Logarithmic Sobolev Inequalities for an Ideal Bose Gas

Fabio Cipriani

Abstract The aim of this work is to derive logarithmic Sobolev inequalities, with respect to the Fock vacuum state and for the second quantized Hamiltonian $d\Gamma(H^\Lambda - \mu\mathbb{I})$ of an ideal Bose gas with Dirichlet boundary conditions in a finite volume Λ, from the free energy variation with respect to a Gibbs temperature state and from the monotonicity of the relative entropy. Hypercontractivity of the semigroup $e^{-\beta d\Gamma(H^\Lambda)}$ is also deduced.

Keywords Free energy • Gibbs state • Hypercontractivity • Ideal bose gas • Logarithmic sobolev inequality • Relative entropy

1 Introduction

In the 1938 the mathematical physicist S.L. Sobolev proved the following inequality

$$\left(\int_{\mathbb{R}^n} |\psi(x)|^p \, dx\right)^{2/p} \leq c_n \int_{\mathbb{R}^n} |\nabla\psi(x)|^2 \, dx, \qquad \psi \in C_c^1(\mathbb{R}^n)\,,$$

for $n \geq 3$, $p = \frac{2n}{n-2}$ and some constant $c_n > 0$. Due to the possible interpretation of the Dirichlet integral on the right hand side as an energy functional, their are of great use in mathematical physics and became such a basic tool of investigation in linear and nonlinear PDE, that is impossible to exaggerate their importance.

In Quantum Mechanics, Dirichlet integrals are the quadratic form of the Laplace operator $H_0 := -\Delta$ that represent the kinetic energy observable of a finite system of particles and the use of the inequality above provides, among other things, classes of possibly unbounded potentials V whose quantum Hamiltonians $H_0 + V$ are self-adjoint on the Lebesgue space $L^2(\mathbb{R}^n, dx)$.

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A. Michelangeli, G. Dell'Antonio (eds.), *Advances in Quantum Mechanics*,
Springer INdAM Series 18, DOI 10.1007/978-3-319-58904-6_7

At a more fundamental level, E. Lieb recognized in the Sobolev inequalities an *uncertainty principle* which is one of the fundamental ingredients to prove the Stability of the Matter [8].

In 1976, L. Gross [7] proved the following Logarithmic Sobolev inequality for $f \in C_c^1(\mathbb{R}^n)$ and $\|f\|_{L^2(\mathbb{R}^n,\gamma)} = 1$

$$\int_{\mathbb{R}^n} \gamma(dx)|f(x)|^2 \log|f(x)|^2 \leq \int_{\mathbb{R}^n} \gamma(dx)|\nabla f(x)|^2$$

with respect to the Gaussian probability measure $\gamma(dx) = (2\pi)^{-n/2}e^{-|x|^2/2}dx$. He demonstrated that this inequality is an infinitesimal version of the Nelson's hypercontractivity

$$\|e^{-t\widetilde{H}}u\|_{L^4(\mathbb{R}^n,\gamma)} \leq \|u\|_{L^2(\mathbb{R}^n,\gamma)}, \qquad t > 0\,, \quad u \in L^2(\mathbb{R}^n,\gamma)$$

of the Ornstein-Uhlenbeck semigroup $e^{-t\widetilde{H}}$ generated by the ground state representation $\widetilde{H}$ of the Hamiltonian $H = \frac{1}{2}(-\Delta + |x|^2 - 1)$ of the quantum harmonic oscillator (see [10]).

A first key difference between SI and LSI is that in the latter, the constant in front the Gaussian Dirichlet integral is dimension independent. This fact allowed Gross to prove LSI on infinite dimensional Gaussian Banach spaces, providing a useful tool to infinite dimensional analysis.

Both E. Nelson and L. Gross were motivated in discovering their results by the problems of constructive Quantum Field Theory where hypercontractivity and logarithmic Sobolev inequalities provide sufficient compactness near the bottom of the spectrum of free Hamiltonians H_0 to prove essential self-adjointness, lower semiboundedness, existence and finite degeneracy of the ground state as well its uniqueness in case of ergodicity, for interacting Hamiltonians $H_0 + V$ (see [6, 9] and also [13, 14]).

Among the applications of infinite dimensional LSI to Mathematical Physics, we recall the work of E. Carlen and D. Stroock [3] on the extension of the Bakry-Emery criterion and its use to prove LSI for non product Gibbs measures for continuous spin systems as well as the work of D. Stroock and B. Zegarlinski [15] about the equivalence of LSI with the Dobrushin-Shlosman mixing condition for lattice gases with compact continuous spin space.

Later, E.B. Davies and B. Simon [5] discovered that families of LSI

$$\int_X dm\,|u|^2 \log|u|^2 \leq \beta\mathscr{E}[u] + b(\beta), \qquad \beta > 0\,, \quad \|u\|_{L^2(X,\mu)} = 1$$

on a locally compact measured space (X,μ), are deeply connected with the ultracontractivity of the heat semigroup associated to the Dirichlet form $\mathscr{E}$, provided the local norm $b(\beta)$ is not too singular as β goes to zero. This theory was subsequently used by E.B. Davies [D] to get sharp off diagonal bounds upon the heat kernel of the Markovian semigroup generated by a Dirichlet form $\mathscr{E}$ satisfying such logarithmic Sobolev inequality.

