Spherical Schrödinger Hamiltonians: Spectral Analysis and Time Decay

Luca Fanelli

Abstract In this survey, we review recent results concerning the canonical dispersive flow e^{itH} led by a Schrödinger Hamiltonian H. We study, in particular, how the time decay of space L^p-norms depends on the frequency localization of the initial datum with respect to the some suitable spherical expansion. A quite complete description of the phenomenon is given in terms of the eigenvalues and eigenfunctions of the restriction of H to the unit sphere, and a comparison with some uncertainty inequality is presented.

Keywords Dispersive estimates • Electromagnetic potentials • Schrödinger equation

1 Introduction

For $\psi = \psi(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, let us consider the free Schrödinger equation

$$\partial_t \psi = i\Delta\psi, \qquad \psi(0,x) = \psi_0(x). \tag{1}$$

Solving (1) with initial datum $\psi_0(x) \in L^2(\mathbb{R}^d)$ is to find a wavefunction $\psi \in \mathscr{C}^1(\mathbb{R}; L^2(\mathbb{R}^d))$ such that $\widehat{\psi}(t,\xi) = e^{-it|\xi|^2}\widehat{\psi}_0(\xi)$, the hat denoting the Fourier transform in the x-variable

$$\widehat{\psi}(t,\xi) := \int_{\mathbb{R}^d} e^{-itx\cdot\xi}\psi(t,x)\,dx.$$

Computing the distributional Fourier transform of $e^{-it|\xi|^2}$, one finds that the unique solution to (1), in the above sense, is given by

$$\psi(t,x) = (4\pi it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} * \psi_0(x) = (4\pi it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^d} e^{i\frac{x\cdot y}{2t}} e^{i\frac{|y|^2}{4t}} \psi_0(y)\,dy. \tag{2}$$

L. Fanelli (✉)
Dipartimento di Matematica, SAPIENZA Università di Roma, P. le Aldo Moro 5, 00185 Roma, Italy
e-mail: fanelli@mat.uniroma1.it

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From now on, we will denote by $e^{it\Delta}$ the one-parameter flow on $L^2(\mathbb{R}^d)$ defined by formula (2), namely $e^{it\Delta}\psi_0(\cdot) = \psi(t,\cdot)$, being ψ as in (2). By Plancherel Theorem it follows that $e^{it\Delta}$ is unitary on $L^2(\mathbb{R}^d)$, namely

$$\left\| e^{it\Delta}\psi_0(\cdot)\right\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}, \qquad \forall t \in \mathbb{R}. \tag{3}$$

By (2), it also immediately follows that

$$\left\| e^{it\Delta}\psi_0(\cdot)\right\|_{L^\infty(\mathbb{R}^d)} \leqslant C|t|^{-\frac{d}{2}}\|\psi_0\|_{L^1(\mathbb{R}^d)}, \qquad \forall t \in \mathbb{R}, \tag{4}$$

with a constant $C > 0$ independent on t and ψ_0. The last inequality, together with (3), gives by Riesz-Thorin the full list of *time decay estimates* for the free Schrödinger equation

$$\left\| e^{it\Delta}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^d)} \leqslant C|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^d)}, \qquad \forall t \in \mathbb{R}, \qquad \forall p \geqslant 2 \tag{5}$$

where the constant C only depends on p and d. Inequalities (5) turn out to be a crucial tool in Scattering Theory and Nonlinear Analysis; in particular, a suitable time average of the same leads to the so called *Strichartz estimates* (see the standard reference [23]), which play a fundamental role both for fixed point results and as Restriction Theorems for the Fourier transform:

$$\left\| e^{it\Delta}\psi_0\right\|_{L^q_t L^r_x} \leqslant C\|\psi_0\|_{L^2(\mathbb{R}^d)}, \tag{6}$$

with $2/q = d/2 - d/r$, $q \geqslant 2$ and $(q,r,d) \neq (2,\infty,2)$, and

$$\left\| e^{it\Delta}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^d)} := \left\| \left\| e^{it\Delta}\psi_0(\cdot)\right\|_{L^r(\mathbb{R}^d)}\right\|_{L^q(\mathbb{R})}.$$

From now on, we point our attention on estimate (4) and try to give it a deeper insight. First of all, it is clear by (2) that a crucial role is played by the plane wave $K(x,y) := e^{i\frac{x\cdot y}{2t}}$ which is uniformly bounded with respect to the x, y variables, for any fixed time $t \neq 0$, i.e.

$$\sup_{x,y\in\mathbb{R}^d}\left| e^{i\frac{x\cdot y}{2t}}\right| = 1 < \infty, \qquad \forall t \neq 0. \tag{7}$$

