{k \geqslant 1 \\ \beta \in \Delta_+}} \hat{Cl}\psi^*_{\beta,k}).$$

It is an irreducible $\hat{Cl}$-module, and as $\mathbb{C}$-vector spaces we have

$$\mathcal{F}_{\mathfrak{n}} \cong \Lambda(\mathfrak{n}^*[t^{-1}]) \otimes \Lambda(\mathfrak{n}[t^{-1}]t^{-1}).$$

There is a unique vertex (super)algebra structure on $\mathcal{F}_{\mathfrak{n}}$ such that the image of 1 is the vacuum $|0\rangle$ and

$$Y(\psi_{\alpha,-1}|0\rangle, z) = \psi_\alpha(z) := \sum_{n\in\mathbb{Z}} \psi_{\alpha,n} z^{-n-1},$$
$$Y(\psi^*_{\alpha,0}|0\rangle, z) = \psi^*_\alpha(z) := \sum_{n\in\mathbb{Z}} \psi^*_{\alpha,n} z^{-n}.$$

We have $F^1\mathcal{F}_{\mathfrak{n}} = \mathfrak{n}^*[t^{-1}]t^{-1}\mathcal{F}_{\mathfrak{n}} + \mathfrak{n}[t^{-1}]t^{-2}\mathcal{F}_{\mathfrak{n}}$, and it follows that there is an isomorphism

$$\begin{array}{rcl} \overline{Cl} & \xrightarrow{\sim} & R_{\mathcal{F}_{\mathfrak{n}}}, \\ x_\alpha & \mapsto & \overline{\psi_{\alpha,-1}|0\rangle}, \\ x^*_\alpha & \mapsto & \overline{\psi^*_{\alpha,0}|0\rangle} \end{array}$$

as Poisson superalgebras. Thus,

$$X_{\mathcal{F}_{\mathfrak{n}}} = T^*\Pi\mathfrak{n},$$

where $\Pi\mathfrak{n}$ is the space $\mathfrak{n}$ considered as a purely odd affine space. The arc space $JT^*\Pi\mathfrak{n}$ is also regarded as a purely odd affine space, such that $\mathbb{C}[JT^*\Pi\mathfrak{n}] = \Lambda(\mathfrak{n}^*[t^{-1}]) \otimes \Lambda(\mathfrak{n}[t^{-1}]t^{-1})$. The map $\mathbb{C}[JX_{\mathcal{F}_{\mathfrak{n}}}] \to \mathrm{gr}\,\mathcal{F}_{\mathfrak{n}}$ is an isomorphism and $\mathcal{F}_{\mathfrak{n}}$ admits a PBW basis. Therefore we have the isomorphism

$$\eta_{\mathcal{F}_{\mathfrak{n}}} : R_{\mathcal{F}_{\mathfrak{n}}} = \overline{Cl} \xrightarrow{\sim} \mathrm{Zhu}(\mathcal{F}_{\mathfrak{n}})$$

by Exercise 30. On the other hand the map

$$\begin{array}{rcl} Cl & \to & \mathrm{Zhu}(\mathcal{F}_{\mathfrak{n}}) \\ x_\alpha & \mapsto & \overline{\psi_{\alpha,-1}|0\rangle}, \\ x^*_\alpha & \mapsto & \overline{\psi^*_{\alpha,0}|0\rangle} \end{array}$$

gives an algebra homomorphism that respects the filtration. Hence we have

$$\mathrm{Zhu}(\mathcal{F}_{\mathfrak{n}}) \cong Cl.$$

That is, $\mathcal{F}_{\mathfrak{n}}$ is a chiralization of Cl.

5 *W*-Algebras

We are now in a position to define W-algebras. We will construct a differential graded vertex algebra, so that its cohomology algebra is a vertex algebra and that will be our main object to study.

For simplicity, we let $\mathfrak{g} = \mathfrak{gl}_n$ and we only consider the principal nilpotent case. However the definition works for any simple Lie algebra. The general definition for an arbitrary nilpotent element will be similar but one does need a new idea (see [60] for the most general definition).

5.1 The BRST Complex

Let $\mathfrak{g}, \mathfrak{n}$ be as in Sect. 2.1. Denote by $\kappa_{\mathfrak{g}}$ the Killing form on $\mathfrak{g}$ and $\kappa_0 = \frac{1}{2n}\kappa_{\mathfrak{g}}$, so that $\kappa_0(\theta, \theta) = 2$.

Choose any symmetric invariant bilinear form κ on $\mathfrak{g}$ and let $V^\kappa(\mathfrak{g})$ be the universal affine vertex algebra associated with $(\mathfrak{g}, \kappa)$ (see Sect. 3.8) and let $\mathcal{F} = \mathcal{F}_{\mathfrak{n}}$ be the fermion Fock space as in Example 3.

We have the following commutative diagrams:

$$\begin{array}{ccc} \mathbb{C}[J\mathfrak{g}^*] & \xleftarrow{\mathrm{gr}(?)} & V^\kappa(\mathfrak{g}) \\ {\scriptstyle \mathrm{Zhu}(?)}\downarrow & \swarrow{\scriptstyle R_?} & \downarrow{\scriptstyle \mathrm{Zhu}(?)} \\ \mathbb{C}[\mathfrak{g}^*] & \xleftarrow[\mathrm{gr}(?)]{} & U(\mathfrak{g}), \end{array} \qquad \begin{array}{ccc} \mathbb{C}[JT^*\Pi\mathfrak{n}] & \xleftarrow{\mathrm{gr}(?)} & \mathcal{F} \\ {\scriptstyle \mathrm{Zhu}(?)}\downarrow & \swarrow{\scriptstyle R_?} & \downarrow{\scriptstyle \mathrm{Zhu}(?)} \\ \overline{Cl} & \xleftarrow[\mathrm{gr}(?)]{} & Cl \end{array}$$

Define

$$C^\kappa(\mathfrak{g}) := V^\kappa(\mathfrak{g}) \otimes \mathcal{F}.$$

Since it is a tensor product of two vertex algebras, $C^\kappa(\mathfrak{g})$ is a vertex algebra. We have

$$R_{C^\kappa(\mathfrak{g})} = R_{V^\kappa(\mathfrak{g})} \otimes R_{\mathcal{F}} = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl} = \bar{C}(\mathfrak{g}),$$

and

$$\mathrm{Zhu}(C^\kappa(\mathfrak{g})) = \mathrm{Zhu}(V^\kappa(\mathfrak{g})) \otimes \mathrm{Zhu}(\mathcal{F}) = U(\mathfrak{g}) \otimes Cl = C(\mathfrak{g}).$$

Thus, $C^\kappa(\mathfrak{g})$ is a chiralization of $C(\mathfrak{g})$ considered in Sect. 2.5. Further we have

$$\mathrm{gr}\, C^\kappa(\mathfrak{g}) = \mathrm{gr}\, V^\kappa(\mathfrak{g}) \otimes \mathrm{gr}\,\mathcal{F} = \mathbb{C}[J\mathfrak{g}^*] \otimes \mathbb{C}[JT^*\Pi\mathfrak{n}].$$

So we have the following commutative diagram:

$$\begin{array}{ccc} & & C^{\kappa}(\mathfrak{g}) \\ & \overset{R_?}{\swarrow} & \big\downarrow \mathrm{Zhu}(?) \\ \bar{C}(\mathfrak{g}) & \xleftarrow[\mathrm{gr}(?)]{} & C(\mathfrak{g}) \end{array}$$

Define a gradation

$$\mathcal{F} = \bigoplus_{p \in \mathbb{Z}} \mathcal{F}^p \tag{38}$$

by setting $\deg \psi_{\alpha,m} = -1, \deg \psi^*_{\alpha,k} = 1, \forall i,j \in I, m,k \in \mathbb{Z}$, $\deg |0\rangle = 0$. This induces a $\mathbb{Z}$-grading (that is different from the conformal grading) on $C^\kappa(\mathfrak{g})$:

$$C^\kappa(\mathfrak{g}) = V^\kappa(\mathfrak{g}) \otimes \mathcal{F} = \bigoplus_{p \in \mathbb{Z}} C^{\kappa,p}(\mathfrak{g}), \quad \text{where} \quad C^{\kappa,p}(\mathfrak{g}) := V^\kappa(\mathfrak{g}) \otimes \mathcal{F}^p. \tag{39}$$

Let $V(\mathfrak{n})$ be the the universal affine vertex algebra associated with $\mathfrak{n}$ and the zero bilinear form, which is identified with the vertex subalgebra of $V^\kappa(\mathfrak{g})$ generated by $x_\alpha(z)$ with $\alpha \in \Delta_+$.

Lemma 21 *The following defines a vertex algebra homomorphism.*

$$\begin{aligned} \hat{\rho} : V(\mathfrak{n}) &\longrightarrow \mathcal{F} \\ x_\alpha(z) &\longmapsto \sum_{\beta,\gamma \in \Delta_+} c_{\alpha\beta}^{\gamma} \psi^*_\beta(z) \psi_\gamma(z). \end{aligned}$$

Remark 8 In the above formula the normally ordered product is not needed because $\mathfrak{n}$ is nilpotent.

The map $\hat{\rho}$ induces an algebra homomorphism

$$\mathrm{Zhu}(V(\mathfrak{n})) = U(\mathfrak{n}) \to \mathrm{Zhu}(\mathcal{F}) = Cl$$

and a Poisson algebra homomorphism

$$R_{V(\mathfrak{n})} = \mathbb{C}[\mathfrak{n}^*] \to R_{\mathcal{F}} = \overline{Cl}$$

that are identical to ρ and $\bar{\rho}$ (see Lemma 3 and 9), respectively.

Recall the character $\chi : \mathfrak{n} \to \mathbb{C}, x \mapsto (f|x)$.

Lemma 22 *The following defines a vertex algebra homomorphism.*

$$\hat{\theta}_\chi : V(\mathfrak{n}) \longrightarrow C^\kappa(\mathfrak{g})$$
$$x_\alpha(z) \longmapsto (x_\alpha(z) + \chi(x_\alpha)) \otimes \mathrm{id} + \mathrm{id} \otimes \hat{\rho}(x_\alpha(z)).$$

The map $\hat{\theta}_\chi$ induces an algebra homomorphism

$$\mathrm{Zhu}(V(\mathfrak{n})) = U(\mathfrak{n}) \to \mathrm{Zhu}C^\kappa(\mathfrak{g}) = C(\mathfrak{g})$$

and a Poisson algebra homomorphism

$$R_{V(\mathfrak{n})} = \mathbb{C}[\mathfrak{n}^*] \to R_{\mathcal{F}} = \overline{C(\mathfrak{g})}$$

that are identical to θ_χ and $\bar{\theta}$, respectively (see Lemmas 5 and 7).

The proof of the following assertion is similar to that of Lemma 6.

Proposition 8 *There exists a unique element $\hat{Q} \in C^{k,1}(g)$ such that*

$$[\hat{Q}_\lambda(1 \otimes \psi_\alpha)] = \hat{\theta}_\chi(x_\alpha), \quad \forall \alpha \in \Delta_+.$$

We have $[\hat{Q}_\lambda \hat{Q}] = 0$.
The field $\hat{Q}(z)$ is given explicitly as

$$\hat{Q}(z) = \sum_{\alpha \in \Delta_+} (x_\alpha + \chi(x_\alpha)) \otimes \psi_\alpha^*(z) - \mathrm{id} \otimes \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \Delta_+} c_{\alpha,\beta}^\gamma \psi_\alpha^*(z)\psi_\beta^*(z)\psi_\gamma(z).$$

Since $\hat{Q}$ is odd and $[\hat{Q}_\lambda \hat{Q}] = 0$, we have

$$\hat{Q}_{(0)}^2 = 0.$$

(Recall that we write $\hat{Q}(z) = \sum_{n \in \mathbb{Z}} \hat{Q}_{(n)} z^{-n-1}$.) So $(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$ is a cochain complex.

Lemma 23 *If it is nonzero, the cohomology $H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$ inherits the vertex algebra structure from $C^\kappa(\mathfrak{g})$.*

Proof Set $Z := \{v \in C^\kappa(\mathfrak{g}) \mid \hat{Q}_{(0)} v = 0\}$, $B = \hat{Q}_{(0)} C^\kappa(\mathfrak{g}) \subset Z$, so that $H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) = Z/B$. From the commutator formula (16), we know that

$$[\hat{Q}_{(0)}, a_{(m)}] = (\hat{Q}_{(0)} a)_{(m)} \quad \forall a \in C^\kappa(\mathfrak{g}), m \in \mathbb{Z}.$$

Thus, if $a, b \in Z$, then $\hat{Q}_{(0)}(a_{(m)} b) = 0$, that is, $a_{(m)} b \in Z$. It follows that Z a vertex subalgebra of $C^\kappa(\mathfrak{g})$. Further, if $a \in Z$ and $b = \hat{Q}_{(0)} b' \in B$, then $a_{(m)} b = a_{(m)} \hat{Q}_{(0)} b' = \hat{Q}_{(0)}(a_{(m)} b) \in B$. Hence B is an ideal of Z. This completes the proof.

Definition 4 The *W-algebra* $\mathscr{W}^\kappa(\mathfrak{g}) = \mathscr{W}^\kappa(\mathfrak{g}, f)$ associated to $(\mathfrak{g}, f, \kappa)$ is defined to be the zero-th cohomology of the cochain complex $(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$, that is,

$$\mathscr{W}^\kappa(\mathfrak{g}) := H^0(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}).$$

This definition of $\mathscr{W}^\kappa(\mathfrak{g})$ is due to Feigin and Frenkel [36]. In Sect. 5.9 we show that the above $\mathscr{W}^\kappa(\mathfrak{g})$ is identical to the original W-algebra defined by Fateev and Lukyanov [34].

5.2 *Cohomology of Associated Graded*

We have $\hat{Q}_{(0)} F^p C^\kappa(\mathfrak{g}) \subset F^p C^\kappa(\mathfrak{g})$, so $(\mathrm{gr}^F C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$ is also a cochain complex. The cohomology $H^\bullet(\mathrm{gr}^F C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$ inherits a Poisson vertex algebra structure from $\mathrm{gr}^F C^\kappa(\mathfrak{g})$.

Theorem 17 *We have* $H^i(\mathrm{gr}^F C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) = 0$ *for* $i \neq 0$ *and*

$$H^0(\mathrm{gr}^F C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) \cong \mathbb{C}[J\mathcal{S}]$$

as Poisson vertex algebras, where $\mathcal{S}$ *is the slice defined in Sect. 2.*

Proof The proof is an arc space analogue of that of Theorem 3.

The moment map $\mu : \mathfrak{g}^* \longrightarrow \mathfrak{n}^*$ for the N-action on $\mathfrak{g}$ induces a JN-equivariant morphism

$$J\mu : J\mathfrak{g}^* \longrightarrow J\mathfrak{n}^*.$$

The pullback $(J\mu)^* : \mathbb{C}[J\mathfrak{n}^*] \to \mathbb{C}[J\mathfrak{g}^*]$ is an embedding of vertex Poisson algebras.

The point $\chi = J\chi$ of $J\mathfrak{n}^*$ corresponds to the arc $\alpha \in \mathrm{Hom}(D, \mathfrak{n}^*) = \mathrm{Hom}(\mathbb{C}[\mathfrak{n}^*], \mathbb{C}[[t]])$ such that $\alpha(f) = \chi(x)$ for $x \in \mathfrak{n} \subset \mathbb{C}[\mathfrak{n}^*]$.

We have

$$(J\mu)^{-1}(\chi) = J(\mu^{-1}(\chi)) = \chi + J\mathfrak{b} \subset J\mathfrak{g}^*,$$

and the adjoint action gives the isomorphism

$$JN \times J\mathcal{S} \xrightarrow{\sim} J\mu^{-1}(\chi) \tag{40}$$

by Theorem 2 and (24).

Now put

$$C := \mathrm{gr}\, C^\kappa(\mathfrak{g}) = \mathbb{C}[J\mathfrak{g}^*] \otimes \Lambda(\mathfrak{n}[t^{-1}]t^{-1}) \otimes \Lambda(\mathfrak{n}^*[t^{-1}]) \tag{41}$$

and define a bigrading on C by

$$C = \bigoplus_{i \leqslant 0, j \geqslant 0} C^{i,j}, \quad \text{where} \quad C^{i,j} = \mathbb{C}[J\mathfrak{g}^*] \otimes \Lambda^{-i}(\mathfrak{n}[t^{-1}]t^{-1}) \otimes \Lambda^{j}(\mathfrak{n}^*[t^{-1}]). \tag{42}$$

As before, we can decompose the operator $\hat{Q}_{(0)}$ as the sum of two suboperators such that each of them preserves one grading but increase the other grading by 1. Namely, we have

$$\hat{Q}_{(0)} = \hat{d}_+ + \hat{d}_-,$$
$$\hat{d}_- : C^{i,j} \longrightarrow C^{i,j+1}, \quad \hat{d}_+ : C^{i,j} \longrightarrow C^{i+1,j}.$$

This shows that

$$(\hat{d}_+)^2 = (\hat{d}_-)^2 = [\hat{d}_+, \hat{d}_-] = 0.$$

Thus we can get a spectral sequence $E_r \Longrightarrow H^\bullet(C, \hat{Q}_{(0)})$ such that

$$E_1 = H^\bullet(C, \hat{d}_-), \quad E_2 = H^\bullet(H^\bullet(C, \hat{d}_-), \hat{d}_+).$$

This is a converging spectral sequence since C is a direct sum of subcomplexes $F^p C^\kappa(\mathfrak{g})/F^{p+1}C^\kappa(\mathfrak{g})$, and the associated filtration is regular on each subcomplex.

The complex $(C, \hat{d}_-)$ is the Koszul complex with respect to the sequence

$$x_1 t^{-1} - \chi(x_1), \ldots, x_N t^{-1} - \chi(x_N), x_1 t^{-2}, x_2 t^{-2}, \ldots, x_N t^{-2}, x_1 t^{-3}, x_2 t^{-3}, \ldots$$

where $N = \dim \mathfrak{n}$. Hence we have

$$H^i(C, \hat{d}_-) = \delta_{i,0} \mathbb{C}[J\mu^{-1}(\chi)] \otimes \Lambda(\mathfrak{n}^*[t^{-1}]). \tag{43}$$

Next, by (43), the complex $(H^0(C, \hat{d}_-), \hat{d}_+)$ is identical to the Chevalley complex for the Lie algebra cohomology $H^\bullet(\mathfrak{n}[t], \mathbb{C}[J\mu^{-1}(\chi)])$. By (40),

$$\begin{aligned} H^i(\mathfrak{n}[t], \mathbb{C}[J\mu^{-1}(\chi)]) &= H^i(\mathfrak{n}[t], \mathbb{C}[JN] \otimes \mathbb{C}[J\mathcal{S}]) \\ &= H^i(\mathfrak{n}[t], \mathbb{C}[JN]) \otimes \mathbb{C}[J\mathcal{S}] = \delta_{i,0} \mathbb{C}[J\mathcal{S}]. \end{aligned}$$

We conclude that

$$H^i(H^j(C, \hat{d}_-), \hat{d}_+) = \delta_{i,0}\delta_{j,0}\mathbb{C}[\mathcal{S}].$$

Thus, the spectral sequence E_r collapses at $E_2 = E_\infty$, and we get the desired isomorphisms.

Theorem 18 ([36, 40]) *We have $H^0(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) = 0$ for $i \neq 0$ and*

$$\operatorname{gr} \mathscr{W}^k(\mathfrak{g}) = \operatorname{gr} H^0(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) \cong H^0(\operatorname{gr} C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) = \mathbb{C}[J\mathcal{S}].$$

In particular, $R_{W^k(\mathfrak{g})} \cong \mathbb{C}[\mathcal{S}] \cong \mathbb{C}[\mathfrak{g}]^G$, so $\tilde{X}_{\mathscr{W}^k(\mathfrak{g})} = \mathcal{S}$, $SS(\mathscr{W}^k(\mathfrak{g})) = J\mathcal{S}$.

The proof of Theorem 18 will be given in Sect. 5.6.

Note that there is a spectral sequence for $H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$ such that $E_1^{\bullet,q} = H^q(\operatorname{gr} C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$. Hence Theorem 18 would immediately follow from Theorem 17 if this spectral sequence converges. However, this is not clear at this point because our algebra is not Noetherian.

Remark 9 The complex $(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$ is identical to Feigin's standard complex for the semi-infinite $\mathfrak{n}[t, t^{-1}]$-cohomology $H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t, t^{-1}], V^\kappa(\mathfrak{g}) \otimes \mathbb{C}_{\hat{\chi}})$ with coefficient in the $\mathfrak{g}[t, t^{-1}]$-module $V^\kappa(\mathfrak{g}) \otimes \mathbb{C}_{\hat{\chi}}$ [35], where $\mathbb{C}_{\hat{\chi}}$ is the one-dimensional representation of $\mathfrak{n}[t, t^{-1}]$ defined by the character $\hat{\chi} : \mathfrak{n}[t, t^{-1}] \to \mathbb{C}$, $xt^n \mapsto \delta_{n,-1}\chi(x)$:

$$H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) \cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}[t, t^{-1}], V^\kappa(\mathfrak{g}) \otimes \mathbb{C}_{\hat{\chi}}). \tag{44}$$

5.3 W-Algebra Associated with $\mathfrak{sl}_n$

It is straightforward to generalize the above definition to an arbitrary simple Lie algebra $\mathfrak{g}$. In particular, by replacing $V^\kappa(\mathfrak{gl}_n)$ with $V^k(\mathfrak{sl}_n)$, $k \in \mathbb{C}$, we define the W-algebra

$$\mathscr{W}^k(\mathfrak{sl}_n) := H^0(C^k(\mathfrak{sl}_n), \hat{Q}_{(0)}) \tag{45}$$

associated with $(\mathfrak{sl}_n, f)$ at level k.

We have $V^\kappa(\mathfrak{gl}_n) = \pi_\kappa \otimes V^\kappa(\mathfrak{sl}_n)$, where $\kappa|_{\mathfrak{sl}_n \times \mathfrak{sl}_n} = k\kappa_0$ and π_κ is the rank 1 *Heisenberg vertex algebra* generated by $I(z) = \sum_{i=1}^n e_{ii}(z)$ with λ-bracket $[I_\lambda I] = \kappa(I, I)\lambda$. It follows that $C^k(\mathfrak{gl}_n) = \pi_\kappa \otimes C^k(\mathfrak{sl}_n)$. As easily seen, $\hat{Q}_{(0)} I = 0$. Hence $H^\bullet(C^\kappa(\mathfrak{gl}_n)) = \pi_\kappa \otimes H^\bullet(C^k(\mathfrak{sl}_n))$, so that

$$\mathscr{W}^\kappa(\mathfrak{gl}_n) = \mathscr{W}^k(\mathfrak{sl}_n) \otimes \pi_\kappa.$$

In particular if we choose the form κ to be $k\kappa_0$, we find that π_κ belongs to the center of $\mathscr{W}^\kappa(\mathfrak{gl}_n)$ as π_κ belongs to the center of $C^\kappa(\mathfrak{gl}_n)$. Thus, $\mathscr{W}^k(\mathfrak{sl}_n)$ is isomorphic to the quotient of $\mathscr{W}^{k\kappa_0}(\mathfrak{gl}_n)$ by the ideal generated by $I_{(-1)}|0\rangle$.

5.4 *The Grading of* $\mathscr{W}^\kappa(\mathfrak{g})$

The standard conformal grading of $C^\kappa(\mathfrak{g})$ is given by the Hamiltonian H defined by

$$H|0\rangle = 0, \quad [H, x_{(n)}] = -nx_{(n)} \quad (x \in \mathfrak{g}),$$
$$[H, \psi_{\alpha,n}] = -n\psi_{\alpha,n}, \quad [H, \psi^*_{\alpha,n}] = -n\psi^*_{\alpha,n}.$$

However H is not well-defined in $\mathscr{W}^\kappa(\mathfrak{g})$ since H does not commute with the action of

$$\hat{Q}_{(0)} = \sum_{\alpha\in\Delta_+}\sum_{k\in\mathbb{Z}}(x_\alpha)_{(-k)}\psi_{\alpha,k} + \sum_{\alpha\in\Delta_+}\chi(x_\alpha)\psi_{\alpha,1} - \sum_{\alpha,\beta,\gamma\in\Delta_+}\sum_{k+l+m=0}c^\gamma_{\alpha,\beta}\psi^*_{\alpha,k}\psi^*_{\beta,l}\psi_{\gamma,m}.$$

Here and below we omit the tensor product sign.

To remedy this, define the linear operator $H_{\mathscr{W}}$ by

$$H_{\mathscr{W}}|0\rangle = 0, \quad [H_{\mathscr{W}}, (x_i)_{(n)}] = -n(x_i)_{(n)} \quad (i \in I),$$
$$[H_{\mathscr{W}}, (x_\alpha)_{(n)}] = (\alpha(\rho^\vee) - n)(x_\alpha)_{(n)} \quad (\alpha \in \Delta),$$
$$[H_{\mathscr{W}}, \psi_{\alpha,n}] = (\alpha(\rho^\vee) - n)\psi_{\alpha,n}, \quad [H_{\mathscr{W}}, \psi^*_{\alpha,n}] = (-\alpha(\rho^\vee) - n)\psi^*_{\alpha,n}, \quad (\alpha \in \Delta_+).$$

Here $\rho^\vee = 1/2h$, where h is defined in (6). Set $C^\kappa(\mathfrak{g})_{\Delta,new} = \{v \in C^\kappa(\mathfrak{g}) \mid H_{\mathscr{W}} c = \Delta c\}$. Then

$$C^\kappa(\mathfrak{g}) = \bigoplus_{\Delta\in\mathbb{Z}} C^\kappa(\mathfrak{g})_{\Delta,new}. \tag{46}$$

Since $[\hat{Q}, H_{\mathscr{W}}] = 0$, $C^\kappa(\mathfrak{g})_{\Delta,new}$ is a subcomplex of $C^\kappa(\mathfrak{g})$. We have

$$H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)}) = \bigoplus_{\Delta\in\mathbb{Z}} H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})_\Delta,$$
$$H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})_\Delta = H^\bullet(C^\kappa(\mathfrak{g})_{\Delta,new}, \hat{Q}_{(0)}).$$

In particular $\mathscr{W}^\kappa(\mathfrak{g}) = \bigoplus_{\Delta\in\mathbb{Z}} \mathscr{W}^\kappa(\mathfrak{g})_\Delta$. Note that the grading (46) is not bounded from below.

If $k \neq -n$ then the action of $H_{\mathscr{W}}$ on the vertex subalgebra $\mathscr{W}^k(\mathfrak{sl}_n)$ of $\mathscr{W}^k(\mathfrak{g})$ is inner: Set

$$L(z) = L_{sug}(z) + \rho^\vee(z) + L_{\mathcal{F}}(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-1},$$

where $L_{sug}(z)$ is the Sugawara field of $V^k(\mathfrak{sl}_n)$:

$$L_{sug}(z) = \frac{1}{2(k+n)} \sum_a : x_a(z) x^a(z) :,$$

and

$$L_{\mathcal{F}}(z) = \sum_{\alpha \in \Delta_+} (\mathrm{ht}(\alpha) : \partial_z \psi_\alpha(z) \psi_\alpha^*(z) : + (1 - \mathrm{ht}(\alpha)) : \partial_z \psi_\alpha^*(z) \psi_\alpha(z) :).$$

Here $\{x_a\}$ is a basis of $\mathfrak{sl}_n$ and $\{x^a\}$ is the dual basis of $\{x_a\}$ with respect to $(\ |\ )$. Then $\hat{Q}_{(0)} L = 0$, and so L defines an element of $\mathscr{W}^k(\mathfrak{sl}_n)$. It is a conformal vector of $\mathscr{W}^k(\mathfrak{sl}_n)$, that is to say, $L_0 = H_{\mathscr{W}}$ and $L_{-1} = T$ and

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m,n} c,$$

where $c \in \mathbb{C}$ is the central charge of L, which is in this case given by

$$(n-1)(1 - n(n+1)(n+k-1)^2/(n+k)).$$

5.5 Decomposition of BRST Complex

We extend the map in Sect. 2.6 to the linear map $\widehat{\theta}_0 : \mathfrak{g}[t, t^{-1}] \to C^\kappa(\mathfrak{g})$ by setting

$$\widehat{\theta}_0(x_a(z)) = x_a(z) + \sum_{\beta, \gamma \in \Delta_+} c_{a,\beta}^{\gamma} : \psi_\gamma(z) \psi_\beta^*(z) : .$$

Proposition 9

1. *The correspondence*

$$x_a(z) \mapsto J_a(z) := \widehat{\theta}_0(x_a(z)) \quad (x_a \in \mathfrak{b}_-)$$

defines a vertex algebra embedding $V^{\kappa_{\mathfrak{b}}}(\mathfrak{b}) \hookrightarrow C^\kappa(\mathfrak{g})$, *where* $\kappa_{\mathfrak{b}}$ *is the bilinear form on* $\mathfrak{b}$ *defined by* $\kappa_{\mathfrak{b}}(x, y) = \kappa(x, y) + \frac{1}{2}\kappa_{\mathfrak{g}}(x, y)$. *We have*

$$[J_{a\lambda} \psi_\alpha^*] = \sum_{\beta \in \Delta_+} c_{a,\beta}^{\alpha} \psi_\beta^*.$$

2. *The correspondence*

$$x_\alpha(z) \mapsto J_\alpha(z) := \widehat{\theta}_0(x_\alpha) \quad (x_\alpha \in \mathfrak{n})$$

defines a vertex algebra embedding $V(\mathfrak{n}) \hookrightarrow C^{\kappa}(\mathfrak{g})$. *We have*

$$[J_{\alpha\lambda}\psi_\beta] = \sum_{\beta\in\Delta_+} c_{\alpha,\beta}^{\gamma}\psi_\gamma^*.$$

Let $C^{\kappa}(\mathfrak{g})_+$ denote the subalgebra of $C^{\kappa}(\mathfrak{g})$ generated by $J_\alpha(z)$ and $\psi_\alpha(z)$ with $\alpha \in \Delta_+$, and let $C^{\kappa}(\mathfrak{g})_-$ denote the subalgebra generated by $J_a(z)$ and $\psi_\alpha^*(z)$ with $a \in \Delta_- \sqcup I$, $\alpha \in \Delta_+$.

The proof of the following assertions are parallel to that of Lemmas 11, 12 and Proposition 3.

Lemma 24 *The multiplication map gives a linear isomorphism*

$$C^{\kappa}(\mathfrak{g})_- \otimes C^{\kappa}(\mathfrak{g})_+ \xrightarrow{\sim} C^{\kappa}(\mathfrak{g}).$$

Lemma 25 *The subspaces* $C^{\kappa}(\mathfrak{g})_-$ *and* $C^{\kappa}(\mathfrak{g})_+$ *are subcomplexes of* $(C^{\kappa}(\mathfrak{g}), \hat{Q}_{(0)})$. *Hence* $C^{\kappa}(\mathfrak{g}) \cong C^{\kappa}(\mathfrak{g})_- \otimes C^{\kappa}(\mathfrak{g})_+$ *as complexes.*

Theorem 19 ([28, 40]) *We have* $H^i(C^{\kappa}(\mathfrak{g})_+, \hat{Q}_{(0)}) = \delta_{i,0}\mathbb{C}$. *Hence* $H^\bullet(C^{\kappa}(\mathfrak{g}), \hat{Q}_{(0)}) = H^\bullet(C^{\kappa}(\mathfrak{g})_-, \hat{Q}_{(0)})$. *In particular* $\mathscr{W}^{\kappa}(\mathfrak{g}) = H^0(C^{\kappa}(\mathfrak{g})_-, \hat{Q}_{(0)})$.

Since the complex $C^{\kappa}(\mathfrak{g})_-$ has no positive cohomological degree, its zeroth cohomology $\mathscr{W}^k(\mathfrak{g}) = H^0(C^{\kappa}(\mathfrak{g})_-, \hat{Q}_{(0)})$ is a vertex *subalgebra* of $C^{\kappa}(\mathfrak{g})_-$. Observe also that $C^{\kappa}(\mathfrak{g})_-$ has no negative degree with respect to the Hamiltonian $H_{\mathscr{W}}$, and each homogeneous space is finite-dimensional:

$$C^{\kappa}(\mathfrak{g})_- = \bigoplus_{\Delta\in\mathbb{Z}_-} C^{\kappa}(\mathfrak{g})_{-,\Delta,new}, \quad \dim C^{\kappa}(\mathfrak{g})_{-,\Delta,new} < \infty. \tag{47}$$

Here $C^{\kappa}(\mathfrak{g})_{-,\Delta,new} = C^{\kappa}(\mathfrak{g})_- \cap C^{\kappa}(\mathfrak{g})_{\Delta,new}$.

5.6 Proof of Theorem 18

As $\hat{Q}_{(0)}F^pC^{\kappa}(\mathfrak{g})_- \subset F^pC^{\kappa}(\mathfrak{g})_-$, one can consider a spectral sequence for $H^\bullet(C^{\kappa}(\mathfrak{g})_-, \hat{Q}_{(0)})$ such that the E_1-term is $H^\bullet(\operatorname{gr} C^{\kappa}(\mathfrak{g})_-, \hat{Q}_{(0)})$. This spectral sequence clearly converges, since $C^{\kappa}(\mathfrak{g})_-$ is a direct sum of finite-dimensional subcomplexes $C^{\kappa}(\mathfrak{g})_{-,\Delta,new}$.

We have $\operatorname{gr} C^{\kappa}(\mathfrak{g})_- \cong S(\mathfrak{b}_-[t^{-1}]t^{-1}) \otimes \Lambda(\mathfrak{n}[t^{-1}]t^{-1}) \cong \mathbb{C}[J\mu^{-1}(\chi)] \otimes \Lambda(\mathfrak{n}[t^{-1}]t^{-1})$, and the complex $(\operatorname{gr} C^{\kappa}(\mathfrak{g})_-, \hat{Q}_{(0)})$ is identical to the Chevalley complex for the Lie algebra cohomology $H^\bullet(\mathfrak{n}[t], \mathbb{C}[J\mu^{-1}(\chi)])$. Therefore

$$H^i(\operatorname{gr} C^{\kappa}(\mathfrak{g})_-, \hat{Q}_{(0)}) \cong \delta_{i,0}\mathbb{C}[J\mathcal{S}]. \tag{48}$$

Thus the spectral sequence collapses at $E_1 = E_\infty$, and we get

$$\mathrm{gr}^G H^i(C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)}) \cong H^i(\mathrm{gr}\, C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)}) \cong \delta_{i,0}\mathbb{C}[J\mathcal{S}].$$

Here $\mathrm{gr}^G H^i(C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)})$ is the associated graded space with respect to the filtration $G^\bullet H^i(C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)})$ induced by the filtration $F^\bullet C^\kappa(\mathfrak{g})_-$, that is,

$$G^p H^i(C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)}) = \mathrm{Im}(H^i(F^p C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)}) \to H^i(C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)})).$$

We claim that the filtration $G^\bullet H^0(C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)})$ coincides with the canonical filtration of $H^0(C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)}) = \mathscr{W}^\kappa(\mathfrak{g})$. Indeed, from the definition of the canonical filtration we have $F^p W^k(\mathfrak{g}) \subset G^p \mathscr{W}^\kappa(\mathfrak{g})$ for all p, and hence, there is a Poisson vertex algebra homomorphism

$$\mathrm{gr}\, \mathscr{W}^\kappa(\mathfrak{g}, f) \to \mathrm{gr}^G \mathscr{W}^\kappa(\mathfrak{g}, f) \cong \mathbb{C}[J\mathcal{S}] \tag{49}$$

that restricts to a surjective homomorphism

$$\mathscr{W}^\kappa(\mathfrak{g})/F^1\mathscr{W}^\kappa(\mathfrak{g}) \twoheadrightarrow \mathscr{W}^\kappa(\mathfrak{g})/G^1\mathscr{W}^\kappa(\mathfrak{g}) \cong \mathbb{C}[\mathcal{S}].$$

Since $\mathbb{C}[J\mathcal{S}]$ is generated by $\mathbb{C}[\mathcal{S}]$ as differential algebras it follows that (49) is surjective. On the other hand the cohomology vanishing and the Euler-Poincaré principle imply that the graded character of $\mathscr{W}^\kappa(\mathfrak{g})$ and $\mathbb{C}[J\mathcal{S}]$ are the same. Therefore (49) is an isomorphism, and thus, $G^p\mathscr{W}^\kappa(\mathfrak{g}) = F^p\mathscr{W}^\kappa(\mathfrak{g})$ for all p.

Finally the embedding $\mathrm{gr}\, C^\kappa(\mathfrak{g})_- \to \mathrm{gr}\, C^\kappa(\mathfrak{g})$ induces an isomorphism

$$H^0(\mathrm{gr}\, C^\kappa(\mathfrak{g})_-, \hat{Q}_{(0)}) \cong H^0(\mathrm{gr}\, C^\kappa(\mathfrak{g}), \hat{Q}_{(0)})$$

by Theorem 17 and (48). This completes the proof. □

5.7 Zhu's Algebra of W-Algebra

Let $\mathrm{Zhu}_{new}(C^\kappa(\mathfrak{g}))$ be Zhu's algebra of $C^\kappa(\mathfrak{g})$ with respect to the Hamiltonian H_W, $\mathrm{Zhu}_{old}(C^\kappa(\mathfrak{g}))$ Zhu's algebra of $C^\kappa(\mathfrak{g})$ with respect to the standard Hamiltonian H. We have

$$\mathrm{Zhu}_{new}(C^\kappa(\mathfrak{g})) \cong \mathrm{Zhu}_{old}(C^\kappa(\mathfrak{g})) \cong C(\mathfrak{g}),$$

see [10, Proposition 5.1] for the details. Then it is legitimate to write $\mathrm{Zhu}(C^{\kappa}(\mathfrak{g}))$ for $\mathrm{Zhu}_{new}(C^{\kappa}(\mathfrak{g}))$ or $\mathrm{Zhu}_{old}(C^{\kappa}(\mathfrak{g}))$.

By the commutation formula, we have

$$\hat{Q}_{(0)}(C^{\kappa}(\mathfrak{g}) \circ C^{\kappa}(\mathfrak{g})) \subset C^{\kappa}(\mathfrak{g}) \circ C^{\kappa}(\mathfrak{g}).$$

Here the circle $\circ$ is defined as in the definition of the Zhu algebra (with respect to the grading $H_{\mathscr{W}}$). So $(\mathrm{Zhu}_{new} C^{\kappa}(\mathfrak{g}), \hat{Q}_{(0)})$ is a differential, graded algebra, which is identical to $(C(\mathfrak{g}), \mathrm{ad}\, Q)$.

Theorem 20 ([5]) *We have*

$$\mathrm{Zhu}\mathscr{W}^{\kappa}(\mathfrak{g}) \cong H^0(\mathrm{Zhu}_{new} C^{\kappa}(\mathfrak{g}), \hat{Q}_{(0)}) \cong \mathcal{Z}(\mathfrak{g}).$$

Proof By Theorem 18, it follows that $\mathscr{W}^{\kappa}(\mathfrak{g})$ admits a PBW basis. Hence $\eta_{\mathscr{W}^{\kappa}(\mathfrak{g})}$: $\mathrm{gr}\,\mathrm{Zhu}\mathscr{W}^{\kappa}(\mathfrak{g}) \to R_{\mathscr{W}_k(\mathfrak{g})}$ is an isomorphism by Theorem 16. On the other hand we have a natural algebra homomorphism $\mathrm{Zhu}\mathscr{W}^{\kappa}(\mathfrak{g}) \longrightarrow H^0(\mathrm{Zhu} C^{\kappa}(\mathfrak{g}), \hat{Q}_{(0)})$ which makes the following diagram commute.

$$\begin{array}{ccc} \mathrm{gr}\,\mathrm{Zhu}\mathscr{W}^{\kappa}(\mathfrak{g}) & \xrightarrow[\cong]{\eta_{\mathscr{W}^{\kappa}(\mathfrak{g},f)}} & R_{\mathscr{W}^{\kappa}(\mathfrak{g})} \\ \downarrow & & \cong \downarrow \text{Theorem 18} \\ \mathrm{gr}\,\mathcal{Z}(\mathfrak{g}) & \xrightarrow{\cong} & \mathbb{C}[\mathcal{S}]. \end{array}$$

Note that we have the isomorphisms $H^0(R_{C^{\kappa}(\mathfrak{g})}, \hat{Q}_{(0)}) \cong H^0(\bar{C}^k(\mathfrak{g}), \mathrm{ad}\,\bar{Q}_{(0)}) \cong \mathbb{C}[\mathcal{S}]$ and $\mathrm{gr}\, H^0(\mathrm{Zhu}_{new} C^{\kappa}(\mathfrak{g}), \hat{Q}_{(0)}) \cong \mathrm{gr}\,\mathcal{Z}(\mathfrak{g})$ in the diagram. Now the other three isomorphisms will give the desired isomorphism.

We conclude that we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[J\mathcal{S}] & \xleftarrow{\mathrm{gr}(?)} & \mathscr{W}^{\kappa}(\mathfrak{g}) \\ \mathrm{Zhu}(?) \downarrow & \swarrow^{R_?} & \downarrow \mathrm{Zhu}(?) \\ \mathbb{C}[\mathcal{S}] & \xleftarrow[\mathrm{gr}(?)]{} & \mathcal{Z}(\mathfrak{g}). \end{array}$$

Remark 10 The same proof applies for an arbitrary simple Lie algebra $\mathfrak{g}$. In particular, we have $\mathrm{Zhu}(\mathscr{W}^k(\mathfrak{sl}_n)) \cong \mathcal{Z}(\mathfrak{sl}_n)$. In fact the same proof applies for the W-algebra associated with a simple Lie algebra $\mathfrak{g}$ and an arbitrary nilpotent element f of $\mathfrak{g}$ to show its Zhu's algebra is isomorphic to the finite W-algebra $U(\mathfrak{g}, f)$ [29].

5.8 Explicit Generators

It is possible to write down the explicit generators of $\mathscr{W}^\kappa(\mathfrak{g}) \subset C^\kappa(\mathfrak{g})_-$.

Recall that the *column-determinant* of a matrix $A = (a_{ij})$ over an associative algebra is defined by

$$\operatorname{cdet} A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot a_{\sigma(1)\,1} a_{\sigma(2)\,2} \cdots a_{\sigma(n)\,n}.$$

Introduce an extended Lie algebra $\mathfrak{b}[t^{-1}]t^{-1} \oplus \mathbb{C}\tau$, where the element τ commutes with $\mathbf{1}$, and

$$\left[\tau, x_{(-n)}\right] = n x_{(-n)} \qquad \text{for} \quad x \in \mathfrak{b}, n \in \mathfrak{n},$$

where $x_{(-n)} = x t^{-n}$. This induces an associative algebra structure on the tensor product space $U\big(\mathfrak{b}[t^{-1}]t^{-1}\big) \oplus \mathbb{C}[\tau]$.

Consider the matrix

$$B = \begin{bmatrix} \alpha\tau + (e_{11})_{(-1)} & -1 & 0 & \dots & 0 \\ (e_{21})_{(-1)} & \alpha\tau + (e_{22})_{(-1)} & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ (e_{n-1\,1})_{(-1)} & (e_{n-1\,2})_{(-1)} & \dots & \alpha\tau + (e_{n-1\,n-1})_{(-1)} & -1 \\ (e_{n\,1})_{(-1)} & (e_{n\,2})_{(-1)} & \dots & \dots & \alpha\tau + (e_{nn})_{(-1)}. \end{bmatrix}$$

with entries in $U(\mathfrak{b}[t^{-1}]t^{-1}]) \otimes \mathbb{C}[\tau] \otimes \mathbb{C}[\alpha]$, where α is a parameter.

For its column-determinant[1] we can write

$$\operatorname{cdet} B = \tau^n + W_\alpha^{(1)} \tau^{n-1} + \cdots + W_\alpha^{(n)}$$

for certain coefficients $W_\alpha^{(r)}$ which are elements of $U(\mathfrak{b}[t^{-1}]t^{-1}]) \otimes \mathbb{C}[\alpha]$. Set

$$W^{(i)} = W_\alpha^{(i)}|_{\alpha = k+n-1}.$$

This is an element of $U(\mathfrak{b}[t^{-1}]t^{-1})$, which we identify with $V^{\kappa_\mathfrak{b}}(\mathfrak{b}) \subset C^\kappa(\mathfrak{g})_-$.

Theorem 21 ([14]) $\mathscr{W}^k(\mathfrak{g})$ *is strongly generated by* $W^{(1)}, \dots, W^{(n)}$.

[1]It is easy to verify that cdet B coincides with the *row-determinant* of B defined in a similar way.

5.9 Miura Map

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts on $C^\kappa(\mathfrak{g})_+$ by $x_i \mapsto (J_i)_{(0)}$, $i \in I$, see Proposition 9. Let $C^\kappa(\mathfrak{g})_+^\lambda$ be the weight space of weight $\lambda \in \mathfrak{h}^*$ with respect to this action. Then

$$C^\kappa(\mathfrak{g})_+ = \bigoplus_{\lambda \leqslant 0} C^\kappa(\mathfrak{g})_+^\lambda, \quad C^\kappa(\mathfrak{g})_+^0 = V^{\kappa_\mathfrak{b}}(\mathfrak{h}) \subset V^{\kappa_k}(\mathfrak{b}).$$

The vertex algebra $V^{\kappa_\mathfrak{h}}(\mathfrak{h})$ is the *Heisenberg vertex algebra* associated with $\mathfrak{h}$ and the bilinear form $\kappa_\mathfrak{h} := \kappa_\mathfrak{b}|_{\mathfrak{h}\times\mathfrak{h}}$.

The projection $C^\kappa(\mathfrak{g})_+ \to C^\kappa(\mathfrak{g})_+^0 = V^{\kappa_\mathfrak{h}}(\mathfrak{h})$ with respect to this decomposition is a vertex algebra homomorphism. Therefore it restriction

$$\hat{\Upsilon} : \mathscr{W}^\kappa(\mathfrak{g}) \to V^{\kappa_\mathfrak{h}}(\mathfrak{h}) \tag{50}$$

is also a vertex algebra homomorphism that is called the *Miura map*.

Theorem 22 *The Miura map is injective for all $k \in \mathbb{C}$.*

Proof The induced Poisson vertex algebra homomorphism

$$\mathrm{gr}\,\hat{\Upsilon} : \mathrm{gr}\,\mathscr{W}^\kappa(\mathfrak{g}) = \mathbb{C}[J\mathcal{S}] \to \mathrm{gr}\,V^{\kappa_\mathfrak{h}}(\mathfrak{h}) = \mathbb{C}[J\mathfrak{h}^*] \cong \mathbb{C}[J(f+\mathfrak{h})] \tag{51}$$

is just a restriction map and coincides with $J\bar{\Upsilon}$, where $\bar{\Upsilon}$ is defined in (13). Clearly, it is sufficient to show that $J\bar{\Upsilon}$ is injective.

Recall that the action map gives an isomorphism

$$N \times (f + \mathfrak{h}_{\mathrm{reg}}) \xrightarrow{\sim} U \subset f + \mathfrak{b},$$

where U is some open subset of $f+\mathfrak{b}$, see the proof of Proposition 4. Therefore, by Lemma 15, the action map $JN \times J(f+\mathfrak{h}) \to J(f+\mathfrak{b})$ is dominant. Thus, the induced map $\mathbb{C}[J(f+\mathfrak{b})] \to \mathbb{C}[JN \times J(f+\mathfrak{h})]$ is injective, and so is $J\bar{\Upsilon} : C[J(f+\mathfrak{b})]^{JN} \to \mathbb{C}[JN \times J(f+\mathfrak{h})]^{JN} = \mathbb{C}[J(f+\mathfrak{h})]$.

Remark 11 It is straightforward to generalize Theorem 22 for the W-algebra $\mathscr{W}^k(\mathfrak{g})$ associated with a general simple Lie algebra $\mathfrak{g}$.

Theorem 23 *Let $x_i = E_{ii} \in \mathfrak{h} \subset \mathfrak{g} = \mathfrak{gl}_n$, and $J_i(z)$ the corresponding field of $V^{\kappa_k}(\mathfrak{h})$. The image $\Upsilon(W^{(i)}(z))$ of $W^{(i)}(z)$ by the Miura map is described by*

$$\sum_{i=0}^{n} \Upsilon(W^{(i)})(z)(\alpha\partial_z)^{n-i} =: (\alpha\partial_z + J_1(z))(\alpha\partial_z + J_2(z))\ldots(\alpha\partial_z + J_N(z)) :,$$

where $\alpha = k+n-1$, $W^{(0)}(z) = 1$, $[\partial_z, J_i(z)] = \frac{d}{dz}J_i(z)$.

Proof It is straightforward from Theorem 21.

Note that if we choose κ to be $k\kappa_0$ and set $\sum_{i=1}^{N} J_i(z) = 0$, we obtain the image of the generators of $\mathscr{W}^k(\mathfrak{sl}_n)$ by the Miura map $\hat{\Upsilon}$. For $k+n \neq 0$, this expression can be written in more symmetric manner: Set $b_i(z) = \frac{1}{\sqrt{k+n}} J_i(z)$, so that $\sum_{i=1}^{n} b_i(z) = 0$, and

$$[(b_i)_\lambda b_j] = \begin{cases} (1-\frac{1}{n})\lambda & \text{if } i=j, \\ -\frac{1}{n}\lambda & \text{if } i \neq j. \end{cases}$$

Then we obtain the following original description of the $\mathscr{W}^k(\mathfrak{sl}_n)$ due to Fateev and Lukyanov [34].

Corollary 4 *Suppose that $k+n \neq 0$. Then the image of $\mathscr{W}^k(\mathfrak{sl}_n)$ by the Miura map is the vertex subalgebra generated by fields $\tilde{W}_2(z) \dots, \tilde{W}_n(z)$ defined by*

$$\sum_{i=0}^{n} \tilde{W}_i(z)(\alpha_0 \partial_z)^{n-i} =: (\alpha_0\partial_z + b_1(z))(\alpha_0\partial_z + b_2(z)) \dots (\alpha_0\partial_z + b_n(z)) :,$$

where $\alpha_0 = \alpha_+ + \alpha_-$, $\alpha_+ = \sqrt{k+n}$, $\alpha_- = -1/\sqrt{k+n}$, $\tilde{W}_0(z) = 1$, $\tilde{W}_1(z) = 0$.

Corollary 5 *Suppose that $k+n \neq 0$. We have*

$$\mathscr{W}^k(\mathfrak{sl}_n) \cong \mathscr{W}^{^Lk}(\mathfrak{sl}_n),$$

where ^Lk is defined by $(k+n)(^Lk+n) = 1$.

Example 4 Let $\mathfrak{g} = \mathfrak{sl}_2$, $k \neq -2$. Set $b(z) = \sqrt{2}b_1(z) = -\sqrt{2}b_2(z)$, so that $[b_\lambda b] = \lambda$. Then the right-hand-side of the formula in Corollary 5 becomes

$$\begin{aligned} &: (\alpha_0\partial_z + \frac{1}{\sqrt{2}}b(z))(\alpha_0\partial_z - \frac{1}{\sqrt{2}}b(z)) : \\ &= \alpha_0^2\partial_z^2 - L(z), \end{aligned}$$

where

$$L(z) = \frac{1}{2} : b(z)^2 : + \frac{\alpha_0}{\sqrt{2}} \partial_z b(z).$$

It is well-known and is straightforward to check that the field generates the Virasoro algebra of central charge $1-6(k+1)^2/(k+2)$. Thus $\mathscr{W}^k(\mathfrak{sl}_2)$, $k \neq -2$, is isomorphic to the universal Virasoro vertex algebra of central charge $1-6(k+1)^2/(k+2)$.

In the case that $\kappa = \kappa_c := -\frac{1}{2}\kappa_{\mathfrak{g}}$, then it follows from Theorem 22 that $\mathscr{W}^{\kappa_c}(\mathfrak{gl}_n)$ is commutative since $V^{(\kappa_c)_{\mathfrak{h}}}(\mathfrak{h})$ is commutative. In fact the following fact is known: Let $Z(V^{\kappa}(\mathfrak{g})) = \{z \in V^{\kappa}(\mathfrak{g}) \mid [z_{(n)}, a_{(n)}] = 0\}$, the center of $V^{\kappa}(\mathfrak{g})$.

Theorem 24 ([37]) *We have the isomorphism*

$$Z(V^{\kappa_c}(\mathfrak{g})) \xrightarrow{\sim} \mathscr{W}^{\kappa_c}(\mathfrak{g}), \quad z \mapsto [z \otimes 1].$$

This is a chiralization of Kostant's Theorem 5 in the sense that we recover Theorem 5 from Theorem 24 by considering the induced map between Zhu's algebras of both sides. The statement of Theorem 24 holds for any simple Lie algebra $\mathfrak{g}$ [37].

Remark 12 For a general simple Lie algebra $\mathfrak{g}$, the image of the Miura map for a generic k is described in terms of *screening operators*, see [40, 15.4]. Theorem 23 for $\mathfrak{g} = \mathfrak{gl}_n$ also follows from this description (the proof reduces to the case $\mathfrak{g} = \mathfrak{sl}_2$). An important application of this realization is the *Feigin-Frenkel duality* which states

$$\mathscr{W}^{k}(\mathfrak{g}) \cong \mathscr{W}^{{}^L k}({}^L\mathfrak{g}),$$

where ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra of $\mathfrak{g}$, $r^{\vee}(k+h^{\vee})({}^Lk+{}^Lh^{\vee}) = 1$. Here $r^{\vee}$ is the maximal number of the edges of the Dynking diagram of $\mathfrak{g}$ and ${}^Lh^{\vee}$ is the dual Coxeter number of ${}^L\mathfrak{g}$. In [37, 40] this isomorphism was stated only for a generic k, but it is not too difficult to see the isomorphism remains valid for an arbitrary k using the injectivity of the Miura map.

The Miura map is defined [60] for the W-algebra $\mathscr{W}^k(\mathfrak{g},f)$ associated with an arbitrary f, which is injective as well since the proof of Theorem 22 applies. Recently Naoki Genra [46] has obtained the description of the image by the Miura map in terms of screening operators for the W-algebra $\mathscr{W}^k(\mathfrak{g},f)$ associated with an arbitrary nilpotent element f.

5.10 Classical W-Algebras

Since the Poisson structure of $\mathbb{C}[\mathcal{S}]$ is trivial, we can give $\operatorname{gr}\mathscr{W}^{\kappa}(\mathfrak{g})$ a Poisson vertex algebra structure by the formula (32). The Poisson structure of $R_{V^{k+n}(\mathfrak{h})} = \mathbb{C}[\mathfrak{h}]$ is also trivial, hence $\operatorname{gr} V^{\kappa_{\mathfrak{h}}}(\mathfrak{h}) = \mathbb{C}[J\mathfrak{h}^*]$ is equipped with the Poisson vertex algebra structure by the formula (32) as well. Then the map $\operatorname{gr}\hat{\Upsilon} : \operatorname{gr}\mathscr{W}^{\kappa}(\mathfrak{g}) \hookrightarrow \operatorname{gr} V^{\kappa_{\mathfrak{h}}}(\mathfrak{h})$ is a homomorphism of Poisson vertex algebras with respect to these structures. Set $\kappa = k\kappa_0$, $k \in \mathbb{C}$, and consider its restriction $\operatorname{gr}\hat{\Upsilon} : \operatorname{gr}\mathscr{W}^{k}(\mathfrak{sl}_n) \hookrightarrow \operatorname{gr} V^{\kappa_{\mathfrak{h}}}(\mathfrak{h}')$, where $\mathfrak{h}'$ is the Cartan subalgebra of $\mathfrak{sl}_n$.

In $\operatorname{gr} V^{\kappa_{\mathfrak{h}}}(\mathfrak{h}')$ we have

$$\{h_\lambda h'\} = \kappa_{\mathfrak{h}}(h, h') = (k+n)\kappa_0(h, h'),$$

and this uniquely determines the λ-bracket of $\operatorname{gr} V^{\kappa_{\mathfrak{h}}}(\mathfrak{h}')$. Hence it is independent of k provided that $k \neq -n$. Since the image of $\operatorname{gr} \mathscr{W}^k(\mathfrak{sl}_n)$ is strongly generated by elements of $\mathbb{C}[(\mathfrak{h}')^*]^W$, it follows that the Poisson vertex algebra structure of $\operatorname{gr} \mathscr{W}^k(\mathfrak{sl}_n)$, $k \neq -n$, is independent of k. We denote this Poisson vertex algebra by $\mathscr{W}^{cl}(\mathfrak{sl}_n)$.

The Poisson vertex algebra $\mathscr{W}^{cl}(\mathfrak{sl}_n)$ is called the *classical W-algebra* associated with $\mathfrak{sl}_n$, which appeared in the works of Adler [3], Gelfand-Dickey [45] and Drinfeld-Sokolov [32]. Thus, the W-algebra $\mathscr{W}^k(\mathfrak{sl}_n)$, $k \neq -n$, is a deformation of $\mathscr{W}^{cl}(\mathfrak{sl}_n)$.

On the other hand the $\mathscr{W}$-algebra $\mathscr{W}^{-n}(\mathfrak{sl}_n)$ at the critical level can be identified with the space of the $\mathfrak{sl}_n$*-opers* [21] on the disk D. We refer to [39, 40] for more on this subject.

6 Representations of *W*-Algebras

From now on we set $\mathfrak{g} = \mathfrak{sl}_n$ and study the representations of $\mathscr{W}^k(\mathfrak{g})$ [see (45)].

6.1 Poisson Modules

Let R be a Poisson algebra. Recall that a *Poisson R-module* is a R-module M in the usual associative sense equipped with a bilinear map

$$R \times M \to M, \quad (r, m) \mapsto \operatorname{ad} r(m) = \{r, m\},$$

which makes M a Lie algebra module over R satisfying

$$\{r_1, r_2 m\} = \{r_1, r_2\}m + r_2\{r_1, m\}, \quad \{r_1 r_2, m\} = r_1\{r_2, m\} + r_2\{r_1, m\}$$

for $r_1, r_2 \in R$, $m \in M$. Let R-PMod be the category of Poisson modules over R.

Lemma 26 *A Poisson module over* $\mathbb{C}[\mathfrak{g}^*]$ *is the same as a* $\mathbb{C}[\mathfrak{g}^*]$*-module* M *in the usual associative sense equipped with a Lie algebra module structure* $\mathfrak{g} \to \operatorname{End} M$, $x \mapsto \operatorname{ad}(x)$, *such that*

$$\operatorname{ad}(x)(fm) = \{x, f\}.m + f.\operatorname{ad}(x)(m)$$

for $x \in \mathfrak{g}, f \in \mathbb{C}[\mathfrak{g}^*]$, $m \in M$.

6.2 Poisson Vertex Modules

Definition 5 A *Poisson vertex module* over a Poisson vertex algebra V is a V-module M as a vertex algebra equipped with a linear map

$$V \mapsto (\mathrm{End}\, M)[[z^{-1}]]z^{-1}, \quad a \mapsto Y_{-}^{M}(a,z) = \sum_{n \geqslant 0} a_{(n)}^{M} z^{-n-1},$$

satisfying

$$a_{(n)}^{M} m = 0 \quad \text{for } n \gg 0, \tag{52}$$

$$(Ta)_{(n)}^{M} = -n a_{(n-1)}^{M}, \tag{53}$$

$$a_{(n)}^{M}(bv) = (a_{(n)}b)v + b(a_{(n)}^{M}v), \tag{54}$$

$$[a_{(m)}^{M}, b_{(n)}^{M}] = \sum_{i \geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}^{M}, \tag{55}$$

$$(ab)_{(n)}^{M} = \sum_{i=0}^{\infty} (a_{(-i-1)}^{M} b_{(n+i)}^{M} + b_{(-i-1)}^{M} a_{(n+i)}^{M}) \tag{56}$$

for all $a, b \in V$, $m, n \geqslant 0$, $v \in M$.

A Poisson vertex algebra R is naturally a Poisson vertex module over itself.

Example 5 Let M be a Poisson vertex module over $\mathbb{C}[J\mathfrak{g}^*]$. Then by (55), the assignment

$$xt^n \mapsto x_{(n)}^{M} \quad x \in \mathfrak{g} \subset \mathbb{C}[\mathfrak{g}^*] \subset \mathbb{C}[J\mathfrak{g}^*],\ n \geqslant 0,$$

defines a $J\mathfrak{g} = \mathfrak{g}[[t]]$-module structure on M. In fact, a Poisson vertex module over $\mathbb{C}[J\mathfrak{g}^*]$ is the same as a $\mathbb{C}[J\mathfrak{g}^*]$-module M in the usual associative sense equipped with an action of the Lie algebra $J\mathfrak{g}$ such that $(xt^n)m = 0$ for $n \gg 0$, $x \in \mathfrak{g}$, $m \in M$, and

$$(xt^n) \cdot (am) = (x_{(n)}a)m + a(xt^n) \cdot m$$

for $x \in \mathfrak{g}$, $n \geqslant 0$, $a \in \mathbb{C}[J\mathfrak{g}^*]$, $m \in M$.

Below we often write $a_{(n)}$ for $a_{(n)}^{M}$.

The proofs of the following assertions are straightforward.

Lemma 27 *Let R be a Poisson algebra, E a Poisson module over R. There is a unique Poisson vertex JR-module structure on $JR \otimes_R E$ such that*

$$a_{(n)}(b \otimes m) = (a_{(n)}b) \otimes m + \delta_{n,0} b \otimes \{a, m\}$$

for $n \geqslant 0$, $a \in R \subset JR$, $b \in JR$. $m \in E$ (Recall that $JR = \mathbb{C}[J\,\mathrm{Spec}\,R]$.)

Lemma 28 *Let R be a Poisson algebra, M a Poisson vertex module over JR. Suppose that there exists a R-submodule E of M (in the usual commutative sense) such that $a_{(n)}E = 0$ for $n > 0$, $a \in R$, and M is generated by E (in the usual commutative sense). Then there exists a surjective homomorphism*

$$JR \otimes_R E \twoheadrightarrow M$$

of Poisson vertex modules.

6.3 Canonical Filtration of Modules Over Vertex Algebras

Let V be a vertex algebra graded by a Hamiltonian H. A *compatible filtration* of a V-module M is a decreasing filtration

$$M = \Gamma^0 M \supset \Gamma^1 M \supset \cdots$$

such that

$$a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n-1}M \quad \text{for } a \in F^p V,\ \forall n \in \mathbb{Z},$$
$$a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n}M \quad \text{for } a \in F^p V,\ n \geqslant 0,$$
$$H.\Gamma^p M \subset \Gamma^p M \quad \text{for all } p \geqslant 0,$$
$$\bigcap_p \Gamma^p M = 0.$$

For a compatible filtration $\Gamma^\bullet M$ the associated graded space

$$\mathrm{gr}^\Gamma M = \bigoplus_{p \geqslant 0} \Gamma^p M / \Gamma^{p+1} M$$

is naturally a graded vertex Poison module over the graded vertex Poisson algebra $\mathrm{gr}^F V$, and hence, it is a graded vertex Poison module over $JR_V = \mathbb{C}[\tilde{X}_V]$ by Theorem 10.

The vertex Poisson JR_V-module structure of $\mathrm{gr}^\Gamma M$ restricts to the Poisson R_V-module structure of $M/\Gamma^1 M = \Gamma^0 M/\Gamma^1 M$, and $a_{(n)}(M/\Gamma^1 M) = 0$ for $a \in R_V \subset$

JR_V, $n > 0$. It follows that there is a homomorphism

$$JR_V \otimes_{R_V} (M/\Gamma^1 M) \to \mathrm{gr}^{\Gamma} M, \quad a \otimes \bar{m} \mapsto a\bar{m},$$

of vertex Poisson modules by Lemma 28.

Suppose that V is positively graded and so is a V-module M. We denote by $F^{\bullet}M$ the Li filtration [68] of M, which is defined by

$$F^p M = \mathrm{span}_{\mathbb{C}}\{a^1_{(-n_1-1)} \dots a^r_{(-n_r-1)} m \mid a^i \in V,\ m \in M,\ n_1 + \dots + n_r \geqslant p\}.$$

It is a compatible filtration of M, and in fact is the finest compatible filtration of M, that is, $F^p M \subset \Gamma^p M$ for all p for any compatible filtration $\Gamma^{\bullet} M$ of M. The subspace $F^1 M$ is spanned by the vectors $a_{(-2)}m$ with $a \in V$, $m \in M$, which is often denoted by $C_2(M)$ in the literature. Set

$$\bar{M} = M/F^1 M (= M/C_2(M)), \tag{57}$$

which is a Poisson module over $R_V = \bar{V}$. By [68, Proposition 4.12], the vertex Poisson module homomorphism

$$JR_V \otimes_{R_V} \bar{M} \to \mathrm{gr}^F M$$

is surjective.

Let $\{a^i; i \in I\}$ be elements of V such that their images generate R_V in usual commutative sense, and let U be a subspace of M such that $M = U + F^1 M$. The surjectivity of the above map is equivalent to that

$$\begin{aligned} &F^p M \\ &= \mathrm{span}_{\mathbb{C}}\{a^{i_1}_{(-n_1-1)} \dots a^{i_r}_{(-n_r-1)} m \mid m \in U,\ n_i \geqslant 0, n_1 + \dots + n_r \geqslant p,\ i_1, \dots, i_r \in I\}. \end{aligned} \tag{58}$$

Lemma 29 *Let V be a vertex algebra, M a V-module. The Poisson vertex algebra module structure of $\mathrm{gr}^F M$ restricts to the Poisson module structure of $\bar{M} := M/F^1 M$ over R_V, that is, $\bar{M}$ is a Poisson R_V-module by*

$$\bar{a} \cdot \bar{m} = \overline{a_{(-1)}m}, \quad \mathrm{ad}(\bar{a})(\bar{m}) = \overline{a_{(0)}m}.$$

A V-module M is called *finitely strongly generated* if $\bar{M}$ is finitely generated as a R_V-module in the usual associative sense.

6.4 Associated Varieties of Modules Over Affine Vertex Algebras

A $\widehat{\mathfrak{g}}$-module M of level k is called smooth if $x(z)$ is a field on M for $x \in \mathfrak{g}$, that is, $xt^n m = 0$ for $n \gg 0$, $x \in \mathfrak{g}$, $m \in M$. Any $V^k(\mathfrak{g})$-module M is naturally a smooth $\widehat{\mathfrak{g}}$-module of level k. Conversely, any smooth $\widehat{\mathfrak{g}}$-module of level k can be regarded as a $V^k(\mathfrak{g})$-module. It follows that a $V^k(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}$-module of level k.

For a $V = V^k(\mathfrak{g})$-module M, or equivalently, a smooth $\widehat{\mathfrak{g}}$-module of level k, we have

$$\bar{M} = M/\mathfrak{g}[t^{-1}]t^{-2}M,$$

and the Poisson $\mathbb{C}[\mathfrak{g}^*]$-module structure is given by

$$x \cdot \bar{m} = \overline{xt^{-1}m}, \quad \mathrm{ad}\,(x)\bar{m} = \overline{xm}.$$

For a $\mathfrak{g}$-module E let

$$V_E^k := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} E,$$

where E is considered as a $\mathfrak{g}[t]\oplus\mathbb{C}K$-module on which $\mathfrak{g}[t]$ acts via the projection $\mathfrak{g}[t] \to \mathfrak{g}$ and K acts as multiplication by k. Then

$$\overline{V_E^k} \cong \mathbb{C}[\mathfrak{g}] \otimes E,$$

where the Poisson $\mathbb{C}[\mathfrak{g}^*]$-module structure is given by

$$f \cdot g \otimes v = (fg) \otimes v, \quad \mathrm{ad}x(f \otimes v) = \{x,f\} \otimes v + f \otimes xv,$$

for $f, g \in \mathbb{C}[\mathfrak{g}^*]$, $v \in V$.

Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ of level k [53], $\mathbf{KL}_k$ the full subcategory of $\mathcal{O}_k$ consisting of modules M which are integrable over $\mathfrak{g}$. Note that V_E^k is a object of $\mathbf{KL}_k$ for a finite-dimensional representation E of $\mathfrak{g}$. Thus, $V^k(\mathfrak{g}) = V_{\mathbb{C}}^k$ and its simple quotient $V_k(\mathfrak{g})$ are also objects of $\mathbf{KL}_k$.

Both $\mathcal{O}_k$ and $\mathbf{KL}_k$ can be regarded as full subcategories of the category of $V^k(\mathfrak{g})$-modules.

Lemma 30 *For $M \in \mathbf{KL}_k$ the following conditions are equivalent.*

1. *M is finitely strongly generated as a $V^k(\mathfrak{g})$-module,*
2. *M is finitely generated as a $\mathfrak{g}[t^{-1}]t^{-1}$-module,*
3. *M is finitely generated as a $\widehat{\mathfrak{g}}$-module.*

For a finitely strongly generated $V^k(\mathfrak{g})$-module M define its *associated variety* X_M by

$$X_M = \operatorname{supp}_{R_V}(\bar{M}) \subset X_V,$$

equipped with a reduced scheme structure.

Example 6 $X_{V_E^k} = \mathfrak{g}^*$ for a finite-dimensional representation E of $\mathfrak{g}$.

6.5 Ginzburg's Correspondence

Let $\overline{\mathcal{HC}}$ be the full subcategory of the category of Poisson $\mathbb{C}[\mathfrak{g}^*]$-modules on which the Lie algebra $\mathfrak{g}$-action (see Lemma 26) is integrable.

Lemma 31 *For $M \in \mathbf{KL}_k$, the Poisson $\mathbb{C}[\mathfrak{g}^*]$-module $\bar{M}$ belongs to $\overline{\mathcal{HC}}$.*
By Lemma 31 we have a right exact functor

$$\mathbf{KL}_k \to \overline{\mathcal{HC}}, \quad M \mapsto \bar{M}.$$

For $M \in \overline{\mathcal{HC}}$, $M \otimes \overline{Cl}$ is naturally a Poisson module over $\bar{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}$. (The notation of Poisson modules natural extends to the Poisson supralgebras.) Thus, $(M \otimes \overline{Cl}, \operatorname{ad} \bar{Q})$ is a differential graded Poisson module over the differential graded Poisson module $(\bar{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$. In particular it cohomology $H^\bullet(\bar{M} \otimes \overline{Cl}, \operatorname{ad} \bar{Q})$ is a Poisson module over $H^\bullet(\bar{C}(\mathfrak{g}), \operatorname{ad} \bar{Q}) = \mathbb{C}[\mathcal{S}]$. So we get a functor

$$\overline{\mathcal{HC}} \to \mathbb{C}[\mathcal{S}]\text{-Mod}, \quad M \mapsto H^0(M) := H^0(M \otimes \overline{Cl},\ ad \bar{Q}).$$

The following assertion is a restatement of a result of Ginzburg [47] (see [10, Theorem 2.3]).

Theorem 25 *Let $M \in \overline{\mathcal{HC}}$. Then $H^i(M) = 0$ for $i \neq 0$, and we have an isomorphism*

$$H^0(M) \cong (M / \sum_i \mathbb{C}[\mathfrak{g}^*](x_i - \chi(x_i))M)^N.$$

In particular if M is finitely generated $H^0(M)$ is finitely generated over $\mathbb{C}[\mathcal{S}]$ and

$$\operatorname{supp}_{\mathbb{C}[\mathcal{S}]} H^0(M) = (\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]} M) \cap \mathcal{S}.$$

Corollary 6 *The functor $\overline{\mathcal{HC}} \to \mathbb{C}[\mathcal{S}]$-Mod, $M \mapsto H^0(M)$, is exact.*
Denote by $\mathcal{N}$ the set of nilpotent elements of $\mathfrak{g}$, which equals to the zero locus of the augmentation ideal $\mathbb{C}[\mathfrak{g}^*]^G_+$ of $\mathbb{C}[\mathfrak{g}^*]^G$ under the identification $\mathfrak{g} = \mathfrak{g}^*$ via $(\ |\ )$.

Since the element f (defined in (3)). is regular (or principal), the orbit

$$\mathbb{O}_{prin} := G.f \subset \mathfrak{g} = \mathfrak{g}^*$$

is dense in $\mathcal{N}$:

$$\mathcal{N} = \overline{\mathbb{O}_{prin}}.$$

The transversality of $\mathcal{S}$ implies that

$$\mathcal{S} \cap \mathcal{N} = \{f\}.$$

Theorem 26 ([47]) *Let M be a finitely generated object in $\overline{\mathcal{HC}}$.*

1. $H^0(M) \neq 0$ *if and only if* $\mathcal{N} \subset \operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]} M$.
2. $H^0(M)$ *is nonzero and finite-dimensional if* $\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]} M = \mathcal{N}$.

Proof (1) Note that $\operatorname{supp}_{\mathbb{C}[\mathcal{S}]} H^0(M)$ is invariant under the $\mathbb{C}^*$-action (7) on $\mathcal{S}$, which contracts the point $\{f\}$. Hence $\operatorname{supp}_{\mathbb{C}[\mathcal{S}]} H^0(M) = (\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]} M) \cap \mathcal{S}$ is nonempty if and only if $f \in \operatorname{supp}_{\mathbb{C}[\mathcal{S}]} H^0(M)$. The assertion follows since $\operatorname{supp}_{\mathbb{C}[\mathcal{S}]} H^0(M)$ is G-invariant and closed. (2) Obvious since the assumption implies that $\operatorname{supp}_{\mathbb{C}[\mathcal{S}]} H^0(M) = \{f\}$.

6.6 Losev's Correspondence

Let $\mathcal{HC}$ be the category of *Harish-Chandra bimodules*, that is, the full subcategory of the category of $U(\mathfrak{g})$-bimodules on which the adjoint action of $\mathfrak{g}$ is integrable.

Lemma 32 *Every finitely generated object M of $\mathcal{HC}$ admits a good filtration, that is, an increasing filtration* $0 = F_0M \subset F_1M \subset \dots$ *such that* $M = \bigcup F_pM$,

$$U_p(\mathfrak{g}) \cdot F_qM \cdot U_r(\mathfrak{g}) \subset F_{p+q+r}M, \quad [U_p(\mathfrak{g}), F_pM] \subset F_{p+q-1}M,$$

and $\operatorname{gr}^F M = \bigoplus_p F_pM/F_{p-1}M$ *is finitely generated over* $\mathbb{C}[\mathfrak{g}^*]$.

If $M \in \mathcal{HC}$ and $F_\bullet M$ is a good filtration, then $\operatorname{gr}^F M$ is naturally a Poisson module over $\mathbb{C}[\mathfrak{g}^*]$. Therefore, it is an object of $\overline{\mathcal{HC}}$.

Let M be a finitely generated object in $\mathcal{HC}$. It is known since Bernstein that

$$\operatorname{Var}(M) := \operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(\operatorname{gr}^F M) \subset \mathfrak{g}^*$$

in independent of the choice of a good filtration $F_\bullet M$ of M.

For $M \in \mathcal{HC}$, $M \otimes Cl$ is naturally a bimodule over $C(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cl$. Thus, $(M \otimes Cl, \operatorname{ad} Q)$ is a differential graded bimodule over $C(\mathfrak{g})$, and its cohomology

$$H^\bullet(M) := H^\bullet(M \otimes Cl, \operatorname{ad} Q)$$

is naturally a module over $H^0(C(\mathfrak{g}), \operatorname{ad} Q)$ that is identified with $Z(\mathfrak{g})$ by Theorem 5. Thus, we have a functor

$$\mathcal{HC} \to Z(\mathfrak{g})\text{-Mod}, \quad M \mapsto H^0(M). \tag{59}$$

Let $M \in \mathcal{HC}$ be finitely generated, $F_\bullet M$ a good filtration. Then $F_p(M \otimes Cl) := \sum_{i+j=p} F_i M \otimes Cl_j$ defines a good filtration of $M \otimes Cl$, and the associated graded space $\operatorname{gr}_F(M \otimes Cl) = \sum_i F_p(M \otimes Cl)/F_{p-1}(M \otimes Cl) = (\operatorname{gr}_F M) \otimes \overline{Cl}$ is a Poisson module over $\operatorname{gr} C(\mathfrak{g}) = \bar{C}(\mathfrak{g})$.

The filtration $F_\bullet(M \otimes Cl)$ induces a filtration $F_\bullet H^\bullet(M)$ on $H^\bullet(M)$, and $\operatorname{gr}_F H^\bullet(M) = \bigoplus_p F_p H^\bullet(M)/F_{p-1} H^0(M)$ is a module over $\operatorname{gr} Z(\mathfrak{g}) = \mathbb{C}[\mathcal{S}]$.

For a finitely generated $\mathcal{Z}(\mathfrak{g})$-module M, set $\operatorname{Var}(M) = \operatorname{supp}_{\mathbb{C}[\mathcal{S}]}(\operatorname{gr} M)$, $\operatorname{gr} M$ is the associated graded M with respect to a good filtration of M.

The following assertion follows from Theorems 25 and 26.

Theorem 27 ([47, 69])

1. *We have $H^i(M) = 0$ for all $i \neq 0$, $M \in \mathcal{HC}$. Therefore the functor (59) is exact.*
2. *Let M be a finitely generated object of $\mathcal{HC}$, $F_\bullet M$ a good filtration. Then $\operatorname{gr}_F H^0(M) \cong H^0(\operatorname{gr}_F M)$. In particular $H^0(M)$ is finitely generated, $F_\bullet H^0(M)$ is a good filtration of $H^0(M)$.*
3. *For a finitely generated object M of $\mathcal{HC}$, $\operatorname{Var}(H^0(M)) = \operatorname{Var}(M) \cap \mathcal{S}$.*

6.7 Frenkel-Zhu's Bimodules

Recall that for a graded vertex algebra V, Zhu's algebra $\operatorname{Zhu}(V) = V/V \circ V$ is defined. There is a similar construction for modules due to Frenkel and Zhu [43]. For a V-module M set

$$\operatorname{Zhu}(M) = M/V \circ M,$$

where $V \circ M$ is the subspace of M spanned by the vectors

$$a \circ m = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} m$$

for $a \in V_{\Delta_a}$, $\Delta_a \in \mathbb{Z}$, and $m \in M$.

Proposition 10 ([43]) Zhu(M) *is a bimodule over* Zhu(V) *by the multiplications*

$$a * m = \sum_{i\geqslant 0} \binom{\Delta_a}{i} a_{(i-1)}b, \quad m * a = \sum_{i\geqslant 0} \binom{\Delta_a - 1}{i} a_{(i-1)}m$$

for $a \in V_{\Delta_a}$, $\Delta_a \in \mathbb{Z}$, *and* $m \in M$.
Thus, we have a right exact functor

$$V\text{-Mod} \to \mathrm{Zhu}(V)\text{-biMod}, \quad M \mapsto \mathrm{Zhu}(M).$$

Lemma 33 *Let* $M = \bigoplus_{d\in h+\mathbb{Z}_+} M_d$ *be a positive energy representation of a* $\mathbb{Z}_+$*-graded vertex algebra* V. *Define an increasing filtration* $\{\mathrm{Zhu}_p(M)\}$ *on* $\mathrm{Zhu}(V)$ *by*

$$\mathrm{Zhu}_p(M) = \mathrm{Im}(\bigoplus_{d=h}^{h+p} M_p \to \mathrm{Zhu}(M)).$$

1. *We have*

$$\mathrm{Zhu}_p(V) \cdot \mathrm{Zhu}_q(M) \cdot \mathrm{Zhu}_r(V) \subset \mathrm{Zhu}_{p+q+r}(M),$$
$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(M)] \subset \mathrm{Zhu}_{p+q-1}(M).$$

   Therefore $\mathrm{gr}\,\mathrm{Zhu}(M) = \bigoplus_p \mathrm{Zhu}_p(M)/\mathrm{Zhu}_{p-1}(M)$ *is a Poisson* $\mathrm{gr}\,\mathrm{Zhu}(V)$*-module, and hence is a Poisson* R_V*-module through the homomorphism* $\eta_V : R_V \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}(V)$.
2. *There is a natural surjective homomorphism*

$$\eta_M : \bar{M}(= M/F^1M) \to \mathrm{gr}\,\mathrm{Zhu}(M)$$

   of Poisson R_V*-modules. This is an isomorphism if* V *admits a PBW basis and* $\mathrm{gr}\,M$ *is free over* $\mathrm{gr}\,V$.

Example 7 Let $M = V_E^k$. Since $\mathrm{gr}\,V_E^k$ is free over $\mathbb{C}[J\mathfrak{g}^*]$, we have the isomorphism

$$\eta_{V_E^k} : \overline{V_E^k} = E \otimes \mathbb{C}[\mathfrak{g}^*] \xrightarrow{\sim} \mathrm{gr}\,\mathrm{Zhu}(V_E^k).$$

On the other hand, there is a $U(\mathfrak{g})$-bimodule homomorphism

$$\begin{aligned} E \otimes U(\mathfrak{g}) &\to \mathrm{Zhu}(V_E^k), \\ v \otimes x_1 \dots x_r &\mapsto (1 \otimes v) * (x_1 t^{-1}) * (x_1 t^{-1}) + V^k(\mathfrak{g}) \circ V_E^k \end{aligned} \tag{60}$$

which respects the filtration. Here the $U(\mathfrak{g})$-bimodule structure of $U(\mathfrak{g}) \otimes E$ is given by

$$x(v \otimes u) = (xv) \otimes u + v \otimes xu, \quad (v \otimes u)x = v \otimes (ux),$$

and the filtration of $U(\mathfrak{g}) \otimes E$ is given by $\{U_i(\mathfrak{g}) \otimes E\}$. Since the induced homomorphism between associated graded spaces (60) coincides with $\eta_{V_E^k}$, (60) is an isomorphism.