The first aim of this work is prove logarithmic Sobolev inequalities LSI(Λ), with respect to the Fock vacuum state ω_F^Λ or measure μ_F^Λ, for the second quantized Hamiltonian $\beta d\Gamma(H^\Lambda - \mu\mathbb{I})$ (at fixed inverse temperature $\beta > 0$ and activity $\mu \in \mathbb{R}$) of a gas of non interacting identical particles obeying the Bose-Einstein statistics and confined in a bounded Euclidean domain where they are subject to Dirichlet boundary conditions.

Our second aim is to introduce a new approach to logarithmic Sobolev inequality based on two fundamental ideas of Quantum Statistical Mechanics, namely, the relation between *Helmholtz free energy, Gibbs states* and *relative entropy*, on one hand, and the *monotonicity of relative entropy*, on the other hand.

2 Logarithmic Sobolev Inequalities for Ideal Bosons Gas in Finite Volume

To properly state the main result of the paper and introduce notations, we start to describe the framework of the work. For the standard fundamental result we will use, we refer to the standard classical monographies [2, 13].

Warning: *Whenever a self-adjoint operator H is semi-bounded, to ease notation the expression $(\psi, H\psi)$ will be denote the value of the lower semicontinuous quadratic form of H at an element ψ of its quadratic form domain.*

Let $\Lambda \subset \mathbb{R}^d$ be a bounded Euclidean domain and H^Λ be the Dirichlet-Laplacian operator on the complex Hilbert space $\mathfrak{h}_\Lambda := L^2(\Lambda)$, considered with respect to the Lebesgue measure on Λ, defined as the closure of $-\Delta$ on the domain $C_c^\infty(\Lambda)$.

Denote by $\mathfrak{F}(\mathfrak{h}_\Lambda)$ the bosonic Fock space and by $\mathfrak{U}(h_\Lambda)$ the CCR algebra builded on $\mathfrak{h}_\Lambda$, when considered as a symplectic real vector space with the symplectic form

$$\sigma(f,g) := \operatorname{Im}(f,g)_{\mathfrak{h}_\Lambda}, \qquad f,g \in \mathfrak{h}_\Lambda .$$

The vacuum vector $\Omega \in \mathfrak{F}(h_\Lambda)$ is cyclic for $\mathfrak{U}(h_\Lambda)$ and defines on it the Fock vacuum state

$$\omega_F^\Lambda(A) := (\Omega, A\Omega)_{\mathfrak{h}_\Lambda} .$$

The annihilation and creation operators $\{a(f), a^*(f) : f \in \mathfrak{h}_\Lambda\}$ define the self-adjoint field operators $\{\Phi(f) : f \in \mathfrak{h}_\Lambda\}$

$$\Phi(f) := \overline{\frac{a(f) + a^*(f)}{\sqrt{2}}}$$

which give rise to the Weyl unitaries

$$W(f) := e^{i\Phi(f)}$$

that satisfy the Weyl's form of the Canonical Commutation Relation

$$W(f)W(g) = W(f+g)e^{-i\sigma(f,g)/2}, \qquad f, g \in \mathfrak{h}_\Lambda .$$

The subspace $L^2_{\mathbb{R}}(\Lambda)$ of real functions is a Lagrangian submanifold of $L^2(\Lambda)$ in the sense that the symplectic form vanishes identically so that the corresponding Weyl operators commute

$$W(f)W(g) = W(f+g) = W(g)W(f), \qquad f, g \in L^2_{\mathbb{R}}(\Lambda)$$

and the (double commutant) von Neumann algebra

$$\mathfrak{M}_\Lambda := \{W(f) \in \mathscr{B}(\mathfrak{F}(\mathfrak{h}_\Lambda)) : f \in L^2_{\mathbb{R}}(\Lambda)\}''$$

is abelian. By a fundamental theorem due to J. von Neumann, $\mathfrak{M}_\Lambda$ is identical with the weak closure of the subspace of linear combinations of Weyl unitaries in the algebra of all bounded operators on the Fock space.

The Fock vacuum state ω_F^Λ is normal on $\mathfrak{M}_\Lambda$ so that the pair $(\mathfrak{M}_\Lambda, \omega_F^\Lambda)$ can be realized as the abelian von Neumann algebra $L^\infty(Q_\Lambda, \mu_F^\Lambda)$ of essentially bounded measurable functions on a suitable measurable space Q_Λ, endowed with a probability measure. The fundamental relation

$$\omega_F^\Lambda(W(f)) = \omega_F^\Lambda(e^{i\Phi(f)}) = e^{-\frac{1}{4}\|f\|^2}, \qquad f \in \mathfrak{h}_\Lambda$$

allow the identification of the system of self-adjoint operators $\{\Phi(f) : f \in \mathfrak{h}_\Lambda^{\mathbb{R}}\}$ as a Gaussian random field (or process) $\{\phi(f) : f \in \mathfrak{h}_\Lambda^{\mathbb{R}}\}$ on a Gaussian space $(Q_\lambda, \mu_F^\Lambda)$, where the following relations hold true for $f, g \in \mathfrak{h}_\Lambda^{\mathbb{R}}$

$$\omega_F^\Lambda(\Phi(f)\Phi(g)) = \int_{Q_\Lambda} \phi(f)\phi(g)\, d\mu_F^\Lambda = \frac{1}{2}(f,g)_{\mathfrak{h}_\Lambda} = \frac{1}{2}\int_\Lambda f(x)g(x)dx.$$