We stress that a completely analogous behavior occurs when one solves, for positive times, the Heat Equation

$$\partial_t u = \Delta u, \qquad u(0,x) = u_0(x) \in L^p(\mathbb{R}^d), \tag{8}$$

since the solution is given by the convolution

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} e^{\frac{-|x|^2}{4t}} * u_0(x), \qquad (t > 0) \tag{9}$$

for all $p \in [1, +\infty]$. This shows that (8) satisfies the same a priori estimates (5) as equation (1). Notice that (1) and (8) enjoy the same scaling invariance: namely, if ψ and u solve (1) and (8), respectively, then the rescaled function ψ_λ, u_λ, where

$$f_\lambda(t,x) := f\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \qquad \lambda > 0.$$

solve the same equations as ψ and u, respectively, for any $\lambda > 0$. In addition, the Gaussian decay in (9) is much smoother than the oscillating character of the fundamental solution in (2), and leads to much stronger phenomena than the ones led by the dispersive flow $e^{it\Delta}$. Nevertheless, from the point of view of estimate (4) the behavior is the same for the flows $e^{t\Delta}, e^{it\Delta}$, when $t > 0$. Our first question is the following:

A *is the time decay of the flows* $e^{t\Delta}, e^{it\Delta}$ *related to the lowest frequency behavior of the corresponding fundamental solutions?*

We now pass to a more precise analysis of the decay estimate in (4), to describe some additional phenomenon which is hidden in formula (2). To this aim, let us recall the *Jacobi-Anger* expansion of plane waves, which combined with the Addition Theorem for spherical harmonics (see for example [21, formula (4.8.3), p. 116] and [2, Corollary 1]) yields

$$e^{ix\cdot y} = (2\pi)^{d/2}\big(|x||y|\big)^{-\frac{d-2}{2}} \sum_{\ell=0}^{\infty} i^\ell J_{\ell+\frac{d-2}{2}}\big(|x||y|\big)\left(\sum_{m=1}^{m_\ell} Y_{\ell,m}\big(\tfrac{x}{|x|}\big)\overline{Y_{\ell,m}\big(\tfrac{y}{|y|}\big)}\right) \tag{10}$$

for all $x, y \in \mathbb{R}^d$. Here J_ν denotes the ν-th Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{t}{2}\right)^{2k}$$

and the $Y_{\ell,m}$ are usual spherical harmonics. Recalling that $J_\nu(t) \sim t^\nu$, for $\nu \geqslant 0$, as t goes to 0, we see that an additional time-decay, for t large is hidden in formula (2), in the term $e^{i\frac{x\cdot y}{t}}$. Roughly speaking, we expect that initial data localized at higher frequencies (with respect to the spherical harmonics expansion) decay polynomially faster along a Schrödinger evolution, in suitable topologies. This leads to our second question:

B *how can the above described phenomenon be quantified, and how stable is it under lower-order perturbations?*

Looking to identity (10), the presence of spherical harmonics and special functions gives the hint that the spherical laplacian is playing an important role in the description of the above mentioned phenomena. The aim of this survey is to describe this role, giving partial answers to the above questions and leaving some open problems, corroborated by some recent results.

2 A Stationary Viewpoint: Hardy's Inequality

We devote a preliminary section to introduce an interesting stationary viewpoint of the above picture, related to some uncertainty inequalities. To this aim, we recall the well known *Hardy's inequality*:

$$\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{|\psi(x)|^2}{|x|^2}\,dx \leqslant \int_{\mathbb{R}^d}|\nabla\psi(x)|^2\,dx, \qquad (d\geqslant 3) \tag{11}$$

which holds for any function $\psi \in \dot{H}^1(\mathbb{R}^d)$, being $\dot{H}^1(\mathbb{R}^d)$ the completion of $\mathscr{C}_c^\infty(\mathbb{R}^d)$ with respect to the seminorm

$$\|f\|^2_{\dot{H}^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d}|\nabla f|^2\,dx,$$

taking its quotient by the equivalence relation

$$f\sim g \text{ if } \exists c\in\mathbb{R}\,:\, f=g+c.$$

The constant in front of inequality (11) is sharp, and it is not attained on any function ψ for which the right-hand side is finite, as we see in a while. Inequality (11) can be rewritten in operator terms as

$$-\Delta-\frac{\lambda}{|x|^2}\geqslant 0, \qquad \forall\lambda\leqslant\frac{(d-2)^2}{4} \qquad (d\geqslant 3). \tag{12}$$

This has to be interpreted in the sense of the associated quadratic form. The proof of (11) relies on the following fact: given a symmetric operator $\mathscr{S}$ and a skew-symmetric operator $\mathscr{A}$ on L^2, one can (formally) compute

$$0\leqslant\int_{\mathbb{R}^d}|(\mathscr{A}+\mathscr{S})\psi|^2\,dx=\int_{\mathbb{R}^d}|\mathscr{A}\psi|^2\,dx+\int_{\mathbb{R}^d}|\mathscr{S}\psi|^2\,dx-\int_{\mathbb{R}^d}\overline{\psi}\,[\mathscr{A},\mathscr{S}]\,\psi\,dx,$$

where $[\mathscr{A},\mathscr{S}]=\mathscr{A}\mathscr{S}-\mathscr{S}\mathscr{A}$. Then the choices

$$\mathscr{A}:=\nabla, \qquad \mathscr{S}:=\frac{d-2}{2}\frac{x}{|x|^2} \qquad \Rightarrow \qquad [\mathscr{A},\mathscr{S}]=\frac{(d-2)^2}{2|x|^2}$$

immediately give (11) for functions ψ smooth enough, and a regularization argument completes the proof. Also notice the equality in (11) is attained when $(\mathscr{A} + \mathscr{S})\psi \equiv 0$, which yields the maximizing function $\psi(x) = |x|^{1-\frac{d}{2}}$, and we see that $|\nabla\psi| \notin L^2$, as mentioned above. In addition, one immediately realizes that, given $\widetilde{\mathscr{A}} = \partial_r = \nabla \cdot \frac{x}{|x|}$, then

$$[\widetilde{\mathscr{A}}, \mathscr{S}] = [\mathscr{A}, \mathscr{S}] = \frac{(d-2)^2}{2|x|^2},$$

which yields the more precise inequality

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\, dx \leqslant \int_{\mathbb{R}^d} |\partial_r \psi(x)|^2\, dx, \qquad (d \geqslant 3) \tag{13}$$