Lemma 34 *For $M \in \mathbf{KL}_k$ we have* $\mathrm{Zhu}(M) \in \mathcal{HC}$. *If M is finitely generated, then so is* $\mathrm{Zhu}(M)$.

6.8 Zhu's Two Functors Commute with BRST Reduction

For a smooth $\widehat{\mathfrak{g}}$-module M over level k, $C(M) := M \otimes \mathcal{F}$ is naturally a module over $C^k(\mathfrak{g}) = V^k(\mathfrak{g}) \otimes \mathcal{F}$. Thus, $(C(M), Q_{(0)})$ is a cochain complex, and its cohomology $H^\bullet(M) := H^\bullet(C(M), Q_{(0)})$ is a module over $\mathscr{W}^k(\mathfrak{g}) = H^\bullet(C^k(\mathfrak{g}), Q_{(0)})$. Thus we have a functor

$$V^k(\mathfrak{g})\text{-Mod} \to \mathscr{W}^k(\mathfrak{g})\text{-Mod}, \quad M \mapsto H^0(M).$$

Here V-Mod denotes the category of modules over a vertex algebra V.

Theorem 28

1. *[9, 41] We have $H^i(M) = 0$ for $i \neq 0$, $M \in \mathbf{KL}_k$. In particular the functor*

$$\mathbf{KL}_k \to \mathscr{W}^k(\mathfrak{g})\text{-Mod}, \quad M \mapsto H^0(M),$$

   is exact.
2. *[9] For a finitely generated object M of* $\mathbf{KL}$,

$$\overline{H^0(M)} \cong H^0(\bar{M})$$

   as Poisson modules over $R_{W^k(\mathfrak{g})} = \mathbb{C}[\mathcal{S}]$. In particular $H^0(M)$ is finitely strongly generated and

$$X_{H^0(M)} = X_M \cap \mathcal{S}.$$

3. *[10] For a finitely generated object M of* $\mathbf{KL}$,

$$\mathrm{Zhu}(H^0(M)) \cong H^0(\mathrm{Zhu}(M))$$

   as bimodules over $\mathrm{Zhu}(\mathscr{W}^k(\mathfrak{g})) = \mathcal{Z}(\mathfrak{g})$.

Let $\mathscr{W}_k(\mathfrak{g})$ denote the unique simple graded quotient of $\mathscr{W}^k(\mathfrak{g})$. Then $X_{\mathscr{W}_k(\mathfrak{g})}$ is a $\mathbb{C}^*$-invariant subvariety of $\mathcal{S}$. Therefore $X_{\mathscr{W}^k(\mathfrak{g})}$ is lisse if and only if $X_{\mathscr{W}_k(\mathfrak{g})} = \{f\}$ since the $\mathbb{C}^*$-action on $\mathcal{S}$ contracts to the point f.

Corollary 7

1. *$H^0(V_k(\mathfrak{g}))$ is a quotient of $\mathscr{W}^k(\mathfrak{g}) = H^0(V^k(\mathfrak{g}))$. In particular $\mathscr{W}_k(\mathfrak{g})$ is a quotient of $H^0(V_k(\mathfrak{g}))$ if $H^0(V_k(\mathfrak{g}))$ is nonzero.*
2. *$H^0(V_k(\mathfrak{g}))$ is nonzero if and only if $X_{V_k(\mathfrak{g})} \supset \overline{G.f} = \mathcal{N}$.*
3. *The simple W-algebra $\mathscr{W}_k(\mathfrak{g})$ is lisse if $X_{V_k(\mathfrak{g})} = \overline{G.f} = \mathcal{N}$.*

Proof

1. follows from the exactness statement of Theorem 28.
2. $H^0(V_k(\mathfrak{g}))$ is nonzero if and only of $X_{H^0(M)} = X_M \cap \mathcal{S}$ is non-empty. This happens if and only if $f \in X_M$ since $X_{H^0(M)}$ is $\mathbb{C}^*$-stable. The assertion follows since X_M is G-invariant and closed.
3. If $X_{V_k(\mathfrak{g})} = \overline{G.f}$, $X_{H^0(V_k(\mathfrak{g}))} = X_M \cap \mathcal{S} = \{f\}$, and thus, $H^0(V_k(\mathfrak{g}))$ is lisse, and thus, so its quotient $\mathscr{W}_k(\mathfrak{g})$.

Remark 13

1. The above results hold for W-algebras associated with any $\mathfrak{g}$ and any $f \in \mathcal{N}$ without any restriction on the level k [9, 10]. In particular we have the vanishing result

$$H_f^i(M) = 0 \quad \text{for } i \neq 0,\ M \in \mathbf{KL}_k, \tag{61}$$

for the BRST cohomology $H_f^i(M)$ of the quantized Drinfeld-Sokolov reduction functor associated with f in the coefficient in an object M of $\mathbf{KL}_k$. Thus the functor

$$\mathbf{KL}_k \to \mathscr{W}^k(\mathfrak{g},f)\text{-Mod}, \quad M \mapsto H_f^0(M),$$

is exact, and moreover,

$$X_{H_f^0(V_k(\mathfrak{g}))} = X_{V_k(\mathfrak{g})} \cap \mathcal{S}_f,$$

where S_f is the Slodowy slice at f (see Sect. 2.8). In particular

$$H_f^0(V_k(\mathfrak{g})) \neq 0 \iff X_{V_k(\mathfrak{g})} \supset \overline{G.f}. \tag{62}$$

2. In the case that $f = f_\theta$, a minimal nilpotent element of $\mathfrak{g}$, then we also have the following result [4]:

$$H_{f_\theta}(V_k(\mathfrak{g})) = \begin{cases} \mathscr{W}_k(\mathfrak{g},f_\theta) & \text{if } k \notin \mathbb{Z}_+, \\ 0 & \text{if } k \in \mathbb{Z}_+. \end{cases}$$

Here $\mathscr{W}_k(\mathfrak{g}, f_\theta)$ is the simple quotient of $\mathscr{W}^k(\mathfrak{g}, f_\theta)$. Together with (62), this proves the "only if" part of Theorem 12. Indeed, if $V_k(\mathfrak{g})$ is lisse, then $H_{f_\theta}(V_k(\mathfrak{g})) = 0$ by (62), and hence, $k \in \mathbb{Z}_+$.

7 Irreducible Representations of W-Algebras

In this section we quickly review results obtained in [5].

Since $\mathrm{Zhu}(\mathscr{W}^k(\mathfrak{g})) \cong \mathcal{Z}(\mathfrak{g})$, by Zhu's theorem irreducible positive energy representations of $\mathscr{W}^k(\mathfrak{g})$ are parametrized by central characters of $\mathcal{Z}(\mathfrak{g})$. For a central character $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$, let $\mathbb{L}(\gamma)$ be the corresponding irreducible positive energy representations of $\mathscr{W}^k(\mathfrak{g})$. This is a simple quotient of the *Verma module* $\mathbb{M}(\gamma)$ of $\mathscr{W}^k(\mathfrak{g})$ with highest weight γ, which has the character

$$\mathrm{ch}\,\mathbb{M}(\gamma) := \mathrm{tr}_{\mathbb{M}(\gamma)}\, q^{L_0} = \frac{q^{\frac{\gamma(\Omega)}{2(k+h^\vee)}}}{\prod_{j\geqslant 1}(1-q^j)^{\mathrm{rk}\,\mathfrak{g}}}$$

in the case that k is non-critical, where Ω is the Casimir element of $U(\mathfrak{g})$.

In Theorem 28 we showed that the functor $\mathbf{KL}_k \to \mathscr{W}^k(\mathfrak{g})\text{-Mod}$, $M \mapsto H^0(M)$, is exact. However in order to obtain all the irreducible positive energy representation we need to extend this functor to the whole category $\mathcal{O}_k$. However the functor $\mathcal{O}_k \to \mathscr{W}^k(\mathfrak{g})\text{-Mod}$, $M \mapsto H^0(M)$, is *not* exact in general except for the case $\mathfrak{g} = \mathfrak{sl}_2$ [4]. Nevertheless, we can [44] modify the functor to obtain the following result.

Theorem 29 ([5]) *There exists an exact functor*

$$\mathcal{O}_k \to \mathscr{W}^k(\mathfrak{g})\text{-Mod}, \quad M \mapsto H^0_-(M)$$

(called the "−"-reduction functor in [44]), which enjoys the following properties.

1. $H^0_-(M(\lambda)) \cong \mathbb{M}(\gamma_{\bar{\lambda}})$, *where* $M(\lambda)$ *is the Verma module of* $\widehat{\mathfrak{g}}$ *with highest weight* λ, *and* $\gamma_{\bar{\lambda}}$ *is the evaluation of* $\mathcal{Z}(\mathfrak{g})$ *at the Verma module* $M_{\mathfrak{g}}(\bar{\lambda})$ *of* $\mathfrak{g}$ *with highest weight* $\bar{\lambda}$.
2. $H^0_-(L(\lambda)) \cong \begin{cases} \mathbb{L}(\gamma_{\bar{\lambda}}) & \text{if } \bar{\lambda} \text{ is anti-dominant (that is, } M_{\mathfrak{g}}(\bar{\lambda}) \text{ is simple)}, \\ 0 & \text{otherwise.} \end{cases}$

Corollary 8 *Write* $\mathrm{ch}\, L(\lambda) = \sum_\mu c_{\lambda,\mu}\, \mathrm{ch}\, M(\mu)$ *with* $c_{\lambda,\mu} \in \mathbb{Z}$. *If* $\bar{\lambda}$ *is anti-dominant, we have*

$$\mathrm{ch}\,\mathbb{L}(\gamma_{\bar{\lambda}}) = \sum_\mu c_{\lambda,\mu}\, \mathrm{ch}\,\mathbb{M}(\gamma_{\bar{\mu}}).$$

In the case that k is non-critical, then it is known by Kashiwara and Tanisaki [62] that the coefficient $c_{\lambda,\mu}$ is expressed in terms of Kazhdan-Lusztig polynomials.

Since any central character of $\mathcal{Z}(\mathfrak{g})$ can be written as $\gamma_{\bar{\lambda}}$ with anti-dominant $\bar{\lambda}$, Corollary 8 determines the character of *all* the irreducible positive energy representations of $\mathscr{W}^k(\mathfrak{g})$ for all non-critical k.

On the other hand, in the case that k is critical, all $\mathbb{L}(\gamma_{\bar{\lambda}})$ are one-dimensional since $\mathscr{W}^{-n}(\mathfrak{g})$ is commutative. This fact with Theorem 29 can be used in the study of the critical level representations of $\widehat{\mathfrak{g}}$, see [13].

The results in this section hold for arbitrary simple Lie algebra $\mathfrak{g}$.

Remark 14 The condition $\bar{\lambda} \in \mathfrak{h}^*$ is anti-dominant does not imply that $\lambda \in \widehat{\mathfrak{h}}^*$ is anti-dominant. In fact this condition is satisfied by all *non-degenerate admissible weights* λ (see below) which are regular dominant.

Remark 15 Theorem 29 has been generalized in [6]. In particular the character of all the simple ordinary representations (=simple positive energy representations with finite-dimensional homogeneous spaces) has been determined for W-algebras associated with all nilpotent elements f in type A.

8 Kac-Wakimoto Admissible Representations and Frenkel-Kac-Wakimoto Conjecture

We continue to assume that $\mathfrak{g} = \mathfrak{sl}_n$, but the results in this section holds for arbitrary simple Lie algebra $\mathfrak{g}$ as well with appropriate modification unless otherwise stated.

8.1 Admissible Affine Vertex Algebras

Let $\widehat{\mathfrak{h}}$ be the Cartan subalgebra $\mathfrak{h} \oplus \mathbb{C}K$ of $\widehat{\mathfrak{g}}$, $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ the extended Cartan subalgebra, $\widehat{\Delta}$ the set of roots of $\widehat{\mathfrak{g}}$ in $\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, where $\Lambda_0(K) = 1 = \delta(d)$, $\Lambda_0(\mathfrak{h} + \mathbb{C}D) = \delta(\mathfrak{h} \oplus \mathbb{C}K) = 0$, $\widehat{\Delta}_+$ the set of positive roots. $\widehat{\Delta}^{re} \subset \widehat{\Delta}$ the set of real roots, $\widehat{\Delta}^{re}_+ = \widehat{\Delta}^{re} \cap \widehat{\Delta}_+$. Let $\widehat{W}$ be the affine Weyl group of $\widehat{\mathfrak{g}}$.

Definition 6 ([57]) A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* if

1. λ is regular dominant, that is,

$$\langle \lambda + \rho, \alpha^\vee \rangle \notin -\mathbb{Z}_+ \quad \text{for all } \alpha \in \widehat{\Delta}^{re}_+,$$

2. $\mathbb{Q}\widehat{\Delta}(\lambda) = \mathbb{Q}\widehat{\Delta}^{re}$, where $\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{re} \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$.

The irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ with highest weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible. Note that an irreducible integrable representations of $\widehat{\mathfrak{g}}$ is admissible.

Clearly, integrable representations of $\widehat{\mathfrak{g}}$ are admissible.

For an admissible representation $L(\lambda)$ we have [56]

$$\mathrm{ch}\, L(\lambda) = \sum_{w \in \widehat{W}(\lambda)} (-1)^{\ell_\lambda(w)}\, \mathrm{ch}\, M(w \circ \lambda) \tag{63}$$

since λ is regular dominant, where $\widehat{W}(\lambda)$ is the *integral Weyl group* [61, 74] of λ, that is, the subgroup of $\widehat{W}$ generated by the reflections s_α associated with $\alpha \in \widehat{\Delta}$ and $w \circ \lambda = w(\lambda + \rho) - \rho$. Further the condition (2) implies that $\mathrm{ch}\, L(\lambda)$ is written in terms of certain theta functions. Kac and Wakimoto [57] showed that admissible representations are *modular invariant*, that is, the characters of admissible representations form an $SL_2(\mathbb{Z})$ invariant subspace.

Let λ, μ be distinct admissible weights. Then the condition (1) implies that

$$\mathrm{Ext}^1_{\widehat{\mathfrak{g}}}(L(\lambda), L(\mu)) = 0.$$

Further, the following fact is known by Gorelik and Kac [49].

Theorem 30 ([49]) *Let λ be admissible. Then* $\mathrm{Ext}_{\widehat{\mathfrak{g}}}(L(\lambda), L(\lambda)) = 0$.
Therefore admissible representations form a semisimple fullsubcategory of the category of $\widehat{\mathfrak{g}}$-modules.

Recall that the simple affine vertex algebra $V_k(\mathfrak{g})$ is isomorphic to $L(k\Lambda_0)$ as an $\widehat{\mathfrak{g}}$-module.

Lemma 35 *The following conditions are equivalent.*

1. *$k\Lambda_0$ is admissible.*
2. *$k\Lambda_0$ is regular dominant and $k \in \mathbb{Q}$.*
3. *$k + h^\vee = p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$, $p \geqslant h^\vee = n$.*

If this is the case, the level k is called admissible for $\widehat{\mathfrak{g}}$, and $V_k(\mathfrak{g})$ is called an admissible affine vertex algebra.

For an admissible number k let Pr_k be the set of admissible weights of $\widehat{\mathfrak{g}}$ of level k. (For $\mathfrak{g} = \mathfrak{sl}_n$, Pr_k is the same as the set of *principal admissible weights* of level k.)

8.2 Feigin-Frenkel Conjecture and Adamović-Milas Conjecture

The following fact was conjectured by Feigin and Frenkel and proved for the case that $\mathfrak{g} = \mathfrak{sl}_2$ by Feigin and Malikov [38].

Theorem 31 ([9]) *The associated variety $X_{V_k(\mathfrak{g})}$ is contained in $\mathcal{N}$ if k is admissible.*

In fact the following holds.

Theorem 32 ([9]) *Let k be admissible, and let $q \in N$ be the denominator of k, that is, $k + h^{\vee} = p/q$, $p \in N$, $(p, q) = 1$. Then*

$$X_{V_k(\mathfrak{g})} = \{x \in \mathfrak{g} \mid (\mathrm{ad}\, x)^{2q} = 0\} = \overline{\mathbb{O}_q},$$

where $\mathbb{O}_q$ is the nilpotent orbit corresponding to the partition

$$\begin{cases} (n) & \text{if } q \geqslant n, \\ (q, q, \dots, q, s) \quad (0 \leqslant s \leqslant n-1) & \text{if } q < n. \end{cases}$$

The following fact was conjectured by Adamović and Milas [2].

Theorem 33 ([11]) *Let k be admissible. Then an irreducible highest weight representation $L(\lambda)$ is a $V_k(\mathfrak{g})$-module if and only if $k \in Pr_k$. Hence if M is a finitely generated $V_k(\mathfrak{g})$-module on which $\widehat{\mathfrak{n}}_+$ acts locally nilpotently and $\widehat{\mathfrak{h}}$ acts locally finitely then M is a direct sum of $L(\lambda)$ with $\lambda \in Pr_k$.*

8.3 Outline of Proofs of Theorems 31, 32 and 33

The idea of the proofs of Theorems 31 and 33 is to reduce to the $\widehat{\mathfrak{sl}_2}$-cases.

Let $\mathfrak{sl}_{2,i} \subset \mathfrak{g}$ be the copy of $\mathfrak{sl}_2$ spanned by $e_i := e_{i,i+1}$, $h_i := e_{i,i} - e_{i+1,i+1}$, $f_i := e_{i+1,i}$, and let $\mathfrak{p}_i = \mathfrak{sl}_{2,i} + \mathfrak{b} \subset \mathfrak{g}$, the associated minimal parabolic subalgebra. Then

$$\mathfrak{p}_i = \mathfrak{l}_i \oplus \mathfrak{m}_i,$$

where $\mathfrak{l}_i$ is the Levi subalgebra $\mathfrak{sl}_{2,i} + \mathfrak{h}$, and $\mathfrak{m}_i$ is the nilradical $\bigoplus_{\substack{1 \leqslant p < q \leqslant n \\ (p,q) \neq (i,i+1)}} \mathbb{C} e_{p,q}$.

Consider the semi-infinite cohomology $H^{\frac{\infty}{2}+0}(\mathfrak{m}_i[t, t^{-1}], M)$. It is defined as a cohomology of Feigin's complex $(C(\mathfrak{m}_i[t, t^{-1}], M), d)$ [35]. There is a natural vertex algebra homomorphism

$$V^{k_i}(\mathfrak{sl}_2) \to H^{\frac{\infty}{2}+0}(\mathfrak{m}_i[t, t^{-1}], M), \tag{64}$$

where $k_i = k + n - 2$, see, e.g. [50]. Note that if k is an admissible number for $\widehat{\mathfrak{g}}$ then k_i is an admissible number for $\widehat{\mathfrak{sl}_2}$.

Theorem 34 ([8]) *Let k be an admissible number. The map (64) factors through the vertex algebra embedding*

$$V_{k_i}(\mathfrak{sl}_2) \hookrightarrow H^{\frac{\infty}{2}+0}(\mathfrak{m}_i[t, t^{-1}], V_k(\mathfrak{g})).$$

Proof (Outline of Proof of Theorem 31) First, consider the case that $\mathfrak{g} = \mathfrak{sl}_2$. Let N_k be the maximal submodule of $V^k(\mathfrak{g})$, and let I_k be the image of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$, so that $R_{V_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_k$. It is known by Kac and Wakimoto [56] that N_k is generated by a singular vector, say v_k. The projection formula [72] implies that the image $[v_k]$ of v_k in I_k is nonzero. Since $[v_k]$ is a singular vector of $\mathbb{C}[\mathfrak{g}^*]$ with respect to the adjoint action of $\mathfrak{g}$, Kostant's Separation Theorem implies that

$$[v_k] = e^m \Omega^n$$

for some $m, n \in \mathbb{N}$ up to constant multiplication, where $\Omega = ef + fe + \frac{1}{2}h^2$. Now suppose that $X_{V_k(\mathfrak{g})} \not\subset \mathcal{N}$ and let $\lambda \in X_{V_k(\mathfrak{g})} \backslash \mathcal{N}$, so that $\Omega(\lambda) \neq 0$. Then $e(\lambda) = 0$. Since $X_{V_k(\mathfrak{g})}$ is G-invariant this implies that $x(\lambda) = 0$ for any nilpotent element x of $\mathfrak{g}$. Because any element of $\mathfrak{g}$ can be written as a sum of nilpotent elements we get that $\lambda = 0$. Contradiction.

Next, consider the case that $\mathfrak{g}$ is general. Note that since $X_{V_k(\mathfrak{g})}$ is G-invariant and closed, the condition $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ is equivalent to that $X_{V_k(\mathfrak{g})} \cap \mathfrak{h}^* = \{0\}$. Now the complex structure of $C(\mathfrak{m}_i[t, t^{-1}], V_k(\mathfrak{g}))$ induces the complex structure on Zhu's C_2-algebra $R_{C(\mathfrak{m}_i[t,t^{-1}], V_k(\mathfrak{g}))}$. The embedding in Theorem 34 induces a homomorphism

$$R_{V_{k_i}(\mathfrak{sl}_2)} \to H^0(R_{C(\mathfrak{m}_i[t,t^{-1}], V_k(\mathfrak{g}))}, d)$$

of Poisson algebra. Since Ω is nilpotent in $R_{V_{k_i}(\mathfrak{sl}_2)}$, so is its image $\Omega_i = e_i f_i + f_i e_i + \frac{1}{2}h_i^2$ in $H^0(R_{C(\mathfrak{m}_i[t,t^{-1}], V_k(\mathfrak{g}))}, d)$. It follows that $h_i^N \equiv 0 \pmod{\mathfrak{n}_+ R_{V_k(\mathfrak{g})} + \mathfrak{n}_- R_{V_k(\mathfrak{g})}}$ in $R_{V_k(\mathfrak{g})}$ for all $i = 1, \dots, n-q$, and we get that $X_{V_k(\mathfrak{g})} \cap \mathfrak{h}^* = \{0\}$ as required.

Proof (Outline of Proof of Theorem 32) The proof is done by determining the variety $X_{V_k(\mathfrak{g})}$. By Theorem 31, $X_{V_k(\mathfrak{g})}$ is a finite union of nilpotent orbits. Thus it is enough to know which nilpotent element orbits is contained in $X_{V_k(\mathfrak{g})}$. On the other hand, (62) says $X_{V_k(\mathfrak{g})} \supset \overline{G.f}$ if and only $H_f^0(V_k(\mathfrak{g})) \neq 0$. Thus, it is sufficient to compute the character of $H_f^0(V_k(\mathfrak{g}))$. This is in fact possible since we know the explicit formula (63) of the character of $V_k(\mathfrak{g})$, thanks to the vanishing theorem (61) and the Euler-Poincaré principle.

Proof (Outline of Proof of Theorem 33) Let $L(\lambda)$ be a $V_k(\mathfrak{g})$-module. Then, the space $H^{\frac{\infty}{2}+i}(\mathfrak{m}_i[t, t^{-1}], L(\lambda))$, $i \in \mathbb{Z}$, is naturally a $H^{\frac{\infty}{2}+i}(\mathfrak{m}_i[t, t^{-1}], V_k(\mathfrak{g}))$-module. By Theorem 34, this means that $H^{\frac{\infty}{2}+i}(\mathfrak{m}_i[t, t^{-1}], L(\lambda))$ is in particular a module over the admissible affine vertex algebra $V_{k_i}(\mathfrak{sl}_2)$. Therefore Theorem 33 for $\mathfrak{g} = \mathfrak{sl}_2$ that was established by Adamović and Milas [2] implies that $H^{\frac{\infty}{2}+i}(\mathfrak{m}_i[t, t^{-1}], L(\lambda))$ must be a direct sum of admissible representations of $\widehat{\mathfrak{sl}}_2$. This information is sufficient to conclude that $L(\lambda)$ is admissible.

Conversely, suppose that $L(\lambda)$ is an admissible representation of level k. If $L(\lambda)$ is integrable over $\mathfrak{g}$, then it has been already proved by Frenkel and Malikov [42] that $L(\lambda)$ is a $V_k(\mathfrak{g})$-module. But then an affine analogue of Duflo-Joseph Lemma [11, Lemma 2.6] implies that this is true for a general admissible representation as well.

8.4 Lisse Property of W-Algebras

An admissible number k is called *non-degenerate* if $X_{V_k(\mathfrak{g})} = \mathcal{N}$. By Theorem 32, this condition is equivalent to that

$$k + n = \frac{p}{q},\ p, q \in \mathbb{N},\ (p, q) = 1,\ p \geqslant n,\ q \geqslant n.$$

The following assertion follows immediately from Corollary 7.

Theorem 35 ([9]) *Let k be a non-degenerate admissible number. Then the W-algebra $\mathscr{W}_k(\mathfrak{g})$ is lisse.*

8.5 Minimal Models of W-Algebras

A vertex algebra V is called *rational* if any V-module is completely reducible. To a lisse and rational conformal vertex algebra V one can associate *rational 2d conformal field theory*, and in particular, the category V-Mod of V-modules forms [51] a *modular tensor category* [19], as in the case of the category of integrable representation of $\widehat{\mathfrak{g}}$ at a positive level and the category of *minimal series representations* [23] of the Virasoro algebra.

An admissible weight λ is called *non-degenerate* if $\bar{\lambda}$ is anti-dominant. Let $Pr_k^{non\text{-}deg}$ be the set of non-degenerate admissible weights of level k of $\widehat{\mathfrak{g}}$. It is known [44] that $Pr_k^{non\text{-}deg}$ is non-empty if and only if k is non-degenerate.

By Theorem 29, for $\lambda \in Pr^k$, $H^0_-(L(\lambda))$ is a (non-zero) simple $\mathscr{W}^k(\mathfrak{g})$-module if and only of $\lambda \in Pr_k^{non\text{-}deg}$, and $H^0_-(L(\lambda)) \cong H^0_-(L(\mu))$ if and only if $\mu \in W \circ \lambda$ for $\lambda, \mu \in Pr_k^{non\text{-}deg}$.

Let $[Pr_k^{non\text{-}deg}] = Pr_k^{non\text{-}deg}/\sim$, where $\lambda \sim \mu \iff \mu \in W \circ \lambda$. It is known [44] that we have a bijection

$$(\widehat{P}_+^{p-n} \times \widehat{P}_+^{q-n})/\mathbb{Z}_n \xrightarrow{\sim} [Pr_k^{non\text{-}deg}], \quad [(\lambda, \mu)] \mapsto [\bar{\lambda} - \frac{p}{q}(\bar{\mu} + \rho) + k\Lambda_0].$$

Here $k + n = p/q$ as before, $\widehat{P}_+^k$ is the set of integral dominant weights of level k of $\widehat{\mathfrak{g}}$, the cyclic group $\mathbb{Z}_n$ acts diagonally on $\widehat{P}_+^{p-n} \times \widehat{P}_+^{q-n}$ as the Dynkin automorphism, and $\rho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$.

The following assertion was conjectured by Frenkel et al. [44].

Theorem 36 ([10]) *Let k be a non-degenerate admissible number. Then the simple W-algebra $\mathscr{W}_k(\mathfrak{g})$ is rational, and $\{\mathbb{L}(\gamma_{\bar{\lambda}}) = H^0_-(L(\lambda)) \mid \lambda \in [Pr_k^{non\text{-}deg}]\}$ forms the complete set of isomorphism classes of simple $\mathscr{W}_k(\mathfrak{g})$-modules.*

In the case that $\mathfrak{g} = \mathfrak{sl}_2$, Theorems 35 and 36 have been proved in [22, 77], and the above representations are exactly the minimal series representations of the Virasoro algebra.

The representations

$$\{\mathbb{L}(\gamma_{\bar{\lambda}}) \mid \lambda \in [Pr_k^{non\text{-}deg}]\}$$

are called the *minimal series representations* of $\mathscr{W}^k(\mathfrak{g})$, and if $k+n = p/q, p, q \in \mathbb{N}$, $(p, q) = 1, p, q, \geqslant n$, then the rational W-algebra $\mathscr{W}_k(\mathfrak{g})$ is called the (p, q)-*minimal model* of $\mathscr{W}^k(\mathfrak{g})$. Note that the (p, q)-minimal model and the (q, p)-minimal model are isomorphic due to the duality, see Corollary 5.

Proof (Outline of the Proof of Theorem 36) Let k be a non-degenerate admissible number. We have

$$H^0(V_k(\mathfrak{g})) \cong \mathscr{W}_k(\mathfrak{g})$$

by [5]. Hence by Theorem 28 (3)

$$\text{Zhu}(\mathscr{W}_k(\mathfrak{g})) = \text{Zhu}(H^0(V_k(\mathfrak{g}))) = H^0(\text{Zhu}(V_k(\mathfrak{g})).$$

From this together with Theorem 33, it is not too difficult to obtain the classification is the simple $\mathscr{W}_k(\mathfrak{g})$-modules as stated in Theorem 36. One sees that the extensions between simple modules are trivial using the linkage principle that follows from Theorem 29.

Remark 16

1. We have $\mathscr{W}_k(\mathfrak{g}) = \mathbb{L}(\gamma_{-(k+n)\rho})$ for a non-degenerate admissible number k. (Note that $k\Lambda_0 \notin Pr_k^{non\text{-}deg}$.)
2. Let $\lambda \in Pr_k$. From Corollary 8 and (63), we get

$$\text{ch}\,\mathbb{L}(\gamma_{\bar{\lambda}}) = \sum_{w \in \widehat{W}(\lambda)} \epsilon(w)\,\text{ch}\,\mathbb{M}(\gamma_{\overline{w \circ \lambda}}). \tag{65}$$

   This was conjectured by [44].
3. When it is trivial (that is, equals to $\mathbb{C}$), $\mathscr{W}_k(\mathfrak{g})$ is obviously lisse and rational. This happens if and only if $\mathscr{W}_k(\mathfrak{g})$ is the $(n, n+1)$-minimal model (=the $(n+1, n)$-minimal model). In this case the character formula (65) for $\mathscr{W}_k(\mathfrak{g}) = \mathbb{L}(\gamma_{\bar{\lambda}})$, $\lambda = -(k+n)\rho + k\Lambda_0$, gives the following *denominator formula*:

$$\sum_{w \in \widehat{W}(\lambda)} \epsilon(w) q^{\frac{(\overline{w\circ\lambda}, \overline{w\circ\lambda}+2\rho)}{2(k+n)}} = \prod_{j=1}^{n-1}(1-q^j)^{n-1}.$$

In the case that $\mathfrak{g} = \mathfrak{sl}_2$, we get the denominator formula for the Virasoro algebra, which is identical to *Euler's pentagonal identity*.

4. As a generalization of the GKO construction [48] it has been conjectured [58] that the (p, q)-minimal model of $\mathscr{W}^k(\mathfrak{g})$, with $p > q$, is isomorphic to the commutant of $V_{l+1}(\mathfrak{g})$ inside $V_l(\mathfrak{g}) \otimes V_1(\mathfrak{g})$, where $l + n = q/(p-q)$. (Note that $V_l(\mathfrak{g})$ and $V_{l+1}(\mathfrak{g})$ are admissible.) This conjecture has been proved in [18] for the special case that $(p, q) = (n+1, n)$.

   A similar conjecture exists in the case that $\mathfrak{g}$ is simply laced.
5. The existence of rational and lisse W-algebras has been conjectured for general W-algebras $\mathscr{W}^k(\mathfrak{g}, f)$ by Kac and Wakimoto [59]. This has been proved in [12] in part including all the cases in type A. See [15, 63] for a recent development in the classification problem of rational and lisse W-algebras.

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Part II
Contributed Papers

Representations of the Framisation of the Temperley–Lieb Algebra

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Abstract In this note, we describe the irreducible representations and give a dimension formula for the Framisation of the Temperley–Lieb algebra.

Keywords Framisation of the temperley-Lieb algebra • Representation theory • Temperley-Lieb algebra • Yokonuma-Hecke algebra

1 Introduction

The Temperley–Lieb algebra was introduced by Temperley and Lieb in [18] for its applications in statistical mechanics. It was later shown by Jones [8, 9] that it can be obtained as a quotient of the Iwahori–Hecke algebra of type A. Both algebras depend on a parameter q. Jones showed that there exists a unique Markov trace, called the Ocneanu trace, on the Iwahori–Hecke algebra, which depends on a parameter z. For a specific value of z, the Ocneanu trace passes to the Temperley–Lieb algebra. Jones used the Ocneanu trace on the Temperley–Lieb algebra to define a polynomial knot invariant, the famous Jones polynomial. Using the Ocneanu trace as defined originally on the Iwahori–Hecke algebra of type A yields another famous polynomial invariant, the HOMFLYPT polynomial, which is also known as the 2-variable Jones polynomial (the two variables being q and z).

Yokonuma–Hecke algebras were introduced by Yokonuma in [22] as generalisations of Iwahori–Hecke algebras in the context of finite Chevalley groups. The Yokonuma–Hecke algebra of type A is the centraliser algebra associated to the permutation representation of the general linear group over a finite field with respect to a maximal unipotent subgroup. Juyumaya has given a generic presentation for

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this algebra, depending on a parameter q, and defined a Markov trace on it, the latter depending on several parameters [10–12]. This trace was subsequently used by Juyumaya and Lambropoulou for the construction of invariants for framed knots and links [13, 16]. They later showed that these invariants can be also used for classical and singular knots and links [14, 15]. The next step was to construct an analogue of the Temperley–Lieb algebra in this case.

As it is explained in more detail in [17], where the technique of framisation is thoroughly discussed, three possible candidates arose. The first candidate was the Yokonuma–Temperley–Lieb algebra, which was defined in [5] as the quotient of the Yokonuma–Hecke algebra by exactly the same ideal as the one used by Jones in the classical case. We studied the representation theory of this algebra and constructed a basis for it in [3]. The values of the parameters for which Juyumaya's Markov trace passes to the Yokonuma–Temperley–Lieb algebra are given in [5]. Unfortunately, for these values, the invariants for classical knots and links obtained from the Yokonuma–Temperley–Lieb algebra are equivalent to the Jones polynomial.

A second candidate, which is more interesting topologically, was suggested in [6]. This is the Framisation of the Temperley–Lieb algebra, whose representation theory we study in this paper. The Framisation of the Temperley–Lieb algebra is defined in a subtler way than the Yokonuma–Temperley–Lieb algebra, as the quotient of the Yokonuma–Hecke algebra by a more elaborate ideal, and it is larger than the Yokonuma–Temperley–Lieb algebra. The values of the parameters for which Juyumaya's Markov trace passes to this quotient are given in [6]. It was recently shown that the Juyumaya–Lambropoulou invariants for classical links are stronger than the HOMFLYPT polynomial [1]. It turns out that, in a similar way, the invariants for classical links obtained from the Framisation of the Temperley–Lieb algebra are stronger than the Jones polynomial.

The third candidate is the so-called Complex Temperley–Lieb algebra, which is larger than the Framisation of the Temperley–Lieb algebra, but provides the same topological information (see [17]).

In this note, we study the representation theory of the Framisation of the Temperley–Lieb algebra. In Proposition 5 we give a complete description of its irreducible representations, by showing which irreducible representations of the Yokonuma–Hecke algebra pass to the quotient. The representations of the Yokonuma–Hecke algebra of type A were first studied by Thiem [19–21], but here we use their explicit description given later in [4]. Our result generalises in a natural way the analogous result in the classical case. We then use the dimensions of the irreducible representations of the Framisation of the Temperley–Lieb algebra in order to compute the dimension of the algebra. We deduce a combinatorial formula involving Catalan numbers, given in Proposition 6.

A reference to the results of this note is included in [6], so we decided to finally provide them in written form. We also take this opportunity to write down the relations between three types of generators used in bibliography so far (Remark 2), and show that the Yokonuma–Hecke algebra is split semisimple over a smaller field than the one considered in [4] (Corollary 2).

2 The Temperley–Lieb Algebra

In this section, we recall the definition of the Temperley–Lieb algebra as a quotient of the Iwahori–Hecke algebra of type A given by Jones [9], and some classical results on its representation theory.

2.1 The Iwahori–Hecke Algebra $\mathcal{H}_n(q)$

Let $n \in \mathbb{N}$ and let q be an indeterminate. The *Iwahori–Hecke algebra of type A*, denoted by $\mathcal{H}_n(q)$, is a $\mathbb{C}[q, q^{-1}]$-associative algebra generated by the elements

$$G_1, \ldots, G_{n-1}$$

subject to the following relations:

$$\begin{aligned} G_iG_j &= G_jG_i & &\text{for all } i,j = 1, \ldots, n-1 \text{ with } |i-j| > 1, \\ G_iG_{i+1}G_i &= G_{i+1}G_iG_{i+1} & &\text{for all } i = 1, \ldots, n-2, \end{aligned} \tag{1}$$

together with the quadratic relations:

$$G_i^2 = q + (q-1)G_i \qquad \text{for all } i = 1, \ldots, n-1. \tag{2}$$

Remark 1 If we specialise q to 1, the defining relations (1)–(2) become the defining relations for the symmetric group $\mathfrak{S}_n$. Thus the algebra $\mathcal{H}_n(q)$ is a deformation of the group algebra over $\mathbb{C}$ of $\mathfrak{S}_n$.

2.2 The Temperley–Lieb Algebra $\mathrm{TL}_n(q)$

Let $i = 1, \ldots, n-1$. We set

$$G_{i,i+1} := 1 + G_i + G_{i+1} + G_iG_{i+1} + G_{i+1}G_i + G_iG_{i+1}G_i.$$

We define the *Temperley–Lieb algebra* $\mathrm{TL}_n(q)$ to be the quotient $\mathcal{H}_n(q)/I$, where I is the ideal generated by the element $G_{1,2}$. We have $G_{i,i+1} \in I$ for all $i = 1, \ldots, n-2$, since

$$G_{i,i+1} = (G_1G_2 \cdots G_{n-1})^{i-1} G_{1,2} (G_1G_2 \cdots G_{n-1})^{-(i-1)}.$$

2.3 Combinatorics of Partitions

Let $\lambda \vdash n$ be a partition of n, that is, $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a string of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ and $|\lambda| := \lambda_1 + \cdots + \lambda_k = n$. We shall also say that λ is a partition *of size n*.

We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of k rows such that the jth row contains λ_j *nodes* for all $j = 1, \ldots, k$. We write $\theta = (x, y)$ for the node in row x and column y.

For a node θ lying in the line x and the column y of λ (that is, $\theta = (x, y)$), we define $\mathrm{c}(\theta) := q^{y-x}$. The number $\mathrm{c}(\theta)$ is called the *(quantum) content* of θ.

Now, a *tableau of shape* λ is a bijection between the set $\{1, \ldots, n\}$ and the set of nodes in λ. In other words, a tableau of shape λ is obtained by placing the numbers $1, \ldots, n$ in the nodes of λ. The *size* of a tableau of shape λ is n, that is, the size of λ. A tableau is *standard* if its entries increase along any row and down any column of the diagram of λ.

For a tableau $\mathcal{T}$, we denote by $\mathrm{c}(\mathcal{T}|i)$ the quantum content of the node with the number i in it. For example, for the standard tableau $\mathcal{T} = \boxed{1}\boxed{2}\boxed{3}$ of size 3, we have

$$\mathrm{c}(\mathcal{T}|1) = 1\,, \quad \mathrm{c}(\mathcal{T}|2) = q \quad \text{and} \quad \mathrm{c}(\mathcal{T}|3) = q^2\,.$$

For any tableau $\mathcal{T}$ of size n and any permutation $\sigma \in \mathfrak{S}_n$, we denote by $\mathcal{T}^\sigma$ the tableau obtained from $\mathcal{T}$ by applying the permutation σ on the numbers contained in the nodes of $\mathcal{T}$. We have

$$\mathrm{c}(\mathcal{T}^\sigma|i) = \mathrm{c}\big(\mathcal{T}|\sigma^{-1}(i)\big) \qquad \text{for all } i = 1, \ldots, n.$$

Note that if the tableau $\mathcal{T}$ is standard, the tableau $\mathcal{T}^\sigma$ is not necessarily standard.

2.4 Formulas for the Irreducible Representations of $\mathbb{C}(q)\mathcal{H}_n(q)$

We set $\mathbb{C}(q)\mathcal{H}_n(q) := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} \mathcal{H}_n(q)$. Let $\mathcal{P}(n)$ be the set of all partitions of n, and let $\lambda \in \mathcal{P}(n)$. Let V_λ be a $\mathbb{C}(q)$-vector space with a basis $\{\mathrm{v}_\mathcal{T}\}$ indexed by the standard tableaux of shape λ. We set $\mathrm{v}_\mathcal{T} := 0$ for any non-standard tableau $\mathcal{T}$ of shape λ. We have the following result on the representations of $\mathbb{C}(q)\mathcal{H}_n(q)$, established in [7]:

Proposition 1 *Let $\mathcal{T}$ be a standard tableau of shape $\lambda \in \mathcal{P}(n)$. For brevity, we set $\mathrm{c}_i := \mathrm{c}(\mathcal{T}|i)$ for $i = 1, \ldots, n$. The vector space V_λ is an irreducible representation* *of* $\mathbb{C}(q)\mathcal{H}_n(q)$ *with the action of the generators on the basis element* $\mathrm{v}_{\mathcal{T}}$ *defined as follows: for* $i = 1, \dots, n-1$,

$$G_i(\mathrm{v}_{\mathcal{T}}) = \frac{q\mathrm{c}_{i+1} - \mathrm{c}_{i+1}}{\mathrm{c}_{i+1} - \mathrm{c}_i}\,\mathrm{v}_{\mathcal{T}} + \frac{q\mathrm{c}_{i+1} - \mathrm{c}_i}{\mathrm{c}_{i+1} - \mathrm{c}_i}\,\mathrm{v}_{\mathcal{T}^{s_i}}\ , \tag{3}$$

where s_i *is the transposition* $(i, i+1)$. *Further, the set* $\{V_\lambda\}_{\lambda \in \mathcal{P}(n)}$ *is a complete set of pairwise non-isomorphic irreducible representations of* $\mathbb{C}(q)\mathcal{H}_n(q)$.

Corollary 1 *The algebra* $\mathbb{C}(q)\mathcal{H}_n(q)$ *is split semisimple.*

2.5 Irreducible Representations of $\mathbb{C}(q)\mathrm{TL}_n(q)$

Since the algebra $\mathbb{C}(q)\mathcal{H}_n(q)$ is semisimple, the algebra $\mathbb{C}(q)\mathrm{TL}_n(q) := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} \mathrm{TL}_n(q)$ is also semisimple. Moreover, the irreducible representations of $\mathbb{C}(q)\mathrm{TL}_n(q)$ are precisely the irreducible representations of $\mathbb{C}(q)\mathcal{H}_n(q)$ that pass to the quotient. That is, V_λ is an irreducible representation of $\mathbb{C}(q)\mathrm{TL}_n(q)$ if and only if $G_{1,2}(\mathrm{v}_{\mathcal{T}}) = 0$ for every standard tableau $\mathcal{T}$ of shape λ. It is easy to check that the latter is equivalent to the trivial representation not being a direct summand of the restriction $\mathrm{Res}^{\mathfrak{S}_n}_{\langle s_1, s_2\rangle}(E^\lambda)$, where E^λ is the irreducible representation of the symmetric group $\mathfrak{S}_n$ (equivalently, the algebra $\mathbb{C}\mathcal{H}_n(1)$) labelled by λ. We thus obtain the following description of the irreducible representations of $\mathbb{C}(q)\mathrm{TL}_n(q)$:

Proposition 2 *We have that* V_λ *is an irreducible representation of* $\mathbb{C}(q)\mathrm{TL}_n(q)$ *if and only if the Young diagram of* λ *has at most two columns.*

2.6 The Dimension of $\mathbb{C}(q)\mathrm{TL}_n(q)$

For $n \in \mathbb{N}$, we denote by C_n the nth Catalan number, that is, the number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}^2.$$

We have the following standard result on the dimension of $\mathbb{C}(q)\mathrm{TL}_n(q)$ (cf. [8, 9]):

Proposition 3 *We have*

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{TL}_n(q)) = C_n.$$

3 The Framisation of the Temperley–Lieb Algebra

In this section, we will look at a generalisation of the Temperley–Lieb algebra, which is obtained as a quotient of the Yokonuma–Hecke algebra of type A. This algebra was introduced in [6], where some of its topological properties were studied. Here we will determine its irreducible representations and calculate its dimension.

3.1 The Yokonuma–Hecke Algebra $\mathrm{Y}_{d,n}(q)$

Let d, $n \in \mathbb{N}$. Let q be an indeterminate. The *Yokonuma–Hecke algebra of type A*, denoted by $\mathrm{Y}_{d,n}(q)$, is a $\mathbb{C}[q, q^{-1}]$-associative algebra generated by the elements

$$g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$\begin{array}{llll} (\mathrm{b}_1) & g_i g_j = g_j g_i & \text{for all } i, j = 1, \ldots, n-1 \text{ with } |i-j| > 1, \\ (\mathrm{b}_2) & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & \text{for all } i = 1, \ldots, n-2, \\ (\mathrm{f}_1) & t_i t_j = t_j t_i & \text{for all } i, j = 1, \ldots, n, \\ (\mathrm{f}_2) & t_j g_i = g_i t_{s_i(j)} & \text{for all } i = 1, \ldots, n-1 \text{ and } j = 1, \ldots, n, \\ (\mathrm{f}_3) & t_j^d = 1 & \text{for all } j = 1, \ldots, n, \end{array} \tag{4}$$

where s_i is the transposition $(i, i+1)$, together with the quadratic relations:

$$g_i^2 = q + (q-1)\, e_i\, g_i \qquad \text{for all } i = 1, \ldots, n-1, \tag{5}$$

where

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}. \tag{6}$$

Note that we have $e_i^2 = e_i$ and $e_i g_i = g_i e_i$ for all $i = 1, \ldots, n-1$. Moreover, we have

$$t_i e_i = t_{i+1} e_i \qquad \text{for all } i = 1, \ldots, n-1. \tag{7}$$

Remark 2 In [4], the first author and Poulain d'Andecy consider the braid generators $\tilde{g}_i := q^{-1/2} g_i$ which satisfy the quadratic relation

$$\tilde{g}_i^2 = 1 + (q^{1/2} - q^{-1/2}) e_i \tilde{g}_i\,. \tag{8}$$

On the other hand, in all the papers [2, 12, 15, 16] prior to [4], the authors consider the braid generators $\bar{g}_i = \tilde{g}_i + (q^{1/2} - 1)\, e_i\tilde{g}_i$ (and thus, $\tilde{g}_i := \bar{g}_i + (q^{-1/2} - 1)\, e_i\bar{g}_i$) which satisfy the quadratic relation

$$\bar{g}_i^2 = 1 + (q-1)\, e_i + (q-1)\, e_i\, \bar{g}_i\ . \tag{9}$$

Note that

$$e_i g_i = e_i \bar{g}_i = q^{1/2} e_i \tilde{g}_i \quad \text{for all } i = 1, \ldots, n-1. \tag{10}$$

Remark 3 If we specialise q to 1, the defining relations (4)–(5) become the defining relations for the complex reflection group $G(d, 1, n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$. Thus the algebra $\mathrm{Y}_{d,n}(q)$ is a deformation of the group algebra over $\mathbb{C}$ of the complex reflection group $G(d, 1, n)$. Moreover, for $d = 1$, the Yokonuma–Hecke algebra $\mathrm{Y}_{1,n}(q)$ coincides with the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ of type A.

Remark 4 The relations (b_1), (b_2), (f_1) and (f_2) are defining relations for the classical framed braid group $\mathcal{F}_n \cong \mathbb{Z} \wr B_n$, where B_n is the classical braid group on n strands, with the t_j's being interpreted as the "elementary framings" (framing 1 on the jth strand). The relations $t_j^d = 1$ mean that the framing of each braid strand is regarded modulo d. Thus, the algebra $\mathrm{Y}_{d,n}(q)$ arises naturally as a quotient of the framed braid group algebra over the modular relations (f_3) and the quadratic relations (5). Moreover, relations (4) are defining relations for the modular framed braid group $\mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z}) \wr B_n$, so the algebra $\mathrm{Y}_{d,n}(q)$ can be also seen as a quotient of the modular framed braid group algebra over the quadratic relations (5).

3.2 *The Framisation of the Temperley–Lieb Algebra* $\mathrm{FTL}_{d,n}(q)$

Let $i = 1, \ldots, n-1$. We set

$$g_{i,i+1} := 1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i.$$

We define the *Framisation of the Temperley–Lieb algebra* to be the quotient $\mathrm{Y}_{d,n}(q)/I$, where I is the ideal generated by the element

$$e_1 e_2\, g_{1,2}.$$

Note that, due to (7), $e_1 e_2$ commutes with g_1 and with g_2, so it commutes with $g_{1,2}$. Further, we have $e_i e_{i+1} g_{i,i+1} \in I$ for all $i = 1, \ldots, n-2$, since

$$e_i e_{i+1} g_{i,i+1} = (g_1 g_2 \ldots g_{n-1})^{i-1} e_1 e_2\, g_{1,2} (g_1 g_2 \ldots g_{n-1})^{-(i-1)}.$$

Remark 5 In [6], the Framisation of the Temperley–Lieb algebra is defined to be the quotient $\mathrm{Y}_{d,n}(q)/J$, where J is the ideal generated by the element $e_1 e_2\, \bar{g}_{1,2}$, where

$$\bar{g}_{1,2} = 1 + \bar{g}_1 + \bar{g}_2 + \bar{g}_1\bar{g}_2 + \bar{g}_2\bar{g}_1 + \bar{g}_1\bar{g}_2\bar{g}_1.$$

Due to (10) and the fact that the e_i's are idempotents, we have $e_1 e_2\, \bar{g}_{1,2} = e_1 e_2\, g_{1,2}$, and so $I = J$.

Remark 6 For $d = 1$, the Framisation of the Temperley–Lieb algebra $\mathrm{FTL}_{1,n}(q)$ coincides with the classical Temperley–Lieb algebra $\mathrm{TL}_n(q)$.

3.3 Combinatorics of d-Partitions

A d-partition $\boldsymbol{\lambda}$, or a Young d-diagram, of size n is a d-tuple of partitions such that the total number of nodes in the associated Young diagrams is equal to n. That is, we have $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(d)})$ with $\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(d)}$ usual partitions such that $|\boldsymbol{\lambda}^{(1)}| + \cdots + |\boldsymbol{\lambda}^{(d)}| = n$.

A pair $\boldsymbol{\theta} = (\theta, k)$ consisting of a node θ and an integer $k \in \{1, \dots, d\}$ is called a *d-node*. The integer k is called the *position* of $\boldsymbol{\theta}$. A d-partition is then a set of d-nodes such that the subset consisting of the d-nodes having position k forms a usual partition, for any $k \in \{1, \dots, d\}$.

For a d-node $\boldsymbol{\theta}$ lying in the line x and the column y of the kth diagram of $\boldsymbol{\lambda}$ (that is, $\boldsymbol{\theta} = (x, y, k)$), we define $\mathrm{p}(\boldsymbol{\theta}) := k$ and $\mathrm{c}(\boldsymbol{\theta}) := q^{y-x}$. The number $\mathrm{p}(\boldsymbol{\theta})$ is the position of $\boldsymbol{\theta}$ and the number $\mathrm{c}(\boldsymbol{\theta})$ is called the *(quantum) content* of $\boldsymbol{\theta}$.

Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(d)})$ be a d-partition of n. A *d-tableau of shape* $\boldsymbol{\lambda}$ is a bijection between the set $\{1, \dots, n\}$ and the set of d-nodes in $\boldsymbol{\lambda}$. In other words, a d-tableau of shape $\boldsymbol{\lambda}$ is obtained by placing the numbers $1, \dots, n$ in the d-nodes of $\boldsymbol{\lambda}$. The *size* of a d-tableau of shape $\boldsymbol{\lambda}$ is n, that is, the size of $\boldsymbol{\lambda}$. A d-tableau is *standard* if its entries increase along any row and down any column of every diagram in $\boldsymbol{\lambda}$. For $d = 1$, a standard 1-tableau is a usual standard tableau.

For a d-tableau $\mathcal{T}$, we denote respectively by $\mathrm{p}(\mathcal{T}|i)$ and $\mathrm{c}(\mathcal{T}|i)$ the position and the quantum content of the d-node with the number i in it. For example, for the standard 3-tableau $\mathcal{T} = \left(\boxed{1}\boxed{3}\,,\ \varnothing\,,\ \boxed{2}\right)$ of size 3, we have

$$\mathrm{p}(\mathcal{T}|1) = 1\,,\ \ \mathrm{p}(\mathcal{T}|2) = 3\,,\ \ \mathrm{p}(\mathcal{T}|3) = 1 \quad \text{and} \quad \mathrm{c}(\mathcal{T}|1) = 1\,,\ \ \mathrm{c}(\mathcal{T}|2) = 1\,,\ \ \mathrm{c}(\mathcal{T}|3) = q\,.$$

For any d-tableau $\mathcal{T}$ of size n and any permutation $\sigma \in \mathfrak{S}_n$, we denote by $\mathcal{T}^\sigma$ the d-tableau obtained from $\mathcal{T}$ by applying the permutation σ on the numbers contained in the d-nodes of $\mathcal{T}$. We have

$$\mathrm{p}(\mathcal{T}^\sigma|i) = \mathrm{p}\big(\mathcal{T}|\sigma^{-1}(i)\big) \quad \text{and} \quad \mathrm{c}(\mathcal{T}^\sigma|i) = \mathrm{c}\big(\mathcal{T}|\sigma^{-1}(i)\big) \qquad \text{for all } i = 1, \dots, n.$$

Note that if the d-tableau $\mathcal{T}$ is standard, the d-tableau $\mathcal{T}^\sigma$ is not necessarily standard.

3.4 Formulas for the Irreducible Representations of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$

The representation theory of $\mathrm{Y}_{d,n}(q)$ has been first studied by Thiem in [19–21] and subsequently in [4], where a description of its irreducible representations in terms of d-partitions and d-tableaux is given.

Let $\mathcal{P}(d,n)$ be the set of all d-partitions of n, and let $\boldsymbol{\lambda} \in \mathcal{P}(d,n)$. Let $\tilde{V}_{\boldsymbol{\lambda}}$ be a $\mathbb{C}(q^{1/2})$-vector space with a basis $\{\tilde{\mathbf{v}}_{\mathcal{T}}\}$ indexed by the standard d-tableaux of shape $\boldsymbol{\lambda}$. In [4, Proposition 5], the authors describe actions of the generators $\tilde{g}_i$, for $i = 1,\dots,n-1$, and t_j, for $j = 1,\dots,n$, on $\{\tilde{\mathbf{v}}_{\mathcal{T}}\}$, which make $\tilde{V}_{\boldsymbol{\lambda}}$ into a representation of $\mathrm{Y}_{d,n}(q)$ over $\mathbb{C}(q^{1/2})$. The matrices describing the action of the generators t_j have complex coefficients, while the ones describing the action of the generators $\tilde{g}_i$ have coefficients in $\mathbb{C}(q^{1/2})$. However, the change of basis

$$\mathbf{v}_{\mathcal{T}} := q^{N_{\mathcal{T}}/2}\,\tilde{\mathbf{v}}_{\mathcal{T}}, \tag{11}$$

where $N_{\mathcal{T}} := \#\{i \in \{1,\dots,n-1\}\,|\,\mathrm{p}(\mathcal{T}|i) < \mathrm{p}(\mathcal{T}|i+1)\}$, and the change of generators

$$g_i = q^{1/2}\,\tilde{g}_i \tag{12}$$

yield a description of the action of $\mathrm{Y}_{d,n}(q)$ on $\tilde{V}_{\boldsymbol{\lambda}}$ which is realised over $\mathbb{C}(q)$ (see proposition below).

Let $V_{\boldsymbol{\lambda}}$ be a $\mathbb{C}(q)$-vector space with a basis $\{\mathbf{v}_{\mathcal{T}}\}$ indexed by the standard d-tableaux of shape $\boldsymbol{\lambda}$. We set $\mathbf{v}_{\mathcal{T}} := 0$ for any non-standard d-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. Let $\{\xi_1,\dots,\xi_d\}$ be the set of all dth roots of unity (ordered arbitrarily). We set $\mathbb{C}(q)\mathrm{Y}_{d,n}(q) := \mathbb{C}(q)\otimes_{\mathbb{C}[q,q^{-1}]}\mathrm{Y}_{d,n}(q)$. The following result is [4, Proposition 5] and [4, Theorem 1], with the change of basis and generators described by (11) and (12).

Proposition 4 *Let $\mathcal{T}$ be a standard d-tableau of shape $\boldsymbol{\lambda} \in \mathcal{P}(d,n)$. For brevity, we set $\mathrm{p}_i := \mathrm{p}(\mathcal{T}|i)$ and $\mathrm{c}_i := \mathrm{c}(\mathcal{T}|i)$ for $i = 1,\dots,n$. The vector space $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ with the action of the generators on the basis element $\mathbf{v}_{\mathcal{T}}$ defined as follows: for $j = 1,\dots,n$,*

$$t_j(\mathbf{v}_{\mathcal{T}}) = \xi_{\mathrm{p}_j}\mathbf{v}_{\mathcal{T}}\ ; \tag{13}$$

for $i = 1,\dots,n-1$, if $\mathrm{p}_i > \mathrm{p}_{i+1}$ then

$$g_i(\mathbf{v}_{\mathcal{T}}) = \mathbf{v}_{\mathcal{T}^{s_i}}\ , \tag{14}$$

if $\mathrm{p}_i < \mathrm{p}_{i+1}$ then

$$g_i(\mathbf{v}_{\mathcal{T}}) = q\,\mathbf{v}_{\mathcal{T}^{s_i}}\ , \tag{15}$$

and if $\mathrm{p}_i = \mathrm{p}_{i+1}$ *then*

$$g_i(\mathbf{v}_{\mathcal{T}}) = \frac{q\mathrm{c}_{i+1} - \mathrm{c}_{i+1}}{\mathrm{c}_{i+1} - \mathrm{c}_i}\,\mathbf{v}_{\mathcal{T}} + \frac{q\mathrm{c}_{i+1} - \mathrm{c}_i}{\mathrm{c}_{i+1} - \mathrm{c}_i}\,\mathbf{v}_{\mathcal{T}^{s_i}}\ , \tag{16}$$

where s_i *is the transposition* $(i, i+1)$*. Further, the set* $\{V_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in\mathcal{P}(d,n)}$ *is a complete set of pairwise non-isomorphic irreducible representations of* $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$*.*

Corollary 2 *The algebra* $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ *is split semisimple.*

Remark 7 Note that

$$e_i(\mathbf{v}_{\mathcal{T}}) = \begin{cases} 0 & \text{, if } \mathrm{p}_i \neq \mathrm{p}_{i+1}; \\ \mathbf{v}_{\mathcal{T}} & \text{, if } \mathrm{p}_i = \mathrm{p}_{i+1}. \end{cases} \tag{17}$$

3.5 Irreducible Representations of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$

Since the algebra $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ is semisimple, the algebra $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q) := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} \mathrm{FTL}_{d,n}(q)$ is also semisimple. Moreover, the irreducible representations of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ are precisely the irreducible representations of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ that pass to the quotient. That is, $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ if and only if $e_1e_2g_{1,2}(\mathbf{v}_{\mathcal{T}}) = 0$ for every standard d-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$.

Proposition 5 *We have that* $V_{\boldsymbol{\lambda}}$ *is an irreducible representation of* $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ *if and only if the Young diagram of* $\boldsymbol{\lambda}^{(i)}$ *has at most two columns for all* $i = 1, \ldots, d$*.*

Proof Let us assume first that $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ and let $i \in \{1, \ldots, d\}$. Set $n_i := |\boldsymbol{\lambda}^{(i)}|$. If $n_i \leqslant 2$, then $\boldsymbol{\lambda}^{(i)}$ has at most two columns. If $n_i \geqslant 3$, let us consider all the standard d-tableaux $\mathcal{T} = (\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(d)})$ of shape $\boldsymbol{\lambda}$ such that

$$\mathrm{p}_1 = \mathrm{p}_2 = \mathrm{p}_3 = \cdots = \mathrm{p}_{n_i} = i.$$

Then, using the notation of Proposition 1 for the Iwahori–Hecke algebra $\mathcal{H}_{n_i}(q)$ and Eq. (17), we obtain

$$G_{1,2}(\mathrm{v}_{\mathcal{T}^{(i)}}) = g_{1,2}(\mathbf{v}_{\mathcal{T}}) = g_{1,2}e_1e_2(\mathbf{v}_{\mathcal{T}}) = e_1e_2g_{1,2}(\mathbf{v}_{\mathcal{T}}) = 0$$

Since $\mathcal{T}^{(i)}$ runs over all the standard tableaux of shape $\boldsymbol{\lambda}^{(i)}$, Proposition 2 yields that $\boldsymbol{\lambda}^{(i)}$ has at most two columns.

Now assume that the Young diagram of $\boldsymbol{\lambda}^{(i)}$ has at most two columns for all $i = 1, \ldots, d$. Let $\mathcal{T} = (\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(d)})$ be a standard d-tableau of shape $\boldsymbol{\lambda}$. If $\mathrm{p}_1 = \mathrm{p}_2 = \mathrm{p}_3 =: \mathrm{p}$, then, by (17), $e_1e_2g_{1,2}(\mathbf{v}_{\mathcal{T}}) = g_{1,2}e_1e_2(\mathbf{v}_{\mathcal{T}}) = g_{1,2}(\mathbf{v}_{\mathcal{T}})$. In this case, $g_{1,2}$ acts on $\mathbf{v}_{\mathcal{T}}$ in the same way that $G_{1,2}$ acts on $\mathrm{v}_{\mathcal{T}^{(\mathrm{p})}}$ (replacing the entries greater than 3 by entries in $\{4, \dots, |\boldsymbol{\lambda}^{(p)}|\}$). Following the result on the classical case, we have $g_{1,2}(\mathbf{v}_{\mathcal{T}}) = 0$. Otherwise, again by (17), we have $e_1e_2(\mathbf{v}_{\mathcal{T}}) = 0$, and $e_1e_2g_{1,2}(\mathbf{v}_{\mathcal{T}}) = g_{1,2}e_1e_2(\mathbf{v}_{\mathcal{T}}) = 0$ as desired.

3.6 The Dimension of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$

We will now use the complete description of the irreducible representations of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ by Proposition 5 to obtain a dimension formula for $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$. Set

$$\mathcal{K}_{d,n} := \{(k_1, k_2, \dots, k_d) \in \mathbb{N}^d \mid k_1 + k_2 + \cdots + k_d = n\}.$$

Proposition 6 *We have*

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)) = \sum_{(k_1,k_2,\dots,k_d)\in\mathcal{K}_{d,n}} \left(\frac{n!}{k_1!k_2!\dots k_d!}\right)^2 C_{k_1}C_{k_2}\cdots C_{k_d}.$$

Proof Let us denote by $\mathcal{P}^{\leqslant 2}(d,n)$ the set of d-partitions $\boldsymbol{\lambda}$ of n such that the Young diagram of $\boldsymbol{\lambda}^{(i)}$ has at most two columns for all $i = 1, \dots, d$. By Proposition 5, and since the algebra $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ is semisimple, we have

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)) = \sum_{\boldsymbol{\lambda}\in\mathcal{P}^{\leqslant 2}(d,n)} \dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}})^2,$$

where $\dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}})$ is the number of standard d-tableaux of shape $\boldsymbol{\lambda}$.

Fix $(k_1, k_2, \dots, k_d) \in \mathcal{K}_{d,n}$. We denote by $\mathcal{P}^{\leqslant 2}(k_1, k_2, \dots, k_d)$ the set of all d-partitions $\boldsymbol{\lambda}$ in $\mathcal{P}^{\leqslant 2}(d,n)$ such that $|\boldsymbol{\lambda}^{(i)}| = k_i$ for all $i = 1, \dots, d$. We have

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)) = \sum_{(k_1,k_2,\dots,k_d)\in\mathcal{K}_{d,n}} \sum_{\boldsymbol{\lambda}\in\mathcal{P}^{\leqslant 2}(k_1,k_2,\dots,k_d)} \dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}})^2.$$

Let $\boldsymbol{\lambda} \in \mathcal{P}^{\leqslant 2}(k_1, k_2, \dots, k_d)$. We have

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{d-1}}{k_d} = \frac{n!}{k_1!k_2!\dots k_d!}$$

ways to choose the numbers in $\{1, \dots, n\}$ that will be placed in the nodes of the Young diagram of $\boldsymbol{\lambda}^{(i)}$ for each $i = 1, \dots, d$. We deduce that

$$\dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}}) = \frac{n!}{k_1!k_2!\dots k_d!}\prod_{i=1}^{d}\dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}^{(i)}}),$$

where $V_{\boldsymbol{\lambda}^{(i)}}$ is the irreducible representation of $\mathbb{C}(q)\mathrm{TL}_{k_i}(q)$ labelled by $\boldsymbol{\lambda}^{(i)}$. We thus obtain that $\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{FTL}_{d,n}(q))$ is equal to

$$\sum_{(k_1,k_2,\ldots,k_d)\in\mathcal{K}_{d,n}} \left(\frac{n!}{k_1!k_2!\ldots k_d!}\right)^2 \sum_{\boldsymbol{\lambda}\in\mathcal{P}^{\leqslant 2}(k_1,k_2,\ldots,k_d)} \prod_{i=1}^{d} \dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}^{(i)}})^2.$$

We now have that

$$\sum_{\boldsymbol{\lambda}\in\mathcal{P}^{\leqslant 2}(k_1,k_2,\ldots,k_d)} \prod_{i=1}^{d} \dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}^{(i)}})^2$$

is equal to

$$\sum_{\boldsymbol{\lambda}^{(1)}\in\mathcal{P}^{\leqslant 2}(1,k_1)} \sum_{\boldsymbol{\lambda}^{(2)}\in\mathcal{P}^{\leqslant 2}(1,k_2)} \cdots \sum_{\boldsymbol{\lambda}^{(d)}\in\mathcal{P}^{\leqslant 2}(1,k_d)} \prod_{i=1}^{d} \dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}^{(i)}})^2,$$

which in turn is equal to

$$\prod_{i=1}^{d} \left(\sum_{\boldsymbol{\lambda}^{(i)}\in\mathcal{P}^{\leqslant 2}(1,k_i)} \dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}^{(i)}})^2 \right).$$

By Proposition 3, we have that

$$\sum_{\boldsymbol{\lambda}^{(i)}\in\mathcal{P}^{\leqslant 2}(1,k_i)} \dim_{\mathbb{C}(q)}(V_{\boldsymbol{\lambda}^{(i)}})^2 = \dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{TL}_{k_i}(q)) = C_{k_i}\ ,$$

for all $i = 1, \ldots, d$. We conclude that

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)) = \sum_{(k_1,k_2,\ldots,k_d)\in\mathcal{K}_{d,n}} \left(\frac{n!}{k_1!k_2!\ldots k_d!}\right)^2 C_{k_1}C_{k_2}\cdots C_{k_d}.$$

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Some Semi-Direct Products with Free Algebras of Symmetric Invariants

Oksana Yakimova

Abstract Let $\mathfrak{g}$ be a complex reductive Lie algebra and V the underlying vector space of a finite-dimensional representation of $\mathfrak{g}$. Then one can consider a new Lie algebra $\mathfrak{q} = \mathfrak{g}\ltimes V$, which is a semi-direct product of $\mathfrak{g}$ and an Abelian ideal V. We outline several results on the algebra $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ of symmetric invariants of $\mathfrak{q}$ and describe all semi-direct products related to the defining representation of $\mathfrak{sl}_n$ with $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ being a free algebra.

Keywords Coadjoint representation • Non-reductive Lie algebras • Polynomial rings • Regular invariants

1 Introduction

Let Q be a connected complex algebraic group. Set $\mathfrak{q} = \operatorname{Lie} Q$. Then $\mathcal{S}(\mathfrak{q}) = \mathbb{C}[\mathfrak{q}^*]$ and $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}[\mathfrak{q}^*]^Q$. We will call the latter object the *algebra of symmetric invariants* of $\mathfrak{q}$. An important property of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is that it is isomorphic to $Z\mathbf{U}(\mathfrak{q})$ as an algebra by a classical result of M. Duflo (here $Z\mathbf{U}(\mathfrak{q})$ is the centre of the universal enveloping algebra of $\mathfrak{q}$).

Let $\mathfrak{g}$ be a reductive Lie algebra. Then by the Chevalley restriction theorem $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[H_1, \ldots, H_{\operatorname{rk}\mathfrak{g}}]$ is a polynomial ring (in $\operatorname{rk}\mathfrak{g}$ variables). A quest for non-reductive Lie algebras with a similar property has recently become a trend in invariant theory. Here we consider finite-dimensional representations $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ and the corresponding semi-direct products $\mathfrak{q} = \mathfrak{g}\ltimes V$. The Lie bracket on $\mathfrak{q}$ is defined by

$$[\xi + v, \eta + u] = [\xi, \eta] + \rho(\xi)u - \rho(\eta)v \tag{1}$$

for all $\xi, \eta \in \mathfrak{g}$, $v, u \in V$. Let G be a connected simply connected Lie group with $\operatorname{Lie} G = \mathfrak{g}$. Then $\mathfrak{q} = \operatorname{Lie} Q$ with $Q = G\ltimes \exp(V)$.

It is easy to see that $\mathbb{C}[V^*]^G \subset \mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ and therefore $\mathbb{C}[V^*]^G$ must be a polynomial ring if $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is, see [10, Section 3]. Classification of the representations

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of complex simple algebraic groups with free algebras of invariants was carried out by Schwarz [7] and independently by Adamovich and Golovina [1]. One such representation is the spin-representation of Spin_7, which leads to $Q = \mathrm{Spin}_7 \ltimes \mathbb{C}^8$. Here $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring in three variables generated by invariants of bi-degrees $(0,2)$, $(2,2)$, $(6,4)$ with respect to the decomposition $\mathfrak{q} = \mathfrak{so}_7 \oplus \mathbb{C}^8$, see [10, Proposition 3.10].

In this paper, we treat another example, $G = \mathrm{SL}_n$, $V = m(\mathbb{C}^n)^* \oplus k\mathbb{C}^n$ with $n \geqslant 2$, $m \geqslant 1$, $m \geqslant k$. Here $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring in exactly the following three cases:

- $k = 0$, $m \leqslant n+1$, and $n \equiv t \pmod{m}$ with $t \in \{-1, 0, 1\}$;
- $m = k$, $k \in \{n-2, n-1\}$;
- $n \geqslant m > k > 0$ and $m - k$ divides $n - m$.

We also briefly discuss semi-direct products arising as $\mathbb{Z}_2$-contractions of reductive Lie algebras.

2 Symmetric Invariants and Generic Stabilisers

Let $\mathfrak{q} = \mathrm{Lie}\, Q$ be an algebraic Lie algebra, Q a connected algebraic group. The index of $\mathfrak{q}$ is defined as

$$\mathrm{ind}\,\mathfrak{q} = \min_{\gamma \in \mathfrak{q}^*} \dim \mathfrak{q}_\gamma,$$

where $\mathfrak{q}_\gamma$ is the stabiliser of γ in $\mathfrak{q}$. In view of Rosenlicht's theorem, $\mathrm{ind}\,\mathfrak{q} = \mathrm{tr.deg}\, \mathbb{C}(\mathfrak{q}^*)^Q$. In case $\mathrm{ind}\,\mathfrak{q} = 0$, we have $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}$. For a reductive $\mathfrak{g}$, $\mathrm{ind}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{g}$. Recall that $(\dim \mathfrak{g} + \mathrm{rk}\,\mathfrak{g})/2$ is the dimension of a Borel subalgebra of $\mathfrak{g}$. For $\mathfrak{q}$, set $\mathbf{b}(\mathfrak{q}) := (\mathrm{ind}\,\mathfrak{q} + \dim \mathfrak{q})/2$.

Let $\{\xi_i\}$ be a basis of $\mathfrak{q}$ and $\mathcal{M}(\mathfrak{q}) = ([\xi_i, \xi_j])$ the structural matrix with entries in $\mathfrak{q}$. This is a skew-symmetric matrix of rank $\dim \mathfrak{q} - \mathrm{ind}\,\mathfrak{q}$. Let us take Pfaffians of the principal minors of $\mathcal{M}(\mathfrak{q})$ of size $\mathrm{rk}\,\mathcal{M}(\mathfrak{q})$ and let $\mathbf{p} = \mathbf{p}_{\mathfrak{q}}$ be their greatest common divisor. Then $\mathbf{p}$ is called the *fundamental semi-invariant* of $\mathfrak{q}$. The zero set of $\mathbf{p}$ is the maximal divisor in the so called *singular set*

$$\mathfrak{q}^*_{\mathrm{sing}} = \{\gamma \in \mathfrak{q}^* \mid \dim \mathfrak{q}_\gamma > \mathrm{ind}\,\mathfrak{q}\}$$

of $\mathfrak{q}$. Since $\mathfrak{q}^*_{\mathrm{sing}}$ is clearly a Q-stable subset, $\mathbf{p}$ is indeed a semi-invariant, $Q{\cdot}\mathbf{p} \subset \mathbb{C}\mathbf{p}$. One says that $\mathfrak{q}$ has the "codim-2" property (satisfies the "codim-2" condition), if $\dim \mathfrak{q}^*_{\mathrm{sing}} \leqslant \dim \mathfrak{q} - 2$ or equivalently if $\mathbf{p} = 1$.

Suppose that $F_1, \ldots, F_r \in \mathcal{S}(\mathfrak{q})$ are homogenous algebraically independent polynomials. The *Jacobian locus* $\mathcal{J}(F_1, \ldots, F_r)$ of these polynomials consists of all $\gamma \in \mathfrak{q}^*$ such that the differentials $d_\gamma F_1, \ldots, d_\gamma F_r$ are linearly dependent. In other words, $\gamma \in \mathcal{J}(F_1, \ldots, F_r)$ if and only if $(dF_1 \wedge \ldots \wedge dF_r)_\gamma = 0$. The set $\mathcal{J}(F_1, \ldots, F_r)$ is a proper Zariski closed subset of $\mathfrak{q}^*$. Suppose that $\mathcal{J}(F_1, \ldots, F_r)$ does not contain divisors. Then by the characteristic zero version of a result of

Skryabin, see [5, Theorem 1.1], $\mathbb{C}[F_1,\dots,F_r]$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{q})$, each $H \in \mathcal{S}(\mathfrak{q})$ that is algebraic over $\mathbb{C}(F_1,\dots,F_r)$ is contained in $\mathbb{C}[F_1,\dots,F_r]$.

Theorem 1 (cf. [3, Section 5.8]) *Suppose that* $\mathbf{p}_{\mathfrak{q}} = 1$ *and suppose that* $H_1,\dots,H_r \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ *are homogeneous algebraically independent polynomials such that* $r = \operatorname{ind}\mathfrak{q}$ *and* $\sum_{i=1}^{r}\deg H_i = \mathbf{b}(\mathfrak{q})$. *Then* $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}[H_1,\dots,H_r]$ *is a polynomial ring in r generators.*

Proof Under our assumptions $\mathcal{J}(H_1,\dots,H_r) = \mathfrak{q}^*_{\rm sing}$, see [5, Theorem 1.2] and [9, Section 2]. Therefore $\mathbb{C}[H_1,\dots,H_r]$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{q})$ by [5, Theorem 1.1]. Since tr.deg $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} \leqslant r$, each symmetric $\mathfrak{q}$-invariant is algebraic over $\mathbb{C}[H_1,\dots,H_r]$ and hence is contained in it. □

For semi-direct products, we have some specific approaches to the symmetric invariants. Suppose now that $\mathfrak{g} = \operatorname{Lie} G$ is a reductive Lie algebra, no non-zero ideal of $\mathfrak{g}$ acts on V trivially, G is connected, and $\mathfrak{q} = \mathfrak{g}\ltimes V$, where V is a finite-dimensional G-module.

The vector space decomposition $\mathfrak{q} = \mathfrak{g}\oplus V$ leads to $\mathfrak{q}^* = \mathfrak{g}\oplus V^*$, where we identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Each element $x \in V^*$ is considered as a point of $\mathfrak{q}^*$ that is zero on $\mathfrak{g}$. We have $\exp(V){\cdot}x = \operatorname{ad}^*(V){\cdot}x + x$, where each element of $\operatorname{ad}^*(V){\cdot}x$ is zero on V. Note that $\operatorname{ad}^*(V){\cdot}x \subset \operatorname{Ann}(\mathfrak{g}_x) \subset \mathfrak{g}$ and $\dim(\operatorname{ad}^*(V){\cdot}x)$ is equal to $\dim(\operatorname{ad}^*(\mathfrak{g}){\cdot}x) = \dim\mathfrak{g} - \dim\mathfrak{g}_x$. Therefore $\operatorname{ad}^*(V){\cdot}x = \operatorname{Ann}(\mathfrak{g}_x)$.

The decomposition $\mathfrak{q} = \mathfrak{g}\oplus V$ defines also a bi-grading on $\mathcal{S}(\mathfrak{q})$ and clearly $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a bi-homogeneous subalgebra, cf. [10, Lemma 2.12].

A statement is true for a "generic x" if and only if this statement is true for all points of a non-empty open subset.

Lemma 1 *A function* $F \in \mathbb{C}[\mathfrak{q}^*]$ *is a V-invariant if and only if* $F(\xi+\operatorname{ad}^*(V){\cdot}x, x) = F(\xi, x)$ *for generic* $x \in V^*$ *and any* $\xi \in \mathfrak{g}$.

Proof Condition of the lemma guaranties that for each $v \in V$, $\exp(v){\cdot}F = F$ on a non-empty open subset of $\mathfrak{q}^*$. Hence F is a V-invariant. □

For $x \in V^*$, let $\varphi_x\colon \mathbb{C}[\mathfrak{q}^*]^Q \to \mathbb{C}[\mathfrak{g}+x]^{G_x\ltimes\exp(V)}$ be the restriction map. By [10, Lemma 2.5] $\mathbb{C}[\mathfrak{g}+x]^{G_x\ltimes\exp(V)} \cong \mathcal{S}(\mathfrak{g}_x)^{G_x}$. Moreover, if we identify $\mathfrak{g}+x$ with $\mathfrak{g}$ choosing x as the origin, then $\varphi_x(F) \in \mathcal{S}(\mathfrak{g}_x)$ for any $\mathfrak{q}$-invariant F [10, Section 2]. Under certain assumptions on G and V the restriction map φ_x is surjective, more details will be given shortly.

There is a non-empty open subset $U \subset V^*$ such that the stabilisers G_x and G_y are conjugate in G for any pair of points $x, y \in U$ see e.g. [8, Theorem 7.2]. Any representative of the conjugacy class $\{hG_xh^{-1} \mid h \in G, x \in U\}$ is said to be a *a generic stabiliser* of the G-action on V^*.

There is one easy to handle case, $\mathfrak{g}_x = 0$ for a generic $x \in V^*$. Here $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G$, see e.g. [10, Example 3.1], and $\xi + y \in \mathfrak{q}^*_{\rm sing}$ only if $\mathfrak{g}_y \neq 0$, where $\xi \in \mathfrak{g}$, $y \in V^*$. The case $\operatorname{ind}\mathfrak{g}_x = 1$ is more involved.

Lemma 2 *Assume that G has no proper semi-invariants in* $\mathbb{C}[V^*]$. *Suppose that* $\operatorname{ind}\mathfrak{g}_x = 1$, $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x} \neq \mathbb{C}$, *and the map* φ_x *is surjective for generic* $x \in V^*$. *Then*

$\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}[V^*]^G[F]$, *where F is a bi-homogeneous preimage of a generator of $\mathcal{S}(\mathfrak{g}_x)^{G_x}$ that is not divisible by any non-constant G-invariant in $\mathbb{C}[V^*]$.*

Proof If we have a Lie algebra of index 1, in our case $\mathfrak{g}_x$, then the algebra of its symmetric invariants is a polynomial ring. There are many possible explanations of this fact. One of them is the following. Suppose that two non-zero homogeneous polynomials f_1, f_2 are algebraically dependent. Then $f_1^a = cf_2^b$ for some coprime integers $a, b > 0$ and some $c \in \mathbb{C}^{\times}$. If f_1 is an invariant, then so is a polynomial function $\sqrt[b]{f_1} = \sqrt[ab]{c}\sqrt[a]{f_2}$.

Since $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x} \neq \mathbb{C}$, it is generated by some homogeneous f. The group G_x has finitely many connected components, hence $\mathcal{S}(\mathfrak{g}_x)^{G_x}$ is generated by a suitable power of f, say $\mathbf{f} = f^d$.

Let $F \in \mathbb{C}[\mathfrak{q}^*]^Q$ be a preimage of $\mathbf{f}$. Each its bi-homogeneous component is again a $\mathfrak{q}$-invariant. Without loss of generality we may assume that F is bi-homogenous. Also if F is divisible by some non-scalar $H \in \mathbb{C}[V^*]^G$, then we replace F with F/H and repeat the process as long as possible.