Under the Segal isomorphism, the complex Hilbert space $L^2(Q_\Lambda, \mu_F^\Lambda)$ is identified with the Fock space $\mathfrak{F}(\mathfrak{h}_\Lambda)$ and the constant function 1 on Q_Λ is identified or with the identity $\mathbb{I}$ operator, when considered as the unit of $L^\infty(Q_\Lambda, \mu_F^\Lambda)$, or with the vacuum vector Ω, when considered as an element of $L^2(Q_\Lambda, \mu_F^\Lambda)$.

We shall make use of the particular realization of the Gaussian random process where Q_Λ is the infinite product of the one-point compactification of the real line $Q_\Lambda := \Pi_{n=1}^\infty \dot{\mathbb{R}}$ and where the Gaussian measure μ_F^Λ is the infinite product of copies of the Gaussian probability measure on $\dot{\mathbb{R}}$

$$\gamma(dx) := \pi^{-\frac{1}{2}} e^{-x^2} dx .$$

Choosing an orthonormal basis $\{f_n : n \geq 1\} \subset \mathfrak{h}_\Lambda^{\mathbb{R}}$, the field operator $\Phi(f_n)$ is identified with the multiplication operator $\phi(f_n)$ on $L^2(Q_\Lambda, \mu_F^\Lambda)$

$$(\phi(f_n)g)(x_1, \dots) = x_n g(x_1, \dots), \qquad (x_1, \dots) \in Q_\Lambda, \qquad g \in L^2(Q_\Lambda, \mu_F^\Lambda).$$

Notice that, using the Segal isomorphism, the relative entropy $H_{\mathfrak{M}_\Lambda}(\omega_1, \omega_2)$ of restrictions to the abelian von Neumann algebra $\mathfrak{M}_\Lambda$ of states ω_1 , ω_2 of the CCR algebra $\mathfrak{U}(\mathfrak{h}_\Lambda)$, appears as

$$H_{\mathfrak{M}_\Lambda}(\omega_2, \omega_1) = \int_{Q_\Lambda} d\mu_2 \ln\Big(\frac{d\mu_2}{d\mu_1}\Big)$$

in terms of the probability measures μ_1, μ_2 on Q_Λ representing ω_1, ω_2 restricted to $\mathfrak{M}_\Lambda$, provided μ_2 is absolutely continuous with respect to μ_1.

We shall denote by

$$\omega_\beta^\Lambda(A) := \frac{\mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda} A)}{\mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda})} \tag{1}$$

the Gibbs grand canonical equilibrium state, at inverse temperature $\beta > 0$ and activity $\mu < \inf \sigma(H^\Lambda)$, over the CCR algebra $\mathfrak{U}(h_\Lambda)$, corresponding to the second quantization Hamiltonian $K_\mu^\Lambda := d\Gamma(H^\Lambda - \mu\mathbb{I})$ on $\mathfrak{F}(\mathfrak{h}_\Lambda)$ of the one-particle Hamiltonian $H^\Lambda - \mu\mathbb{I}$ on $L^2(\Lambda)$ [2, 5.2.5]. Concerning the existence of the Gibbs state above, notice that, since Λ is bounded then $e^{-\beta H^\Lambda}$ is trace class for any $\beta > 0$ and consequently, by [2] Proposition 5.2.27, $e^{-\beta K_\mu^\Lambda}$ is trace class too for any $\beta > 0$ (and in fact for any real μ). We shall denote by $N^\Lambda := d\Gamma(\mathbb{I})$ the number operator on $\mathfrak{F}(\mathfrak{h}_\Lambda)$. For a unit vector $\psi \in \mathfrak{F}(\mathfrak{h}_\Lambda)$, we shall denote by ω_ψ the corresponding vector state on $\mathfrak{U}(\mathfrak{h}_\Lambda)$, as well as its restriction to $\mathfrak{M}_\Lambda$.

The first step to the main result of the work is the following observation.