In other words, inequality (13) shows that the angular component of $-\Delta$ is not playing a role in (11)–(12). To understand this fact, it is convenient to use spherical coordinates and write

$$\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{d-1}}, \tag{14}$$

being $\Delta_{\mathbb{S}^{d-1}}$ the spherical laplacian, i.e. the Laplace-Beltrami operator on the $(d-1)$-dimensional unit sphere. We recall that $-\Delta_{\mathbb{S}^{d-1}}$ is a (positive) operator with compact inverse, hence it has purely point spectrum which accumulates at infinity, which is explicitly given by the set

$$\sigma\left(-\Delta_{\mathbb{S}^{d-1}}\right) = \sigma_{\mathrm{p}}\left(-\Delta_{\mathbb{S}^{d-1}}\right) = \{\ell(\ell + d - 2)\}_{\ell=0,1,2,\dots}. \tag{15}$$

Spherical harmonics $\{Y_{\ell,m}\}$ are associated eigenfunctions, which form a complete orthonormal set in $L^2(\mathbb{S}^{d-1})$. Denoting by H_ℓ the eigenspace associated to the ℓ-th eigenvalue of $-\Delta_{\mathbb{S}^{d-1}}$, by D_ℓ its algebraic dimension, and by $H_{\ell,m}$ the space generated by $Y_{\ell,m}$, we have the well known decomposition

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{\substack{l \geqslant 0 \\ 1 \leqslant m \leqslant D_\ell}} H_{\ell,m}$$

Therefore any function $\psi \in L^2(\mathbb{R}^d)$ has a (unique) expansion

$$\psi(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{D_\ell} \psi_{\ell,m}(r) Y_{\ell,m}(\omega) \qquad x = r\omega, \quad r := |x| \tag{16}$$

and moreover

$$\|f(r\omega)\|_{L^2(\mathbb{S}^{d-1})} = \sum_{\substack{\ell\geqslant 0\\ 1\leqslant m\leqslant D_\ell}} |f_{\ell,m}|^2.$$

We can hence use (14) to write

$$\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx = -\int_{\mathbb{R}^d} \overline{\psi}\Delta\psi\,dx \tag{17}$$
$$= \underbrace{-\int_{\mathbb{R}^d} \overline{\psi}\left(\partial_r^2\psi + \frac{d-1}{r}\partial_r\psi\right)dx}_{=:I} + \underbrace{\int_{\mathbb{R}^d} \frac{1}{|x|^2}\langle\psi, -\Delta_{\mathbb{S}^{d-1}}\psi\rangle_{L^2(\mathbb{S}^{d-1})}\,dx.}_{=:II}$$

where the brackets $\langle\cdot,\cdot\rangle_{L^2(\mathbb{S}^{d-1})}$ denote the inner product in $L^2(\mathbb{S}^{d-1})$. Arguing as above we see that

$$I \geqslant \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\,dx, \qquad (d\geqslant 3)$$

which is inequality (13). On the other hand, it follows by (15) that

$$II \geqslant 0,$$

therefore no additional contribution to (11) is given by $-\Delta_{\mathbb{S}^{d-1}}$. Nevertheless, given $\psi \in L^2(\mathbb{R}^{d-1})$, if $\psi_{0,1} = 0$ in the expansion (16) (notice that $H_{0,1}$ coincides with the space of L^2-radial functions), then by (15) it follows that

$$\langle\psi, -\Delta_{\mathbb{S}^{d-1}}\psi\rangle_{L^2(\mathbb{S}^{d-1})} \geqslant (d-1)\|\psi(\omega)\|_{L^2(\mathbb{S}^{d-1})} \qquad \text{if } \psi_{0,1} = 0$$

and inequality (13) improves:

$$\int_{\mathbb{R}^d} |\partial_r\psi(x)|^2\,dx \geqslant \left(\frac{(d-2)^2}{4} + (d-1)\right)\int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\,dx, \quad (d\geqslant 2) \quad \psi_{0,1} = 0. \tag{18}$$

Notice that the previous gives a non trivial 2D-inequality, holding on functions ψ which are orthogonal to L^2-radial functions. More in general, given $\psi \in L^2(\mathbb{R}^d)$, let

$$\ell_0 := \min\{\ell \in \mathbb{N} \text{ such that } \exists m = 1,\dots,D_\ell \ :\ \psi_{\ell,m} \neq 0\}.$$

Then, by (17), the following Hardy's inequality holds:

$$\int_{\mathbb{R}^d} |\partial_r\psi(x)|^2\,dx \geqslant \left(\frac{(d-2)^2}{4} + \ell_0(\ell_0+d-2)\right)\int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\,dx. \qquad (d\geqslant 1) \tag{19}$$

Inequality (19) is a quantitative stationary manifestation of the phenomenon described by question **B** in the Introduction. Here it is clear that the improvement comes from the angular component of the free Hamiltonian. In addition, the above arguments clearly suggest that the sharp constant in front of inequality (19) only depends the *lowest energies*, which is reminiscent of question **A** in the Introduction.