Whenever G_y (with $y \in V^*$) is conjugate to G_x and $\varphi_y(F) \neq 0$, $\varphi_y(F)$ is a G_y-invariant of the same degree as $\mathbf{f}$ and therefore is a generator of $\mathcal{S}(\mathfrak{g}_y)^{G_y}$. Clearly $\mathbb{C}(V^*)^G[F] \subset \mathbb{C}[\mathfrak{q}^*]^Q \otimes_{\mathbb{C}[V^*]^G} \mathbb{C}(V^*)^G =: \mathcal{A}$ and $\mathcal{A} \subset \mathcal{S}(\mathfrak{g}) \otimes \mathbb{C}(V^*)^G$. If $\mathcal{A}$ contains a homogeneous in $\mathfrak{g}$ polynomial T that is not proportional (over $\mathbb{C}(V^*)^G$) to a power of F, then $\varphi_u(T)$ is not proportional to a power of $\varphi_u(F)$ for generic $u \in V^*$. But $\varphi_u(T) \in \mathcal{S}(\mathfrak{g}_u)^{G_u}$. This implies that $\mathcal{A} = \mathbb{C}(V^*)^G[F]$. It remains to notice that $\mathbb{C}(V^*)^G = \operatorname{Quot}\mathbb{C}[V^*]^G$, since G has no proper semi-invariants in $\mathbb{C}[V^*]$, and by the same reason $\mathbb{C}(V^*)^G[F] \cap \mathbb{C}[\mathfrak{q}] = \mathbb{C}[V^*]^G[F]$ in case F is not divisible by any non-constant G-invariant in $\mathbb{C}[V^*]$. □

It is time to recall the Raïs' formula [6] for the index of a semi-direct product:

$$\operatorname{ind}\mathfrak{q} = \dim V - (\dim\mathfrak{g} - \dim\mathfrak{g}_x) + \operatorname{ind}\mathfrak{g}_x \quad \text{with } x \in V^* \text{ generic.} \tag{2}$$

Lemma 3 *Suppose that $H_1, \ldots, H_r \in \mathcal{S}(\mathfrak{q})^Q$ are homogenous polynomials such that $\varphi_x(H_i)$ with $i \leqslant \operatorname{ind}\mathfrak{g}_x$ freely generate $\mathcal{S}(\mathfrak{g}_x)^{G_x} = \mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ for generic $x \in V^*$ and $H_j \in \mathbb{C}[V^*]^G$ for $j > \operatorname{ind}\mathfrak{g}_x$; and suppose that $\sum\limits_{i=1}^{\operatorname{ind}\mathfrak{g}_x} \deg_{\mathfrak{g}} H_i = \mathbf{b}(\mathfrak{g}_x)$. Then $\sum\limits_{i=1}^{r} \deg H_i = \mathbf{b}(\mathfrak{q})$ if and only if $\sum\limits_{i=1}^{r} \deg_V H_i = \dim V$.*

Proof In view of the assumptions, we have $\sum\limits_{i=1}^{r} \deg H_i = \mathbf{b}(\mathfrak{g}_x) + \sum\limits_{i=1}^{r} \deg_V H_i$. Further, by Eq. (2)

$$\begin{aligned}\mathbf{b}(\mathfrak{q}) &= (\dim\mathfrak{q} + \dim V - (\dim\mathfrak{g} - \dim\mathfrak{g}_x) + \operatorname{ind}\mathfrak{g}_x)/2 = \\ &= \dim V + (\dim\mathfrak{g}_x + \operatorname{ind}\mathfrak{g}_x)/2 = \mathbf{b}(\mathfrak{g}_x) + \dim V.\end{aligned}$$

The result follows. □

From now on suppose that G is semisimple. Then both G and Q have only trivial characters and hence cannot have proper semi-invariants. In particular, the fundamental semi-invariant is an invariant. We also have $\mathrm{tr.deg}\,\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathrm{ind}\,\mathfrak{q}$. Set $r = \mathrm{ind}\,\mathfrak{q}$ and let $x \in V^*$ be generic. If $\mathbb{C}[\mathfrak{q}^*]^Q$ is a polynomial ring, then there are bi-homogenous generators $H_1, \dots, H_r$ such that H_i with $i > \mathrm{ind}\,\mathfrak{g}_x$ freely generate $\mathbb{C}[V^*]^G$ and the invariants H_i with $i \leqslant \mathrm{ind}\,\mathfrak{g}_x$ are *mixed*, they have positive degrees in $\mathfrak{g}$ and V.

Theorem 2 ([3, Theorem 5.7] and [10, Proposition 3.11]) *Suppose that G is semisimple and $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring with homogeneous generators $H_1, \dots, H_r$. Then*

(i) $\sum_{i=1}^{r} \deg H_i = \mathbf{b}(\mathfrak{q}) + \deg \mathbf{p}_{\mathfrak{q}}$;
(ii) for generic $x \in V^*$, the restriction map $\varphi_x \colon \mathbb{C}[\mathfrak{q}^*]^Q \to \mathbb{C}[\mathfrak{g}+x]^{G_x \ltimes V} \cong \mathcal{S}(\mathfrak{g}_x)^{G_x}$ is surjective, $\mathcal{S}(\mathfrak{g}_x)^{G_x} = \mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$, and $\mathcal{S}(\mathfrak{g}_x)^{G_x}$ is a polynomial ring in $\mathrm{ind}\,\mathfrak{g}_x$ variables.

It is worth mentioning that φ_x is also surjective for stable actions. An action of G on V is called *stable* if generic G-orbits in V are closed, for more details see [8, Sections 2.4 and 7.5]. By [10, Theorem 2.8] φ_x is surjective for generic $x \in V^*$ if the G-action on V^* is stable.

3 $\mathbb{Z}/2\mathbb{Z}$-contractions

The initial motivation for studying symmetric invariants of semi-direct products was related to a conjecture of D. Panyushev on $\mathbb{Z}_2$-contractions of reductive Lie algebras. The results of [10], briefly outlined in Sect. 2, have settled the problem.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a *symmetric decomposition*, i.e., a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$. A semi-direct product, $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ is an Abelian ideal, can be seen as a *contraction*, in this case a $\mathbb{Z}_2$-*contraction*, of $\mathfrak{g}$. For example, starting with a symmetric pair $(\mathfrak{so}_{n+1}, \mathfrak{so}_n)$, one arrives at $\tilde{\mathfrak{g}} = \mathfrak{so}_n \ltimes \mathbb{C}^n$. In [4], it was conjectured that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring (in $\mathrm{rk}\,\mathfrak{g}$ variables).

Theorem 3 ([4, 9, 10]) *Let $\tilde{\mathfrak{g}}$ be a $\mathbb{Z}_2$-contraction of a reductive Lie algebra $\mathfrak{g}$. Then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring (in $\mathrm{rk}\,\mathfrak{g}$ variables) if and only if the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ is surjective.*

If we are in one of the "surjective" cases, then one can describe the generators of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$. Let $H_1, \dots, H_r$ be suitably chosen homogeneous generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and let $H_i^{\bullet}$ be the bi-homogeneous (w.r.t. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$) component of H_i of the highest $\mathfrak{g}_1$-degree. Then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by the polynomials $H_i^{\bullet}$ (of course, providing the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ is surjective) [4, 9].

Unfortunately, this construction of generators cannot work if the restriction homomorphism is not surjective, see [4, Remark 4.3]. As was found out by Helgason [2], there are four "non-surjective" irreducible symmetric pairs, namely, (E_6, F_4),

$(E_7, E_6{\oplus}\mathbb{C})$, $(E_8, E_7{\oplus}\mathfrak{sl}_2)$, and $(E_6, \mathfrak{so}_{10}{\oplus}\mathfrak{so}_2)$. The approach to semi-direct products developed in [10] showed that Panyushev's conjecture does not hold for them. Next we outline some ideas of the proof.

Let $G_0 \subset G$ be a connected subgroup with Lie $G_0 = \mathfrak{g}_0$. Then G_0 is reductive, it acts on $\mathfrak{g}_1 \cong \mathfrak{g}_1^*$, and this action is stable. Let $x \in \mathfrak{g}_1$ be a generic element and $G_{0,x}$ be its stabiliser in G_0. The groups $G_{0,x}$ are reductive and they are known for all symmetric pairs. In particular, $\mathcal{S}(\mathfrak{g}_{0,x})^{G_{0,x}}$ is a polynomial ring. It is also known that $\mathbb{C}[\mathfrak{g}_1]^{G_0}$ is a polynomial ring. By [4] $\tilde{\mathfrak{g}}$ has the "codim-2" property and $\operatorname{ind}\tilde{\mathfrak{g}} = \operatorname{rk}\mathfrak{g}$.

Making use of the surjectivity of φ_x one can show that if $\mathbb{C}[\tilde{\mathfrak{g}}^*]^{\tilde{\mathfrak{g}}}$ is freely generated by some $H_1, \ldots, H_r$, then necessary $\sum_{i=1}^{r} \deg H_i > \mathbf{b}(\tilde{\mathfrak{g}})$ for $\tilde{\mathfrak{g}}$ coming from one of the "non-surjective" pairs [10]. In view of some results from [3] this leads to a contradiction.

Note that in case of $(\mathfrak{g}, \mathfrak{g}_0) = (E_6, F_4)$, $\mathfrak{g}_0 = F_4$ is simple and $\tilde{\mathfrak{g}}$ is a semi-direct product of F_4 and $\mathbb{C}^{26}$, which, of course, comes from one of the representations in Schwarz's list [7].

4 Examples Related to the Defining Representation of $\mathfrak{sl}_n$

Form now assume that $\mathfrak{g} = \mathfrak{sl}_n$ and $V = m(\mathbb{C}^n)^* {\oplus} k\mathbb{C}^n$ with $n \geqslant 2$, $m \geqslant 1$, $m \geqslant k$. According to [7] $\mathbb{C}[V]^G$ is a polynomial ring if either $k = 0$ and $m \leqslant n+1$ or $m \leqslant n$, $k \leqslant n-1$. One finds also the description of the generators of $\mathbb{C}[V^*]^G$ and their degrees in [7]. In this section, we classify all cases, where $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ is a polynomial ring and for each of them give the fundamental semi-invariant.

Example 1 Suppose that either $m \geqslant n$ or $m = k = n-1$. Then $\mathfrak{g}_x = 0$ for generic $x \in V^*$ and therefore $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G$, i.e., $\mathbb{C}[\mathfrak{q}^*]^Q$ is a polynomial ring if and only if $\mathbb{C}[V^*]^G$ is. The latter takes place for $(m, k) = (n+1, 0)$, for $m = n$ and any $k < n$, as well as for $m = k = n-1$. Non-scalar fundamental semi-invariants appear here only for

- $m = n$, where $\mathbf{p}$ is given by $\det(v)^{n-1-k}$ with $v \in n\mathbb{C}^n$;
- $m = k = n-1$, where $\mathbf{p}$ is the sum of the principal $2k{\times}2k$-minors of

$$\left(\begin{array}{c|c} 0 & v \\ \hline w & 0 \end{array}\right) \text{ with } v \in k\mathbb{C}^n, w \in k(\mathbb{C}^n)^*.$$

In the rest of the section, we assume that $\mathfrak{g}_x \neq 0$ for generic $x \in V^*$.

4.1 The Case k = 0

Here the ring of G-invariants on V^* is generated by

$$\{\Delta_I \mid I \subset \{1,\dots,m\}, |I| = n\} \ \ [8, \text{Section 9}],$$

where each $\Delta_I(v)$ is the determinant of the corresponding submatrix of $v \in V^*$. The generators are algebraically independent if and only if $m \leqslant n+1$, see also [7].

We are interested only in m that are smaller than n. Let $n = qm + r$, where $0 < r \leqslant m$, and let $I \subset \{1,\dots,m\}$ be a subset of cardinality r. By choosing the corresponding r columns of v we get a matrix $w = v_I$. Set

$$F_I(A,v) := \det\left(v|Av|\dots|A^{q-1}v|A^q w\right), \ \ \text{where } A \in \mathfrak{g}, v \in V^*. \tag{3}$$

Clearly each F_I is an SL_n-invariant. Below we will see that they are also V-invariants. If $r = m$, then there is just one invariant, $F = F_{\{1,\dots,m\}}$. If r is either 1 or $m-1$, we get m invariants.

Lemma 4 *Each F_I defined by Eq. (3) is a V-invariant.*

Proof According to Lemma 1 we have to show that $F_I(\xi + \mathrm{ad}^*(V){\cdot}x, x) = F(\xi, x)$ for generic $x \in V^*$ and any $\xi \in \mathfrak{sl}_n$. Since $m < n$, there is an open SL_n-orbit in V^* and we can take x as E_m. Let $\mathfrak{p} \subset \mathfrak{gl}_n$ be the standard parabolic subalgebra corresponding to the composition $(m, n-m)$ and let $\mathfrak{n}_-$ be the nilpotent radical of the opposite parabolic. Each element (matrix) $\xi \in \mathfrak{gl}_n$ is a sum $\xi = \xi_- + \xi_p$ with $\xi_- \in \mathfrak{n}_-$, $\xi_p \in \mathfrak{p}$. In this notation $F_I(A, E_m) = \det\left(A_-|(A^2)_-|\dots|(A^{q-1})_-|(A^q)_{-,I}\right)$.

Let $\alpha = \alpha_A$ and $\beta = \beta_A$ be $m{\times}m$ and $(n-m){\times}(n-m)$-submatrices of A standing in the upper left and lower right corner, respectively. Then $(A^{s+1})_- = \sum_{t=0}^{s} \beta^t A_- \alpha^{s-t}$. Each column of $A_-\alpha$ is a linear combination of columns of A_- and each column of $\beta^t A_- \alpha^{j+1}$ is a linear combination of columns of $\beta^t A_- \alpha^j$. Therefore

$$\begin{aligned} F_I(A, E_m) = \det\left(A_-|\dots|(A^{q-1})_-|(A^q)_{-,I}\right) = \\ = \det\left(A_-|\beta A_-|\dots|\beta^{q-2}A_-|\beta^{q-1}A_{-,I}\right). \end{aligned} \tag{4}$$

Notice that $\mathfrak{g}_x \subset \mathfrak{p}$ and the nilpotent radical of $\mathfrak{p}$ is contained in $\mathfrak{g}_x$ (with $x = E_m$). Since $\mathrm{ad}^*(V){\cdot}x = \mathrm{Ann}(\mathfrak{g}_x) = \mathfrak{g}_x^{\perp} \subset \mathfrak{g}$ (after the identification $\mathfrak{g} \cong \mathfrak{g}^*$), $A_- = 0$ for any $A \in \mathfrak{g}_x^{\perp}$; and we have $\beta_A = cE_{n-m}$ with $c \in \mathbb{C}$ for this A. An easy observation is that

$$\begin{aligned} \det\left(\xi_-|(\beta_\xi + cE_{n-m})\xi_-|\dots|(\beta_\xi + cE_{n-m})^{q-1}\xi_{-,I}\right) = \\ = \det\left(\xi_-|\beta_\xi\xi_-|\dots|\beta_\xi^{q-1}\xi_{-,I}\right). \end{aligned}$$

Hence $F_I(\xi + A, E_m) = F_I(\xi, E_m)$ for all $A \in \mathrm{ad}^*(V){\cdot}E_m$ and all $\xi \in \mathfrak{sl}_n$. □

Theorem 4 *Suppose that* $\mathfrak{q} = \mathfrak{sl}_n \ltimes m(\mathbb{C}^n)^*$. *Then* $\mathbb{C}[\mathfrak{q}^*]^Q$ *is a polynomial ring if and only if* $m \leqslant n+1$ *and* m *divides either* $n-1$, n *or* $n+1$. *Under these assumptions on* m, $\mathbf{p}_\mathfrak{q} = 1$ *exactly then, when* m *divides either* $n-1$ *or* $n+1$.

Proof Note that the statement is true for $m \geqslant n$ by Example 1. Assume that $m \leqslant n-1$. Suppose that $n = mq + r$ as above. A generic stabiliser in $\mathfrak{g}$ is $\mathfrak{g}_x = \mathfrak{sl}_{n-m} \ltimes m\mathbb{C}^{n-m}$. On the group level it is connected. Notice that $\operatorname{ind}\mathfrak{g}_x = \operatorname{tr.deg} \mathcal{S}(\mathfrak{g}_x)^{G_x}$, since G_x has no non-trivial characters. Note also that $\mathbb{C}[V^*]^G = \mathbb{C}$, since $m < n$. If $\mathbb{C}[\mathfrak{q}^*]^Q$ is a polynomial ring, then so is $\mathbb{C}[\mathfrak{g}_x^*]^{G_x}$ by Theorem 2(ii) and either $n - m = 1$ or, arguing by induction, $n - m \equiv t \pmod m$ with $t \in \{-1, 0, 1\}$.

Next we show that the ring of symmetric invariants is freely generated by the polynomials F_I for the indicated m. Each element $\gamma \in \mathfrak{g}_x^*$ can be presented as $\gamma = \beta_0 + A_-$, where $\beta_0 \in \mathfrak{sl}_{n-m}$. Each restriction $\varphi_x(F_I)$ can be regarded as an element of $\mathcal{S}(\mathfrak{g}_x)$. Equation (4) combined with Lemma 4 and the observation that $\mathfrak{g}_x^* \cong \mathfrak{g}/\operatorname{Ann}(\mathfrak{g}_x)$ shows that $\varphi_x(F_I)$ is either Δ_I of $\mathfrak{g}_x$ (in case $q = 1$, where $F_I(A, E_m) = \det A_{-,I}$) or F_I of $\mathfrak{g}_x$. Arguing by induction on n, we prove that the restrictions $\varphi_x(F_I)$ freely generate $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ for $x = E_m$ (i.e., for a generic point in V^*). Notice that $n - m = (q-1)m + r$.

The group SL_n acts on V^* with an open orbit $\mathrm{SL}_n{\cdot}E_m$. Therefore the restriction map φ_x is injective. By the inductive hypothesis it is also surjective and therefore is an isomorphism. This proves that the polynomials F_I freely generate $\mathbb{C}[\mathfrak{q}^*]^Q$.

If m divides n, then $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[F]$ and the fundamental semi-invariant is a power of F. As follows from the equality in Theorem 2(i), $\mathbf{p} = F^{m-1}$.

Suppose that m divides either $n-1$ or $n+1$. Then we have m different invariants F_I. By induction on n, $\mathfrak{g}_x$ has the "codim-2" property, therefore the sum of $\deg \varphi_x(F_I)$ is equal to $\mathbf{b}(\mathfrak{g}_x)$ by Theorem 2(i). The sum of V-degrees is $m \times n = \dim V$ and hence by Lemma 3 $\sum \deg F_I = \mathbf{b}(\mathfrak{q})$. Thus, $\mathfrak{q}$ has the "codim-2" property. □

Remark 1 Using induction on n one can show that the restriction map φ_x is an isomorphism for all $m < n$. Therefore the polynomials F_I generate $\mathbb{C}[\mathfrak{q}^*]^Q$ for all $m < n$.

4.2 The Case $m = k$

Here $\mathbb{C}[V^*]^G$ is a polynomial ring if and only if $k \leqslant n-1$; a generic stabiliser is $\mathfrak{sl}_{n-k}$, and the G-action on $V \cong V^*$ is stable. We assume that $\mathfrak{g}_x \neq 0$ for generic $x \in V^*$ and therefore $k \leqslant n-2$.

For an $N \times N$-matrix C, let $\Delta_i(C)$ with $1 \leqslant i \leqslant N$ be coefficients of its characteristic polynomial, each Δ_i being a homogeneous polynomial of degree i. Let $\gamma = A + v + w \in \mathfrak{q}^*$ with $A \in \mathfrak{g}$, $v \in k\mathbb{C}^n$, $w \in k(\mathbb{C}^n)^*$. Having these objects we form an $(n+k) \times (n+k)$-matrix

$$Y_\gamma := \left(\begin{array}{c|c} A & v \\ \hline w & 0 \end{array}\right)$$

and set $F_i(\gamma) = \Delta_i(Y_\gamma)$ for each $i \in \{2k+1, 2k+2, 2k+3, \dots, n+k\}$. Each F_i is an $\mathrm{SL}_n \times \mathrm{GL}_k$-invariant. Unfortunately, these polynomials are not V-invariants.

Remark 2 If we repeat the same construction for $\tilde{\mathfrak{q}} = \mathfrak{gl}_n \ltimes V$ with $k \leqslant n-1$, then $\mathbb{C}[\tilde{\mathfrak{q}}^*]^{\tilde{Q}} = \mathbb{C}[V^*]^{\mathrm{GL}_n}[\{F_i \mid 2k+1 \leqslant i \leqslant n+k\}]$ and it is a polynomial ring in $\operatorname{ind}\tilde{\mathfrak{q}} = n - k + k^2$ generators.

Theorem 5 *Suppose that* $m = k \leqslant n-1$. *Then* $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ *is a polynomial ring if and only if* $k \in \{n-2, n-1\}$. *In case* $k = n-2$, $\mathfrak{q}$ *has the "codim-2" property.*

Proof Suppose that $k = n-2$. Then a generic stabiliser $\mathfrak{g}_x = \mathfrak{sl}_2$ is of index 1 and since the G-action on V is stable, $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}}$ has to be a polynomial ring by [10, Example 3.6]. One can show that the unique mixed generator is of the form $F_{2k+2}H_{2k} - F_{2k+1}^2$, where H_{2k} is a certain $\mathrm{SL}_n \times \mathrm{GL}_k$-invariant on V of degree $2k$ and then see that the sum of degrees is $\mathbf{b}(\mathfrak{q})$.

More generally, $\mathfrak{q}$ has the "codim-2" property for all $k \leqslant n-2$. Here each G-invariant divisor in V^* contains a G-orbit of maximal dimension, say Gy. Set $u = n-k-1$. If G_y is not SL_{n-k}, then $\mathfrak{g}_y = \mathfrak{sl}_u \ltimes (\mathbb{C}^u \oplus (\mathbb{C}^u)^* \oplus \mathbb{C})$ is a semi-direct product with a Heisenberg Lie algebra. Following the proof of [4, Theorem 3.3], one has to show that $\operatorname{ind}\mathfrak{g}_y = u$ in order to prove that $\mathfrak{q}$ has the "codim-2" property. This is indeed the case, $\operatorname{ind}\mathfrak{g}_y = 1 + \operatorname{ind}\mathfrak{sl}_u$.

Suppose that $0 < k < n-2$ and assume that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a polynomial ring. Then there are bi-homogeneous generators $\mathbf{h}_2, \dots, \mathbf{h}_{n-k}$ of $\mathbb{C}[\mathfrak{q}^*]^Q$ over $\mathbb{C}[V^*]^G$ such that their restrictions to $\mathfrak{g} + x$ form a generating set of $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ for a generic x (with $\mathfrak{g}_x \cong \mathfrak{sl}_{n-k}$), see Theorem 2(ii). In particular, $\deg_{\mathfrak{g}} \mathbf{h}_t = t$.

Take $\tilde{\mathfrak{g}} = (\mathfrak{sl}_n \oplus \mathfrak{gl}_k) \ltimes V$, which is a $\mathbb{Z}_2$-contraction of $\mathfrak{sl}_{n+k}$. Then $\mathfrak{q}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$. Note that GL_k acts on $\mathfrak{q}$ via automorphisms and therefore we may assume that the $\mathbb{C}$-linear span of $\{\mathbf{h}_t\}$ is GL_k-stable. By degree considerations, each $\mathbf{h}_t$ is an SL_k-invariant as well. The Weyl involution of SL_n acts on V and has to preserve each line $\mathbb{C}\mathbf{h}_t$. Since this involution interchanges $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$, each $\mathbf{h}_t$ is also a GL_k-invariant. Thus,

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{q})^{\tilde{\mathfrak{g}}} = \mathcal{S}(\mathfrak{q}) \cap \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}.$$

Since $\tilde{\mathfrak{g}}$ is a "surjective" $\mathbb{Z}_2$-contraction, its symmetric invariants are known [4, Theorem 4.5]. The generators of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ are $\Delta_j^\bullet$ with $2 \leqslant j \leqslant n+k$. Here $\deg \Delta_j^\bullet = j$ and the generators of $(\mathfrak{sl}_n \oplus \mathfrak{gl}_k)$-degrees $2, 3, \dots, n-k$ are $\Delta_{2k+2}^\bullet, \Delta_{2k+3}^\bullet, \dots, \Delta_{n+k}^\bullet$. As the restriction to $\mathfrak{sl}_n \oplus \mathfrak{gl}_k + x$ shows, none of the generators $\Delta_j^\bullet$ with $j \geqslant 2k+2$ lies in $\mathcal{S}(\mathfrak{q})$. This means that $\mathbf{h}_t$ cannot be equal or even proportional over $\mathbb{C}[V^*]^G$ to $\Delta_{2k+t}^\bullet$ and hence has a more complicated expression. More precisely, a product $\Delta_{2k+1}^\bullet \Delta_{2k+t-1}^\bullet$ necessary appears in $\mathbf{h}_t$ with a non-zero coefficient from $\mathbb{C}[V^*]^G$ for $t \geqslant 2$. Since $\deg_V \Delta_{2k+1}^\bullet = 2k$, we have $\deg_V \mathbf{h}_t \geqslant 4k$ for every $t \geqslant 2$. The ring $\mathbb{C}[V^*]^G$ is freely generated by k^2 polynomials of degree two. Therefore, the total sum of degrees over all generators of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is greater than or equal to

$$\mathbf{b}(\mathfrak{sl}_{n-k}) + 4k(n-k-1) + 2k^2 = \mathbf{b}(\mathfrak{q}) + 2k(n-k-2).$$

This contradicts Theorem 2(i) in view of the fact that $\mathbf{p}_{\mathfrak{q}} = 1$.

□

4.3 The Case $0 < k < m$

Here $\mathbb{C}[V^*]^G$ is a polynomial ring if and only if $m \leqslant n$, [7]. If $n = m$, then $\mathfrak{g}_x = 0$ for generic $x \in V^*$. For $m < n$, our construction of invariants is rather intricate.

Let $\pi_1, \dots, \pi_{n-1}$ be the fundamental weights of $\mathfrak{sl}_n$. We use the standard convention, $\pi_i = \varepsilon_1 + \dots + \varepsilon_i$, $\varepsilon_n = -\sum_{i=1}^{n-1} \varepsilon_i$. Recall that for any t, $1 \leqslant t < n$, $\Lambda^t\mathbb{C}^n$ is irreducible with the highest weight π_t. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathbb{C}^n$ such that each e_i is a weight vector and $\ell_t := e_1 \wedge \dots \wedge e_t$ is a highest weight vector of $\Lambda^t\mathbb{C}^n$. Clearly $\Lambda^t\mathbb{C}^n \subset \mathcal{S}^t(t\mathbb{C}^n)$. Write $n-k = d(m-k)+r$ with $0 < r \leqslant (m-k)$. Let $\varphi : m\mathbb{C}^n \to \Lambda^m\mathbb{C}^n$ be a non-zero m-linear G-equivariant map. Such a map is unique up to a scalar and one can take φ with $\varphi(v_1 + \dots + v_m) = v_1 \wedge \dots \wedge v_m$. In case $r \neq m-k$, for any subset $I \subset \{1, \dots, m\}$ with $|I| = k + r$, let $\varphi_I : m\mathbb{C}^n \to (k+r)\mathbb{C}^n \to \Lambda^{k+r}\mathbb{C}^n$ be the corresponding (almost) canonical map. By the same principle we construct $\tilde{\varphi} : k(\mathbb{C}^n)^* \to \Lambda^k(\mathbb{C}^n)^*$.

Let us consider the tensor product $\mathbb{W} := (\Lambda^m\mathbb{C}^n)^{\otimes d} \otimes \Lambda^{k+r}\mathbb{C}^n$ and its weight subspace $\mathbb{W}_{d\pi_k}$. One can easily see that $\mathbb{W}_{d\pi_k}$ contains a unique up to a scalar non-zero highest weight vector, namely

$$w_{d\pi_k} = \sum_{\sigma \in S_{n-k}} \mathrm{sgn}(\sigma)(\ell_k \wedge e_{\sigma(k+1)} \wedge \dots \wedge e_{\sigma(m)}) \otimes \dots \otimes (\ell_k \wedge e_{\sigma(n-r+1)} \dots \wedge e_{\sigma(n)}).$$

This means that $\mathbb{W}$ contains a unique copy of $V_{d\pi_k}$, where $V_{d\pi_k}$ is an irreducible $\mathfrak{sl}_n$-module with the highest weight $d\pi_k$. We let ρ denote the representation of $\mathfrak{gl}_n$ on $\Lambda^m\mathbb{C}^n$ and ρ_r the representation of $\mathfrak{gl}_n$ on $\Lambda^{k+r}\mathbb{C}^n$. Let $\xi = A + v + w$ be a point in $\mathfrak{q}^*$. (It is assumed that $A \in \mathfrak{sl}_n$.) Finally let (,) denote a non-zero $\mathfrak{sl}_n$-invariant scalar product between $\mathbb{W}$ and $\mathcal{S}^d(\Lambda^k(\mathbb{C}^n)^*)$ that is zero on the $\mathfrak{sl}_n$-invariant complement of $V_{d\pi_k}$ in $\mathbb{W}$. Depending on r, set

$$\mathbf{F}(\xi) := (\varphi(v) \otimes \rho(A)^{m-k}\varphi(v) \otimes \rho(A^2)^{m-k}\varphi(v) \otimes \dots \otimes \rho(A^d)^{m-k}\varphi(v), \tilde{\varphi}(w)^d)$$

$$\text{for } r = m-k;$$

$$\mathbf{F}_I(\xi) := (\varphi(v) \otimes \rho(A)^{m-k}\varphi(v) \otimes \dots \otimes \rho(A^{d-1})^{m-k}\varphi(v) \otimes \rho_r(A^d)^r\varphi_I(v), \tilde{\varphi}(w)^d)$$

for each I as above in case $r < m - k$. By the constructions the polynomials $\mathbf{F}$ and $\mathbf{F}_I$ are SL_n-invariants.

Lemma 5 *The polynomials* $\mathbf{F}$ *and* $\mathbf{F}_I$ *are* V*-invariants.*

Proof We restrict $\mathbf{F}$ and $\mathbf{F}_I$ to $\mathfrak{g}^* + x$ with $x \in V^*$ generic. Changing a basis in V if necessary, we may assume that $x = E_m + E_k$. If $r < m - k$, some of the invariants $\mathbf{F}_I$ may become linear combinations of such polynomials under the change of basis, but this does not interfere with V-invariance. Now $\varphi(v)$ is a vector of weight π_m and $\tilde{\varphi}(w)^d$ of weight $-d\pi_k$. Notice that $dm + (k + r) = n + kd$. If $\sum_{i=1}^{n+kd} \lambda_i = d \sum_{i=1}^{k} \varepsilon_i$ and each λ_i is one of the ε_j, $1 \leqslant j \leqslant n$, then in the sequence $(\lambda_1, \dots, \lambda_{n+kd})$ we must have exactly one ε_j for each $k < j \leqslant n$ and $d + 1$ copies of each ε_i with $1 \leqslant i \leqslant k$. Hence the only summand of $\rho(A^s)^{m-k}\varphi(E_m)$ that plays any rôle in $\mathbf{F}$ or $\mathbf{F}_I$ is $\ell_k \wedge A^s e_{k+1} \wedge \dots \wedge A^s e_m$. Moreover, in $A^s e_{k+1} \wedge \dots \wedge A^s e_m$ we are interested only in vectors lying in $\Lambda^{m-k}\mathrm{span}(e_{k+1}, \dots, e_n)$.

Let us choose blocks α, U, β of A as shown in Fig. 1. Then up to a non-zero scalar $\mathbf{F}(A, E_m + E_k)$ is the determinant of

$$(U|\beta U + U\alpha|P_2(\alpha, U, \beta)| \dots |P_{d-1}(\alpha, U, \beta)),$$

$$\text{where } P_s(\alpha, U, \beta) = \sum_{t=0}^{s} \beta^t U \alpha^{s-t}.$$

Each column of $U\alpha$ is a linear combination of the columns of U, a similar relation exists between $\beta^t U \alpha^{s+1}$ and $\beta^t U \alpha^s$. Therefore

$$\mathbf{F}(A, E_m + E_k) = \det\left(U|\beta U|\beta^2 U| \dots |\beta^{d-1} U\right). \tag{5}$$

We have to check that $\mathbf{F}(\xi + A, x) = \mathbf{F}(\xi, x)$ for any $A \in \mathrm{ad}^*(V)\cdot x$ and any $\xi \in \mathfrak{g}$, see Lemma 1. Recall that $\mathrm{ad}^*(V)\cdot x = \mathrm{Ann}(\mathfrak{g}_x) = \mathfrak{g}_x^\perp \subset \mathfrak{g}$. In case $x = E_m + E_k$, U is zero in each $A \in \mathfrak{g}_x^\perp$ and β corresponding to such A is a scalar matrix. Therefore $\mathbf{F}(\xi + \mathrm{ad}^*(V)\cdot x, x) = \mathbf{F}(\xi, x)$.

The case $r < m-k$ is more complicated. If $\{1, \dots, k\} \subset I$, then $I = \tilde{I} \sqcup \{1, \dots, k\}$. Let $U_{\tilde{I}}$ be the corresponding submatrix of U and $\alpha_{\tilde{I}\times\tilde{I}}$ of α. One just has to replace

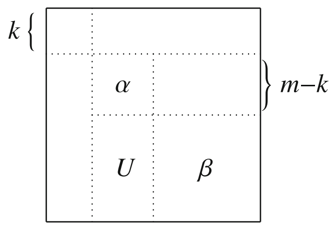

Fig. 1 Submatrices of $A \in \mathfrak{sl}_n$

U by $U_{\tilde{I}}$ and α by $\alpha_{\tilde{I}\times\tilde{I}}$ in the last polynomial $P_{d-1}(\alpha, U, \beta)$ obtaining

$$\mathbf{F}_I(A,x) = \det\left(U|\beta U|\beta^2 U|\ldots|\beta^{d-2}U|\beta^{d-1}U_{\tilde{I}}\right).$$

These are $\binom{m-k}{r}$ linearly independent invariants in $\mathcal{S}(\mathfrak{g}_x)$.

Suppose that $\{1,\ldots,k\} \not\subset I$. Then $\rho_I(A^d)^r$ has to move more than r vectors e_i with $k+1 \leqslant i \leqslant m$, which is impossible. Thus, $\mathbf{F}_I(A,x) = 0$ for such I. □

Theorem 6 *Suppose that* $0 < k < m < n$ *and* $m-k$ *divides* $n-m$*, then* $\operatorname{ind}\mathfrak{g}_x = 1$ *for generic* $x \in V^*$ *and* $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G[\mathbf{F}]$ *is a polynomial ring, the fundamental semi-invariant is equal to* $\mathbf{F}^{m-k-1}$.

Proof A generic stabiliser $\mathfrak{g}_x$ is $\mathfrak{sl}_{n-m}\ltimes(m-k)\mathbb{C}^{n-m}$. Its ring of symmetric invariants is generated by $F = \varphi_x(\mathbf{F})$, see Theorem 4 and Eq. (5). We also have $\operatorname{ind}\mathfrak{g}_x = 1$. It remains to see that $\mathbf{F}$ is not divisible by a non-constant G-invariant polynomial on V^*. By the construction, $\mathbf{F}$ is also invariant with respect to the action of $\mathrm{SL}_m\times\mathrm{SL}_k$. The group $L = \mathrm{SL}_n\times\mathrm{SL}_m\times\mathrm{SL}_k$ act on V^* with an open orbit. As long as $\operatorname{rk} w = k$, $\operatorname{rk} v = m$, the L-orbit of $y = v + w$ contains a point $v' + E_k$, where also $\operatorname{rk} v' = m$. If in addition the upper $k\times m$-part of v has rank k, then $L{\cdot}y$ contains $x = E_m + E_k$. Here $\mathbf{F}$ is non-zero on $\mathfrak{g} + y$. Since the group L is semisimple, the complement of $L{\cdot}(E_m + E_k)$ contains no divisors and $\mathbf{F}$ is not divisible by any non-constant G-invariant in $\mathbb{C}[V^*]$. This is enough to conclude that $\mathbb{C}[\mathfrak{q}^*]^Q = \mathbb{C}[V^*]^G[\mathbf{F}]$, see Theorem 2.

The singular set $\mathfrak{q}^*_{\rm sing}$ is L-stable. And therefore $\mathbf{p}_{\mathfrak{q}}$ is also an $\mathrm{SL}_m\times\mathrm{SL}_k$-invariant. Hence $\mathbf{p}$ is a power of $\mathbf{F}$. In view of Theorem 2(i), $\mathbf{p} = \mathbf{F}^{m-k-1}$. □

Theorem 7 *Suppose that* $0 < k < m < n$ *and* $m-k$ *does not divide* $n-m$*, then* $\mathbb{C}[\mathfrak{q}^*]^Q$ *is not a polynomial ring.*

Proof The reason for this misfortune is that $\binom{m}{k+r} > \binom{m-k}{r}$ for $r < m-k$. One could prove that each $\mathbf{F}_I$ must be in the set of generators and thereby show that $\mathbb{C}[\mathfrak{q}^*]^Q$ is not a polynomial ring. But we present a different argument.

Assume that the ring of symmetric invariants is polynomial. It is bi-graded and SL_m acts on it preserving the bi-grading. Since SL_m is reductive, we can assume that there is a set $\{H_1,\ldots,H_s\}$ of bi-homogeneous mixed generators such that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}[V^*]^G[H_1,\ldots,H_s]$ and the $\mathbb{C}$-linear span $\mathcal{H} := \operatorname{span}(H_1,\ldots,H_s)$ is SL_m-stable. The polynomiality implies that a generic stabiliser $\mathfrak{g}_x = \mathfrak{sl}_{n-m}\ltimes(m-k)\mathbb{C}^{n-m}$ has a free algebra of symmetric invariants, see Theorem 2(ii), and by the same statement φ_x is surjective. This means that r is either 1 or $m-k-1$, see Theorem 4, $s = m-k$, and φ_x is injective on $\mathcal{H}$. Taking our favourite (generic) $x = E_m + E_k$, we see that there is SL_{m-k} embedded diagonally into $G\times\mathrm{SL}_m$, which acts on $\varphi_x(\mathcal{H})$ as on $\Lambda^r\mathbb{C}^{m-k}$. The group SL_{m-k} acts on $\mathcal{H}$ in the same way. Since $m-k$ does not divide $n-m$, we have $m-k \geqslant 2$. The group SL_m cannot act on an irreducible module $\Lambda^r\mathbb{C}^{m-k}$ of its non-trivial subgroup SL_{m-k}, this is especially obvious in our two cases of interest, $r = 1$ and $r = m-k-1$. A contradiction. □

Conjecture 1 It is very probable that $\mathbb{C}[\mathfrak{q}^*]^{\mathfrak{q}} = \mathbb{C}[V^*]^G[\{\mathbf{F}_I\}]$ for all $n > m > k \geqslant 1$.

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On Extensions of Affine Vertex Algebras at Half-Integer Levels

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Abstract We construct certain new extensions of vertex operator algebras by its simple module. We show that extensions of certain affine vertex operator algebras at admissible half-integer levels have the structure of simple vertex operator algebras. We also discuss some methods for determination of simple current modules for affine vertex algebras.

Keywords Fusion rules • Simple current extension • Vertex operator algebra • Virasoro algebra

1 Introduction

Extensions of vertex operator algebras are an important tool for constructing new vertex operator algebras. Many important vertex operator algebras can be constructed by using extensions. One of the best-known examples is the Moonshine module vertex operator algebra, which is an extension of the $\mathbb{Z}_2$-orbifold of the Leech lattice vertex operator algebra by its simple module (cf. [16]). Extensions of affine vertex operator algebras have mostly been studied in the case of positive integer levels, and simple current modules (cf. [13]). A connection between simple current extensions and conformal embeddings have been studied in [26]. In that case, the associated simple affine vertex operator algebra is rational and C_2-cofinite. Another significant class are affine vertex operator algebras with admissible levels (cf. [23, 24]), which are connected with the minimal series $\mathcal{W}$-algebras via quantum Drinfeld-Sokolov reduction (cf. [8, 19]). Although admissible affine vertex operator algebras are rational in the category $\mathcal{O}$ (cf. [2, 9]), they are neither rational (in the usual sense) nor C_2-cofinite.

Some extensions of admissible affine vertex operator algebras can be constructed using explicit realizations of these vertex operator algebras and their modules. Recently, some extensions of admissible affine vertex operator algebras have been constructed using the notion of conformal embedding (see [4, 5]). These extensions

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of affine vertex operator algebras are associated to lowest admissible levels with given denominator (which is equal to 2 or 3 in all these cases). In the present paper, we will consider the higher level case. Let us explain our results in more detail.

Denote by $L_{\mathfrak{g}}(k,0)$ the simple affine vertex operator algebra associated to (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$, of level k. We show that the $\hat{\mathfrak{sl}}_2$-module

$$L_{A_1}(n-\tfrac{3}{2},0) \oplus L_{A_1}(n-\tfrac{3}{2},2n-1), \tag{1}$$

and the $C_\ell^{(1)}$-module

$$L_{C_\ell}(n-\tfrac{3}{2},0) \oplus L_{C_\ell}(n-\tfrac{3}{2},(2n-1)\omega_1) \tag{2}$$

have the structure of simple $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator algebras, for any $n \in \mathbb{Z}_{>0}$. Note that in the case $n = 1$, this is in fact a vertex operator algebra associated to the Weyl algebra. We also show that the $B_4^{(1)}$-module

$$L_{B_4}(n-\tfrac{7}{2},0) \oplus L_{B_4}(n-\tfrac{7}{2},(2n-1)\omega_4) \tag{3}$$

has the structure of a vertex operator algebra, for $n = 2$ and 3. We conjecture that the claim holds for any $n \in \mathbb{Z}_{>0}$. In the case $n = 1$, this extension is isomorphic to affine vertex operator algebra $L_{F_4}(-\frac{5}{2},0)$.

Our method is in some aspect similar to the fusion rules methods developed in the case of conformal embeddings (cf. [4–6]). We conformally embed a simple affine vertex algebra V into certain larger vertex algebra S which contains a (conjectural) simple-current module M. Then the fusion rules analysis gives the fusion

$$M \times M = V.$$

So this shows that $V \oplus M$ is a subalgebra of a larger vertex algebra S, and proves our extension problem. Such vertex algebra S is usually defined as double commutant. In particular, in the cases studied in Sect. 6, S is an extension of

$$Com_{L_{B_4}(n-\frac{7}{2},0)\otimes L_{B_4}(\Lambda_0)}(Com_{L_{B_4}(n-\frac{7}{2},0)\otimes L_{B_4}(\Lambda_0)}(L_{B_4}(n-\tfrac{5}{2},0))).$$

A new approach for studying extensions of vertex operator algebras have been developed in [20] by using a connections between braided tensor categories and extensions. So far such approach can be applied on rational vertex operator algebras. New interesting results on the extension problem are also obtained in [36] and [33].

Another obstacle in dealing with extensions of affine vertex algebras at admissible levels is the calculation of fusion rules. In Sect. 7, we introduce a new method for determining some fusion rules (up to a certain conjecture), using semi-infinite restriction functor from [9]. We notice that all extensions from (2) and (3) have the form $V \oplus M$, where V is a simple affine vertex algebra, and M is a simple V-module which semi-infinite restriction functor from [9] sends to a simple current

$L_{A_1}(k, 0)$-module, where k is a certain positive integer. This enables us to show that the module M has certain simple-current property in a suitable category of V-modules (for details see Sect. 7). We believe that such approach works for a larger class of affine vertex algebras and their extensions.

Throughout the paper, we fix the root vectors for simple finite-dimensional symplectic and orthogonal Lie algebras as in [11, 15] (see also [21] for details on root systems and weights for simple and affine Lie algebras).

We would like to thank the referee for useful comments and in particular for bringing articles [36] and [33] to our attention.

2 Preliminaries

In this section we recall the definition of vertex operator (super)algebra. We assume that the reader is familiar with associated notions of modules and intertwining operators (cf. [16, 17, 22, 25, 28, 37]).

Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be any $\mathbb{Z}_2$-graded vector space. Then any element $u \in V^{\bar{0}}$ (resp. $u \in V^{\bar{1}}$) is said to be even (resp. odd). We define $|u| = 0$ if u is even and $|u| = 1$ if u is odd. Elements in $V^{\bar{0}}$ or $V^{\bar{1}}$ are called homogeneous. Whenever $|u|$ is written, it is understood that u is homogeneous.

Definition 1 A vertex superalgebra is a quadruple $(V, Y, \mathbf{1}, D)$ where V is a $\mathbb{Z}_2$-graded vector space, D is an endomorphism of V, $\mathbf{1}$ is a specified element called the vacuum of V, and Y is a linear map

$$\begin{aligned} Y(\cdot, z):\ & V \to (\mathrm{End}\, V)[[z, z^{-1}]]; \\ & a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in (\mathrm{End}\, V)[[z, z^{-1}]] \end{aligned}$$

satisfying the following conditions for $a, b \in V$:

(V1) $|a_n b| = |a| + |b|$.
(V2) $a_n b = 0$ for n sufficiently large.
(V3) $[D, Y(a, z)] = Y(D(a), z) = \frac{d}{dz} Y(a, z)$.
(V4) $Y(\mathbf{1}, z) = I_V$ (the identity operator on V).
(V5) $Y(a, z)\mathbf{1} \in (\mathrm{End} V)[[z]]$ and $\lim_{z\to 0} Y(a, z)\mathbf{1} = a$.
(V6) The following Jacobi identity holds

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)Y(b, z_2) - (-1)^{|a||b|} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(b, z_2)Y(a, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2).$$

A vertex superalgebra V is called a vertex operator superalgebra if there is a special element $\omega \in V$ (called the *Virasoro element*, or *conformal vector*) whose vertex operator we write in the form $Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$, such that

(V7) $[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}c, \quad c = \text{rank}\, V \in \mathbb{C}$.

(V8) $L(-1) = D$.

(V9) $V = \oplus_{n\in\frac{1}{2}\mathbb{Z}} V_{(n)}$ is $\frac{1}{2}\mathbb{Z}$-graded so that $V^{\bar{0}} = \oplus_{n\in\mathbb{Z}} V_{(n)}$, $V^{\bar{1}} = \oplus_{n\in\frac{1}{2}+\mathbb{Z}} V_{(n)}$, $L(0)\,|_{V(n)} = nI\,|_{V_{(n)}}$, $\dim V_{(n)} < \infty$, and $V_{(n)} = 0$ for n sufficiently small.

Remark 1 If in the definition of vertex (operator) superalgebra the odd subspace $V^{\bar{1}} = 0$, we get the usual definition of vertex (operator) algebra.

We shall also need the concept of $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra.

Definition 2 A vertex algebra V is called a $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra if there is a Virasoro element $\omega \in V$ such that $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, $L(-1) = D$ and $V = \oplus_{n\in\frac{1}{2}\mathbb{Z}_{\geqslant 0}} V_{(n)}$ is $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded so that $L(0)\,|_{V(n)} = nI\,|_{V_{(n)}}$, $\dim V_{(n)} < \infty$.

Definition 3 Assume that V is a vertex algebra (not necessarily vertex operator algebra). Let U be a subalgebra of V. The following subalgebra of V

$$\text{Com}_V(U) = \{v \in V \mid a_i v = 0, \ \forall a \in U, \ i \geqslant 0\}$$

is called the commutant (or coset) subalgebra.

Suppose that U and V are vertex operator algebras with conformal vectors ω' and ω, respectively, and denote by $L'(z)$ and $L(z)$ the associated Virasoro fields. The following result is from [27]:

Proposition 1 *Let $U \subset V$ be vertex operator algebras. We have:*

$$Com_V(U) = Ker_V L'(-1).$$

For a simple Lie algebra $\mathfrak{g}$ and $k \in \mathbb{C}$, $k \neq -h^\vee$, let $N_{\mathfrak{g}}(k, 0) = N_{\mathfrak{g}}(k\Lambda_0)$ be the universal affine vertex operator algebra associated to (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$, of level k, and let $L_{\mathfrak{g}}(k, 0) = L_{\mathfrak{g}}(k\Lambda_0)$ be its simple quotient. We will also use similar notation for modules for affine vertex operator algebras.

3 Virasoro Vertex Operator Algebras

In this section we recall some results on vertex operator algebras associated to minimal models for Virasoro algebra. Rationality of these vertex operator algebras was proved by Wang in [34], and regularity by Dong et al. in [14].

Let $Vir = \oplus_{n\in\mathbb{Z}} \mathbb{C}L(n) \oplus \mathbb{C}C$ be the Virasoro algebra. For any $(c, h) \in \mathbb{C}^2$ let $L^{Vir}(c, h)$ be the irreducible highest weight Vir-module with central charge c and highest weight h (cf. [18, 34], [12]). Then $L^{Vir}(c,0)$ is a simple vertex operator algebra. Set

$$d_{p,q} = 1 - 6\frac{(p-q)^2}{pq}, \quad k_{p,q}^{m,n} = \frac{(np-mq)^2-(p-q)^2}{4pq}.$$

Whenever we mention $d_{p,q}$ again, we always assume that p and q are relatively prime positive integers larger than 1. Define

$$S_{p,q}^{Vir} = \{k_{p,q}^{m,n} \mid 0 < m < p,\ 0 < n < q\}.$$

Theorem 1 ([14, 34]) *The vertex operator algebra $L^{Vir}(d_{p,q},0)$ is regular, and the set*

$$\{L^{Vir}(d_{p,q},h) \mid h \in S_{p,q}^{Vir}\}$$

provides all irreducible $L^{Vir}(d_{p,q},0)$-modules.

In the case when a vertex operator algebra V contains a subalgebra $U \cong L^{Vir}(c,0)$ we have the following description of the commutant

$$\mathrm{Com}_V(L^{Vir}(c,0)) = \{v \in V \mid L(n)v = 0,\ \forall n \geqslant -1\}. \tag{4}$$

4 Extensions of the $\hat{\mathfrak{sl}}_2$-Vertex Operator Algebra $L_{A_1}(m,0)$ at Half Integer Levels

We shall now study certain coset constructions for representations of the vertex operator algebra $L(m,0) = L_{A_1}(m,0)$ associated to the affine Kac-Moody Lie algebra $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_2$.

Definition 4 A rational number $m = t/u$ is called admissible if $u \in \mathbb{Z}_{>0}$, $t \in \mathbb{Z}$, $(t,u) = 1$ and $2u + t - 2 \geqslant 0$.
Let $m = t/u \in \mathbb{Q}$ be admissible, and let

$$P^m = \{\lambda_{m,k,n} = (m - n + k(m+2))\Lambda_0 + (n - k(m+2))\Lambda_1,$$
$$k,n \in \mathbb{Z}_{\geqslant 0},\ n \leqslant 2u + t - 2,\ k \leqslant u - 1\}.$$

The modules $L(\lambda)$, $\lambda \in P^m$ are all modular invariant modules for affine Lie algebra $\hat{\mathfrak{sl}}_2$ of a level m (cf. [23]).

When m is admissible, the $\hat{\mathfrak{sl}}_2$-module $L(m\Lambda_0) = L(m,0)$ carries a structure of a vertex operator algebra. The classification of irreducible $L(m,0)$-modules was given in [2]. It was proved that the set $\{L(\lambda) \mid \lambda \in P^m\}$ provides all irreducible $L(m,0)$-modules from the category $\mathcal{O}$. So the admissible representations of level m for $\hat{\mathfrak{sl}}_2$

can be identified with the irreducible $L(m,0)$-modules in the category $\mathcal{O}$. We note that the generalization of this claim to arbitrary affine Lie algebra $\hat{\mathfrak{g}}$ (and associated vertex algebra) was proved in [9] using the semi-infinite restriction functor, and the results on $\hat{\mathfrak{sl}_2}$ from [2]. We will study that functor in Sect. 7 in more detail.

Let $m = \frac{t}{u} \in \mathbb{Q}$ be admissible. Set $p = t + 3u, \quad q = t + 2u$,

Then $d_m = d_{p,q}$.

Theorem 2 ([23]) *The $\hat{\mathfrak{g}}$-module $L(\Lambda_i) \otimes L(\lambda_{m,n,k})$, $i = 0, 1$, is a module for the vertex operator algebra $L^{Vir}(d_{p,q}, 0) \otimes L(m+1, 0)$, and the following decomposition holds:*

$$L(\Lambda_i) \otimes L(\lambda_{m,n,k}) \cong \bigoplus_{\substack{0 \leqslant n' \leqslant p-2 \\ n' \equiv n+i \bmod 2}} L^{Vir}(d_{p,q}, k_{p,q}^{n'+1,n+1}) \otimes L(\lambda_{m+1,k,n'}).$$

Theorem 3 *For every $n \in \mathbb{Z}_{>0}$, the $\hat{\mathfrak{sl}_2}$-module*

$$W_n = L(n - \tfrac{3}{2}, 0) \bigoplus L(n - \tfrac{3}{2}, 2n-1)$$

has the structure of a $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra.

Proof We shall prove the theorem by induction. For $n = 1$, the structure of a $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra on W_1 was explicitly obtained in [35]. W_1 is in fact the vertex operator algebra associated to the $\beta\gamma$-system.

Assume now that W_n is a $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra. Then the vertex operator algebra $L(\Lambda_0) \otimes W_n$ contains a subalgebra isomorphic to $L^{Vir}(d_{2n+3,2n+1}, 0)$. Applying Theorem 2 we obtain the following relations

$$L(\Lambda_0) \otimes L(n - \tfrac{3}{2}, 0) \cong \bigoplus_{i=0}^{n} L^{Vir}(d_{2n+3,2n+1}, k_{2n+3,2n+1}^{2i+1,1}) \otimes L(n - \tfrac{1}{2}, 2i),$$

$$L(\Lambda_0) \otimes L(n - \tfrac{3}{2}, 2n-1) \cong \bigoplus_{i=0}^{n} L^{Vir}(d_{2n+3,2n+1}, k_{2n+3,2n+1}^{2i+2,2n}) \otimes L(n - \tfrac{1}{2}, 2i+1).$$

Since $k_{2n+3,2n+1}^{2i+1,1} = 0$ if and only if $i = 0$, and $k_{2n+3,2n+1}^{2i+2,2n} = 0$ if and only if $i = n$, we obtain that the commutant of $L^{Vir}(d_{2n+3,2n+1}, 0)$ in

$$L(\Lambda_0) \otimes W_n = L(\Lambda_0) \otimes L(n - \tfrac{3}{2}, 0) \oplus L(\Lambda_0) \otimes L(n - \tfrac{3}{2}, 2n-1)$$

is exactly $L(n - \frac{1}{2}, 0) \oplus L(n - \frac{1}{2}, 2n+1) = W_{n+1}$. So W_{n+1} is also a $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra. This finishes the inductive proof of the Theorem. □

We shall now see that the vertex operator (super)algebras constructed above are simple. From the construction easily follows that our extended vertex operator (super)algebras have the form

$$V = V^{\bar{0}} \oplus V^{\bar{1}} \tag{5}$$

where

$$V^{\bar{0}} = L(m, 0), \quad V^{\bar{1}} \text{ is a simple } L(m, 0)\text{-module}, \tag{6}$$

and

$$0 \neq V^{\bar{i}} \cdot V^{\bar{j}} \subseteq V^{\overline{i+j}}, \quad \text{for every } \bar{i}, \bar{j} \in \mathbb{Z}_2. \tag{7}$$

(Here $V^{\bar{i}} \cdot V^{\bar{j}} = \text{span}_{\mathbb{C}}\{u_n v \mid u \in V^{\bar{i}},\ v \in V^{\bar{j}},\ n \in \mathbb{Z}\}$.)

By using language of [29, 32] one can say that V is a $\mathbb{Z}_2$-extension of the vertex operator algebra $L(m, 0)$. Since $V^{\bar{0}}$ is a simple vertex operator algebra and $V^{\bar{1}}$ is its simple module, we have:

Corollary 1 *All extended vertex operator (super)algebras constructed in Theorem 3 are simple.*

5 The Extension of the Vertex Operator Algebra $L_{C_\ell}(n - \frac{3}{2}, 0)$

In this section we shall construct extension of symplectic affine vertex algebra $L_{C_\ell}(n - \frac{3}{2}, 0)$ by its simple module $L_{C_\ell}(n - \frac{3}{2}, (2n-1)\omega_1)$. Our approach is using results from previous section. It is important to notice the following lemma:

Lemma 1 *The vertex operator algebra $L_{A_1}(n - \frac{3}{2}, 0)$ is isomorphic to the subalgebra of $L_{C_\ell}(n - \frac{3}{2}, 0)$ generated by the vectors $X_{\pm 2\epsilon_1}(-1)\mathbf{1}$.*

Proof In the case $n = 1$, the proof follows from the explicit realizations of the vertex operator algebras $L_{A_1}(-\frac{1}{2}, 0)$ and $L_{C_\ell}(-\frac{1}{2}, 0)$ using Weyl vertex operator algebra (or $\beta\gamma$-system) (cf. [15, 35]), also called (super)fermions in [22].

Note also that $L_{A_1}(1, 0)$ is a subalgebra of $L_{C_\ell}(1, 0)$ generated by the vectors $X_{\pm 2\epsilon_1}(-1)\mathbf{1}$. The claim of the lemma now easily follows by induction and the fact that $L_{C_\ell}(n - \frac{1}{2}, 0)$ (resp. $L_{A_1}(n - \frac{1}{2}, 0)$) is a subalgebra of $L_{C_\ell}(n - \frac{3}{2}, 0) \otimes L_{C_\ell}(1, 0)$ (resp. $L_{A_1}(n - \frac{3}{2}, 0) \otimes L_{A_1}(1, 0)$).

Theorem 4 *For every $n \in \mathbb{Z}_{>0}$, the $C_\ell^{(1)}$-module*

$$W_{n,\ell} = L_{C_\ell}(n - \tfrac{3}{2}, 0) \bigoplus L_{C_\ell}(n - \tfrac{3}{2}, (2n-1)\omega_1)$$

has the structure of a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator algebra.

Proof We shall prove this theorem by induction.

First we notice that in the case $n = 1$, $W_{1,\ell}$ is the Weyl vertex operator algebra explicitly constructed in [35].

In the inductive proof we shall follow the approach from [1]. Assume now that $W_{n,\ell}$ is a $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra. Then the vertex operator algebra $W_{n,\ell} \otimes L_{C_\ell}(\Lambda_0)$ contains a subalgebra isomorphic to $L_{C_\ell}(n - \frac{1}{2}, 0)$. Let $\mu = -(n + \frac{1}{2})\Lambda_0 + (2n - 1)\Lambda_1$. Then

$$v = 2X_{2\epsilon_1}(-1)v_\mu \otimes v_{\Lambda_0} + (2n + 1)v_\mu \otimes X_{2\epsilon_1}(-1)v_{\Lambda_0}$$

is a singular vector for $C_\ell^{(1)}$ in $W_{n,\ell} \otimes L_{C_\ell}(\Lambda_0)$, of weight $(2n + 1)\omega_1$ for C_ℓ. Since $W_{n,\ell} \otimes L_{C_\ell}(\Lambda_0) = \big(L_{C_\ell}(n - \frac{3}{2}, 0) \oplus L_{C_\ell}(n - \frac{3}{2}, (2n - 1)\omega_1)\big) \otimes L_{C_\ell}(\Lambda_0)$ is a completely reducible $C_\ell^{(1)}$-module (cf. [24]), we obtain:

$$U(\hat{\mathfrak{g}})v \cong L_{C_\ell}(n - \tfrac{1}{2}, (2n + 1)\omega_1).$$

Note that vector v belongs to a $A_1^{(1)}$-submodule $W_{n,1} \otimes L_{A_1}(\Lambda_0)$. Then Theorem 3 implies that

$$v_j v \in L_{A_1}(n - \tfrac{1}{2}, 0) \subset L_{C_\ell}(n - \tfrac{1}{2}, 0) \qquad (j \in \mathbb{Z}).$$

This easily gives that $W_{n+1,\ell}$ is a vertex operator algebra.

Similarly as in Corollary 1, we conclude:

Corollary 2 *All extended vertex operator (super)algebras constructed in Theorem 4 are simple.*

6 Extensions of the Vertex Operator Algebra $L_{B_4}(n - \frac{7}{2}, 0)$

In this section we consider the extension of affine vertex algebra $L_{B_4}(n - \frac{7}{2}, 0)$ by its simple module $L_{B_4}(n - \frac{7}{2}, (2n - 1)\omega_4)$. Our approach is using techniques from previous sections. We have the following conjecture:

Conjecture 1 For every $n \in \mathbb{Z}_{>0}$, the $B_4^{(1)}$-module

$$W_n = L_{B_4}(n - \tfrac{7}{2}, 0) \bigoplus L_{B_4}(n - \tfrac{7}{2}, (2n - 1)\omega_4)$$

has the structure of a vertex operator algebra.

First we notice that in the case $n = 1$, W_1 is affine vertex operator algebra $L_{F_4}(-\frac{5}{2}, 0)$ (cf. [31]). In this paper we prove this conjecture for $n = 2$ and 3.

We consider the structure of $B_4^{(1)}$-module $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)$. Denote by $\mu=-(n+\frac{5}{2})\Lambda_0+(2n-1)\Lambda_4$ its highest weight and by v_μ the highest weight vector of associated generalized Verma module $N_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)$.

Lemma 2

(a) We have:

$$L_{B_4}(n-\tfrac{7}{2},(2n-1)\omega_4)\cong\frac{N_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)}{U(\hat{\mathfrak{g}})u},$$

where

$$\begin{aligned}u&=(n-\tfrac{1}{2})X_{\epsilon_1}(-1)v_\mu+X_{\epsilon_1+\epsilon_2}(-1)X_{-\epsilon_2}(0)v_\mu\\&+X_{\epsilon_1+\epsilon_3}(-1)X_{-\epsilon_3}(0)v_\mu+X_{\epsilon_1+\epsilon_4}(-1)X_{-\epsilon_4}(0)v_\mu\end{aligned}$$

is a singular vector for $B_4^{(1)}$.

(b) Relation

$$\begin{aligned}&-(n-\tfrac{1}{2})\sum_{i=1}^{4}h_{\epsilon_1}(-1)v_\mu+(n+\tfrac{5}{2})\sum_{i=1}^{4}X_{\epsilon_i}(-1)X_{-\epsilon_i}(0)v_\mu-2\\&\times\sum_{1\leqslant i<j\leqslant 4}X_{\epsilon_i+\epsilon_j}(-1)X_{-\epsilon_i-\epsilon_j}(0)v_\mu=0\end{aligned}$$

holds in $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)$

Proof (a) Since μ is an admissible weight, it follows from [23] that $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)$ is a quotient of $N_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)$ modulo the maximal submodule generated by a singular vector u of weight $\mu-\delta+\epsilon_1$. The formula for vector u follows by direct calculation. Part (b) follows from relation

$$\big(X_{-\epsilon_1}(0)+X_{-\epsilon_2}(0)X_{-\epsilon_1+\epsilon_2}(0)+X_{-\epsilon_3}(0)X_{-\epsilon_1+\epsilon_3}(0)+X_{-\epsilon_4}(0)X_{-\epsilon_1+\epsilon_4}(0)\big)u=0,$$

which holds in $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)$ by part (a).

We will also use the following lemma:

Lemma 3 *The $B_4^{(1)}$-module $L_{B_4}(n-\frac{5}{2},(2n+1)\omega_4)$ is a submodule of $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)\otimes L_{B_4}(\Lambda_0)$.*

Proof Denote now by v_μ the highest weight vector in $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)$. Then

$$\begin{aligned}
v &= X_{\epsilon_1+\epsilon_2}(-1)v_\mu \otimes X_{\epsilon_3+\epsilon_4}(-1)v_{\Lambda_0} - X_{\epsilon_1+\epsilon_3}(-1)v_\mu \otimes X_{\epsilon_2+\epsilon_4}(-1)v_{\Lambda_0} \\
&+X_{\epsilon_1+\epsilon_4}(-1)v_\mu \otimes X_{\epsilon_2+\epsilon_3}(-1)v_{\Lambda_0} + X_{\epsilon_2+\epsilon_3}(-1)v_\mu \otimes X_{\epsilon_1+\epsilon_4}(-1)v_{\Lambda_0} \\
&-X_{\epsilon_2+\epsilon_4}(-1)v_\mu \otimes X_{\epsilon_1+\epsilon_3}(-1)v_{\Lambda_0} + X_{\epsilon_3+\epsilon_4}(-1)v_\mu \otimes X_{\epsilon_1+\epsilon_2}(-1)v_{\Lambda_0} \\
&+(n+\tfrac{5}{2})v_\mu \otimes X_{\epsilon_1+\epsilon_2}(-1)X_{\epsilon_3+\epsilon_4}(-1)v_{\Lambda_0} + \frac{2}{2n+1}X_{\epsilon_1+\epsilon_2}(-1)X_{\epsilon_3+\epsilon_4}(-1)v_\mu \otimes v_{\Lambda_0} \\
&-\frac{2}{2n+1}X_{\epsilon_1+\epsilon_3}(-1)X_{\epsilon_2+\epsilon_4}(-1)v_\mu \otimes v_{\Lambda_0} + \frac{2}{2n+1}X_{\epsilon_1+\epsilon_4}(-1)X_{\epsilon_2+\epsilon_3}(-1)v_\mu \otimes v_{\Lambda_0}
\end{aligned}$$

is a singular vector for $B_4^{(1)}$ in $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)\otimes L_{B_4}(\Lambda_0)$, of weight $(2n+1)\omega_4$ for B_4. Since $L_{B_4}(n-\frac{7}{2},(2n-1)\omega_4)\otimes L_{B_4}(\Lambda_0)$ is a completely reducible $B_4^{(1)}$-module (cf. [24]), we obtain:

$$U(\hat{\mathfrak{g}})v \cong L_{B_4}(n-\tfrac{5}{2},(2n+1)\omega_4).$$

Denote by ω' the conformal vector for $L_{B_4}(n-\frac{5}{2},0)$ obtained from Sugawara construction, and by ω the conformal vector for the vertex operator algebra $L_{B_4}(n-\frac{7}{2},0)\otimes L_{B_4}(\Lambda_0)$, which is a sum of two Sugawara conformal vectors. Denote by $L'(z)$ and $L(z)$ the associated Virasoro fields.

It follows from [18] that the commutant $Com_{L_{B_4}(n-\frac{7}{2},0)\otimes L_{B_4}(\Lambda_0)}(L_{B_4}(n-\frac{5}{2},0))$ is a vertex operator algebra with the conformal vector $\omega-\omega'$, and that the double commutant

$$Com_{L_{B_4}(n-\frac{7}{2},0)\otimes L_{B_4}(\Lambda_0)}(Com_{L_{B_4}(n-\frac{7}{2},0)\otimes L_{B_4}(\Lambda_0)}(L_{B_4}(n-\tfrac{5}{2},0)))$$

is a vertex operator algebra with conformal vector ω'. Thus, $L_{B_4}(n-\frac{5}{2},0)$ is conformally embedded in this double commutant. In what follows, we will consider the conformal embedding of $L_{B_4}(n-\frac{5}{2},0)$ into the double commutant of $L_{B_4}(n-\frac{5}{2},0)$ in the extension of $L_{B_4}(n-\frac{7}{2},0)\otimes L_{B_4}(\Lambda_0)$, for certain $n\in\mathbb{Z}_{>0}$. This is basically the same idea as in Theorem 3, where the commutant is Virasoro vertex algebra. But in this case, the structure of the commutant is more complicated.

Lemma 4 *We have:*

$$L(0)v = L'(0)v \tag{8}$$

$$L(-1)v = L'(-1)v. \tag{9}$$

Proof Formula for the lowest conformal weight of $L_{B_4}(n-\frac{5}{2},0)$-module $L_{B_4}(n-\frac{5}{2},(2n+1)\omega_4)$ implies that $L'(0)v=(2n+1)v$. Relation $L(0)v=(2n+1)v$ follows from the formula for v from Lemma 3 by induction. This proves relation (8).

To prove (9), first note that the explicit formula for Sugawara conformal vector ω' and relation from Lemma 2(b) (in $L_{B_4}(n-\frac{5}{2},(2n+1)\omega_4)$) imply that

$$L'(-1)v = \frac{1}{2}\sum_{i=1}^{4} X_{\epsilon_i}(-1)X_{-\epsilon_i}(0)v. \tag{10}$$

Denote now by $\bar{\omega}$ the Sugawara conformal vector in $L_{B_4}(n-\frac{7}{2},0)$ and by $\tilde{\omega}$ the Sugawara conformal vector in $L_{B_4}(\Lambda_0)$. Then

$$L(-1) = \bar{L}(-1)\otimes 1 + 1\otimes \tilde{L}(-1).$$

Using relations

$$[\bar{L}(-1), X(-1)] = X(-2), \quad [\tilde{L}(-1), X(-1)] = X(-2) \quad (X\in\mathfrak{g})$$

and

$$\bar{L}(-1)v_\mu = \frac{1}{2}\sum_{i=1}^{4} X_{\epsilon_i}(-1)X_{-\epsilon_i}(0)v_\mu,$$

and explicit formula for vector v from Lemma 3, one easily obtains the formula for $L(-1)v$. One the other hand, relation (10), the explicit formula for vector v, along with relation from Lemma 2(a) and relations in $L_{B_4}(\Lambda_0)$ (using fermionic realization of $L_{B_4}(\Lambda_0)$, for example), give (after lengthy calculations) the formula for $L'(-1)v$ and that $L'(-1)v = L(-1)v$. We omit details of these calculations.

Lemma 3 implies that W_{n+1} is $B_4^{(1)}$-submodule of $W_n \otimes L_{B_4}(\Lambda_0)$. Moreover, the results from [24] imply that $W_n\otimes L_{B_4}(\Lambda_0)$ is a direct sum of $B_4^{(1)}$-modules $L_{B_4}(n-\frac{5}{2},\mu)$ such that

$$\langle \mu, 2\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee\rangle \leqslant 2n-1. \tag{11}$$

Theorem 5 *W_n are vertex operator algebras, for $n=2$ and 3.*

Proof First, we consider the case $n=2$. Denote by V the vertex operator algebra

$$V = W_1\otimes L_{B_4}(\Lambda_0) = L_{B_4}(-\tfrac{5}{2},0)\otimes L_{B_4}(\Lambda_0)\oplus L_{B_4}(-\tfrac{5}{2},\omega_4)\otimes L_{B_4}(\Lambda_0).$$

Let us also denote by S the vertex operator algebra

$$S = \mathrm{Com}_V(Com_V(L_{B_4}(-\tfrac{3}{2},0))).$$

It follows from [18] that $L_{B_4}(-\frac{3}{2},0)$ is conformally embedded in S. Furthermore, Lemma 4 and Proposition 1 imply that $v\in S$, where v is a highest weight vector for

$L_{B_4}(-\frac{3}{2}, 3\omega_4)$, which implies that $L_{B_4}(-\frac{3}{2}, 0)$-module $L_{B_4}(-\frac{3}{2}, 3\omega_4)$ is a submodule of S. To prove that

$$L_{B_4}(-\tfrac{3}{2}, 0) \bigoplus L_{B_4}(-\tfrac{3}{2}, 3\omega_4) \quad (= W_2)$$

is a vertex subalgebra of S, we use fusion rules and conformal weights arguments as in [3] and [4]. The only highest weight B_4-modules $V_{B_4}(\mu)$ appearing in the tensor product decomposition $V_{B_4}(3\omega_4) \otimes V_{B_4}(3\omega_4)$ satisfying relation (11) are for $\mu = 2\omega_4, \omega_3, \omega_2, \omega_1$ and 0, and the lowest conformal weights for the first four modules are non-integer ($\frac{20}{11}$, $\frac{18}{11}$, $\frac{14}{11}$ and $\frac{8}{11}$, respectively). This immediately implies that

$$v_j v \in L_{B_4}(-\tfrac{3}{2}, 0) \qquad (j \in \mathbb{Z}),$$

which gives that W_2 is vertex subalgebra of S.

Now, $W_2 \otimes L_{B_4}(\Lambda_0)$ is a vertex operator algebra and one can give a similar proof for $n = 3$. The tensor product decomposition of $V_{B_4}(5\omega_4) \otimes V_{B_4}(5\omega_4)$ and relation (11) give the weights $\mu = \omega_1, \omega_2, 2\omega_1, \omega_3, 2\omega_4, \omega_1+\omega_2, \omega_1+\omega_3, 2\omega_2, \omega_1+2\omega_4, \omega_2+\omega_3, \omega_2+2\omega_4, 2\omega_3, \omega_3+2\omega_4, 4\omega_4$ and 0 (the corresponding lowest conformal weights are $\frac{8}{13}$, $\frac{14}{13}$, $\frac{18}{13}$, $\frac{18}{13}$, $\frac{20}{13}$, $\frac{24}{13}$, $\frac{28}{13}$, $\frac{32}{13}$, $\frac{30}{13}$, $\frac{36}{13}$, $\frac{38}{13}$, $\frac{42}{13}$, $\frac{44}{13}$ and $\frac{48}{13}$, respectively).

Similarly as in Corollary 1, we conclude:

Corollary 3 *All extended vertex operator algebras constructed in Theorem 5 are simple.*

Remark 2 In order to prove Theorem 5 for general $n \in \mathbb{Z}_{>0}$ (i.e. to prove Conjecture 1) using our methods, one has to understand the fusion rules of certain $L_{B_4}(n - \frac{7}{2}, 0)$-modules. More precisely, one has to prove certain "simple current property" for the module $L_{B_4}(n - \frac{7}{2}, (2n-1)\omega_4)$ (see Theorem 7 for the precise statement, for general simple Lie algebra of type B_ℓ). In the next section, we propose a new method for studying these fusion rules.

7 Semi-Infinite Restriction Functor and Fusion Rules

In this section, we recall certain results from [9]. We also use the notation from that paper. Our goal is to use the semi-infinite restriction functor for determining fusion rules for certain affine vertex operator algebras.

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}_-$, and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}_-$ be the direct sum decomposition of $\mathfrak{p}$ with the Levi subalgebra $\mathfrak{l}$ containing $\mathfrak{h}$ and the nilpotent radical $\mathfrak{m}_-$. Denote by $\mathfrak{m}$ the opposite algebra of $\mathfrak{m}_-$, so that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$. The claim of following theorem follows directly from the construction from [9]:

Theorem 6 *Let k be an admissible number for $\hat{\mathfrak{g}}$, and M_1, M_2, M_3 modules for $L_{\mathfrak{g}}(k\Lambda_0)$. Then, any intertwining operator I of type $\binom{M_3}{M_1\ M_2}$ induces an intertwining operator $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, I)$ of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, L_{\mathfrak{g}}(k\Lambda_0))$-modules of type*

$$\begin{pmatrix} & H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, M_3) & \\ H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, M_1) & & H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, M_2) \end{pmatrix}.$$

We have the following conjecture:

Conjecture 2 If $I \neq 0$, then $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, I) \neq 0$, i.e. the map $I \mapsto H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, I)$ is injective.

Now we consider a special case of the parabolic subalgebra $\mathfrak{p}$ determined by a root vector $X_{-\alpha}$, for a positive root α of $\mathfrak{g}$. Then

$$\mathfrak{l} = \mathfrak{sl}_2{}^{(\alpha)} + \mathfrak{h}.$$

Denote by k_α the rational number given by the formula

$$k_\alpha + 2 = \frac{2}{(\alpha, \alpha)}(k + h^\vee). \tag{12}$$

Since $L_{\mathfrak{sl}_2{}^{(\alpha)}}(k_\alpha \Lambda_0)$ is a vertex subalgebra of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, L_{\mathfrak{g}}(k\Lambda_0))$, we obtain:

Corollary 4 *$H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, I)$ induces an intertwining operator of $L_{\mathfrak{sl}_2{}^{(\alpha)}}(k_\alpha \Lambda_0)$-modules.*

The following decomposition follows from [9], Theorem A.2 (see also [7], Theorem 7.7):

Proposition 2 *Let k be an admissible number for $\hat{\mathfrak{g}}$, $\lambda \in Pr_k^+$. Then*

$$H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, L_{\mathfrak{g}}(\lambda)) \cong \bigoplus_{w \in W^{\mathfrak{l}}(\lambda)} L_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}}) \tag{13}$$

as $\hat{\mathfrak{l}}$-modules.

Now, we apply these results to determine certain fusion rules for affine vertex operator algebra $L_{B_\ell}(n - \ell + \frac{1}{2}, 0) = L_{B_\ell}((n - \ell + \frac{1}{2})\Lambda_0)$, for $n \in \mathbb{Z}_{>0}$. Recall from [9, 30] that irreducible (ordinary) modules for that vertex operator algebra are given by $L_{B_\ell}(\lambda) = L_{B_\ell}(n - \ell + \frac{1}{2}, \bar{\lambda})$, where $\lambda = (n - \ell + \frac{1}{2})\Lambda_0 + \bar{\lambda}$, such that

$$\langle \bar{\lambda}, \varepsilon_1^\vee \rangle \leqslant 2n - 1.$$

Recall also, that for such weight λ, the associated set of simple coroots $\Pi_\lambda^\vee$ is equal to:

$$\Pi_\lambda^\vee = \{(\delta - \epsilon_1)^\vee, \alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_\ell^\vee\}.$$

Then, we have:

$$\langle \lambda + \rho, \alpha_i^\vee \rangle = \langle \bar{\lambda}, \alpha_i^\vee \rangle + 1, \text{ for } i = 1, \ldots, \ell, \tag{14}$$

$$\langle \lambda + \rho, (\delta - \epsilon_1)^\vee \rangle = 2n - \langle \bar{\lambda}, \epsilon_1^\vee \rangle. \tag{15}$$

In particular, we are interested in the irreducible $L_{B_\ell}(n-\ell+\frac{1}{2},0)$-module $L_{B_\ell}(n-\ell+\frac{1}{2},(2n-1)\omega_\ell)$. Note that we studied this module in Sect. 6, in the special case $\ell = 4$.

To determine the fusion rules, we consider semi-infinite restriction of $L_{B_\ell}(n-\ell+\frac{1}{2},0)$ associated to the root $\alpha = \varepsilon_i$, for every $i = 1,\ldots \ell$. Let us denote $k_{(i)} = k_{\varepsilon_i}$, for $i = 1,\ldots \ell$.

Relation (12) gives that the level

$$k_{(i)} = 2n + 2\ell - 3$$

is a positive integer, for every $i = 1,\ldots \ell$. Thus, the rational vertex operator algebra $L_{\mathfrak{sl}_2^{(\varepsilon_i)}}((2n+2\ell-3)\Lambda_0)$ is a vertex subalgebra of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)}, L_{B_\ell}(n-\ell+\frac{1}{2},0))$. The construction from [13] implies that $L_{\mathfrak{sl}_2^{(\varepsilon_i)}}((2n+2\ell-3)\Lambda_1)$ is a simple current module for $L_{\mathfrak{sl}_2^{(\varepsilon_i)}}((2n+2\ell-3)\Lambda_0)$. More precisely,

$$\dim I\begin{pmatrix} L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(k_{(i)}\Lambda_0) \\ L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(k_{(i)}\Lambda_1) \quad L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(k_{(i)}\Lambda_1)\end{pmatrix} = 1, \tag{16}$$

and

$$\dim I\begin{pmatrix} L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(\lambda) \\ L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(k_{(i)}\Lambda_1) \quad L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(k_{(i)}\Lambda_1)\end{pmatrix} = 0, \tag{17}$$

for any other $L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(k_{(i)}\Lambda_0)$-module $L_{\mathfrak{sl}_2^{(\varepsilon_i)}}(\lambda)$.

Proposition 3 *We have:*

(a) Module $L_{\mathfrak{sl}_2^{(\varepsilon_i)}}((2n+2\ell-3)\Lambda_1)$ appears in the decomposition of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)}, L_{B_\ell}(n-\ell+\frac{1}{2},(2n-1)\omega_\ell)$ for every $i = 1,\ldots \ell$.

(b) If $L_{\mathfrak{sl}_2^{(\varepsilon_i)}}((2n+2\ell-3)\Lambda_0)$ appears in the decomposition of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)}, L_{B_\ell}(n-\ell+\frac{1}{2},\bar{\lambda}))$, for some $L_{B_\ell}(n-\ell+\frac{1}{2},0)$-module $L_{B_\ell}(n-\ell+\frac{1}{2},\bar{\lambda})$, then there exists $k \in \{i,\ldots,\ell\}$ such that

$$\langle \bar{\lambda}, \varepsilon_k^\vee \rangle = 2(k-i).$$

Proof

(a) First, let $i = \ell$. By direct calculation, one can easily verify that

$$\langle s_{\varepsilon_1-\varepsilon_\ell} \circ \big((n-\ell+\tfrac{1}{2})\Lambda_0 + (2n-1)\omega_\ell\big), \varepsilon_\ell^\vee \rangle = 2n + 2\ell - 3.$$

Furthermore, $s_{\varepsilon_1-\varepsilon_\ell}(\varepsilon_\ell) = \varepsilon_1 \in \hat{\Delta}^{re}_+$, which implies that $s_{\varepsilon_1-\varepsilon_\ell} \in W^{\mathfrak{l}}(\lambda)$. Now, the claim follows from relation (13). Now, let $i \in \{1, \dots, \ell-1\}$. Let $w \in W(\lambda)$ be given by relation $w = s_{\varepsilon_i-\varepsilon_{\ell-i}} s_{\delta-\varepsilon_{\ell-i}}$. If ℓ is even and $i = \frac{\ell}{2}$, we put $s_{\varepsilon_i-\varepsilon_{\ell-i}} = 1$. Then one easily obtains that

$$\langle w \circ \big((n-\ell+\tfrac{1}{2})\Lambda_0 + (2n-1)\omega_\ell\big), \varepsilon_i^\vee \rangle = 2n + 2\ell - 3.$$

Since $w^{-1}(\varepsilon_i) = 2\delta - \varepsilon_{\ell-i} \in \hat{\Delta}^{re}_+$, we have that $w \in W^{\mathfrak{l}}(\lambda)$, and the claim follows from relation (13).

