Dissipatively Generated Entanglement

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Abstract Given two non-interacting 2-level systems weakly coupled to an environment and thus evolving according to a statistically mixing dissipative reduced dynamics, we provide necessary and sufficient conditions for the generator of the time-evolution to entangle the two systems.

Keywords Entanglement • Markovian quantum evolutions • Open quantum systems

1 Introduction

Quantum systems always interact with the environment in which they are immersed; when the coupling to the environment is negligible, they evolve reversibly. Otherwise, when the interaction with the environment is weak, but cannot be neglected, quantum systems are called *open* [1, 2]. The weakness of the interaction allows one to derive a *reduced dynamics* that describes the noisy and dissipative effects due to the presence of the environment after it has been eliminated by tracing out its degrees of freedom. Usually, this operation yields an irreversible time-evolution characterised by memory effects that can be eliminated by suitable Markovian approximations that lead to master equations of the form

$$\partial_t \varrho_t = \mathbb{L}[\varrho_t] \, , \tag{1}$$

for all $t \geq 0$, where $\mathbb{L}$ is a time-independent generator. Assuming the system to be a d-level system, then $\varrho_t \in M_d(\mathbb{C})$ must be a (positive and normalized) $d \times d$ density matrix describing the state of the open quantum system at time $t \geq 0$.

Therefore, the dynamical maps $\Lambda_t = \mathrm{e}^{t\mathbb{L}}$ generated by (1) must preserve the positivity of any initial ϱ, so that the eigenvalues of $\varrho_t = \Lambda_t[\varrho] \geq 0$ might be interpretable as probabilities at all times $t \geq 0$; namely, Λ_t must be positivity

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preserving, positive in short, for all $t \geq 0$. This condition is necessary but not sufficient to ensure the full physical consistency of Λ_t; indeed, one can always statistically couple an open quantum system S with another inert d-level system S, so that the states $\varrho \in M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ of the compound bipartite system $S+S$ would evolve under the action of a fully admissible dynamical map $\Lambda_t \otimes \mathrm{id}$. If the states ϱ were all of the form

$$\rho_{sep} = \sum_k \lambda_k \rho_k^{(1)} \otimes \rho_k^{(2)} , \quad \lambda_k \geq 0 , \sum_k \lambda_k = 1 , \tag{2}$$

with their correlations only due to the mixing with weights λ_k of the uncorrelated tensor products of constituent system states $\rho_k^{(1)} \otimes \rho_k^{(2)}$, the positivity of Λ_t would clearly be sufficient to guarantee that $\Lambda_t \otimes \mathrm{id}[\rho_{sep}] \geq 0$. However, not all bipartite states are expressible in a separable form as in (2): those which cannot are called *entangled* [3]. It turns out that, when Λ_t is positive, but not completely positive, there surely exists an entangled state $\varrho_{ent} \in M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ such that $\Lambda_t \otimes \mathrm{id}[\varrho_{ent}]$ assumes negative eigenvalues in the course of time [4]. Summarizing, complete positivity of Λ_t is necessary (and sufficient) to guarantee that both Λ_t and $\Lambda_t \otimes \mathrm{id}$ be positivity preserving and thus physically consistent.

In the Markovian case, the dynamical maps Λ_t are completely positive if and only if the generator is of the so-called Gorini-Kossakowski-Sudarshan-Lindblad form [5, 6]

$$\mathbb{L}[\varrho_t] = -i[H\,,\,\varrho_t] + \sum_{j=1}^{d^2-1} K_{ij}\Big(F_i\,\varrho_t\,F_j^\dagger - \frac{1}{2}\Big\{F_j^\dagger F_i\,,\,\varrho_t\Big\}\Big)\,, \tag{3}$$

with traceless matrices such that $\{F_j\}_{j=1}^{d^2-1}$, $\mathrm{Tr}(F_i^\dagger F_j) = \delta_{ij}$, which, together with $F_{d^2} = 1/\sqrt{d}$, constitute an orthonormal basis in $M_d(\mathbb{C})$ and the $(d^2-1) \times (d^2-1)$ matrix $K = [K_{ij}]$, known as *Kossakowski matrix*, being positive semi-definite.

Markovian semigroups of completely positive maps are used to describe decoherence processes detrimental to the persistence of non-classical correlations, like entanglement, and to their use to perform classically impossible informational tasks like teleportation and quantum cryptography [7]. However, not always the presence of an environment is negative; sometimes, it is also possible to engineer the environment in such a way that two non-directly interacting systems immersed in it become entangled [8–10].