Having this in mind, we now see how linear lower-order perturbations of the free spherical Hamiltonian can perturb the spectral picture in (15), with consequences on the Hardy's inequality (19).

Example 1 (0-Order Perturbations) For $a \in \mathbb{R}$, consider the shifted Hamiltonians in dimension $d \geqslant 3$

$$H = -\Delta + \frac{a}{|x|^2}, \qquad L = -\Delta_{\mathbb{S}^{d-1}} + a.$$

Clearly L only has point spectrum, which is just a shift of (15)

$$\sigma(L) = \sigma_{\mathrm{p}}(L) = \{\ell(\ell + d - 2) + a\}_{\ell=0,1,2,\ldots}$$

and spherical harmonics are still eigenfunctions. The corresponding Hardy's inequality is trivially

$$\left(\frac{(d-2)^2}{4} + a\right) \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\, dx \leqslant \int_{\mathbb{R}^d} |\nabla\psi(x)|^2\, dx + a \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\, dx. \qquad (d \geqslant 3) \tag{20}$$

More in general, if $a = a(\omega) : \mathbb{S}^{d-1} \to \mathbb{R}$, then it is still true that L as only point spectrum, but the picture is more complicated. A typical phenomenon is the formation of clusters of eigenvalues around the (shifted) free eigenvalues. The size of the clusters depends on some universal dimensional quantity related to $a(\omega)$ (see e.g. the standard references [3, 20, 29, 30, 33] and Lemma 1 below). Moreover, for the lowest eigenvalue of L we have

$$\mu_0 := \min \sigma(L) = \inf_{\omega \in \mathbb{S}^{d-1}} a(\omega).$$

One easily see by the same arguments as above that the following Hardy's inequality holds

$$\left(\frac{(d-2)^2}{4} + \mu_0\right) \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\, dx \leqslant \int_{\mathbb{R}^d} \overline{\psi} H \psi\, dx. \tag{21}$$

Example 2 (1st-Order Perturbations) Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and recall the *diamagnetic inequality*

$$|(-i\nabla + A)\psi(x)| \geqslant |\nabla|\psi(x)||.$$

This gives for free, together with (11), that

$$\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{|\psi(x)|^2}{|x|^2}\,dx \leqslant \int_{\mathbb{R}^d}|(-i\nabla + A)\psi(x)|\,dx, \qquad (d \geqslant 3). \tag{22}$$

We wonder if an improvement to the best constant of inequality (22) can occur, due to the presence of an angular perturbation of the associated Hamiltonian, in the same style as in the above example. The main example we have in mind is given by the 2D-*Aharonov-Bohm* vector potential: for $\lambda \in \mathbb{R}$, consider let us denote by

$$A : \mathbb{R}^2 \to \mathbb{R}^2, \qquad A(x,y) := \lambda\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

and consider the following quadratic form

$$q[\psi] := \int_{\mathbb{R}^2}|(-i\nabla + A)\psi|^2\,dx.$$

Since q is positive, we can consider the *Friedrichs' extension* of the self-adjoint Hamiltonian $H := -\nabla_A^2$, on the natural form domain induced by q (see Sect. 3 below for details). The angular component of H is the operator

$$L := \left(-i\nabla_{\mathbb{S}^1} + \mathscr{A}(\omega)\right)^2, \qquad \mathscr{A} : \mathbb{S}^1 \to \mathbb{S}^1, \qquad \mathscr{A}(x,y) = \lambda\left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right).$$

As above, L has compact inverse and its spectrum is explicitly given by

$$\sigma(L) = \sigma_{\mathrm{p}}(L) = \{(\lambda - z)^2\}_{z\in\mathbb{Z}}.$$

Therefore, the lowest eigenvalue is given by

$$\mu_0 := \min\sigma(L) = \mathrm{dist}\,(\lambda,\mathbb{Z})^2 \geqslant 0$$

and we gain the following 2D-Hardy's inequality, proved in [24]

$$\mu_0\int_{\mathbb{R}^2}\frac{|\psi(x)|^2}{|x|^2}\,dx \leqslant \int_{\mathbb{R}^2}|(-i\nabla + A)\psi|^2\,dx. \tag{23}$$

As soon as $\lambda \notin \mathbb{Z}$, this is an improvement with respect to the free case $A \equiv 0$, in which such an inequality cannot hold for any function ψ such that $|\nabla\psi| \in L^2(\mathbb{R}^2)$ (since the weight $|x|^{-2}$ is not locally integrable in 2D).
In view of the above considerations, we will restrict our attention, from now on, to some scaling-critical electromagnetic Hamiltonians and we will present some recent results which partially answer to questions **A** and **B** in the Introduction of this survey.