Lemma 1 (Free Energy Variation, Gibbs State and Relative Entropy) *Denote by N the von Neumann algebra $\mathscr{B}(\mathfrak{F}(\mathfrak{h}_\Lambda))$ of all bounded operators on the Fock space. On its normal state space $N_{*,1}$, identified with the space of nonnegative trace class operators ρ such that* $\mathrm{Tr}\,(\rho) = 1$ *(called density matrices), define the energy functional*

$$E : N_{*,1} \to [0, +\infty], \qquad E(\rho) := \mathrm{Tr}\,(\rho^{1/2} K_\mu^\Lambda \rho^{1/2}), \qquad \rho \in N_{*,1},$$

the von Neumann entropy functional

$$S_N : N_{*,1} \to [0, +\infty], \qquad S_N(\rho) := -\mathrm{Tr}\,(\rho \ln \rho), \qquad \rho \in N_{*,1},$$

and the Helmholtz free energy functional at inverse temperature $\beta > 0$

$$F_\beta : N_{*,1} \to [0, +\infty], \qquad F_\beta(\rho) := E(\rho) - \frac{1}{\beta} S(\rho), \qquad \rho \in N_{*,1}\,.$$

The free energy functional attains its minimum value $F_\beta(\rho_\beta) = -\beta^{-1} \ln \mathrm{Tr}\,(e^{-\beta K^\Lambda_\mu})$ *at the Gibbs state* ω^Λ_β*, represented by the density matrix* $\rho_\beta := e^{-\beta K^\Lambda_\mu} / \mathrm{Tr}\,(e^{-\beta K^\Lambda_\mu})$*. Moreover, the variation of the free energy with respect to the Gibbs state, is proportional by* β*, to the relative entropy* H_N *of the states*

$$0 \le H_N(\rho, \rho_\beta) = \beta(F(\rho) - F(\rho_\beta)), \qquad \rho \in N_{*,1}\,. \tag{2}$$

Proof We may assume that $\beta = 1$ and that $\mathrm{Tr}\,(e^{-\beta K^\Lambda_\mu}) = 1$. By the cyclicity of the trace

$$\begin{aligned} F_1(\rho_1) = E(\rho_1) - S_N(\rho_1) &= \mathrm{Tr}\,(\rho_1^{1/2} K^\Lambda_\mu \rho_1^{1/2}) + \mathrm{Tr}\,(\rho_1 \ln \rho_1) \\ &= -\mathrm{Tr}\,(\rho_1^{1/2} (\ln e^{-K^\Lambda_\mu}) \rho_1^{1/2}) + \mathrm{Tr}\,(\rho_1 \ln \rho_1) \\ &= -\mathrm{Tr}\,(\rho_1^{1/2} (\ln \rho_1) \rho_1^{1/2}) + \mathrm{Tr}\,(\rho_1 \ln \rho_1) \\ &= -\mathrm{Tr}\,(\rho_1 \ln \rho_1) + \mathrm{Tr}\,(\rho_1 \ln \rho_1) = 0 \end{aligned}$$

and, for all $\rho \in N_{*,1}$, by the definition of the relative entropy H_N (see [16]) we have

$$\begin{aligned} F_1(\rho) = E(\rho) - S_N(\rho) &= \mathrm{Tr}\,(\rho^{1/2} K^\Lambda_\mu \rho^{1/2}) + \mathrm{Tr}\,(\rho \ln \rho) \\ &= -\mathrm{Tr}\,(\rho^{1/2} (\ln \rho_1) \rho^{1/2}) + \mathrm{Tr}\,(\rho \ln \rho) \\ &= \mathrm{Tr}\,(\rho^{1/2} (\ln \rho - \ln \rho_1) \rho^{1/2}) \\ &= H_N(\rho, \rho_1)\,. \end{aligned}$$

The second step in the proof of the our main result is the following fundamental property.

Theorem 1 (Relative Entropy Monotonicity, [16] Theorem 4') *Let* $\omega_1\,, \omega_2$ *be normal states on* $N := \mathscr{B}(\mathfrak{F}(\mathfrak{h}_\Lambda))$ *and* ω'_1, ω'_2 *their restriction to the von Neumann subalgebra* $\mathfrak{M}_\Lambda$*. Denoting by* H_N *and* $H_{\mathfrak{M}_\Lambda}$*, the relative entropy on* N *and* $\mathfrak{M}_\Lambda$*, respectively, one has*

$$H_{\mathfrak{M}_\Lambda}(\omega'_1, \omega'_2) \le H_N(\omega_1, \omega_2)\,. \tag{3}$$

More explicitly, if ρ_1, ρ_2 *are the density matrices representing* ω_1, ω_2 *and* μ_1, μ_2 *are the probability measures on* Q_Λ *representing the restrictions* ω_1', ω_2'*, then one has*

$$\int_{Q_\Lambda} d\mu_2 \Big(\frac{d\mu_1}{d\mu_2}\Big) \ln\Big(\frac{d\mu_1}{d\mu_2}\Big) \leq \mathrm{Tr}\,(\rho_1^{1/2}(\ln \rho_1 - \ln \rho_2)\rho_1^{1/2})\,. \tag{4}$$

The following is the main result of the work.