For two 2-level systems, a sufficient condition for such a possibility to occur was provided in [9] in the case of a purely dissipative generator of the form

$$\mathbb{L}[\varrho_t] = -i[H\,,\,\varrho_t] + \sum_{j,k=1}^{6} K_{j,k}\Big(S_j\,\varrho_t\,S_k - \frac{1}{2}\Big\{S_k S_j\,,\,\varrho_t\Big\}\Big)$$

$$H = \sum_{j=1} H_j\,S_j\,, \qquad H_j = H_j^*\,,$$

where $S_j = \sigma_j \otimes \mathbf{1}$ for $j = 1, 2, 3$, $S_j = \mathbf{1} \otimes \sigma_{j-3}$ for $j = 4, 5, 6$, with $\sigma_{1,2,3}$ the Pauli matrices and $\mathbf{1}$ the identity 2×2 matrix, and the hermitean Kossakowski matrix $K = [K_{jk}]$ is positive semi-dedefinite. Notice the absence in the above generator of operators pertaining simultaneously to the two qubits like $\sigma_i \otimes \sigma_j$. Then, the emergence of entanglement during the time-evolution may only be due to the mixing properties of the dissipation and not to dynamical effects.

In the following, we provide necessary and sufficient conditions for the above generator to create entanglement by focussing on just one part of the generator and proving the following result.

Theorem 1 *Let two* 2*-level systems immersed in a common environment evolve according to a master equation* $\partial_t \varrho_t = \mathbb{L}[\varrho_t]$ *generated by* $\mathbb{L}$ *as in (5). Given an initially separable state* ϱ_{sep}*, the generated dynamical maps* Λ_t *on* $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ *turns it into an entangled state, if and only if so does the dynamics generated by*

$$\mathbb{Z}[\varrho] = -i\left(H - \frac{i}{2}\Gamma\right)\varrho + i\,\varrho\left(H + \frac{i}{2}\Gamma\right)$$

$$\Gamma = \sum_{j,k=1}^{6} K_{jk}\, S_k S_j \geq 0 \, .$$

2 Dissipative Entanglement Generation

The simplest introduction to the notion of entanglement is by means of two 2-level systems, or in the jargon of quantum information, by systems consisting of two qubits. We shall denote by $\{|i\rangle\}_{i=0}^{1}$ the orthonormal basis of the eigenvectors of σ_3 in the single qubit Hilbert space $\mathbb{C}^2$: $\sigma|i\rangle = (-)^i|i\rangle$.

Then, two qubit vector states $|\Psi_{12}\rangle \in \mathbb{C}^4$ are entangled if they cannot be written as tensor products $|\psi\rangle \otimes |\phi\rangle$ of single qubit vector states, the prototype of such states being the so-called symmetric state

$$|\Psi_+\rangle = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}} \, . \qquad (4)$$

Entanglement as a property of quantum states is strictly related to positive, but not completely positive maps on quantum observables [4, 11], the prototype of such maps being the transposition map $\mathbb{T}$ (defined with respect to the chosen representation). Indeed, the latter is a positive map as it does not alter the spectrum of the matrices on which it acts; however, the partial transposition $\mathbb{T} \otimes \mathrm{id}$, transposing only the first factor of a bipartite tensor product of operators, fails to be positive.

Indeed, the symmetric projector $P_+ = |\Psi_+\rangle\langle\Psi_+|$ changes from

$$P_+ = \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0|\Big)$$

into

$$(\mathbb{T} \otimes \mathrm{id})[P_+] \\ = \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0|\Big)$$

which has the anti-symmetric state

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\Big)$$

as eigenvector relative to the negative eigenvalue $-1/2$. Therefore, though $\mathbb{T}$ is a sensible, positivity preserving map on single qubits, its so-called lifting $\mathbb{T} \otimes \mathrm{id}$ fails to be such when acting on systems consisting of two qubits due to the existence of entangled states. In practice, transposition acts as a witness for the entanglement of P_+; actually for two qubits $\mathbb{T}$ is an exhaustive entanglement witness [12].