3 Decay Estimates: Main Results

From now on, for any $x \in \mathbb{R}^d$, we denote by $x = r\omega$, $r = |x|$. Let

$$\mathbf{A} = \mathbf{A}(\omega) : \mathbb{S}^{d-1} \to \mathbb{R}^d, \qquad a = a(\omega) : \mathbb{S}^{d-1} \to \mathbb{R}$$

be 0-degree homogeneous functions, and consider the quadratic form

$$q[\psi] := \int_{\mathbb{R}^d} \left| \left(-i\nabla + \frac{A(\omega)}{r} \right) \psi(x) \right|^2 dx + \int_{\mathbb{R}^d} \frac{a(\omega)}{r^2} |\psi(x)|^2 \, dx. \tag{24}$$

As we see in the sequel, under suitable conditions, a self-adjoint Hamiltonian

$$H := \left(-i\nabla + \frac{A(\omega)}{r} \right)^2 + \frac{a(\omega)}{r^2}, \tag{25}$$

associated to q (Friedrichs' Extension) is well defined on a domain containing $L^2(\mathbb{R}^d)$, therefore the L^2-initial value problem

$$\begin{cases} i\partial_t \psi = -iH\psi, \\ \psi(0) = \psi_0 \in L^2(\mathbb{R}^d), \end{cases} \tag{26}$$

for the wavefunction $\psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ makes sense. Here $d \geqslant 2$, and we choose a *transversal gauge* for the magnetic vector potential, i.e. we assume

$$\mathbf{A}(\omega) \cdot \omega = 0 \quad \text{for all } \omega \in \mathbb{S}^{d-1}. \tag{27}$$

Notice that equation (26) is invariant under the scaling $u_\lambda(x, t) := u(x/\lambda, t/\lambda^2)$, which is the same of the free Schrödinger equation.

The aim is to understand the role of the spherical operator L associated to H, defined by

$$L = \big(-i\,\nabla_{\mathbb{S}^{d-1}} + \mathbf{A} \big)^2 + a(\omega), \tag{28}$$

where $\nabla_{\mathbb{S}^{d-1}}$ is the spherical gradient on the unit sphere $\mathbb{S}^{d-1}$. Assuming $a \in L^\infty(\mathbb{S}^{d-1}; \mathbb{R}), \mathbf{A} \in \mathscr{C}^1(\mathbb{S}^{d-1}; \mathbb{R}^d)$, then the spectrum of the operator L is formed by a diverging sequence of real eigenvalues with finite multiplicity $\mu_0(\mathbf{A}, a) \leqslant \mu_1(\mathbf{A}, a) \leqslant \cdots \leqslant \mu_k(\mathbf{A}, a) \leqslant \cdots$ (see e.g. [16, Lemma A.5]), where each eigenvalue is repeated according to its multiplicity. Moreover we have that $\lim_{k\to\infty} \mu_k(\mathbf{A}, a) = +\infty$. To each $k \geqslant 1$, we can associate a $L^2\big(\mathbb{S}^{d-1}, \mathbb{C}\big)$-normalized eigenfunction φ_k of the operator L on $\mathbb{S}^{d-1}$ corresponding to the k-th eigenvalue $\mu_k(\mathbf{A}, a)$, i.e. satisfying

$$\begin{cases} L\varphi_k = \mu_k(\mathbf{A}, a)\,\varphi_k, & \text{in } \mathbb{S}^{d-1}, \\ \int_{\mathbb{S}^{d-1}} |\varphi_k|^2\, dS(\theta) = 1. \end{cases} \tag{29}$$

In particular, if $d = 2$, φ_k are one-variable 2π-periodic functions, i.e. $\varphi_k(0) = \varphi_k(2\pi)$. Since the eigenvalues $\mu_k(\mathbf{A}, a)$ are repeated according to their multiplicity, exactly one eigenfunction φ_k corresponds to each index $k \geqslant 1$. We can choose the functions φ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{d-1}, \mathbb{C})$. We also introduce the numbers

$$\alpha_k := \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}, \quad \beta_k := \sqrt{\left(\frac{d-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}, \tag{30}$$

so that $\beta_k = \frac{d-2}{2} - \alpha_k$, for $k = 1, 2, \ldots$.

Under the condition

$$\mu_0(\mathbf{A}, a) > -\frac{(d-2)^2}{4} \tag{31}$$

the quadratic form q in (24) associated to H is positive definite, and the Friedrichs' extension of H is well defined, with domain

$$\mathscr{D} := \left\{ f \in H^1_*(\mathbb{R}^d) : \ Hf \in L^2(\mathbb{R}^d) \right\}, \tag{32}$$

where $H^1_*(\mathbb{R}^d)$ is the completion of $C^\infty_c(\mathbb{R}^d \setminus \{0\}, \mathbb{C})$ with respect to the norm

$$\|f\|_{H^1_*(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^N} \left(|\nabla f(x)|^2 + \frac{|f(x)|^2}{|x|^2} + |f(x)|^2 \right) dx \right)^{1/2}.$$