(b) If $L_{\widehat{\mathfrak{sl}_2}(\varepsilon_i)}((2n+2\ell-3)\Lambda_0)$ appears in the decomposition of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)}, L_{B_\ell}(n-\ell+\frac{1}{2}, \bar{\lambda}))$, relation (13) implies that there exists $w \in W^{\mathfrak{l}}(\lambda)$ such that

$$\langle w \circ \lambda, \varepsilon_i^\vee \rangle = 0, \tag{18}$$

where $\lambda = (n-\ell+\frac{1}{2})\Lambda_0 + \bar{\lambda}$. Since $w \in W^{\mathfrak{l}}(\lambda)$, we have that $w^{-1}(\varepsilon_i) = \beta \in \hat{\Delta}^{re}_+$, i.e. β is a positive real short root of $\hat{\mathfrak{g}}$. Now, relation (18) implies that

$$\langle \lambda + \rho, \beta^\vee \rangle = 2\ell - 2i + 1.$$

Relations (14) and (15) imply that we have the following possibilities:

(i) $\beta = \varepsilon_k$, for some $k \in \{1, \dots, \ell\}$. This easily implies that $k \in \{i, \dots, \ell\}$ and $\langle \bar{\lambda}, \varepsilon_k^\vee \rangle = 2(k-i)$.

(ii) In all other cases of positive short real roots $\beta = w^{-1}(\varepsilon_i)$, for $w \in W^{\mathfrak{l}}(\lambda)$, we have $\langle \lambda + \rho, \beta^\vee \rangle > 2\ell - 2i + 1$.

The claim (b) follows.

Proposition 4 *Assume that the Conjecture 2 holds. We have:*

$$\dim I \begin{pmatrix} & L_{B_\ell}(n-\ell+\frac{1}{2}, \bar{\lambda}) & \\ L_{B_\ell}(n-\ell+\frac{1}{2}, (2n-1)\omega_\ell) & & L_{B_\ell}(n-\ell+\frac{1}{2}, (2n-1)\omega_\ell) \end{pmatrix} = 0,$$

for any irreducible $L_{B_\ell}(n-\ell+\frac{1}{2}, 0)$-module $L_{B_\ell}(n-\ell+\frac{1}{2}, \bar{\lambda})$, such that $\bar{\lambda} \neq 0$.

Proof Let us denote

$$M = L_{B_\ell}(n-\ell+\tfrac{1}{2}, (2n-1)\omega_\ell).$$

Let I be a non-trivial intertwining operator of type

$$\begin{pmatrix} L_{B_\ell}(n-\ell+\frac{1}{2}, \bar{\lambda}) \\ M \quad M \end{pmatrix}$$

for some irreducible $L_{B_\ell}((n-\ell+\frac{1}{2})\Lambda_0)$-module $L_{B_\ell}(n-\ell+\frac{1}{2},\bar{\lambda})$. Conjecture 2 and Corollary 4 now imply that I induces a non-trivial intertwining operator $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},I)$ of $L_{\mathfrak{sl}_2(\varepsilon_i)}((2n+2\ell-3)\Lambda_0)$-modules of type

$$\begin{pmatrix} H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},L_{B_\ell}(n-\ell+\frac{1}{2},\bar{\lambda})) \\ H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},M) \quad H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},M) \end{pmatrix},$$

for every $i=1,\ldots\ell$. Proposition 3(a) gives that $L_{\mathfrak{sl}_2(\varepsilon_i)}((2n+2\ell-3)\Lambda_0)$-module $L_{\mathfrak{sl}_2(\varepsilon_i)}((2n+2\ell-3)\Lambda_1)$ appears in the decomposition of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},M)$, for every $i=1,\ldots\ell$. Let v be the highest weight vector of $L_{\mathfrak{sl}_2(\varepsilon_i)}((2n+2\ell-3)\Lambda_1)$. Since $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},I)$ is non-trivial, there exists a non-trivial coefficient in $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},I)(v,z)v$. Relations (16) and (17) now imply that this coefficient generates a copy of $L_{\mathfrak{sl}_2(\varepsilon_i)}((2n+2\ell-3)\Lambda_0)$ in $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)},L_{B_\ell}(n-\ell+\frac{1}{2},\bar{\lambda}))$. It follows from Proposition 3(b) that for every $i=1,\ldots\ell$ there exists $k\in\{i,\ldots,\ell\}$ such that

$$\langle\bar{\lambda},\varepsilon_k^\vee\rangle=2(k-i).$$

For $i=\ell$, we obtain $\langle\bar{\lambda},\varepsilon_\ell^\vee\rangle=0$. For $i=\ell-1$, we obtain $\langle\bar{\lambda},\varepsilon_{\ell-1}^\vee\rangle=0$ or $\langle\bar{\lambda},\varepsilon_\ell^\vee\rangle=2$. Since $\langle\bar{\lambda},\varepsilon_\ell^\vee\rangle=0$, we obtain that necessarily $\langle\bar{\lambda},\varepsilon_{\ell-1}^\vee\rangle=0$. Similarly, for $i=\ell-2,\ldots,1$, we obtain

$$\langle\bar{\lambda},\varepsilon_i^\vee\rangle=0,$$

for every $i=1,\ldots\ell$. Thus $\bar{\lambda}=0$, which finishes the proof of proposition.

Proposition 4 and general results on intertwining operators from [17] now imply:

Theorem 7 *Assume that the Conjecture 2 holds. We have the following fusion rules of $L_{B_\ell}(n-\ell+\frac{1}{2},0)$-modules:*

$$\dim I\begin{pmatrix} L_{B_\ell}(n-\ell+\frac{1}{2},0) \\ L_{B_\ell}(n-\ell+\frac{1}{2},(2n-1)\omega_\ell) \quad L_{B_\ell}(n-\ell+\frac{1}{2},(2n-1)\omega_\ell) \end{pmatrix}=1,$$

and

$$\dim I\begin{pmatrix} L_{B_\ell}(n-\ell+\frac{1}{2},\bar{\lambda}) \\ L_{B_\ell}(n-\ell+\frac{1}{2},(2n-1)\omega_\ell) \quad L_{B_\ell}(n-\ell+\frac{1}{2},(2n-1)\omega_\ell) \end{pmatrix}=0,$$

for any irreducible (ordinary) $L_{B_\ell}(n-\ell+\frac{1}{2},0)$-module $L_{B_\ell}(n-\ell+\frac{1}{2},\bar{\lambda})$, such that $\bar{\lambda}\neq 0$.

Remark 3 Assuming that the Conjecture 2 holds, Theorem 7 and results from Sect. 6 imply that Conjecture 1 holds for any $n \in \mathbb{Z}_{>0}$, that is:

$$L_{B_4}(n - \tfrac{7}{2}, 0) \bigoplus L_{B_4}(n - \tfrac{7}{2}, (2n-1)\omega_4)$$

has the structure of a (simple) vertex operator algebra, for any $n \in \mathbb{Z}_{>0}$.

Remark 4 We were informed by the referee that there is a direct proof that

$$L_{B_4}(n - \tfrac{7}{2}, 0) \bigoplus L_{B_4}(n - \tfrac{7}{2}, (2n-1)\omega_4)$$

is a vertex operator algebra, which uses the theory of tensor categories, results of Bantay [10] and methods developed by Yamauchi [36] and van Ekeren et al. [33]. So one can expect that in order to classify all simple current extensions of admissible vertex operator algebras, one can use restriction functors and show that this functor sends a module for admissible affine vertex operator algebras to a simple current module for affine $\widehat{sl}_2$ vertex operator algebras. We hope to study this and related topics in our forthcoming publications.

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Dirac Cohomology in Representation Theory

Jing-Song Huang

Abstract This article is an introduction to Dirac cohomology for reductive Lie groups, reductive Lie algebras and rational Cherednik algebras. We also survey recent results focusing particularly on Dirac cohomology of unitary representations and its connection with Lie algebra cohomology.

Keywords Category $\mathcal{O}$ • Dirac cohomology • Harish-Chandra module • Rational Cherednik algebra • Reductive Lie group and Lie algebra

1 Introduction

Consider a possibly indefinite inner product $\langle x, y\rangle = \sum_i \epsilon_i x_i y_i$, for $x, y \in \mathbb{R}^n$ with $n \geqslant 2$ and $\epsilon_i = \pm 1$. Let $\Delta = \sum_i \epsilon_i \partial_i^2$ be the corresponding Laplace operator. We look for a first order differential operator D such that $D^2 = \Delta$. If we write $D = \sum_i e_i \partial_i$ for some scalars e_i, then $D^2 = \sum_i e_i^2 \partial_i^2 + \sum_{i<j}(e_i e_j + e_j e_i)\partial_i \partial_j$. It leads to require the relations

$$e_i^2 = \epsilon_i \qquad \text{and} \qquad e_i e_j + e_j e_i = 0 \quad \text{for} \quad i \neq j.$$

This is clearly impossible for real or complex scalars e_i's. Nevertheless, we can consider an algebra generated by $e_1, \ldots, e_n$, satisfying the same relations. If we allow e_i's to be in the Clifford algebra, then we do get a Dirac operator D which squares to Δ.

In representation theory Dirac operators were employed in 1970s by Parthasarathy [50] and Atiyah-Schmid [5] for purpose of constructing the discrete series representations [20]. It turns out that they can be constructed as kernels of Dirac operators acting on certain spin bundles on the symmetric space G/K. In 1990s, Vogan made a conjecture on the property of the Dirac operator in the setting of a reductive Lie algebra and its associated Clifford algebra [53]. This property

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implies that the standard parameter of the infinitesimal character of a Harish-Chandra module X and the infinitesimal character of its Dirac cohomology $H_D(X)$ are conjugate under the Weyl group. Vogan's conjecture was consequently verified in [26], and it has been playing a key role in the theory of Dirac cohomology. Dirac cohomology offers new perspectives for understanding irreducible unitary representations and proofs of some classical theorems. It is a basic invariant related to $(\mathfrak{g}, K)$-cohomology, $\mathfrak{u}$-cohomology, the K-characters and the global characters. It has interesting applications in harmonic analysis such as branching laws and endoscopy. We summarize some recent results here.

1. Dirac cohomology provides a new point of view for understanding classic theory. The geometric construction of discrete series representations initially did not use Dirac cohomology and Vogan's conjecture, but using Dirac cohomology makes some of the proofs easier [28]. Dirac cohomology is further used for geometric quantization [11, 22]. Simpler proofs of the generalized Weyl character formula [39] and generalized Bott-Borel-Weil theorem [40] are given in [28]. Moreover, Dirac cohomology is used to extend the Langlands formula on dimensions of automorphic forms [45] to a slightly more general setting [28].
2. The Dirac cohomology of several families of Harish-Chandra modules has been determined. These modules include finite-dimensional modules and irreducible unitary $A_{\mathfrak{q}}(\lambda)$-modules [25]. It was proved that if X is a unitary Harish-Chandra module, then

$$H^*(\mathfrak{g}, K; X \otimes F^*) \cong \mathrm{Hom}(H_D(F), H_D(X))$$

   for any irreducible finite-dimensional module F. More precisely, Dirac cohomology determines the $(\mathfrak{g}, K)$-cohomology when the latter exists, and can be thought of as a generalization of $(\mathfrak{g}, K)$ cohomology when the latter no longer exists. It is evident that unitary representations with nonzero Dirac cohomology are closely related to automorphic representations [54].
3. Another aspect of Dirac cohomology is its connection with $\mathfrak{u}$-cohomology. Kostant has extended Vogan's conjecture to the setting of the cubic Dirac operator and proved a nonvanishing result on Dirac cohomology for highest weight modules in the most general setting [41]. He also determined the Dirac cohomology of finite-dimensional modules in the equal rank case. The Dirac cohomology for all irreducible highest weight modules was determined in [34] in terms of coefficients of Kazhdan–Lusztig polynomials. It is proved Dirac cohomology and $\mathfrak{u}$-cohomology are isomorphic up to a one-dimensional character for irreducible highest weight modules [34].
4. Dirac cohomology, or rather its Euler characteristic, or the Dirac index, gives the K-characters of representations. It leads to a generalization of certain classical branching formulas due to Littlewood [32] which describe how a finite dimensional representation of $GL(n, \mathbb{C})$ decomposes under orthogonal or symplectic subgroups. We also generalize some of the other classical branching rules in [23]. When G is Hermitian symmetric and $\mathfrak{u}$ is unipotent radical of

a parabolic subalgebra with Levi subgroup K, [30] showed that for a unitary representation its Dirac cohomology is isomorphic to its $\mathfrak{u}$-cohomology up to a twist of a one-dimensional character. In particular, Enright's calculation of $\mathfrak{u}$-cohomology [16] gives the Dirac cohomology of the irreducible unitary highest weight modules. The Dirac cohomology of unitary lowest weight modules of scalar type is calculated more explicitly in [31]. Dirac cohomology of more families of unitary representation are determined in [6, 7] and [48].

5. Dirac index and the K-character are intimately related to the global characters on the set of elliptic elements. Dirac cohomology is employed as a tool to study a class of irreducible unitary representations, called elliptic representations [24]. More precisely, Harish-Chandra showed that the characters of irreducible or more generally admissible representations are locally integrable functions and smooth on the open dense subset of regular elements [19]. An elliptic representation has a global character that does not vanish on the elliptic elements in the set of regular elements. It is proved that an irreducible admissible (not necessarily unitary) representation is elliptic if and only if its Dirac index is nonzero [24, Theorem 8.3]. Dirac index is nonzero implies that Dirac cohomology is nonzero. Note that under the condition of regular infinitesimal character, the Dirac index is zero if and only if the Dirac cohomology is zero [24, Theorem 10.1]. This equivalence is conjectured to hold in general without the regularity condition [24, Conjecture 10.3]. In particular, an irreducible tempered elliptic representation has nonzero Dirac cohomology, and therefore it is a discrete series or a limit of discrete series representation [15, Theorem 7.5]. The characters of the irreducible tempered elliptic representations are associated in a natural way to the supertempered distributions defined by Harish-Chandra [21].
6. Better understanding of the endoscopic transfer factor for real groups [47] is the first of the 'problems for real groups' raised by Arthur [4]. It is observed [24] there is a connection between Labesse's calculation [44] of the endoscopic transfer of pseudo-coefficients of discrete series [43] and the calculation of the characters of the Dirac index of discrete series. This offers a new point of view for understanding the endoscopic transfer in the framework of Dirac cohomology and the Dirac index.
7. Vogan's conjecture has been extended to several other settings by many authors as follows:

   (i) Kostant considered the case when the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is replaced by any reductive subalgebra $\mathfrak{r}$ such that the form B remains nondegenerate when restricted to $\mathfrak{r}$. The appropriate analogue of D is then Kostant's cubic Dirac operator. He generalized Vogan's conjecture to this setting of the cubic Dirac operator [41].

   (ii) Alekseev and Meinrenken proved a version of Vogan's conjecture in their study of *Lie theory and the Chern–Weil homomorphism* [2].

   (iii) Kumar proved a similar version of Vogan's conjecture in *Induction functor in non-commutative equivariant cohomology and Dirac cohomology* [42].

(iv) Pandžić and I defined an analogue of D and prove an analogue of Vogan's conjecture in case when $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a basic classical Lie superalgebra. We extended Vogan's conjecture to the symplectic Dirac operator in Lie superalgebras [27].

(v) Kac, Möseneder Frajria and Papi extended Vogan's conjecture to the affine cubic Dirac operator in affine Lie algebras [37].

(vi) Barbasch, Ciubotaru and Trapa extended Vogan's conjecture to the setting of Luszting's graded affine Hecke algebras [8]. They also found applications of Dirac cohomology to unitary representations of p-adic groups.

(vii) Ciubotaru and Trapa proved a version of Vogan's conjecture for studying Weyl group representations in connection with Springer theory [13].

Mostly recently, Ciubotaru extended the definition of Dirac operator and Vogan's conjecture to the setting of Drinfeld's graded Hecke algebras including symplectic reflection algebras [17] and particularly rational Cherednik algebras [12]. Many results on Dirac cohomology and Lie algebra cohomology for Hermitian symmetric Lie groups have analogues for rational Cherednik algebras [33].

2 Dirac Cohomology of Harish-Chandra Modules

A complex Lie algebra $\mathfrak{g}$ is called reductive if its adjoint representation is completely reducible [35]. A Lie group G is called reductive if the complexification of the Lie algebra of G is reductive. Typical examples of reductive Lie groups include various matrix groups, i.e., closed subgroups of the general linear group $GL(n,\mathbb{C})$, for instance, $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $U(p,q)$, $O(p,q)$, $Sp(p,q)$ and $Sp(2n,\mathbb{R})$. Each reductive Lie group G comes with a Cartan involution Θ. In the above matrix examples, one can take Θ to be the transpose inverse of the complex conjugate matrix, i.e., $\Theta(g) = (\bar{g}^{-1})^t$. In what follows we assume that the group $K = G^\Theta$ of fixed points of Θ is a maximal compact subgroup of G. The involution Θ induces a decomposition of the complexified Lie algebra of G, called the Cartan decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{1}$$

where $\mathfrak{k}$ is the complexified Lie algebra of K, and $\mathfrak{p}$ is the (-1)-eigenspace for the differential of Θ.

A topological vector space $\mathcal{H}$ over $\mathbb{C}$ is a representation of G if there is a continuous action of G on $\mathcal{H}$ by linear operators. Assume now that G is reductive with Cartan involution Θ and maximal compact subgroup $K = G^\Theta$. Then we can consider the subspace $\mathcal{H}_K$ of the representation space $\mathcal{H}$ consisting of K-finite vectors, i.e., vectors $h \in \mathcal{H}$ such that the subspace of $\mathcal{H}$ spanned by $K \cdot h$ is finite-dimensional. One can show that the G-action on $\mathcal{H}$ induces an action of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}_K$. Thus $\mathcal{H}_K$ becomes an example of a *Harish-Chandra* *module* for the pair $(\mathfrak{g}, K)$, which is by definition a vector space with a Lie algebra action of $\mathfrak{g}$ and a finite action of the group K, with certain natural compatibility conditions. The space $\mathcal{H}_K$ can be decomposed into a direct sum of irreducible (finite-dimensional) representations of K, each appearing with certain multiplicity. If all these multiplicities are finite, then $\mathcal{H}_K$ and $\mathcal{H}$ are called admissible. An important special class of representations of G consists of *unitary representations*, for which the space $\mathcal{H}$ is a Hilbert space, and G acts on $\mathcal{H}$ by unitary operators. Harish-Chandra showed that irreducible unitary representations are automatically admissible. Irreducible admissible representations were classified by Langlands [46]. We refer to [52] and [38] for the theory of representations of real reductive Lie groups.

Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, which restricts to the Killing form on the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ the Clifford algebra of $\mathfrak{p}$ with respect to B. Then one can consider the following version of the Dirac operator:

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p});$$

here $Z_1, \ldots, Z_n$ is an orthonormal basis of $\mathfrak{p}$ with respect to the symmetric bilinear form B. It follows that D is independent of the choice of the orthonomal basis $Z_1, \ldots, Z_n$ and it is invariant under the diagonal adjoint action of K.

The Dirac operator D is a square root of the Laplace operator associated to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. To explain this, we start with a Lie algebra map

$$\alpha : \mathfrak{k} \to C(\mathfrak{p}),$$

which is defined by the adjoint map $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ composed with the embedding of $\mathfrak{so}(\mathfrak{p})$ into $C(\mathfrak{p})$ using the identification $\mathfrak{so}(\mathfrak{p}) \simeq \bigwedge^2 \mathfrak{p}$. The explicit formula for α is (see [28, §2.3.3])

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j. \tag{2}$$

Using α we can embed the Lie algebra $\mathfrak{k}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, by

$$X \mapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{k})$. We denote the image of $\mathfrak{k}$ by $\mathfrak{k}_\Delta$, and then the image of $U(\mathfrak{k})$ is the enveloping algebra $U(\mathfrak{k}_\Delta)$ of $\mathfrak{k}_\Delta$.

Let $\Omega_\mathfrak{g}$ be the Casimir operator for $\mathfrak{g}$, given by $\Omega_\mathfrak{g} = \sum Z_i^2 - \sum W_j^2$, where W_j is an orthonormal basis for $\mathfrak{k}_0$ with respect to the inner product $-B$, where B is the Killing form. Let $\Omega_\mathfrak{k} = -\sum W_j^2$ be the Casimir operator for $\mathfrak{k}$. The image of $\Omega_\mathfrak{k}$ under Δ is denoted by $\Omega_{\mathfrak{k}_\Delta}$. Fix a positive root system $\Delta^+(\mathfrak{g})$ for $\mathfrak{t}$ in $\mathfrak{g}$. Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Write $\rho = \rho(\Delta^+(\mathfrak{g}))$, $\rho_c = \rho(\Delta^+(\mathfrak{k}))$ and $\rho_n = \rho - \rho_c$. Then

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + (||\rho_c||^2 - ||\rho||^2) 1 \otimes 1. \tag{3}$$

The Parthasarathy's Dirac inequality for unitary Harish-Chandra modules is an important criteria for irreducible unitary representations of reductive Lie groups. Let X be an irreducible Harish-Chandra module with infinitesimal character Λ. Consider the action of the Dirac operator D on $X \otimes S$, with S the spinor module for the Clifford algebra $C(\mathfrak{p})$. If X is unitary, then D is self-adjoint with respect to a natural Hermitian inner product on $X \otimes S$. Let E_μ be any $\widetilde{K}$-module occurring in $X \otimes S$ with a highest weight $\mu \in \mathfrak{t}^*$, then

$$\langle \mu + \rho_c, \mu + \rho_c \rangle \geq \langle \Lambda, \Lambda \rangle.$$

The Dirac cohomology are defined to be those E_μ so that the equality holds, namely $H_D(X) = \operatorname{Ker} D = \operatorname{Ker} D^2$.

For better understanding these E_μ in $\operatorname{Ker} D$, Vogan formulated a conjecture saying that every element $z \otimes 1$ of $Z(\mathfrak{g}) \otimes 1 \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$ can be written as

$$\zeta(z) + Da + bD$$

where $\zeta(z)$ is in $Z(\mathfrak{k}_\Delta)$, and $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Vogan's conjecture implies a refinement of Parthasarathy's Dirac inequality, namely the equality holds if and only if conjugate of Λ is equal to $\mu + \rho_c$.

A main result in [26] is introducing a differential d on the K-invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by a super bracket with D, and determining the cohomology of this differential complex. As a consequence, Pandžić and I proved the following theorem. In the following we denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ so that $\mathfrak{t}^*$ is embedded into $\mathfrak{h}^*$, and by W and W_K the Weyl groups of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively.

Theorem 1 ([26]) *Let $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}) \cong Z(\mathfrak{k}_\Delta)$ be the algebra homomorphism that is determined by the following commutative diagram:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\ \zeta\ } & Z(\mathfrak{k}) \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta_{\mathfrak{k}}} \\ P(\mathfrak{h}^*)^W & \xrightarrow{\ \mathrm{Res}\ } & P(\mathfrak{t}^*)^{W_K}, \end{array}$$

where P denotes the polynomial algebra, and vertical maps η and $\eta_{\mathfrak{k}}$ are Harish-Chandra isomorphisms. Then for each $z \in Z(\mathfrak{g})$ one has

$$z \otimes 1 - \zeta(z) = Da + aD, \text{ for some } a \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

The Dirac cohomology is defined as follows:

$$H_D(X) := \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D.$$

It follows from the identity (3) that $H_D(X)$ is a finite-dimensional module for the spin double cover $\widetilde{K}$ of K. As a consequence of the above theorem, we have that $H_D(X)$, if nonzero, determines the infinitesimal character of X.

Theorem 2 ([26]) *Let X be an admissible $(\mathfrak{g}, K)$-module with standard infinitesimal character parameter $\Lambda \in \mathfrak{h}^*$. Suppose that $H_D(X)$ contains a representation of $\widetilde{K}$ with infinitesimal character λ. Then Λ and $\lambda \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$ are conjugate under W.*

The above theorem is proved in [26] for a connected semisimple Lie group G. It is straightforward to extend the result to a possibly disconnected reductive Lie group in Harish-Chandra's class [15].

Let G be a connected reductive algebraic group over a local field F of characteristic 0. Arthur [3] studied a subset $\Pi_{\text{temp, ell}}(G(F))$ of tempered representations of $G(F)$, namely elliptic tempered representations. The set of tempered representations $\Pi_{\text{temp}}(G(F))$ includes the discrete series and in general the irreducible constituents of representations induced from the discrete series. These are exactly the representations which occur in the Plancherel formula for $G(F)$.

In Harish-Chandra's theory, the character of an infinite-dimensional representation π is defined as a distribution

$$\Theta(\pi,f) = \operatorname{tr}\Big(\int_{G(F)} f(x)\pi(x)dx\Big), \quad f \in C_c^\infty(G(F)),$$

which can be identified with a function on $G(F)$. In other words,

$$\Theta(\pi,f) = \int_{G(F)} f(x)\Theta(\pi,x)dx, \quad f \in C_c^\infty(G(F)),$$

where $\Theta(\pi,x)$ is a locally integrable function on $G(F)$ that is smooth on the open dense subset $G_{\text{reg}}(F)$ of regular elements. A representation π is called elliptic if $\Theta(\pi,x)$ does not vanish on the set of elliptic elements in $G_{\text{reg}}(F)$. Elliptic representations are precisely those representations with nonzero Dirac index (see 5. in Sect. 1).

We note that a real reductive group $G(\mathbb{R})$ has elliptic elements if and only if it is of equal rank with $K(\mathbb{R})$. We also assume this equal rank condition. Induced representations from proper parabolic subgroups are not elliptic. Consider the quotient of the Grothendieck group of the category of finite length Harish-Chandra modules by the subspace generated by induced representations. Let us call this quotient group the elliptic Grothendieck group. Arthur [3] found an orthonormal basis of this elliptic Grothendieck group in terms of elliptic tempered (possibly virtual) characters. Those characters are the supertempered distributions defined by Harish-Chandra [21].

For a real reductive algebraic group $G(\mathbb{R})$, the Harish-Chandra modules of irreducible elliptic unitary representations with regular infinitesimal characters are showed to be strongly regular (in the sense of [51]) and hence they are $A_{\mathfrak{q}}(\lambda)$-modules.

An irreducible tempered representation is either elliptic or induced from an elliptic tempered representation by parabolic induction. If $G(\mathbb{R})$ is not of equal rank, then there is no elliptic representation for $G(\mathbb{R})$. Still, we know that $G(\mathbb{R})$ has representations with nonzero Dirac cohomology.

Conjecture 1 ([24]) A unitary representation either has nonzero Dirac cohomology or is induced from a unitary representation with nonzero Dirac cohomology by parabolic induction (including complement series).
This conjecture holds for $GL(n,\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the twofolds covering group of $GL(n,\mathbb{R})$. A recent preprint of Adams–van Leeuwen–Trapa–Vogan [1] gives an algorithm to determine the irreducible unitary representations. The above conjecture means that one may regard unitary representations with nonzero Dirac cohomology as 'cuspidal' ones. Classification of irreducible unitary representations with nonzero Dirac cohomology remains to be an open problem.

3 Dirac Cohomology in Category $\mathcal{O}$

Let $\mathfrak{g}$ be a complex reductive Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ in a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. The category $\mathcal{O}$ introduced by Bernstein, et al. [9, 36] is the category of all $\mathfrak{g}$-modules, which are finitely generated, locally $\mathfrak{b}$-finite and semisimple under the $\mathfrak{h}$-action. Kostant proved a nonvanishing result on Dirac cohomology for highest weight modules in the most general setting. His theorem implies that for the equal rank case all highest weight modules have nonzero Dirac cohomology. He also determined the Dirac cohomology of finite-dimensional modules in this case. The connection of Dirac cohomology of $(\mathfrak{g}, K)$-modules and that of highest weight modules was studied in [14] using the Jacquet functor. In [34] we determined the Dirac cohomology of all irreducible highest weight modules in terms of Kazhdan–Lusztig polynomials.

We first recall the definition of Kostant's cubic Dirac operator and the basic properties of the corresponding Dirac cohomology. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Killing form B. Let $\mathfrak{r} \subset \mathfrak{g}$ be a reductive Lie subalgebra such that $B|_{\mathfrak{r}\times\mathfrak{r}}$ is nondegenerate. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be the orthogonal decomposition with respect to B. Then the restriction $B|_{\mathfrak{s}}$ is also nondegenerate. Denote by $C(\mathfrak{s})$ the Clifford algebra of $\mathfrak{s}$ with

$$uu' + u'u = -2B(u, u')$$

for all $u, u' \in \mathfrak{s}$. The above choice of sign is the same as in [28], but different from the definition in [39], as well as in [30]. The two different choices of signs make no essential difference since the two bilinear forms are equivalent over $\mathbb{C}$. Now fix an orthonormal basis $Z_1, \ldots, Z_m$ of $\mathfrak{s}$. Kostant [39] defines the cubic Dirac operator D by

$$D = \sum_{i=1}^{m} Z_i \otimes Z_i + 1 \otimes v \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Here $v \in C(\mathfrak{s})$ is the image of the fundamental 3-form $w \in \bigwedge^3(\mathfrak{s}^*)$,

$$w(X, Y, Z) = \frac{1}{2} B(X, [Y, Z]),$$

under the Chevalley map $\bigwedge(\mathfrak{s}^*) \to C(\mathfrak{s})$ and the identification of $\mathfrak{s}^*$ with $\mathfrak{s}$ by the Killing form B. Explicitly,

$$v = \frac{1}{2} \sum_{1 \leqslant i < j < k \leqslant m} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k.$$

The cubic Dirac operator has a good square in analogy with the Dirac operator associated with the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ in Sect. 2. We have a similar Lie algebra map

$$\alpha : \mathfrak{r} \to C(\mathfrak{s})$$

which is defined by the adjoint map ad $: \mathfrak{r} \to \mathfrak{so}(\mathfrak{s})$ composed with the embedding of $\mathfrak{so}(\mathfrak{s})$ into $C(\mathfrak{s})$ using the identification $\mathfrak{so}(\mathfrak{s}) \simeq \bigwedge^2 \mathfrak{s}$. The explicit formula for α is (see [28, §2.3.3])

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j, \qquad X \in \mathfrak{r}. \tag{4}$$

Using α we can embed the Lie algebra $\mathfrak{r}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, by

$$X \mapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{r})$. We denote the image of $\mathfrak{r}$ by $\mathfrak{r}_\Delta$, and then the image of $U(\mathfrak{r})$ is the enveloping algebra $U(\mathfrak{r}_\Delta)$ of $\mathfrak{r}_\Delta$. Let $\Omega_\mathfrak{g}$ (resp. $\Omega_\mathfrak{r}$) be the Casimir elements for $\mathfrak{g}$ (resp. $\mathfrak{r}$). The image of $\Omega_\mathfrak{r}$ under Δ is denoted by $\Omega_{\mathfrak{r}_\Delta}$.

Let $\mathfrak{h}_\mathfrak{r}$ be a Cartan subalgebra of $\mathfrak{r}$ which is contained in $\mathfrak{h}$. It follows from Kostant's calculation ([39, Theorem 2.16]) that

$$D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{r}_\Delta} - (\|\rho\|^2 - \|\rho_\mathfrak{r}\|^2) 1 \otimes 1, \tag{5}$$

where $\rho_{\mathfrak{r}}$ denotes the half sum of positive roots for $(\mathfrak{r}, \mathfrak{h}_{\mathfrak{r}})$. We also note the sign difference with Kostant's formula due to our choice of bilinear form for the definition of the Clifford algebra $C(\mathfrak{s})$.

We denote by W the Weyl group associated to the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ and $W_{\mathfrak{r}}$ the Weyl group associated to the root system $\Delta(\mathfrak{r}, \mathfrak{h}_{\mathfrak{r}})$. The following theorem due to Kostant is an extension of Vogan's conjecture on the symmetric pair case which is proved in [26]. (See also [41, Theorems 4.1 and 4.2] or [28, Theorem 4.1.4]).

Theorem 3 *There is an algebra homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{r}) \cong Z(\mathfrak{r}_\Delta)$ such that for any $z \in Z(\mathfrak{g})$ one has*

$$z \otimes 1 - \zeta(z) = Da + aD \text{ for some } a \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Moreover, ζ is determined by the following commutative diagram:

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{r}) \\ \eta\downarrow & & \eta_{\mathfrak{r}}\downarrow \\ P(\mathfrak{h}^*)^W & \xrightarrow{\text{Res}} & P(\mathfrak{h}_{\mathfrak{r}}^*)^{W_{\mathfrak{r}}}. \end{array}$$

Here the vertical maps η and $\eta_{\mathfrak{r}}$ are Harish-Chandra isomorphisms.

Let S be a spin module of $C(\mathfrak{s})$. Consider the action of D on $V \otimes S$

$$D : V \otimes S \to V \otimes S \tag{6}$$

with $\mathfrak{g}$ acting on V and $C(\mathfrak{s})$ on S. The *Dirac cohomology* of V is defined to be the $\mathfrak{r}$-module

$$H_D(V) := \operatorname{Ker} D / \operatorname{Ker} D \cap \operatorname{Im} D.$$

The following theorem is a consequence of the above theorem.

Theorem 4 ([28, 41]) *Let V be a $\mathfrak{g}$-module with $Z(\mathfrak{g})$ infinitesimal character χ_Λ. Suppose that an $\mathfrak{r}$-module N is contained in the Dirac cohomology $H_D(V)$ and has $Z(\mathfrak{r})$ infinitesimal character χ_λ . Then $\lambda = w\Lambda$ for some $w \in W$.*

Suppose that V_λ is a finite-dimensional representation with highest weight $\lambda \in \mathfrak{h}^*$. Kostant [40] calculated the Dirac cohomology of V_λ with respect to any equal rank quadratic subalgebra $\mathfrak{r}$ of $\mathfrak{g}$. Assume that $\mathfrak{h} \subset \mathfrak{r} \subset \mathfrak{g}$ is the Cartan subalgebra for both $\mathfrak{r}$ and $\mathfrak{g}$. Define $W(\mathfrak{g}, \mathfrak{h})^1$ to be the subset of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ defined by

$$W(\mathfrak{g}, \mathfrak{h})^1 = \{w \in W(\mathfrak{g}, \mathfrak{h}) \mid w(\rho) \text{ is } \Delta^+(\mathfrak{r}, \mathfrak{h})\text{-dominant}\}.$$

This is the same as the subset of elements $w \in W(\mathfrak{g}, \mathfrak{h})$ that map the positive Weyl $\mathfrak{g}$-chamber into the positive $\mathfrak{r}$-chamber. There is a bijection

$$W(\mathfrak{r}, \mathfrak{h}) \times W(\mathfrak{g}, \mathfrak{h})^1 \to W(\mathfrak{g}, \mathfrak{h})$$

given by $(w, \tau) \mapsto w\tau$. Kostant proved [40] that

$$H_D(V_\lambda) = \bigoplus_{w \in W(\mathfrak{g}, \mathfrak{h})^1} E_{w(\lambda+\rho)-\rho_{\mathfrak{r}}}.$$

This result has been extended to the unequal rank case by Mehdi and Zierau [49]. Dirac cohomology of a simple highest weight module of possibly infinite dimension and its relation with nilpotent Lie algebra cohomology are determined in [34].

4 Rational Cherednik Algebras

Ciubotaru has extended Dirac cohomology and Vogan's conjecture to very general setting for Drinfeld's graded Hecke algebras including symplectic reflection algebras [17] and particularly rational Cherednik algebras [12]. The case for rational Cherednik algebras is particularly interesting to us, since it has Lie algebra cohomology defined by half Dirac operators [33].

Let W be a finite complex reflection group acting on a complex vector space $\mathfrak{h}$, i.e., W is a finite group generated by the pseudo-reflections $s \in \mathcal{R}$ fixing a hyperplane $H_s \in \mathfrak{h}$. Let $\alpha_s \in \mathfrak{h}^*$ be a non-zero vector so that the W-invariant symmetric pairing $\langle\ ,\ \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$ gives $\langle y, \alpha_s\rangle = 0$ for all $y \in H_s$. Similarly, we define $\alpha_s^\vee \in \mathfrak{h}$ corresponding to the action of s on $\mathfrak{h}^*$. Set $V = \mathfrak{h} \oplus \mathfrak{h}^*$.

The *rational Cherednik algebra* $\mathbf{H}_{t,c}$ associated to $\mathfrak{h}$, W, with parameters $t \in \mathbb{C}$ and W-invariant functions $c : \mathcal{R} \to \mathbb{C}$ is defined as the quotient of $S(V) \rtimes \mathbb{C}[W]$ by the relation

$$[y, x] = t\langle y, x\rangle - \sum_{s \in \mathcal{R}} c(s) \frac{\langle y, \alpha_s\rangle \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} s$$

for all $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$.

Let $\{y_1, \ldots, y_n\}$ be a basis of $\mathfrak{h}$, and $\{x_1, \ldots, x_n\}$ be the corresponding dual basis of $\mathfrak{h}^*$. Set

$$\mathbf{h} := \sum_i (x_i y_i + y_i x_i) = 2 \sum_i x_i y_i + nt - \sum_{s \in \mathcal{R}} c(s)s \in \mathbf{H}_{t,c}^W,$$

where $\mathbf{H}_{t,c}^W$ denotes the W-invariants in $\mathbf{H}_{t,c}$. Clearly, $\mathbf{h}$ does not depend on choice of bases. Denote by

$$\Omega_{\mathbf{H}_{t,c}} := \mathbf{h} - \sum_{s\in\mathcal{R}} c(s)\frac{1+\lambda_s}{1-\lambda_s}s = 2\sum_i x_i y_i + nt - \sum_{s\in\mathcal{R}} \frac{2c(s)}{1-\lambda_s}s,$$

where $\lambda_s = \det_{\mathfrak{h}}(s) \in \mathbb{C}$. Then $\Omega_{\mathbf{H}_{t,c}}$ is in $\mathbf{H}_{t,c}^W$ and it satisfies (see [12] (4.12))

$$[\Omega_{\mathbf{H}_{t,c}}, x] = 2tx,\ [\Omega_{\mathbf{H}_{t,c}}, y] = -2ty, \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

Let $\langle\ ,\ \rangle$ be a W-invariant bilinear product on V given by $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = 0$, $\langle x_i, y_j\rangle = \delta_{ij}$. The *Clifford algebra* $C(V)$ with respect to $\langle\ ,\ \rangle$ is the tensor algebra of V subject to the relations

$$x_i x_j + x_j x_i = y_i y_j + y_j y_i = 0,\ x_i y_j + y_j x_i = -2\delta_{ij}.$$

The *spinor module* S corresponding to the Clifford algebra $C(V)$ can be realized as $S \cong \wedge^\bullet\mathfrak{h}$ as vector spaces. The $C(V)$ action on S is defined by

$$x(y_{i_1} \wedge \cdots \wedge y_{i_p}) = 2\sum_j (-1)^j \langle x, y_{i_j}\rangle y_{i_1} \wedge \cdots \wedge \widehat{y_{i_j}} \wedge \cdots \wedge y_{i_p},\quad x \in \mathfrak{h}^*;$$

$$y(y_{i_1} \wedge \cdots \wedge y_{i_p}) = y \wedge y_{i_1} \wedge \cdots \wedge y_{i_p},\quad y \in \mathfrak{h}.$$

We denote by $O(V) = O(V, \langle\ ,\ \rangle)$ the complex orthogonal group preserving the symmetric form $\langle\ ,\ \rangle$. Let $\widetilde{W}$ be the twofolds cover of W defined by the pull back of the covering map $p : \mathrm{Pin}(V) \to O(V)$ via $W \hookrightarrow O(V)$. Then one has

$$\widetilde{W} \hookrightarrow Pin(V) \hookrightarrow C(V)^\times.$$

We note that the covering map $p : \widetilde{W} \to W$ factors through

$$W < GL(\mathfrak{h}) \hookrightarrow O(V) \text{ and } W < GL(\mathfrak{h}^*) \hookrightarrow O(V).$$

There is a well-defined genuine character

$$\chi\colon \widetilde{W} \to \mathbb{C},\ \text{such that } \chi^2(\tilde{w}) = \det_{\mathfrak{h}^*}(p(\tilde{w})).$$

We have the $\widetilde{W}$-module isomorphism

$$S \cong \wedge^\bullet\mathfrak{h} \otimes \chi,$$

where $\widetilde{W}$-action on $\wedge^\bullet\mathfrak{h}$ factors through the natural action of W on $\mathfrak{h}$.

We define the half Dirac operators D_x, D_y and the Dirac operator D by

$$D_x = \sum_i x_i \otimes y_i,\ D_y = \sum_i y_i \otimes x_i \text{ and } D = D_x + D_y \ \in \mathbf{H}_{t,c} \otimes C(V).$$

Clearly, these definitions are independent of the choice of bases.

Proposition 1 (Proposition 4.9 [12]) *We have*

(i) *Let $\Delta : \mathbb{C}[\widetilde{W}] \to \mathbf{H}_{t,c} \otimes C(V)$ be the diagonal embedding $\widetilde{w} \mapsto p(\widetilde{w}) \otimes \widetilde{w}$. Then D, D_x and D_y commute with $\Delta(\mathbb{C}[\widetilde{W}])$.*
(ii) $D_x^2 = D_y^2 = 0.$
(iii) *Let $\Omega_{\widetilde{W},c} \in \mathbb{C}[\widetilde{W}]$ be the Casimir element of $\mathbb{C}[\widetilde{W}]$ defined by (2.3.12) in [12].*

Then $\Delta(\Omega_{\widetilde{W},c}) \in (\mathbf{H}_{t,c} \otimes C(V))^{\widetilde{W}}$, and

$$D^2 = \widetilde{\Omega}_{\mathbf{H}_{t,c}} - \Delta(\Omega_{\widetilde{W},c}),$$

where $\widetilde{\Omega}_{\mathbf{H}_{t,c}} = -\Omega_{\mathbf{H}_{t,c}} \otimes 1 + 1 \otimes \frac{t}{2}(\sum_i x_i y_i + n) \in (\mathbf{H}_{t,c} \otimes C(V))^{\widetilde{W}}$.
For a $\mathbf{H}_{t,c}$-module M, the action of D (and D_x and D_y) on $M \otimes S$ is given by

$$D(m \otimes s)(= D_x(m \otimes s) + D_y(m \otimes s)) = \sum_i x_i \cdot m \otimes y_i s + \sum_j y_j \cdot m \otimes x_j s.$$

The *Dirac cohomology* $H_D(M)$ of M is defined by

$$H_D(M) = \ker D/(\ker D \cap \mathrm{im} D).$$

Regarding $\mathfrak{h}$ and $\mathfrak{h}^*$ as Abelian Lie algebras, one can define the $\mathfrak{h}^*$-cohomology $H^\bullet(\mathfrak{h}^*, M)$ and $\mathfrak{h}$-homology $H_\bullet(\mathfrak{h}, M)$ as W-modules [33]. By the above identification $S = \wedge^\bullet \mathfrak{h} \otimes \chi$ and the differentials with the action of D_x and D_y on the complexes, we have $\widetilde{W}$-module isomorphisms:

$$\ker D_x/\mathrm{im}\, D_x \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi \text{ and } \ker D_y/\mathrm{im}\, D_y \cong H_\bullet(\mathfrak{h}, M) \otimes \chi.$$

A $\mathbf{H}_{t,c}$-module M is said to be $\Omega_{\mathbf{H}_{t,c}}$*-admissible* if M can be decomposed into a direct sum of generalized $\Omega_{\mathbf{H}_{t,c}}$-eigenspaces, i.e.

$$M = \bigoplus_\lambda M_\lambda,\ M_\lambda = \{m \in M |\ (\Omega_{\mathbf{H}_{t,c}} - \lambda)^n m = 0\}$$

with each generalized $\Omega_{\mathbf{H}_{t,c}}$-eigenspace M_λ being finite-dimensional. Let M be a $\mathbf{H}_{t,c}$-module that is $\Omega_{\mathbf{H}_{t,c}}$-admissible. Then the Dirac cohomology $H_D(M)$ is a finite-dimensional $\widetilde{W}$-module (see Lemma 3.13 of [12]).

Etingof and Stoica [18] define and study unitary $\mathbf{H}_{t,c}$-modules with respect to a star operation $*$. Let M be such a unitary module. It follows that we have on $M \otimes S$

$$D_x^* = -D_y, D_y^* = -D_x \text{ and } D^* = -D.$$

The following theorem is the analogue of the Hodge decomposition theorem for Dirac cohomology of unitary representations of a reductive Lie group of Hermitian symmetric type [30]. We do not assume that M is $\Omega_{\mathbf{H}_{t,c}}$-admissible in the following theorem.

Theorem 5 ([33]) *Let M be a unitary $\mathbf{H}_{t,c}$-module. Then*

(i) $H_D(M) = \ker D = \ker D^2$.
(ii) $M \otimes S = \ker D \oplus \operatorname{im} D_x \oplus \operatorname{im} D_y$.
(iii) $\ker D_x = \ker D \oplus \operatorname{im} D_x$, $\ker D_y = \ker D \oplus \operatorname{im} D_y$. *Consequently,*

$$H_D(M) \cong H^\bullet(\mathfrak{h}^*, M) \otimes \chi \cong H_\bullet(\mathfrak{h}, M) \otimes \chi.$$

We note that for $t \neq 0$ the center of $\mathbf{H}_{t,c}$ consists of scalar $\mathbb{C}$ only. One can however consider a larger commutative subalgebra $\mathcal{B} \subset \mathbf{H}_{t,c} \otimes C(V)$ (see sect. 5.5 [12]) for the extension of Vogan's conjecture in this case. We also refer to Theorem 5.8 [12] for the case $\mathbf{H}_{t,c}$ with $t = 0$. An extension of Vogan's conjecture to more general setting of Drinfeld's Hecke algebras is proved in Theorem 3.5 and Theorem 3.14 [12].

An analogue of the Casselman-Osborne Lemma [10] is proved by generalizing Vogan's conjecture to the setting of half Dirac operators D_x and D_y [33]. This is based on the ideas for the similar results for reductive Lie algebras in [29].

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Superconformal Vertex Algebras and Jacobi Forms

Jethro van Ekeren

Abstract We discuss the appearance of Jacobi automorphic forms in the theory of superconformal vertex algebras, explaining it by way of supercurves and formal geometry. We touch on some related topics such as Ramanujan's differential equations for Eisenstein series.

Keywords Jacobi forms • Superconformal algebras • Vertex algebras

1 Introduction

This paper is about the link between automorphic forms and infinite dimensional algebras. It is primarily an exposition of joint work [13] of the author with R. Heluani, which specifically relates certain automorphic forms on the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ called Jacobi forms to vertex algebras equipped with an $N = 2$ structure.

In the broad sense this theory goes back to Kac and Peterson [16], who used the Weyl-Kac character formula to express characters of integrable modules over affine Kac-Moody algebras in terms of theta functions. Another perspective was adopted by Zhu [27], who proved that characters of suitable conformal vertex algebras are classical modular forms on the group $SL_2(\mathbb{Z})$. Zhu proceeded by analysing $\mathcal{D}$-modules associated with the vertex algebra over families of elliptic curves, establishing in particular a certain $SL_2(\mathbb{Z})$-equivariance. We study vertex algebras which admit the richer structure of $N = 2$ superconformal symmetry. These give rise to $\mathcal{D}$-modules over families of elliptic supercurves, and we show these to be equivariant under a certain $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$-action.

The $N = 2$ superconformal symmetry algebra, which arose in theoretical physics, is the Lie superalgebra $\widehat{W}^{1|1}$ with explicit basis and relations as in Sect. 5. There is an associated family $L(\widehat{W}^{1|1})_c$ of simple vertex algebras (see Example 1) depending on an auxiliary parameter c called the central charge. For generic c the representation theory of $L(\widehat{W}^{1|1})_c$ is rather complicated. However

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for $c(u) = 3 - 6/u$, where $u \in \mathbb{Z}_{\geq 2}$, it turns out that $L(\widehat{W}^{1|1})_{c(u)}$ has precisely $u(u-1)/2$ irreducible modules $L_u(j,k)$ which are parameterised by the set of pairs $(j,k) \in \mathbb{Z}^2$ where $j \geq 0$, $k \geq 1$ and $j + k < u$. There is an explicit formula for the graded superdimensions of these modules too [17, 21], viz.

$$\mathrm{STr}_{L_u(j,k)}\, q^{L_0} y^{J_0} = q^{\frac{jk}{u}} y^{\frac{j-k+1}{u}} P^{(u)}_{j,k} / P^{(2)}_{1/2,1/2}, \tag{1}$$

where

$$P^{(u)}_{j,k} = \prod_{n=1}^{\infty} \frac{(1-q^{u(n-1)+j+k})(1-q^{un-j-k})(1-q^{un})^2}{(1-q^{un-j}y)(1-q^{u(n-1)+j}y^{-1})(1-q^{un-k}y^{-1})(1-q^{u(n-1)+k}y)}.$$

Now the normalised functions $y^{c(u)/6}\, \mathrm{STr}_{L_u(j,k)}\, q^{L_0} y^{J_0}$ span a vector space which turns out to be invariant under an action of the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ (specifically the weight 0 index $c/6$ Jacobi action (11)). In other words the span of the normalised graded superdimensions is a vector valued Jacobi form. The question is to explain this fact conceptually.

In [13] we showed that the picture outlined above is an instance a general phenomenon. More precisely, for any vertex algebra equipped with an '$N = 2$ superconformal structure' (of which $L(\widehat{W}^{1|1})_c$ above is an example) the normalised graded superdimensions satisfy $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$-invariant differential equations. The key observations are the following:

1. Jacobi forms are essentially sections of vector bundles over the moduli space of pairs $(E, \mathcal{L})$ where E is an elliptic curve and $\mathcal{L}$ a holomorphic line bundle over E.
2. Such pairs can be reinterpreted as certain special $1|1$-dimensional supercurves.
3. A vertex algebra V equipped with a suitable $N = 2$ superconformal structure 'localises' nicely to give a $\mathcal{D}$-module $\mathcal{C}$ on the moduli space of such supercurves.
4. If V is well-behaved, the normalised graded superdimensions of V-modules (generalising the left of (1)) converge in the analytic topology and yield horizontal sections of $\mathcal{C}$.

The issue of convergence is technical. In the appendix to [13] we establish the convergence subject to the well known (to vertex algebraists) condition of C_2-cofiniteness. The proof involves analysis of the coefficients of the differential equations corresponding to $\mathcal{C}$. The key points are to show that these coefficients lie in a certain ring of quasi-Jacobi forms, and to establish that this ring is Noetherian.

For careful statements of results and complete proofs we refer the reader to [13]. In this note we have taken the opportunity to adopt a more discursive style. In particular we digress to discuss an interpretation (Sect. 7) of Ramanujan's differential equations as an expression of the 'Virasoro uniformisation' of the moduli space of elliptic curves.

2 Notation

Aside from standard symbols such as $\mathbb{C}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\partial_z = \frac{\partial}{\partial z}$, etc., we shall use the following notation without further comment: $\mathcal{O} = \mathbb{C}[[z]]$ the ring of formal power series in one variable, $\mathfrak{m} = z\mathbb{C}[[z]]$ its maximal ideal, and $\mathcal{K} = \mathbb{C}((z))$ the ring of Laurent series. The supercommutative algebra $\mathcal{O}^{1|1}$ is by definition $\mathcal{O} \otimes \bigwedge[\theta]$, i.e., is obtained by adjoining to $\mathcal{O}$ a single odd variable θ satisfying $\theta^2 = 0$. Similarly we have $\mathfrak{m}^{1|1} = \mathfrak{m} \otimes \bigwedge[\theta]$ and $\mathcal{K}^{1|1} = \mathcal{K} \otimes \bigwedge[\theta]$. The structure sheaf, tangent sheaf, cotangent sheaf, and sheaf of differential operators of a (super)scheme X are denoted $\mathcal{O}_X$, Θ_X, Ω_X, and $\mathcal{D}_X$, respectively.

3 Superschemes and Elliptic Supercurves

The picture to keep in mind of a complex supermanifold (of dimension $m|n$) is that of a space on which the Taylor expansion of a function in terms of local coordinates $z_1, \ldots, z_m, \theta_1, \ldots, \theta_n$ lies in the supercommutative ring $\mathbb{C}[[z_i]] \otimes \bigwedge[\theta_j]$. We refer the reader to [20] for background on superalgebra and supergeometry. A superscheme is formally defined [20, Chapt. 4] to be a topological space X_{top} together with a sheaf $\mathcal{O}_X$ of supercommutative local rings such that the even part $(X_{\text{top}}, \mathcal{O}_{X,0})$ is a scheme. Morphisms are required to be $\mathbb{Z}_2$-graded. The bulk X_{rd} of a superscheme X is the scheme $(X_{\text{top}}, \mathcal{O}_X/\mathcal{J})$ where $\mathcal{J} = \mathcal{O}_{X,1} + \mathcal{O}_{X,1}^2$. A (complex) supercurve is a smooth superscheme over Spec $\mathbb{C}$ of dimension $1|n$. In this article we shall concern ourselves with $1|1$-dimensional complex supercurves, and we shall generally work in the analytic topology.

Let X_0 be a smooth curve and $\mathcal{L}$ a holomorphic line bundle over X_0. We may construct a $1|1$-dimensional supercurve X from this data by putting

$$\mathcal{O}_X = \bigwedge \mathcal{L}[-1] = \mathcal{O}_{X_0} \oplus \mathcal{L},$$

with $\mathbb{Z}/2\mathbb{Z}$-grading induced by cohomological degree.

In fact any even family of $1|1$-dimensional complex supercurves is of the above form. Indeed, for a $1|1$-dimensional supercurve defined over a base superscheme Spec R, transformations between coordinate charts take the general form

$$\begin{aligned} z' &= f_{11}(z) + f_{12}(z)\theta, \\ \theta' &= f_{21}(z) + f_{22}(z)\theta, \end{aligned} \tag{2}$$

where f_{11}, f_{22} are power series whose coefficients are even elements of R, and f_{12}, f_{21} are power series whose coefficients are odd elements of R. If the base ring R contains no odd elements then f_{12} and f_{21} vanish, (2) is linear in θ and comprises the Čech cocycle description of a line bundle $\mathcal{L}$, and the supercurve is consequently of the form $\bigwedge \mathcal{L}[-1]$.

Recall the set $\mathrm{Pic}(X)$ of isomorphism classes of holomorphic line bundles over a smooth curve X, and its subset $\mathrm{Pic}_0(X)$ of line bundles of degree 0. As is well known [10, Appendix B.5] there is a natural bijection $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$, and the exponential exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

yields the following morphisms in cohomology

$$H^1(X, \underline{\mathbb{Z}}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \underline{\mathbb{Z}}).$$

The last map here assigns a line bundle its degree, and the kernel $\mathrm{Pic}_0(X)$ is identified with the quotient

$$H^1(X, \mathcal{O}_X)/H^1(X, \underline{\mathbb{Z}})$$

which is a complex torus of dimension g, where g is the genus of X.

An elliptic curve is a smooth complex curve of genus 1, together with a marked point. We shall define an elliptic supercurve to be a supercurve X of dimension $1|1$ whose bulk X_{rd} has genus 1, together with a marked point.

Let $\mathcal{H}$ denote the complex upper half plane, and let z be the standard coordinate on $\mathbb{C}$ which we fix once and for all. The trivial family $\mathcal{H} \times \mathbb{C} \to \mathcal{H}$ carries the action $(m, n) : (z, \tau) \mapsto (z + m\tau + n, \tau)$ of $\mathbb{Z}^2$ and the quotient together with marked point $z = 0$ is a family of elliptic curves, which we denote $E \to \mathcal{H}$.

Quite generally [23, Appendix to §2], for X a topological space with a free discontinuous action of a discrete group G, and $\mathcal{F}$ a sheaf on the quotient space X/G (and with $\pi : X \to X/G$ the quotient), there is a natural map

$$H^\bullet(G, \Gamma(X, \pi^*\mathcal{F})) \to H^\bullet(X/G, \mathcal{F}),$$

from group cohomology to sheaf cohomology. In case X is a fibre $\mathbb{C}_\tau$ of the trivial family above, this map is an isomorphism. An element $\alpha \in \mathbb{C}$ defines a group 1-cocycle $c_\alpha : \mathbb{Z}^2 \to \Gamma(\mathbb{C}_\tau, \mathcal{O}^*)$ by $(m, n) \mapsto e^{2\pi i m\alpha}$. We denote by $\mathcal{L}_\alpha \in \mathrm{Pic}_0(E_\tau)$ the corresponding line bundle on E_τ. Let $S^\circ = \mathcal{H} \times \mathbb{C}$. We denote by $E^\circ \to S^\circ$ the family whose fibre over (τ, α) is the elliptic supercurve corresponding to $(E_\tau, \mathcal{L}_\alpha)$.

The group $SL_2(\mathbb{Z})$ acts on $E \to \mathcal{H}$ in such a way as to identify fibres isomorphic as elliptic curves. We now have the following $1|1$-dimensional analogue.

Proposition 1 *The formulas*

$$A : (t, \zeta, \tau, \alpha) \mapsto \left(\frac{t}{c\tau + d}, e^{-2\pi i t \frac{c\alpha}{c\tau+d}} \zeta, \frac{a\tau + b}{c\tau + d}, \frac{\alpha}{c\tau + d} \right)$$

$$(m, n) : (t, \zeta, \tau, \alpha) \mapsto (t, e^{2\pi i m t} \zeta, \tau, \alpha + m\tau + n),$$

where $A \in SL_2(\mathbb{Z})$ and $m, n \in \mathbb{Z}$, extend to a left action on $E^\circ \to S^\circ$ of the semidirect product group

$$SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \quad \textit{where} \quad (A, x) \cdot (A', x') = (AA', xA' + x').$$

The restriction of the action of $g \in SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ to the fibre $E_{(\tau,\alpha)}$ is an isomorphism $E_{(\tau,\alpha)} \cong E_{g\cdot(\tau,\alpha)}$ of supercurves.

Following the comments above, every elliptic supercurve over $\mathbb{C}$ appears as a fibre of $E^\circ \to S^\circ$. However $E^\circ \to S^\circ$ is not a universal family in the sense that it does not 'see' families over odd base schemes.

We denote by $\mathbb{A}^{1|1}$ the superscheme whose set of R-points is Spec $R[z, \theta]$. In fact we distort convention a little by fixing a choice z, θ of coordinates, in particular our $\mathbb{A}^{1|1}$ has a distinguished origin, and we denote by $(\mathbb{A}^{1|1})^\times$ the subscheme with origin removed. We then have the algebraic supergroup $GL(1|1)$ of linear automorphisms acting on $(\mathbb{A}^{1|1})^\times$. The trivial family $(\mathbb{A}^{1|1})^\times \times GL(1|1) \to GL(1|1)$ carries the action $n : (x, \mathbf{q}) \mapsto (\mathbf{q}^n x, \mathbf{q})$ of $\mathbb{Z}$. We restrict to the subscheme $S^\bullet \subset GL(1|1)$ consisting of automorphisms with nonzero even reduction, then the quotient by $\mathbb{Z}$ is a family $E^\bullet \to S^\bullet$ of elliptic supercurves. The distinguished point is $(z, \theta) = (1, 0)$.

We introduce the morphism sexp : $E^\circ \to E^\bullet(\mathbb{C})$ of $\mathbb{C}$-schemes defined by

$$(t, \zeta, \tau, \alpha) \mapsto \left(e^{2\pi i t}, e^{2\pi i t}\zeta, \left(\begin{smallmatrix} q & 0 \\ 0 & qy \end{smallmatrix}\right)\right). \tag{3}$$

The notation $q = e^{2\pi i \tau}$, $y = e^{2\pi i \alpha}$ used here will be in force throughout the paper.

Remark 1 There is a quite distinct notion of supercurve, which we recall here for the sake of avoiding confusion. A SUSY$_n$ curve [19, Chapt. 2, Definition 1.10] consists of a $1|n$-dimensional supercurve X together with the extra data of a rank $0|n$ subbundle $T \subset \Theta_X$ such that the alternating form

$$\varphi : T \otimes T \xrightarrow{[\cdot,\cdot]} \Theta_X \twoheadrightarrow \Theta_X/T$$

is nondegenerate and split. There is a forgetful functor from the category of SUSY$_n$ curves to that of $1|n$-dimensional supercurves. On the other hand, there turns out to be a nontrivial *equivalence* (due to Deligne [19, pp. 47]) between the category of all $1|1$-dimensional supercurves, and the category of 'orientable' SUSY$_2$ curves. We describe the correspondence briefly.

Let X, T be a SUSY$_2$ curve. Locally there is a splitting of T as a direct sum of rank $0|1$-subbundles, each isotropic with respect to φ. If this can be extended to a global splitting, then we say X, T is orientable. Suppose this is the case, and let $T_1 \subset T$ be an isotropic subbundle. Set $\overline{X}$ to be the superscheme $(X_{\text{top}}, \mathcal{O}_X/T_1 \cdot \mathcal{O}_X)$. Then $\overline{X}$ is a $1|1$-dimensional supermanifold, and X can be recovered uniquely from $\overline{X}$.

Much of the theory discussed below extends straightforwardly to $1|n$-dimensional supercurves, and to SUSY$_n$ curves.

4 The Bundle of Coordinates

In this section and the two subsequent ones we outline the basics of 'formal geometry'. This theory, which goes back to [9], provides a bridge between representation theory of infinite dimensional algebras and geometry of algebraic varieties. The book [8] contains a good introduction for the case of curves. We focus on the case of $1|1$-dimensional supercurves.

The basic object of formal geometry is the 'set of all coordinates' on a variety X, denoted here by Coord_X. It may be defined precisely either as the subscheme of the jet scheme [7] consisting of jets with nonzero differential, or as the fibre bundle with fibre at $x \in X$ the set of choices of generator of $\mathfrak{m}_x$ (where $\mathfrak{m}_x$ is the unique maximal ideal of the local ring $\mathcal{O}_x$ at x).

For the case of X a supercurve of dimension $1|1$ we have the noncanonical isomorphism $\mathcal{O}_x \cong \mathcal{O}^{1|1}$ at each point $x \in X$. Each fibre therefore carries a simply transitive action of the supergroup $\mathrm{Aut}\,\mathcal{O}^{1|1}$ by changes of coordinates, in other words Coord_X is a principal $\mathrm{Aut}\,\mathcal{O}^{1|1}$-bundle. This supergroup consists of transformations

$$
\begin{aligned}
z &\mapsto a_{0,1}\theta + a_{1,0}z + a_{1,1}z\theta + a_{2,0}z^2 + a_{2,1}z^2\theta + \cdots,\\
\theta &\mapsto b_{0,1}\theta + b_{1,0}z + b_{1,1}z\theta + b_{2,0}z^2 + b_{2,1}z^2\theta + \cdots
\end{aligned}
$$

where $\left(\begin{smallmatrix} a_{0,1} & a_{1,0} \\ b_{0,1} & b_{1,0} \end{smallmatrix}\right) \in GL(1|1)$. As such the corresponding Lie superalgebra $\mathrm{Der}_0\,\mathcal{O}^{1|1}$ of derivations preserving $\mathfrak{m}^{1|1}$ has basis

$$
\begin{aligned}
L_n &= -z^{n+1}\partial_z - (n+1)z^n\theta\partial_\theta, & J_n &= -z^n\theta\partial_\theta,\\
Q_n &= -z^{n+1}\partial_\theta, & H_n &= z^n\theta\partial_\theta,
\end{aligned}
\tag{4}
$$

where $n \in \mathbb{Z}_+$. The Lie bracket is the usual bracket of vector fields [11].

5 Superconformal Algebras and SUSY Vertex Algebras

The Lie superalgebra $\mathrm{Der}_0\,\mathcal{O}^{1|1}$ embeds naturally into $\mathrm{Der}\,\mathcal{K}^{1|1}$ by extending the basis (4) to $n \in \mathbb{Z}$. This algebra admits a central extension

$$
0 \to \mathbb{C}C \to \widehat{W}^{1|1} \to \mathrm{Der}\,\mathcal{K}^{1|1} \to 0,
$$

which splits over $\mathrm{Der}_0\, \mathcal{O}^{1|1}$. Explicit relations in $\widehat{W}^{1|1}$ are as follows [12, (2.5.1c)]:

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [L_m, J_n] = -nJ_{m+n} + \delta_{m,-n}\frac{m^2+m}{6}C,$$
$$[L_m, H_n] = -nH_{m+n}, \qquad [L_m, Q_n] = (m-n)Q_{m+n},$$
$$[J_m, J_n] = \delta_{m,-n}\frac{m}{3}C, \qquad [J_m, Q_n] = Q_{m+n},$$
$$[J_m, H_n] = -H_{m+n}, \qquad [H_m, Q_n] = L_{m+n} - mJ_{m+n} + \delta_{m,-n}\frac{m^2-m}{6}C.$$

Remark 2 In the 1|0-dimensional setting we have analogously the Virasoro extension

$$0 \to \mathbb{C}C \to \mathrm{Vir} \to \mathrm{Der}\,\mathcal{K} \to 0,$$

which is conventionally given by

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}C.$$

There is an embedding $\mathrm{Vir} \hookrightarrow \widehat{W}^{1|1}$ via the map $L_n \mapsto L_n - \frac{1}{2}(n+1)J_n$ and $C \mapsto C$.

Though we will not be using vertex algebras until Sect. 8, this is a convenient place to give their definition. To avoid clutter we present only the definition of '$N_W = 1$ SUSY vertex algebra', which is the relevant variant for us. See [15] and [12] for the general picture.

Definition 1 A SUSY vertex algebra is a vector superspace V, a vector $|0\rangle \in V$, linear operators $S, T : V \to V$, and an even linear map $V \otimes V \to V \widehat{\otimes} \mathcal{K}^{1|1}$ which is denoted

$$a \otimes b \mapsto Y(a, Z)b = Y(a, z, \theta)b.$$

These structures are to satisfy the following axioms.

1. $Y(|0\rangle, Z) = \mathrm{Id}_V$, and $Y(a, Z)|0\rangle = a \bmod (V\widehat{\otimes}\mathfrak{m}^{1|1})$.
2. The series $Y(a, Z)Y(b, W)c$, $(-1)^{p(a)p(b)}Y(b, W)Y(a, Z)c$ and $Y(Y(a, Z-W)b, W)c$ are expansions of a single element of $V\widehat{\otimes}\mathcal{K}^{1|1} \otimes_{\mathbb{C}[z,w]} \mathbb{C}[(z-w)^{-1}]$.
3. $[T, Y(a, Z)] = \partial_z Y(a, Z)$ and $[S, Y(a, Z)] = \partial_\theta Y(a, Z)$.

The notion of conformal structure (i.e., compatible Vir-action) on a vertex algebra permits connection with the geometry of algebraic curves via formal geometry [8, Chapt. 6]. Similarly important in the context of 1|1-dimensional supercurves is the notion of superconformal structure on a SUSY vertex algebra [11].

Definition 2 ([12]) A superconformal structure on the SUSY vertex algebra V is a pair of vectors j and h (even and odd respectively) such that the following associations furnish V with a $\widehat{W}^{1|1}$-module structure:

$$Y(j,Z) = J(z) - \theta Q(z), \quad Y(h,Z) = H(z) + \theta[L(z) + \partial_z J(z)],$$

and

$$J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \qquad Q(z) = \sum_{n\in\mathbb{Z}} Q_n z^{-n-2},$$
$$H(z) = \sum_{n\in\mathbb{Z}} H_n z^{-n-1}, \qquad L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

Furthermore it is required that $T = L_{-1}$, $S = Q_{-1}$, and that V be graded by finite dimensional eigenspaces of L_0, J_0, with integral eigenvalues bounded below. Let $b \in V$ such that $L_0 b = \Delta b$. We write $o(b) \in \operatorname{End} V$ for the $z^{-\Delta}\theta$ coefficient of $Y(b,z,\theta)$.
There is a natural notion of module over a SUSY vertex algebra. In the superconformal case we shall include in the definition the L_0, J_0-grading conditions of Definition 2.

Example 1 Let $M(h,m,c)$ denote the Verma module $U(\widehat{W}^{1|1}) \otimes_{U(\widehat{W}^{1|1}_+)} \mathbb{C}v$, where the action on v is by $C = c$, $L_0 = h$, $J_0 = m$, $Q_0 = 0$, and all positive modes acting by 0. Let $L(h,m,c)$ denote the unique irreducible quotient of $M(h,m,c)$. Then $M(0,0,c)$ and $L(\widehat{W}^{1|1})_c = L(0,0,c)$ have unique superconformal vertex algebra structures such that $v = |0\rangle, j = J_{-1}v, h = H_{-1}v$.

6 Harish-Chandra Localisation

Let K be a Lie group, Z a principal K-bundle over a smooth manifold S, and V a left K-module. The familiar associated bundle construction produces a vector bundle $\mathbb{V} = Z \times_K V$ over S (recall by definition $Z \times_K V$ is $Z \times V$ modulo the relation $(zg, v) = (z, gv)$). If $\dim S = n$ then S carries a canonical $GL(n)$-bundle, namely the frame bundle whose fibre at $s \in S$ if the set of all bases of the tangent space $T_s S$. Associated with the defining $GL(n)$-module ${R_{\mathrm{red}}}^n$ is the tangent bundle Θ_S, and with its dual the cotangent bundle Ω_S.

The functor of Harish-Chandra localisation extends the associated bundle construction, enabling the construction of vector bundles with connection (more properly $\mathcal{D}$-modules) from K-modules with the action of an additional Lie algebra. See [8, Chapt. 17] and [4, Sect. 1.2] for the general theory.

Definition 3 A Harish-Chandra pair $(\mathfrak{g}, K)$ consists of a Lie algebra $\mathfrak{g}$, a Lie group K, an action Ad of K on $\mathfrak{g}$, and a Lie algebra embedding $\operatorname{Lie} K \hookrightarrow \mathfrak{g}$ compatible

with Ad. A $(\mathfrak{g}, K)$-module is a vector space with compatible left $\mathfrak{g}$- and K-module structures. A $(\mathfrak{g}, K)$-structure on a space S is a principal K-bundle $Z \to S$ together with a transitive action $\mathfrak{g} \to \Theta_Z$ satisfying certain compatibilities.
Let $Z \to S$ be a $(\mathfrak{g}, K)$-structure, and V a $(\mathfrak{g}, K)$-module. The fibre $\mathbb{V}_s$ of the associated bundle $\mathbb{V} = Z \times_K V$ over the point $s \in S$ carries an action of the Lie algebra $\mathfrak{g}_s = Z_s \times_K \mathfrak{g}$. Inside $\mathfrak{g}_s$ we have the pointwise stabiliser $\mathfrak{g}_s^0$ of Z_s. We denote by $\Delta(V)$ the sheaf whose fibre over s is the space of coinvariants $\mathbb{V}_s/\mathfrak{g}_s^0 \cdot \mathbb{V}_s$. The $\mathfrak{g}$-action on V translates into a flat connection (more precisely a left $\mathcal{D}_S$-module structure) on $\Delta(V)$.

Now let $\widehat{\mathfrak{g}}$ be a central extension of $\mathfrak{g}$ split over Lie$K \subset \mathfrak{g}$. If V is a $\widehat{\mathfrak{g}}$-module then a variation on the construction above yields $\Delta(V)$ a twisted $\mathcal{D}_S$-module. That is to say, there is a certain sheaf $\mathcal{F}$ on S (which depends on the central extension) such that $\Delta(V)$ is a module over $\mathcal{D}_{\mathcal{F}}$ the sheaf of differential operators on $\mathcal{F}$.

The Harish-Chandra pairs of particular importance in our context are (Vir, Aut $\mathcal{O}$) and $(\widehat{W}^{1|1}, \mathrm{Aut}\,\mathcal{O}^{1|1})$. Their relevance stems from the fact that moduli spaces of curves and $1|1$-dimensional supercurves carry natural $(\mathfrak{g}, K)$-structures for these respective pairs [2, 5] (see also [8, Chapt. 17] for an overview). This fact frequently goes by the name 'Virasoro Uniformisation'.

Let $\widehat{\mathcal{M}}$ denote the moduli space of triples (X, x, t) consisting of a smooth algebraic curve X (of genus $g \geqslant 1$), a point $x \in X$, and a local coordinate $t \in \mathrm{Coord}_{X,x}$, and let $\mathcal{M}$ denote the moduli space of pairs (X, x). Let $\pi : \widehat{X} \to \widehat{\mathcal{M}}$ be the universal curve, and $Y \subset \widehat{X}$ the section of points $(X, x, t; x)$.

Let $\pi : X \to S$ be a morphism of schemes. In the sequence

$$0 \to \Theta_{X/S} \to \Theta_X \to \pi^*\Theta_S \to 0$$

(which defines the relative tangent bundle $\Theta_{X/S}$ in fact), we denote by Θ_π the preimage of $\pi^{-1}\Theta_S$ in Θ_X. Intuitively Θ_π consists of vector fields on X of the shape $f(s)\partial_s + g(s, x)\partial_x$.

The following theorem can be viewed as a refinement of the Kodaira-Spencer isomorphism.

Theorem 1 ([5, Lemma 4.1.1]) *There is a canonical* (Der $\mathcal{K}$, Aut $\mathcal{O}$)*-structure on* $\widehat{\mathcal{M}} \to \mathcal{M}$ *induced by the isomorphism*

$$\Theta_\pi(\widehat{X} \backslash Y) \to \mathcal{O}_{\widehat{\mathcal{M}}} \otimes \mathrm{Der}\,\mathcal{K}$$

of $\mathcal{O}_{\widehat{\mathcal{M}}}$*-modules, which sends a vector field to the expansion of its vertical component at* x *in powers of* t.

The $1|1$-dimensional analogue (along with other cases) is [25, Theorem 6.1]. It follows that any (Vir, Aut $\mathcal{O}$)-module gives rise to a twisted $\mathcal{D}$-module on $\mathcal{M}$ (or on any family of smooth curves). Similarly any $(\widehat{W}^{1|1}, \mathrm{Aut}\,\mathcal{O}^{1|1})$-module gives rise to a twisted $\mathcal{D}$-module on any family of smooth $1|1$-dimensional supercurves.

7 Elliptic Curves and Ramanujan Differential Equations

It is instructive to flesh out this construction a little for the case of elliptic curves. Let $E \to \mathcal{H}$ be the family of elliptic curves introduced in Sect. 3, and let V be a $(\mathrm{Der}\,\mathcal{K}, \mathrm{Aut}\,\mathcal{O})$-module, so that we obtain a $\mathcal{D}$-module $\Delta(V)$ over $\mathcal{H}$.

Recall the Weierstrass (quasi)elliptic functions

$$\overline{\zeta}(z) = \frac{-1}{2\pi i}\left[z^{-1} - \sum_{k\in\mathbb{Z}_{>0}} z^{2k-1}G_{2k}\right] \quad \text{and} \quad \wp(z,\tau) = z^{-2} + \sum_{k\in\mathbb{Z}_{>0}} (2k-1)z^{2k-2}G_{2k}$$

where the Eisenstein series G_{2k} is defined by

$$G_{2k} = \frac{(-1)^{k+1}B_{2k}}{(2k)!}(2\pi)^{2k}E_{2k}, \quad \text{where} \quad E_{2k} = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n},$$

and $x/(e^x - 1) = \sum_{n=1}^{\infty} B_n x^n/n!$ defines the Bernoulli numbers B_n [1]. The nonstandard normalisation $\overline{\zeta}$ is chosen so that

$$\overline{\zeta}(z+1) = \overline{\zeta}(z) \quad \text{and} \quad \overline{\zeta}(z+\tau) = \overline{\zeta}(z) + 1. \tag{5}$$

We now have:

Lemma 1 *Flat sections s of $\Delta(V)$ satisfy the differential equation*

$$\frac{\partial s}{\partial \tau} + \left(\mathrm{Res}_t\, \overline{\zeta}(t)L(t)dt\right)\cdot s = 0.$$

Proof The proof is an exercise in unwinding the definitions of the previous section. All that needs to be checked is that the vector field $\partial_\tau + \overline{\zeta}(z)\partial_z$ is well defined on E (being *a priori* well defined only on the universal cover, since $\overline{\zeta}$ is not elliptic).

Under the transformation $(z', \tau') = (z+\tau, \tau)$ we have $\partial_{\tau'} = \partial_\tau - \partial_z$ and $\partial_{z'} = \partial_z$. This together with (5) shows that $\partial_\tau + \overline{\zeta}(z)\partial_z$ is well defined. The same check on the transformation $(z, \tau) \mapsto (z+1, \tau)$ is immediate.
Another incarnation of Lemma 1 is the following PDE satisfied by the Weierstrass function $\wp$.

Proposition 2 *The Weierstrass functions satisfy*

$$\frac{\partial}{\partial \tau}\wp + \overline{\zeta}\frac{\partial}{\partial z}\wp = \frac{1}{2\pi i}(2\wp^2 - 2G_2\wp - 20G_4). \tag{6}$$

Proof Differentiating

$$\wp(z+\tau,\tau) - \wp(z,\tau) = 0$$

with respect to τ yields

$$\dot{\wp}(z+\tau,\tau) - \dot{\wp}(z,\tau) = -\wp'(z+\tau,\tau)$$

(where $\wp'$ and $\dot{\wp}$ are the derivatives with respect to the first and second entries). Similarly $\dot{\wp}(z+1,\tau) - \dot{\wp}(z,\tau) = 0$. It is clear then that $\dot{\wp} + \overline{\zeta}\wp'$ is an elliptic function with pole of order 4 at $z = 0$, hence a polynomial in $\wp$. Comparing leading coefficients yields the result.

Equating coefficients of (6) yields an infinite list of differential equations on Eisenstein series. The first few of these

$$q\partial E_2/\partial q = (E_2^2 - E_4)/12,$$
$$q\partial E_4/\partial q = (E_2 E_4 - E_6)/3,$$
$$q\partial E_6/\partial q = (E_2 E_6 - E_4^2)/2,$$

were discovered by Ramanujan [24] (see also [26] and [22]).

8 Conformal Blocks and Trace Functions

A conformal vertex algebra carries a (Vir, Aut $\mathcal{O}$)-module structure, so the machinery of Sect. 6 can be applied. Let V be a conformal vertex algebra and X a smooth algebraic curve. Let $\mathcal{A} = \mathbb{V} \otimes \Omega_X$ be the dual of $\mathbb{V} = \mathrm{Coord}_X \times_{\mathrm{Aut}\,\mathcal{O}} V$. Then $\mathcal{A}$ acquires the following extra structure deriving from the vertex operation: an action μ of the space of sections $\Gamma(D_x^\times, \mathcal{A})$ on the fibre $\mathcal{A}_x$, for each $x \in X$ (here $D_x^\times = \mathrm{Spec}\,\mathcal{K}_x$ is the punctured infinitesimal disc at x). In fact this structure makes $\mathcal{A}$ into a chiral algebra over X [3] [8, Theorem 19.3.3]. Underlying this construction is the following formula due to Huang [14]

$$R(\rho)Y(a,z)R(\rho)^{-1} = Y(R(\rho_z)a, \rho(z)), \tag{7}$$

valid for all $\rho \in \mathrm{Aut}\,\mathcal{O}$. Here by definition $\rho_z \in \mathrm{Aut}\,\mathcal{O}$ is the automorphism defined by $\rho_z(t) = \rho(z+t) - \rho(t)$, and $R(\rho)$ is the action of ρ on the conformal vertex algebra V (obtained by exponentiating $\mathrm{Der}_0\,\mathcal{O} \subset \mathrm{Vir}$).

Applying the Harish-Chandra formalism to V and a family X, x of pointed curves over base S yields the $\mathcal{D}_S$-module $\Delta(V)$, with fibres

$$\frac{\mathcal{A}_x}{\Gamma(X \backslash x, \mathcal{A}) \cdot \mathcal{A}_x}.$$

The dual of this vector space is called the space of conformal blocks associated with X, x, V, and is denoted $C(X, x, V)$.

A superconformal SUSY vertex algebra carries a $(\widehat{W}^{1|1}, \operatorname{Aut} O^{1|1})$-module structure, and can therefore be similarly localised on $1|1$-dimensional supercurves. These sheaves are again chiral algebras, using [11, Theorem 3.4] which is a general SUSY analogue of (7) above.

The theorems of this section and the next concern construction of horizontal sections of the conformal blocks bundle C for elliptic supercurves, and the modular properties of these sections. They are super-analogues of fundamental results of Zhu [27].

Theorem 2 ([13, Proposition 7.10]) *Let V be a superconformal vertex algebra and M its module. Let $X = (\mathbb{A}^{1|1})^\times/\mathbf{q}$ be an elliptic supercurve with marked point $x = (z, \theta) = (1, 0)$ as in Sect. 3. Then the element of V^* defined by*

$$\varphi_M : b \mapsto \operatorname{STr}_M o(b) R(\mathbf{q})$$

is a conformal block, i.e., $\varphi_M \in C(X, x, V)$.

Proof (Sketch) Let a, b be sections of a chiral algebra $\mathcal{A}, \mu$ over $(\mathbb{A}^{1|1})^\times$. Huang's formula (7) can be written schematically as

$$\rho\mu(a)\rho^{-1} = \mu(\rho \cdot a).$$

Item (2) of Definition 1 may be reformulated [12, Theorem 3.3.17] as the following relation (again expressed only schematically here):

$$\mu(a)\mu(b) - \mu(b)\mu(a) = \mu(\mu(a)b).$$

Let $\mathbf{q} \in GL(1|1)$, and suppose a is $\mathbf{q}$-equivariant. Then (super)symmetry of the (super)trace, equivariance of a, and the relations above combine to yield

$$\begin{aligned} \operatorname{tr} \mu(\mu(a)b)\mathbf{q} &= \operatorname{tr} \left[\mu(a)\mu(b) - \mu(b)\mu(a)\right] \mathbf{q} \\ &= \operatorname{tr} \left[\mu(a)\mu(b)\mathbf{q} - \mu(b)\mathbf{q}\mu(\mathbf{q} \cdot a)\right] \\ &= \operatorname{tr} \left[\mu(a)\mu(b)\mathbf{q} - \mu(b)\mathbf{q}\mu(a)\right] \\ &= \operatorname{tr} \left[\mu(a)\mu(b)\mathbf{q} - \mu(a)\mu(b)\mathbf{q}\right] \\ &= 0. \end{aligned} \tag{8}$$

In other words $b \mapsto \operatorname{STr} \mu(b)\mathbf{q}$ annihilates the action of global $\mathbf{q}$-equivariant sections, and hence is a conformal block on $(\mathbb{A}^{1|1})^\times/\mathbf{q}$. This sketch can be made precise either in the language of chiral algebras or of vertex algebras (and is done so in [13] Sects. 7.10 and 7.11, respectively).