Theorem 2 *Let* $\Lambda \subset \mathbb{R}^d$ *be a bounded Euclidean domain and* H^Λ *be the Dirichlet-Laplacian operator on* $\mathfrak{h}_\Lambda := L^2(\Lambda)$*. Denote by* $K_\mu^\Lambda := d\Gamma(H^\Lambda - \mu\mathbb{I})$ *its second quantization on the Fock space* $\mathfrak{F}(\mathfrak{h}_\Lambda)$*, with activity* $\mu < \inf\sigma(H^\Lambda)$*.*

Then the following logarithmic Sobolev inequalities hold true for any $\beta > 0$ *and* $\|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi, \omega_F^\Lambda) \leq \beta(\psi, K_\mu^\Lambda\psi) + \ln \mathrm{Tr}\, e^{-\beta K_\mu^\Lambda} + 4d(\beta,\mu)(\psi, N^\Lambda\psi) + d(\beta,\mu) \tag{5}$$

where $z := e^{\beta\mu}$ *and* $d(\beta,\mu) := \mathrm{Tr}\,(ze^{-\beta H^\Lambda}(\mathbb{I} + ze^{-\beta H^\Lambda})^{-1})$*. In terms of the free energy of the system the inequality reads as follows*

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi, \omega_F^\Lambda) \leq \beta(F(\omega_\psi) - F(\omega_\beta^\Lambda)) + 4d(\beta,\mu)(\psi, N^\Lambda\psi) + d(\beta,\mu)\,. \tag{6}$$

Notice that, when $\mathfrak{M}_\Lambda$ is identified with $L^\infty(Q_\Lambda, d\mu_F^\Lambda)$, we have

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi, \omega_F^\Lambda) = \int_{Q_\Lambda} d\mu_F^\Lambda |\psi|^2 \ln|\psi|^2, \qquad \|\psi\|_{L^2(Q_\Lambda, \mu_F^\Lambda)} = 1\,.$$

Notice also that, by a classical result [4, 1.9], the following bound holds true

$$d(\beta,\mu) \leq z(1 + ze^{-\beta\lambda_0})^{-1}(4\pi\beta)^{-d/2}|\Lambda|, \qquad \beta > 0\,.$$

Proof Denoting by μ and μ_β the probability measures on Q_Λ representing the restriction to $\mathfrak{M}_\Lambda \simeq L^\infty(Q_\Lambda, \mu_F)$ of the normal states represented by the density matrix ρ and ρ_β, by Uhlmann's monotonicity theorem [16] or Theorem 1 above, we have

$$H_{\mathfrak{M}_\Lambda}(\mu, \mu_\beta) \leq H_N(\rho, \rho_\beta), \qquad \rho \in N_{*,1}$$

so that, by Lemma 1 above, the following inequality holds true

$$H_{\mathfrak{M}_\Lambda}(\mu, \mu_\beta) \leq \beta(F(\rho) - F(\rho_\beta)), \qquad \rho \in N_{*,1}\,.$$

The density matrix ρ_ψ representing a vector state ω_ψ is the orthogonal projection onto the subspace generated by ψ. On it the von Neumann entropy vanishes $S(\rho_\psi) = 0$ and the value of the energy functional is given by $E(\rho_\psi) =$

$\mathrm{Tr}\,(\rho_\psi^{1/2} K_\mu^\Lambda \rho_\psi^{1/2}) = (\psi, K_\mu^\Lambda \psi)$ so that $F(\rho_\psi) = \beta(\psi, K_\mu^\Lambda \psi)$. Denoting by $\mu_\psi = |\psi|^2 \cdot \mu_F$ the probability measure on Q_Λ representing the restriction to $\mathfrak{M}_\Lambda$ of the state represented by ρ_ψ, we obtain the following logarithmic Sobolev inequality with respect to the Gaussian measure μ_β

$$H_{\mathfrak{M}_\Lambda}(\mu_\psi, \mu_\beta) \le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}), \qquad \|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$$

which can be written as

$$\int_{Q_\Lambda} d\mu_\psi \ln\Big(\frac{d\mu_\psi}{d\mu_\beta}\Big) \le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}), \qquad \|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$$

and as

$$\int_{Q_\Lambda} d\mu_\beta \Big(\frac{d\mu_\psi}{d\mu_\beta}\Big) \ln\Big(\frac{d\mu_\psi}{d\mu_\beta}\Big) \le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}), \qquad \|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1\,.$$