By the Hardy's inequality (11), $H^1_*(\mathbb{R}^d) = H^1(\mathbb{R}^d)$ with equivalent norms if $d \geqslant 3$, while $H^1_*(\mathbb{R}^d)$ is strictly smaller than $H^1(\mathbb{R}^d)$ if $d = 2$. Furthermore, from condition (31) and [16, Lemma 2.2], it follows that $H^1_*(\mathbb{R}^d)$ coincides with the space obtained by completion of $C^\infty_c(\mathbb{R}^d \setminus \{0\}, \mathbb{C})$ with respect to the norm naturally associated to H, i.e.

$$q[\psi] + \|\psi\|_2^2.$$

We remark that H could be not essentially self-adjoint. Indeed, in the case $\mathbf{A} \equiv 0$, Kalf, Schmincke, Walter, and Wüst [22] and Simon [28] proved that H is essentially self-adjoint if and only if $\mu_0(\mathbf{0}, a) \geqslant -\left(\frac{d-2}{2}\right)^2 + 1$ and, consequently, admits a unique self-adjoint extension (which coincides with the Friedrichs' extension); otherwise, i.e. if $\mu_0(\mathbf{0}, a) < -\left(\frac{d-2}{2}\right)^2 + 1$, H is not essentially self-adjoint and admits infinitely many self-adjoint extensions, among which the Friedrichs' extension is the only one whose domain is included in the domain of the associated quadratic form (see also [9, Remark 2.5]).

The Friedrichs' extension H naturally extends to a self adjoint operator on the dual $\mathscr{D}^\star$ of $\mathscr{D}$ and the unitary group e^{-itH} extends to a group of isometries on the dual of $\mathscr{D}$ which will be still denoted as e^{-itH} (see [6], Section 1.6 for further details). Then for every $\psi_0 \in L^2(\mathbb{R}^d)$,

$$\psi(t, x) := e^{-itH}\psi_0(x) \in \mathscr{C}(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathscr{C}^1(\mathbb{R}; \mathscr{D}^\star),$$

is the unique solution to (26).

Now, by means of (29) and (30) define the following kernel:

$$K(x, y) = \sum_{k=-\infty}^{\infty} i^{-\beta_k} j_{-\alpha_k}(|x||y|)\varphi_k\left(\tfrac{x}{|x|}\right)\overline{\varphi_k\left(\tfrac{y}{|y|}\right)}, \tag{33}$$

where

$$j_\nu(r) := r^{-\frac{d-2}{2}} J_{\nu+\frac{d-2}{2}}(r)$$

and J_ν denotes the usual Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}.$$

Notice that (33) reduces to (10), in the free case $\mathbf{A} \equiv a \equiv 0$. The first result we mention in this survey is the following representation formula for e^{-itH}:

Theorem 1 (L. Fanelli, V. Felli, M. Fontelos, A. Primo—[12]) *Let $d \geqslant 3$, $a \in L^\infty(\mathbb{S}^{d-1}, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{d-1}, \mathbb{R}^N)$, and assume (27) and (31). Then, for any $\psi_0 \in L^2(\mathbb{R}^d)$,*

$$e^{-itH}\psi_0(x) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{d/2}} \int_{\mathbb{R}^d} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} \psi_0(y)\, dy. \tag{34}$$

As an immediate consequence, we see by (34) that the analog to condition (7) gives for H the complete list of usual time decay estimates (5):

Corollary 1 *Let $d \geqslant 3$, $a \in L^\infty(\mathbb{S}^{d-1}, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{d-1}, \mathbb{R}^N)$, and assume (27) and (31). If*

$$\sup_{x,y\in\mathbb{R}^d} |K(x, y)| < \infty, \tag{35}$$

then

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^d)} \leqslant C|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^d)}, \qquad \forall t \in \mathbb{R}, \qquad \forall p \geqslant 2, \tag{36}$$

for some $C > 0$ *independent on* ψ_0.

In the two last decades, estimates (36) were intensively studied by several authors. The following is an incomplete list of results about this topic [1, 7, 8, 10, 11, 17, 18, 25–27, 31, 32, 34–37]. In all these papers, the potentials are sub-critical with respect to the functional scale of the Hardy's inequality (11): in other words, the critical potentials in (25) are never considered, and it does not seem that one could handle them by perturbation techniques, which are a common factor of all the above mentioned papers. Now, formula (34) and Corollary 1 give a usual tool to reduce matters to prove time decay, to a spectral analysis problem. This allowed us to prove some new positive results concerning with estimates (36). In 2D, the picture is quite well understood, thanks to the following theorem.

Theorem 2 (L. Fanelli, V. Felli, M. Fontelos, A. Primo—[13]) *Let* $d = 2$, $a \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R})$, $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$ *satisfying* (27) *and* $\mu_1(\mathbf{A}, a) > 0$, *and H be given by* (25). *Then, for any* $\psi_0 \in L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)$,

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^2)} \leqslant C|t|^{-2\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^2)}, \qquad \forall t \in \mathbb{R}, \qquad \forall p \geqslant 2, \tag{37}$$

for some $C > 0$ *independent on* ψ_0.