In [27] the convergence in the analytic topology of the series defining φ_M is important, and is derived from a finiteness condition on V called C_2-cofiniteness.

The superconformal analogue is proved in [13, Appendix A], also using C_2-cofiniteness.

We may now regard the element $\varphi_M \in C(E^{\bullet}(\mathbf{q}), V)$ as a section of the sheaf C of conformal blocks over $S^{\bullet}$. As we have seen this sheaf is a twisted $\mathcal{D}$-module. The φ_M are flat sections of C, as we shall see in the next theorem via a variation on the proof of Theorem 2. Though the argument applies generally, we restrict attention to even $\mathbf{q} = \begin{pmatrix} q & 0 \\ 0 & qy \end{pmatrix}$ for the sake of clarity. In this case the operator $R(\mathbf{q})$ on V is simply $q^{L_0} y^{J_0}$ and we recover the supercharacter

$$\varphi_M(b) = \mathrm{STr}_M \, o(b) q^{L_0} y^{J_0}. \tag{9}$$

Expressed in terms of $x = e^{2\pi i t}$ we have the following expression for the Weierstrass function

$$\overline{\zeta}(t) = \xi(x) = \frac{1}{2} + \frac{1}{x-1} + \sum_{n \in \mathbb{Z} \backslash 0} \left(\frac{1}{q^n x - 1} - \frac{1}{q^n - 1} \right). \tag{10}$$

We remark that the relation $\xi(qx) = \xi(x) + 1$ is easily deduced from (10) via a telescoping sum argument.

Theorem 3 ([13, Theorem 8.15]) *The function φ_M satisfies the following (in general infinite) system of PDEs:*

$$q \frac{\partial}{\partial q} \varphi_M(b) = \varphi_M(\mathrm{Res}_{x=1} \, x \xi(x) L(x-1) b),$$
$$y \frac{\partial}{\partial y} \varphi_M(b) = \varphi_M(\mathrm{Res}_{x=1} \, \xi(x) J(x-1) b).$$

Proof (Sketch) As in Theorem 2 we work on $(\mathbb{A}^{1|1})^{\times}$ with coordinates (z, θ) fixed. We repeat the calculation (8) with the section a of $\mathcal{A}$ no longer $\mathbf{q}$-equivariant, but satisfying instead $\mathbf{q} \cdot a = a - s$, where s will be one of the explicit sections hz or $j\theta$. We obtain

$$\mathrm{STr} \, \mu(\mu(a) b) \mathbf{q} = \mathrm{STr} \, \mu(b) \mathbf{q} \mu(s)$$

in general.

The function ξ may be used to construct the appropriate section a because of the key relation $\xi(qx) = \xi(x) + 1$. A precise calculation (in, for instance, the case $s = hz$) yields

$$\varphi_M(\mathrm{Res}_{x=1} \, x \zeta(x) L(x-1) b) = \mathrm{STr}_M \, o(b) q^{L_0} y^{J_0} L_0 = q \frac{\partial}{\partial q} \varphi_M(b).$$

The other relation derives in the same way from $s = j\theta$.

Remark 3 By the same reasoning as in Lemma 1, we see that the differential equations of Theorem 3 are essentially the explicit expressions of the canonical Harish-Chandra connection.

9 Jacobi Modular Invariance

We now study the pullbacks of the sections φ_M via the morphism sexp defined by formula (3). We show that (after a normalisation) they are horizontal with respect to a certain $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$-equivariant connection. Explicitly:

Theorem 4 ([13, Theorem 9.10]) *The normalised section*

$$\widetilde{\varphi}_M = e^{2\pi i \alpha \cdot (C/6)} \operatorname{sexp}^*(\varphi_M)$$

is flat with respect to the connection

$$\nabla = d + \left(\operatorname{Res}_t \overline{\zeta}(t) J(t) dt\right) d\alpha + \frac{1}{2\pi i}\left(\operatorname{Res}_z \overline{\zeta}(z)\left[L(z) + \partial_z J(z)\right]\right) d\tau.$$

Furthermore ∇ is equivariant with respect to the $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$-action on $E^\circ \to S^\circ$ of Proposition 1.

This theorem is proved by analysing the behaviour of the PDEs of Theorem 3 under $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ transformations, which is an explicit computation.

It is possible to write the (projective) $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$-action on flat sections $\widetilde{\varphi}$ of C explicitly [13, Theorem 1.2 (c)]. The specialisation to $b = |0\rangle$ is

$$\begin{aligned} [\widetilde{\varphi} \cdot (m,n)](|0\rangle, \tau, \alpha) &= \exp 2\pi i \frac{C}{6}\left[m^2\tau + 2m\alpha + 2n\right] \widetilde{\varphi}(|0\rangle, \tau, \alpha + m\tau + n) \\ [\widetilde{\varphi} \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)](|0\rangle, \tau, \alpha) &= \exp 2\pi i \frac{C}{6}\left[\frac{-c\alpha^2}{c\tau + d}\right] \widetilde{\varphi}\left(|0\rangle, \frac{a\tau + b}{c\tau + d}, \frac{\alpha}{c\tau + d}\right). \end{aligned} \tag{11}$$

This recovers the well known transformation law [6, Theorem 1.4] for Jacobi forms of weight 0 and index $C/6$. Evaluation at other elements $b \in V$ yields Jacobi forms of higher weight, as well as more complicated 'quasi-Jacobi' forms.

In order to deduce Jacobi invariance of the (normalised) supercharacters (9) it suffices to show that they span the fibre of C. This can presumably be done following the method of Zhu [27, Sect. 5], assuming V is a rational vertex algebra. Alternatively Jacobi invariance can be proved by an extension of [18] to the supersymmetric case.

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Centralizers of Nilpotent Elements and Related Problems, a Survey

Anne Moreau

Abstract Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero. This note is a survey on several results, obtained jointly with Jean-Yves Charbonnel, concerning the centralizer $\mathfrak{g}^e$ of a nilpotent element e of $\mathfrak{g}$. First, we take interest in a famous conjecture by Elashvili on the index of $\mathfrak{g}^e$. Second, we study the question of whether the algebra $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ of symmetric invariants of $\mathfrak{g}^e$ is a polynomial algebra. Our main result stipulates that if for some homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$, the initial homogeneous components of their restrictions to $e + \mathfrak{g}^f$ are algebraically independent, with (e, h, f) an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. As applications, we pursue the investigations of Panyushev et al. (J Algebra 313:343–391, 2007) and produce new examples of nilpotent elements that verify the above polynomiality condition.

We also present a recent result of Arakawa-Premet related to the above problems.

Keywords Centralizer • Elashvili conjecture • Slodowy grading • Symmetric invariant

This note is a survey on several results, obtained jointly with Jean-Yves Charbonnel, on centralizers of elements in a reductive Lie algebra. These results are mostly based on the articles [5, 6].

1 Elashvili's Conjecture and Consequences

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. The *index* of $\mathfrak{g}$, denoted by $\operatorname{ind}\mathfrak{g}$, is the minimal dimension of the stabilizers of linear forms on $\mathfrak{g}$ for the coadjoint representation, [11]:

$$\operatorname{ind}\mathfrak{g} := \min\{\dim \mathfrak{g}^{\xi} \ ; \ \xi \in \mathfrak{g}^*\}$$

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where $\mathfrak{g}^\xi = \{x \in \mathfrak{g} \; ; \; \xi([x,\mathfrak{g}]) = 0\}$. The notion of the index is important in representation theory and invariant theory. By Rosenlicht [28], if $\mathfrak{g}$ is algebraic, i.e., $\mathfrak{g}$ is the Lie algebra of some algebraic linear group G, then the index of $\mathfrak{g}$ is the transcendence degree of the field of G-invariant rational functions on $\mathfrak{g}^*$. The index of a reductive algebra is equal to its rank. In general, computing the index of an arbitrary Lie algebra is a wild problem. However, there are a large number of interesting results for several classes of non-reductive subalgebras of reductive Lie algebras. For example, the centralizers of elements form an interesting class of subalgebras (cf. e.g.,[13, 22, 32]). This topic is closely related to the theory of integrable Hamiltonian systems [2, 3]. Let us make this link precise.

The symmetric algebra $\mathrm{S}(\mathfrak{g})$ carries a natural Poisson structure. Let $\xi \in \mathfrak{g}^*$ and consider the *Mishchenko-Fomenko subalgebra* $\mathcal{A}_\xi$ of $\mathrm{S}(\mathfrak{g}) \cong \Bbbk[\mathfrak{g}^*]$, constructed by the so-called *argument shift method*, [20]. It is generated by the ξ-shifts of p for p in the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants of $\mathrm{S}(\mathfrak{g})$, that is, $\mathcal{A}_\xi$ is generated by all the derivatives $D_\xi^i(p)$ for $p \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and $i \in \{0,\dots,\deg p - 1\}$, where

$$D_\xi^i(p)(x) := \frac{\mathrm{d}^i}{\mathrm{d}t} p(x + t\xi)|_{t=0}, \qquad x \in \mathfrak{g}^*.$$

It is well-known that $\mathcal{A}_\xi$ is a Poisson-commutative subalgebra of $S(\mathfrak{g})$.

Let $\mathfrak{g}^*_{\mathrm{sing}}$ be the set of nonregular linear forms $x \in \mathfrak{g}^*$, i.e.,

$$\mathfrak{g}^*_{\mathrm{sing}} := \{x \in \mathfrak{g}^* \mid \dim \mathfrak{g}^x > \operatorname{ind} \mathfrak{g}\}.$$

If $\mathfrak{g}^*_{\mathrm{sing}}$ has codimension at least 2 in $\mathfrak{g}^*$, we say that $\mathfrak{g}$ is *nonsingular*.

Theorem 1 (Bolsinov, [3, Theorem 2.1 and 3.2]) *Assume that* $\mathfrak{g}$ *is* nonsingular, *and let* $x \in \mathfrak{g}^*$. *For some* $\xi \in \mathfrak{g}^*$, *there is a Mishchenko-Fomenko subalgebra* $\mathcal{A}_\xi$ *in* $S(\mathfrak{g}) \cong \Bbbk[\mathfrak{g}^*]$ *such that its restriction to* Gx *contains* $\frac{1}{2}\dim(Gx)$ *algebraically independent functions if and only if* $\operatorname{ind} \mathfrak{g}^x = \operatorname{ind} \mathfrak{g}$.

If $\mathfrak{g}$ is reductive, then $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic as Lie algebras and its index $\operatorname{ind} \mathfrak{g}$ is equal to its rank, denoted by ℓ. The stabilizer of an element $x \in \mathfrak{g}^*$ identifies with the centralizer $\mathfrak{g}^x$ of x viewed as an element of $\mathfrak{g}$ through this isomorphism. Motivated by the preceding result of Bolsinov, Elashvili formulated a conjecture:

Conjecture 1 *Assume that* $\mathfrak{g}$ *is reductive. Then the index of* $\mathfrak{g}^x$ *is equal to* $\operatorname{ind} \mathfrak{g} = \ell$ *for any* $x \in \mathfrak{g}$.

Elashvili's conjecture has attracted the interest of many invariant theorists (e.g. [10, 22, 26, 32]). Thanks to Jordan decomposition, the conjecture reduces to the case where $x \in \mathfrak{g}$ is a nilpotent element. Also, it reduces to the case where $\mathfrak{g}$ is simple. Then the conjecture was proven by Yakimova for $\mathfrak{g}$ a simple Lie algebra of classical type, [32], and checked by a computer program by De Graaf for $\mathfrak{g}$ a simple Lie algebra of exceptional type, [10]. Before that, the result was established for some particular classes of nilpotent elements by Panyushev, [22, 23].

In a joint work with Jean-Yves Charbonnel, [5], we gave an almost case-free proof of Elashvili's conjecture using Bolsinov's criterion (cf. Theorem 1). Our approach was totally different from the previous ones. In more detail, this criterion is used to reduce the conjecture to the case of *rigid nilpotent elements*, that is those whose nilpotent G-orbit cannot be properly induced in the sense of Lusztig-Spaltenstein, [18]. For the rigid nilpotent elements, we have developed other methods that cover all cases except one case in type $\mathbf{E}_7$ and six cases in type $\mathbf{E}_8$. These remaining cases have been dealt with the computer program `GAP4`.

To summarize, we can state:

Theorem 2 ([5, Theorem 1.2]) *Assume that* $\mathfrak{g}$ *is reductive. Then the index of* $\mathfrak{g}^x$ *is equal to* $\operatorname{ind}\mathfrak{g} = \ell$ *for any* $x \in \mathfrak{g}$.

Assume from now on that $\mathfrak{g}$ is simple of rank ℓ. Denote by G its adjoint group and by $\langle . , . \rangle$ the Killing form of $\mathfrak{g}$.

Elashvili's conjecture also appears in invariant theory through the following interesting question, first raised by Premet:

Question 1 *Let* $x \in \mathfrak{g}$. *Is* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ *a polynomial algebra in* ℓ *variables?*

In order to answer this question, we can assume that x is nilpotent. Besides, if $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial for some $x \in \mathfrak{g}$, then it is so for any element in the adjoint orbit Gx of x. If $x = 0$, it is well-known since Chevalley that $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is polynomial in ℓ variables. At the extreme opposite, if x is a regular nilpotent element of $\mathfrak{g}$, then $\mathfrak{g}^x$ is abelian of dimension ℓ, [12], and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)$ is polynomial in ℓ variables too.

A positive answer to Question 1 was suggested in [26, Conjecture 0.1] for any simple $\mathfrak{g}$ and any $x \in \mathfrak{g}$. Yakimova has since discovered a counter-example in type $\mathbf{E}_8$, [33], disconfirming the conjecture. More precisely, the elements of the minimal nilpotent orbit in $\mathbf{E}_8$ do not satisfy the polynomiality condition. The present note contains another counter-example in type $\mathbf{D}_7$ (cf. Example 6). Question 1 still remains interesting and has a positive answer for a large number of nilpotent elements $e \in \mathfrak{g}$ as it is explained below.

Elashvili's conjecture (cf. Theorem 2) is deeply related to Question 1. First of all, it implies that if $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, it is so in ℓ variables. Further, according to Theorem 2, the main results of [26], that we summarize below, can be applied (see Theorem 3).

Let e be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov Theorem, e is embedded into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Denote by $\mathcal{S}_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with* e. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $(\mathfrak{g}^e)^*$ with $\mathfrak{g}^f$, through the Killing form $\langle . , . \rangle$. For p in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}^*] \simeq \Bbbk[\mathfrak{g}]$, denote by ${}^e p$ the initial homogenous component of its restriction to $\mathcal{S}_e$. According to [26, Proposition 0.1], if p is in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Theorem 3 (Panyushev-Premet-Yakimova, [26, Theorem 0.3]) *Suppose that the following two conditions are satisfied:*

(1) for some homogenous generators $q_1, \ldots, q_\ell$ *of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$*, the polynomial functions* ${}^e q_1, \ldots, {}^e q_\ell$ *are algebraically independent,*

(2) $\mathfrak{g}^e$ *is nonsingular.*

Then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is a polynomial algebra with generators* ${}^e q_1, \ldots, {}^e q_\ell$*.*

As a consequence of Theorem 3, if $\mathfrak{g}$ is simple of type $\mathbf{A}_\ell$ or $\mathbf{C}_\ell$, then all nilpotent elements of $\mathfrak{g}$ verify the polynomiality condition, cf. [26, Theorem 4.2 and 4.4]. The result for the type $\mathbf{A}_\ell$ was independently obtained by Brown and Brundan, [4]. In [26], the authors also provide some examples of nilpotent elements satisfying the polynomiality condition in the simple Lie algebras of types $\mathbf{B}_\ell$ and $\mathbf{D}_\ell$, and a few ones in the simple exceptional Lie algebras. At last, note that the analogue question to Question 1 for the positive characteristic was dealt with by Topley for the simple Lie algebras of types $\mathbf{A}_\ell$ and $\mathbf{C}_\ell$, [31].

2 Characterization of Good Elements

We now summarize the main result of [6], which continues the investigations of [26]. The following definition will be central:

Definition 1 An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some graded sequence $(p_1, \ldots, p_\ell)$ in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$.

For example, regular nilpotent elements are good. Indeed, for e in the regular nilpotent orbit of $\mathfrak{g}$ and $(q_1, \ldots q_\ell)$ a homogenous generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, it is well-known that ${}^e q_i = \mathrm{d}_e q_i$ for $i = 1, \ldots, \ell$ and that $(\mathrm{d}_e q_1, \ldots, \mathrm{d}_e q_\ell)$ forms a basis of $\mathfrak{g}^e$, [17]. Hence e is good.

Also, by Panyushev et al. [26, Theorem 5.4], all nilpotent elements of a simple Lie algebra of type $\mathbf{A}_\ell$ are good. Moreover, according to [34, Corollary 8.2], *even*[1] nilpotent elements without odd (respectively even) Jordan blocks of $\mathfrak{g}$ are good if $\mathfrak{g}$ is of type $\mathbf{C}_\ell$ (respectively $\mathbf{B}_\ell$ or $\mathbf{D}_\ell$). We generalize these results (cf. Proposition 3).

Our first result is the following:

Proposition 1 ([6]) *Let* x *be a good element of* $\mathfrak{g}$*. Then* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ *is a polynomial algebra and* $\mathrm{S}(\mathfrak{g}^x)$ *is a free extension of* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$*.*

Furthermore, we show that x is good if and only if so is its nilpotent component in the Jordan decomposition. As a consequence, we can restrict the study to the case of nilpotent elements.

[1] i.e., this means that the Dynkin grading of $\mathfrak{g}$ associated with the nilpotent element has no odd term.

Our main result is the following theorem whose proof is outlined in Sect. 3:

Theorem 4 ([6]) *Suppose that for some homogeneous generators* $q_1, \ldots, q_\ell$ *of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$*, the polynomial functions* ${}^e q_1, \ldots, {}^e q_\ell$ *are algebraically independent. Then* e *is a good element of* $\mathfrak{g}$*. In particular,* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is a polynomial algebra and* $\mathrm{S}(\mathfrak{g}^e)$ *is a free extension of* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$*. Moreover,* $({}^e q_1, \ldots, {}^e q_\ell)$ *is a regular sequence in* $\mathrm{S}(\mathfrak{g}^e)$*.*

In other words, Theorem 4 says that Condition (1) of Theorem 3 is sufficient to ensure the polynomiality of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. However, if only Condition (1) of Theorem 3 is satisfied, the (polynomial) algebra $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not necessarily generated by the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$. As a matter of fact, there are nilpotent elements e satisfying Condition (1) and for which $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not generated by some ${}^e q_1, \ldots, {}^e q_\ell$, for any choice of homogenous generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ (cf. Remark 1).

Theorem 4 can be applied to a great number of nilpotent orbits in the simple classical Lie algebras (cf. Sect. 4), and for some nilpotent orbits in the exceptional Lie algebras (cf. Sect. 5). For example, according to [25, Proposition 6.3] and Theorem 4, the elements of the subregular nilpotent orbit of $\mathfrak{g}$ are good.

All examples of good elements we encounter satisfy the hypothesis of Theorem 4. In fact, we have recently proven that the converse of Theorem 4 is true (see [7]).

Theorem 5 *Let* e *be a nilpotent of* $\mathfrak{g}$*. If* e *is good then for some graded generating sequence* $(q_1, \ldots, q_\ell)$ *in* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$*,* ${}^e q_1, \ldots, {}^e q_\ell$ *are algebraically independent over* $\mathbb{k}$*. In other words, the converse implication of Theorem 4 holds.*

Notice that it may happen that for some $r_1, \ldots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent over $\mathbb{k}$, but e is not good. This is the case for instance for the nilpotent elements in $\mathfrak{so}_{12}(\mathbb{k})$ associated with the partition $(5, 3, 2^2)$, see Example 5.

In fact, according to [26, Corollary 2.3], for any nilpotent e of $\mathfrak{g}$, there exist $r_1, \ldots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent over $\mathbb{k}$.

3 Outline of the Proof of Theorem 4

Let $q_1, \ldots, q_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \ldots, d_\ell$ respectively. The sequence $(q_1, \ldots, q_\ell)$ is ordered so that $d_1 \leqslant \cdots \leqslant d_\ell$. Assume that the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent.

According to Proposition 1, it suffices to show that e is good, and so it suffices to show that the nullvariety of ${}^e q_1, \ldots, {}^e q_\ell$ in $\mathfrak{g}^f$ has codimension ℓ since ${}^e q_1, \ldots, {}^e q_\ell$ are invariant homogeneous polynomials. To this end, it suffices to prove that

$$S := \mathrm{S}(\mathfrak{g}^e)$$

is a free extension of the $\Bbbk$-algebra generated by ${}^eq_1, \ldots, {}^eq_\ell$. We are led to find a subspace V_0 of S such that the linear map

$$V_0 \otimes_\Bbbk \Bbbk[{}^eq_1, \ldots, {}^eq_\ell] \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism. We explain below the construction of the subspace V_0.

Let $x_1, \ldots, x_r$ be a basis of $\mathfrak{g}^e$ such that for $i = 1, \ldots, r$, $[h, x_i] = n_i x_i$ for some nonnegative integer n_i. For $\mathbf{j} = (j_1, \ldots, j_r)$ in $\mathbb{N}^r$, set:

$$|\mathbf{j}| := j_1 + \cdots + j_r, \qquad |\mathbf{j}|_e := j_1 n_1 + \cdots + j_r n_r + 2|\mathbf{j}|, \qquad x^{\mathbf{j}} = x_1^{j_1} \cdots x_r^{j_r}.$$

The algebra S has two gradations: the standard one and the *Slodowy gradation*. For all $\mathbf{j}$ in $\mathbb{N}^r$, $x^{\mathbf{j}}$ is homogeneous with respect to these two gradations. It has standard degree $|\mathbf{j}|$ and, by definition, it has Slodowy degree $|\mathbf{j}|_e$. For m nonnegative integer, denote by $S^{[m]}$ the subspace of S of Slodowy degree m.

For any subspace V of S, set:

$$V[t] := \Bbbk[t] \otimes_\Bbbk V, \qquad V[[t]] := \Bbbk[[t]] \otimes_\Bbbk V, \qquad V((t)) := \Bbbk((t)) \otimes_\Bbbk V.$$

For V a subspace of $S[[t]]$, denote by $V(0)$ the image of V by the quotient morphism

$$S[[t]] \longrightarrow S, \qquad a(t) \longmapsto a(0).$$

The Slodowy grading of S induces a grading of the algebra $S((t))$ with t having degree 0. Let τ be the morphism of algebras

$$S \longrightarrow S[t], \qquad x_i \mapsto t x_i, \quad i = 1, \ldots, r.$$

The morphism τ is a morphism of graded algebras. Denote by $\delta_1, \ldots, \delta_\ell$ the standard degrees of ${}^eq_1, \ldots, {}^eq_\ell$ respectively, and set for $i = 1, \ldots, \ell$:

$$Q_i := t^{-\delta_i} \tau(\kappa(q_i)) \quad \text{with} \quad \kappa(q_i)(x) := q_i(e + x), \quad \forall\, x \in \mathfrak{g}^f.$$

Let A be the subalgebra of $S[t]$ generated by $Q_1, \ldots, Q_\ell$. Then $A(0)$ is the subalgebra of S generated by ${}^eq_1, \ldots, {}^eq_\ell$. For $(j_1, \ldots, j_\ell)$ in $\mathbb{N}^\ell$, $\kappa(q_1^{j_1}) \cdots \kappa(q_\ell^{j_\ell})$ and ${}^eq_1^{j_1} \cdots {}^eq_\ell^{j_\ell}$ are Slodowy homogenous of Slodowy degree $2d_1 j_1 + \cdots + 2d_\ell j_\ell$ (cf. e.g [26, 27]). Hence, A and $A(0)$ are graded subalgebras of $S[t]$ and S respectively. Denote by $A(0)_+$ the augmentation ideal of $A(0)$, and let V_0 be a graded complement to $SA(0)_+$ in S. The main properties of our data A and $A(0)$ are the following ones:

(1) A is a graded polynomial algebra,
(2) the canonical morphism $A \to A(0)$ is a homogenous isomorphism,
(3) the algebra $S[t, t^{-1}]$ is a free extension of A,
(4) the ideal $S[t, t^{-1}]A_+$ of $S[t, t^{-1}]$ is radical.

With these properties we first obtain that $S[[t]]$ is a free extension of A and that $S[[t]]$ is a free extension of the subalgebra $\tilde{A}$ of $S[[t]]$ generated by $\Bbbk[[t]]$ and A. From these results, we deduce that the linear map

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism, as expected.

4 Consequences of Theorem 4 for the Simple Classical Lie Algebras

By Panyushev et al. [26, Theorem 4.2 and 4.4], the first consequence of Proposition 1 and Theorem 4 is the following.

Proposition 2 *Assume that* $\mathfrak{g}$ *is simple of type* $\mathbf{A}_\ell$ *or* $\mathbf{C}_\ell$*. Then all the elements of* $\mathfrak{g}$ *are good.*

To state our results for the simple Lie algebras of types $\mathbf{B}_\ell$ and $\mathbf{D}_\ell$, let us introduce some more notations. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V}) \subset \mathfrak{gl}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $n = 2\ell + 1$ or $n = 2\ell$. For an endomorphism x of $\mathbb{V}$ and for $i \in \{1, \dots, n\}$, denote by $Q_i(x)$ the coefficient of degree $n - i$ of the characteristic polynomial of x. Then for any x in $\mathfrak{g}$, $Q_i(x) = 0$ whenever i is odd. Define a generating family $q_1, \dots, q_\ell$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell - 1$, set $q_i := Q_{2i}$. If $n = 2\ell + 1$, set $q_\ell := Q_{2\ell}$, and if $n = 2\ell$, let q_ℓ be the Pfaffian that is a homogenous element of degree ℓ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$.

Following the notations of Sects. 2 and 3, denote by ${}^e q_i$ the initial homogeneous component of the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto q_i(e + x)$, and by δ_i the degree of ${}^e q_i$.[2] According to [26, Theorem 2.1], ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent if and only if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0.$$

In that event, by Theorem 4, e is good and we say that e is *very good*.

The very good nilpotent elements of $\mathfrak{g}$ can be characterized in term of their associated partitions of n as follows. Assume that $\lambda = (\lambda_1, \dots, \lambda_k)$, with $\lambda_1 \geqslant \cdots \geqslant \lambda_k$, is the partition of n associated with the nilpotent orbit Ge. Then the even integers of λ have an even multiplicity, [9, §5.1]. Thus k and n have the same parity. Consider the following conditions on a sequence $\mu = (\mu_1, \dots, \mu_j)$ with $\mu_1 \geqslant \cdots \geqslant \mu_j$:

1) μ_{j-1} and μ_j are odd,
2) μ_{j-1} and μ_j are even,
3) $j > 3$, μ_1 and μ_j are odd and μ_i is even for any $i \in \{2, \dots, j-1\}$.

[2] The sequence of the degrees $(\delta_1, \dots, \delta_\ell)$ is described by [26, Remark 4.2].

Lemma 1

(i) *If n is odd, then λ is very good if and only if λ_1 is odd and if $(\lambda_2, \ldots, \lambda_k)$ is a concatenation of sequences verifying Conditions* (1) *or* (2) *with $j = 2$.*
(ii) *If n is even, then λ is very good if and only if λ is a concatenation of sequences verifying Condition* (3) *or Condition* (1) *with $j = 2$.*

For example, the partitions $(5, 3^2, 2^2)$ and $(7, 5^2, 4^2, 3, 1^2)$ of 15 and 30 respectively are very good. In particular, by Lemma 1, all even nilpotent elements in type $\mathbf{B}_\ell$, or in type $\mathbf{D}_\ell$ with odd rank ℓ, correspond to very good partitions and so are good.

Theorem 4 also allows to obtain examples of good, but not very good, nilpotent elements of $\mathfrak{g}$. For them, there are a few more work to do. Let us state one result as an illustration:

Proposition 3 ([6])

(i) *Assume that for some $k' \in \{2, \ldots, k\}$, λ_i is even for all $i \leqslant k'$, that $(\lambda_{k'+1}, \ldots, \lambda_k)$ is very good and that $\lambda_1 = \cdots = \lambda_{k'}$. Then λ is not very good, but e is good.*
(ii) *Assume that $k = 4$ and that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even. Then e is good.*

For example, $(6^4, 5, 3)$ satisfies the hypothesis of (i), and $(6^2, 4^2)$ satisfies the hypothesis of (ii).

There are also examples of elements that verify the polynomiality condition but that are not good; see Examples 4 and 5. To deal with them, we use different techniques, more similar to those used in [26] and that we present in Sect. 6.

As a result of all this, we observe for example that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^n)$, with $n \leqslant 8$, are good, and that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^n)$, with $n \leqslant 13$, verify the polynomiality condition.

Our results do not cover all nilpotent orbits in type $\mathbf{B}_\ell$ and $\mathbf{D}_\ell$ for larger ℓ (cf. Example 6).

Remark 1 Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$, with $\dim \mathbb{V} = 12$, and that $\lambda = (5^2, 1^2)$. Then the degrees of ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$ are $1, 1, 2, 2, 2, 2$ respectively. Since $10 = 1 + 1 + 2 + 2 + 2 + 2 = (\dim \mathfrak{g}^e + \ell)/2$, the polynomial functions ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$ are algebraically independent, and by Theorem 4, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial. One can verify that ${}^eq_5 = z^2$ for some z in the center $\mathfrak{z}(\mathfrak{g}^e)$ of $\mathfrak{g}^e$. Since $\mathfrak{z}(\mathfrak{g}^e)$ has dimension 3, for any other choice of homogenous generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ cannot be generated by the elements ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$ for degree reasons.

This shows that Condition (2) of Theorem 3 cannot be removed to ensure that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra in the variables ${}^eq_1, {}^eq_2, {}^eq_3, {}^eq_4, {}^eq_5, {}^eq_6$. However one can sometimes, as in this example, provide explicit generators.

5 Examples in Simple Exceptional Lie Algebras

We give in this section examples of good nilpotent elements in simple exceptional Lie algebras of type $\mathbf{E}_6$, $\mathbf{F}_4$ or $\mathbf{G}_2$ which are not covered by Panyushev et al. [26]. These examples are all obtained through Theorem 4.

According to [26, Theorem 0.4] and Theorem 4, the elements of the minimal nilpotent orbit of $\mathfrak{g}$, for $\mathfrak{g}$ not of type $\mathbf{E}_8$, are good. In addition, as it is explained in Sect. 2, the elements of the regular, or subregular, nilpotent orbit of $\mathfrak{g}$ are good. So we do not consider here these cases.

Example 1 Suppose that $\mathfrak{g}$ has type $\mathbf{E}_6$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 in the notation of Bourbaki. It has dimension 27 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{27}(\mathbb{k})$. For x in $\mathfrak{sl}_{27}(\mathbb{k})$ and for $i = 2, \ldots, 27$, let $p_i(x)$ be the coefficient of T^{27-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then $(q_2, q_5, q_6, q_8, q_9, q_{12})$ is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, [19]. Note that (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{27}(\mathbb{k})$. We denote by ${}^e p_i$ the initial homogeneous component of the restriction to $e + \tilde{\mathfrak{g}}^f$ of p_i where $\tilde{\mathfrak{g}}^f$ is the centralizer of f in $\mathfrak{sl}_{27}(\mathbb{k})$. For $i = 2, 5, 6, 8, 9, 12$,

$$\deg {}^e p_i \leqslant \deg {}^e q_i.$$

On the other hand,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} \leqslant \frac{1}{2}(\dim \mathfrak{g}^e + 6),$$

with equality if and only if ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent. So if the sum

$$\Sigma := \deg {}^e p_2 + \deg {}^e p_5 + \deg {}^e p_6 + \deg {}^e p_8 + \deg {}^e p_9 + \deg {}^e p_{12}$$

is equal to

$$\Sigma' := \frac{1}{2}(\dim \mathfrak{g}^e + 6),$$

then we can directly conclude that e is good. Otherwise, from the knowledge of the maximal eigenvalue $\nu_{\max}$ of the restriction of $\mathrm{ad}\,h$ to $\mathfrak{g}$ and the $\mathrm{ad}\,h$-weight of ${}^e p_i$, it is sometimes possible to deduce that $\deg {}^e p_i < \deg {}^e q_i$ and that e is good. We list in Table 1 the cases where we able to conclude in this way. The details are omitted. In Table 1, the fourth column gives the partition of 27 corresponding to the nilpotent element e of $\mathfrak{sl}_{27}(\mathbb{k})$, and the sixth one gives the $\mathrm{ad}\,h$-weights of ${}^e p_2$, ${}^e p_5$, ${}^e p_6$, ${}^e p_8$, ${}^e p_9$, ${}^e p_{12}$.

In conclusion, there remain nine unsolved nilpotent orbits in type $\mathbf{E}_6$.

Example 2 Suppose that $\mathfrak{g}$ is simple of type $\mathbf{F}_4$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_4 in the notation of Bourbaki. Then $\mathbb{V}$ has

Table 1 Data for $\mathbf{E}_6$

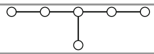
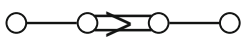
Label		$\dim \mathfrak{g}^e$	Partition	$\deg {}^e p_i$	Weights	$\nu_{\max}$	Σ	Σ'
D_5	2 0 2 0 2 2	10	(11,9,5,1,1)	1,1,1,1,1,1	2,8,10,14,16,22	14	6	8
$E_6(a_3)$	2 0 2 0 2 0	12	$(9, 7, 5^2, 1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	9
$D_5(a_1)$	1 1 0 1 1 2	14	(8,7,6,3,2,1)	1,1,1,1,2,2	2,8,10,14,14,20	10	8	10
A_5	2 1 0 1 2 1	14	$(9, 6^2, 5, 1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	10
$A_4 + A_1$	1 1 0 1 1 1	16	(7, 6, 5, 4, 3, 2)	1,1,1,2,2,2	2,8,10,12,14,20	8	9	11
D_4	0 0 2 0 0 2	18	$(7^3, 1^6)$	1,1,1,2,2,2	2,8,10,12,14,20	10	9	12
$D_4(a_1)$	0 0 2 0 0 0	20	$(5^3, 3^3, 1^3)$	1,1,2,2,2,3	2,8,8,12,14,18	6	11	13
$2A_2 + A_1$	1 0 1 0 1 0	24	$(5, 4^2, 3^3, 2^2, 1)$	1,1,2,2,2,3	2,8,8,12,14,18	5	11	15

Table 2 Data for $\mathbf{F}_4$

Label		$\dim \mathfrak{g}^e$	Partition	$\deg {}^e p_i$	Weights	$\nu_{\max}$	Σ	Σ'
$F_4(a_2)$	0 2 0 2	8	$(9, 7, 5^2)$	1,1,1,2	2,10,14,20	10	5	6
C_3	1 0 1 2	10	$(9, 6^2, 5)$	1,1,1,2	2,10,14,20	10	5	7
B_3	2 2 0 0	10	$(7^3, 1^5)$	1,1,2,2	2,10,12,20	10	6	7
$F_4(a_3)$	0 2 0 0	12	$(5^3, 3^3, 1^2)$	1,2,2,3	2,8,12,18	6	8	8
$C_3(a_1)$	1 0 1 0	14	$(5^2, 4^2, 3, 2^2, 1)$	1,2,2,3	2,8,12,18	6	8	9
$\tilde{A}_2 + A_1$	0 1 0 1	16	$(5, 4^2, 3^3, 2^2)$	1,2,2,3	2,8,12,18	5	8	10

dimension 26 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{26}(\mathbb{k})$. For x in $\mathfrak{sl}_{26}(\mathbb{k})$ and for $i = 2, \dots, 26$, let $p_i(x)$ be the coefficient of T^{26-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then (q_2, q_6, q_8, q_{12}) is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, [19]. As in Example 1, in some cases, it is possible to prove that e is good. These cases are listed in Table 2, indexed as in Example 1.

In conclusion, there remain six unsolved nilpotent orbits in type $\mathbf{F}_4$.

Example 3 Suppose that $\mathfrak{g}$ is simple of type $\mathbf{G}_2$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 in the notation of Bourbaki. Then $\mathbb{V}$ has dimension 7 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_7(\mathbb{k})$. For x in $\mathfrak{sl}_7(\mathbb{k})$ and for $i = 2, \dots, 7$, let $p_i(x)$ be the coefficient of T^{7-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then q_2, q_6 is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, [19]. There is only one nonzero nilpotent orbit which is neither regular, subregular or minimal. For e in it, we can show that e is good from Table 3, indexed as in Example 1.

In conclusion, all elements are good in type $\mathbf{G}_2$.

Table 3 Data for $\mathbf{G}_2$

Label		$\dim \mathfrak{g}^e$	Partition	$\deg {}^e p_i$	Weights	$\nu_{\max}$	Σ	Σ'
$\tilde{A}_1$	1 0	6	$(3, 2^2)$	1,3	2,6	3	4	4

6 A Result of Joseph-Shafrir and Applications

In this section, we provide, in a different way, examples of nilpotent elements which verify the polynomiality condition but that are not good, using techniques developed by Joseph-Shafrir, [16]. We also obtain an example of nilpotent element in type $\mathbf{D}_7$ which does not verify the polynomiality condition.

Let $\eta_0 \in \mathfrak{g}^e \otimes_{\Bbbk} \bigwedge^2 \mathfrak{g}^f$ be the bivector defining the Poisson bracket on $\mathrm{S}(\mathfrak{g}^e)$ induced from the Lie bracket. According to the main theorem of [27], $\mathrm{S}(\mathfrak{g}^e)$ is the graded algebra relative to the Kazhdan filtration of the *finite W-algebra* associated with e so that $\mathrm{S}(\mathfrak{g}^e)$ inherits another Poisson structure. The graded algebra structure so-obtained is the Slodowy graded algebra structure. Let $\eta \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^2 \mathfrak{g}^f$ be the bivector defining this other Poisson structure. According to [27, Prop. 6.3] (see also [26, §2.4]), η_0 is the initial homogeneous component of η. Denote by r the dimension of $\mathfrak{g}^e$ and set:

$$\omega := \eta^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge{}^{r-\ell} \mathfrak{g}^f, \quad \omega_0 := \eta_0^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge{}^{r-\ell} \mathfrak{g}^f.$$

Then ω_0 is the initial homogeneous component of ω. This fact is the key point in the proof of the results we state now.

Theorem 6 ([6]) *Let $q_1, \ldots, q_\ell$ be some homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, and let $r_1, \ldots, r_\ell$ be algebraically independent homogeneous elements of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent.*

(i) *For some homogeneous element p of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,*

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell = p\, \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_\ell$$

and we have,

$$\sum_{i=1}^{\ell} \deg {}^e r_i = \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

(ii) *If ${}^e p$ is a greatest divisor of $\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^{\ell} \mathfrak{g}^e$, then $\mathfrak{g}^e$ is nonsingular.*

(iii) *Assume that for some homogeneous polynomials $p_1, \ldots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, ${}^e r_1, \ldots, {}^e r_\ell$ are in $\Bbbk[p_1, \ldots, p_\ell]$ and that*

$$\deg p_1 + \cdots + \deg p_\ell = d + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

where d is the degree of a greatest divisor of $\mathrm{d}p_1\wedge\cdots\wedge\mathrm{d}p_\ell$ *in* $\mathrm{S}(\mathfrak{g}^e)$. *Then* $\mathfrak{g}^e$ *is nonsingular.*

The following proposition is a particular case of [16, §5.7].

Proposition 4 (Joseph-Shafrir, [16]) *Suppose that $\mathfrak{g}^e$ is nonsingular.*

(i) *If there exist algebraically independent homogeneous polynomials $p_1,\ldots,p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that*

$$\deg p_1+\cdots+\deg p_\ell=\frac{1}{2}(\dim\mathfrak{g}^e+\ell),$$

then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1,\ldots,p_\ell$.

(ii) *Suppose that the semiinvariant elements of $\mathrm{S}(\mathfrak{g}^e)$ are invariant. If $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra, then it is generated by homogeneous polynomials $p_1,\ldots,p_\ell$ such that*

$$\deg p_1+\cdots+\deg p_\ell=\frac{1}{2}(\dim\mathfrak{g}^e+\ell).$$

To produce new examples, our general strategy is the following: We first apply Theorem 6,(ii), in order to prove that $\mathfrak{g}^e$ is nonsingular. Next, we search for independent homogeneous polynomials $p_1,\ldots,p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ satisfying the conditions of Theorem 6,(iii), with $d=0$. Then we can apply Proposition 4,(i).

Proposition 4,(ii), is useful to construct counter-examples (cf. Example 6).

Example 4 Let e be a nilpotent element of $\mathfrak{so}_{10}(\Bbbk)$ associated with the partition $(3^2,2^2)$. Then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra but e is not good as explained below. Let $q_1,\ldots,q_5$ be as in Sect. 4. The degrees of ${}^eq_1,\ldots,{}^eq_5$ are $1,2,2,3,2$ respectively. Using the computer program Maple, we get the algebraic relation:

$${}^eq_4^2-4\,{}^eq_3\,{}^eq_5^2=0.$$

Set:

$$r_i:=\begin{cases} q_i & \text{if } i=1,2,3,5\\ q_4^2-4q_3q_5^2 & \text{if } \quad i=4.\end{cases}$$

The polynomials $r_1,\ldots,r_5$ are algebraically independent over $\Bbbk$ and

$$\mathrm{d}r_1\wedge\cdots\wedge\mathrm{d}r_5=2\,q_4\,\mathrm{d}q_1\wedge\cdots\wedge\mathrm{d}q_5$$

Moreover, ${}^er_4$ has degree at least 7, ${}^er_1,\ldots,{}^er_5$ are algebraically independent, and ${}^er_4$ has degree 7.

A precise computation shows that ${}^er_3=p_3^2$ for some p_3 in the center of $\mathfrak{g}^e$, and that ${}^er_4=p_4\,{}^er_5$ for some polynomial p_4 of degree 5 in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i:={}^er_i$ for

$i = 1, 2, 5$, the polynomials $p_1, \ldots, p_5$ are algebraically independent homogeneous polynomials of degree $1, 2, 1, 5, 2$ respectively. Furthermore, the greatest divisors of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_5$ in $\mathrm{S}(\mathfrak{g}^e)$ have degree 0, and p_4 is in the ideal of $\mathrm{S}(\mathfrak{g}^e)$ generated by p_3 and p_5. So, by Theorem 6,(iii), $\mathfrak{g}^e$ is nonsingular, and by Proposition 4,(i), $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1, \ldots, p_5$.

At last, e is not good since the nullvariety of $p_1, \ldots, p_5$ in $(\mathfrak{g}^e)^*$ has codimension at most 4.

Example 5 In the same way, for the nilpotent element e of $\mathfrak{so}_{11}(\Bbbk)$ associated with the partition $(3^2, 2^2, 1)$, we can prove that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by polynomials of degree $1, 1, 2, 2, 7$, $\mathfrak{g}^e$ is nonsingular but e is not good.

Remark 2 Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or 2ℓ and let $e \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$. Our results imply that if $\ell \leqslant 6$, then either e is good, or e is not good but $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathfrak{g}^e$ is nonsingular.

In particular, there are good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1. Indeed, by Panyushev et al. [26, §3.9], for some nilpotent element e' in $\mathbf{B}_3$, the codimension of $(\mathfrak{g}^{e'})^*_{\mathrm{sing}}$ in $(\mathfrak{g}^{e'})^*$ is one but, in $\mathbf{B}_3$, all nilpotent elements are good. For such nilpotent elements, note that [26, Theorem 0.3] (cf. Theorem 3) cannot be applied.

Example 6 From the rank 7, there are elements that do no satisfy the polynomiality condition. Let e be a nilpotent element of $\mathfrak{so}_{14}(\Bbbk)$ associated with the partition $(3^2, 2^4)$. Then $\ell = 7$ and the degrees of ${}^e q_1, \ldots, {}^e q_7$ are $1, 2, 2, 3, 4, 5, 3$ respectively, with $q_1, \ldots, q_7$ as in Sect. 4. Using Proposition 4,(ii), we can prove that e does not satisfy the polynomiality condition.

7 A Result of Arakawa-Premet

Let us mention a recent result of Arakawa and Premet, [1], which is related to the problems addressed in the previous sections.

We assume in this section that $\Bbbk$ is the field of complex numbers $\mathbb{C}$. Let $\xi \in (\mathfrak{g}^e)^*$ and denote by $\mathcal{A}_{e,\xi}$ be the Mishchenko-Fomenko subalgebra of $\mathrm{S}(\mathfrak{g}^e)$ generated by the derivatives $D_\xi^i({}^e p)$ for $p \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and $i \in \{0, \ldots, \deg {}^e p - 1\}$.

Theorem 7 (Panyushev-Yakimova, [24]) *Suppose that the conditions (1) and (2) of Theorem 3 are satisfied and that* $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ *has codimension* $\geqslant 3$ *in* $(\mathfrak{g}^e)^*$. *Then for a regular element* $\xi \in (\mathfrak{g}^e)^*$, $\mathcal{A}_{e,\xi}$ *is a polynomial algebra in the variables* $D_\xi^j({}^e q_i)$, *for* $i \in \{1, \ldots, \ell\}$ *and* $j \in \{0, \ldots, \deg {}^e q_i - 1\}$. *Moreover,* $\mathcal{A}_{e,\xi}$ *is a maximal Poisson-commutative subalgebra of* $\mathrm{S}(\mathfrak{g}^e)$.

Arakawa and Premet proved the following.

Theorem 8 (Arakawa-Premet, [1]) *Under the assumption of Theorem 7, there exists a maximal commutative subalgebra* $\hat{\mathcal{A}}_{e,\xi}$ *of* $U(\mathfrak{g})$ *such that* $\mathrm{gr}\, \hat{\mathcal{A}}_{e,\xi} \cong \mathcal{A}_{e,\xi}$.

Theorem 8 was known in the case where $e = 0$. It has been proven by Tarasov [30], and independently by Cherednik, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, by Nazarov-Olshanski [21] for classical $\mathfrak{g}$, and by Rybnikov [29], Chervov-Falqui-Rybnikov [8] and Feigin-Frenkel-Toledano-Laredo [15] for an arbitrary $\mathfrak{g}$.

The main step to prove Theorem 8 is to establish a chiralization of Theorem 3. Namely, Arakawa and Premet proved the following.

Let $\hat{\mathfrak{g}}^e$ be the affine Kac-Moody algebra associated with $\mathfrak{g}^e$ and a certain invariant bilinear form $\langle \cdot, \cdot \rangle_e$ on $\mathfrak{g}^e$, that is $\hat{\mathfrak{g}}_e$ is the central extension of the Lie algebra $\mathfrak{g}^e((t)) = \mathfrak{g}^e \otimes \mathbb{C}((t))$ by the one-dimensional center $\mathbb{C}\mathbf{1}$ with commutation relations:

$$[x(m), y(n)] = [x, y](m + n) + m\langle x, y \rangle_e \delta_{m+n,0}\mathbf{1},$$

where $x(m) = x \otimes t^m$ for $m \in \mathbb{Z}$. For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}^e) := U(\hat{\mathfrak{g}}^e) \otimes_{U(\mathfrak{g}^e[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}^e[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{g}^e[t]$ acts trivially and $\mathbf{1}$ acts as multiplication by k. The space $V^k(\mathfrak{g}^e)$ is naturally a vertex algebra, and it is called the *universal affine vertex algebra associated with* $\mathfrak{g}^e$ *at level* k. By the PBW theorem, $V^k(\mathfrak{g}^e) \cong U(\mathfrak{g}^e[t^{-1}]t^{-1})$ as $\mathbb{C}$-vector spaces and there is a natural filtration on $V^k(\mathfrak{g}^e)$ such that $\operatorname{gr} V^k(\mathfrak{g}^e) = \mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$. For $a \in V^k(\mathfrak{g}^e)$, denote by $\sigma(a) \in \mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ its symbol. We regard $\mathrm{S}(\mathfrak{g}^e)$ as a subring of $\mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ via the embedding defined by $x \mapsto x \otimes t^{-1}$, $x \in \mathfrak{g}^e$. The translation operator T on the Vertex Poisson algebra $\mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ is the derivation of the ring $\mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$ defined by

$$Tx(-m) = mx(-m-1), \qquad T1 = 0.$$

Assume from now that $k = cri$ is the critical level cri, and let $Z(V^{cri}(\mathfrak{g}^e))$ be the center of $V^{cri}(\mathfrak{g}^e)$.

Theorem 9 (Arakawa-Premet, [1]) *Assume that Conditions (1) et (2) of Theorem 3 are satisfied. Then there exist homogeneous elements* ${}^e\hat{q}_1, \ldots, {}^e\hat{q}_\ell$ *in* $Z(V^{cri}(\mathfrak{g}^e))$ *such that* $\sigma(\hat{q}_i) = {}^e q_i \in \mathrm{S}(\mathfrak{g}^e) \subset \mathrm{S}(\mathfrak{g}^e[t^{-1}]t^{-1})$. *Moreover,* $Z(V^{cri}(\mathfrak{g}^e))$ *is a polynomial algebra in the variables* $T^j({}^e\hat{q}_i)$ *with* $j \in \{1, \ldots, \ell\}$ *and* $j \in \{0, 1, \ldots\}$.

The particular case where $e = 0$ is an old result of Feigin-Frenkel, [14]. Arakawa and Premet have used *affine W-algebras* to prove the general case.

Remark 3 It would be interesting to extend the results of Arakawa and Premet to the setting of Theorem 4, that is to the case where only Condition (1) of Theorem 3 is satisfied. We can hope such a generalization at least in the case where we have explicit generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}^e}$, not necessarily of the form ${}^e q_1, \ldots, {}^e q_\ell$ for some generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, as in Remark 1.

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Pluri-Canonical Models of Supersymmetric Curves

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Abstract This paper is about pluri-canonical models of supersymmetric (susy) curves. Susy curves are generalisations of Riemann surfaces in the realm of super geometry. Their moduli space is a key object in supersymmetric string theory. We study the pluri-canonical models of a susy curve, and we make some considerations about Hilbert schemes and moduli spaces of susy curves.

Keywords Canonical models • Super geometry • Supersymmetric curves

1 Introduction

This paper is about supersymmetric (susy) curves and their pluri-canonical models. Susy curves are generalisations of Riemann surfaces in the realm of super geometry; because of this, they are also called super Riemann surfaces. Their moduli space is a key object in supersymmetric string theory.

Our goal is to construct the pluri-canonical model of a supersymmetric curve. In super geometry, the generalisation of the canonical bundle is the Berezinian bundle. Our result shows that an appropriate power of the Berezinian line bundle embeds all genus g susy curves in a projective space of fixed dimension. The embedded susy curve is called the pluri-canonical model of the curve.

The prefix "pluri" indicates that we are using powers of the Berezinian bundle. As explained in Remark 2, a canonical model of a susy curve does not exist.

Theorem 1 (Theorem 4, Pluri-Canonical Model) *Let*

$$f\colon C \to B$$

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be a susy curve of genus $g \geqslant 2$ over a super-manifold B. Let Ber(C) *be the Berezinian of the relative cotangent bundle. Fix an integer $\nu \geqslant 3$. Then, $f_*(\mathrm{Ber}(C)^{\otimes \nu})$ is a vector bundle on B.*

Moreover, if the genus g of C is at least 4*, the line bundle* $\mathrm{Ber}(C)^{\otimes \nu}$ *is relatively very ample; in other words, it embeds C into $\mathbb{P}(f_*(\mathrm{Ber}(C)^{\otimes \nu}))$. If $g = 3$ or $g = 2$, we need respectively $\nu \geqslant 4$ or $\nu \geqslant 5$ for the very ampleness of* $\mathrm{Ber}(C)^{\otimes \nu}$.

In Theorem 4, the reader can find also the rank of $f_*(\mathrm{Ber}(C)^{\otimes \nu})$. The relevant definitions are given in Sects. 2 and 3.1. The argument mimics the proof of the classical Kodaira's embedding theorem, see also [14]. In Sect. 2 we prove some general criteria to show that the push-forward of a line bundle is a vector bundle, and to check its very ampleness.

Our result is a prerequisites for the study of moduli space of susy curves using Hilbert schemes; we make some speculations in this direction in Sect. 4.

For the reader convenience, let us finish this introduction with a (non- comprehensive) review of the literature about the topics of this paper. General references about super geometry are [3, 6, 16, 21] and the first section of [9]; let us mention also the slightly more technical [14]. Two important references about supersymmetric curves are [15] and [22]; other sources are [2, 10, 18] and [12]. Reference about moduli of susy curves are [13] and [7], two recent important contributions are [9] and [8]. Recently, P. Deligne posted on his webpage a manuscript letter that he sent to Manin in 1987: this document contains a number of deep results probably still unexploited [5].

In the forthcoming paper [4], we will present three different versions of the moduli space of susy curves, and discuss the super-analogue of the period map.

2 Preliminaries on Line Bundles and Push-Forward

First, let us recall the definition of super-manifold. Let $(X, \mathcal{O}_X)$ be a $\mathbb{Z}_2$-graded commutative ringed space, i.e. X is a topological space and $\mathcal{O}_X$ is a sheaf of super commutative algebras. Let $\mathrm{n}_X \subset \mathcal{O}_X$ (or simply n) be the ideal sheaf generated by odd elements. We say that $(X, \mathcal{O}_X)$ (or simply X) is a smooth super-variety,[1] or *super-manifold*, if the following conditions hold:

1. $X^{\mathrm{bos}} := (X, \mathcal{O}_X/\mathrm{n}_X)$ is a complex manifold;
2. the sheaf $\mathrm{n}_X/\mathrm{n}_X^2$ is a locally free sheaf of $\mathcal{O}_X/\mathrm{n}_X$-modules;
3. $\mathcal{O}_X$ is locally isomorphic to the $\mathbb{Z}/2\mathbb{Z}$-graded exterior algebra $\Lambda^\bullet(\mathrm{n}_X/\mathrm{n}_X^2)$ over $\mathrm{n}_X/\mathrm{n}_X^2$.

The dimension of a connected super-manifold, written as $n|m$, is a pair of numbers: the first, the even dimension, is the dimension of X^{bos}; the second, the odd

[1] We refrain from defining more general super-varieties since in this paper we will only deal with smooth super-varieties.

dimension, is the rank of $\mathfrak{n}_X/\mathfrak{n}_X^2$. Note that $n|0$ super-manifolds are ordinary manifolds. For any super-manifold X, there is a natural closed embedding of $X^{\mathrm{bos}} \hookrightarrow X$ which is an identity on the underlying topological spaces and it is the quotient map $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X/\mathfrak{n}_X$ on the sheaf of functions. The manifold X^{bos} is sometime called the *bosonic reduction*, or just the reduction when no confusion arise, of X.

It is possible to construct a super-manifold out of a complex manifold Y and a vector bundle $\pi: E \to Y$: the topological space is Y and the structure sheaf is $\mathcal{O}_Y \otimes \bigwedge^{\bullet} E$. We denote this new super-manifold by Y_E. In this case, $Y_E^{\mathrm{bos}} = Y$ and $\mathfrak{n}_{Y_E}/\mathfrak{n}_{Y_E}^2 = E$. The projection π induces a section of the natural inclusion $Y \hookrightarrow Y_E$.

A super-manifold X is called *split* if it is isomorphic to Y_E, where Y has to be isomorphic to X^{bos} and E to $\mathfrak{n}_X/\mathfrak{n}_X^2$. Any $n|1$ dimensional super manifold is split, see e.g. [9, Corollary 2.2]; on the other hand, super-manifolds with odd dimension strictly bigger than one are not, in general, split.

A *vector bundle* E on a super-manifold X is a locally free sheaf of $\mathcal{O}_X$-modules. Let U be an open subset over which E is free. On U, E is of the form $A^p|A^q$, where $A = \mathcal{O}_X(U)$. We define the rank of E to be $p|q$. The definition does not depend on U. More details can be found in [16, Sect. IV.9] or [3, Appendix B3]. We define E^{bos} to be the pull-back of E to X^{bos}. More concretely, E^{bos} is $E/\mathfrak{n}E$. The bundle E^{bos} is a $\mathbb{Z}_2$-graded vector bundle on X^{bos}; its graded rank equals the rank of E. A *line bundle*ăE on X is a vector bundle of rank either $1|0$ or $0|1$; in particular, E^{bos} is a $\mathbb{Z}_2$ graded line bundle on X^{bos}.

In this paper, we are interested in the relative setting. Let $f: X \to B$ be a morphisms between smooth manifolds and let $f^{\mathrm{bos}} : X^{\mathrm{bos}} \to B^{\mathrm{bos}}$ be the induced morphism on the corresponding reduced manifolds. We say that f is proper if f^{bos} is proper. We say that f is smooth if its differential is surjective, i.e. the morphism is a submersion, see [3, Sect. 5.2]. The definition of flatness is as in the classical case, see [20, Sect. 1.5] and [1, Sect. 2.1.3]; in particular, a submersion is flat. Through all this paper, we assume that f is proper and smooth and that the relative dimension is $1|1$.

We define the split model X^{split} of X to be the cartesian product of B^{bos} and X; let us draw the resulting commutative diagram.

$$\begin{array}{ccccc} X^{\mathrm{bos}} & \xrightarrow{j} & X^{\mathrm{split}} & \xrightarrow{\iota} & X \\ & \searrow^{f^{\mathrm{bos}}} & \downarrow^{f^{\mathrm{split}}} & & \downarrow^{f} \\ & & B^{\mathrm{bos}} & \xrightarrow{i} & B \end{array}$$

The idea is that X^{split} is still a super-manifold in between X and X^{bos}, where we have killed all the odd functions coming from the base B—the odd moduli—but we still keep the odd functions living on the fibres of f. When the relative dimension is $1|1$, as in this paper, X^{split} is really a split manifold; this false in a more general setting. Being split, we can write $X^{\mathrm{split}} = C_L$, where $C = X^{\mathrm{bos}}$ is a classical curve over B^{bos} and L is a line bundle on C. Let $\pi\colon X^{\mathrm{split}} \to X^{\mathrm{bos}}$ be a section of j. Such a π exists because X^{split} is a split variety; this splitting is unique up to a scalar, but we do not need and we do not prove this fact.

Being $X^{\mathrm{split}} \cong C_L$, we have $\pi_* \mathcal{O}_{X^{\mathrm{split}}} = \mathcal{O}_{X^{\mathrm{bos}}} \oplus L$. For a line bundle E on X, we have

$$\pi_* \iota^* E = E^{\mathrm{bos}} \oplus (E^{\mathrm{bos}} \otimes L)$$

In other words, this is a splitting of $E/\mathrm{n}_B E$ as $\mathcal{O}_{X^{\mathrm{bos}}}$-module. Remark also that $\pi \circ f^{\mathrm{bos}} = f^{\mathrm{split}}$. The following criterion is a generalisation of [2, Sect. 2.6].

Criterion 2 *Let $f\colon X \to B$ be a proper smooth morphism of relative dimension* $1|1$ *between super manifolds; let E be a line bundle on X. Write $X^{\mathrm{split}} = C_L$. Then, the sheaf $f_* E$ is a vector bundle on B if both $f_*^{\mathrm{bos}} E^{\mathrm{bos}}$ and $f_*^{\mathrm{bos}}(E^{\mathrm{bos}} \otimes L)$ are vector bundle on B^{bos}, and*

$$R^1 f_*^{\mathrm{bos}} E^{\mathrm{bos}} = R^1 f_*^{\mathrm{bos}}(E^{\mathrm{bos}} \otimes L) = 0\,.$$

Moreover, under these hypotheses, the rank of $f_* E$ is equal to

$$\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}})\,|\,\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}} \otimes L)$$

if E is of rank $1|0$ *, and to*

$$\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}} \otimes L)\,|\,\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}})$$

if E is of rank $0|1$.

Proof We work at a local ring of a point of B; recall that Nakayama's Lemma and all its consequences hold in the super setting, cf. [3, Appendix B3]. We denote by n_B we denote the ideal sheaf on B generated by odd elements. We also fix a section $\pi\colon X^{\mathrm{split}} \to X^{\mathrm{bos}}$ of $j\colon X^{\mathrm{bos}} \to X^{\mathrm{split}}$.

We are going to show by induction on l that, for every l, the sheaf $f_*(E \otimes_{\mathcal{O}_X} f^* \mathcal{O}_B/\mathrm{n}_B^l)$ is a locally free sheaf of $\mathcal{O}_B/\mathrm{n}_B^l$-modules of rank either $\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}})\,|\,\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}} \otimes L)$ if E is of rank $1|0$, or $\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}} \otimes L)\,|\,\mathrm{rk}\,(f_*^{\mathrm{bos}} E^{\mathrm{bos}})$ if E is of rank $0|1$. This is enough to conclude because, for l big enough, the ideal sheaf n_B^l is trivial.

We first take $l = 1$. We have

$$\begin{aligned} f_*(E \otimes_{\mathcal{O}_X} f^* \mathcal{O}_B/\mathrm{n}_B) &= f_* \iota_* \iota^* E = i_* f_*^{\mathrm{split}} \iota^* E = \\ &= i_* f_*^{\mathrm{bos}} \pi_* \iota^* E = i_*(f_*^{\mathrm{bos}} E^{\mathrm{bos}} \oplus f_*^{\mathrm{bos}}(E^{\mathrm{bos}} \otimes L)) \end{aligned} \tag{1}$$

The right hand side is a locally free sheaf of $\mathcal{O}_{B^{\mathrm{bos}}}$ modules of the requested rank by hypothesis.

We now assume the statement for l and we prove it for $l+1$. Consider the exact sequence of sheaves on X

$$0 \to f^*(\mathrm{n}_B^l/\mathrm{n}_B^{l+1}) \to f^*(O_B/\mathrm{n}_B^{l+1}) \to f^*(O_B/\mathrm{n}_B^l) \to 0$$

We tensor by E; since E is flat on X, the sequence remains exact, so we obtain

$$0 \to f^*(\mathrm{n}_B^l/\mathrm{n}_B^{l+1}) \otimes_{O_X} E \to f^*(O_B/\mathrm{n}_B^{l+1}) \otimes_{O_X} E \to f^*(O_B/\mathrm{n}_B^l) \otimes_{O_X} E \to 0$$

Remark that the action of n_B on $\mathrm{n}_B^l/\mathrm{n}_B^{l+1}$ is trivial, so we can write

$$f^*(\mathrm{n}_B^l/\mathrm{n}_B^{l+1}) \otimes_{O_X} E = \iota_*((f^{\mathrm{split}})^*(\mathrm{n}_B^l/\mathrm{n}_B^{l+1}) \otimes_{O_{X^{\mathrm{split}}}} \iota^* E)$$

Here, we are viewing $\mathrm{n}_B^l/\mathrm{n}_B^{l+1}$ as a locally free sheaf of $O_{B^{\mathrm{bos}}}$-modules. We now apply Rf_*. Since both ι and i are affine, we have

$$(Rf_*)(\iota_*(-)) = R(f \circ \iota)_*(-) = R(i \circ f^{\mathrm{split}})_*(-) = i_* R((f^{\mathrm{split}})^*)(-)$$

Since $\mathrm{n}_B^l/\mathrm{n}_B^{l+1}$ is locally free on B^{bos}, we can apply the projection formula to Rf^{split}. A part of the resulting long exact sequence is

$$\begin{aligned} 0 \to i_*(\mathrm{n}_B^l/\mathrm{n}_B^{l+1} \otimes_{O_{B^{\mathrm{bos}}}} f_*^{\mathrm{split}} \iota^* E) \to f_*(f^* O_B/\mathrm{n}_B^{l+1} \otimes_{O_X} E) \to f_*(f^* O_B/\mathrm{n}_B^l \otimes_{O_X} E) \to \\ \to i_*(\mathrm{n}_B^l/\mathrm{n}_B^{l+1} \otimes_{O_{B^{\mathrm{bos}}}} (R^1 f_*^{\mathrm{split}}) \iota^* E)\,. \end{aligned}$$

Note that the last term vanishes:

$$\begin{aligned} (R^1 f_*^{\mathrm{split}}) \iota^* E = (R^1 f_*^{\mathrm{bos}} \pi_*) \iota^* E = (R^1 f_*^{\mathrm{bos}}) \pi_* \iota^* E = \\ R^1 f_*^{\mathrm{bos}} E^{\mathrm{bos}} \oplus R^1 f_*^{\mathrm{bos}} (E^{\mathrm{bos}} \otimes L) = 0\,, \end{aligned}$$

where we used that π is an affine maps, and the vanishing assumed in the hypotheses of the criterion.

We are now left with the short exact sequence

$$0 \to i_*(\mathrm{n}_B^l/\mathrm{n}_B^{l+1} \otimes_{O_{B^{\mathrm{bos}}}} f_*^{\mathrm{split}} \iota^* E) \to f_*(f^* O_B/\mathrm{n}_B^{l+1} \otimes_{O_X} E) \to f_*(f^* O_B/\mathrm{n}_B^l \otimes_{O_X} E) \to 0 \tag{2}$$

As discussed above, we have

$$f_*^{\mathrm{split}} \iota^* E = f_*^{\mathrm{bos}} E^{\mathrm{bos}} \oplus f_*^{\mathrm{bos}} (E^{\mathrm{bos}} \otimes L)$$

so, the first term of the sequence (2) is a locally free sheaf of O_B/n_B-modules by hypothesis; the last term is, by induction, a locally free sheaf of O_B/n_B^l-modules of the requested rank. One can now apply a Nakayama-type argument to show that the central term is a locally-free sheaf of O_B/n_B^{l+1}-modules of the requested rank;

see for example the proof of [19, Tag 051H] with $R = \mathcal{O}_B/\mathfrak{n}_B^{l+1}$ and $I = \mathfrak{n}_B^l$. More explicitly, one takes a basis for the last term of the sequence (2); lift these elements to the central term, and shows that they generate it because of Nakayama's Lemma. Then, arguing again as in the proof of [19, Tag 051H], one shows that, since $\mathfrak{n}_B$ is nilpotent, the central term is locally free. □

From now on, we assume that f_*L is locally free. The line bundle L gives a morphism relative to B from X to $\mathbb{P}f_*L$; we say that L is *f-very ample* if this morphism is an embedding. Following the proof of [14, Theorem 1], we have the following criterion.

Criterion 3 *Let $f\colon X \to B$ be a proper smooth morphism of relative dimension $1|1$ between super manifolds; write $X^{\text{split}} = C_L$. Let E be a line bundle of rank $1|0$ on X. Assume that f_*E is a locally free sheaf. Then L is f-very ample if, for every pair of points x and y in X^{bos} such that $f^{\text{bos}}(x) = f^{\text{bos}}(y)$ we have*

$$R^1 f_*^{\text{bos}}(E^{\text{bos}} \otimes I_x \otimes I_y) = 0$$

and

$$R^1 f_*^{\text{bos}}(E^{\text{bos}} \otimes L \otimes I_x) = 0\,,$$

where $I_x \subset \mathcal{O}_{X^{\text{bos}}}$ (resp. I_y) is the ideal sheaf of x (resp. y). If E is of rank $0|1$, the same statement holds replacing in the second condition $E^{\text{bos}} \otimes L$ with E^{bos}.

Proof We have to show that the map given by E embeds X into $\mathbb{P}(f_*E)$. The first condition, by the classical proof of the Kodaira embedding theorem, guarantees that E embeds X^{bos} into $\mathbb{P}(f_*^{\text{bos}}E^{\text{bos}})$. We now have to verify the statement in the odd directions. The second condition means that the odd differential is injective in the vertical direction, and this is enough to conclude.

Let us be more explicit; to fix the notation, assume that the rank of E is $1|0$. We also shrink B around $f(x)$, so that f_*E is free. The second vanishing in the hypothesis guarantees that the morphism

$$f_*^{\text{bos}}(E^{\text{bos}} \otimes L) \to f_*^{\text{bos}}(E_x^{\text{bos}} \otimes L_x)$$

is surjective, where the sub-index x means the stalk. This means that there exists a global section $\bar{s}$ of $E^{\text{bos}} \otimes L$ which does not vanish at x.

Using Eq. (1), we can identify $\bar{s}$ with an *odd* global section of $\iota^*E = f^*\mathcal{O}_B/\mathfrak{n}_B \otimes E$. Using sequence (2) enough times, we can show that there exists an odd global section s of E which restricts to $\bar{s}$ modulo $\mathfrak{n}_B$. In particular, s is not zero at x.

If we trivialise E around x on X, the global section s restrict to a vertical odd co-ordinate at x. (By vertical odd co-ordinate, we mean an odd co-ordinate which is not zero modulo $\mathfrak{n}_B$.) This is enough to conclude that E gives an embedding at x. More details in the proof of [14, Theorem 1]. □

3 Pluri-Canonical Models of Susy Curves

3.1 Definition of Susy Curves

We give the definition of susy curve over a smooth base B; the reason for this set up is twofold: it is suitable for moduli theory; and a curve defined over a point is always split, so there is not much super-geometry going on.

Definition 1 (Susy Curves) A genus g susy curve over a smooth base B is a proper smooth morphism

$$f: C \to B$$

between super manifolds together with a susy structure $\mathcal{D}$ satisfying the following conditions. The morphism f has relative dimension is $1|1$ and the fibres are, as topological spaces, genus g Riemann surfaces. The susy structure $\mathcal{D}$ is a sub bundle

$$\mathcal{D} \hookrightarrow Ker(df) \hookrightarrow TC$$

which is of rank $0|1$, and such that $\mathcal{D}$ and $[\mathcal{D}, \mathcal{D}]$ span $Ker(df)$.
Smooth morphisms are discussed in [3, Sect. 5.2], where they are called submersions. Recall that the coordinates on the base B are sometime called the moduli of the curve.

3.2 Pluri-Canonical Model

Let

$$f: C \to B$$

be a susy curve; write $\mathrm{Ber}(C) := \mathrm{Ber}(T_f C^\vee)$ for the Berezinian of the relative cotangent bundle: this is rank $0|1$ line bundle. Since $\mathcal{D}$ and $[\mathcal{D}, \mathcal{D}] \cong \mathcal{D}^2$ spans $T_f C$, we have an exact sequence

$$0 \to \mathcal{D}^{-2} \to T_f C^\vee \to \mathcal{D}^{-1} \to 0$$

The Berezinian of this sequence gives an isomorphism $\mathrm{Ber}(C) = \mathcal{D}^{-1}$; cf. [22, Eq. 2.28].

Lemma 1 *Writing* $C^{\mathrm{split}} = C_L$, *we have*

$$\mathrm{Ber}(C)^{\mathrm{bos}} = L$$

Proof We know that $\mathrm{Ber}(C)$ is isomorphic to $\mathcal{D}^{-1}$. In local super conformal co-ordinates, $\mathcal{D}$ is generated by $\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$. Recall that θ is an odd co-ordinate, so a section of L, and $\frac{\partial}{\partial\theta}$ is a section of $L^\vee$. Setting $\theta = 0$ we obtain $\mathcal{D}^{\mathrm{bos}} = L^\vee$, so the conclusion. More details about super-conformal co-ordinates can be found in [22] and [9, Lemma 3.1]. □

Theorem 4 (Pluri-Canonical Model) *Let*

$$f\colon C \to B$$

be a susy curve over a super-manifold B. Fix an integer $\nu \geqslant 3$. *Then* $f_*(\mathrm{Ber}(C)^{\otimes\nu})$ *is a locally free sheaf of* $\mathcal{O}_B$*-modules. If* ν *is even, the rank of* $f_*\,\mathrm{Ber}(C)$ *is*

$$(\nu-1)g-\nu+1\,|\,(2\nu-1)g-2\nu+1\ .$$

If ν *is odd, the rank of* $f_*\,\mathrm{Ber}(C)$ *is*

$$(2\nu-1)g-2\nu+1\,|\,(\nu-1)g-\nu+1\ .$$

If the genus g of C is at least 4*, then* $\mathrm{Ber}(C)^{\otimes\nu}$ *is relatively very ample. If* $g = 3$ *or* $g = 2$*, we need respectively* $\nu \geqslant 4$ *or* $\nu \geqslant 5$ *for the very ampleness.*

Proof In this proof we follow the notations of Sect. 2. To show that $f_*\,\mathrm{Ber}(C)^\nu$ is locally free we apply Criterion 2 and Lemma 1. Writing $C^{\mathrm{split}} = C_L$, we need to show that $f_*^{\mathrm{bos}}L^\nu$ and $f_*^{\mathrm{bos}}L^{\nu+1}$ are vector bundles on B^{bos}, and that

$$R^1f_*^{\mathrm{bos}}L^\nu = R^1f_*^{\mathrm{bos}}L^{\nu+1} = 0\,.$$

The degree of L being $g-1$, these conditions are true if $\nu \geqslant 3$, but fail for $\nu \leqslant 2$. The statement about the rank also follows from Criterion 2.

Let us show that the bundle if f-very ample. We apply Criterion 3. To fix the notation, assume that ν is even. First, we remark that

$$R^1f_*[(\mathrm{Ber}(C)^\nu)^{\mathrm{bos}} \otimes I_x \otimes I_y] = R^1f_*[L^\nu \otimes I_x \otimes I_y]$$

By the classical Serre duality theorem, the right hand side vanishes for $\nu \geqslant 3$ if $g \geqslant 4$, for $\nu \geqslant 4$ if $g = 3$, and for $\nu \geqslant 5$ if $g = 2$. The second condition of Criterion 2 equally follows, because $(\mathrm{Ber}(C)^\nu)^{\mathrm{bos}} \otimes L = L^{\nu+1}$. The same computation applies when ν is odd. □

Remark 1 Theorem 4 holds in a more general set-up: we have never used the susy structure. What we need is just that the line bundle $L = \mathrm{Ber}(C)^{\mathrm{bos}}$ is of degree $g-1$.

Remark 2 (Canonical Model) The line bundle $\mathrm{Ber}(C)$ is not f-very ample for a generic curve. For example, if B is a point, $X = C_L$ and $h^0(C, L) = 0$, it is well known that the rank of $f_*\,\mathrm{Ber}(C) = H^0(X, \mathrm{Ber}(C))$ is $0|g$, see for instance [22, Sect. 8]; in this case, $\mathrm{Ber}(C)$ can not be very ample for dimensional reasons. This means that we can not speak about the canonical model of a susy curve, but just about its pluri-canonical models.

4 Moduli Space and Hilbert Scheme

This section is purely speculative, because a theory of stacks and Hilbert schemes in the super setting has not be fully developed yet.

The moduli space $s\mathcal{M}_g$ of genus g Super Riemann surfaces can be defined with the usual machinery of stacks: it is the pseudo-functor mapping a smooth super variety B to the groupoid of susy curves over B. In other words, for any super-manifold B, the B-points of $s\mathcal{M}_g$, usually denoted by $s\mathcal{M}_g(B)$, are the susy curves over B. See [1] for an introduction to stacks in the super setting.

To simplify the notations, take $g \geqslant 4$ and $\nu = 3$. In Theorem 4, we showed that $\mathrm{Ber}(C)^{\otimes 3}$ embeds every susy curve in a super projective space of dimension $2g-1|5g-5$. We thus want to consider the Hilbert scheme H_g parametrising smooth connected sub-varieties of $\mathbb{P} := \mathbb{P}^{2g-1|5g-5}$ with appropriate degree, dimension $1|1$, and embedded with $\mathrm{Ber}(C)^{\otimes 3}$.

The Hilbert scheme H_g can be defined as a functor; we do not know if it can be represented by a projective super-scheme, as in the classical case.

Theorem 4 means that we have a morphism

$$\iota : s\mathcal{M}_g \to H_g / \operatorname{Aut}(\mathbb{P})$$

The group $\operatorname{Aut}(\mathbb{P})$ is studied in [11] and [17].

Theorem 5[2] *The morphism ι defined above is a closed embedding.*

Proof Following [19, Tag 04XV], we have to show that ι is a monomorphism and universally closed.

To show that ι is a monomorphism, we have to show that for any super-manifold B, the morphism $\iota : s\mathcal{M}_g(B) \to H_g / \operatorname{Aut}(\mathbb{P})(B)$ is injective. To this end, we have to show that, given a $1|1$ dimensional super manifold over B, the susy structure $\mathcal{D}$, if it exists, it is unique. This is shown in [5, Proposition 1.7]. Let us sketch a proof that is slightly closer to [22].

First, we prove the result in the split case C_L. A split $1|1$ manifold is equivalent to a curve C with a line bundle L; to have a susy structure $\mathcal{D}$ on C, L must be a theta characteristic. The distribution $\mathcal{D}$ is uniquely determined by L and vice versa, so the claim.

Once the split case is established, we have to show that we can not deform $\mathcal{D}$ keeping the underlining super-manifold fixed. It is enough to prove it for a first order

[2] This result was well known to the experts, we thank R. Donagi for discussing this theorem with us.

deformation. To do this we argue as follows. Let $\mathcal{S}$ be the sheaf of super-conformal vector field; that is the vector field that commute with $\mathcal{D}$. A deformation of a susy curve is given by an element of $H^1(C, \mathcal{S})$. We can forget the susy structure and see the deformation just as a deformation of a $1|1$ super-manifold. This corresponds to look at the map

$$i: H^1(C, \mathcal{S}) \to H^1(C, TC)$$

induced by the inclusion of $\mathcal{S}$ in TC. The claim is equivalent to the injectivity of i; indeed, this means that if we deform the susy structure then we deform also the underlying super-manifold. As explained in [22, p. 11], the projection

$$\pi: TC \to TC/\mathcal{D} \cong \mathcal{D}^2$$

induces an isomorphism between $\mathcal{S}$ and $\mathcal{D}^2$. In other words, in the (non-exact) sequence

$$\mathcal{S} \xrightarrow{i} TC \xrightarrow{\pi} \mathcal{D}^2 \cong \mathcal{S}$$

we have $i \circ \pi = Id$. If we take to cohomology, we have

$$H^1(C, \mathcal{S}) \xrightarrow{i} H^1(C, TC) \xrightarrow{\pi} H^1(C, \mathcal{S})$$

with again $i \circ \pi = Id$, so i is injective at the level of H^1.

To prove that ι is universally closed, we show that the image is the fixed locus of an involution of the co-domain; to this end, we introduce duality. Given a $1|1$ super-manifold X we can define the dual curve $X^\vee$. There are a few ways to define $X^\vee$. A possibility is the moduli space of $0|1$ dimensional sub-manifold of X whose intersection with the reduced space is a point. For more details see e.g. [22, Sect. 9] or [2, Sects. 2.2 and 2.3].

Because of Remark 1, we can see $H_g/\operatorname{Aut}(\mathbb{P})$ as the moduli space of dimension $1|1$ super-manifolds such that $\deg \operatorname{Ber}(X)^{\text{bos}} = g-1$. The duality preserves these invariants, in particular $\deg \operatorname{Ber}(X^\vee)^{\text{bos}} = 2g-2-\deg \operatorname{Ber}(X)^{\text{bos}} = g-1$, cf. [2, Example 2.2.3]. We conclude that the duality gives an order 2 automorphism of H_g. Susy curves are exactly the curves which are auto-dual, cf. [2, Sect. 2.3] or [22, Sect. 9]; this means that $s\mathcal{M}_g$ is the fixed locus of this automorphism of $H_g/\operatorname{Aut}(\mathbb{P})$. □

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Report on the Broué–Malle–Rouquier Conjectures

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Abstract This paper is a short survey on the state-of-the-art concerning the main 1998 Broué–Malle–Rouquier conjectures about 'Complex reflection groups, Braid groups, Hecke algebras'.

Keywords Braid groups • Complex reflection groups • Hecke algebras

1 Introduction

About two decades ago, M. Broué, G. Malle and R. Rouquier published a programmatic paper [14] entitled *Complex reflection groups, Braid groups, Hecke algebras* (see also [13]). Motivated by earlier prospections on generalizations of reductive groups, they managed to associate to every *complex* reflection group two objects which were classically associated to *real* reflexion groups (a.k.a. finite Coxeter groups): a generalized braid group and a Iwahori-Hecke algebra. Moreover, they put forward good reasons to believe that the nice properties of these objects in the classical case could be extended to the general one. This paper was followed by a couple of others (most notably [15]) adding precisions on what could be expected. The present paper aims at reporting on the progression of this program. However, it is not possible to explore, in a short text like this one, all the ramifications of the program, because it is connected to a whole area in representation theory (Cherednik algebras and related topics). Therefore, one has to make a choice in order to provide a potentially useful review of it.

In this paper, we made the following choice. We decided to focus on what we regard as the most fundamental properties of *the objects* at the core of [14], that is braid groups and Hecke algebras, disregarding the context in which these objects have been first introduced (an attempt to generalize reductive groups and related objects), and disregarding as well specific properties that might be of use for specific representation-theoretic problems.

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This focus implies that we are going to emphasize what is now actually *proven* of the ‘main’ (in the above sense) conjectures concerning the objects appearing in [14], and that we shall try to provide a hopefully handy text for people interested in these objects, who would probably appreciate pointers to the literature concerning practical aspects (homology groups, matrix models for representations, etc.).