Since

$$\frac{d\mu_\psi}{d\mu_\beta} = |\psi|^2 \frac{d\mu_F^\Lambda}{d\mu_\beta}$$

we have, for $\|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$,

$$\begin{aligned}
\int_{Q_\Lambda} d\mu_F^\Lambda \Big(\frac{d\mu_\beta}{d\mu_F^\Lambda}\Big)\Big(\frac{d\mu_\psi}{d\mu_\beta}\Big) \ln\Big(\frac{d\mu_\psi}{d\mu_\beta}\Big) &\le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}) \\
\int_{Q_\Lambda} d\mu_F^\Lambda \Big(\frac{d\mu_\psi}{d\mu_F^\Lambda}\Big) \ln\Big(\frac{d\mu_\psi}{d\mu_\beta}\Big) &\le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}) \\
\int_{Q_\Lambda} d\mu_F^\Lambda \Big(\frac{d\mu_\psi}{d\mu_F^\Lambda}\Big) \ln\Big(\frac{d\mu_\psi}{d\mu_F^\Lambda}\frac{d\mu_F^\Lambda}{d\mu_\beta}\Big) &\le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}) \\
\int_{Q_\Lambda} d\mu_F^\Lambda |\psi|^2 \ln\Big(|\psi|^2 \frac{d\mu_F^\Lambda}{d\mu_\beta}\Big) &\le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}) \\
\int_{Q_\Lambda} d\mu_F^\Lambda |\psi|^2 \ln |\psi|^2 &\le \beta(\psi, K_\mu^\Lambda \psi) + \ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}) + \\
&\quad + \int_{Q_\Lambda} d\mu_F^\Lambda |\psi|^2 \ln\Big(\frac{d\mu_\beta}{d\mu_F^\Lambda}\Big)
\end{aligned} \tag{7}$$

provided we show that measure associated to the Gibbs state is absolutely continuous with respect to the one associated to the Fock vacuum state.

Since $e^{-\beta H^{\Lambda}}$ is trace class on $\mathfrak{h}_{\Lambda}$, by Proposition 5.2.7 and Theorem 5.2.8 in [2], the operator $e^{-\beta K_{\mu}^{\Lambda}}$ is trace class over the Fock space $\mathfrak{F}(\mathfrak{h}_{\Lambda})$ and the Gibbs grand canonical equilibrium state ω_{β}^{Λ} is a gauge-invariant quasi-free state over the CCR algebra $\mathfrak{U}(\mathfrak{h}_{\Lambda})$ with two-point function

$$\omega_{\beta}^{\Lambda}(a^{*}(f)a(g)) = (g, T_{\beta,\mu}^{\Lambda} f), \qquad f, g \in \mathfrak{h}_{\Lambda}$$

where $T_{\beta,\mu}^{\Lambda} := ze^{-\beta H^{\Lambda}}(\mathbb{I} - ze^{-\beta H^{\Lambda}})^{-1}$ and $z := e^{\beta\mu}$. Since $\omega_{F}^{\Lambda}(W(f)) = e^{-\frac{1}{4}\|f\|^{2}_{\mathfrak{h}_{\Lambda}}}$, by Example 5.2.18 in [2] Example 5.2.18 we have that the two-point function of the Fock vacuum state vanishes identically so that the operator T_{F}^{Λ} defined by its two-point function vanishes too

$$0 = \omega_{F}^{\Lambda}(a^{*}(f)a(g)) =: (g, T_{F}^{\Lambda} f), \qquad f, g \in \mathfrak{h}_{\Lambda}\,.$$

Since $T_{\beta,\mu}^{\Lambda} \leq ze^{-\beta H^{\Lambda}}(\mathbb{I} - ze^{-\beta\lambda_{0}})^{-1}$ then $T_{\beta,\mu}^{\Lambda}$ is a trace class operator and

$$\sqrt{T_{\beta,\mu}^{\Lambda}} - \sqrt{T_{F}^{\Lambda}} = \sqrt{T_{\beta,\mu}^{\Lambda}}$$

is an Hilbert-Schmidt operator. By [1], main Theorem p. 285, the state ω_{β}^{Λ} is quasi-equivalent to the Fock vacuum state ω_{F}^{Λ} in the sense that they have quasi-equivalent GNS representation and thus give rise to the same (abelian) von Neumann algebra $\mathfrak{M}_{\Lambda}$ which can be identified with $L^{\infty}(Q_{\Lambda}, \mu_{F}^{\Lambda})$. We thus have the mutual absolute continuity of the Gaussian measures μ_{β}^{Λ} and μ_{F}^{Λ} on Q_{Λ} representing the states ω_{β}^{Λ} and ω_{F}^{Λ}. By [12], Theorem 3, the Radon-Nikodym derivative $d\mu_{\beta}^{\Lambda}/d\mu_{F}^{\Lambda}$ is given by

$$\frac{d\mu_{\beta}^{\Lambda}}{d\mu_{F}^{\Lambda}}(f) = (\det(A))^{-1/2}\exp[(f, (\mathbb{I} - A^{-1})f)], \qquad f \in \mathfrak{h}_{\Lambda}\,,$$

where

$$A := \frac{\mathbb{I} + ze^{-\beta H^{\Lambda}}}{\mathbb{I} - ze^{-\beta H^{\Lambda}}}\,, \qquad \mathbb{I} - A^{-1} = \mathbb{I} - \frac{\mathbb{I} - ze^{-\beta H^{\Lambda}}}{\mathbb{I} + ze^{-\beta H^{\Lambda}}} = \frac{2ze^{-\beta H^{\Lambda}}}{\mathbb{I} + ze^{-\beta H^{\Lambda}}}\,,$$

provided we show that $\det A$ is well defined. In fact, since

$$0 \leq A - \mathbb{I} = \frac{2ze^{-\beta H^{\Lambda}}}{\mathbb{I} - ze^{-\beta H^{\Lambda}}} \leq \frac{2e^{\beta\mu}}{1 - e^{-\beta(\lambda_{0}-\mu)}}e^{-\beta H^{\Lambda}}\,,$$

the trace class property of $e^{-\beta H^{\Lambda}}$ implies the same property for $A - \mathbb{I}$ and then

$$\det A \leq e^{\mathrm{Tr}\,(A-\mathbb{I})} < +\infty\,.$$

In particular

$$\ln \frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda}(f) = -\frac{1}{2}\ln\det A + (f, (\mathbb{I} - A^{-1})f)$$
$$= -\frac{1}{2}\ln\det A + (f, 2ze^{-\beta H^\Lambda}(\mathbb{I} + ze^{-\beta H^\Lambda})^{-1}f)\,.$$