Theorem 2 is proved in [13]. The core consists in proving that (35) holds, and a crucial role is played by the following Lemma, which gives a quite explicit expansion of eigenvalues and eigenfunctions of L, generalizing the results in [20]:

Lemma 1 (L. Fanelli, V. Felli, M. Fontelos, A. Primo—[13]) *Let* $a \in W^{1,\infty}(\mathbb{S}^1)$, $\widetilde{a} := \frac{1}{2\pi}\int_0^{2\pi} a(s)\,ds$, $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1)$ *such that*

$$\widetilde{\mathbf{A}} = \frac{1}{2\pi}\int_0^{2\pi} A(s)\,ds \notin \frac{1}{2}\mathbb{Z}. \tag{38}$$

Then there exist $k^*, \ell \in \mathbb{N}$ *such that* $\{\mu_k : k > k^*\} = \{\lambda_j : j \in \mathbb{Z}, |j| \geqslant \ell\}$,

$$\sqrt{\lambda_j - \widetilde{a}} = (\operatorname{sgn} j)\left(\widetilde{\mathbf{A}} - \left\lfloor \widetilde{\mathbf{A}} + \tfrac{1}{2} \right\rfloor\right) + |j| + O\left(\tfrac{1}{|j|^3}\right), \quad \text{as } |j| \to +\infty$$

and

$$\lambda_j = \widetilde{a} + \left(j + \widetilde{\mathbf{A}} - \left\lfloor \widetilde{\mathbf{A}} + \tfrac{1}{2} \right\rfloor\right)^2 + O\left(\tfrac{1}{j^2}\right), \quad \text{as } |j| \to +\infty. \tag{39}$$

Furthermore, for all $j \in \mathbb{Z}$, $|j| \geqslant \ell$, *there exists a* $L^2(\mathbb{S}^1, \mathbb{C})$*-normalized eigenfunction* φ_j *of the operator* L *on* $\mathbb{S}^1$ *corresponding to the eigenvalue* λ_j *such that*

$$\varphi_j(\theta) = \frac{1}{\sqrt{2\pi}} e^{-i\left(\lfloor\widetilde{\mathbf{A}}+1/2\rfloor\theta + \int_0^\theta A(t)\,dt\right)} \left(e^{i(\widetilde{\mathbf{A}}+j)\theta} + R_j(\theta)\right), \tag{40}$$

where $\|R_j\|_{L^\infty(\mathbb{S}^1)} = O\left(\frac{1}{|j|^3}\right)$ *as* $|j| \to \infty$*. In the above formula* $\lfloor\cdot\rfloor$ *denotes the floor function* $\lfloor x\rfloor = \max\{k \in \mathbb{Z} : k \leqslant x\}$.

Analogous results to Lemma 1 can be proved (and are in part available) in higher dimension $d \geqslant 3$. Nevertheless, the higher dimensional scenario is quite more complicate, and some chaotic behavior of the eigenvalues of L can occur. This makes the generic validity of (36) completely unclear in dimension $d \geqslant 3$. In this direction, the only result which is available at the moment is concerned with the 3D-inverse square electric potential, and reads as follows:

Theorem 3 (L. Fanelli, V. Felli, M. Fontelos, A. Primo—[12]) *Let* $d = 3$, $\mathbf{A} \equiv 0$ *and* $a(\omega) \equiv a \in \mathbb{R}$*, with* $a > -\frac{1}{4}$.

i) If $a \geqslant 0$*, then, for any* $\psi_0 \in L^2(\mathbb{R}^3) \cap L^{p'}(\mathbb{R}^3)$,

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{L^p(\mathbb{R}^2)} \leqslant C|t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^2)}, \qquad \forall t \in \mathbb{R}, \qquad \forall p \geqslant 2, \tag{41}$$

for some $C > 0$ *which does not depend on* ψ_0.

ii) If $-\frac{1}{4} < a < 0$*, let* α_1 *as in* (30)*, and define*

$$\|\psi\|_{p,\alpha_1} := \left(\int_{\mathbb{R}^3} (1 + |x|^{-\alpha_1})^{2-p}|\psi(x)|^p\,dx\right)^{1/p}, \quad p \geqslant 1.$$

Then the following estimates hold

$$\left\|e^{-itH}\psi_0(\cdot)\right\|_{p,\alpha_1} \leqslant \frac{C(1+|t|^{\alpha_0})^{1-\frac{2}{p}}}{|t|^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}\|\psi\|_{p',\alpha_0}, \quad p \geqslant 2, \quad \frac{1}{p}+\frac{1}{p'} = 1, \tag{42}$$

for some constant $C > 0$ *which does not depend on* ψ_0.

Remark 1 It is interesting to remark that, in the range $-1/4 < a < 0$, (41) does not hold, while the full set of usual Strichartz estimates hold (see [4, 5]). This is now clearly understood in terms of formula (34): notice that, if $a = \mu_0 < 0$, then $\alpha_0 > 0$ and a negative-index Bessel function appears in the kernel K given by (33); since negative-index functions J_ν are singular at the origin, one cannot either expect the solution (34) to be in L^∞.