After reading this text, one impression could be that the progresses have not been that spectacular in the past twenty years. After all, among the exceptional complex reflection groups, the smaller one (G_4) is the only one for which *all* the natural questions mentioned below are settled by now ! Moreover, *all* the results below, when they are proved for every reflection group, need to use the classification of the complex reflection groups in their proof. One should keep in mind however that all these exceptional complex reflection groups are the fundamental symmetric groups arising in low-dimensional phenomena, and therefore, although one might spend time dreaming at a ‘general, conceptual proof’ (if it exists), ad-hoc proofs should not be regarded as a waste of time. Not only do they provide the basis for applying the conjectures to the particular phenomenon controlled by a given reflection group, but they usually provide additional results on the specific group that are sometimes crucial for applications.

As an example of the first aspect, we mention that the case of the Hecke algebra of the smallest reflection group G_4 itself was successfully applied in studying a potentially new invariant of knots [29], improving our understanding of the Links-Gould invariant and of the Birman-Wenzl-Murakami algebra [42, 43]. Similarly, the cases of G_4 and G_5 were used in [39] and [37] to identify up to isomorphism two different constructions of the same representations of the usual braid groups, while the cases of G_8 and G_{16} are used in [18] to recover and explain a classification due to Tuba-Wenzl of small-dimensional irreducible representations of the braid groups.

This text is an expanded version of a talk I gave in Pisa in February 2015, during the intensive research period ‘Perspectives in Lie theory’, more precisely during the session on ‘Algebraic topology, geometric and combinatorial group theory’ at Centro de Giorgi. I am very grateful to the organizers of this research period for this opportunity.

2 Complex Reflection Groups

Recall that a complex reflection group is a finite subgroup of some general linear group over the complex numbers $\mathrm{GL}_r(\mathbb{C})$, which has the property of being generated by (pseudo)-reflections, namely endomorphisms whose invariant subspace is an hyperplane. If W is such a group, we denote by $\mathcal{R} \subset W$ the set of all pseudo-reflections belonging to W, and call r the *rank* of W. A recent reference on such objects is [31]. It is convenient to introduce a subset $\mathcal{R}^* \subset \mathcal{R}$ of so-called *distinguished reflections*. If $s \in \mathcal{R}$, the set of elements in W fixing $\mathrm{Ker}(s-1)$ is a cyclic group of some order m, and there is only one element in this set with eigenvalue $\exp(-2\pi \mathrm{i}/m)$. This is the distinguished pseudo-reflection attached to $\mathrm{Ker}(s-1)$, and $\mathcal{R}^*$ is the collection of all such elements.

A basic property of a complex reflection group is that it can be canonically decomposed as a direct product of *irreducible* ones—meaning that $W \subset \mathrm{GL}_r(\mathbb{C})$ acts irreducibly on $\mathbb{C}^r$. By Schur's lemma, such groups have cyclic center. Moreover, by a change of basis one can always assume $W \subset \mathrm{GL}_r(K)$, where K is the subfield of $\mathbb{C}$ generated by the traces of elements of W. Finally, a fundamental result of Steinberg says that, if S is any subspace of $\mathbb{C}^r$, then the subgroup W_S of W made of all the elements fixing S is a reflection subgroup, generated by $\mathcal{R} \cap W_S$. Such subgroups are called parabolic subgroups.

Completing a quest of several decades, irreducible complex reflection groups were classified by Shephard and Todd, in [47]. They either belong to an infinite series $G(de, e, r) < \mathrm{GL}_r(\mathbb{C})$ of groups of monomial matrices, or to a finite set of 34 exceptions. These 34 exceptions were labelled $G_4, G_5, \dots$ up to G_{37}. Using this classification, a general result due to M. Benard (see [5]) is that all the irreducible representations of W can be defined over K.

Inside this list of exceptions, some of the groups can be realized as *real* reflection groups. Since they are the geometric realization of finite Coxeter groups, they are pretty well understood. In the Coxeter-Dynkin classification, the correspondance is $G_{23} = H_3$, $G_{28} = F_4$, $G_{30} = H_4$, $G_{35} = E_6$, $G_{36} = E_7$, $G_{37} = E_8$. In this case, the rank of W is equal to the minimum number of reflections which are necessary to generate the group. In the general case, it can be shown that this number is either r or $r + 1$, where r is the rank of W. In the former case W is called *well-generated*, in the latter *badly generated*. Among the (exceptional) groups of rank at least 3, only G_{31} is badly generated. We denote $n(W) \in \{r, r + 1\}$ the minimal number of reflections needed to generate W.

Since most of the conjectures that we are interested in have been proven early enough for the general series of the $G(de, e, r)$ (see Ariki-Koike [3], Broué-Malle [12], Ariki [2], Bremke-Malle [10]) we will concentrate on the exceptional groups. While the details of the classification are cumbersome, its general scheme is clear enough. We recall it because the proof of many results on the BMR conjectures uses separation of cases in families that originate from the proof of the classification. It proceeds as follows:

1. If the action of the group on $\mathbb{C}^r$ is imprimitive, one proves that W has to belong to the infinite series of the $G(de, e, r)$
2. If it is primitive of rank 2, then $W/Z(W)$ can be identified through the isomorphism $\mathrm{PSU}_2 \simeq \mathrm{SO}_3(R_{\mathrm{red}})$ with the group of rotations of the tetrahedron, of the octahedron or of the icosaedron, thus splitting this case in three families. All such W are then obtained as a cyclic central extension of such a group.
3. If it is primitive of higher rank, then a theorem of Blichtfeld asserts that the pseudo-reflections of W can have order only 2 or 3. The rest of the proof is based on this result, and needs a lot of careful analysis to be completed. Actually it turns out that only three groups, G_{25}, G_{26} and G_{32} admits pseudo-reflections of order 3. Moreover, only one of these groups (G_{31}) is badly generated.

A convenient software for dealing with these exceptional groups is the (development version of the) CHEVIE package for GAP3 (see [20, 44]). In particular it contains an increasing number of matrix models for irreducible representations of the Hecke algebras (an therefore of the associated braid group).

3 Braid Groups

A crucial consequence of Steinberg's theorem is that the action of W on the complement $\mathbb{C}^r \setminus \bigcup \mathcal{A}$ of the attached reflection arrangement $\mathcal{A} = \{\mathrm{Ker}(s-1); s \in \mathcal{R}\}$ is free. Broué, Malle and Rouquier defined a (generalized) braid group attached to $W \subset \mathrm{GL}_r(\mathbb{C})$ as $B = \pi_1(X/W)$. It fits into a short exact sequence $1 \to P \to B \to W \to 1$, where $P = \pi_1(X)$ is the fundamental group of the hyperplane complement. They defined (conjugacy) classes of distinguished generators for B, called *braided reflections*, which map onto (pseudo-)reflections, and which generate the group. More generally, they proved that every parabolic subgroup W_S of W can be lifted to a 'parabolic' subgroup B_S of B, isomorphic to the braid group of W_S, in a way which is well-defined up to P-conjugacy.

When W is a real reflection group, there are distinguished 'base-points', namely the (contractible) components of the real hyperplane complement, and distinguished generators attached to a choice of such a component (so-called Weyl chamber), namely the straight loops around the walls of the Weyl chamber inside the orbit space X/W (see [11]). The corresponding braid group is known as an Artin group of finite Coxeter type, and these groups are well-understood: there is a nice presentation mimicking the Coxeter presentation for W, and all the conjectures that we are exploring in this paper are known for them. Therefore, the list of exceptional groups which need to be taken care of actually ends at G_{34}.

One thing which is useful to keep in mind when studying B is that, up to (abstract) group isomorphisms, the correspondence $W \mapsto B$ is not 1–1. Actually, every B can be obtained by only considering the 2-reflections groups (that is, complex reflection groups whose pseudo-reflections all have order 2). Therefore, when proving purely group-theoretic properties of B (not of P !), one can assume that W is a 2-reflections group.

3.1 Center

The center of P obviously contains the homotopy class $\boldsymbol{\pi}$ of the loop $t \mapsto e^{2\pi i t} x_0$, where $x_0 \in X$ is the chosen base-point. Since $e^{\frac{2\pi i}{|Z(W)|}}$ is a generator of the cyclic group $Z(W)$, Broué, Malle and Rouquier defined a central element $\boldsymbol{\beta}$ of B as the homotopy class of the loop $t \mapsto W.(e^{\frac{2\pi i t}{|Z(W)|}} x_0)$. They conjectured

1. $Z(P)$ is infinite cyclic, generated by $\boldsymbol{\pi}$.
2. $Z(B)$ is infinite cyclic, generated by $\boldsymbol{\beta}$.

3. The exact sequence $1 \to P \to B \to W \to 1$ induces a short exact sequence $1 \to Z(P) \to Z(B) \to Z(W) \to 1$.

Item (ii) was proven by Bessis in [7]. Items (i) and (iii) were proven by Digne, Marin and Michel in [26].

3.2 Presentations

It was conjectured in [14] that there exists 'nice' finite presentations, similar to the Artin presentations in the Coxeter case, and in particular satisfying the following requirements:

- the generators are $n(W)$ braided reflexions, where $n(W)$ is the minimal number of reflections needed to generated W
- the relations are homogeneous and positive.

A theorem of Bessis states that such a presentation exists as soon as the highest degree of W is regular in the sense of Springer (see [6]). This applies to all exceptional reflection groups except G_{15}, for which such a presentation was already known (see [14]). The case of the general series had also been established in [14], and therefore the result is known in all cases.

For practical purposes however, one needs more precise results, specific to each of the exceptional cases. To this end, Bessis and Michel manufactured a GAP3 package, additional to CHEVIE, called VKCURVE (see [8]). This software computes presentations for complements of algebraic curves and can be useful also in other contexts. Experimental presentations for all the exceptional groups were presented in [8], their rigorousness were subsequently justified in [7], and additional presentations are given in [35].

3.3 Additional Properties

We gather here a few results obtained on these generalized braid groups since [14].

3.3.1 Word Problem, Conjugacy Problem

It is known how to solve the word problem and the conjugacy problem for these groups. One major tool for this is the construction by Bessis in [7] of so-called Garside monoids, as introduced in [23], for all well-generated groups. This proves that the corresponding braid groups are the groups of fractions of such a monoid, and therefore are torsion-free, and have decidable word and conjugacy problem. For the general series, these consequences were clear from the description of B in

[14], while for the exceptional groups of rank 2 they can be easily deduced from the description in [4] of the braid group (see also e.g. [26, 46] for some Garside monoids aspects in these cases, too). In the case of G_{31}, one needs more involved tools: the group B, viewed as groupoid, is equivalent to a Garside groupoid in the sense of [24] (see again [7]) and therefore has a solvable word and conjugacy problem, too.

3.3.2 Homology

It was known earlier that, for the general series (see [45]), the Coxeter groups and the groups of rank 2, the spaces X and X/W are $K(\pi, 1)$. It was proved by Bessis in [7], that this result is also true for all exceptional groups. This reproves that B is always torsion-free, and also provides a way to compute the homology of B from X/W. The introduction of Garside monoids in [7] also provides, using the work of Dehornoy-Lafont in [22], several complexes from which the homology of B can be in principle computed. From this, the homology of B for all exceptional groups but the higher homology groups of G_{34} was computed in [16]. The integral homology for the general series still remains a bit mysterious (see [16] for partial results).

3.3.3 Linearity?

The proof of the linearity of the usual braid group [9, 30], and its subsequent extension to the Artin groups of finite Coxeter type [21, 25], has been a major breakthrough. It was achieved using a linear representation that we will call the Krammer representation. A detailed study of this representation provided additional properties of the group: that they can be seen as Zariski-dense subgroups of the general linear group (and therefore essentially cannot be decomposed as direct products), and that the pure braid groups are residually torsion-free nilpotent [36, 37].

A natural question is then whether the similar properties hold in the general case. This was conjectured in [39], where a generalization of the Krammer representation has been constructed. It is shown there that the faithfulness of this representation would have the same consequences on the structure of the group as in the real case. The construction of [39] focus on 2-reflection groups. It has been generalized by Chen in [19] to arbitrary reflection groups.

4 Hecke Algebras

One can attach to W a Laurent polynomial ring $R = \mathbb{Z}[u_{i,c}^{\pm}]$, where c runs among the conjugacy classes of distinguished pseudo-reflections, and $0 \leqslant i < e_c$ where e_c is the order of an arbitrary pseudo-reflection inside the conjugacy class c. There is a natural specialization morphism $\theta : R \to K$, $u_{j,c} \mapsto \exp(-2\pi \mathrm{i} j/e_c)$, and a useful ring automorphism of R induced by $u_{i,c} \mapsto u_{i,c}^{-1}$, that we denote $z \mapsto \bar{z}$.

The Hecke algebra associated to W is defined as the quotient of the group algebra RB of B by the relations $\prod_{0\leqslant i<e_c}(\sigma - u_{i,c}) = 0$, where c runs among the conjugacy classes of pseudo-reflections, and σ runs among the braid reflections mapping to a pseudo-reflection in c under the natural mapping $B \to W$. Since two braided reflections mapping to the same pseudo-reflection are conjugated, it is enough to impose only one relation per conjugacy class. We have a natural isomorphism $H \otimes_\theta K \simeq KW$, and therefore H can be seen as a deformation of KW.

The automorphism $z \mapsto \bar{z}$ of R can be extended to an anti-automorphism of RB by putting $\bar{b} = b^{-1}$ for all $b \in B$, and this induces an anti-automorphism of H, since the defining ideal of H is easily checked to be invariant under this anti-automorphism.

4.1 Freeness Conjecture

The basic conjecture about H is that it should be a free module over R, of rank $|W|$. It has been proven in [14] that it is enough to show that H is spanned over R by $|W|$ elements. A weak version of this conjecture states, as an important first step, that H should be at least finitely generated as a R-module. By general arguments based on Tits' deformation theorem, this weak version is strong enough to imply that, after extension of scalars to an algebraic closure F of the field of fractions of R, there exists an isomorphism $H \otimes_R F \simeq FW$ (see e.g. [40]). A more difficult result due to Losev (see [32]) states that it also implies that every specialization $\varphi : R \to \mathbb{C}$ of H to the complex numbers has the same dimension: $\dim H \otimes_\varphi \mathbb{C} = |W|$. An account on the earlier works on this conjecture can be found in the introduction of [40]. We just recall from there that the case of the general series (strong version) is proved in [1, 3, 12]. We focus here on the most recent and inclusive results on exceptional groups.

The weak version is now known for every group, thanks to results of Etingof-Rains ([27]; see also [40]) for the groups of rank 2 (from G_4 to G_{22}), Marin [38, 40] for the groups G_{25}, G_{26}, G_{32}, and Marin-Pfeiffer (see [41]) for the remaining groups $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$.

The full version is, for now, known for all the groups of rank at least 3 [38, 40, 41], and all the groups belonging to the first two families of groups of rank 2 (from G_4 to G_{15}) by work of E. Chavli (in the course of writing, see [17]) plus the groups G_{16} (Chavli, see [18]) and G_{22} (Marin-Pfeiffer, see [41]). The remaining groups are G_{17}, G_{18}, G_{19}, G_{20}, G_{21}. For these five groups, the freeness conjecture is still open, but seems now to be within reach. In all the cases for which this version is proved, one can actually find a basis originating from the braid group itself, as expected in [15, 1.17].

4.2 Trace Conjecture(s)

It is conjectured that H admits the structure of a *symmetric algebra* over R. This means that there should exist a *symetrizing trace* $t : H \to R$, that is a R-linear form satisfying $t(ab) = t(ba)$, such that the associated map $H \to \mathrm{Hom}_R(H, R)$, $x \mapsto (y \mapsto t(xy))$ is an isomorphism. It was proved in [10] that H satisfies this conjecture for the general series.

This property is important in particular in order to understand the possible specializations. A computational understanding of such a trace is related to the knowledge of the so-called *Schur elements* associated to it. These elements are essentially (the inverse of) the coefficients of the decomposition of such a trace as a linear combination of the matrix traces associated to the irreductible representations of $H \otimes_R F$, here assumed by the freeness conjecture to be isomorphic to FW.

It was proved in [15] that, if the freeness conjecture is true, then there exists at most one trace satisfying the following properties.

1. t is a symmetrizing trace
2. $t_0 = \theta \otimes t : KW \simeq H \otimes_\theta K \to R \otimes_\theta K \simeq K$ is the usual symmetrizing trace on KW, defined by $t_0(w) = 0$ if $w \in W \setminus \{1\}$, $t_0(1) = 1$.
3. for all $b \in B$, we have $t(\boldsymbol{\pi})\overline{t(b^{-1})} = t(b\boldsymbol{\pi})$.

If there is such a trace, these conditions define a canonical trace on H. However, the fact that the trace constructed in [10] for the general series satisfies this condition is apparently still conjectural and this is an exception to the usual motto that 'everything is known for the infinite series'. So far, the only (non-Coxeter) exceptional groups for which this conjecture has been proved are G_4, G_{12}, G_{22} and G_{24}, in [35] (the case of G_4 was later independently checked by the author in [43]), under the freeness assumption. Moreover, in these three cases, the trace used satisfies the characterization above. Since the freeness conjecture is now known to hold for these three groups, this solves the trace conjectures for these cases.

We also mention that, under the validity of the freeness conjecture, Malle constructed the Schur elements associated to a potential trace satisfying similarly nice properties, for every exceptional complex reflection groups in [33, 34].

4.3 Additional (Conjectural) Properties

In the case of Coxeter groups, much more structural properties of the Hecke algebras are known. A natural and widely unanswered question is whether these properties can be extended to the general case. Among these, two probably deserve a natural interest. Before stating them, we recall that, as a corollary of the lifting of parabolic subgroups to the braid group, the choice of a parabolic subgroup W_0 of W endows H with the structure of a H_0-module, where H_0 denotes the Hecke algebra. This structure is well-defined up to P-conjugacy.

The two properties in question, for Coxeter groups, are the following ones:

1. if W_0 is a parabolic subgroup of W, then H is a free H_0-module of rank $|W|/|W_0|$
2. if S is an R-algebra, then the center of $S \otimes_R H$ is a free S-module.

Among the exceptional groups, the first property has been proved only for a couple of inclusions (W, W_0), in the course of proving the freeness conjecture. These are: (G_{32}, G_{25}), (G_{25}, G_4), $(G_4, \mathbb{Z}_3)$, $(G_{25}, \mathbb{Z}_3 \times \mathbb{Z}_3)$, (G_{26}, G_4) (see [38, 40]), $(G_8, \mathbb{Z}_4)$, $(G_{16}, \mathbb{Z}_5)$ (see [18]), $(G_{12}, \mathbb{Z}_2)$, $(G_{22}, \mathbb{Z}_2)$, (G_{24}, B_2), (G_{27}, B_2), (G_{29}, B_3), (G_{31}, A_3), (G_{33}, A_4), (G_{33}, D_4) (G_{34}, G_{33}) (see [41]). Concerning the second property, it has been proven for $S = R$ by Francis [28] for the groups G_4 and $G(4,1,2)$. We are not aware of any other result in this direction.

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A Generalization of the Davis-Januszkiewicz Construction and Applications to Toric Manifolds and Iterated Polyhedral Products

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Abstract The fundamental Davis–Januszkiewicz construction of toric manifolds is reinterpreted in order to allow for an explanation, within their formalism, of the authors' previous construction of infinite families of toric manifolds. The latter uses the simplicial wedge J-construction. Consequences of a recent generalization of the J-construction by Ayzenberg (Trans. Moscow Math. Soc. **74**, 175–202, 2013)

Keywords Davis-Januszkiewicz construction • J construction • Moment-angle complex • Non-singular toric variety • Polyhedral product • Quasitoric manifold • Simplicial wedge • Smooth toric variety • Toric manifold

1 Introduction

The topological approach to non-singular toric varieties requires two ingredients:

1. a simple polytope P^n of dimension n having a set $\mathcal{F}$ of m facets and
2. a *characteristic* function $\lambda: \mathcal{F} \longrightarrow \mathbb{Z}^n$ which assigns an integer vector to each facet of the simple polytope P^n.

S. Gitler was deceased at the time of publication.

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The latter can be considered as an $(n \times m)$-matrix $\lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$ with integer entries and columns indexed by the facets of P^n. A regularity condition, which ensures the smoothness of the toric manifold, requires all $n \times n$ minors of λ corresponding to the vertices of P^n to be $+1$ or -1.

Associated to the pair (P^n, λ), Davis and Januszkiewicz [6], constructed two spaces:

$$\mathcal{L} = T^m \times P^n \big/ \sim$$

and a toric manifold[1]

$$M^{2n} = T^n \times P^n \big/ \sim_\lambda \; .$$

The properties of the spaces $\mathcal{L}$ have been studied extensively via an alternative general construction developed by Buchstaber and Panov [5], who gave them the name "moment-angle complexes"

$$\mathcal{L} \;=\; T^m \times P^n \big/ \sim \quad \cong \quad Z(K_P; (D^2, S^1)).$$

In the notation used here, K_P represents the simplicial complex dual to the boundary of a simple polytope P^n. From this point of view, the toric manifold M^{2n} is recovered as the quotient $Z(K_P; (D^2, S^1))\big/\ker \lambda$.

The main results presented in the authors' earlier work [4], arise from a construction on a simplicial complex K_P having m vertices. For each sequence $J = (j_1, j_2, \ldots, j_m)$ of positive integers, a new simplicial complex $K_P(J)$ is constructed,

$$K_P \rightsquigarrow K_P(J).$$

Also, associated to P^n is another simple polytope $P(J)$ and $K_P(J) = K_{P(J)}$. Everything fits together in such a way that, from the toric manifold (P^n, λ, M^{2n}), it is possible to construct another toric manifold $\big(P(J), \lambda(J), M(J)\big)$. In the context of moment-angle complexes and polyhedral products [2, 3], it is shown in [4] that there is a diffeomorphism of orbit spaces

$$Z(K_P; (\underline{D}^{2J}, \underline{S}^{2J-1}))\big/\ker \lambda \;\longrightarrow\; Z\big(K_P(J); (D^2, S^1))\big)\big/\ker \lambda(J) \tag{1}$$

[1] Sometimes, this manifold is called a "quasitoric manifold".

which defines $M(J)$. Here, the left hand side uses the notation of a polyhedral product, (*generalized* moment-angle complex), associated to the family of pairs

$$(\underline{D}^{2J}, \underline{S}^{2J-1}) \;=\; \{(D^{2j_i}, S^{2j_i})\}_{i=1}^{m}.$$

These ideas are elaborated upon in Sect. 5. The right hand side of (1) involves an ordinary moment-angle complex and fits in with the formalism of Davis–Januszkiewicz and Buchstaber–Panov. A natural question arises about the left hand side which involves the combinatorics of the smaller simplicial complex K_P: *where is this visible in the standard Davis–Januszkiewicz construction of toric manifolds*? One of the primary goals here is to answer this question.

The details of the Davis–Januszkiewicz construction are reviewed in Sect. 2 and the modifications necessary to generalize the construction are described in Sect. 3. The modified construction is interpreted in terms of polyhedral products in Sect. 4 and the answer to the question posed above is presented in Sect. 5. Additional applications involving the "composed complex" constructions of Ayzenberg [1], are discussed in Sects. 6 and 7.

2 A Review of the Davis-Januszkiewicz Construction

A toric manifold M^{2n} is a manifold covered by local charts $\mathbb{C}^n$, each with the standard action of a real n-dimensional torus T^n, compatible in such a way that the quotient M^{2n}/T^n has the structure of a *simple* polytope P^n. Here, "simple" means that P^n has the property that at each vertex, exactly n facets intersect. Under the T^n action, each copy of $\mathbb{C}^n$ must project to an $\mathbb{R}^n_+$ neighborhood of a vertex of P^n. The fundamental construction of Davis and Januszkiewicz [6, Sect. 1.5] is described briefly below. It realizes all toric manifolds and, in particular, all smooth projective toric varieties. Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$$

denote the set of facets of P^n. The fact that P^n is simple implies that every codimension-l face F can be written uniquely as

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

where the F_{i_j} are the facets containing F. Let

$$\lambda : \mathcal{F} \longrightarrow \mathbb{Z}^n \tag{2}$$

be a function into an n-dimensional integer lattice satisfying the *regularity* condition that whenever $F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$ then $\{\lambda(F_{i_1}), \lambda(F_{i_2}), \ldots, \lambda(F_{i_l})\}$ span an l-dimensional submodule of $\mathbb{Z}^n$ which is a direct summand. Such a map is called a *characteristic function* associated to P^n. Next, regarding $\mathbb{R}^n$ as the Lie algebra of T^n, the map λ is used to associate to each codimension-l face F of P^n a rank-l subgroup $G_F \subset T^n$. Specifically, writing

$$\lambda(F_{i_j}) = (\lambda_{1i_j}, \lambda_{2i_j}, \ldots, \lambda_{ni_j})$$

gives

$$G_F = \left\{\left(e^{2\pi i(\lambda_{1i_1}t_1+\lambda_{1i_2}t_2+\cdots+\lambda_{1i_l}t_l)}, \ldots, e^{2\pi i(\lambda_{ni_1}t_1+\lambda_{ni_2}t_2+\cdots+\lambda_{ni_l}t_l)}\right) \in T^n\right\}$$

where $t_i \in \mathbb{R}$, $i = 1, 2, \ldots, l$. Finally, let $p \in P^n$ and $F(p)$ be the unique face with p in its relative interior. Define an equivalence relation $\sim_\lambda$ on $T^n \times P^n$ by $(g, p) \sim_\lambda (h, q)$ if and only if $p = q$ and $g^{-1}h \in G_{F(p)} \cong T^l$. Then

$$M^{2n} \cong M^{2n}(\lambda) = T^n \times P^n \big/ \sim_\lambda \tag{3}$$

is a smooth, closed, connected, $2n$-dimensional manifold with T^n action induced by left translation [6, p. 423]. A projection $\pi: M^{2n} \to P^n$ onto the polytope is induced from the projection $T^n \times P^n \to P^n$.

Remark 1 In the cases when M^{2n} is a projective non-singular toric variety, P^n and λ encode topologically the information in the defining fan, [5, Chap. 5].

Let K_P denote the simplicial complex dual to the boundary of a simple polytope P^n having m facets. Recall that the duality here is in the sense that the facets of P^n correspond to the vertices of K_P. A set of vertices in K_P is a simplex if and only if the corresponding facets in P^n all intersect. Davis and Januszkiewicz constructed a second space in [6], which came to be known as a *moment-angle manifold*, by

$$\mathcal{L} = T^m \times P^n \big/ \sim \tag{4}$$

where here $\sim$ does not involve the characteristic λ but the combinatorics of the simplicial complex K_P only. Here also, the circles in T^m are indexed by the facets of P^n. The equivalence relation $\sim$ is defined by analogy with that of (3). Specifically, λ in (2) is replaced by

$$\theta: \mathcal{F} \longrightarrow \mathbb{Z}^m \tag{5}$$

where $\theta(F_i) = \underline{e}_i \in \mathbb{Z}^m$.

Constructions (4) and (3) are related by a quotient map given by the free action of $\ker\theta$ on $\mathcal{L}$

$$T^m \times P^n \big/ \sim \longrightarrow \left(T^m \times P^n \big/ \sim\right) \big/ \ker\lambda \;\cong\; T^n \times P^n \big/ \sim_\lambda \tag{6}$$

as described in [5, Sect. 6.1].

3 Modifying the Equivalence Relations

As above, let P^n be simple polytope. The construction of (4) is generalized easily by first replacing each of the circles in T^m by spaces $X_1, X_2, \dots, X_m$, indexed by the facets of P^n.

Construction 1 *Define an equivalence relation $\sim_1$ on the Cartesian product*

$$X_1 \times X_2 \times \cdots \times X_m \times P^n$$

as follows:

$$(x_1, x_2, \dots, x_m, p) \;\sim_1\; (y_1, y_2, \dots, y_m, q)$$

if and only if:

(a) $p = q$ *and*
(b) *when p is in the relative interior of the face* $F(p) = F_{j_1} \cap F_{j_2} \cap \cdots F_{j_k}$ *given as the intersection of the k facets which are complementary to* $\{F_{i_1}, F_{i_2}, \dots, F_{i_{m-k}}\}$, *then* $x_{i_s} = y_{i_s}$ *for all* $s \in \{1, 2, \dots, m-k\}$.

Equivalence classes of points in $(X_1 \times X_2 \times \cdots \times X_m) \times P^n \big/ {\sim_1}$ *are denoted by the symbol* $\big[(x_1, x_2, \dots, x_m, p)\big]_1$.

Suppose now that S^1 acts *freely* on the spaces $X_1, X_2, \dots, X_m$, giving an action of T^m on $X_1 \times X_2 \times \cdots \times X_m$ in the obvious way. Recall that the function θ of (5) indexes the "coordinate" circles in T^m by the facets of P^n. Also, each space X_i is associated with the facet F_i, So, an intersection of k facets in P^n determines a projection $T^m \longrightarrow T^{m-k}$ and, by this projection, T^m acts on the product $X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}}$.

Next, let λ be a characteristic map specified for the polytope P^n. Then

$$\ker\lambda \;\cong\; T^{m-n} \;\subset\; T^m.$$

For $k \leqslant n$, there is the induced action of $\ker\lambda \subset T^m$ on the product

$$X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}}$$

and a projection

$$\pi_{i_1, i_2, \dots, i_{m-k}}\colon X_1 \times X_2 \times \cdots \times X_m \big/ \ker\lambda \;\longrightarrow X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}} \big/ \ker\lambda \qquad (7)$$

corresponding to each intersection of k facets.

Equivalence classes of points in $X_1 \times X_2 \times \cdots \times X_m \big/ \ker\lambda$ are denoted by the symbol $[x_1, x_2, \dots, x_m]_\lambda$. The next construction generalizes that of (3).

Construction 2 *Define an equivalence relation* $\sim_2$ *on the Cartesian product*

$$\big(X_1 \times X_2 \times \cdots \times X_m \big/ \ker\lambda\big) \times P^n$$

as follows:

$$\big([x_1, x_2, \ldots, x_m]_\lambda, p\big) \sim_2 \big([y_1, y_2, \ldots, y_m]_\lambda, q\big)$$

if and only if:

(i) $p = q$
(ii) when p *is in the relative interior of the face* $F(p) = F_{j_1} \cap F_{j_2} \cap \cdots F_{j_k}$ *given as the intersection of the* k *facets which are complementary to* $\{F_{i_1}, F_{i_2}, \ldots, F_{i_{m-k}}\}$, *then*

$$\pi_{i_1, i_2, \ldots, i_{m-k}}([x_1, x_2, \ldots, x_m]_\lambda) = \pi_{i_1, i_2, \ldots, i_{m-k}}([y_1, y_2, \ldots, y_m]_\lambda).$$

Equivalence classes of points in $\big(X_1 \times X_2 \times \cdots \times X_m \big/ \ker\lambda\big) \times P^n \big/ {\sim_2}$ are denoted by the symbol $\big[([x_1, x_2, \ldots, x_m]_\lambda, p)\big]_2$.

Construction 2 can be reinterpreted as follows. The group $\ker\lambda$ acts on the space $(X_1 \times X_2 \times \cdots \times X_m) \times P^n \big/ {\sim_1}$ by

$$t \cdot [(x_1, x_2, \ldots, x_m, p)]_1 \;=\; [t \cdot (x_1, x_2, \ldots, x_m), p)]_1.$$

Property (b) in Construction 1 ensures that the action is well defined. The next lemma, the analogue of (6), follows naturally.

Lemma 1 *There is a homeomorphism*

$$h\colon \big(X_1 \times X_2 \times \cdots \times X_m \big/ \ker\lambda\big) \times P^n \big/ {\sim_2} \quad\longrightarrow\quad \big((X_1 \times X_2 \times \cdots \times X_m) \times P^n \big/ {\sim_1}\big) \big/ \ker\lambda$$

given by

$$h\Big(\big[([x_1, x_2, \ldots, x_m]_\lambda, p)\big]_2\Big) \;=\; \big[[(x_1, x_2, \ldots, x_m, p)]_1\big]_\lambda.$$

Proof To see that h is well defined, suppose

$$\big[([x_1, x_2, \ldots, x_m]_\lambda, p)\big]_2 \;=\; \big[([y_1, y_2, \ldots, y_m]_\lambda, p)\big]_2$$

with p is in the relative interior of the face $F(p) = F_{j_1} \cap F_{j_2} \cap \cdots F_{j_k}$ given as the intersection of the k facets which are complementary to $\{F_{i_1}, F_{i_2}, \ldots, F_{i_{m-k}}\}$, as in Construction 2. Then, $\pi_{i_1, i_2, \ldots, i_{m-k}}([x_1, x_2, \ldots, x_m]) = \pi_{i_1, i_2, \ldots, i_{m-k}}([y_1, y_2, \ldots, y_m])$ and hence,

$$t \cdot (x_{i_1}, x_{i_2}, \ldots, x_{i_{m-k}}) \;=\; (y_{i_1}, y_{i_2}, \ldots, y_{i_{m-k}})$$

for some $t \in \ker \lambda$. It follows now from Construction 1 that

$$\big[([x_1, x_2, \ldots, x_m]_\lambda, p)\big]_2 \;=\; \big[([y_1, y_2, \ldots, y_m]_\lambda, p)\big]_2$$

as required.

To check that h is an injection, suppose that

$$h\Big(\big[([x_1, x_2, \ldots, x_m]_\lambda, p)\big]_2\Big) \;=\; h\Big(\big[([y_1, y_2, \ldots, y_m]_\lambda, p)\big]_2\Big).$$

Then $t \in \ker \lambda$ exists so that

$$t \cdot [(x_1, x_2, \ldots, x_m, p)]_1 \;=\; [t \cdot (x_1, x_2, \ldots, x_m), p)]_1 \;=\; [(y_1, y_2, \ldots, y_m, p)]_1.$$

Next, write $t \cdot (x_1, x_2, \ldots, x_m) = (u_1, u_2, \ldots, u_m)$. Since p is in the relative interior of the face $F(p) = F_{j_1} \cap F_{j_2} \cap \cdots F_{j_k}$, it follows that

$$u_{i_s} = y_{i_s} \quad \text{for all} \quad s \in \{1, 2, \ldots, m-k\}$$

corresponding to the complementary facets. It follows that

$$\pi_{i_1, i_2, \ldots, i_{m-k}}([x_1, x_2, \ldots, x_m]_\lambda) = \pi_{i_1, i_2, \ldots, i_{m-k}}([y_1, y_2, \ldots, y_m]_\lambda)$$

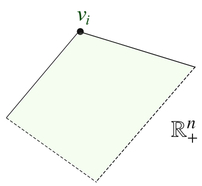
and hence

$$\big[([x_1, x_2, \ldots, x_m]_\lambda, p)\big]_2 \;=\; \big[([y_1, y_2, \ldots, y_m]_\lambda, p)\big]_2.$$

It is easy now to see that h is a homeomorphism.

4 Interpreting the Generalizations in the Buchstaber–Panov Formalism

Construction (4) can be analyzed locally. In the neighbourhood of a vertex v_i, a simple polytope P^n looks like $\mathbb{R}^n_+$.



The polytope can be given a *cubical* structure as in [5, Construction 5.8 and Lemma 6.6]. The cube I^n, *anchored* by the vertex v_i, sits inside the copy of $\mathbb{R}^n_+$ obtained by deleting all faces of P^n which do not contain v_i.

Locally, $T^m \times P^n$ is

$$T^m \times I^n \;\cong\; (S^1 \times I)^n \times (S^1)^{m-n} \tag{8}$$

Recall that all the circles in T^m are indexed by the facets of the polytope so here, the order of factors has been shuffled naturally. The factors S^1 which are paired with a copy of I are those corresponding to the facets of P^n which meet at v_i. The effect of the equivalence relation $\sim$ in $T^m \times I^n/\sim$, (4), is to convert every $S^1 \times I$ on the right hand side of (8) into a disc by collapsing $S^1 \times \{0\}$ to a point. So

$$T^m \times I/\sim \quad\cong\quad (D^2)^n \times (S^1)^{m-n}. \tag{9}$$

The vertices of P^n correspond to the maximal simplices of the simplicial complex K_P so, assembling the blocks (9) gives the moment-angle manifold

$$\mathcal{L} \;=\; T^m \times P^n/\sim \quad\cong\quad Z(K_P;(D^2,S^1)).$$

As described in [6] and [5], the map λ determines a subtorus $\ker\lambda = T^{m-n} \subset T^m$ and a commutative diagram of quotient maps

$$\begin{array}{ccc} T^m \times P^n/\sim & \longrightarrow & T^n \times P^n/\sim_\lambda \\ \big\downarrow\cong & & \big\downarrow\cong \\ Z(K_P;(D^2,S^1)) & \longrightarrow & Z(K_P;(D^2,S^1))/\ker\lambda\,. \end{array} \tag{10}$$

The construction above is generalized easily by replacing each of the circles in T^m by spaces $X_1, X_2, \ldots, X_m$ indexed by the facets of P^n. Again, locally in the neighbourhood of a vertex v_i, at which facets $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ meet, $X_1 \times X_2 \times \cdots \times X_m \times P^n$ is

$$\begin{aligned} &X_1 \times X_2 \times \cdots \times X_m \times I^n \\ &\qquad = (X_{i_1} \times X_{i_2} \times \cdots \times X_{i_n} \times I^n) \;\times\; X_{i_{n+1}} \times X_{i_{n+2}} \times \cdots \times X_{i_m} \\ &\qquad = (X_{i_1} \times I) \times (X_{i_2} \times I) \times \cdots \times (X_{i_n} \times I) \;\times\; X_{i_{n+1}} \times X_{i_{n+2}} \times \cdots \times X_{i_m} \end{aligned}$$

As everything in the Cartesian product is indexed by the facets of the polytope, the order of the factors here has been shuffled naturally. Finally, the equivalence relation $\sim_1$ on

$$(X_1 \times X_2 \times \cdots \times X_m) \times I^n$$

converts every $X_{i_k} \times I$ into the cone CX_{i_k}. in a natural way. So,

$$(X_1 \times X_2 \times \cdots \times X_m) \times I^n / \sim_1 \\ \cong \quad CX_{i_1} \times CX_{i_2} \times \cdots \times CX_{i_n} \times (X_{i_{n+1}} \times X_{i_{n+2}} \times \cdots \times X_{i_m})$$

Assembling over all the vertices of P^n along the common intersection determined by the cubical structure on P^n, gives the polyhedral product

$$Z\Big(K_P; \big(\underline{CX_i},\ \underline{X_i}\big)\Big) \quad \subseteq \quad CX_1 \times CX_2 \times \cdots \times CX_m.$$

just as in the case $X_i = S^1$ for standard moment-angle complexes, [5, Sect. 6.1].

A choice of cubical structure for the simple polytope P^n allows a choice of homeomorphism

$$\alpha\colon X_1 \times X_2 \times \cdots \times X_m \times P^n / \sim_1 \quad \longrightarrow \quad Z\big(K_P; \big(\underline{CX},\ \underline{X}\big)\big). \tag{11}$$

The fact that S^1 acts (freely) on X_i is now used again. The action extends to CX_i by preserving the cone parameter. Consequently, there is an action of T^m on $CX_1 \times CX_2 \times \cdots \times CX_m$ which extends to

$$Z\Big(K_P; \big(\underline{CX},\ \underline{X}\big)\Big) \ \subseteq \ CX_1 \times CX_2 \times \cdots \times CX_m.$$

In exactly the same way as it does for the case of $\ker\lambda$ acting on $Z(K_P; (D^2, S^1))$, the regularity condition on the characteristic map λ, (Sect. 2), ensures that $\ker\lambda \subset T^m$ acts freely on $Z\Big(K_P; \big(\underline{CX},\ \underline{X}\big)\Big)$.

Remark 2 With respect to this action of $\ker\lambda$, the homeomorphism α above, is equivariant and induces a homeomorphism of orbit spaces

$$\overline{\alpha}\colon \big((X_1 \times X_2 \times \cdots \times X_m) \times P^n / \sim_1 \big) / \ker\lambda \ \longrightarrow \ Z\big(K_P; (\underline{CX},\ \underline{X})\big) / \ker\lambda\,.$$

The next theorem is the analogue of (10).

Theorem 3 *The following diagram commutes:*

$$\begin{array}{ccc} X_1 \times X_2 \times \cdots \times X_m \times P^n / \sim_1 & \xrightarrow{\beta} & \big(X_1 \times X_2 \times \cdots \times X_m / \ker\lambda\big) \times P^n / \sim_2 \\ \alpha \big\downarrow \cong & & \gamma \big\downarrow \cong \\ Z\big(K_P; (\underline{CX},\ \underline{X})\big) & \xrightarrow{\delta} & Z\big(K_P; (\underline{CX},\ \underline{X})\big) / \ker\lambda \end{array} \tag{12}$$

where the map β is the composite of the quotient map

$$\big((X_1 \times X_2 \times \cdots \times X_m) \times P^n / \sim_1 \big) \ \longrightarrow \big((X_1 \times X_2 \times \cdots \times X_m) \times P^n / \sim_1 \big) / \ker\lambda$$

with the map h^{-1} of Lemma 1, and γ is the composite $\overline{\alpha} \circ h^{-1}$.

Proof The homeomorphism h can be used to replace the space in the top right hand corner with the space $\big((X_1 \times X_2 \times \cdots \times X_m) \times P^n \big/ \sim_1\big) \big/ \ker\lambda$. This gives a new commutative diagram by the equivariance of the homeomorphism α. The maps β and δ are defined in terms of the map h^{-1} and so the diagram commutes as given.

The next remark confirms that the original Davis-Januszkiewicz constructions are preserved.

Remark 3 For the case $X_i = S^1$ and S^1 acting on itself in the usual way, Constructions 1 and 2 agree with those of (4) and (3).

5 Application to the Construction of Infinite Families of Toric Manifolds

5.1 The Case of Odd Spheres

The first application is to the infinite families of toric manifolds constructed in [4] and summarized briefly below.

Let K be a simplicial complex of dimension $n-1$ on vertices $\{v_1, v_2, \ldots, v_m\}$. Given a sequence of positive integers $J = (j_1, j_2, \ldots, j_m)$, define a new simplicial complex $K(J)$ on new vertices

$$\big\{v_{11}, \ldots, v_{1j_1}, v_{21}, \ldots, v_{2j_2}, \ldots, v_{m1}, \ldots, v_{mj_m}\big\},$$

with the property that

$$\big\{v_{i_1 1}, \ldots, v_{i_1 j_{i_1}}, \quad \ldots \quad , v_{i_k 1}, \ldots, v_{i_k j_{i_k}}\big\}$$

is a minimal non-face of $K(J)$ if and only if $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimal non-face of K. Moreover, all minimal non-faces of $K(J)$ have this form. A result of Provan and Billera, [8, p. 578], ensures that if $K = K_P$ is dual to the boundary of a simple polytope P, then $K(J)$ is dual to the boundary of another simple polytope $P(J)$. That is

$$K_P(J) = K_{P(J)}. \tag{13}$$

More details may be found in [4, Sect. 2].

Remark 4 It is the case also that the polytope $P(J)$ can be constructed *directly* from P in a straightforward way.

Let (P^n, λ, M^{2n}) specify a toric manifold as in (3). From this, it is possible to construct another toric manifold $\big(P(J), \lambda(J), M(J)\big)$ where the numbers m and n,

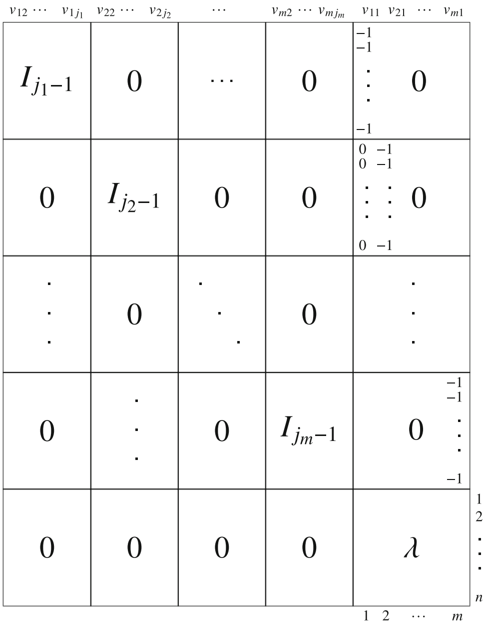


Fig. 1 The matrix $\lambda(J)$

from Sect. 2, transform as follows.

$$\begin{bmatrix} m \\ n \\ m-n \end{bmatrix} \rightsquigarrow \begin{bmatrix} d(J) = j_1 + j_2 + \cdots + j_m \\ d(J) - m + n \\ m - n \end{bmatrix} \tag{14}$$

In terms of the original characteristic map λ, the matrix specifying the characteristic map $\lambda(J)$ is given in Fig. 1 below. In that figure, I_{j_i-1} represents the identity matrix of size $j_i - 1$. It is clear from the form of the matrix $\lambda(J)$ and the definition of $K(J)$, that the Davis-Januszkiewicz cohomology calculation, [6, Theorem 4.14], expresses the integral cohomology of $M(J)$ in terms of the sequence J, the original matrix λ and the combinatorics of K.

Also in [4] is an interpretation of this construction of the toric manifolds $M(J)$ in terms of generalized moment-angle complexes. To see this, consider the family of CW pairs

$$(\underline{D}^{2J}, \underline{S}^{2J-1}) \;=\; \{(D^{2j_i}, S^{2j_i-1})\}_{i=1}^{m}$$

and the associated generalized moment-angle complex $Z(K_P; (\underline{D}^{2J}, \underline{S}^{2J-1}))$. There is an inclusion of tori $T^m \longrightarrow T^{d(J)}$ which includes the ith circle in T^m by the diagonal

$$S^1 \longrightarrow (S^1)^{j_i}. \tag{15}$$

This gives an action of T^m on $Z\big(K_P(J); (D^2, S^1))\big)$. Also, via a choice of diffeomorphism $D^{2j_i} \cong (D^2)^{j_i}$, there is an action of $T^{d(J)}$ on the moment-angle complex $Z(K_P; (\underline{D}^{2J}, \underline{S}^{2J-1}))$. With this understood, there are T^m and $T^{d(J)}$-equivariant diffeomorphisms

$$Z(K_P; (\underline{D}^{2J}, \underline{S}^{2J-1})) \longrightarrow Z\big(K_P(J); (D^2, S^1)\big) \tag{16}$$

from which arises a diffeomorphism of orbit spaces

$$Z(K_P; (\underline{D}^{2J}, \underline{S}^{2J-1}))\big/\ker\lambda \longrightarrow Z\big(K_P(J); (D^2, S^1))\big)\big/\ker\lambda(J) \tag{17}$$

which defines $M(J)$. (Here, $\ker\lambda$ and $\ker\lambda(J)$ are isomorphic subgroups of $T^{d(J)}$.) The appearance of the toric manifold $M(J)$ as the right hand side of (17) is perplexing because that space is not reflected in either the fundamental construction (3) or in diagram (10). The matter is resolved by the diagram of Theorem 3 where the space appears in the bottom right of the diagram with $X_i = S^{2j_i-1}$, and $CX_i = D^{2j_i}$; the group S^1 acts freely on S^{2j_i-1} in the usual way. this observation is formalized in the next theorem.

Theorem 4 *The toric manifolds $M(J)$, defined by either the original Davis-Januszkiewicz construction (3) or equivalently, by the quotients (17), are examples of Construction 2 as follows:*

$$M(J) \;=\; \big(T^{d(J)-m+n}\times P(J)\big)\big/\!\sim_{\lambda(J)} \;\cong\; \big(S^{2j_1-1}\times S^{2j_2-1}\times\cdots\times S^{2j_m-1}\big/\ker\lambda\big)\times P^n\big/\!\sim_2 .$$

Proof This result follows immediately from the right hand side of diagram (12).

Notice here that the advantage of the right hand side is the use of the (generalized) Davis- Januszkiewicz construction with the polytope P^n, which is in general much smaller than $P(J)$ and has simpler combinatorics.

Remark 5 Notice that the part of $\lambda(J)$ in Fig. 1 reproduced below (Fig. 2), is essentially the $(j_2-1)\times j_2$ matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \cdot & & \cdots & \cdot & \cdot \\ \cdot & & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \tag{18}$$

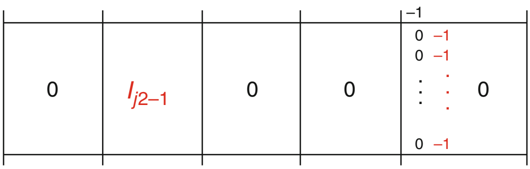


Fig. 2 A typical row of the matrix $\lambda(J)$ from Figure 1.

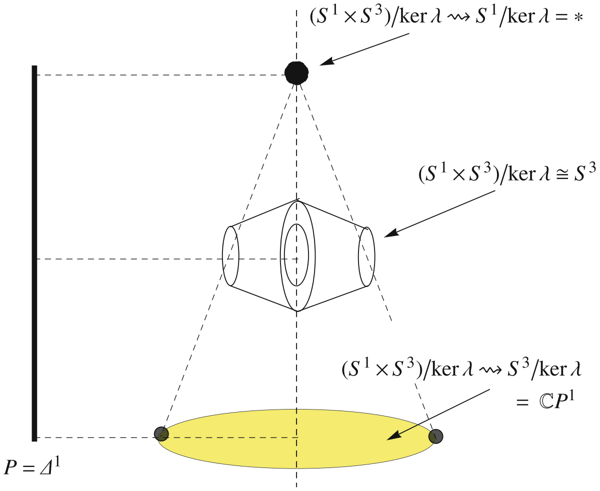
which is the characteristic matrix for the diagonal S^1 action on S^{2j_2-1}. Indeed, the ith "block row" of $\lambda(J)$ is the characteristic matrix for the diagonal S^1 action on S^{2j_i-1}. This particular connection to odd spheres becomes evident in the light of Construction 2 but was not obvious when [4] was written. This observation becomes relevant in the next section.

5.2 A Simple Illustration

The toric manifold $\mathbb{C}P^2$ is made usually by the construction (3) using a two-simplex as the simple polytope. The diagram below illustrates Construction 2 in this case. The ingredients are as follows:

(1) $P^n = \Delta^1$ a one-simplex. Here $n = 1$ and $m = 2$.
(2) $J = (1,2)$ so that $X_1 = S^1$ and $X_2 = S^3$ with the usual free S^1 action.
(3) the characteristic map $\lambda\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ is given by the matrix $[1,-1]$ and $\ker\lambda \cong T^1$ sits inside T^2 as $t \mapsto (t, t^{-1})$.

Remark 6 Here, $P(J) = \Delta^1(1,2) = \Delta^2$ a two-simplex, the usual polytope used to construct $\mathbb{C}P^2$ as a toric manifold from (3).



$$\mathbb{C}P^2 \cong (S^1 \times S^3/\ker\lambda) \times \Delta^1/\sim_2$$

In the diagram, the symbol $\rightsquigarrow$ represents the projection (7) which appears in part (ii) of Construction 2. The diagram presents $\mathbb{C}P^2$ as the cone on S^3 attached to $\mathbb{C}P^1$ via the Hopf map

$$(S^1 \times S^3)/\ker\lambda \xrightarrow{\pi_2} S^3/\ker\lambda.$$

Remark 7 It has been brought to the authors' attention by Michael Wiemeler that the presentation of the manifolds $M(J)$ given by Theorem 4 reveals an action of the group

$$\Big(\prod_{i=1}^{m} U(j_i)\Big)\big/\ker\lambda$$

where here, the group $\ker\lambda$ is naturally a subgroup of the center of $\prod_{i=1}^{m} U(j_i)$. Moreover, classification results, [9, Theorem 1.2, Corollary 5.14], suggest then that the manifolds can be obtained from an iterated construction involving bundles over certain projective spaces specified by the sequence J.

6 Application to Iterated Polyhedral Products

The odd spheres of Sect. 5 are themselves examples of moment-angle complexes

$$S^{2j_i-1} \cong Z\big(K_{\Delta^{j_i-1}};\ (D^2,S^1)\big)$$

where $K_{\Delta^{j_i-1}}$ is the simplicial complex dual to the boundary of the simplex Δ^{j_i-1}. Every moment-angle complex $Z\big(K;(D^2,S^1)\big)$ supports a free circle action and so it's natural to ask about the case $X_i = Z\big(K_i;(D^2,S^1)\big)$ in Construction 2 for a collection $\{K_1,K_2,\dots,K_m\}$ of arbitrary simplicial complexes. In this case, (11) becomes

$$(X_1\times X_2\times\cdots\times X_m)\times P^n\big/\!\sim_1\ \cong\ Z\big(K_P;\big(\underline{CZ(K_i;(D^2,S^1))},\ \underline{Z(K_i;(D^2,S^1))}\big)\big). \tag{19}$$

6.1 A Generalization of the Construction K(J)

The problem of finding an analogue of (16) and (17) now presents itself. Those diffeomorphisms follow from [4, Theorem 7.2] which is a more general result about the behaviour of polyhedral products with respect to "exponentiation" of CW pairs. Recent work by Ayzenberg [1], generalizing this exponentiation construction becomes relevant to understanding the problem further. A brief description of Ayzenberg's construction, tailored to the context here, follows.

Let K be a simplicial complex on m vertices and $\{K_1,K_2,\dots,K_m\}$ a collection of m simplicial complexes on $j_1,j_2,\dots,j_m$ vertices respectively. From these ingredients, a new simplicial complex $K(K_1,K_2,\dots,K_m)$, on $j_1+j_2+\cdots+j_m$ vertices, is constructed by

$$K(K_1,K_2,\dots,K_m)\ =\ \bigcup_{\sigma\in K} V_\sigma\ \subset\ \Delta^{j_1-1}*\Delta^{j_2-1}*\cdots*\Delta^{j_m-1} \tag{20}$$

where

$$V_\sigma = B_1 * B_2 * \cdots * B_m \quad \text{with} \quad B_i = \begin{cases} \Delta^{j_i - 1} & \text{if } i \in \sigma \\ K_i & \text{if } i \notin \sigma. \end{cases}$$

Remark 8 In this language, the construction $K(J)$ at the beginning of Sect. 5.1 is just $K\big(\partial\Delta^{j_1-1}, \partial\Delta^{j_2-1}, \ldots, \partial\Delta^{j_m-1}\big)$ where $\partial\Delta^{j_i-1}$ is the boundary of the $(j_i - 1)$-simplex.

For $K = K_P$, the result analogous to (16) is the following.

Theorem 5 ([1, Proposition 5.1])

$$Z\big(K_P; \big(\underline{(D^2)^{j_i}}, \underline{Z(K_i; (D^2, S^1))}\big)\big) = Z\big(K_P(K_1, K_2, \ldots, K_m); (D^2, S^1)\big).$$

The next task is to relate these spaces to the right hand side of (19). The following proposition addresses this point. As usual, set $d(J) = j_1 + j_2 + \cdots + j_m$ and, to simplify the notation, set

$$Z(K_i) := Z(K_i; (D^2, S^1)).$$

Proposition 1 *There is a $T^{d(J)}$ -equivariant homotopy equivalence of polyhedral products*

$$Z\big(K_P; \big(\underline{CZ(K_i)}, \underline{Z(K_i)}\big)\big) \simeq Z\big(K_P; \big(\underline{(D^2)^{j_i}}, \underline{Z(K_i)}\big)\big). \tag{21}$$

Proof Let K be a simplicial complex on j vertices. Then if K is not the $(j-1)$-simplex Δ^{j-1}, there is simplicial embedding $K \longrightarrow \partial\Delta^{j-1}$ into the boundary. This induces an inclusion

$$Z\big(K; (D^2, S^1)\big) \longrightarrow Z\big(\partial\Delta^{j-1}; (D^2, S^1)\big) = \partial\big((D^2)^j\big) \cong S^{2j-1} \tag{22}$$

equivariant with respect to the action of the j-torus T^j. In turn, this extends to an equivariant homotopy equivalence on cones:

$$CZ\big(K; (D^2, S^1)\big) \longrightarrow CS^{2j-1} \simeq D^{2j}$$

where the action of T^j preserves the cone parameter. Next, choose a standard T^j—equivariant diffeomorphism $h\colon D^{2j} \longrightarrow (D^2)^j$ to get an equivariant homotopy equivalence of CW pairs

$$\big(CZ\big(K; (D^2, S^1)\big), Z\big(K; (D^2, S^1)\big)\big) \xrightarrow{h} \big((D^2)^j, Z\big(K; (D^2, S^1)\big)\big). \tag{23}$$

The functorial properties of the polyhedral product [7, Lemma 2.2.1] and an application of (23) for each $i = 1, 2, \ldots, m$ complete the proof.

Remark 9 In the case that $j = 4$ and K is dual to the boundary of the square, the inclusion (22) is

$$Z\big(K;(D^2,S^1)\big) \;=\; S^3\times S^3 \;\longrightarrow\; Z\big(\partial\Delta^3;(D^2,S^1)\big) \;\simeq\; S^7 \qquad \big(\simeq\; S^3 * S^3\big)$$

and the corresponding homotopy equivalence of pairs is

$$\big(C(S^3\times S^3),\; S^3\times S^3\big) \;\longrightarrow\; \big((D^2)^4,\; S^3\times S^3\big)$$

equivariant with respect to the action of T^4.

6.2 The Case of Moment-Angle Complexes

As before, let P^n be a simple polytope having m facets, equipped with a *characteristic* function

$$\lambda\colon \mathcal{F} \longrightarrow \mathbb{Z}^n$$

satisfying the regularity condition following (2). Regarding

$$\ker\lambda \hookrightarrow T^m \hookrightarrow T^{d(J)}$$

as in (15) and the case of odd spheres, there is a natural *free* action of $\ker\lambda$ on both sides of (21) yielding a homotopy equivalence of orbit spaces

$$Z\big(K_P;\big(\underline{CZ(K_i)},\ \underline{Z(K_i)}\big)\big)\big/\ker\lambda \;\simeq\; Z\big(K_P;\big(\underline{(D^2)^{j_i}},\ \underline{Z(K_i)}\big)\big)\big/\ker\lambda\,. \tag{24}$$

Combining Theorem 3 and (24) gives now the main observation of this section.

Theorem 6 *For a simple polytope P^n, characteristic function λ and $X_i = Z(K_i)$, Construction 2 corresponds, up to homotopy, to a quotient of a moment-angle complex by a free action of $\ker\lambda$ as follows:*

$$\begin{aligned}\big(X_1\times X_2\times\cdots\times X_m\big/\ker\lambda\big)\times P^n\big/\!\sim_2 \;&\cong\; Z\big(K_P;\big(\underline{CZ(K_i)},\underline{Z(K_i)}\big)\big)\big/\ker\lambda\\ &\simeq\; Z\big(K_P;\big(\underline{(D^2)^{j_i}},\underline{Z(K_i)}\big)\big)\big/\ker\lambda\\ &\cong\; Z\big(K_P(K_1,K_2,\ldots,K_m);(D^2,S^1)\big)\big/\ker\lambda\,.\end{aligned}$$

It should be noted that in general, the space $Z\big(K_P(K_1,K_2,\ldots,K_m);(D^2,S^1)\big)\big/\ker\lambda$ might not be smooth because $K_P(K_1,K_2,\ldots,K_m)$ is dual to the boundary of a simple polytope only in the case that all the K_i are boundaries of simplices.

7 Further Generalizations

Away from the diagonal circle action, the situation becomes a little more complicated. Ayzenberg's construction [1], can be done in the realm of polytopes. In particular, given a simple polytope P^n having m facets and a sequence of simple polytopes $\{P_1, P_2, \ldots, P_m\}$ where $P_i = P_i^{n_i}$ is a simple polytope of dimension n_i having j_i facets, the construction yields a new polytope

$$P^n \longrightarrow P^n\big(P_1, P_2, \ldots, P_m\big).$$

Though the new polytope is not simple when the P_i differ from simplices, it does retain some nice properties. A simplicial complex $K_{P(P_1,P_2,\ldots,P_m)}$ is associated to it as the *nerve complex*. On the level of simplicial complexes, the construction is written

$$K_P \longrightarrow K_P(K_{P_1}, K_{P_2}, \ldots, K_{P_m})$$

as in (20). As expected, it is shown in [1] that

$$K_{P(P_1,P_2,\ldots,P_m)} \;=\; K_P(K_{P_1}, K_{P_2}, \ldots, K_{P_m}).$$

Under this operation, the numbers m and n transform by the analogue of (14):

$$\begin{bmatrix} m \\ n \\ m-n \end{bmatrix} \rightsquigarrow \begin{bmatrix} d(J) = j_1 + j_2 + \cdots + j_m \\ n+N \\ d(J) - n - N \end{bmatrix} \tag{25}$$

where $N = n_1 + n_2 + \cdots + n_m$. Notice that (25) reduces to (14) for the case $P_i = \Delta^{j_i - 1}$.

7.1 A Full Torus Action

For each $i = 1, 2, \ldots, m$, let

$$\lambda_i \colon \mathbb{Z}^{j_i} \longrightarrow \mathbb{Z}^{n_i}$$

be a characteristic function on P_i. It has $\ker \lambda_i \cong T^{j_i - n_i}$ which acts freely on $Z\big(K_{P_i}; (D^2, S^1)\big)$. To ease the notation, we set

$$\Lambda \;=\; \big\{\lambda_1, \lambda_2, \ldots, \lambda_m\big\}.$$

The next step is to mimic the construction of $\lambda(J)$ in Sect. 5.1. To this end, denote by $\overline{\lambda}_i$ the first $j_i - 1$ columns of the $(n_i \times j_i)$—matrix (λ_i^{lk}) corresponding to λ_i. The last column of (λ_i^{lk}) is

$$\begin{bmatrix} \lambda_i^{1j_i} \\ \lambda_i^{2j_i} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_i^{n_ij_i} \end{bmatrix} \tag{26}$$

Entirely by analogy with the case of odd spheres in Sect. 5.1, in particular Remark 5, the matrices $\lambda,\ \lambda_1, \lambda_2, \dots, \lambda_m$ are used to construct an $\big((n+N) \times d(J)\big)$—matrix $\lambda(\Lambda, J, N)$, shown in Fig. 3, which defines a map

$$\lambda(\Lambda, J, N)\colon\ \mathbb{Z}^{d(J)} \longrightarrow \mathbb{Z}^{n+N}.$$

The characteristic matrix corresponding to the diagonal S^1 action on the odd sphere S^{2j_i-1} which appears in (18), is replaced by λ_i which has $\ker\lambda_i \cong T^{j_i-n_i}$ acting freely on $Z\big(K_{P_i}; (D^2, S^1)\big)$. The blocks I_{j_i-1} in Fig. 1 are replaced by $\overline{\lambda}_i$ and the last columns containing "-1", by (26).

Fig. 3 The matrix $\lambda(\Lambda, J, N)$

$\overline{\lambda}_1$	0	$\cdots$	0	$\lambda_1^{1j_1}$, $\lambda_1^{2j_1}$, $\vdots$, $\lambda_1^{n_1j_1}$ 0	
0	$\overline{\lambda}_2$	0	0	0 $\lambda_2^{1j_2}$, 0 $\lambda_2^{2j_2}$, $\vdots$ $\vdots$, 0 $\lambda_2^{n_2j_2}$ 0	
$\vdots$	0	$\ddots$	0	$\vdots$	
0	$\vdots$	0	$\overline{\lambda}_m$	0 $\lambda_m^{1j_m}$, $\lambda_m^{2j_m}$, $\vdots$, $\lambda_m^{n_mj_m}$	
0	0	0	0	λ	1 2 $\vdots$ n
				1 2 $\cdots$ m	

7.2 The Rank of the Matrix $\lambda(\Lambda, J, N)$

The matrix corresponding to λ is a characteristic matrix and so can be written in *refined* block form as:

$$\lambda = I_n \big| S$$

where I_n is the $n \times n$-identity matrix and S is of size $n \times (m-n)$. Similarly, the matrix corresponding to $\overline{\lambda}_i$ can be written in the block form as

$$I_{n_i} \big| S_i$$

where I_{n_i} is the $n_i \times n_i$-identity matrix and S_i is of size $n_i \times (j_i - 1 - n_i)$. This observation allows the conclusion that the row rank of $\lambda(\Lambda, J, N)$ is $N + n$ and the next proposition follows.

Proposition 2 *The row rank of the matrix $\lambda(\Lambda, J, N)$ is $N + n$ and so*

$$\ker \lambda(J, N) \;\cong\; T^{d(J)-N-n}.$$

7.3 A New Toric Space Construction

The inclusion $\ker \lambda(J, N) \longrightarrow T^{d(J)}$ gives an action of $\ker \lambda(J, N)$ on the $T^{d(J)}$-equivariantly homotopy equivalent spaces

$$Z\big(K_P; \big(\underline{CZ(K_{P_i})},\ \underline{Z(K_{P_i})}\big)\big) \;\simeq\; Z\big(K_P; \big(\underline{(D^2)^{j_i}},\ \underline{Z(K_{P_i})}\big)\big).$$

In this context of simple polytopes, Theorem 5 gives

$$Z\big(K_P; \big(\underline{(D^2)^{j_i}},\ \underline{Z(K_{P_i})}\big)\big) \;=\; Z\big(K_{P(P_1,P_2,\ldots,P_m)}; (D^2, S^1)\big).$$

The next theorem follows from the assembly of this information.

Theorem 7 *There is a homotopy equivalence of orbit spaces*

$$Z\big(K_P; \big(\underline{CZ(K_{P_i})},\ \underline{Z(K_{P_i})}\big)\big)\big/\ker \lambda(J, N) \;\simeq\; Z\big(K_{P(P_1,P_2,\ldots,P_m)}; (D^2, S^1)\big)\big/\ker \lambda(J, N)$$

where $K_{P(P_1,P_2,\ldots,P_m)}$ is the nerve complex of the $d(J)$—faceted, $(n+N)$—dimensional polytope $P(P_1, P_2, \ldots, P_m)$ and $\ker \lambda(J, N)$ is isomorphic to a torus of dimension equal to $d(J) - (n + N)$.

Remark 10 In a natural way, the form of the matrix $\lambda(\Lambda, J, N)$ indicates that

$$Q = \ker\lambda_1 \oplus \ker\lambda_2 \oplus \cdots \oplus \ker\lambda_m$$

can be considered as a $(d(J) - N)$-dimensional subspace of $\mathbb{Z}^{d(J)}$. Characterizing the cases when $\ker \lambda(J,N)$ is an $(d(J) - N - n)$-dimensional subspace of Q would characterize cases in which the action in Theorem 7 is free. There would follow a natural generalization of Construction 2 in which the free S^1 action of X_i is replaced with a free $T^{j_i - n_i}$ action on each X_i. In the case above, $X_i = Z(K_{P_i})$ with $\ker \lambda_i \cong T^{j_i - n_i}$ acting freely. The group $\ker \lambda(J,N)$ acts on $Z(K_{P_1}) \times Z(K_{P_2}) \times \cdots \times Z(K_{P_m})$ via the inclusion $\ker \lambda(J,N) \longrightarrow T^{d(J)}$. So, with very little change in the definitions, the left hand side of the equivalence in Theorem 7 would be identified as

$$\big(Z(K_{P_1}) \times Z(K_{P_2}) \times \cdots \times Z(K_{P_m})\big/\ker\lambda(J,N)\big) \times P^n\big/{\sim_2} \, .$$

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Restrictions of Free Arrangements and the Division Theorem

Takuro Abe

Abstract This is a survey and research note on the modified Orlik conjecture derived from the division theorem introduced in Abe (Invent. Math. **204**(1), 317–346, 2016). The division theorem is a generalization of classical addition-deletion theorems for free arrangements. The division theorem can be regarded as a modified converse of the Orlik's conjecture with a combinatorial condition, i.e., an arrangement is free if the restriction is free and the characteristic polynomial of the restriction divides that of an arrangement. In this article we recall, summarize, pose and re-formulate some of results and problems related to the division theorem based on Abe (Invent. Math. **204**(1), 317–346, 2016), and study the modified Orlik's conjecture with partial answers.

Keywords Arrangements of hyperplanes • Divison theorems • Divisionally free arrangements • Free arrangements • Orlik's conjecture

1 Introduction

Let $\mathbb{K}$ be an arbitrary field, $V = \mathbb{K}^\ell$ and $S = \mathbb{K}[x_1, \dots, x_\ell]$ the coordinate ring of V^*. Let $\mathcal{A}$ be an arrangement of hyperplanes in V, i.e., a finite collection of linear hyperplanes in V. For $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. For $\operatorname{Der} S := \oplus_{i=1}^{\ell} S\partial_{x_i}$, a logarithmic derivation module $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in S\alpha_H \ (\forall H \in \mathcal{A})\}.$$

$D(\mathcal{A})$ is a reflexive S-module, and not free in general. We say $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$ if there is derivations $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ such that $D(\mathcal{A}) = \oplus_{i=1}^{\ell} S\theta_i$ and $\deg \theta_i(\beta_i) = d_i$ $(i = 1, \dots, \ell)$ for some linear form $\beta_1, \dots, \beta_\ell$ such that $\theta_i(\beta_i) \neq 0$.

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Let $L(\mathcal{A}) := \{\cap_{H\in\mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A}\}$ be the intersection lattice of $\mathcal{A}$, and $L_i(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \operatorname{codim}_V X = i\}$. A flag of $\mathcal{A}$ is a set $\{X_i\}_{i=0}^{\ell}$ such that $X_0 \subset \cdots \subset X_\ell$ and $X_i \in L_i(\mathcal{A})$. The Möbius function $\mu : L(\mathcal{A}) \to \mathbb{Z}$ is defined by, $\mu(V) = 1$ and by $\mu(X) := -\sum_{X\subsetneq Y\subset V} \mu(Y)$ for $X \neq V$. The Poincarè polynomial $\pi(\mathcal{A}; t)$ of $\mathcal{A}$ is defined by $\pi(\mathcal{A}; t) := \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{codim}_V X}$. Also, the characteristic polynomial $\chi(\mathcal{A}; t)$ of $\mathcal{A}$ is defined by $\chi(\mathcal{A}; t) := t^\ell \pi(\mathcal{A}; -t^{-1})$. It is known that $\pi(\mathcal{A}; t)$ coincides with the topological Poincarè polynomial of the complement $M(\mathcal{A}) := V \setminus \cup_{H\in\mathcal{A}} H$ when $\mathbb{K} = \mathbb{C}$. Hence the coefficient $b_i(\mathcal{A})$ of t^i in $\pi(\mathcal{A}; t)$ is nothing but the i-th Betti number of $M(\mathcal{A})$. For $X \in L(\mathcal{A})$, a localization $\mathcal{A}_X$ of $\mathcal{A}$ at X is defined by $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$, and the restriction $\mathcal{A}^X$ of $\mathcal{A}$ onto X is defined by $\mathcal{A}^X := \{H \cap X \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$. Note that $\mathcal{A}_X$ is an arrangement in V, but $\mathcal{A}^X$ is that in $X \simeq \mathbb{K}^{\dim X}$.

Free arrangements have been intensively studied by several mathematicians, and that research has been the most important among the study of algebraic aspects of an arrangement. To check the freeness of given arrangement, or to construct a new free arrangement is very difficult though that is very fundamental. For that purpose, Terao's addition-deletion and restriction theorems have been the most useful and important.

Theorem 1 ([8], Addition-Deletion and Restriction Theorems) *For $H \in \mathcal{A}$, let $\mathcal{A}' := \mathcal{A} \setminus \{H\}$. Then for the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$, two of the following three implies the third:*

(1) $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-1}, d_\ell)$.
(2) $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.
(3) $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (d_1, \ldots, d_{\ell-1})$.

Moreover, all the three above hold if both $\mathcal{A}$ and $\mathcal{A}'$ are free.

In [2] the division theorem for free arrangements was introduced, which is a generalization of Terao's addition-deletion theorem 1.

Theorem 2 (Theorem 1.1, [2], Division Theorem) *$\mathcal{A}$ is free if $\mathcal{A}^H$ is free and $\chi(\mathcal{A}^H; t)$ divides $\chi(\mathcal{A}; t)$ for some $H \in \mathcal{A}$.*

Theorem 2 can be regarded as a converse of modified Orlik's conjecture. Orlik's conjecture asserted that $\mathcal{A}^H$ is free if $\mathcal{A}$ is free, the counter example to which was found by Edelman and Reiner in [6]. Theorem 2 is a converse of this conjecture with one more condition that $\chi(\mathcal{A}^H; t)$ divides $\chi(\mathcal{A}; t)$. Then it is natural to ask whether this modified Orlik's conjecture is true or not.

Problem 1 (Modified Orlik's Conjecture) Let $\mathcal{A}$ be an ℓ-arrangement and $H \in \mathcal{A}$. Assume that $\mathcal{A}$ is free and $\pi(\mathcal{A}^H; t)$ divides $\pi(\mathcal{A}; t)$. (Equivalently, $\pi(\mathcal{A} \setminus \{H\}; t)$ divides $\pi(\mathcal{A}; t)$.) Then is $\mathcal{A}^H$ (and hence $\mathcal{A} \setminus \{H\}$) a free arrangement?

It seems that what is stated in Problem 1 is too strong, hence we believe that there will be a counter example to Problem 1. In other words, we believe the following conjecture.

Conjecture 1 There exists an arrangement $\mathcal{A}$ and $H \in \mathcal{A}$ such that, for the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$, it holds that

(1) $\pi(\mathcal{A}^H; t) = \prod_{i=1}^{\ell-1}(1 + d_i t)$ with $d_1, \ldots, d_{\ell-1} \in \mathbb{Z}$,
(2) $\pi(\mathcal{A}^H; t)$ divides both $\pi(\mathcal{A}; t)$ and $\pi(\mathcal{A}'; t)$, and
(3) Neither $\mathcal{A}$ nor $\mathcal{A}'$ are free (or, $\mathcal{A}$ is free and $\mathcal{A}'$ is not free).

By Theorem 2, there are no triples as in Conjecture 1 if '$\mathcal{A}^H$ is free. Also, there are no such triple when $\ell \leqslant 3$ due to [1]. Hence to show Conjecture 1, the assumption $\ell \geqslant 4$ is essential. However, and surprisingly, there have been no example as in Conjecture 1.

The purpose of this article is to consider in which condition Problem 1 is true. The key role is played by the second Betti number $b_2(\mathcal{A})$ of the complement of $\mathcal{A}$ when $\mathbb{K} = \mathbb{C}$. Namely, $b_2(\mathcal{A})$ is the coefficient of $t^{\ell-2}$ of $\chi(\mathcal{A}; t)$. One of the answer is the following, which is a main result in this article.

Theorem 3 *Assume that $\mathcal{A}$ is free and $b_2(\mathcal{A}) = b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$. Then $\mathcal{A}^H$ is free if there is $L \in \mathcal{A} \setminus \{H\}$ such that $\mathcal{A} \setminus \{L\}$ is free and $|\mathcal{A}_{L\cap H}| \geqslant 3$.*

Note that the equation $b_2(\mathcal{A}) = b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$ holds when $\chi(\mathcal{A}^H; t)$ divides $\chi(\mathcal{A}; t)$. What is interesting in Theorem 3 is, to determine the freeness of the restriction, a freeness of some other restriction works.

The other main result in this article is to give an easy sufficient condition for $\mathcal{A}'$ not to be free even when $\mathcal{A}$ is free. This gives us an easy sufficient condition for the equation on the second Betti numbers above not to be true.

Theorem 4 *Assume that $\mathcal{A}$ is a free ℓ-arrangement with $\ell \geqslant 3$. Let $H \in \mathcal{A}$ and $\mathcal{A}' := \mathcal{A} \setminus \{H\}$. Then $\mathcal{A}'$ is not free if there is $X \in L_2(\mathcal{A}^H)$ such that one of the following three holds:*

(1) $\mathcal{A}_X^H = \{K_1, K_2\}$ and $m^H(K_1) > 1,\ m^H(K_2) > 1$.
(2) $\mathcal{A}_X^H = \{K_1, K_2, K_3\}$ and $m^H(K_i) \geqslant 2$ for $i = 1, 2, 3$.
(3) $\mathcal{A}_X^H = \{K_1, K_2, K_3\}$ and $m^H(K_1) \geqslant 3,\ m^H(K_2) \geqslant 2$.

Here $m^H : \mathcal{A}^H \to \mathbb{Z}_{>0}$ is the Ziegler multiplicity on $\mathcal{A}^H$ defined by

$$m^H(K) := |\{L \in \mathcal{A}' \mid H \cap L = K\}|$$

for $K \in \mathcal{A}^H$. In particular, if one of the three above holds, then it holds that $b_2(\mathcal{A}) > b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$.

Hence the arrangement $\mathcal{A}$ in Problem 1 has a very special geometry. Let us show an application of Theorem 4.

Example 1 Let $\mathcal{A}$ be an arrangement in $V = {R_{\mathrm{red}}}^4$ defined by

$$\prod_{i=1}^{4} x_i \prod_{1\leqslant i<j\leqslant 4,\ (i,j)\neq(3,4)} (x_i - x_j) = 0.$$

Then it is easy to see that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 2, 3, 3)$. Let $H := \{x_1 = 0\} \in \mathcal{A}$, and show that $\mathcal{A} \setminus \{H\}$ is not free. The Ziegler restriction $(\mathcal{A}^H, m^H)$ of $\mathcal{A}$ onto H is defined by

$$(\prod_{i=2}^{4} x_i^2)(x_2 - x_3)(x_2 - x_4) = 0.$$

Let $X := \{x_3 = x_4 = 0\} \in L_2(\mathcal{A}^H)$. Then $\mathcal{A}_X^H = \{x_3^2 x_4^2 = 0\}$, which satisfies the condition (1) in Theorem 4. Hence $b_2(\mathcal{A}) > b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$ and $\mathcal{A} \setminus \{H\}$ is not free.

The organization of this article is as follows. In Sect. 2 we recall several results and definitions for the proof. This section contains some re-formulation of results in [2]. In Sect. 3, first we give some partial answers to Problem 1 which follows immediately from the division theorem and other results in [2]. After that, we show Theorems 3 and 4. In Sect. 4 we observe the similarity of supersolvable and divisionally free arrangements.

2 Preliminaries

In this section let us recall several results we will use for the proof of main results. The first one is the most important result among the theory of free arrangements.

Theorem 5 ([9], Terao's Factorization) *Assume that $\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$. *Then $\chi(\mathcal{A}; t) = \prod_{i=1}^{\ell}(t - d_i)$. In particular, $\mathcal{A}$ is not free if $\chi(\mathcal{A}; t)$ is irreducible over $\mathbb{Z}$.*

Next let us recall some fundamental definitions and results for multiarrangements. For an arrangement $\mathcal{A}$, let $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ be a multiplicity function. Then the pair $(\mathcal{A}, m)$ is called a multiarrangement, and we can define the logarithmic derivation module $D(\mathcal{A}, m)$ by

$$D(\mathcal{A}, m) := \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in S\alpha_H^{m(H)} \ (\forall H \in \mathcal{A})\}.$$

Then we can define the freeness and exponents for multiarrangements in the same manner as for arrangements.

From an arrangement, we may define a multiarrangement canonically. For an arrangement $\mathcal{A}$ and $H \in \mathcal{A}$, define a multiarrangement $(\mathcal{A}^H, m^H)$, called the Ziegler restriction of $\mathcal{A}$ onto H, by $m^H(X) := |\mathcal{A}_X| - 1$ for $X \in \mathcal{A}^H$. Then the following is the most fundamental.

Theorem 6 ([11]) *Assume that $\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$. *Then $(\mathcal{A}^H, m^H)$ is free with* $\exp(\mathcal{A}^H, m^H) = (d_2, \ldots, d_\ell)$.

The next result is a generalization of Theorem 1 to multiarrangements. For details and definitions on the Euler restriction, see [5].

Theorem 7 ([5], Theorem 0.8) *Let $(\mathcal{A}, m)$ be a multiarrangement, $H \in \mathcal{A}$ and let $\delta_H : \mathcal{A} \to \{0, 1\}$ be a multiplicity such that $\delta_H(L) = 1$ only when $H = L$. Then any two of the following imply the third:*

(1) $(\mathcal{A}, m)$ *is free with* $\exp(\mathcal{A}, m) = (d_1, \ldots, d_{\ell-1}, d_\ell)$.
(2) $(\mathcal{A}, m - \delta_H)$ *is free with* $\exp(\mathcal{A}, m - \delta_H) = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.
(3) $(\mathcal{A}^H, m^*)$ *is free with* $\exp(\mathcal{A}^H, m^*) = (d_1, \ldots, d_{\ell-1})$,

where $(\mathcal{A}^H, m^*)$ is the Euler restriction of $(\mathcal{A}, m)$ onto $H \in \mathcal{A}$. Moreover, all the three above hold if both $(\mathcal{A}, m)$ and $(\mathcal{A}, m - \delta_H)$ are free.

The following is a freeness criterion by using the second Betti number and the Ziegler restriction. For details, see [3]. Also, for the definition of the second Betti number of a multiarrangement, see [4].

Theorem 8 ([3], Theorem 5.1) *Let $\mathcal{A}$ be a central ℓ-arrangement, $H \in \mathcal{A}$ and $(\mathcal{A}^H, m^H)$ the Ziegler restriction of $\mathcal{A}$ onto H. Then $\mathcal{A}$ is free if and only if $(\mathcal{A}^H, m^H)$ is free and $b_2(\mathcal{A}) = |\mathcal{A}| - 1 + b_2(\mathcal{A}^H, m^H)$. In particular, $b_2(\mathcal{A}) \geqslant |\mathcal{A}| - 1 + b_2(\mathcal{A}^H, m^H)$.*

Let us introduce two more results from [2]. Since the formulations of these results are slight different from those in the original version in [2], we give proofs for the completeness.

The first one is the following proposition, which says that the Ziegler and Euler restriction commutes if there is a division $\chi(\mathcal{A}^H; t) \mid \chi(\mathcal{A}; t)$.