Theorem 2 *A state ϱ in $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is entangled if and only if it does not remain positive semi-definite under partial transposition, namely if and only if*

$$(\mathbb{T} \otimes \mathrm{id})[\varrho] \ngeq 0 \ .$$

The issue at stake in the following is the role of the dissipative part of the generator in Theorem 1,

$$\mathbb{L}[\varrho_t] = -i[H\,,\,\varrho_t] + \sum_{j,k=1}^{6} K_{jk}\Big(S_j\,\varrho_t\,S_k - \frac{1}{2}\Big\{S_k S_j\,,\,\varrho_t\Big\}\Big) \tag{5}$$

$$H = \sum_{j=1}^{6} H_j\,S_j\ , \qquad H_j = H_j^*\ , \tag{6}$$

with $S_j = \sigma_j \otimes \mathbf{1}$ for $j = 1, 2, 3$ and $S_j = \mathbf{1} \otimes \sigma_{j-3}$ for $j = 4, 5, 6$, in transforming an initial separable state (2) into an entangled state.

Notice that the Hamiltonian part splits into two terms acting independently on the two qubits and cannot thus entangle them, being thus only the dissipative contribution that can achieve it.

The generator can be subdivided into two terms: the first one, $\mathbb{Z}$, consists of a pseudo-commutator

$$\mathbb{Z}[\varrho] = -i\left(H - \frac{i}{2}\Gamma\right)\varrho \; + i\,\varrho\left(H + \frac{i}{2}\Gamma\right) \tag{7}$$

$$\Gamma = \sum_{j,k=1}^{6} K_{jk}\, S_k\, S_j \;, \tag{8}$$

with respect to a non-hermitean Hamiltonian. Since $\Gamma \geq 0$ because the Kossakowski matirx $K \geq 0$, $\mathbb{Z}$ generates a damped quantum dynamics sending projections into non-normalized projections:

$$\mathrm{e}^{t\mathbb{Z}}[|\psi\rangle\langle\psi|] = \mathrm{e}^{-it(H-i\Gamma/2)}\,|\psi\rangle\langle\psi|\,\mathrm{e}^{it(H+i\Gamma/2)} \;. \tag{9}$$

As to the remaining contribution to the generator,

$$\mathbb{B}[\varrho] = \sum_{j,k=1}^{6} K_{jk}\, S_j\, \varrho\, S_k \;, \tag{10}$$

by using the spectral representation of the Kossakowski matrix $K = [K_{jk}] \geq 0$, it can be expressed in the standard Kraus-Stinespring form of completely positive maps

$$\mathbb{B}[\varrho] = \sum_{\ell=1}^{6} V_\ell\, \varrho\, V_\ell^\dagger \;. \tag{11}$$

Unlike the damping term, $\mathbb{B}$ transforms projectors into mixtures of projections, thus representing a so-called *noisy channel*[1] The standard lore has it that entanglement comes from mutual interactions between the qubits described by the Hamiltonian H, while the remaining dissipative contributions are responsible for its depletion in time due to damping and noise.

This conclusion is not always true: suitably engineered dissipative dynamics may lead to dissipatively generated entanglement even in absence of direct qubit interactions and this entanglement can also persist asymptotically in time [8–10, 13].

Regarding the generator (5), in [9] a sufficient condition for entanglement generation was provided that was related to the structure of the 6×6 Kossakowski matrix $C = [C_{jk}]$. In the following, we shall show that the actual source of entanglement might only be due to the pseudo-commutative contribution, the noise term being unable to counteract this fact.

[1]Notice that the trace is preserved since $\mathrm{Tr}(\mathbb{B}[\varrho] + \mathrm{Tr}(\mathbb{Z}[\varrho]) = 0$.

2.1 Checking Entanglement Generation

In order to ascertain whether $\Lambda_t = \mathrm{e}^{t\mathbb{L}}$ acting on an initially separable two qubit state may or not entangle it in the course of time, we base our strategy upon Theorem 2 and the following observations:

- since general separable density matrices are liner convex combination of pure separable states, one need just study the action of Λ_t on projections of the form $P_\psi \otimes P_\phi$;
- one need check whether there exist separable projections such that

$$(\mathbb{T} \otimes \mathrm{id}) \circ \Lambda_t[P_\psi \otimes P_\phi] \ngeq 0 ;$$

- one can focus upon very small times; indeed, in order to become negative an eigenvalue of $(\mathbb{T} \otimes \mathrm{id}) \circ \Lambda_t[P_\psi \otimes P_\phi]$ must first become zero at some time $t_* \geq 0$ and then < 0 at $t_* + \varepsilon$, for $\varepsilon > 0$ sufficiently small.