This can be proved as a general fact:

Theorem 4 (L. Fanelli, V. Felli, M. Fontelos, A. Primo—[13]) *Let $d \geqslant 3$, $a \in L^\infty(\mathbb{S}^{d-1}, \mathbb{R})$, $\mathbf{A} \in C^1(\mathbb{S}^{d-1}, \mathbb{R}^d)$, and assume (27), (31), and $\mu_0 < 0$. Then, for almost every $t \in \mathbb{R}$, $e^{-itH}(L^1) \nsubseteq L^\infty$; in particular e^{-itH} is not a bounded operator from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.*

The above phenomenon can be quantified. To this aim, let us restrict our attention to the case

$$H = -\Delta + \frac{a}{|x|^2}, \qquad x \in \mathbb{R}^3.$$

Let us define

$$V_{n,j}(x) = |x|^{-\alpha_j} e^{-\frac{|x|^2}{4}} P_{j,n}\Big(\frac{|x|^2}{2}\Big)\psi_j\Big(\frac{x}{|x|}\Big), \quad n, j \in \mathbb{N},\ j \geqslant 1, \tag{43}$$

where $P_{j,n}$ is the polynomial of degree n given by

$$P_{j,n}(t) = \sum_{i=0}^{n} \frac{(-n)_i}{\big(\frac{d}{2} - \alpha_j\big)_i} \frac{t^i}{i!},$$

denoting as $(s)_i$, for all $s \in \mathbb{R}$, the Pochhammer's symbol

$$(s)_i = \prod_{j=0}^{i-1}(s + j), \qquad (s)_0 = 1.$$

Moreover, for all $k > 1$, define

$$\mathscr{U}_k = \operatorname{span}\big\{V_{n,j} : n \in \mathbb{N}, 1 \leqslant j < k\big\} \subset L^2(\mathbb{R}^N).$$

The functions $V_{n,j}$ spans $L^2(\mathbb{R}^3)$ (see [14] for details). Moreover, as initial data for (1), these functions have a quite explicit evolution: indeed, denoting by $\widetilde{V}_{n,j} := V_{n,j}/\|V_{n,j}\|_2$, the following identity holds:

$$\begin{aligned} e^{-itH}\widetilde{V}_{n,j}(x) &= e^{it\left(-\Delta + \frac{a}{|x|^2}\right)} V_{n,j}(x) \\ &= (1 + t^2)^{-\frac{d}{4} + \frac{\alpha_j}{2}} |x|^{-\alpha_j} \frac{e^{\frac{-|x|^2}{4(1+t^2)}}}{\|V_{n,j}\|_{L^2(\mathbb{R}^d)}} e^{i\frac{|x|^2 t}{4(1+t^2)}} e^{-i\gamma_{n,j}\arctan t} \psi_j\big(\tfrac{x}{|x|}\big) P_{n,j}\big(\tfrac{|x|^2}{2(1+t^2)}\big). \end{aligned} \tag{44}$$

Formula (44) has been proved in [14]. Clearly, if $a = \mu_0 \geqslant 0$, then $\alpha_0 \leqslant 0$ and the first function $\widetilde{V}_{1,0}$ decays polynomially faster than usual, in a weighted space. This is reminiscent to question **B** in the Introduction, and gives us the following evolution version of the frequency-dependent Hardy's inequality (19):

Theorem 5 (L. Fanelli, V. Felli, M. Fontelos, A. Primo—[14]) *Let* $d = 3$, $a = \mu_0 \geqslant 0$, α_0 *as in* (30).

(i) There exists $C > 0$ *such that, for all* $\psi_0 \in L^2(\mathbb{R}^3)$ *with* $|x|^{-\alpha_0}\psi_0 \in L^1(\mathbb{R}^3)$,

$$\left\| |x|^{\alpha_0} e^{-itH}\psi_0(\cdot)\right\|_{L^\infty} \leqslant C|t|^{-\frac{3}{2}+\alpha_0} \| |x|^{-\alpha_0}\psi_0\|_{L^1}.$$

(ii) For all $k \in \mathbb{N}$, $k \geqslant 1$, *there exists* $C_k > 0$ *such that, for all* $\psi_0 \in \mathscr{U}_k^\perp$ *with* $|x|^{-\alpha_k}\psi_0 \in L^1(\mathbb{R}^3)$,

$$\left\| |x|^{\alpha_k} e^{-itH}\psi_0(\cdot)\right\|_{L^\infty} \leqslant C_k|t|^{-\frac{3}{2}+\alpha_k} \| |x|^{-\alpha_k}\psi_0\|_{L^1}.$$

Some analogous results, only concerning with the decay of the first frequency space, had been previously proven in [15, 19].

To complete the survey, we leave some open questions.

(i) Concerning Theorems 2, 3, does any general result hold in dimension $d \geqslant 3$?

(ii) In what extent can one perturb the models in (25)? What is the real role played by the scaling invariance?

(iii) The proof of formula (34) strongly relies on some pseudoconformal law associated to the free Schrödinger flow (Appell transform; see [12]). Is there any analog for other dispersive models, e.g. the wave equation?

(iv) One can use formula (34) to represent the wave operators

$$W_\pm := L^2 - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}, \qquad H_0 := -\Delta.$$

What can one prove about the boundedness of $W_\pm$ in $L^p(\mathbb{R}^d)$, in the same style as in [31, 32, 34–37] (at least in 2D, having in mind Theorem 2.

(v) By standard $TT^\star$-arguments, one can obtain some weighted Strichartz estimates by Theorem 5. Which kind of informations do these estimates give for nonlinear Schrödinger equations associated to H?

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