Proposition 1 ([2], cf. Theorem 1.7) *Let $\mathcal{A}$ be an ℓ-arrangement, $H \in \mathcal{A}$ and $(\mathcal{A}^H, m^H)$ be the Ziegler restriction of $\mathcal{A}$ onto H. Let $X \in \mathcal{A}^H$ with $m^H(X) \geqslant 2$. Assume that $b_2(\mathcal{A}) = b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$. Then*

(1) the Ziegler restriction of $\mathcal{A}^H$ onto X coincides with the Euler restriction of $(\mathcal{A}^H, m^H)$ onto X, and
(2) $b_2(\mathcal{A}^H, m^H) - b_2(\mathcal{A}^H, m^H - \delta_X) = |\mathcal{A}^H| - 1$.

Proof Immediate from Theorem 1.7 and its proof in [2]. □

Proposition 2 ((b_1, b_2)-Inequality, cf., [2], Corollary 4.10) *Let $H \in \mathcal{A}$. Then*

$$b_2(\mathcal{A}) \geqslant b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|.$$

In particular,

$$b_2(\mathcal{A}) \geqslant \sum_{i=0}^{\ell-2} (|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}|)|\mathcal{A}^{X_{i+1}}|$$

for any flag $\{X_i\}_{i=0}^{\ell-1}$ of $\mathcal{A}$.

Proof Let $b_2(d\mathcal{A})$ denote the coefficient of t^2 in $\pi(\mathcal{A};t)/(1+t)$. Then the equation (4.1) in [2] shows that

$$b_2(d\mathcal{A}) \geqslant b_2(d\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)(|\mathcal{A}^H| - 1).$$

Since $b_2(d\mathcal{A}) + |\mathcal{A}| - 1 = b_2(\mathcal{A})$ and $|\mathcal{A}| - 1 = (|\mathcal{A}| - |\mathcal{A}^H|) + (|\mathcal{A}^H| - 1)$, we have $b_2(\mathcal{A}) \geqslant b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$. The next inequality holds by applying the same argument to the Corollary 4.10 in [2]. □

Here we give an observation. As in Theorem 1, the freeness of $\mathcal{A}$ and $\mathcal{A}'$ implies the freeness of each member of the triple. Here we do not have to consider the freeness of $\mathcal{A}^H$. Then, when each member of the triple is free if we assume the freeness of $\mathcal{A}^H$? The answer is immediate from Theorem 2.

Proposition 3 *Let $\mathcal{A}$ be an arrangement and $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$ the triple with respect to $H \in \mathcal{A}$. Then each member of the triple is free if and only if $\mathcal{A}^H$ is free and $b_2(\mathcal{A}) = b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$.*

Proof The "only if" part is nothing but Theorem 1. The "if" part follows also immediately by Theorem 2. □

3 A Partial Results and the Proof of Main Results

Before the proof of Theorem 3, let us give some partial answer which follows immediately from the division theorem and (b_1, b_2)-inequality.

Theorem 9 *Let $\mathcal{A}$ be a free arrangement with* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$. *Take $H \in \mathcal{A}$ and $X \in \mathcal{A}^H$ such that*

(1) $\mathcal{A}^X$ is free with $\exp(\mathcal{A}^X) = (d_1, \ldots, d_{\ell-2})$*, and*
(2) $|\mathcal{A}| - |\mathcal{A}^H| = d_\ell$, (hence automatically, $|\mathcal{A}^H| - |\mathcal{A}^X| = d_{\ell-1}$).

Then $\mathcal{A}^H$ is also free with $\exp(\mathcal{A}^H) = (d_1, \ldots, d_{\ell-1})$.

Proof By Theorem 2, it suffices to show that $b_2(\mathcal{A}^H) = \sum_{1 \leqslant i<j \leqslant \ell-1} d_i d_j$. By Proposition 2, it holds that

$$b_2(\mathcal{A}) = \sum_{1 \leqslant i<j \leqslant \ell} d_i d_j \geqslant b_2(\mathcal{A}^H) + d_\ell(d_1 + \cdots + d_{\ell-1}), \text{ and}$$

$$b_2(\mathcal{A}^H) \geqslant b_2(\mathcal{A}^X) + d_{\ell-1}(d_1 + \cdots + d_{\ell-2}) = \sum_{1 \leqslant i<j \leqslant \ell-2} d_i d_j + d_{\ell-1}(d_1 + \cdots + d_{\ell-2}).$$

Hence

$$\sum_{1 \leqslant i<j \leqslant \ell-1} d_i d_j \leqslant b_2(\mathcal{A}^H) \leqslant \sum_{1 \leqslant i<j \leqslant \ell-1} d_i d_j.$$

Hence $b_2(\mathcal{A}^H) = b_2(\mathcal{A}^X) + (|\mathcal{A}^H| - |\mathcal{A}^X|)|\mathcal{A}^X|$, and Theorem 2 completes the proof. □

In Theorem 9, we apply the proof of Theorem 2 conversely. Hence Theorem 9 may be regarded as an application of the proof of Theorem 2 and Proposition 2. A useful part of Theorem 9 is, if we know the exponents of $\mathcal{A}$ and $\mathcal{A}^X$, then we can check the freeness between them just by computing the number of hyperplanes in it (we do not need any information on the second Betti number!). Hence practically, or when we want to check some hereditary freeness (see [7]), Theorem 9 and the following corollaries are useful.

Corollary 1 *Let $\mathcal{A}$ be a free arrangement with* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$. *Take $X_i \in L_i(\mathcal{A})$ $(i = 1, \ldots, k)$ with $X_1 \supset \cdots \supset X_k$ such that*

(1) $\mathcal{A}^{X_k}$ *is free with* $\exp(\mathcal{A}^{X_k}) = (d_1, \ldots, d_{\ell-k})$.
(2) $|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}| = d_{\ell-i}$ $(i = 0, \ldots, k-1)$.

Then $\mathcal{A}^{X_i}$ is also free with $\exp(\mathcal{A}^{X_i}) = (d_1, \ldots, d_{\ell-i})$ *for $i = 1, \ldots, k-1$.*

Proof By Proposition 2, it holds that

$$b_2(\mathcal{A}^{X_{k-1}}) \geqslant (|\mathcal{A}^{X_{k-1}}| - |\mathcal{A}^{X_k}|)|\mathcal{A}^{X_k}| + b_2(\mathcal{A}^{X_k}) = \sum_{1 \leqslant i < j \leqslant \ell-k+1} d_i d_j.$$

On the other hand, again by applying (b_1, b_2)-inequality, we have

$$\begin{aligned}
\sum_{1 \leqslant i < j \leqslant \ell} d_j d_i &= b_2(\mathcal{A}) \\
&\geqslant b_2(\mathcal{A}^{X_1}) + (|\mathcal{A}| - |\mathcal{A}^{X_1}|)|\mathcal{A}^{X_1}| = b_2(\mathcal{A}^{X_1}) + d_\ell \sum_{i=1}^{\ell-1} d_i \\
&\geqslant b_2(\mathcal{A}^{X_2}) + (|\mathcal{A}^{X_1}| - |\mathcal{A}^{X_2}|)|\mathcal{A}^{X_2}| + d_\ell \sum_{i=1}^{\ell-1} d_i \\
&\geqslant \cdots \\
&\geqslant b_2(\mathcal{A}^{X_{k-1}}) + \sum_{i=\ell-k+2}^{\ell} d_i \Big(\sum_{j=1}^{i-1} d_j\Big).
\end{aligned}$$

Hence it holds that

$$b_2(\mathcal{A}^{X_{k-1}}) \leqslant \sum_{1 \leqslant i < j \leqslant \ell} d_j d_i - \sum_{i=\ell-k+2}^{\ell} d_i \Big(\sum_{j=1}^{i-1} d_j\Big) = \sum_{1 \leqslant i < j \leqslant \ell-k+1} d_i d_j.$$

Combine these two inequalities to obtain

$$b_2(\mathcal{A}^{X_{k-1}}) = \sum_{1 \leqslant i<j \leqslant \ell-k+1} d_i d_j.$$

Hence Theorem 2 shows that $\mathcal{A}^{X_{k-1}}$ is free. Apply the same argument to all $\mathcal{A}^{X_1}, \ldots, \mathcal{A}^{X_{k-2}}$ to complete the proof. □

Moreover, we do not need to assume the freeness of $\mathcal{A}$ as follows:

Corollary 2 *Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| = b_1(\mathcal{A}) = d_1+\cdots+d_\ell$, $b_2(\mathcal{A}) = \sum_{1 \leqslant i<j \leqslant \ell} d_i d_j$ for some positive integers $d_1, \ldots, d_\ell$. Take $X_i \in L_i(\mathcal{A})$ $(i = 1, \ldots, k)$ with $X_1 \supset \cdots \supset X_k$ such that*

(1) $\mathcal{A}^{X_k}$ is free with $\exp(\mathcal{A}^{X_k}) = (d_1, \ldots, d_{\ell-k})$.
(2) $|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}| = d_{\ell-i}$ $(i = 0, \ldots, k-1)$.

Then $\mathcal{A}^{X_i}$ is also free with $\exp(\mathcal{A}^{X_i}) = (d_1, \ldots, d_{\ell-i})$ for $i = 0, \ldots, k-1$. In particular, we do not need the freeness of $\mathcal{A}^{X_k}$ if $k = \ell - 2$.

Proof Apply the same argument as in the proof of Corollary 1 repeatedly. When $k = \ell-2$, this is nothing but the divisional freeness in Definition 1.5 in [2] (see also Definition 2).

Now let us prove Theorems 3.

Proof of Theorem 3 Let $X := L \cap H$. Let $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$ with $d_1 = 1$. Also, let $\exp(\mathcal{A}') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$, where $\mathcal{A}' := \mathcal{A} \setminus \{L\}$. Then $\mathcal{A}^L$ is free with $\exp(\mathcal{A}^L) = (d_1, \ldots, d_{\ell-1})$ by Theorem 1. Hence Theorem 6 shows that both $(\mathcal{A}^H, m^H)$ and $(\mathcal{A}^H, m^H - \delta_X)$ are free with exponents $(d_2, \ldots, d_\ell)$ and $(d_2, \ldots, d_\ell - 1)$ respectively, where $\delta_X : \mathcal{A}^H \to \{0, 1\}$ is a multiplicity such that $\delta_X^{-1}(1) = X \in \mathcal{A}^H$. Then Theorem 7 shows that the Euler restriction $(\mathcal{A}^X, m^*)$ of $(\mathcal{A}^H, m^H)$ onto X is also free with $\exp(\mathcal{A}^X, m^*) = (d_2, \ldots, d_{\ell-1})$ by Theorem 6. Now recall that $m^H(X) \geqslant 2$ by the fact that $|\mathcal{A}_X| \geqslant 3$. Hence Proposition 1 and the equality $b_2(\mathcal{A}) = b_2(\mathcal{A}^H)+(|\mathcal{A}|-|\mathcal{A}^H|)|\mathcal{A}^H|$ show that $(\mathcal{A}^X, m^*) = (\mathcal{A}^X, m^X)$, where $(\mathcal{A}^X, m^X)$ is the Ziegler restriction of $\mathcal{A}^H$ onto X. Hence $(\mathcal{A}^X, m^X)$ is also free with $\exp(\mathcal{A}^X, m^X) = (d_2, \ldots, d_{\ell-1})$. In particular, $|m^X| = d_2+\cdots+d_{\ell-1} = |\mathcal{A}|-d_\ell-1$. On the other hand, again Proposition 1 shows that $|m^*| = b_2(\mathcal{A}^H, m^H) - b_2(\mathcal{A}^H, m^H - \delta_X) = |\mathcal{A}^H| - 1$. Since $|m^*| = |m^X|$, we have $|\mathcal{A}| - d_\ell - 1 = |\mathcal{A}^H| - 1$. Hence $|\mathcal{A}| - |\mathcal{A}^H| = d_\ell$, and $|\mathcal{A}^H| = \sum_{i=1}^{\ell} d_i - d_\ell = d_1 + \cdots + d_{\ell-1}$. So the equation $b_2(\mathcal{A}) = b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H|)|\mathcal{A}^H|$ shows that $b_2(\mathcal{A}^H) = \sum_{1 \leqslant i<j \leqslant \ell-1} d_i d_j$. In particular, the coefficient of t^2 in $\pi_0(\mathcal{A}^H; t)$ is $\sum_{2 \leqslant i<j \leqslant \ell-1} d_i d_j = b_2(\mathcal{A}^X, m^X)$. Hence Theorem 8 shows that $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (d_1, \ldots, d_{\ell-1})$. □

Theorem 3 has the following corollary.

Corollary 3 *Let $\mathcal{A}$ be an ℓ-arrangement and $H_1, \ldots, H_s \in \mathcal{A}$ be distinct hyperplanes such that* $\operatorname{codim} \cap_{i=1}^{s} H_i = 2$. *Assume that $b_2(\mathcal{A}) = b_2(\mathcal{A}^{H_i})$*

$+ (|\mathcal{A}| - |\mathcal{A}^{H_i}|)|\mathcal{A}^{H_i}|$ for $i = 2, \ldots, s$. Then all the $\mathcal{A}'_i := \mathcal{A} \setminus \{H_i\}$ $(i = 2, \ldots, s)$ and $\mathcal{B} := \mathcal{A} \setminus \{H_1, \ldots, H_s\}$ are free if H_1 satisfies the following conditions:

(1) $\mathcal{A}$ and $\mathcal{A} \setminus \{H_1\}$ are free, and
(2) $|\mathcal{A}_X| \geqslant s + 1$ for $X := \cap_{i=1}^{s} H_i$.

Proof Apply Theorem 3 to each pair H_1, H_i to obtain the statement in Theorem 3. Hence $\mathcal{A}^{H_i}$ is free for each $i \geqslant 2$ with exponents$(d_1, d_2, \ldots, d_{\ell-1})$, here we assume that $\exp(\mathcal{A}) = (d_1 \ldots, d_\ell),\ d_1 = 1$ and $|\mathcal{A}| - |\mathcal{A}^{H_1}| = d_\ell$. Since $|\mathcal{A}_X| \geqslant s + 1$, it holds that

$$(\mathcal{A} \setminus \{H_1, \ldots, H_i\})^{H_{i+1}} = \mathcal{A}^{H_{i+1}}.$$

Hence Theorem 1 shows that $\mathcal{A} \setminus \{H_1, \ldots, H_i\}$ is free with exponents $(1, d_2, \ldots, d_{\ell-1}, d_\ell - i)$, which completes the proof, ☐

As we can see from the proof of Theorem 3, the following general fact holds.

Corollary 4 *Assume that $\mathcal{A}$ is free and $\chi(\mathcal{A}^H; t)$ divides $\chi(\mathcal{A}; t)$ for $H \in \mathcal{A}$. Then $\mathcal{A}^H$ is free if there is $X \in \mathcal{A}^H$ such that $m^H(X) \geqslant 2$ and $(\mathcal{A}^H, m^H - \delta_X)$ is free.*

Proof Immediate from the proof of Theorem 3. ☐

Proof of Theorem 4 Since the proof for each given condition is the same, we show only the case (1). Assume that $\mathcal{A}'$ is free. Then Theorem 1 shows that $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) \subset \exp(\mathcal{A})$. In particular, the equation $b_2(d\mathcal{A}) = b_2(\mathcal{A}^H) + (|\mathcal{A}| - |\mathcal{A}^H| - 1)(|\mathcal{A}^H| - 1)$ holds. By the proof of Theorem 2 in [2], for every multiplicity $m : \mathcal{A} \to \mathbb{Z}_{>0}$ with $m(Y) \leqslant m^H(Y)$ $(Y \in \mathcal{A}^H)$, it holds that $b_2(\mathcal{A}^H, m) = b_2(\mathcal{A}^H) + (|m| - |\mathcal{A}^H|)(|\mathcal{A}^H| - 1)$. In particular, for any $Y \in \mathcal{A}^H$ with $m(Y) \geqslant 2$, it holds that $b_2(\mathcal{A}^H, m) - b_2(\mathcal{A}^H, m - \delta_Y) = |\mathcal{A}^H| - 1$.

By assumption (1), we may pick a multiplicity m such that $m(K_1) = m(K_2) = 2$. Let m^* be the Euler multiplicity of $(\mathcal{A}^H, m)$ onto K_1. Then it follows that $m^*(X) = 2$. Hence

$$\begin{aligned} b_2(\mathcal{A}^H, m) - b_2(\mathcal{A}^H, m - \delta_{K_1}) &= \sum_{Y \in (\mathcal{A}^H)^{K_1}} (b_2(\mathcal{A}^H_Y, m) - b_2(\mathcal{A}^H_Y, m - \delta_{K_1})) \\ &= \sum_{Y \in (\mathcal{A}^H)^{K_1}} m^*(Y) \\ &> \sum_{Y \in (\mathcal{A}^H)^{K_1}} (|\mathcal{A}^H_Y| - 1) \\ &= |\mathcal{A}^H| - 1 \end{aligned}$$

by Lemma 3 (2), Lemma 4 and the assumption that $m^*(X) = 2 > 1$, which is a contradiction. For other cases, use the same argument with the result in [10]. ☐

Let us see an example how to apply Corollary 1.

Example 2 Let $\mathcal{A}$ be an arrangement in $V = \mathbb{R}^6$ defined by

$$\prod_{1 \leqslant i < j \leqslant 6} (x_i^2 - x_j^2) = 0.$$

This is the Weyl arrangement of the type D_6, hence free with $\exp(\mathcal{A}) = (1, 3, 5, 5, 7, 9)$. In general, to investigate the freeness of restrictions is very difficult. In the case of Weyl arrangements, it is proved by Orlik and Terao in [7] that all restrictions are free, and such a free arrangement is called hereditarily free. Here let us check freeness of some restrictions of $\mathcal{A}$ by applying Corollary 1.

Let $X_1 = \{x_1 = x_6\}$, $X_2 = \{x_1 = x_6, x_2 = x_5\}$, $X_3 = \{x_1 = x_6, x_2 = x_5, x_3 = x_4\}$, and consider the freeness of $\mathcal{A}^{X_i}$ for $i = 1, 2, 3$. Then it is easy to show that

$$\mathcal{A}^{X_1} : x_6 \prod_{2 \leqslant i < j \leqslant 6} (x_i^2 - x_j^2)) = 0,$$

$$\mathcal{A}^{X_2} : x_5 x_6 \prod_{3 \leqslant i < j \leqslant 6} (x_i^2 - x_j^2) = 0,$$

$$\mathcal{A}^{X_3} : x_4 x_5 x_6 \prod_{4 \leqslant i < j \leqslant 6} (x_i^2 - x_j^2) = 0.$$

Since $\mathcal{A}^{X_3}$ is the Weyl arrangement of the type B_3, it is free with $\exp(\mathcal{A}^{X_3}) = (1, 3, 5)$. Hence we may apply Corollary 1 to check the freeness of these three arrangements.

By the equations, we can see that $|\mathcal{A}| = 30$, $|\mathcal{A}^{X_1}| = 21$, $|\mathcal{A}^{X_2}| = 14$, $|\mathcal{A}^{X_3}| = 9$. Hence Corollary 1 shows that $\mathcal{A}^{X_1}$ and $\mathcal{A}^{X_2}$ are both free with $\exp(\mathcal{A}^{X_1}) = (1, 3, 5, 5, 7)$ and $\exp(\mathcal{A}^{X_2}) = (1, 3, 5, 5)$.

4 Supersolvable and Divisionally Free Arrangements

First recall the definition of the supersolvable arrangement.

Definition 1 $\mathcal{A}$ is **supersolvable** if and only if there is a flag $\{X_i\}$ such that, $\mathcal{A}_{X_i}$ is of rank i for $i = 0, \ldots, \ell - 1$ and for every $H \neq L \in \mathcal{A}_{X_{i+1}} \setminus \mathcal{A}_{X_i}$, there is $K \in \mathcal{A}_{X_i}$ such that $H \cap L \subset K$. In this case, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (|\mathcal{A}| - |\mathcal{A}_{X_{\ell-1}}|, \ldots, |\mathcal{A}_{X_2}| - |\mathcal{A}_{X_1}|, |\mathcal{A}_{X_1}|)$.

Second, let us introduce a different definition of a supersolvable arrangement. We do not know whether it has been already known. Here we give a proof for the completeness.

Proposition 4 *$\mathcal{A}$ is supersolvable if and only if there is a flag $\{X_i\}$ such that*

$$b_2(\mathcal{A}) = \sum_{i=0}^{\ell-1}(|\mathcal{A}_{X_{i+1}}| - |\mathcal{A}_{X_i}|)|\mathcal{A}_{X_i}|.$$

In this case, $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (|\mathcal{A}_{X_\ell}| - |\mathcal{A}_{X_{\ell-1}}|, |\mathcal{A}_{X_{\ell-1}}| - |\mathcal{A}_{X_{\ell-2}}|, \dots, |\mathcal{A}_{X_2}| - |\mathcal{A}_{X_1}|, |\mathcal{A}_{X_1}|)$.

Proof Let $\mathcal{A}_i := \mathcal{A}_{X_i}$. Since a supersolvable arrangement is free with exponents in Definition 1, the "only if" part is immediate. Assume that $\mathcal{A}$ satisfies the equality in Proposition 4. Assume that the assumption for supersolvable arrangements holds true for $\mathcal{A}_0, \dots, \mathcal{A}_i$. We show that, for any distinct $H, L \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$, there is $K \in \mathcal{A}_i$ such that $H \cap L \subset K$.

By the induction hypothesis, we know that $\mathcal{A}_i$ is supersolvable with $b_2(\mathcal{A}_i) = \sum_{j=0}^{i-1}(|\mathcal{A}_{j+1}| - |\mathcal{A}_j|)|\mathcal{A}_j|$. Let $H \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$. Since $\mathcal{A}_i \cup \{H\}$ is of rank $i+1$ by definition of localization, it holds that

$$b_2(\mathcal{A}_i \cup \{H\}) \geqslant b_2(\mathcal{A}_i) + |\mathcal{A}_i|.$$

Hence

$$\begin{aligned} b_2(\mathcal{A}_{i+1}) &\geqslant b_2(\mathcal{A}_i) + (|\mathcal{A}_{i+1}| - |\mathcal{A}_i|)|\mathcal{A}_i| \\ &= \sum_{j=0}^{i}(|\mathcal{A}_{j+1}| - |\mathcal{A}_j|)|\mathcal{A}_j|. \end{aligned}$$

At every i, this has to be equal since we have the equation

$$b_2(\mathcal{A}) = \sum_{j=0}^{\ell-1}(|\mathcal{A}_{j+1}| - |\mathcal{A}_j|)|\mathcal{A}_j|.$$

Hence it holds that $b_2(\mathcal{A}_i \cup \{H\}) = b_2(\mathcal{A}_i) + |\mathcal{A}_i|$ for any $H \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$ and $i = 0, \dots, \ell-1$.

Now assume that there are no $K \in \mathcal{A}_i$ such that $H \cap L \subset K$. Since $|(\mathcal{A}_i \cup \{H\})^H| \geqslant |\mathcal{A}_i|$ by the definition of the localization, the above implies that $b_2(\mathcal{A} \cup \{H\}) > b_2(\mathcal{A}_i) + |\mathcal{A}_i|$, which is a contradiction. Hence for any distinct $H, L \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$, there is $K \in \mathcal{A}_i$ such that $H \cap L \subset K$. ☐

The reason why we introduced another characterization of supersolvable arrangements in Proposition 4 is to point out the similarity of the supersolvability to the divisional freeness introduced in [2].

Definition 2 (Divisionally Free Arrangement, [2], Definition 1.5) $\mathcal{A}$ is divisionally free if there is a flag $\{X_i\}$ such that

$$b_2(\mathcal{A}) = \sum_{i=0}^{\ell-2}(|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}|)|\mathcal{A}^{X_{i+1}}|.$$

In this case, $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (|\mathcal{A}^{X_0}| - |\mathcal{A}^{X_1}|, |\mathcal{A}^{X_1}| - |\mathcal{A}^{X_2}|, \ldots, |\mathcal{A}^{X_{\ell-2}}| - |\mathcal{A}^{X_{\ell-1}}|, |\mathcal{A}^{X_{\ell-1}}|)$. Such a flag is called a divisional flag.

It is also shown in [2] that all inductively free arrangements are divisionally free [2, Theorem 1.6]. Since supersolvable arrangements are inductively free, they are of course divisionally free. Here we give an another proof of the fact that supersolvable arrangements are divisionally free by using Proposition 4 and Definition 2 to see their similarity.

Proposition 5 (cf. [2], Theorem 1.6) *A supersolvable arrangement $\mathcal{A}$ is divisionally free.*

Proof Let $\mathcal{A}$ be a supersolvable arrangement with a flag $\{X_i\}$ as in Definition 4. Let $\alpha_1, \ldots, \alpha_\ell$ be linear forms such that $X_i = \{\alpha_1 = \cdots = \alpha_i = 0\}$. Then define the flag $\{Y_i\}$ by $Y_i := \{\alpha_\ell = \cdots = \alpha_{\ell-i+1} = 0\}$. Then it is clear that this flag becomes a divisional flag. □

Remark 1 By Proposition 4, Definition 2 and Proposition 5, supersolvable and divisionally free arrangements are similar. They both use flags for localizations and restrictions respectively. Since there are a lot of nice properties for supersolvable arrangements, it is natural to ask whether some special properties which hold for supersolvable arrangements also hold true for divisionally free arrangements.

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The Pure Braid Groups and Their Relatives

Alexander I. Suciu and He Wang

Abstract In this mostly survey paper, we investigate the resonance varieties, the lower central series ranks, and the Chen ranks, as well as the residual and formality properties of several families of braid-like groups: the pure braid groups P_n, the welded pure braid groups wP_n, the virtual pure braid groups vP_n, as well as their 'upper' variants, wP_n^+ and vP_n^+. We also discuss several natural homomorphisms between these groups, and various ways to distinguish among the pure braid groups and their relatives.

Keywords Chen ranks • Formality • Lower central series • Pure braid groups • Residually nilpotent • Resonance varieties • Virtual pure braid groups • Welded pure braid group

1 Introduction

1.1 Cast of Characters

Let F_n be the free group on generators $x_1, \ldots, x_n$, and let $\mathrm{Aut}(F_n)$ be its automorphism group. Magnus [51] showed that the map $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ which sends an automorphism to the induced map on the abelianization $(F_n)^{\mathrm{ab}} = \mathbb{Z}^n$ is surjective. Furthermore, the kernel of this homomorphism, denoted by IA_n, is generated by automorphisms α_{ij} and α_{ijk} $(1 \leqslant i \neq j \neq k \leqslant n)$ which send x_i to $x_j x_i x_j^{-1}$ and $x_i x_j x_k x_j^{-1} x_k^{-1}$, respectively, and leave invariant the remaining generators of F_n. The subgroup generated by the automorphisms α_{ij} and α_{ijk} with $i < j < k$ is denoted by IA_n^+.

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An automorphism of F_n is called a 'permutation-conjugacy' if it sends each generator x_i to a conjugate of $x_{\tau(i)}$, for some permutation $\tau \in S_n$. The classical Artin braid group B_n is the subgroup of $\mathrm{Aut}(F_n)$ consisting of those permutation-conjugacy automorphisms which fix the word $x_1 \cdots x_n \in F_n$, see for instance Birman's book [17]. The kernel of the canonical projection from B_n to the symmetric group S_n is the pure braid group P_n on n strings. As shown by Fadell, Fox, and Neuwirth [33, 34], a classifying space for P_n is $\mathrm{Conf}_n(\mathbb{C})$, the configuration space of n ordered points on the complex line.

The set of all permutation-conjugacy automorphisms of F_n forms a subgroup of $\mathrm{Aut}(F_n)$, denoted by $B\Sigma_n$. The subgroup $P\Sigma_n = B\Sigma_n \cap \mathrm{IA}_n$ is generated by the Magnus automorphisms α_{ij} ($1 \leqslant i \neq j \leqslant n$), while the subgroup $P\Sigma_n^+ = P\Sigma_n \cap \mathrm{IA}_n^+$ is generated by the automorphisms α_{ij} with $i < j$. In [56], McCool gave presentations for the groups $P\Sigma_n$ and $P\Sigma_n^+$; these groups are now also called the McCool groups and the upper McCool groups, respectively.

The *welded braid groups* were introduced by Fenn et al. in [38], who showed that the welded braid group wB_n is isomorphic to $B\Sigma_n$. These groups, together with the *welded pure braid groups* $wP_n \cong P\Sigma_n$ and the *upper welded pure braid groups* $wP_n^+ \cong P\Sigma_n^+$ have generated quite a bit of interest since then, see for instance [3, 4, 8, 15, 30] and references therein. The welded pure braid group wP_n can be identified with group of motions of n unknotted, unlinked circles in the 3-sphere. As shown by Brendle and Hatcher in [18], this group can be realized as the fundamental group of the space of configurations of parallel rings in $\mathbb{R}^3$.

A related class of groups are the *virtual braid groups* vB_n, which were introduced by Kauffman in [45] in the context of virtual knot theory, see also [42]. The kernel of the canonical epimorphism $vB_n \to S_n$ is called the *virtual pure braid group* vP_n. In [5], Bardakov found a concise presentation for vP_n, and defined accordingly the *upper virtual pure braid* group vP_n^+. Whether or not the virtual (pure) braid groups are subgroups of $\mathrm{Aut}(F_n)$ is an open question that goes back to [5].

The groups vP_n and vP_n^+ were also independently studied by Bartholdi et al. [11] and P. Lee [50] as groups arising from the Yang-Baxter equations. Classifying spaces for these groups (also known as the quasi-triangular groups and the triangular groups, respectively) can be constructed by taking quotients of permutahedra by suitable actions of the symmetric groups.

The groups mentioned so far fit into the diagram from Fig. 1 (a related diagram can be found in [3]). We will discuss presentations for these groups, various extensions and homomorphisms between them, as well as their centers in Sect. 2.

1.2 Lie Algebras, LCS Ranks, and Formality

To any finitely generated group G, there corresponds a graded Lie algebra, $\mathrm{gr}(G)$, obtained by taking the direct sum of the successive quotients of the lower central series of G, and tensoring with $\mathbb{C}$. The *LCS ranks* of the group G are defined as the dimensions, $\phi_k(G) = \dim \mathrm{gr}_k(G)$, of the graded pieces of this Lie algebra. As

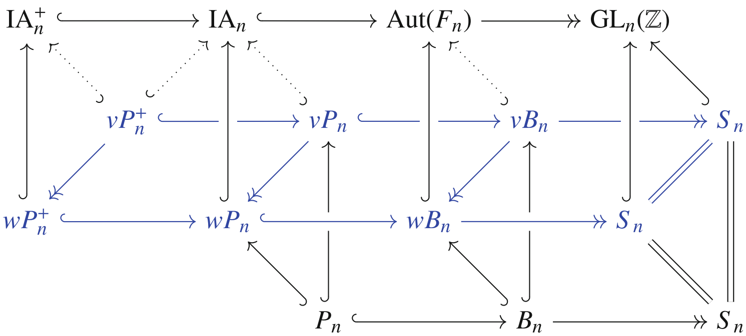


Fig. 1 Braid-like groups and automorphism groups of free groups

explained in Theorem 4, the computation of these ranks is greatly simplified if the group G satisfies certain formality properties, and its cohomology algebra is Koszul.

The set of primitive elements of the completed group algebra $\widehat{\mathbb{C}G}$ is a complete, filtered Lie algebra over $\mathbb{C}$, called the *Malcev Lie algebra* of G, and denoted by $\mathfrak{m}(G)$. By a theorem of Quillen [62], there exists an isomorphism of graded Lie algebras between $\operatorname{gr}(G)$ and $\operatorname{gr}(\mathfrak{m}(G))$.

The group G is said to be *graded-formal* if its associated graded Lie algebra, $\operatorname{gr}(G)$, admits a quadratic presentation. The group G is said to be *filtered-formal* if there exists an isomorphism of filtered Lie algebras between $\mathfrak{m}(G)$ and the degree completion of $\operatorname{gr}(G)$. Furthermore, the group G is called 1*-formal* if it is graded-formal and filtered-formal, or, equivalently, if there is a 1-quasi-isomorphism between the 1-minimal model of G and the cohomology algebra $H^*(G, \mathbb{C})$ endowed with the zero differential. We refer to [69] for a comprehensive study of these formality notions for groups.

A presentation for the Malcev Lie algebra of P_n was given by Kohno in [48], while the associated graded Lie algebra $\operatorname{gr}(P_n)$ and its graded ranks were computed by Kohno [49] and Falk–Randell [36]. It was also realized around that time that the pure braid groups P_n are 1-formal. As shown by Berceanu and Papadima in [15], the Malcev Lie algebras of wP_n and wP_n^+ admit quadratic presentations, that is, the groups wP_n and wP_n^+ are 1-formal. Furthermore, as shown in [11, 50], the groups vP_n and vP_n^+ are graded-formal. On the other hand, we show in [71] that the virtual pure braid groups vP_n and vP_n^+ are 1-formal if and only if $n \leqslant 3$.

A lot is also known about the residual properties of the pure braid-like groups, especially as they relate to the lower central series. For instance, a theorem of Berceanu and Papadima [15], which uses work of Andreadakis [1] and an idea of Hain [43], shows that the groups IA_n are residually torsion-free nilpotent, for all n. Thus, the groups P_n, wP_n, and wP_n^+ also enjoy this property. The fact that the pure braid groups P_n are residually torsion-free nilpotent also follows from the work of Falk and Randell [36, 37]. It is also known that the virtual pure braid groups vP_n and vP_n^+ are residually torsion-free nilpotent for $n \leqslant 3$, but it is not known whether this is the case for $n \geqslant 4$.

1.3 Resonance Varieties and Chen Ranks

We conclude our survey with a discussion of the cohomology algebras of the pure braid-like groups, and of two other related objects: the resonance varieties attached to these graded algebras, and the Chen ranks associated to the groups themselves.

The cohomology algebra of the classical pure braid group P_n was computed by Arnol′d in his seminal paper on the subject, [2]. An explicit presentation for the cohomology algebra of the McCool group wP_n was given by Jensen et al. [44], thereby confirming a conjecture of A. Brownstein and R. Lee. Using different methods, F. Cohen et al. [28] determined the cohomology algebra of the upper McCool group wP_n^+. Finally, the cohomology algebras of the virtual pure braid groups vP_n and vP_n^+ were computed by Bartholdi et al. [11] and Lee [50].

For all these groups G, the cohomology algebra $A = H^*(G, \mathbb{C})$ is quadratic, i.e., it is generated in degree 1 and the ideal of relations is generated in degree 2. In fact, for all but the groups wP_n, $n \geqslant 4$, the ideal of relations admits a quadratic Gröbner basis, and so the algebra A is Koszul. For more details and references regarding this topic, we direct the reader to Table 1 and to Sect. 3.1.

Given a group G satisfying appropriate finiteness conditions, the resonance varieties $\mathcal{R}_s^i(G)$ are certain closed, homogeneous subvarieties of the affine space $A^1 = H^1(G; \mathbb{C})$, defined by means of the vanishing cup products in the cohomology algebra $A = H^*(G, \mathbb{C})$. We restrict our attention here to the first resonance variety $\mathcal{R}_1(G) = \mathcal{R}_1^1(G)$ attached to a finitely generated group G. This variety consists of all elements $a \in A^1$ for which there exists an element $b \in A^1$ such that $a \cup b = 0$, yet b is not proportional to a.

The aforementioned computations of the cohomology algebras of the various pure braid-like groups allows one to determine the corresponding resonance varieties, at least in principle. In the case of the first resonance varieties of the groups P_n, wP_n, and wP_n^+, complete answers can be found in [21, 27], and [71], respectively,

Table 1 Hilbert series, Koszulness, and formality of pure braid-like groups

G	Hilbert series $\mathrm{Hilb}(H^*(G;\mathbb{C}), t)$		Koszulness		1-Formality	
P_n	$\prod_{j=1}^{n-1}(1+jt)$	[49]	Yes	[2, 49, 64]	Yes	[48]
wP_n	$(1+nt)^{n-1}$	[44]	No (for $n \geqslant 4$)	[29]	Yes	[15]
wP_n^+	$\prod_{j=1}^{n-1}(1+jt)$	[28]	Yes	[22]	Yes	[15]
vP_n	$\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{n!}{(n-i)!}t^i$	[11]	Yes	[11, 50]	No (for $n \geqslant 4$)	[70]
vP_n^+	$\sum_{j=1}^{n}\frac{\left(\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(k-i)^n\right)t^{n-j}}{j!}$	[11]	Yes	[11, 50]	No (for $n \geqslant 4$)	[70]

Table 2 Resonance and Chen ranks of braid-like groups

G	First resonance variety $\mathcal{R}_1(G) \subseteq H^1(G;\mathbb{C})$	Chen ranks $\theta_k(G)$, $k \geqslant 3$	Resonance–Chen ranks formula
P_n	$\binom{n}{3} + \binom{n}{4}$ planes [27]	$(k-1)\binom{n+1}{4}$ [24]	Yes [27]
wP_n	$\binom{n}{2}$ planes and $\binom{n}{4}$ linear spaces of dimension 3 [21]	$(k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$ for $k \gg 3$ [23]	Yes [23]
wP_n^+	$(n-i)$ linear spaces of dimension $i \geqslant 2$ [71]	$\sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$ [71]	No [71]
vP_3	$H^1(vP_3, \mathbb{C}) = \mathbb{C}^6$ [10, 70]	$\binom{k+3}{5} + \binom{k+2}{4} + \binom{k+1}{3} + 6\binom{k}{2} + k - 2$ [70]	No [70]
vP_4^+	3-dimensional non-linear subvariety of degree 6 [70]	$(k^3-1) + \binom{k}{2}$ [70]	No [70]

while for the virtual pure braid groups, partial answers are given in [70]. We list some of the features of these varieties in Table 2.

By comparing the resonance varieties of the groups P_n and wP_n^+, it can be shown that these groups are not isomorphic for $n \geqslant 4$ (cf. [71]); this answers a question of F. Cohen et al. [28], see Remark 2. By computing the resonance variety $\mathcal{R}_1(vP_4^+)$, and using the Tangent Cone Theorem from [32], we prove that the group vP_4^+ is not 1-formal. In view of the retraction property for 1-formality established in [69], we conclude that the groups vP_n and vP_n^+ are not 1-formal for $n \geqslant 4$.

The *Chen ranks* of a finitely generated group G are the dimensions, $\theta_k(G) = \dim \operatorname{gr}_k(G/G'')$, of the graded pieces of the graded Lie algebra associated to the maximal metabelian quotient of G. In [19], K.-T. Chen computed the Chen ranks of the free groups F_n, while in [54], W.S. Massey gave an alternative method for computing the Chen ranks of a group G in terms of the Alexander invariant G'/G''.

The Chen ranks of the pure braid groups P_n were computed in [24], while an explicit relation between the Chen ranks and the resonance varieties of an arrangement group was conjectured in [67]. Building on work from [26, 63, 65] and especially [58], Cohen and Schenck confirmed this conjecture in [23] for a class of 1-formal groups which includes arrangement groups. In the process, they also computed the Chen ranks $\theta_k(wP_n)$ for k sufficiently large.

Using the Gröbner basis algorithm from [24, 26], we compute in [71] all the Chen ranks of the upper McCool groups wP_n^+. This computation, recorded here in Theorem 11, shows that, for each $n \geqslant 4$, the group wP_n^+ is not isomorphic to either the pure braid group P_n, or to the product $\prod_{i=1}^{n-1} F_i$, although these three groups share the same LCS ranks and the same Betti numbers. We also provide the Chen ranks of the groups vP_n and vP_n^+ for low values of n. The full computation of the Chen ranks of the virtual pure braid groups remains to be done.

2 Braid Groups and Their Relatives

2.1 Braid Groups and Pure Braid Groups

Let $\mathrm{Aut}(F_n)$ be the group of (right) automorphisms of the free group F_n on generators $x_1, \dots, x_n$. Recall that the Artin braid group B_n consists of those permutation-conjugacy automorphisms which fix the word $x_1 \cdots x_n \in F_n$. In particular, $B_1 = \{1\}$ and $B_2 = \mathbb{Z}$. The natural inclusion $\alpha_n \colon B_n \hookrightarrow \mathrm{Aut}(F_n)$ is also known as the Artin representation of the braid group.

For each $1 \leqslant i < n$, let σ_i be the braid automorphism which sends x_i to $x_i x_{i+1} x_i^{-1}$ and x_{i+1} to x_i, while leaving the other generators of F_n fixed. As shown for instance in [17], the braid group B_n is generated by the elementary braids $\sigma_1, \dots, \sigma_{n-1}$, subject to the well-known relations

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leqslant i \leqslant n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geqslant 2. \end{cases} \tag{R1}$$

On the other hand, the symmetric group S_n has a presentation with generators s_i for $1 \leqslant i \leqslant n-1$ and relations

$$\begin{cases} s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, & 1 \leqslant i \leqslant n-2, \\ s_is_j = s_js_i, & |i-j| \geqslant 2. \\ s_i^2 = 1, & 1 \leqslant i \leqslant n-1; \end{cases} \tag{R2}$$

The canonical projection from the braid group to the symmetric group, which sends the elementary braid σ_i to the transposition s_i, has kernel the *pure braid group* on n strings,

$$P_n = \ker(\phi \colon B_n \twoheadrightarrow S_n) = B_n \cap \mathrm{IA}_n, \tag{1}$$

where $\phi(\sigma_i) = s_i$ for $1 \leqslant i \leqslant n-1$. The group P_n is generated by the n-stranded braids

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \tag{2}$$

for $1 \leqslant i < j \leqslant n$. It is readily seen that $P_1 = \{1\}$, $P_2 = \mathbb{Z}$, and $P_3 \cong F_2 \times \mathbb{Z}$. More generally, as shown by Fadell and Neuwirth [34] (see also [25, 36, 37]), the pure braid group P_n can be decomposed as an iterated semi-direct product of free groups,

$$P_n = F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1} = F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_1, \tag{3}$$

where $\alpha_{n-1} \colon P_{n-1} \hookrightarrow \mathrm{Aut}(F_n)$ is the restriction of the Artin representation of the braid group B_{n-1} to the pure braid subgroup P_{n-1}.

Work of Chow [20] and Birman [17] shows that the center $Z(P_n)$ of the pure braid group on $n \geqslant 2$ strands is infinite cyclic, generated by the full twist braid $\prod_{1\leqslant i<j\leqslant n} A_{ij}$. It follows that $P_n \cong \overline{P}_n \times \mathbb{Z}$, where $\overline{P}_n = P_n/Z(P_n)$.

2.2 Welded Braid Groups

The set of all permutation-conjugacy automorphisms of the free group of rank n forms the braid-permutation group wB_n. This group has a presentation with generators σ_i and s_i ($1 \leqslant i < n$) and relations (R1) and (R2), as well as

$$\begin{cases} s_i\sigma_j = \sigma_j s_i, & |i-j| \geqslant 2, \\ \sigma_i s_{i+1} s_i = s_{i+1} s_i \sigma_{i+1}, & 1 \leqslant i \leqslant n-2, \end{cases} \tag{R3}$$

and

$$s_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i s_{i+1}, \quad 1 \leqslant i \leqslant n-2. \tag{R4}$$

The three types of braid crossings mentioned above are depicted in Fig. 2.

The welded pure braid group wP_n, also known as the group of basis-conjugating automorphisms in [4, 21, 38], or the McCool group in [15], is defined as

$$wP_n = \ker(\rho\colon wB_n \twoheadrightarrow S_n) = wB_n \cap \mathrm{IA}_n, \tag{4}$$

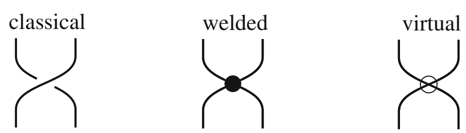
where $\rho(\sigma_i) = \rho(s_i) = s_i$ for $1 \leqslant i \leqslant n-1$. As shown by McCool in [56], this group is generated by the Magnus automorphisms α_{ij}, for all $1 \leqslant i \neq j \leqslant n$, subject to the relations

$$\begin{aligned} &\alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{jk}\alpha_{ik}\alpha_{ij}, && \text{for } i,j,k \text{ distinct},\\ &[\alpha_{ij},\alpha_{st}] = 1, && \text{if } \{i,j\}\cap\{s,t\} = \emptyset,\\ &[\alpha_{ik},\alpha_{jk}] = 1, && \text{for } i,j,k \text{ distinct}. \end{aligned}$$

In particular, $wP_1 = \{1\}$ and $wP_2 = F_2$.

Consider now the upper welded pure braid group (or, the upper McCool group) $wP_n^+ = wP_n \cap \mathrm{IA}_n^+$. This is the subgroup of wP_n generated by all the automorphisms α_{ij} with $i < j$. It readily seen that $wP_1^+ = \{1\}$, $wP_2^+ = \mathbb{Z}$, and $wP_3^+ \cong F_2 \times \mathbb{Z}$.

Fig. 2 Braid crossings



Furthermore, as shown by F. Cohen et al. [28], the group wP_n^+ can be decomposed as an iterated semi-direct product of free groups,

$$wP_n^+ = F_{n-1} \rtimes_{\alpha_{n-1}^+} wP_{n-1}^+ = F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_1, \tag{5}$$

where $\alpha_{n-1}^+ \colon wP_{n-1}^+ \hookrightarrow \mathrm{Aut}(F_{n-1})$ is the restriction of the Artin representation of B_{n-1} to wP_{n-1}^+.

It follows from the previous discussion that $P_n \cong wP_n^+$ for $n \leqslant 3$. In view of this fact, a natural question (asked by F. Cohen et al. in [28]) is whether the groups P_n and wP_n^+ are isomorphic for $n \geqslant 4$. A negative answer will be given in Corollary 1. In the same circle of ideas, let us also mention the following result from [71].

Proposition 1 ([71]) *For each $n \geqslant 4$, the inclusion map $wP_n^+ \hookrightarrow wP_n$ is not a split monomorphism.*

The proof of this proposition is based upon the contrasting nature of the resonance varieties of the two groups. We will come back to this point in Sect. 3.

Cohen and Pruidze showed in [22] that the center of the group wP_n^+ ($n \geqslant 2$) is infinite cyclic, generated by the automorphism $\prod_{1\leqslant j\leqslant n-1} \alpha_{j,n}$. On the other hand, Dies and Nicas showed in [31] that the center of the group wP_n is trivial for $n \geqslant 2$.

2.3 Virtual Braid Groups

Closely related are the virtual braid groups vB_n, the virtual pure braid groups vP_n, and their upper triangular subgroups, vP_n^+, obtained by omitting certain commutation relations from the respective McCool groups. The group vB_n admits a presentation with generators σ_i and s_i for $i = 1, \dots, n-1$, subject to the relations (R1), (R2), and (R3). The virtual pure braid group vP_n is defined as the kernel of the canonical epimorphism $\psi \colon vB_n \to S_n$ given by $\psi(\sigma_i) = \psi(s_i) = s_i$ for $1 \leqslant i \leqslant n-1$, see [5].

A finite presentation for vP_n was given by Bardakov [5]. The virtual pure braid group vP_n and its 'upper' subgroup, vP_n^+, were both studied in depth (under different names) by Bartholdi et al. and Lee in [11, 50]. These groups are generated by elements x_{ij} for $i \neq j$ (respectively, for $i < j$), subject to the relations

$$x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, \qquad \text{for } i,j,k \text{ distinct},$$

$$[x_{ij}, x_{st}] = 1, \qquad \text{if } \{i,j\} \cap \{s,t\} = \emptyset.$$

Unlike the inclusion map $wP_n^+ \hookrightarrow wP_n$ from Proposition 1, the inclusion $vP_n^+ \hookrightarrow vP_n$ *does* admit a splitting, see [11, 70].

Proposition 2 *There exist monomorphisms and epimorphisms making the following diagram commute.*

$$\begin{array}{ccccc} B_n & \xhookrightarrow{\varphi_n} & vB_n & \xrightarrow{\pi_n} & wB_n \\ \uparrow & & \uparrow & & \uparrow \\ P_n & \hookrightarrow & vP_n & \twoheadrightarrow & wP_n \end{array}$$

Furthermore, the compositions of the horizontal homomorphisms are also injective.

Proof There are natural inclusions $\varphi_n: B_n \hookrightarrow vB_n$ and $\psi_n: B_n \hookrightarrow wB_n$ that send σ_i to σ_i, as well as a canonical projection $\pi_n: vB_n \twoheadrightarrow wB_n$, that matches the generators σ_i and s_i of the respective groups. By construction, we have that $\pi_n \circ \varphi_n = \psi_n$.

We claim that these homomorphisms restrict to homomorphisms between the respective pure-braid like groups. Indeed, as shown Bartholdi et al. in [11], the homomorphism φ_n restricts to a map $P_n \hookrightarrow vP_n$, given by

$$A_{ij} \mapsto x_{j-1,j} \dots x_{i+1,j} x_{i,j} x_{j,i} (x_{j-1,j} \dots x_{i+1,j})^{-1}. \tag{6}$$

Clearly, the projection π_n restricts to a map $vP_n \twoheadrightarrow wP_n$ that sends x_{ij} to α_{ij}. Using these observations, together with work of Bardakov [5], we see that the homomorphism ψ_n restricts to an injective map $P_n \hookrightarrow wP_n$. □

From the defining presentations, it is readily seen that $vP_2^+ \cong \mathbb{Z}$ and $vP_2 \cong F_2$, while $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$. Moreover, using a computation of Bardakov et al. [10], we show in [70] that $vP_3 \cong \overline{P}_4 * \mathbb{Z}$. Consequently vP_2, vP_3 and vP_3^+ have trivial centers.

More generally, Dies and Nicas showed in [31] that the center of the group vP_n is trivial for $n \geqslant 2$; furthermore, the center of vP_n^+ is trivial for $n \geqslant 3$, with one possible exception (and no exception if Wilf's conjecture is true).

Remark 1 In Lemma 6 from [5], Bardakov states that the group vP_n splits as a semi-direct product of the form $V_n^* \rtimes vP_{n-1}$, where V_n^* is a free group. This lemma would imply, via an easy induction argument, that $Z(vP_n) = \{1\}$ for $n \geqslant 2$ and $Z(vP_n^+) = \{1\}$ for $n \geqslant 3$. Unfortunately, there seems to be a problem with this lemma, according to the penultimate remark from [39, §6].

2.4 Topological Interpretations

All the braid-like groups mentioned previously admit nice topological interpretations. For instance, the braid group B_n can be realized as the mapping class group of the 2-disk with n marked points, $\mathrm{Mod}_{0,n}^1$, while the pure braid group P_n can be

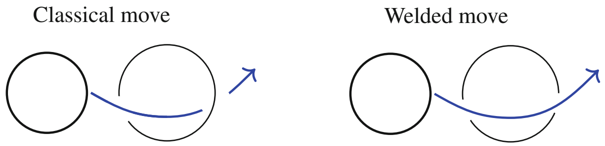


Fig. 3 Untwisted flying rings

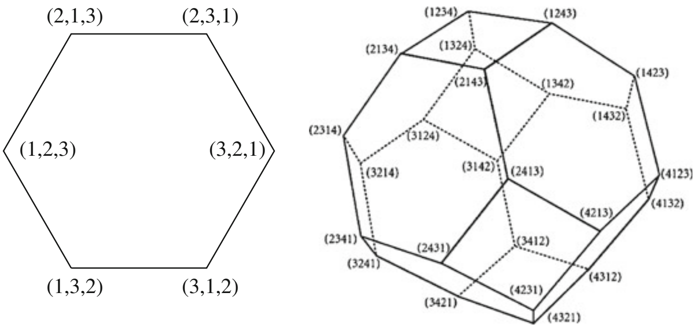


Fig. 4 $\mathbf{P}_3$ and $\mathbf{P}_4$

viewed as the fundamental group of the configuration space of n ordered points on the complex line, $\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$, see for instance [17].

The welded braid group wB_n is the fundamental group of the 'untwisted ring space,' which consists of all configurations of n parallel rings (i.e., unknotted circles) in $\mathbb{R}^3$, see Fig. 3. However, as shown in [18], this space is not aspherical. The welded pure braid group wP_n can be viewed as the pure motion group of n unknotted, unlinked circles in $\mathbb{R}^3$, cf. Goldsmith [40]. The group wP_n^+ is the fundamental group of the subspace consisting of all configurations of circles of unequal diameters in the 'untwisted ring space,' see Brendle–Hatcher [18] and Bellingeri–Bodin [13].

A classifying space for the group vP_n^+ is identified in [11] as the quotient space of the $(n-1)$-dimensional permutahedron $\mathbf{P}_n$ by actions of certain symmetric groups. More precisely, let $\mathbf{P}_n$ be the convex hull of the orbit of a generic point in $\mathbb{R}^n$ under the permutation action of the symmetric group S_n on its coordinates. Then $\mathbf{P}_n$ is a polytope whose faces are indexed by all ordered partitions of the set $[n] = \{1, \ldots, n\}$; see Fig. 4. For each $r \in [n]$, there is a natural action of S_r on the disjoint union of all $(n-r)$-dimensional faces of this polytope, $C_1 \sqcup \cdots \sqcup C_r$. Similarly, a classifying space for vP_n can be constructed as a quotient space of $\mathbf{P}_n \times S_n$.

3 Cohomology Algebras and Resonance Varieties

3.1 Cohomology Algebras

Recall that the pure braid group P_n is the fundamental group of the configuration space $\mathrm{Conf}_n(\mathbb{C})$. As shown by Arnol′d in [2], the cohomology ring $H^*(P_n;\mathbb{Z})$ is the skew-commutative ring generated by degree 1 elements u_{ij} ($1 \leqslant i < j \leqslant n$), identified with the de Rham cocycles $\frac{dz_i - dz_j}{z_i - z_j}$. The cohomology algebras of the other pure braid-like groups were computed by several groups of researchers over the last 10 years, see [11, 28, 44, 50].

We summarize these computations, as follows. To start with, we denote the standard (degree 1) generators of $H^*(wP_n;\mathbb{C})$ and $H^*(vP_n;\mathbb{C})$ by a_{ij} for $1 \leqslant i \neq j \leqslant n$, and we denote the generators of $H^*(wP_n^+;\mathbb{C})$ and $H^*(vP_n^+;\mathbb{C})$ by e_{ij} for $1 \leqslant i < j \leqslant n$. Next, we list several types of relations that occur in these algebras.

$$u_{jk}u_{ik} = u_{ij}(u_{ik} - u_{jk}) \quad \text{for } i < j < k, \tag{I1}$$

$$a_{ij}a_{ji} = 0 \quad \text{for } i \neq j, \tag{I2}$$

$$a_{kj}a_{ik} = a_{ij}(a_{ik} - a_{jk}) \quad \text{for } i, j, k \text{ distinct}, \tag{I3}$$

$$a_{ji}a_{ik} = (a_{ij} - a_{ik})a_{jk} \quad \text{for } i, j, k \text{ distinct}, \tag{I4}$$

$$e_{ij}(e_{ik} - e_{jk}) = 0 \quad \text{for } i < j < k, \tag{I5}$$

$$(e_{ij} - e_{ik})e_{jk} = 0 \quad \text{for } i < j < k. \tag{I6}$$

Finally, we record in Table 3 the cohomology algebras of the pure braid groups, the welded pure braid groups, the virtual pure braid groups, and their upper triangular subgroups.

The above presentations of the cohomology algebras differ slightly from those given in the original papers. Using these presentations, it is easily seen that the aforementioned homomorphisms, $f_n\colon vP_n \hookrightarrow wP_n$ and $g_n\colon vP_n^+ \hookrightarrow wP_n^+$, induce epimorphisms in cohomology with $\mathbb{C}$-coefficients.

Let us also highlight the fact that the cohomology algebras of the pure braid-like groups are *quadratic algebras*. More precisely, they are all algebras of the form

Table 3 Cohomology algebras of the pure braid-like groups

	$H^*(P_n;\mathbb{C})$ [2]	$H^*(wP_n;\mathbb{C})$ [44]	$H^*(wP_n^+;\mathbb{C})$ [28]	$H^*(vP_n;\mathbb{C})$ [11, 50]	$H^*(vP_n^+;\mathbb{C})$ [11, 50]
Generators	u_{ij} $1 \leqslant i < j \leqslant n$	a_{ij} $1 \leqslant i \neq j \leqslant n$	e_{ij} $1 \leqslant i < j \leqslant n$	a_{ij} $1 \leqslant i \neq j \leqslant n$	e_{ij} $1 \leqslant i < j \leqslant n$
Relations	(I1)	(I2) (I3)	(I5)	(I2)–(I4)	(I5) (I6)

$A = E/I$, where E is an exterior algebra generated in degree 1, and I is an ideal generated in degree 2.

It is also known that the cohomology algebras of all these groups (with the exception of wP_n for $n \geqslant 4$) are *Koszul algebras*. That is to say, for each such algebra A, the ground field $\mathbb{C}$ admits a free, *linear* resolution over A, or equivalently, $\mathrm{Tor}_i^A(\mathbb{C}, \mathbb{C})_j = 0$ for $i \neq j$. In fact, it is known that in all these cases, the relations specified in Table 3 form a quadratic Gröbner basis for the ideal of relations I, a fact which implies Koszulness for the algebra $A = E/I$. On the other hand, it was recently shown by Conner and Goetz [29] that the cohomology algebras of the groups wP_n are not Koszul for $n \geqslant 4$.

For a summary of the above discussion, as well as detailed references for the various statements therein we refer to Table 1.

3.2 Resonance Varieties

Now let $A = \bigoplus_{i \geqslant 0} A^i$ be a graded, graded-commutative algebra over $\mathbb{C}$. We shall assume that A is connected, i.e., $A^0 = \mathbb{C}$, generated by the unit 1. For each element $a \in A^1$, one can form a cochain complex, (A, δ_a), known as the *Aomoto complex*, with differentials $\delta_a^i \colon A^i \to A^{i+1}$ given by $\delta_a^i(u) = a \cdot u$.

The *resonance varieties* of the graded algebra A are the jump loci for the cohomology of the Aomoto complexes parametrized by the vector space A^1. More precisely, for each $i \geqslant 0$ and $s \geqslant 1$, the (degree i, depth s) resonance variety of A is the set

$$\mathcal{R}_s^i(A) = \{a \in A^1 \mid \dim H^i(A, \delta_a) \geqslant s\}. \tag{7}$$

If A is locally finite (i.e., each graded piece A^i is finite-dimensional), then all these sets are closed, homogeneous subvarieties of the affine space A^1.

The resonance varieties of a finitely-generated group G are defined as $\mathcal{R}_s^i(G) := \mathcal{R}_s^i(A)$, where $A = H^*(G, \mathbb{C})$ is the cohomology algebra of the group. If G admits a classifying space with finite k-skeleton, for some $k \geqslant 1$, then the sets $\mathcal{R}_s^i(G)$ are algebraic subvarieties of the affine space A^1, for all $i \leqslant k$, see [60, Corollary 4.2]. We will focus here on the first resonance variety $\mathcal{R}_1(G) := \mathcal{R}_1^1(G)$, which can be described more succinctly as

$$\mathcal{R}_1(G) = \{a \in A^1 \mid \exists\, b \in A^1 \setminus \mathbb{C} \cdot a \text{ such that } ab = 0 \in A^2\}. \tag{8}$$

The idea of studying a family of cochain complexes parametrized by the cohomology ring in degree 1 originated from the theory of hyperplane arrangements, in the work of Falk [35], while the more general concept from (8) first appeared in work of Matei and Suciu [55].

The resonance varieties of a variety spaces and groups have been studied intensively from many perspectives and in varying degrees of generality, see for instance [32, 60, 68] and reference therein. In particular, the first resonance varieties of the groups P_n, wP_n, and wP_n^+ have been completely determined in [21, 27, 71]. We summarize those results as follows.

Theorem 1 ([27]) *For each $n \geqslant 3$, the first resonance variety of the pure braid group P_n has decomposition into irreducible components given by*

$$\mathcal{R}_1(P_n) = \bigcup_{1\leqslant i<j<k\leqslant n} L_{ijk} \cup \bigcup_{1\leqslant i<j<k<l\leqslant n} L_{ijkl},$$

where

- *L_{ijk} is the plane defined by the equations $\sum_{\{p,q\}\subset\{i,j,k\}} x_{pq} = 0$ and $x_{st} = 0$ if $\{s,t\} \not\subseteq \{i,j,k\}$;*
- *L_{ijkl} is the plane defined by the equations $\sum_{\{p,q\}\subset\{i,j,k,l\}} x_{pq} = 0$, $x_{ij} = x_{kl}$, $x_{jk} = x_{il}$, $x_{ik} = x_{jl}$ and $x_{st} = 0$ if $\{s,t\} \not\subseteq \{i,j,k,l\}$.*

Theorem 2 ([21]) *For each $n \geqslant 2$, the first resonance variety of the group wP_n has decomposition into irreducible components given by*

$$\mathcal{R}_1(wP_n) = \bigcup_{1\leqslant i<j\leqslant n} L_{ij} \cup \bigcup_{1\leqslant i<j<k\leqslant n} L_{ijk},$$

where L_{ij} is the plane defined by the equations $x_{pq} = 0$ if $\{p,q\} \neq \{i,j\}$ and L_{ijk} is the 3-dimensional linear subspace defined by the equations $x_{ji} + x_{ki} = x_{ij} + x_{kj} = x_{ik} + x_{jk} = 0$ and $x_{st} = 0$ if $\{s,t\} \not\subseteq \{i,j,k\}$.

Theorem 3 ([71]) *For each $n \geqslant 2$, the first resonance variety of the group wP_n^+ has decomposition into irreducible components given by*

$$\mathcal{R}_1(wP_n^+) = \bigcup_{n-1\geqslant i>j\geqslant 1} L_{i,j},$$

where $L_{i,j}$ is the linear subspace of dimension $j+1$ spanned by $e_{i+1,j+1}$ and $e_{j+1,k} - e_{i+1,k}$, for $1 \leqslant k \leqslant j$.

Much less is known about the resonance varieties of the virtual pure braid groups. For low values of n, the varieties $\mathcal{R}_1(vP_n)$ and $\mathcal{R}_1(vP_n^+)$ have been determined in [70]. For instance, the decomposition $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ and known properties of resonance varieties of free products leads to the equality $\mathcal{R}_1(vP_3) = H^1(vP_3, \mathbb{C})$. We will need the following computation later on.

Proposition 3 ([70]) *The resonance variety* $\mathcal{R}_1(vP_4^+)$ *is the degree* 6*, irreducible,* 4*-dimensional subvariety of the affine space* $H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6$ *defined by the equations*

$$x_{12}x_{24}(x_{13}+x_{23}) + x_{13}x_{34}(x_{12}-x_{23}) - x_{24}x_{34}(x_{12}+x_{13}) = 0,$$
$$x_{12}x_{23}(x_{14}+x_{24}) + x_{12}x_{34}(x_{23}-x_{14}) + x_{14}x_{34}(x_{23}+x_{24}) = 0,$$
$$x_{13}x_{23}(x_{14}+x_{24}) + x_{14}x_{24}(x_{13}+x_{23}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0,$$
$$x_{12}(x_{13}x_{14}-x_{23}x_{24}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0.$$

4 Lie Algebras and Formality

4.1 The Associated Graded Lie Algebra of a Group

The *lower central series* of a group G is a descending sequence of normal subgroups, $\{\gamma_k G\}_{k\geqslant 1}$, defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1}G = [\gamma_k G, G]$. The successive quotients of this series, $\gamma_k G/\gamma_{k+1}G$, are abelian groups. The direct sum of these groups,

$$\operatorname{gr}(G;\mathbb{Z}) = \bigoplus_{k\geqslant 1} \gamma_k G/\gamma_{k+1}G, \tag{9}$$

endowed with the Lie bracket $[x, y]$ induced from the group commutator, has the structure of a graded Lie algebra over $\mathbb{Z}$. The *associated graded Lie algebra* of G over $\mathbb{C}$ is defined as $\operatorname{gr}(G) = \operatorname{gr}(G;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. Before proceeding, let us recall the following lemma due to Falk and Randell [37].

Lemma 1 ([37]) *Let* $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ *be a split exact sequence of groups, and suppose* Q *acts trivially on* $N/[N,N]$*. Then, for each* $k \geqslant 1$*, there is an induced split exact sequence,* $1 \longrightarrow \gamma_k(N) \longrightarrow \gamma_k(G) \longrightarrow \gamma_k(Q) \longrightarrow 1$.

If a group G is finitely generated, its lower central series quotients are finitely generated abelian groups. Set

$$\phi_k(G) = \operatorname{rank} \gamma_k G/\gamma_{k+1}G, \tag{10}$$

or, equivalently, $\phi_k(G) = \dim \operatorname{gr}_k(G)$. These LCS ranks can be computed from the Hilbert series of the universal enveloping algebra of $\operatorname{gr}(G)$ by means of the Poincaré–Birkhoff–Witt theorem, as follows:

$$\prod_{k\geqslant 1}(1-t^k)^{-\phi_k(G)} = \operatorname{Hilb}(U(\operatorname{gr}(G)), t). \tag{11}$$

A finitely generated group G is said to be *graded-formal* if the associated graded Lie algebra $\mathrm{gr}(G)$ is quadratic, that is, admits a presentation with generators in degree 1 and relators in degree 2. The next theorem was proved under some stronger hypothesis by Papadima and Yuzvinsky in [61], and essentially in this form in [69]. For completeness, we sketch the proof.

Theorem 4 ([61, 69]) *If G is a finitely generated, graded-formal group, and its cohomology algebra, $A = H^*(G;\mathbb{C})$, is Koszul, then the LCS ranks of G are given by*

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k(G)} = \mathrm{Hilb}(A,-t). \tag{12}$$

Proof By assumption, the graded Lie algebra $\mathfrak{g} = \mathrm{gr}(G)$ is quadratic, while the algebra $A = H^*(G;\mathbb{C})$ is Koszul, hence quadratic. Write $A = T(V)/I$, where $T(V)$ is the tensor algebra on a finite-dimensional $\mathbb{C}$-vector space V concentrated in degree 1, and I is an ideal generated in degree 2. Define the quadratic dual of A as $A^! = T(V^*)/I^{\perp}$, where V^* is the vector space dual to V and $I^{\perp}$ is the ideal generated by all $\alpha \in V^* \wedge V^*$ with $\alpha(I_2) = 0$, see [64]. It follows from [61, Lemma 4.1] that $A^!$ is isomorphic to $U(\mathfrak{g})$.

Now, since A is Koszul, its quadratic dual is also Koszul. Thus, the following Koszul duality formula holds:

$$\mathrm{Hilb}(A,t)\cdot \mathrm{Hilb}(A^!,-t) = 1. \tag{13}$$

Putting things together and using (11) completes the proof. □

4.2 *The LCS Ranks of the Pure-Braid Like Groups*

We now turn to the associated graded Lie algebras of the pure braid-like groups. We start by listing the types of relations occurring in these Lie algebras:

$$[x_{ij},x_{ik}] + [x_{ij},x_{jk}] + [x_{ik},x_{jk}] = 0 \quad \text{for distinct } i,j,k, \tag{L1}$$

$$[x_{ij},x_{kl}] = 0 \quad \text{for } \{i,j\}\cap\{k,l\} = \emptyset, \tag{L2}$$

$$[x_{ik},x_{jk}] = 0 \quad \text{for distinct } i,j,k, \tag{L3}$$

$$[x_{im},x_{ij}+x_{ik}+x_{jk}] = 0 \quad \text{for } m=j,k \text{ and } i,j,m \text{ distinct.} \tag{L4}$$

The corresponding presentations for the associated graded Lie algebras of the groups P_n, wP_n, wP_n^+, vP_n, and vP_n^+ are summarized in Table 4. It is readily seen that all these graded Lie algebras are quadratic. Consequently, all aforementioned pure braid-like groups are graded-formal.

Table 4 Associated graded Lie algebras of the pure braid-like groups

	$\mathrm{gr}(P_n)$ [36, 48]	$\mathrm{gr}(wP_n)$ [44]	$\mathrm{gr}(wP_n^+)$ [28]	$\mathrm{gr}(vP_n)$ [11, 50]	$\mathrm{gr}(vP_n^+)$ [11, 50]
Generators	x_{ij}, $1 \leqslant i < j \leqslant n$	x_{ij}, $1 \leqslant i \neq j \leqslant n$	x_{ij}, $1 \leqslant i < j \leqslant n$	x_{ij}, $1 \leqslant i \neq j \leqslant n$	x_{ij}, $1 \leqslant i < j \leqslant n$
Relations	(L2) (L4)	(L1) (L2) (L3)	(L1) (L2) (L3)	(L1) (L2)	(L1) (L2)

The various homomorphisms between the pure-braid like groups defined previously induce morphisms between the corresponding associated graded Lie algebras. These morphisms fit into the following commuting diagram.

$$\begin{array}{ccccc} \mathrm{gr}(vP_n^+) & \hookrightarrow & \mathrm{gr}(vP_n) & & \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & \hookleftarrow & \mathrm{gr}(P_n)\,. \\ \mathrm{gr}(wP_n^+) & \hookrightarrow & \mathrm{gr}(wP_n) & \leftarrow & \end{array} \tag{14}$$

As shown by Bartholdi et al. in [11], the morphism $\mathrm{gr}(\varphi_n)\colon \mathrm{gr}(P_n) \to \mathrm{gr}(vP_n)$ is injective. Using the presentations given in Table 4, we see that the remaining morphisms are either injective or surjective (as indicated in the diagram), with the possible exception of $\mathrm{gr}(\psi_n)\colon \mathrm{gr}(P_n) \to \mathrm{gr}(wP_n)$, whose injectivity has not been established, as far as we know.

The LCS ranks of the pure braid groups P_n were computed by Kohno [49], using methods from rational homotopy theory, and by Falk and Randell [36], using the decomposition (3) and Lemma 1. The LCS ranks of the upper welded braid groups wP_n^+ were computed by F. Cohen et al. [28], using the decomposition (5) and again Lemma 1. Finally, work of Bartholdi et al. [11] and P. Lee [50] gives the LCS ranks of vP_n and vP_n^+. We summarize these results in the next theorem.

Theorem 5 *The LCS ranks of the groups $G = P_n$, wP_n^+, vP_n, and vP_n^+ are given by the identity $\prod_{k=1}^{\infty}(1-t^k)^{\phi_k(G)} = \mathrm{Hilb}(H^*(G;\mathbb{C}), -t)$, with the relevant Hilbert series given in Table 1.*

Alternatively, this result follows from Theorem 4 once it is shown that, in all these cases, the Lie algebra $\mathrm{gr}(G)$ is quadratic and the cohomology algebra $A = H^*(G;\mathbb{C})$ is Koszul.

On the other hand, as mentioned previously, the cohomology algebras $H^*(wP_n,\mathbb{C})$ are not Koszul for $n \geqslant 4$. The (computer-aided) proof of this fact by Conner and Goetz [29] implies that the LCS ranks of wP_n are *not* given by formula (12). For instance, the formula would say that the first eight LCS ranks of wP_4 are $12, 18, 60, 180, 612, 2\,010, 7\,020$, and $24\,480$. Using the computations from [29], we see that the first seven values of $\phi_k(wP_4)$ are correct, but that $\phi_8(wP_4) = 24\,490$.

4.3 Residual Properties

Let $\mathcal{P}$ be a group-theoretic property. A group G is said to be *residually* $\mathcal{P}$ if for any $g \in G$, $g \neq 1$, there exists a group Q satisfying property $\mathcal{P}$, and an epimorphism $\psi\colon G \to Q$ such that $\psi(g) \neq 1$.

We are mainly interested here in the residual properties related to the lower central series of G. For instance, we say that the group G is *residually nilpotent* if every non-trivial element can be detected in a nilpotent quotient. This happens precisely when the nilpotent radical of G is trivial, that is, $\bigcap_{k\geqslant 1} \gamma_k G = \{1\}$.

Likewise, we say that a group G is *residually torsion-free nilpotent* if every non-trivial element can be detected in a torsion-free nilpotent quotient. This happens precisely when $\bigcap_{k\geqslant 1} \tau_k G = \{1\}$, where

$$\tau_k G = \{g \in G \mid g^n \in \gamma_k G, \text{ for some } n \in \mathbb{N}\}. \tag{15}$$

Clearly, the second property is stronger than the first. Nevertheless, the following holds: if G is residually nilpotent and $\mathrm{gr}_k(G, \mathbb{Z})$ is torsion-free, for each $k \geqslant 1$, then G is residually torsion-free nilpotent. In turn, this last property implies that G is torsion-free. Moreover, residually nilpotent groups are residually finite.

For a group G, the properties of being residually nilpotent or residually torsion-free nilpotent are inherited by subgroups $H < G$, since $\gamma_k H < \gamma_k G$ and $\tau_k H < \tau_k G$. Moreover, both properties are preserved under direct products and free products, see Malcev [52] and Baumslag [12]. For more on this subject, see also [6, 7, 46].

In [1], Andreadakis introduced a new filtration on the automorphism group of a group G, nowadays called the Andreadakis–Johnson filtration. This filtration is defined by setting

$$\Phi_k(\mathrm{Aut}(G)) = \ker(\mathrm{Aut}(G) \to \mathrm{Aut}(G/\gamma_{k+1}(G)), \tag{16}$$

for all $k \geqslant 0$. Note that $\Phi_0(\mathrm{Aut}(G)) = \mathrm{Aut}(G)$; the group $\mathcal{I}(G) = \Phi_1(\mathrm{Aut}(G))$ is called the *Torelli group* of G.

As shown by Andreadakis, if the intersection $\bigcap_{k\geqslant 1} \gamma_k G$ is trivial then the intersection $\bigcap_{k\geqslant 1} \Phi_k(\mathrm{Aut}(G))$ is also trivial. Furthermore, a theorem of L. Kaloujnine implies that $\gamma_k(\mathcal{I}(G)) < \Phi_k(\mathrm{Aut}(G))$ for all $k \geqslant 1$, see e.g. [59]. We thus have the following basic result.

Theorem 6 ([1]) *Let G be a residually nilpotent group. Then the Torelli group $\mathcal{I}(G)$ is also residually nilpotent.*

As noted by Hain [43] in the case of the Torelli group of a Riemann surface and proved by Berceanu and Papadima [15] in full generality, a stronger assumption leads to a stronger conclusion.

Theorem 7 ([15, 43]) *Let G be a finitely generated, residually nilpotent group, and suppose $\mathrm{gr}_k(G, \mathbb{Z})$ is torsion-free for all $k \geqslant 1$. Then the Torelli group $\mathcal{I}(G)$ is residually torsion-free nilpotent.*

We specialize now to the case $G = F_n$. In [51] Magnus showed that all free groups are residually torsion-free nilpotent (this also follows from the aforementioned results of Malcev and Baumslag). Furthermore, P. Hall and Magnus showed that $\mathrm{gr}(F_n, \mathbb{Z})$ is the free Lie algebra on n generators, and thus is torsion-free (see [66, Chap. IV, §6]). Therefore, by Theorem 7, the Torelli group $\mathrm{IA}_n = \mathcal{I}(F_n)$ is residually torsion-free nilpotent. Hence, all its subgroups, such as IA_n^+, P_n, wP_n, and wP_n^+ also enjoy this property.

Let us now look in more detail at the residual properties of the braid groups and their relatives. We start with the classical braid groups. As shown by Krammer [47] and Bigelow [16], the braid groups B_n admit faithful linear representations, and thus, by a theorem of Malcev, they are residually finite. On the other hand, it was shown in [41] by Gorin and Lin that $\gamma_2 B_n = \gamma_3 B_n$ for $n \geqslant 3$ (see [14] for an alternative proof); thus, the braid groups B_n are not residually nilpotent for $n \geqslant 3$. Since both the welded braid group wB_n and the virtual braid group vB_n contain the braid group B_n as a subgroup, we conclude that wB_n and vB_n are not residually nilpotent for $n \geqslant 3$, either (see also [6]).

A different approach to the residual properties of the pure braid groups was taken by Falk and Randell in [37]. Using the semi-direct product decomposition (3) and Lemma 1, these authors gave another proof that the groups P_n are residually nilpotent; in fact, their proof shows that P_n is residually torsion-free nilpotent, see [6]. A similar approach, based on decomposition (5) provides another proof that the upper McCool groups wP_n^+ are residually torsion-free nilpotent.

From the work of Berceanu and Papadima [15] mentioned above, we know that the full McCool groups wP_n are residually torsion-free nilpotent. For $n = 3$, an even stronger result was proved by Metaftsis and Papistas [57], who showed that $\mathrm{gr}_k(wP_3, \mathbb{Z})$ is torsion-free for all k. Whether an analogous statement holds for the McCool groups wP_n with $n \geqslant 4$ is an open problem.

Finally, let us consider the virtual pure braid groups. Clearly, the groups $vP_2^+ = \mathbb{Z}$, $vP_2 = F_2$, and $vP_3^+ = \mathbb{Z} * \mathbb{Z}^2$ are all residually torsion-free nilpotent. Bardakov et al. [10] show that the group vP_3 also enjoys this property. Alternatively, this can be seen from our decomposition $vP_3 \cong \overline{P}_4 * \mathbb{Z}$, the fact that both $\overline{P}_4$ and $\mathbb{Z}$ have this property, and the aforementioned result of Malcev. Whether or not the groups vP_n and vP_n^+ are residually torsion-free nilpotent for $n \geqslant 4$ is an open problem, see [6].

4.4 Malcev Lie Algebras and Formality Properties

For each finitely-generated, torsion-free nilpotent group N, A.I. Malcev [53] constructed a filtered Lie algebra $\mathfrak{m}(N)$ over $\mathbb{Q}$, which is now called the Malcev Lie algebra of N. Given a finitely generated group G, the inverse limit of the Malcev Lie algebras of the torsion free parts of the nilpotent quotients $G/\gamma_i G$ for $i \geqslant 2$ defines the *Malcev Lie algebra* of the group G, which we denote by $\mathfrak{m}(G)$. This Lie algebra coincides with the dual Lie algebra of the 1-minimal model of G defined by D. Sullivan. For relevant background, we refer to [69] and references therein.

We will use another equivalent definition of the Malcev Lie algebra which was given by Quillen in [62]. The group-algebra $\mathbb{C}G$ has a natural Hopf algebra structure with comultiplication $\Delta: \mathbb{C}G \otimes \mathbb{C}G \to \mathbb{C}G$ given by $\Delta(x) = x \otimes x$ for $x \in G$. Let $\widehat{\mathbb{C}G} = \varprojlim_r \mathbb{C}G/I^r$ be the completion of $\mathbb{C}G$ with respect to the I-adic filtration, where I is the augmentation ideal. An element $x \in \widehat{\mathbb{C}G}$ is called *primitive* if $\widehat{\Delta}x = x\hat{\otimes}1 + 1\hat{\otimes}x$. The Malcev Lie algebra $\mathfrak{m}(G)$ is then the set of all primitive elements in $\widehat{\mathbb{C}G}$, with Lie bracket $[x, y] = xy - yx$, and endowed with the induced filtration.

The group G is said to be *filtered-formal* if there exists an isomorphism of filtered Lie algebras, $\mathfrak{m}(G) \cong \widehat{\mathrm{gr}}(G)$. The group G is 1*-formal* if there exists an filtered Lie algebra isomorphism $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}}$, where $\mathfrak{h}$ is a quadratic Lie algebra. As shown in [69], a finitely generated group G is 1-formal if and only if it is both graded-formal and filtered-formal.

Theorem 8 *For each $n \geqslant 1$, the following hold.*

1. ([2, 48]) *The pure braid group P_n is* 1*-formal.*
2. ([15]) *The welded pure braid groups wP_n and wP_n^+ are* 1*-formal.*
3. ([11, 50]) *The virtual pure braid groups vP_n and vP_n^+ are graded-formal.*

Let us now recall the following consequence of the 'Tangent Cone Theorem' from [32].

Theorem 9 ([32]) *Let G be a finitely generated,* 1*-formal group. Then all irreducible components of $\mathcal{R}_1(G)$ are rationally defined linear subspaces of $H^1(G, \mathbb{C})$.*

In view of Proposition 3 and Theorem 9, the group vP_4^+ is not 1-formal. In fact, we have the following theorem.

Theorem 10 ([71]) *For the virtual pure braid group vP_n and its upper-triangular subgroup, vP_n^+, the following hold: both are* 1*-formal for $n \leqslant 3$, and both are non-filtered-formal for $n \geqslant 4$.*

4.5 Chen Ranks

The *Chen Lie algebra* of a finitely generated group G is defined to be the associated graded Lie algebra of its second derived quotient, G/G''. The projection $\pi: G \twoheadrightarrow G/G''$ induces an epimorphism, $\mathrm{gr}(\pi): \mathrm{gr}(G) \twoheadrightarrow \mathrm{gr}(G/G'')$. It is readily verified that $\mathrm{gr}_k(\pi)$ is an isomorphism for $k \leqslant 3$.

In [19], K.-T. Chen gave a method for finding a basis for $\mathrm{gr}(G/G'')$ via a path integral technique for free groups. In the process, he showed that the Chen ranks of the free group of rank n are given by $\theta_1(F_n) = n$ and

$$\theta_k(F_n) = \binom{n+k-2}{k}(k-1), \quad \text{for } k \geqslant 2. \tag{17}$$

As shown by Massey in [54], the Chen ranks of a group G can be computed from the Alexander invariant G'/G''. In [24, 26], Cohen and Suciu developed this method, by introducing the use of Gröbner basis techniques in this context. As an application, they showed in [24] that the Chen ranks of the pure braid groups P_n are given by

$$\theta_k(P_n) = (k-1)\binom{n+1}{4}, \quad \text{for } k \geqslant 3. \tag{18}$$

Based on this and many other similar computations, the first author conjectured in [67] that for $k \gg 0$, the Chen ranks of an arrangement group G are given by

$$\theta_k(G) = \sum_{m \geqslant 2} c_m \cdot \theta_k(F_m), \tag{19}$$

where c_m is the number of m-dimensional components of $\mathcal{R}_1(G)$. Much work has gone into proving this conjecture, with special cases being verified in [58, 63, 65]. A key advance was made in [58], where it was shown that the Chen ranks of a finitely presented, 1-formal group G are determined by the truncated cohomology ring $H^{\leqslant 2}(G, \mathbb{C})$.