Then, the maps Λ_t are entanglement generating if and only if there exists a separable pure state projection $P_\psi \otimes P_\phi$ onto $|\psi\rangle \otimes |\phi\rangle$ such that, at first order in t,

$$(\mathbb{T} \otimes \mathrm{id}) \circ \Lambda_t[P_\psi \otimes P_\phi] \simeq P_{\psi^*} \otimes P_\phi + t\,(\mathbb{T} \otimes \mathrm{id}) \circ \mathbb{L}[P_\psi \otimes P_\phi] \tag{12}$$

is not positive semi-definite. Here, $|\psi^*\rangle$ is the conjugate of $|\psi\rangle$ with respect to the orthonormal basis where σ_3 is diagonal so that, under transposition, $\sigma_j^T = \epsilon_j\,\sigma_j$ with ϵ_j, $j = 1, 2, 3$, determined by

$$\sigma_1^T = \sigma_1 , \qquad \sigma_2^T = -\sigma_2 , \qquad \sigma_3^T = \sigma_3 . \tag{13}$$

For later use, we then introduce the following 3×3 diagonal matrix

$$\mathscr{E} := \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{14}$$

The lack of positive semi-definiteness of $(\mathbb{T} \otimes \mathrm{id}) \circ \Lambda_t[P_\psi \otimes P_\phi]$ can be studied by considering its expectation values with respect to an (entangled) pure state $|\Psi\rangle$ orthogonal to $|\psi^*\rangle \otimes |\phi\rangle$, so that

$$\langle\Psi|(\mathbb{T} \otimes \mathrm{id}) \circ \Lambda_t[P_\psi \otimes P_\phi]\Psi\rangle \simeq t\,\langle\Psi|(\mathbb{T} \otimes \mathrm{id}) \circ \mathbb{L}[P_\psi \otimes P_\phi]|\Psi\rangle =: \Delta(t) . \tag{15}$$

Remark 1 The vector $|\Psi\rangle$ must be entangled: if $P_\Psi = |\Psi\rangle\langle\Psi| = P_{\psi_1} \otimes P_{\psi_2}$, where $P_{\psi_1} = |\psi_1\rangle\langle\psi_1|$, $P_{\psi_2} = |\psi_2\rangle\langle\psi_2|$, by transferring the partial transposition from

$\Lambda_t[P_\psi \otimes P_\phi]$ onto P_Ψ, one would obtain

$$\begin{aligned}\langle \Psi|(\mathbb{T}\otimes \mathrm{id})\circ \Lambda_t[P_\psi \otimes P_\phi]\Psi\rangle &= \mathrm{Tr}\Big((\mathbb{T}\otimes \mathrm{id})[P_\Psi]\Lambda_t[P_\psi \otimes P_\phi]\Big)\\ &= \langle \psi_1^* \otimes \psi_2|\Lambda_t[P_\psi \otimes P_\phi]\psi_1^* \otimes \psi_2\rangle \geq 0\,,\end{aligned}$$

where $|\psi_1^*\rangle$ is the conjugate of $|\psi_1\rangle$ that comes from transposing P_{ψ_1} in the fixed representation.

The action of the partial transposition on the generator $\mathbb{L}$ is better understood by rewriting the 6×6 Kossakowski matrix $K = [K_{jk}] \geq 0$ as

$$K = \begin{pmatrix} A & B \\ B^\dagger & C\end{pmatrix} \tag{16}$$

with $A, B, C \in M_3(\mathbb{C})$ and A and C necessarily positive semi-definite, and then, recasting Γ as

$$\begin{aligned}\Gamma = &\sum_{j,k=1}^{3} A_{jk}\,\sigma_k\sigma_j \otimes \mathbf{1} + \sum_{j,k=1}^{3} C_{jk}\,\mathbf{1}\otimes \sigma_k\sigma_j\\ &+ \sum_{j,k=1}^{3} B_{jk}\,\sigma_j\otimes\sigma_k + \sum_{j,k=1}^{3} B^*_{kj}\,\sigma_k\otimes\sigma_j\,,\end{aligned} \tag{17}$$

and $\mathbb{B}[\varrho]$ in (10) as

$$\mathbb{B}[\varrho] = \sum_{j,k=1}^{3} A_{jk}\,\sigma_j\otimes\mathbf{1}\,\varrho\,\sigma_k\otimes\mathbf{1} + \sum_{j,k=1}^{3} C_{jk}\,\mathbf{1}\otimes\sigma_j\,\varrho\,\mathbf{1}\otimes\sigma_k \tag{18}$$

$$+ \sum_{j,k=1}^{3} B_{jk}\,\sigma_j\otimes\mathbf{1}\,\varrho\,\mathbf{1}\otimes\sigma_k + \sum_{j,k=1}^{3} B^*_{jk}\,\mathbf{1}\otimes\sigma_k\,\varrho\,\sigma_j\otimes\mathbf{1}\,. \tag{19}$$