Recall now that, in the model where the space Q_Λ of the Gaussian random process associated to $\mathfrak{h}_\Lambda$ is identified with the infinite product of the one-point compactification of the real line $Q_\Lambda := \Pi_{n=1}^\infty \dot{\mathbb{R}}$, the logarithm of the Radon-Nikodym derivative above is the random variable which associates to $(x_1, x_2, \ldots) \in Q_\Lambda$ the value

$$\ln \frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda}(x_1, x_2, \ldots) = -\frac{1}{2}\ln\det A + \sum_{n=1}^\infty 2ze^{-\beta\lambda_n}(1 + ze^{-\beta\lambda_n})^{-1}x_n^2\,.$$

If we choose as a basis for $\mathfrak{h}_\Lambda$ the normalized eigenfunctions $\{f_n \in \mathfrak{h}_\Lambda : n \geq 1\}$ of H^Λ corresponding to the eigenvalues $\{\lambda_n \in (0, +\infty) : n \geq 1\}$, $H^\Lambda f_n = \lambda_n f_n$, then the self-adjoint operator on the Fock space corresponding to the real random variable $\ln d\mu_\beta^\Lambda / d\mu_F^\Lambda$ is given by

$$\ln \frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda} = -\frac{1}{2}\Big(\ln\det A\Big)\mathbb{I} + \sum_{n=1}^\infty 2ze^{-\beta\lambda_n}(1 + ze^{-\beta\lambda_n})^{-1}\phi(f_n)^2\,.$$

Since by [2] Lemma 5.2.12

$$\phi(f_n)^2 \leq 2a^*(f_n)a(f_n) + \mathbb{I} = 2N(f_n) + \mathbb{I}\,,$$

we have

$$\ln \frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda} \leq -\frac{1}{2}\Big(\ln\det A\Big)\mathbb{I} + \sum_{n=1}^\infty 2ze^{-\beta\lambda_n}(1 + ze^{-\beta\lambda_n})^{-1}(2N(f_n) + \mathbb{I})\,.$$

Since moreover

$$\ln\det A = \sum_{n=1}^\infty \ln\Big(\frac{1 + ze^{-\beta\lambda_n}}{1 - ze^{-\beta\lambda_n}}\Big)\,,$$

setting

$$b(\beta,\mu) := \sum_{n=1}^{\infty} \frac{2ze^{-\beta\lambda_n}}{1+ze^{-\beta\lambda_n}} + \frac{1}{2}\ln\Big(1-\frac{2ze^{-\beta\lambda_n}}{1+ze^{-\beta\lambda_n}}\Big),$$

$$c(\beta,\mu) := 4\sum_{n=1}^{\infty} ze^{-\beta\lambda_n}(1+ze^{-\beta\lambda_n})^{-1}$$

we have

$$\ln\frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda} \le b(\beta,\mu)\mathbb{I} + c(\beta,\mu)N\,.$$

If $d(\beta,\mu) := \sum_{n=1}^{\infty} ze^{-\beta\lambda_n}(1+ze^{-\beta\lambda_n})^{-1}$ then $b(\beta,\mu) \le d(\beta,\mu)$ and $c(\beta,\mu) = 4d(\beta,\mu)$ so that

$$\ln\frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda} \le d(\beta,\mu)\mathbb{I} + 4d(\beta,\mu)N\,. \tag{8}$$

From the *intrinsic logarithmic Sobolev inequality* (2.8) for the operator K_μ^Λ on the Gaussian space $L^2(Q_\Lambda,\mu_F^\Lambda)$ obtained above, we have

$$\int_{Q_\Lambda} d\mu_F^\Lambda\,|\psi|^2\ln|\psi|^2 \le \beta(\psi,K_\mu^\Lambda\psi) + \ln\operatorname{Tr}e^{-\beta K_\mu^\Lambda} + \int_{Q_\Lambda} d\mu_F^\Lambda\,|\psi|^2\ln\frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda} \tag{9}$$

for $\|\psi\|_{L^2(Q_\Lambda,\mu_F^\Lambda)} = 1$, which, on the Fock space, reads as follows

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi,\omega_F^\Lambda) \le \beta(\psi,K_\mu^\Lambda\psi) + \ln\operatorname{Tr}e^{-\beta K_\mu^\Lambda} + (\psi,\ln\frac{d\mu_\beta^\Lambda}{d\mu_F^\Lambda}\psi) \tag{10}$$

for $\|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$. By the bound (2.8) above we have the desired logarithmic Sobolev inequalities (2.5)

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi,\omega_F^\Lambda) \le \beta(\psi,K_\mu^\Lambda\psi) + \ln\operatorname{Tr}e^{-\beta K_\mu^\Lambda} + 4d(\beta,\mu)(\psi,N^\Lambda\psi) + d(\beta,\mu)$$

for $\|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$.