Using this fact, Cohen and Schenck show in [23] that, for a finitely presented, commutator-relators 1-formal group G, the Chen ranks formula (19) holds, provided the components of $\mathcal{R}_1(G)$ are isotropic, projectively disjoint, and reduced as a scheme. They also verify that arrangement groups and the welded pure braid groups wP_n satisfy these conditions. From the Chen ranks formula (19) and the first resonance varieties of wP_n in (2), they deduce that for $k \gg 1$, the Chen ranks of wP_n are given by

$$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}. \tag{20}$$

We conjecture that formula (20) holds for all n and all $k \geqslant 4$. We have verified this conjecture for $n \leqslant 8$, based on direct computations of the Chen ranks of the groups wP_n in that range.

However, the resonance varieties of wP_n^+ do not satisfy the isotropicity hypothesis. Nevertheless, we compute in [71] the Chen ranks of these groups, using the Gröbner basis algorithm outlined in [24]. The result reads as follows.

Theorem 11 ([71]) *The Chen ranks of wP_n^+ are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, $\theta_3 = 2\binom{n+1}{4}$, and*

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$$

for $k \geqslant 4$.

We have seen previously that the pure braid group P_n, the upper McCool group wP_n^+, and the group $\Pi_n = \prod_{i=1}^{n-1} F_i$ share the same LCS ranks and the same Betti numbers. Furthermore, the centers of all these groups are infinite cyclic, provided $n \geqslant 2$. However, the Chen ranks can distinguish these groups.

Corollary 1 ([71]) *For $n \geqslant 4$, the pure braid group P_n, the upper McCool group wP_n^+, and the group Π_n are all pairwise non-isomorphic.*

Remark 2 The fact that $P_n \ncong wP_n^+$ for $n \geqslant 4$ answers in the negative Problem 1 from [28, §10]. An alternate solution for $n = 4$ was given by Bardakov and Mikhailov in [9], but that solution relies on the claim that the single-variable Alexander polynomial of a finitely presented group G is an invariant of the group, a claim which is far from being true if $b_1(G) > 1$.

The Chen ranks of the virtual pure braid groups and their upper triangular subgroups are more complicated. We summarize some of our computations of these ranks, as follows.

$$\sum_{k\geqslant 2} \theta_k(vP_3^+)t^{k-2} = (2-t)/(1-t)^3,$$

$$\sum_{k\geqslant 2} \theta_k(vP_4^+)t^{k-2} = (8-3t+t^2)/(1-t)^4,$$

$$\sum_{k\geqslant 2} \theta_k(vP_5^+)t^{k-2} = (20+15t+5t^2)/(1-t)^4,$$

$$\sum_{k\geqslant 2} \theta_k(vP_6^+)t^{k-2} = (40+35t-40t^2-20t^3)/(1-t)^5.$$

$$\sum_{k\geqslant 2} \theta_k(vP_3)t^{k-2} = (9-20t+15t^2-4t^4+t^5)/(1-t)^6.$$

It would be interesting to find closed formulas for the Chen ranks of the groups vP_n^+ and vP_n, but this seems to be a very challenging undertaking.

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Homological Representations of Braid Groups and the Space of Conformal Blocks

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Abstract We compare homological representations of the braid groups and the monodromy representations of the KZ connection by means of hypergeometric integrals. Then we discuss a relationship to the space of conformal blocks.

Keywords Braid groups • Configuration spaces • Conformal field theory • Hypergeometric integrals • Hyperplane arrangements • Quantum groups

1 Introduction

The purpose of this paper is to give a review on recent developments on the homological representations of the braid groups and the monodromy representations of the Knizhnik-Zamolodchikov (KZ) connection. The homological representations of the braid groups are defined as the action of the braid groups on the homology of abelian coverings of certain configuration spaces. They were extensively investigated by Bigelow [2] and Krammer [12]. It was shown independently by Bigelow and Krammer that they provide faithful representations of braid groups.

On the other hand, it was shown by Schechtman-Varchenko [14] and others that the solutions of the KZ equation are expressed by hypergeometric integrals. We consider the KZ equation with values in the space of null vectors in the tensor product of Verma modules of $sl_2(\mathbf{C})$ and show that a specialization of the homological representation is equivalent to the monodromy representation of such KZ equation for generic parameters. This result was obtained in [9, 10] and [11].

Then we describe a relation between homological representations and the space of conformal blocks in the case of complex simple Lie algebras. We study a period map from the homology of local systems over the configuration spaces to the space of conformal blocks. This construction is based on [4, 5, 15] and [11].

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2 Local Systems on the Complement of Hyperplane Arrangement

Let $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ be an arrangement of affine hyperplanes in the complex vector space $\mathbf{C}^n$. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

First, we recall some basic definition for local systems. Let M be a smooth manifold and V a complex vector space. Given a linear representation of the fundamental group

$$r : \pi_1(M, x_0) \longrightarrow GL(V)$$

there is an associated flat vector bundle E over M. The local system $\mathcal{L}$ associated to the representation r is the sheaf of horizontal sections of the flat bundle E. Let $\pi : \widetilde{M} \to M$ be the universal covering. We denote by $\mathbf{Z}\pi_1$ the group ring of the fundamental group $\pi_1(M, x_0)$. We consider the chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbf{Z}\pi_1} V$$

with the boundary map defined by $\partial(c \otimes v) = \partial c \otimes v$. Here $\mathbf{Z}\pi_1$ acts on $C_*(\widetilde{M})$ via the deck transformations and on V via the representation r. The homology of this chain complex is called the homology of M with coefficients in the local system $\mathcal{L}$ and is denoted by $H_*(M, \mathcal{L})$.

Let $\mathcal{L}$ be a complex rank one local system over $M(\mathcal{A})$ associated with a representation of the fundamental group

$$r : \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

For an arrangement $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ we denote by f_j a linear form defining the hyperplane H_j, $1 \leqslant j \leqslant \ell$. We associate a complex number $a_j = a(H_j)$ called an exponent to each hyperplane and consider the multivalued function

$$\Phi = f_1^{a_1} \cdots f_\ell^{a_\ell}.$$

The homology $H_1(M(\mathcal{A}); \mathbf{Z})$ is isomorphic to $\mathbf{Z}^{\oplus \ell}$, where each generator corresponds to a hyperplane. By associating to the generator of $H_1(M(\mathcal{A}); \mathbf{Z})$ corresponding to the hyperplane H_j the complex number $e^{2\pi\sqrt{-1}a_j}$ we obtain a homomorphism $H_1(M(\mathcal{A}); \mathbf{Z}) \to \mathbf{C}^*$. Combining with the abelianization map

$\pi_1(M(\mathcal{A}), x_0) \to H_1(M(\mathcal{A}); \mathbf{Z})$ we obtain a homomorphism

$$\rho_\Phi : \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

The associated local system is denoted by $\mathcal{L}_\Phi$.

We shall investigate $H_*(M(\mathcal{A}), \mathcal{L})$ the homology of $M(\mathcal{A})$ with coefficients in the local system $\mathcal{L}$. For our purpose the homology of locally finite chains $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ also plays an important role.

We briefly summarize basic properties of the above homology groups. Let $\mathcal{A}$ be an essential hyperplane arrangement. Namely, we suppose that maximal codimension of a non-empty intersection of some subfamily of $\mathcal{A}$ is equal to n. We choose a smooth compactification $i : M(\mathcal{A}) \longrightarrow X$. Namely, $M(\mathcal{A})$ is written as $X \setminus D$, where X is a smooth projective variety and D is a divisor with normal crossings.

We shall say that the local system $\mathcal{L}$ is generic if and only if there is an isomorphism

$$i_*\mathcal{L} \cong i_!\mathcal{L}$$

where i_* is the direct image and $i_!$ is the extension by 0. This means that the monodromy of $\mathcal{L}$ along any divisor at infinity is not equal to 1. The following theorem was shown in [7].

Theorem 1 *If the local system $\mathcal{L}$ is generic in the above sense, then there is an isomorphism*

$$H_*(M(\mathcal{A}), \mathcal{L}) \cong H_*^{lf}(M(\mathcal{A}), \mathcal{L}).$$

Moreover, we have $H_k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$.

Let us suppose that each hyperplane in $\mathcal{A}$ is defined over $\mathbf{R}$. We set $M(\mathcal{A})_\mathbf{R} = M(\mathcal{A}) \cap \mathbf{R}^n$ and denote by Δ_ν, $1 \leqslant \nu \leqslant s$, the bounded chambers in $M(\mathcal{A})_\mathbf{R}$. Let

$$j : M(\mathcal{A}) \setminus \cup_\nu \Delta_\nu \longrightarrow X$$

be the inclusion map. We denote by $\mathcal{L}_0$ the restriction of the local system $\mathcal{L}$ on $M(\mathcal{A}) \setminus \cup_\nu \Delta_\nu$. In this situation we have the following theorem.

Theorem 2 ([11]) *In addition to the condition that the local system $\mathcal{L}$ is generic we suppose that there is an isomorphism*

$$j_*\mathcal{L}_0 \cong j_!\mathcal{L}_0.$$

Then the homology with locally finite chains $H_n^{lf}(M(\mathcal{A}), \mathcal{L})$ is spanned by the homology class of bounded chambers Δ_ν, $1 \leqslant \nu \leqslant s$.

3 Homological Representations of Braid Groups

We denote by B_n the braid group with n strands. We fix a positive integer n and a set of distinct n points in $\mathbf{R}^2$ as

$$Q = \{(0,0), \cdots, (n-1,0)\},$$

where we set $p_\ell = (\ell - 1, 0)$, $\ell = 1, \cdots, n$. We take a 2-dimensional disk D in $\mathbf{R}^2$ containing Q in the interior. We fix a positive integer m and consider the configuration space of ordered distinct m points in $\Sigma = D \setminus Q$ defined by

$$\mathcal{F}_m(\Sigma) = \{(t_1, \cdots, t_m) \in \Sigma \ ; \ t_i \neq t_j \text{ if } i \neq j\},$$

which is also denoted by $\mathcal{F}_{n,m}(D)$. The symmetric group $\mathfrak{S}_m$ acts freely on $\mathcal{F}_m(\Sigma)$ by the permutations of distinct m points. The quotient space of $\mathcal{F}_m(\Sigma)$ by this action is by definition the configuration space of unordered distinct m points in Σ and is denoted by $C_m(\Sigma)$. We also denote this configuration space by $C_{n,m}(D)$.

In the original papers by Bigelow [2] and by Krammer [12] the case $m = 2$ was extensively studied, but for our purpose it is convenient to consider the case when m is an arbitrary positive integer such that $m \geqslant 2$.

We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$. The quotient space $\mathbf{C}^m/\mathfrak{S}_m$ defined by the action of $\mathfrak{S}_m$ by the permutations of coordinates is analytically isomorphic to $\mathbf{C}^m$ by means of the elementary symmetric polynomials. Now the image of the hyperplanes defined by $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the diagonal hyperplanes $t_i = t_j$, $1 \leqslant i \leqslant j \leqslant m$, are complex codimension one irreducible subvarieties of the quotient space $D^m/\mathfrak{S}_m$. This allows us to give a description of the first homology group of $C_{n,m}(D)$ as

$$H_1(C_{n,m}(D); \mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z} \tag{1}$$

where the first n components correspond to meridians of the images of hyperplanes $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the last component corresponds to the meridian of the image of the diagonal hyperplanes $t_i = t_j$, $1 \leqslant i \leqslant j \leqslant m$, namely, the discriminant set. We consider the homomorphism

$$\alpha : H_1(C_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \tag{2}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$. Composing with the abelianization map $\pi_1(C_{n,m}(D), x_0) \to H_1(C_{n,m}(D); \mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(C_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}. \tag{3}$$

Let $\pi : \widetilde{C}_{n,m}(D) \to C_{n,m}(D)$ be the covering corresponding to $\operatorname{Ker} \beta$. Now the group $\mathbf{Z} \oplus \mathbf{Z}$ acts as the deck transformations of the covering π. We identify the group ring of $\mathbf{Z} \oplus \mathbf{Z}$ with the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. We consider the homology group

$$H_{n,m} = H_m(\widetilde{C}_{n,m}(D); \mathbf{Z})$$

as an R-module by the action of the deck transformations.

As is explained in the case of $m = 2$ in [2] it can be shown that $H_{n,m}$ is a free R-module of rank

$$d_{n,m} = \binom{m+n-2}{m}. \tag{4}$$

A basis of $H_{n,m}$ as a free R-module is discussed in relation with the homology of local systems in the next sections. Let $\mathcal{M}(D, Q)$ denote the mapping class group of the pair (D, Q), which consists of the isotopy classes of homeomorphisms of D which fix Q setwise and fix the boundary ∂D pointwise. The braid group B_n is naturally isomorphic to the mapping class group $\mathcal{M}(D, Q)$. Now a homeomorphism f representing a class in $\mathcal{M}(D, Q)$ induces a homeomorphism $\tilde{f} : C_{n,m}(D) \to C_{n,m}(D)$, which is uniquely lifted to a homeomorphism of $\widetilde{C}_{n,m}(D)$. This homeomorphism commutes with the deck transformations.

Therefore, for $m \geqslant 2$ we obtain a representation of the braid group

$$\rho_{n,m} : B_n \longrightarrow \mathrm{Aut}_R\, H_{n,m} \tag{5}$$

which is called the homological representation of the braid group or the Lawrence-Krammer-Bigelow (LKB) representation. Let us remark that in the case $m = 1$ the above construction gives the reduced Burau representation over $\mathbf{Z}[q^{\pm 1}]$.

4 Homology of Local Systems on Configuration Spaces

Let us consider the configuration space of ordered distinct n points in the complex plane defined by

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \; ; \; z_i \neq z_j \text{ if } i \neq j\}.$$

The configuration space X_n is also denoted by $\mathcal{F}_n(\mathbf{C})$ as in the previous section. The fundamental group of X_n is the pure braid group with n strands denoted by P_n. For a positive integer m we consider the projection map

$$\pi_{n,m} : X_{n+m} \longrightarrow X_n \tag{6}$$

given by $\pi_{n,m}(z_1, \cdots, z_n, t_1, \cdots, t_m) = (z_1, \cdots, z_n)$, which defines a fiber bundle over X_n. For $p \in X_n$ the fiber $\pi_{n,m}^{-1}(p)$ is denoted by $X_{n,m}$, which is also written as

$\mathcal{F}_{n,m}(\mathbf{C})$. Let $(z_1, \cdots, z_n)$ be the coordinates for p. Then, $X_{n,m}$ is the complement of hyperplanes defined by

$$t_i = z_\ell, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant \ell \leqslant n, \quad t_i = t_j, \quad 1 \leqslant i < j \leqslant m. \tag{7}$$

We call these hyperplanes $H_{i\ell}$, $1 \leqslant i \leqslant m$, $1 \leqslant \ell \leqslant n$, and D_{ij}, $1 \leqslant i < j \leqslant m$. Such arrangement of hyperplanes is called a discriminantal arrangement. The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinates functions $t_1, \cdots, t_m$. We put $Y_{n,m} = X_{n,m}/\mathfrak{S}_m$, which is also denoted by $C_{n,m}(\mathbf{C})$.

Identifying $\mathbf{R}^2$ with the complex plane $\mathbf{C}$, we have the inclusion map

$$\iota : \mathcal{F}_{n,m}(D) \longrightarrow X_{n,m}, \tag{8}$$

which is a homotopy equivalence. By taking the quotient by the action of the symmetric group $\mathfrak{S}_m$, we have the inclusion map

$$\bar{\iota} : C_{n,m}(D) \longrightarrow Y_{n,m}, \tag{9}$$

which is also a homotopy equivalence.

We fix $p = (z_1, z_2 \cdots, z_n)$ as a base point. We consider a rank one local system $\mathcal{L}$ associated with a representation $r : \pi_1(X_{n,m}, x_0) \longrightarrow \mathbf{C}^*$.

Let us consider the compactification

$$i_0 : X_{n,m} \longrightarrow (\mathbf{C}P^1)^m = \underbrace{\mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1}_{m}.$$

Then we take blowing-ups at multiple points $\pi : (\widehat{\mathbf{C}P^1})^m \longrightarrow (\mathbf{C}P^1)^m$ and obtain a smooth compactification $i : X_{n,m} \to (\widehat{\mathbf{C}P^1})^m$ with normal crossing divisors. We are able to write down the condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ explicitly by computing the monodromy of the local system $\mathcal{L}$ along divisors at infinity.

We consider the local system associated with the multivalued function of the form

$$\Phi = \prod_{1 \leqslant i \leqslant m, 1 \leqslant \ell \leqslant n} (t_i - z_\ell)^{\alpha_\ell} \prod_{1 \leqslant i < j \leqslant m} (t_i - t_j)^{2\gamma}. \tag{10}$$

The local system $\mathcal{L}$ on $X_{n,m}$ is invariant under the action of the symmetric group $\mathfrak{S}_m$ and induces the local system $\overline{\mathcal{L}}$ on $Y_{n,m}$.

We have the following proposition.

Proposition 1 *There is an open dense subset V in $\mathbf{C}^{\ell+1}$ such that for $(\alpha_1, \cdots \alpha_\ell, \gamma) \in V$ the associated local system $\overline{\mathcal{L}}$ on $Y_{n,m}$ satisfies*

$$H_*(Y_{n,m}, \overline{\mathcal{L}}) \cong H_*^{lf}(Y_{n,m}, \overline{\mathcal{L}})$$

and $H_k(Y_{n,m}, \overline{\mathcal{L}}) = 0$ *for any* $k \neq m$*. Moreover, we have*

$$\dim H_m(Y_{n,m}, \overline{\mathcal{L}}^*) = d_{n,m}, \tag{11}$$

where we use the same notation as in Eq. (4) for $d_{n,m}$*.*

For the purpose of describing the homology group $H_m^{lf}(X_{n,m}, \mathcal{L})$ and $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$ we introduce the following notation. We take the base point $p = (1, \cdots, n)$. For non-negative integers $m_1, \cdots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m$$

we define a bounded chamber $\Delta_{m_1,\cdots,m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned}
&1 < t_1 < \cdots < t_{m_1} < 2 \\
&2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3 \\
&\cdots \\
&n-1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n.
\end{aligned}$$

We put $M = (m_1, \cdots, m_{n-1})$ and we write Δ_M for $\Delta_{m_1,\cdots,m_{n-1}}$. We denote by $\overline{\Delta}_M$ the image of Δ_M by the projection map $\pi_{n,m}$. The bounded chamber Δ_M defines a homology class $[\Delta_M] \in H_m^{lf}(X_{n,m}, \mathcal{L})$ and its image $\overline{\Delta}_M$ defines a homology class $[\overline{\Delta}_M] \in H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$.

5 KZ Connection

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i : \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leqslant i \leqslant n$, be representations of the Lie algebra $\mathfrak{g}$. We denote by Ω_{ij} the action of Ω on the i-th and j-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$. It is known that the Casimir element $c = \sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra $U\mathfrak{g}$. By means of this fact it can be shown that the infinitesimal pure braid relations

$$[\Omega_{ik}, \Omega_{ij} + \Omega_{jk}] = 0, \quad (i, j, k \text{ distinct}), \tag{12}$$

$$[\Omega_{ij}, \Omega_{k\ell}] = 0, \quad (i, j, k, \ell \text{ distinct}) \tag{13}$$

hold.

We define the Knizhnik-Zamolodchikov (KZ) connection as the 1-form

$$\omega = \frac{1}{\kappa} \sum_{1 \leqslant i < j \leqslant n} \Omega_{ij} d \log(z_i - z_j) \tag{14}$$

with values in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ for a non-zero complex parameter κ. We set $\omega_{ij} = d \log(z_i - z_j)$, $1 \leqslant i, j \leqslant n$. It follows from the above infinitesimal pure braid relations among Ω_{ij} together with Arnold's relation

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{k\ell} + \omega_{k\ell} \wedge \omega_{ij} = 0$$

that $\omega \wedge \omega = 0$ holds. This implies that ω defines a flat connection for a trivial vector bundle over the configuration space $X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}$ with fiber $V_1 \otimes \cdots \otimes V_n$. A horizontal section of the above flat bundle is a solution of the total differential equation $d\varphi = \omega\varphi$ for a function $\varphi(z_1, \cdots, z_n)$ with values in $V_1 \otimes \cdots \otimes V_n$. This total differential equation can be expressed as a system of partial differential equations

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j, j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi, \quad 1 \leqslant i \leqslant n, \tag{15}$$

which is called the KZ equation. The KZ equation was first introduced in [6] as the differential equation satisfied by n-point functions in Wess-Zumino-Witten conformal field theory.

Let $\phi(z_1, \cdots, z_n)$ be the matrix whose columns are linearly independent solutions of the KZ equation. By considering the analytic continuation of the solutions with respect to a loop γ in X_n with base point x_0 we obtain the matrix $\theta(\gamma)$ defined by

$$\phi(z_1, \cdots, z_n) \mapsto \phi(z_1, \cdots, z_n)\theta(\gamma).$$

Since the KZ connection ω is flat the matrix $\theta(\gamma)$ depends only on the homotopy class of γ. The fundamental group $\pi_1(X_n, x_0)$ is the pure braid group P_n. As the above holonomy of the connection ω we have a one-parameter family of linear representations of the pure braid group

$$\theta : P_n \to \mathrm{GL}(V_1 \otimes \cdots \otimes V_n).$$

The symmetric group $\mathfrak{S}_n$ acts on X_n by the permutations of coordinates. We denote the quotient space $X_n/\mathfrak{S}_n$ by Y_n. The fundamental group of Y_n is the braid group B_n. In the case $V_1 = \cdots = V_n = V$, the symmetric group $\mathfrak{S}_n$ acts diagonally on the trivial vector bundle over X_n with fiber $V^{\otimes n}$ and the connection ω is invariant by this action. Thus we have one-parameter family of linear representations of the braid group $\theta : B_n \to \mathrm{GL}(V^{\otimes n})$.

6 Solutions of KZ Equation by Hypergeometric Integrals

In this section we describe solutions of the KZ equation for the case $\mathfrak{g} = sl_2(\mathbf{C})$ by means of hypergeometric integrals following Schechtman and Varchenko [14]. A description of the solutions of the KZ equation was also given by Date, Jimbo, Matsuo and Miwa [3]. We refer the reader to [1] and [13] for general treatments of hypergeometric integrals.

Let us recall basic facts about the Lie algebra $sl_2(\mathbf{C})$ and its Verma modules. As a complex vector space the Lie algebra $sl_2(\mathbf{C})$ has a basis H, E and F satisfying the relations:

$$[H, E] = 2E, \ \ [H, F] = -2F, \ \ [E, F] = H. \tag{16}$$

For a complex number λ we denote by M_λ the Verma module of $sl_2(\mathbf{C})$ with highest weight λ. Namely, there is a non-zero vector $v_\lambda \in M_\lambda$ called the highest weight vector satisfying

$$Hv_\lambda = \lambda v_\lambda, \ Ev_\lambda = 0$$

and M_λ is spanned by $F^j v_\lambda, j \geqslant 0$. It is known that if $\lambda \in \mathbf{C}$ is not a non-negative integer, then the Verma module M_λ is irreducible.

For $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ we put $|\Lambda| = \lambda_1 + \cdots + \lambda_n$ and consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$. For a non-negative integer m we define the space of weight vectors with weight $|\Lambda| - 2m$ by

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (|\Lambda| - 2m)x\}$$

and consider the space of null vectors defined by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m] \ ; \ Ex = 0\}.$$

The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$.

For parameters κ and λ we consider the multi-valued function

$$\Phi_{n,m} = \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{2\kappa}} \prod_{1 \leqslant i \leqslant m, 1 \leqslant \ell \leqslant n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \prod_{1 \leqslant i < j \leqslant m} (t_i - t_j)^{\frac{2}{\kappa}} \tag{17}$$

defined over X_{n+m}. The function $\Phi_{n,m}$ is called the master function. Let $\mathcal{L}$ denote the local system associated to the multi-valued function $\Phi_{n,m}$.

The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinate functions $t_1, \cdots, t_m$. The function $\Phi_{n,m}$ is invariant by the action of $\mathfrak{S}_m$. The local system $\mathcal{L}$ over $X_{n,m}$ defines a local system on $Y_{n,m}$, which we denote by $\overline{\mathcal{L}}$. The local system dual to $\mathcal{L}$ is denoted by $\mathcal{L}^*$.

We put $v = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$ and for $J = (j_1, \cdots, j_n)$ set $F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}$, where $j_1, \cdots, j_n$ are non-negative integers. The weight space $W[|\Lambda| - 2m]$ has a basis $F^J v$ for each J with $|J| = j_1 + \cdots + j_n = m$. For the sequence of integers $(i_1, \cdots, i_m) = (\underbrace{1, \cdots, 1}_{j_1}, \cdots, \underbrace{n, \cdots, n}_{j_n})$ we set

$$S_J(z, t) = \frac{1}{(t_1 - z_{i_1}) \cdots (t_m - z_{i_m})}$$

and define the rational function $R_J(z, t)$ by

$$R_J(z, t) = \frac{1}{j_1! \cdots j_n!} \sum_{\sigma \in \mathfrak{S}_m} S_J(z_1, \cdots, z_n, t_{\sigma(1)}, \cdots, t_{\sigma(m)}).$$

Since $\pi_{n,m} : X_{m+n} \to X_n$ is a fiber bundle with fiber $X_{n,m}$ the fundamental group of the base space X_n acts naturally on the homology group $H_m(X_{n,m}, \mathcal{L}^*)$. Thus we obtain a representation of the pure braid group $r_{n,m} : P_n \longrightarrow \operatorname{Aut} H_m(X_{n,m}, \mathcal{L}^*)$ which defines a local system on X_n denoted by $\mathcal{H}_{n,m}$. In the case $\lambda_1 = \cdots = \lambda_n$ there is a representation of the braid group $r_{n,m} : B_n \longrightarrow \operatorname{Aut} H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ which defines a local system $\overline{\mathcal{H}}_{n,m}$ on $Y_{n,m}$.

The twisted de Rham complex $(\Omega^*(X_{n,m}), \nabla)$ is a complex with differential $\nabla : \Omega^j(X_{n,m}) \to \Omega^{j+1}(X_{n,m})$ defined by

$$\nabla \omega = d\omega + d \log \Phi_{n,m} \wedge \omega.$$

for $\omega \in \Omega^j(X_{n,m})$. We define a map

$$\rho : W[\lambda - 2m] \to \Omega^m(X_{n,m})$$

given by

$$\rho(w) = R_J(t, z) dt_1 \wedge \cdots \wedge dt_m$$

for $w = F^J v$ using the rational function $R_J(t, z)$. It turns out that ρ induces a map to the cohomology of the twisted de Rham complex

$$N[\lambda - 2m] \longrightarrow H^m(\Omega^*(X_{n,m}), \nabla).$$

By this construction we obtain a map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow N[\lambda - 2m]^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \Phi_{n,m} \rho(w).$$

A lot of works have been done on the expression of the solutions of the KZ equation by means of hypergeometric type integrals (see [3] and [14]). For any horizontal section $c(z)$ of the local system $\mathcal{H}_{n,m}$ we have the following.

Theorem 3 (Schechtman and Varchenko [14]) *The integral*

$$\sum_{|J|=m} \left(\int_{c(z)} \Phi_{n,m} R_J(z,t)\, dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

lies in the space of null vectors $N[|\Lambda| - 2m]$ *and is a solution of the KZ equation.*

7 Relation Between Homological Representation and KZ Connection

We fix a complex number λ and consider the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}$$

by putting $\lambda_1 = \cdots = \lambda_n = \lambda$ in the definition of Sect. 6. As the monodromy of the KZ connection

$$\omega = \frac{1}{\kappa} \sum_{1 \leqslant i < j \leqslant n} \Omega_{ij} d \log(z_i - z_j)$$

with values in $N[n\lambda - 2m]$ we obtain the linear representation of the braid group

$$\theta_{\lambda,\kappa} : B_n \longrightarrow \operatorname{Aut} N[n\lambda - 2m].$$

We put

$$F(z_1, \cdots, z_n) = \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^{\frac{\lambda^2}{2\kappa}}.$$

The multivalued function F gives an abelian representation of the braid group

$$a_n : B_n \longrightarrow \mathbf{C}^*$$

and the representation $\theta_{\lambda,\kappa}$ is expressed in the form $a_n \otimes \tilde{\theta}_{\lambda,\kappa}$.

By comparing the action of the braid group on $H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ and the monodromy representations of the KZ connection by means of an explicit description of the horizontal sections by hypergeometric integrals we obtain the following theorem.

Theorem 4 ([9, 10]) *There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the homological representation $\rho_{n,m}$ with the specialization*

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\tilde{\theta}_{\lambda,\kappa}$ with values in the space of null vectors $N[n\lambda - 2m] \subset M_\lambda^{\otimes n}$.

8 Space of Conformal Blocks

First, we recall the definition of the space of conformal blocks. We refer the reader to [8] for an introductory treatment of this subject. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra. We denote by Δ the set of all roots and for a root α set

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g}\ ;\ [h, v] = \alpha(h)v \text{ for any } h \in \mathfrak{h}\}.$$

We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

We denote by $\mathfrak{g}_+$ the direct sum of $\mathfrak{g}_\alpha$ for all positive roots and $\mathfrak{g}_-$ the direct sum of $\mathfrak{g}_\alpha$ for all negative roots. We have a decomposition of Lie algebras

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-.$$

Let θ be the longest root and $\langle\cdot,\cdot\rangle$ be the invariant bilinear form normalized by $\langle\theta, \theta\rangle = 2$. We take $\alpha_1, \cdots, \alpha_r$ a system of simple roots and $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ the half sum of positive roots.

We fix a positive integer K called the level. Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight satisfying $\langle\lambda, \theta\rangle \leqslant K$. We call such weight a dominant integral weight of level K. We denote by V_λ the irreducible representation of $\mathfrak{g}$ with highest weight λ.

Let us recall the notion of affine Lie algebras. We start from the loop algebra $\mathfrak{g} \otimes \mathbf{C}((\xi))$, where $\mathbf{C}((\xi))$ denotes the ring of Laurent series. We consider the central extension $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$ defined by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \langle X, Y\rangle\ \mathrm{Res}_{\xi=0}(df\, g)$$

We call $\widehat{\mathfrak{g}}$ the affine Lie algebra. We denote by A_+ the subalgebra of $\mathbf{C}((\xi))$ consisting of the series with only positive powers. Similarly, A_- denotes the subalgebra consisting of the series with only negative powers. We define Lie subalgebras N_+, N_0, N_- by

$$N_+ = [\mathfrak{g} \otimes A_+] \oplus \mathfrak{g}_+, \quad N_0 = \mathfrak{h} \oplus \mathbf{C}c, \quad N_+ = [\mathfrak{g} \otimes A_-] \oplus \mathfrak{g}_-.$$

We have a direct sum decomposition

$$\widehat{\mathfrak{g}} = N_+ \oplus N_0 \oplus N_-$$

as Lie algebras.

Let λ be a dominant integral weight of level K. We start from the finite dimensional irreducible representation V_λ of $\mathfrak{g}$. We consider the Verma module $\mathcal{M}_\lambda$ defined as $\mathcal{M}_\lambda = U(N_-)V_\lambda$ satisfying $N_+V_\lambda = 0$, where the action of $U(N_-)$ is free and the central elements c acts as the multiplication by K. It turns out that the Verma module $\mathcal{M}_\lambda$ contains a null vector, which means that there exists a non-zero vector $\chi \in \mathcal{M}_\lambda$ such that $N_+\chi = 0$. We consider the quotient module

$$\mathcal{H}_\lambda = \mathcal{M}_\lambda / U(N_-)\chi$$

and it can be shown that $\mathcal{H}_\lambda$ is an irreducible $\widehat{\mathfrak{g}}$-module. We call $\mathcal{H}_\lambda$ the integral highest weight module of $\widehat{\mathfrak{g}}$ with highest weight λ of level K.

We regard the Riemann sphere $\mathbf{C}P^1$ as the one point compactification $\mathbf{C} \cup \{\infty\}$ and fix an affine coordinate function z for $\mathbf{C}$. We take local coordinates around p_j, $1 \leqslant j \leqslant n$ as $\xi_j = z - z(p_j)$ and take $\xi_{n+1} = 1/t$ as a local coordinate around $p_{n+1} = \infty$. We associate to the points $p_1, \cdots, p_{n+1}$ dominant integral weights $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$ of level K.

We denote by $\mathcal{M}_p$ the set of meromorphic functions on $\mathbf{C}P^1$ with poles of any order at most at $p_1, \cdots, p_{n+1}$. Then $\mathfrak{g} \otimes \mathcal{M}_p$ has a structure of a Lie algebra and acts diagonally on the tensor product $\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes \mathcal{H}^*_{\lambda_{n+1}}$ by means of the Laurent expansions of a meromorphic function at the points $p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with respect to the above local coordinates. Here we notice that this action is well-defined since the affine Lie algebra is defined by means of a central extension given by a 2-cocycle expressed by the residue of a 1-form and the sum of the residues is zero on $\mathbf{C}P^1$.

The space of conformal blocks is defined as the space of coinvariants

$$\mathcal{H}(p, \lambda) = (\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes \mathcal{H}^*_{\lambda_{n+1}})/(\mathfrak{g} \otimes \mathcal{M}_p).$$

There is also a dual formulation as follows. We define $\mathcal{H}(p, \lambda)^*$ as the space of invariant multilinear forms by

$$\mathcal{H}(p, \lambda)^* = \mathrm{Hom}_{\mathfrak{g} \otimes \mathcal{M}_p}(\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes \mathcal{H}^*_{\lambda_{n+1}}, \mathbf{C})$$

where the action of $\mathfrak{g} \otimes \mathcal{M}_p$ on $\mathbf{C}$ is supposed to be trivial. This means that the dual space of conformal blocks $\mathcal{H}(p, \lambda)^*$ is defined as the space of invariant multilinear forms by the action of $\mathfrak{g} \otimes \mathcal{M}_p$.

It is a basic result in conformal field theory that the spaces of conformal blocks form a vector bundle over the configuration space X_n equipped with a flat connection. This connection is explicitly give by the KZ connection.

$$\omega = \frac{1}{\kappa} \sum_{i<j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa = K + h$$

where h is the dual Coxeter number. Therefore, the pure braid group P_n acts on the space of conformal blocks $\mathcal{H}(p, \lambda)^*$ by means of the holonomy of this connection.

It turns out that there is a surjective map

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \otimes V^*_{\lambda_{n+1}})/\mathfrak{g} \longrightarrow \mathcal{H}(p, \lambda)$$

and the kernel is described by some algebraic equations coming from the definition of the space of conformal blocks. The reason that the above map is not an isomorphism is as follows. The integrable highest module $\mathcal{H}_\lambda$ is not a Verma module and there exists a null vector in $\mathcal{M}_\lambda$. The existence of such null vectors yields the above algebraic equations.

By generalizing the master function (17) for $sl_2(\mathbf{C})$ we introduce the following function for the Lie algebra $\mathfrak{g}$ and the dominant integral weights $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$. We write

$$\sum_{i=1}^{n} \lambda_i - \lambda_{n+1} = \sum_{j=1}^{r} k_j \alpha_j$$

and put $m = \sum_{j=1}^{r} k_j$. We set

$$\Phi^{\mathfrak{g}}_{n,m} = \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^{\frac{\langle \lambda_i, \lambda_j \rangle}{\kappa}} \prod_{1 \leqslant i \leqslant m, 1 \leqslant \ell \leqslant n} (t_i - z_\ell)^{-\frac{\langle \alpha_i, \lambda_\ell \rangle}{\kappa}} \prod_{1 \leqslant i < j \leqslant m} (t_i - t_j)^{\frac{\langle \alpha_i, \alpha_j \rangle}{\kappa}}$$

and denote by $\mathcal{L}$ the associated local system on the configuration space $X_{n,m}$.

As is shown in [5] one can construct horizontal sections of the KZ connections by means of hypergeometric integrals of the form

$$\int_{\Delta} \Phi^{\mathfrak{g}}_{n,m} R_w(z, t) dt_1 \wedge \cdots \wedge dt_m$$

with some rational function $R_w(z, t)$ for $w \in (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \otimes V^*_{\lambda_{n+1}})/\mathfrak{g}$. Here Δ is a cycle in $H_m(X_{n,m}, \mathcal{L}^*)$. There is an action of the symmetric group $\mathfrak{S}_m$ on

$H_m(X_{n,m}, \mathcal{L}^*)$ and we define the subspace $H_m(X_{n,m}, \mathcal{L}^*)^-$ by

$$H_m(X_{n,m}, \mathcal{L}^*)^- = \{c \in H_m(X_{n,m}, \mathcal{L}^*)\ ;\ \sigma \cdot c = \mathrm{sig}(\sigma)c,\ \sigma \in \mathfrak{S}_m\}.$$

It follows from [5] that we have a well-defined period map

$$\phi : H_m(X_{n,m}, \mathcal{L}^*)^- \longrightarrow \mathcal{H}(p, \lambda)^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \Phi^{\mathfrak{g}}_{n,m} R_w(z, t) dt_1 \wedge \cdots \wedge dt_m.$$

Combining with results by A. Varchenko [16], we can summarize the properties of the period map ϕ as follows.

Theorem 5 *The period map*

$$\phi : H_m(X_{n,m}, \mathcal{L}^*)^- \longrightarrow \mathcal{H}^*(p, \lambda)$$

is equivariant with respect to the action of the pure braid group P_n. If K is sufficiently large relative to $\lambda_1, \cdots, \lambda_n$ the period map ϕ gives an isomorphism.

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From Totally Nonnegative Matrices to Quantum Matrices and Back, via Poisson Geometry

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Abstract In this survey article, we describe recent work that connects three separate objects of interest: totally nonnegative matrices; quantum matrices; and matrix Poisson varieties.

Keywords Cells • Poisson algebras • Quantum matrices • Symplectic leaves • Torus-invariant prime ideals • Totally nonnegative matrices • Totally positive matrices

1 Introduction

In recent publications, the same combinatorial description has arisen for three separate objects of interest: $\mathcal{H}$-prime ideals in quantum matrices, $\mathcal{H}$-orbits of symplectic leaves in matrix Poisson varieties and totally nonnegative cells in the space of totally nonnegative matrices.

Many quantum algebras have a natural action by a torus and a key ingredient in the study of the structure of these algebras is an understanding of the torus-invariant objects. For example, the Stratification Theory of Goodearl and Letzter shows that, in the generic case, a complete understanding of the prime spectrum of quantum matrices would start by classifying the (finitely many) torus-invariant prime ideals. In [8] Cauchon succeeded in counting the number of torus-invariant prime ideals in

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quantum matrices. His method involved a bijection between certain diagrams, now known as Cauchon diagrams, and the torus-invariant primes. Considerable progress in the understanding of quantum matrices has been made since that time by using Cauchon diagrams.

The semiclassical limit of quantum matrices is the classical coordinate ring of the variety of matrices endowed with a Poisson bracket that encodes the nature of the quantum deformation which leads to quantum matrices. As a result, the variety of matrices is endowed with a Poisson structure. A natural torus action leads to a stratification of the variety via torus-orbits of symplectic leaves. In [5], Brown, Goodearl and Yakimov showed that there are finitely many such torus-orbits of symplectic leaves. Each torus orbit is defined by certain rank conditions on submatrices. The classification is given in terms of certain permutations from the relevant symmetric group with restrictions arising from the Bruhat order.

The totally nonnegative part of the space of real matrices consists of those matrices whose minors are all nonnegative. One can specify a cell decomposition of the set of totally nonnegative matrices by specifying exactly which minors are to be zero/non-zero. In [26], Postnikov classified the nonempty cells by means of a bijection with certain diagrams, known as Le-diagrams. The work of Postnikov has recently found applications in Integrable Systems [18] and Theoretical Physics [2].

The interesting observation from the point of view of this work is that in each of the above three sets of results the combinatorial objects that arise turn out to be the same! The definitions of Cauchon diagrams and Le-diagrams are the same, and the restricted permutations arising in the Brown-Goodearl-Yakimov study can be seen to lead to Cauchon/Le diagrams via the notion of pipe dreams.

Once one is aware of these connections, this suggests that there should be a connection between torus-invariant prime ideals, torus-orbits of symplectic leaves and totally nonnegative cells. This connection has been investigated in recent papers by Goodearl and the present authors, [15, 16]. In particular, we have shown that the Restoration Algorithm, developed by the first author for use in quantum matrices, can also be used in the other two settings to answer questions concerning the torus-orbits of symplectic leaves and totally nonnegative cells. The detailed proofs of the results that were obtained in [15, 16] are very technical, and our aim in this survey is to describe the results informally and to compute some examples to illuminate our results. We also present applications of these connections to testing for total nonnegativity through lacunary minors, see Sect. 7.

2 Totally Nonnegative Matrices

A real matrix is *totally positive* (TP for short) if each of its minors is positive and is *totally nonnegative* (TNN for short) if each of its minors is nonnegative.

The minor formed by using rows from a set I and columns from a set J is denoted by $[I \mid J]$, or $[I \mid J](M)$ if we need to specify the matrix.

An excellent survey of totally positive and totally nonnegative matrices can be found in [11]. In this survey, the authors draw attention to appearance of TP and TNN matrices in many areas of mathematics, including: oscillations in mechanical systems, stochastic processes and approximation theory, Pólya frequency sequences, representation theory, planar networks, ... A good source of examples, especially illustrating the important link with planar networks (discussed below) is [27].

2.1 Checking Total Positivity and Total Nonnegativity

In order to specify a $k \times k$ minor of an $n \times n$ matrix, we must choose k rows and k columns. Hence the number of $k \times k$ minors of an $n \times n$ matrix is $\binom{n}{k}^2$; and so the total number of minors is

$$\sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n} - 1 \approx \frac{4^n}{\sqrt{\pi n}}$$

by using Stirling's approximation $n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$. This suggests that we do not want to calculate all of the minors to check for total nonnegativity.

Luckily, for total positivity, we can get away with much less. The simplest example is the 2×2 case.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has *five* minors: $a, b, c, d, \Delta = ad - bc$. Moreover, if $a, b, c, \Delta = ad - bc > 0$ then $d = \frac{\Delta + bc}{a} > 0$, so it is sufficient to check *four* minors.

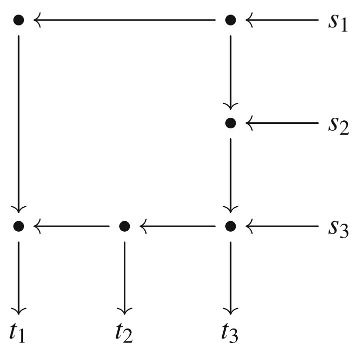
This observation actually extends to the general situation, and the optimal result is due to Gasca and Peña, [13, Theorem 4.1]: for an $n \times n$ matrix, it is only necessary to check n^2 specified minors.

Definition 1 A minor is said to be an *initial minor* if it is formed of consecutive rows and columns, one of which being the first row or the first column.

For example, a 2×2 matrix has 4 initial minors: a, b, c and Δ. More generally, an initial minor is specified by its bottom right entry; so an $n \times n$ matrix has n^2 initial minors.

Theorem 1 (Gasca and Peña) *The $n \times n$ matrix A is totally positive if and only if each of its initial minors is positive.*

In contrast, there is no such family to check whether a matrix is TNN, see, for example, [10, Example 3.3.1]. However Gasca and Peña do give an efficient algorithm to check TNN, see the comment after [13, Theorem 5.4].



$$M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Fig. 1 An example of a planar network and its associated path matrix

2.2 Planar Networks

We refer the reader to [27] for the definition of a planar network. Consider a directed planar graph with no directed cycles, m sources, s_i, and n sinks, t_j. Set $M = (m_{ij})$ where m_{ij} is the number of paths from source s_i to sink t_j. The matrix M is called the *path matrix* of this planar network. See Fig. 1 for an example. Planar networks give an easy way to construct TNN matrices.

Theorem 2 (Lindström's Lemma, [24]) *The path matrix of any planar network is totally nonnegative. In fact, the minor $[I \mid J](M)$ is equal to the number of families of non-intersecting paths from sources indexed by I and sinks indexed by J.*

For example, the matrix M from Fig. 1 is totally nonnegative, by Lindström's Lemma. If we allow weights on paths then even more is true.

Theorem 3 (Brenti, [3]) *Every totally nonnegative matrix is the weighted path matrix of some planar network.*

2.3 Cell Decomposition

Our main concern in this section is to consider the possible patterns of zeros that can occur as the values of the minors of a totally nonnegative matrix. The following example shows that one cannot choose a subset of minors arbitrarily and hope to find a totally nonnegative matrix for which the chosen subset is precisely the subset of minors with value zero.

Example 1 There is no 2×2 totally nonnegative matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $d = 0$, but the other four minors nonzero. For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is TNN and $d = 0$. Then

$a, b, c \geqslant 0$ and also $ad - bc \geqslant 0$. Thus, $-bc \geqslant 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$. □

Remark 1 The argument of Example 1 can be used to prove that if $M = (x_{ij})$ is a totally nonnegative matrix and $x_{st} = 0$ for some s, t, then, either $x_{it} = 0$ for all $i < s$, or $x_{sj} = 0$ for all $j < t$. For, suppose not, and that there are entries $x_{it} \neq 0, x_{st} = 0$ for some $i < s$, and consider x_{sj} for any $j < t$. If $x_{sj} > 0$ then the minor coming from rows i, s and columns j, t is equal to $-x_{it}x_{sj} < 0$, a contradiction. This observation motivates the definition of Le-diagram which is given below.

Let $\mathcal{M}^{\text{tnn}}_{m,p}$ be the set of totally nonnegative $m \times p$ real matrices. Let Z be a subset of minors. The *cell* S^o_Z is the set of matrices in $\mathcal{M}^{\text{tnn}}_{m,p}$ for which the minors in Z are zero (and those not in Z are nonzero). Some cells may be empty. The space $\mathcal{M}^{\text{tnn}}_{m,p}$ is partitioned by the nonempty cells.

Example 2 It can easily be checked that of the 32 zero patterns for minors in $\mathcal{M}^{\text{tnn}}_2$, only 14 produce nonempty cells.

The question is then to describe the patterns of minors that represent nonempty cells in the space of totally nonnegative matrices. In [26], Postnikov defines *Le-diagrams* to solve this problem. An $m \times p$ array with entries either 0 or 1 is said to be a *Le-diagram* if it satisfies the following rule: if there is a 0 in a given square then either each square to the left is also filled with 0 or each square above is also filled with 0. (Compare with Remark 1.)

Figure 2 shows an example and a non-example of a Le-diagram on a 3×3 array.

Theorem 4 (Postnikov) *There is a bijection between Le-diagrams on an $m \times p$ array and nonempty cells S°_Z in $\mathcal{M}^{\text{tnn}}_{m,p}$.*

In fact, Postnikov proves this theorem for the totally nonnegative grassmannian, and we are interpreting the result on the big cell, which is the space of totally nonnegative matrices.

In view of Example 2, there should be 14 2×2 Le-diagrams. At this stage, the interested reader should draw the 16 possible fillings of a 2×2 array with either 0 or 1 and identify the two non-Le-diagrams.

In [26], Postnikov describes an algorithm that starts with a Le-diagram and produces a planar network from which one generates a totally nonnegative matrix which defines a nonempty cell in the space of totally nonnegative matrices. The procedure to produce the planar network is as follows. In each 1 box of the Le-diagram, place a black dot. From each black dot draw a hook which goes to the right end of the diagram and the bottom of the diagram. Label the right ends of the horizontal part of the hooks as the sources of a planar network, numbered from top to bottom, and label the bottom ends of the vertical part of the hooks as the sinks, numbered from left to right. Then consider the resulting graph to be directed

Fig. 2 An example and a non-example of a Le-diagram on a 3×3 array

1	0	1
0	0	1
1	1	1

1	0	1
0	0	0
1	0	0

by allowing movement from right to left along horizontal lines and top to bottom along vertical lines. By Lindström's Lemma (see Theorem 2) the path matrix of this planar network is a totally nonnegative matrix, and so the pattern of its zero minors produces a nonempty cell in the space of totally nonnegative matrices. The above procedure that associates to any Le-diagram a nonempty cell provides a bijection between the set of $m \times p$ Le-diagrams and nonempty cells in the space of totally nonnegative $m \times p$ matrices (see Theorem 4).

Example 3 One can easily check that Postnikov's procedure applied to the 3×3 Le-diagram on the left of Fig. 2 leads to the planar network and the path matrix from Fig. 1.

The minors that vanish on this path matrix are:

$$[1,2|2,3], \quad [1,3|2,3], \quad [2,3|2,3], \quad [2,3|1,3], \quad [2,3|1,2], \quad [1,2,3|1,2,3].$$

The cell associated to this family of minors is nonempty and this is the nonempty cell associated to the Le-diagram above. □

In fact, by allowing suitable weights on the edges of the above planar network, one can obtain all of the matrices in this cell as weighted path matrices of the planar network.

3 Quantum Matrices

We denote by $R := O_q\left(\mathcal{M}_{m,p}(\mathbb{K})\right)$ the standard quantisation of the ring of regular functions on $m \times p$ matrices with entries in $\mathbb{K}$; the algebra R is the $\mathbb{K}$-algebra generated by the $m \times p$ indeterminates $X_{i,\alpha}$, for $1 \leqslant i \leqslant m$ and $1 \leqslant \alpha \leqslant p$, subject to the following relations:

$$\begin{array}{ll} X_{i,\alpha}X_{i,\beta} = qX_{i,\beta}X_{i,\alpha} & (\alpha < \beta) \\ X_{i,\alpha}X_{j,\alpha} = qX_{j,\alpha}X_{i,\alpha} & (i < j) \\ X_{j,\beta}X_{i,\alpha} = X_{i,\alpha}X_{j,\beta} & (i < j,\ \alpha > \beta) \\ X_{i,\alpha}X_{j,\beta} - X_{j,\beta}X_{i,\alpha} = (q - q^{-1})X_{i,\beta}X_{j,\alpha} & (i < j,\ \alpha < \beta). \end{array}$$

It is well known that R can be presented as an iterated Ore extension over $\mathbb{K}$, with the generators $X_{i,\alpha}$ adjoined in lexicographic order. Thus, the ring R is a noetherian domain; its skew-field of fractions is denoted by F.

It is easy to check that the torus $\mathcal{H} := (\mathbb{K}^{\times})^{m+p}$ acts on R by $\mathbb{K}$-algebra automorphisms via:

$$(a_1, \ldots, a_m, b_1, \ldots, b_p).X_{i,\alpha} = a_i b_\alpha X_{i,\alpha} \quad \text{for all} \quad (i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!].$$

We refer to this action as the *standard action* of $(\mathbb{K}^{\times})^{m+p}$ on $O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$.

The algebra R possesses a set of distinguished elements called *quantum minors* that we now define. If $I \subseteq C1, m]\!]$ and $\Lambda \subseteq C1, p]\!]$ with $|I| = |\Lambda| = k \geqslant 1$, then we denote by $[I|\Lambda]_q$ the corresponding *quantum minor* of R. This is the element of R defined by:

$$[I|\Lambda]_q = [i_1, \ldots, i_k|\alpha_1, \ldots, \alpha_k]_q := \sum_{\sigma \in S_k} (-q)^{l(\sigma)} X_{i_1,\alpha_{\sigma(1)}} \cdots X_{i_k,\alpha_{\sigma(k)}},$$

where $I = \{i_1, \ldots, i_k\}$, $\Lambda = \{\alpha_1, \ldots, \alpha_k\}$ and $l(\sigma)$ denotes the length of the k-permutation σ. Note that quantum minors are $\mathcal{H}$-eigenvectors.

In this survey, we will assume that q is not a root of unity. As a consequence, the stratification theory of Goodearl and Letzter applies, so that the prime spectrum of R is controlled by those prime ideals of R that are $\mathcal{H}$-invariant, the so-called *$\mathcal{H}$-primes*. We denote by $\mathcal{H} - \mathrm{Spec}(R)$ the set of $\mathcal{H}$-primes of R. The set $\mathcal{H} - \mathrm{Spec}(R)$ is finite and all $\mathcal{H}$-primes are completely prime, see [4, Theorem II.5.14].

The aim is to parameterise/study the $\mathcal{H}$-prime ideals in quantum matrices.

In [8], Cauchon showed that his theory of deleting derivations can be applied to the iterated Ore extension R. As a consequence, he was able to parametrise the set $\mathcal{H}$-Spec(R) in terms of combinatorial objects called *Cauchon diagrams*.

Definition 2 ([8]) An $m \times p$ *Cauchon diagram* C is simply an $m \times p$ grid consisting of mp squares in which certain squares are coloured black. We require that the collection of black squares have the following property: if a square is black, then either every square strictly to its left is black or every square strictly above it is black. See Fig. 3 for an example and a non-example.

We denote by $C_{m,p}$ the set of $m \times p$ Cauchon diagrams.

Note that we will often identify an $m \times p$ Cauchon diagram with the set of coordinates of its black boxes. Indeed, if $C \in C_{m,p}$ and $(i, \alpha) \in [\![1, m]\!] \times [\![1, p]\!]$, we will say that $(i, \alpha) \in C$ if the box in row i and column α of C is black. Recall [8, Corollary 3.2.1] that Cauchon has constructed (using the deleting derivations algorithm) a bijection between $\mathcal{H}$-Spec$(O_q(\mathcal{M}_{m,p}(\mathbb{C})))$ and the collection $C_{m,p}$. As a consequence, Cauchon [8] was able to give a formula for the size of $\mathcal{H} - \mathrm{Spec}(O_q(\mathcal{M}_{m,p}(\mathbb{C})))$. This formula was later re-written by Goodearl and McCammond (see [21]) in terms of Stirling numbers of second kind and poly-Bernoulli numbers as defined by Kaneko (see [17]).

Notice that the definitions of Le-diagrams and Cauchon diagrams are the same! Thus, the nonempty cells in totally nonnegative matrices and the $\mathcal{H}$-prime ideals in quantum matrices are parameterised by the same combinatorial objects. Much more is true, as we will now see in the 2×2 case.

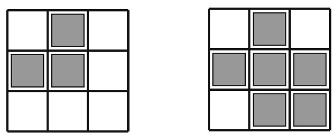


Fig. 3 An example and a non-example of a 3×3 Cauchon diagram

The algebra of 2×2 quantum matrices may be presented as

$$\mathcal{O}_q(\mathcal{M}_2(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with relations

$$ab = qba \qquad ac = qca \qquad bc = cb$$

$$bd = qdb \qquad cd = qdc \qquad ad - da = (q - q^{-1})bc.$$

This algebra has 5 quantum minors: a, b, c, d and the *quantum determinant* $D_q := [12|12]_q = ad - qbc$.

Example 4 Let P be an $\mathcal{H}$-prime ideal that contains d. Then

$$(q - q^{-1})bc = ad - da \in P$$

and, as $0 \neq (q - q^{-1}) \in \mathbb{C}$ and P is completely prime, we deduce that either $b \in P$ or $c \in P$. Thus, there is no $\mathcal{H}$-prime ideal in $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ such that d is the only quantum minor that is in P. □

You should notice the analogy with the corresponding result in the space of 2×2 totally nonnegative matrices: the cell corresponding to d being the only vanishing minor is empty (see Example 1).

The algebra $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ has 14 $\mathcal{H}$-prime ideals, as there are 14 Cauchon/Le-diagrams. It is relatively easy to identify these $\mathcal{H}$-primes: they are $\langle 0 \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle D_q \rangle$, $\langle a, b \rangle$, $\langle a, c \rangle$, $\langle b, c \rangle$, $\langle b, d \rangle$, $\langle c, d \rangle$, $\langle a, b, c \rangle$, $\langle a, b, d \rangle$, $\langle a, c, d \rangle$, $\langle b, c, d \rangle$, $\langle a, b, c, d \rangle$,

It is easy to check that 13 of the ideals are prime. For example, let P be the ideal generated by b and d. Then $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))/P \cong \mathbb{C}[a, c]$ and $\mathbb{C}[a, c]$ is an iterated Ore extension of $\mathbb{C}$ and so a domain. The only problem is to show that the determinant generates a prime ideal. This was originally proved by Jordan, and, independently, by Levasseur and Stafford. A general result that includes this as a special case is [14, Theorem 2.5].

Recently, Casteels, [7], has shown that all $\mathcal{H}$-prime ideals are generated by the quantum minors that they contain, following on from a similar result by the first author with the restriction that the parameter q be transcendental over $\mathbb{Q}$, [20] (see also [28]).

Comparing Example 2 with the above list reveals that the sets of all quantum minors that define $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ are exactly the quantum versions of the sets of vanishing minors for nonempty cells in the space of 2×2 totally nonnegative matrices. This coincidence also occurs in the general case and an explanation of this coincidence is obtained in [15, 16]. However, in order to explain the coincidence, we need to introduce a third setting, that of Poisson matrices, and this is done in the next section.

4 Poisson Matrix Varieties

In this section, we study the standard Poisson structure of the coordinate ring $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ coming from the commutators of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Recall that a *Poisson algebra* (over $\mathbb{C}$) is a commutative $\mathbb{C}$-algebra A equipped with a Lie bracket $\{-,-\}$ which is a derivation (for the associative multiplication) in each variable. The derivations $\{a,-\}$ on A are called *Hamiltonian derivations*. When A is the algebra of complex-valued C^∞ functions on a smooth affine variety V, one can use Hamiltonian derivations in order to define Hamiltonian paths in V. A smooth path $\gamma : [0,1] \to V$ is a *Hamiltonian path in V* if there exists $H \in C^\infty(V)$ such that for all $f \in C^\infty(V)$:

$$\frac{d}{dt}(f \circ \gamma)(t) = \{H, f\} \circ \gamma(t), \tag{1}$$

for all $0 < t < 1$. In other words, Hamiltonian paths are the integral curves (or flows) of the Hamiltonian vector fields induced by the Poisson bracket. It is easy to check that the relation "connected by a piecewise Hamiltonian path" is an equivalence relation. The equivalence classes of this relation are called the *symplectic leaves* of V; they form a partition of V.

4.1 The Poisson Algebra $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$

Denote by $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ the coordinate ring of the variety $\mathcal{M}_{m,p}(\mathbb{C})$; note that $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ is a (commutative) polynomial algebra in mp indeterminates $Y_{i,\alpha}$ with $1 \leqslant i \leqslant m$ and $1 \leqslant \alpha \leqslant p$.

The variety $\mathcal{M}_{m,p}(\mathbb{C})$ is a Poisson variety: there is a unique Poisson bracket on the coordinate ring $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ determined by the following data. For all $(i,\alpha) < (k,\gamma)$, we set:

$$\{Y_{i,\alpha}, Y_{k,\gamma}\} = \begin{cases} Y_{i,\alpha}Y_{k,\gamma} & \text{if } i = k \text{ and } \alpha < \gamma \\ Y_{i,\alpha}Y_{k,\gamma} & \text{if } i < k \text{ and } \alpha = \gamma \\ 0 & \text{if } i < k \text{ and } \alpha > \gamma \\ 2Y_{i,\gamma}Y_{k,\alpha} & \text{if } i < k \text{ and } \alpha < \gamma. \end{cases}$$

This is the standard Poisson structure on the affine variety $\mathcal{M}_{m,p}(\mathbb{C})$ (cf. [5, §1.5]); the Poisson algebra structure on $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ is the semiclassical limit of the noncommutative algebras $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Indeed one can easily check that

$$\{Y_{i,\alpha}, Y_{k,\gamma}\} = \frac{[X_{i,\alpha}, X_{k,\gamma}]}{q-1} \Big|_{q=1} .$$

In particular, the Poisson bracket on $\mathcal{O}(\mathcal{M}_2(\mathbb{C})) = \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined by:

$$\{a,b\} = ab \qquad \{a,c\} = ac \qquad \{b,c\} = 0$$
$$\{b,d\} = bd \qquad \{c,d\} = cd \qquad \{a,d\} = 2bc.$$

Note that the Poisson bracket on $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ extends uniquely to a Poisson bracket on $\mathcal{C}^\infty(\mathcal{M}_{m,p}(\mathbb{C}))$, so that $\mathcal{M}_{m,p}(\mathbb{C})$ can be viewed as a Poisson manifold. Hence, $\mathcal{M}_{m,p}(\mathbb{C})$ can be decomposed as the disjoint union of its symplectic leaves.

We finish this section by stating an analogue of Examples 1 and 4 in the Poisson setting.

Proposition 1 *Let $\mathcal{L}$ be a symplectic leaf such that $d(M) = 0$ for all $M \in \mathcal{L}$. Then, either $b(M) = 0$ for all $M \in \mathcal{L}$ or $c(M) = 0$ for all $M \in \mathcal{L}$.*

4.2 $\mathcal{H}$-Orbits of Symplectic Leaves in $\mathcal{M}_{m,p}(\mathbb{C})$

Notice that the torus $\mathcal{H} := (\mathbb{C}^\times)^{m+p}$ acts rationally on $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ by Poisson automorphisms via:

$$(a_1,\dots,a_m,b_1,\dots,b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \text{for all} \quad (i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!].$$

At the geometric level, this action of the algebraic torus $\mathcal{H}$ comes from the left action of $\mathcal{H}$ on $\mathcal{M}_{m,p}(\mathbb{C})$ by Poisson isomorphisms via:

$$(a_1,\dots,a_m,b_1,\dots,b_p).M := \operatorname{diag}(a_1,\dots,a_m)^{-1}\cdot M \cdot \operatorname{diag}(b_1,\dots,b_p)^{-1}.$$

This action of $\mathcal{H}$ on $\mathcal{M}_{m,p}(\mathbb{C})$ induces an action of $\mathcal{H}$ on the set $\operatorname{Sympl}(\mathcal{M}_{m,p}(\mathbb{C}))$ of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ (cf. [5, §0.1]). As in [5], we view the $\mathcal{H}$-orbit of a symplectic leaf $\mathcal{L}$ as the set-theoretic union $\bigcup_{h\in\mathcal{H}} h.\mathcal{L} \subseteq \mathcal{M}_{m,p}(\mathbb{C})$, rather than as the family $\{h.\mathcal{L} \mid h \in \mathcal{H}\}$. We denote the set of such orbits by $\mathcal{H}$-$\operatorname{Sympl}(\mathcal{M}_{m,p}(\mathbb{C}))$.

As the symplectic leaves of $\mathcal{M}_{m,p}(\mathbb{C})$ form a partition of $\mathcal{M}_{m,p}(\mathbb{C})$, so too do the $\mathcal{H}$-orbits of symplectic leaves.

Example 5 The symplectic leaf $\mathcal{L}$ containing $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the set $\mathcal{E}$ of those 2×2 complex matrices $M = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ with $y - z = 0$, $xt - yz = 0$ and $y \neq 0$. In other words,

$$\mathcal{E} := \{M \in \mathcal{M}_2(\mathbb{C}) \mid \Delta(M) = 0,\ (b-c)(M) = 0 \text{ and } b(M) \neq 0\},$$

where a, b, c, d denote the canonical generators of the coordinate ring of $\mathcal{M}_2(\mathbb{C})$ and $\Delta := ad - bc$ is the determinant function. It easily follows from this that the $\mathcal{H}$-orbit of symplectic leaves in $\mathcal{M}_2(\mathbb{C})$ that contains the point $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the set of those 2×2 matrices M with $\Delta(M) = 0$ and $b(M)c(M) \neq 0$. Moreover the closure of this $\mathcal{H}$-orbit coincides with the set of those 2×2 matrices M with $\Delta(M) = 0$. □

The $\mathcal{H}$-orbits of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ have been explicitly described by Brown, Goodearl and Yakimov, [5, Theorems 3.9, 3.13, 4.2].

Theorem 5 *Set* $\mathcal{S} := \{w \in S_{m+p} \mid -p \leqslant w(i) - i \leqslant m$ *for all* $i = 1, 2, \ldots, m+p\}$.

1. *The* $\mathcal{H}$*-orbits of symplectic leaves in* $\mathcal{M}_{m,p}(\mathbb{C})$ *are smooth irreducible locally closed subvarieties.*
2. *There is an explicit* $1:1$ *correspondence between* $\mathcal{S}$ *and* $\mathcal{H}$-Sympl$(\mathcal{M}_{m,p}(\mathbb{C}))$.
3. *Each* $\mathcal{H}$*-orbit is defined by some rank conditions.*

Before going any further let us look at the 2×2 case: by the above theorem, there is a $1:1$ correspondence between $\mathcal{H}$-Sympl$(\mathcal{M}_2(\mathbb{C}))$ and

$$\mathcal{S} = \{w \in S_4 \mid -2 \leqslant w(i) - i \leqslant 2 \text{ for all } i = 1, 2, 3, 4\},$$

in other words, with the set of those permutations w in S_4 such that $w(1) \neq 4$ and $w(4) \neq 1$. One may be disappointed not to retrieve 2×2 Cauchon diagrams, but a direct inspection shows that there are exactly 14 such restricted permutations in the 2×2 case! This is not at all a coincidence as we will see in the following section.

The rank conditions that define the $\mathcal{H}$-orbits of symplectic leaves and their closures are explicit in [5]. The reader is referred to [5] for more details.

For $w \in \mathcal{S}$, we denote by $\mathcal{P}_w$ the $\mathcal{H}$-orbit of symplectic leaves associated to the restricted permutation w. To finish, let us mention that the set $\mathcal{M}(w)$ of all minors that vanish on the closure of $\mathcal{P}_w$ has been described in [15, Definition 2.6].

Example 6 When $m = p = 3$ and $w = (2\ 3\ 5\ 4)$, then we obtain

$$\mathcal{M}(w) = \{[1,2|2,3], [1,3|2,3], [2,3|2,3], [2,3|1,3], [2,3|1,2], [1,2,3|1,2,3]\}.$$

Observe that this family of minors defines a nonempty cell in $\mathcal{M}_3^{\text{tnn}}(\mathbb{R})$ by Example 3. □

In [15], the following result was obtained thanks to previous results of [5] and [12].

Theorem 6 *Let* $w \in \mathcal{S}$*. The closure of the* $\mathcal{H}$*-orbit* $\mathcal{P}_w$ *is given by:*

$$\overline{\mathcal{P}_w} = \{x \in \mathcal{M}_{m,p}(\mathbb{C}) \mid [I|J](x) = 0 \text{ for all } [I|J] \in \mathcal{M}(w)\}.$$

Moreover, the minor $[I|J]$ *vanishes on* $\overline{\mathcal{P}_w}$ *if and only if* $[I|J] \in \mathcal{M}(w)$.

5 From Cauchon Diagrams to Restricted Permutations, via Pipe Dreams

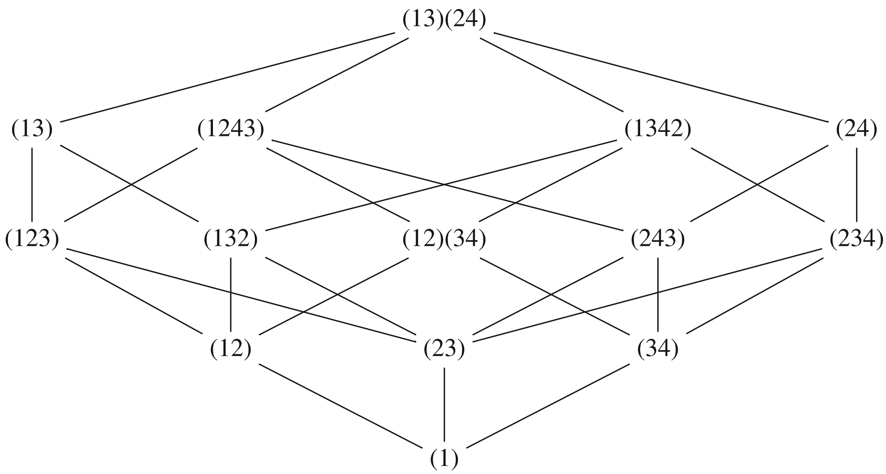
In the previous section, we have seen that the $\mathcal{H}$-orbits of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ are parameterised by the restricted permutations in S_{m+p} given by

$$\mathcal{S} = \{w \in S_{m+p} \mid -p \leqslant w(i) - i \leqslant m \text{ for all } i = 1, 2, \ldots, m+p\}.$$

In the 2×2 case, this subposet of the Bruhat poset of S_4 is

$$\mathcal{S} = \{w \in S_4 \mid -2 \leqslant w(i) - i \leqslant 2 \text{ for all } i = 1, 2, 3, 4\}.$$

and is shown below.

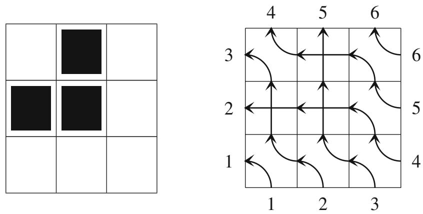


Inspection of this poset reveals that it is isomorphic to the poset of the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ in Sect. 3, partially ordered by inclusion; and so to a similar poset of the Cauchon diagrams corresponding to the $\mathcal{H}$-prime ideals.

More generally, it is known that the numbers of $\mathcal{H}$-primes in $\mathcal{O}_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ (and so the number of $m \times p$ Cauchon diagrams) is equal to $|\mathcal{S}|$ (see [22]). This is no coincidence, and the connection between the two posets can be illuminated by using *Pipe Dreams*.

The procedure to produce a restricted permutation from a Cauchon diagram goes as follows. Given a Cauchon diagram, replace each black box by a cross, and each white box by an elbow joint, that is:

For example, the Cauchon diagram on the left below produces the pipe dream on the right



We obtain a permutation σ from the pipe dream in the following way. To calculate $\sigma(i)$, locate the i either on the right hand side or the bottom of the pipe dream and trace through the pipe dream to find the number $\sigma(i)$ that is at the end of the pipe starting at i. For example, in the pipe dream displayed above, we locate 2 on the bottom line and follow the pipe from 2 to 3, similarly, we locate 5 at the right hand side and follow the pipe from 5 to 4. Continuing in this way, we find that $\sigma = 135246$ (in one line notation).

It is easy to check that this produces a restricted permutation of the required type by using the observation that as you move along a pipe from source to image, you can only move upwards and leftwards; so, for example, in any 3×3 example $\sigma(2)$ is at most 5 (the number directly above 2).

This procedure provides an explicit bijection between the set of $m \times p$ Cauchon diagrams and the poset $\mathcal{S}$ (see [9, 26]).

6 The Unifying Theory

In the previous sections we have seen that the nonempty cells in $\mathcal{M}_{m,p}^{\text{tnn}}$, the torus-invariant prime ideals in $O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ and the closure of the $\mathcal{H}$-orbits of symplectic leaves are all parametrised by $m \times p$ Cauchon diagrams. This suggests that there might be a connection between these objects. Going a step further, all these objects are characterised by certain families of (quantum) minors.

Totally nonnegative cells are defined by the vanishing of families of minors. Some of the TNN cells are empty. So it is natural to introduce the following definition. A family of minors is *admissible* if the corresponding TNN cell is nonempty. Three obvious questions, which we discuss in the next section, are:

Question 1 **what are the admissible families of minors?**

Question 2 **which families of quantum minors generate $\mathcal{H}$-prime ideals?**

Question 3 **which families of minors define closures of $\mathcal{H}$-orbits of symplectic leaves?**

6.1 An Algorithm to Rule Them All

In [8], Cauchon developed and used an algorithm, called the *deleting derivations algorithm* in order to study the $\mathcal{H}$-invariant prime ideals in $O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$. Roughly speaking, in the 2×2 case, this algorithm consists in the following change of variable:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a - bd^{-1}c & b \\ c & d \end{pmatrix}.$$

Let us now give a precise definition of the deleting derivations algorithm.

If $M = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$ with K a skew-field, then we set $g_{j,\beta}(M) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} - x_{i,\beta}x_{j,\beta}^{-1}x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

We set $M^{(j,\beta)} := g_{j,\beta} \circ \cdots \circ g_{m,p-1} \circ g_{m,p}(M)$ where the indices are taken in lexicographic order.

The matrix $M^{(1,1)}$ is called the matrix obtained from M at the end of the deleting derivations algorithm.

The deleting derivations algorithm has an inverse that is called the *restoration algorithm*. It was originally developed in [19] to study $\mathcal{H}$-primes in quantum matrices. Roughly speaking, in the 2×2 case, the restoration algorithm consists of making the following change of variable:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a + bd^{-1}c & b \\ c & d \end{pmatrix}.$$

Let us now give a precise definition of the restoration algorithm.

If $M = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, then we set $f_{j,\beta}(M) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} + x_{i,\beta}x_{j,\beta}^{-1}x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

We set $M^{(j,\beta)^+} := f_{j,\beta} \circ \cdots \circ f_{1,2} \circ f_{1,1}(M)$ where the indices are taken in the reverse of the lexicographic order and where $(j,\beta)^+ \in \{1,\ldots,m\} \times \{1,\ldots,p\} \cup \{(m+1,p)\}$ is the successor of (j,β) in the lexicographic order.

The matrix $M^{(m,p)^+} = M^{(m+1,p)}$ is called the matrix obtained from M at the end of the restoration algorithm.

Example 7 Set $M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then, applying the restoration algorithm to M, we get $M^{(3,3)^+} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which is the matrix obtained from M at the end of the restoration algorithm. □

6.2 The Restoration Algorithm and TNN Matrices

The matrix $M^{(3,3)^+}$ obtained from M by the restoration algorithm in Example 7 is not TNN, as the minor $[1,2|2,3]$ is negative. The reason for this failure to be TNN is that the starting matrix M has a negative entry. In general, one can express the (quantum) minors of $M^{(j,\beta)^+}$ in terms of the (quantum) minors of $M^{(j,\beta)}$ (see [15, 16]). As a consequence, one is able to prove the following result that gives a necessary and sufficient condition for a real matrix to be TNN.

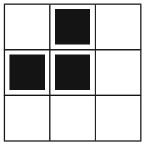
Theorem 7 ([15]) *Let M be a matrix with real entries.*

1. *If the entries of M are nonnegative and its zeros form a Cauchon diagram, then the matrix $M^{(m,p)^+}$ obtained from M at the end of the restoration algorithm is TNN.*
2. *Let N be the matrix obtained at the end of the deleting derivations algorithm applied to M. Then M is TNN if and only if the matrix N is nonnegative (i.e. the entries of N are nonnegative) and its zeros form a Cauchon diagram. (That is, the zeros of N correspond to the black boxes of a Cauchon diagram.)*

Example 8 Use the deleting derivations algorithm to test whether the following matrices are TNN: $M_1 = \begin{pmatrix} 11 & 7 & 4 & 1 \\ 7 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 7 & 5 & 4 & 1 \\ 6 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

6.3 Main Result

Let C be an $m \times p$ Cauchon diagram and $T = (t_{i,\alpha})$ be a matrix with entries in a skew-field K. Assume that $t_{i,\alpha} = 0$ if and only if (i,α) is a black box of C. Set $T_C := f_{m,p} \circ \cdots \circ f_{1,2} \circ f_{1,1}(T)$, so that T_C is the matrix obtained from T by the restoration algorithm.

Example 9 Let $m = p = 3$ and consider the Cauchon diagram  .

Then $T = \begin{pmatrix} t_{1,1} & 0 & t_{1,3} \\ 0 & 0 & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}$ and $T^{(j,\beta)^+} := f_{j,\beta} \circ \cdots \circ f_{1,1}(T)$, so that

$$T_C = T^{(3,3)^+} = \begin{pmatrix} t_{1,1} + t_{1,3}t_{3,3}^{-1}t_{3,1} & t_{1,3}t_{3,3}^{-1}t_{3,2} & t_{1,3} \\ t_{2,3}t_{3,3}^{-1}t_{3,1} & t_{2,3}t_{3,3}^{-1}t_{3,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}.$$

The above construction can be applied in a variety of situations. In particular, we have the following.

- If $K = \mathbb{R}$ and T is nonnegative, then T_C is TNN.
- If the nonzero entries of T commute and are algebraically independent, and if $K = \mathbb{C}(t_{ij})$, then the minors of T_C that are equal to zero are exactly those that vanish on the closure of a given $\mathcal{H}$-orbit of symplectic leaves. (See [15].)
- If the nonzero entries of T are the generators of a certain quantum affine space over $\mathbb{C}$ and K is the skew-field of fractions of this quantum affine space, then the quantum minors of T_C that are equal to zero are exactly those belonging to the unique $\mathcal{H}$-prime in $O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ associated to the Cauchon diagram C. (See [19] for more details.)
- The families of (quantum) minors we get depend only on C in these three cases, and if we start from the same Cauchon diagram in these three cases, then we get exactly the same families.

As a consequence, we get the following comparison result (see [15, 16]).

Theorem 8 *Let $\mathcal{F}$ be a family of minors in the coordinate ring of $\mathcal{M}_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Then the following are equivalent:*

1. *The totally nonnegative cell associated to $\mathcal{F}$ is nonempty.*
2. *$\mathcal{F}$ is the set of minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$.*
3. *$\mathcal{F}_q$ is the set of quantum minors that belong to an $\mathcal{H}$-prime in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$.*

This result has several interesting consequences.

First, it easily follows from Theorem 8 that the TNN cells in $\mathcal{M}^{\text{tnn}}_{m,p}$ are the traces of the closure of $\mathcal{H}$-orbits of symplectic leaves on $\mathcal{M}^{\text{tnn}}_{m,p}$.

Next, the sets of all minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ have been explicitly described in [15] (see also Theorem 6). So, as a consequence of the previous theorem, *the sets of minors* *that define nonempty totally nonnegative cells are explicitly described:* these are the families $\mathcal{M}(w)$ of [15, Definition 2.6] for $w \in \mathcal{S}$.

On the other hand, the torus-invariant primes in $O(\mathcal{M}_{m,p}(\mathbb{C}))$ are generated by the quantum minors that they contain, and so we deduce from the above theorem *explicit generating sets of quantum minors for the torus-invariant prime ideals of* $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Recently and independently, Yakimov [28] also described explicit families of quantum minors that generate $\mathcal{H}$-primes. However his families are smaller than ours and so are not adapted to the TNN world. The problem of deciding whether a given quantum minor belongs to the $\mathcal{H}$-prime associated to a Cauchon diagram C was studied by Casteels [6] who gave a combinatorial criterion inspired by Lindström's Lemma.

7 Lacunary Sequences

It follows from Theorem 7 that each totally nonnegative matrix is associated to a Cauchon diagram via the deleting derivation algorithm. In other words, we have a mapping $\pi : M \mapsto C$ from $\mathcal{M}^{\text{tnn}}_{m,p}$ to the set $C_{m,p}$ of $m \times p$ Cauchon diagrams, where C is the Cauchon diagram associated to the matrix N deduced from M by the deleting derivations algorithm. In [23], we have proved that the nonempty totally nonnegative cells are precisely the fibres of π, so that the nonempty totally nonnegative cells in $\mathcal{M}^{\text{tnn}}_{m,p}$ are precisely the sets

$$S^0_C := \{M \in \mathcal{M}^{\text{tnn}}_{m,p} \mid \pi(M) = C\},$$

where C runs through the set of $m \times p$ Cauchon diagrams.

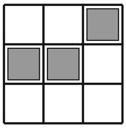
When C is the all white diagram, S^0_C is the cell of all totally positive matrix. Gasca and Peña's result, Theorem 1, specifies a set of n^2 minors that need to be checked to guarantee that an $n \times n$ matrix is totally positive; that is, in S^0_C.

Here we outline a generalisation of this result to arbitrary nonnegative cells: full details are in [23]. The inspiration for lacunary minors comes from Cauchon's work on $\mathcal{H}$-primes in quantum matrices and Theorem 8.

Given an $m \times p$ Cauchon diagram C, we specify for each box (i, j) a *lacunary minor*, $\Delta_{i,j}$, given by Launois and Lenagan [23, Algorithm 1]. Then a matrix A is totally nonnegative and in the cell corresponding to the Cauchon diagram C if and only if each of the lacunary minors of A corresponding to a black box is zero, while each of the lacunary minors of A corresponding to a white box is greater than zero.

Note that this test only involves mp minors. In the case that C is the Cauchon diagram with all boxes coloured white, the test states that a real matrix M is totally positive if and only if each final minor of M is strictly positive. (A minor $[I|J]$ is a final minor if I and J consist of consecutive entries and either $m \in I$ or $p \in J$.) This is the well-known Gasca and Peña test, but applied to final minors rather than initial minors.

Example 10 A real matrix M is TNN and belongs to the cell associated to the Cauchon diagram C on the left below if and only if the nine lacunary conditions on the right below are satisfied. (The lacunary minors have all been obtained by using [23, Algorithm 1].)

$C =$ 

$$\Delta_{1,1} = [13|12] > 0, \Delta_{1,2} = [12|23] > 0, \Delta_{1,3} = [1|3] = 0$$

$$\Delta_{2,1} = [23|12] = 0, \Delta_{2,2} = [23|23] = 0, \Delta_{2,3} = [2|3] > 0$$

$$\Delta_{3,1} = [3|1] > 0, \quad \Delta_{3,2} = [3|2] > 0, \quad \Delta_{3,3} = [3|3] > 0.$$

It is easy to check that the matrix $M = \begin{pmatrix} 16 & 5 & 0 \\ 12 & 6 & 3 \\ 4 & 2 & 1 \end{pmatrix}$ satisfies the above nine conditions. Hence, we deduce from the comments above that M is TNN and belongs to the TNN cell S_C^0 associated to C. □

Lacunary minors have been used in recent work by Adm and Garloff on intervals of totally nonnegative matrices, [1].

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