Then, using (13) one computes

$$(\mathbb{T}\otimes\mathrm{id})\circ\mathbb{Z}[P_\psi\otimes P_\phi] = -i\sum_{j=1}^{3} H_j\epsilon_j\,\big[P_{\psi^*}\,,\,\sigma_j\big]\otimes P_\phi - i\sum_{j=4}^{6} H_j\,P_{\psi^*}\otimes\big[\sigma_j\,,\,P_\phi\big] \tag{20}$$

$$-\frac{1}{2}\sum_{j,k=1}^{3} A_{jk}\epsilon_j\epsilon_k\,\big\{P_{\psi^*}\,,\,\sigma_j\sigma_k\big\}\otimes P_\phi - \frac{1}{2}\sum_{j,k=1}^{3} C_{jk}\,P_{\psi^*}\otimes\big\{\sigma_k\sigma_j\,,\,P_\phi\big\} \tag{21}$$

$$-\sum_{j,k=1}^{3}\mathscr{R}e(B_{jk})\,\epsilon_j\Big(\sigma_j\otimes\mathbf{1}\,P_{\psi^*}\otimes P_\phi\,\mathbf{1}\otimes\sigma_k + \mathbf{1}\otimes\sigma_k P_{\psi^*}\otimes P_\phi\,\sigma_j\otimes\mathbf{1}\Big)\,, \tag{22}$$

and

$$(\mathbb{T}\otimes \mathrm{id})\circ \mathbb{B}[P_\psi\otimes P_\phi] = \sum_{j,k=1}^{3} A_{jk}\epsilon_j\epsilon_k\,\sigma_k\otimes \mathbf{1}\left(P_{\psi^*}\otimes P_\phi\right)\sigma_j\otimes\mathbf{1} \tag{23}$$

$$+\sum_{j,k=1}^{3} C_{jk}\,\mathbf{1}\otimes\sigma_j\left(P_{\psi^*}\otimes P_\phi\right)\sigma_k\otimes\mathbf{1} \tag{24}$$

$$+\sum_{j,k=1}^{3} B_{jk}\epsilon_j\left(P_{\psi^*}\otimes P_\phi\right)\sigma_j\otimes\sigma_k \tag{25}$$

$$+\sum_{j,k=1}^{3} B^*_{jk}\,\epsilon_j\,\sigma_j\otimes\sigma_k\left(P_{\psi*}\otimes P_\phi\right). \tag{26}$$

Notice that, by putting together the above expressions, it thus turns out that partial transposition transforms the generator $\mathbb{L}$ into a linear map $\mathbb{N} := (\mathbb{T}\otimes\mathrm{id})\circ\mathbb{L}\circ(\mathbb{T}\otimes\mathrm{id})$ such that

$$\mathbb{N}[\varrho] = -i[\widetilde{H},\varrho] + \sum_{j,k=1}^{6} N_{jk}\left(S_j\,\varrho\,S_k - \frac{1}{2}\left\{S_k\,S_j\,,\,\varrho\right\}\right) \tag{27}$$

$$\widetilde{H} := -\sum_{j=1}^{3}\epsilon_j H_j\,S_j + \sum_{j=4}^{6} H_j\,S_j + \sum_{j,k=1}^{3}\epsilon_j\,\mathscr{I}m(B_{jk})\,\sigma_j\otimes\sigma_k \tag{28}$$

$$N = [N_{jk}] := \mathscr{E}\begin{pmatrix} A^T & (B+B^*)/2 \\ (B^\dagger+B^T)/2 & C\end{pmatrix}\mathscr{E}\,, \tag{29}$$

where $\mathscr{E}$ is the matrix introduced in (14), B^* is the matrix with entries B^*_{jk} and $B+B^*$ is the 3×3 matrix with real entries $2\,\mathscr{R}e(B_{jk})$.

Remark 2 The linear map $\mathbb{N}$ generates a semigroup of maps $\mathscr{N}_t = \exp(t\mathbb{N})$. Since, the matrix $N = [N_{jk}]$ need not be positive semi-definite, the partially transposed semigroup need not have any physical meaning.