Corollary 1 *There exists $\beta_0 > 0$ depending on $0 < \mu < \lambda_0$ such that the following logarithmic Sobolev inequalities hold true for all $\beta \ge \beta_0$*

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi,\mu_F^\Lambda) \le \beta(\psi,d\Gamma(H^\Lambda)\psi) + \ln\operatorname{Tr}e^{-\beta K_\mu^\Lambda} + z\operatorname{Tr}(e^{-\beta H^\Lambda}), \tag{11}$$

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi,\mu_F^\Lambda) \le \beta(\psi,d\Gamma(H^\Lambda)\psi) + \frac{z}{1-ze^{-\beta\lambda_0}}\operatorname{Tr}(e^{-\beta H^\Lambda}) \tag{12}$$

with $\|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$.

Proof Since $K_\mu^\Lambda = d\Gamma(H^\Lambda - \mu\mathbb{I}) = d\Gamma(H^\Lambda) - \mu N$, from the theorem above we have

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi, \omega_F^\Lambda) \leq$$
$$\leq \beta(\psi, d\Gamma(H_\Lambda)\psi) + \ln \mathrm{Tr}\, e^{-\beta K_\mu^\Lambda} + d(\beta,\mu) + (4d(\beta,\mu) - \beta\mu)(\psi, N\psi)$$

for all $\|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1$. Since $d(\beta,\mu) \leq z\mathrm{Tr}\,(e^{-\beta H^\Lambda})$ and $d(\beta,\mu)$ is decreasing to 0 as β increase to $+\infty$, there exists $\beta_0 > 0$ such that $4d(\beta,\mu) - \beta\mu \leq 0$ for all $\beta \geq \beta_0$ and we get (2.11). Finally, by Proposition 5.2.27 in [2] we have

$$\ln \mathrm{Tr}\,(e^{-\beta K_\mu^\Lambda}) \leq z(1 - ze^{-\beta\lambda_0})^{-1}\mathrm{Tr}\,(e^{-\beta H^\Lambda})$$

so that

$$\ln \mathrm{Tr}\, e^{-\beta K_\mu^\Lambda} + z\mathrm{Tr}\,(e^{-\beta H^\Lambda}) \leq [z + z(1 - ze^{-\beta\lambda_0})^{-1}]\mathrm{Tr}\,(e^{-\beta H^\Lambda}) \leq$$
$$\leq \frac{z}{1 - ze^{-\beta\lambda_0}}\mathrm{Tr}\,(e^{-\beta H^\Lambda})$$

from which (2.12) follows.

Corollary 2 *The semigroup* $\{e^{-\beta d\Gamma(H^\Lambda)} : \beta > 0\}$ *is hypercontractive, i.e. it is Markovian in the sense that it is positivity preserving and contractive on* $L^p(Q_\Lambda, \mu_F^\Lambda)$ *for any* $p \in [0, +\infty]$ *and* $e^{-\beta_0 H^\Lambda}$ *is bounded from* $L^2(Q_\Lambda, \mu_F^\Lambda)$ *to* $L^4(Q_\Lambda, \mu_F^\Lambda)$. *In particular, the following logarithmic Sobolev inequality holds true for some* $\beta_h > \beta_0$

$$H_{\mathfrak{M}_\Lambda}(\omega_\psi, \mu_F^\Lambda) \leq \beta_h(\psi, d\Gamma(H^\Lambda)\psi), \qquad \|\psi\|_{\mathfrak{F}(\mathfrak{h}_\Lambda)} = 1\,. \tag{13}$$

Proof Since $\beta H^\Lambda \geq 0$ for all $\beta > 0$, then $e^{-\beta d\Gamma(H^\Lambda)} = d\Gamma(e^{-\beta H^\Lambda})$ is positive preserving (see [11]). Since, by construction, $e^{-\beta d\Gamma(H^\Lambda)}\Omega = \Omega$ for all $\beta > 0$, the semigroup is also contractive on $\mathfrak{M}_\Lambda \simeq L^\infty(Q_\Lambda, \mu_F^\Lambda)$, hence Markovian.

Fix now $0 < \mu < \lambda_0$ and consider the value β_0 determined in Corollary 2. Since, by construction, the spectrum of $d\Gamma(H^\Lambda)$ is discrete, $0 = \inf\sigma(d\Gamma(H^\Lambda))$ and the logarithmic Sobolev inequality (2.12) holds true, the stated results follow from Theorem 6.1.22 ii) in [5].

Acknowledgements This work has been supported by GREFI-GENCO INDAM Italy-CNRS France and MIUR PRIN 2012 Project No 2012TC7588-003.

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