We now expand $|\Psi\rangle$ along the orthonormal basis of $\mathbb{C}^2\otimes\mathbb{C}^2$ given by the orthogonal pairs $\{|\psi^*\rangle, |(\psi^*)^\perp\rangle\}$ and $\{|\phi\rangle, |\phi^\perp\rangle\}$,

$$|\Psi\rangle = a\,|\psi^*\rangle\otimes|\phi\rangle + b\,|(\psi^*)^\perp\rangle\otimes|\phi\rangle + c\,|(\psi^*)^\perp\rangle\otimes|\phi^\perp\rangle\,. \tag{30}$$

Then, the orthogonality between $|\Psi\rangle$ and $|\psi^*\rangle\otimes|\phi_1\rangle$ reduces to one the contributions to $\Delta(t)$ in (15):

$$\Delta(t)=t\langle\Psi|\mathbb{N}[P_{\psi^*}\otimes P_\phi]|\Psi\rangle=\sum_{j,k=1}^{6}\langle\Psi|\,S_j\,P_{\psi^*}\otimes P_\phi\,S_k\,|\Psi\rangle\;. \tag{31}$$

From these considerations the following result ensues [14]).

Proposition 1 *The dissipative semigroup of completely positive maps* $\Lambda_t=\exp(t\mathbb{L})$*, with* $\mathbb{L}$ *as in (5), is entanglement generating if and only if there exist vectors* $|\psi\rangle\,,\,|\phi\rangle\in\mathbb{C}^2$ *such that* $\Delta(t)$ *in (31) becomes negative at some* $t>0$.

2.2 Case 1: $\mathbb{Z}$ Does Not Generate Entanglement

We shall first show that, if $\mathbb{Z}$ in (7) and (8) cannot generate entanglement, then $\mathbb{B}$ in (10) is such that $\mathbb{L}=\mathbb{Z}+\mathbb{B}$ cannot either.

Proposition 2 *Suppose the pseudo-commutator* $\mathbb{Z}$ *in (7) does not entangle any initial separable projection* $P_\psi\otimes P_\phi$*; then, in (29),* $B+B^*=0$.

Proof The argument leading to Proposition 1 applies also when the time-evolution is generated by $\mathbb{Z}$ only, the difference being that the evolving state is a non-normalized projection (see (9)). Then, to the corresponding quantity $\Delta(t)$ there contributes only the term (22), so that $\Delta(t)$ in (15) becomes

$$\begin{aligned}\Delta(t)&=t\,\langle\Psi|(\mathbb{T}\otimes\mathrm{id})\circ\mathbb{Z}[P_{\psi^*}\otimes P_\phi]|\Psi\rangle\\&=-2\,\mathscr{I}m\Big(a\,b^*\sum_{j,k=1}^{3}\varepsilon_j\,\mathscr{R}e(B_{jk})\,\langle\psi^*|\sigma_j(\psi^*)^\perp\rangle\langle\phi^\perp|\sigma_k\phi\rangle\Big)\\&=-2\mathscr{I}m\left(a\,b^*\left\langle u\left|\,\mathscr{E}\,\frac{B+B^*}{2}\,v\right.\right\rangle\right)\end{aligned}$$

where $|\Psi\rangle$ is the entangled state in (30) orthogonal to $|\psi^*\rangle\otimes|\phi\rangle$ and

$$|u\rangle=\begin{pmatrix}\langle(\psi^*)^\perp|\sigma_1\psi^*\rangle\\-\langle(\psi^*)^\perp|\sigma_2\psi^*\rangle\\\langle(\psi^*)^\perp|\sigma_3\psi^*\rangle\end{pmatrix}\,,\quad|v\rangle=\begin{pmatrix}\langle\phi^\perp|\sigma_1\phi\rangle\\\langle\phi^\perp|\sigma_2\phi\rangle\\\langle\phi^\perp|\sigma_3\phi\rangle\end{pmatrix}\,.$$

The assumption that $\mathbb{Z}$ generate no entanglement amounts to the request that $\Delta(t)$ be non negative for all possible $a, b \in \mathbb{C}$ and all $|u\rangle$, $|v\rangle$. This in turn asks for

$$\left\langle u \,\middle|\, \mathscr{E}\,\frac{B+B^*}{2}\,v\right\rangle = 0$$

for all $|u\rangle\,|v\rangle \in \mathbb{C}^3$. By choosing $|\psi\rangle$ to be an eigenstate of σ_1, then of σ_2 and finally of σ_3, one gets three linearly independent $|u\rangle \in \mathbb{C}^3$ and analogously for $|v\rangle$. Then, the request that $\Delta(t)$ be non-negative for all $|\psi\rangle$, $|\phi\rangle \in \mathbb{C}^2$, together with the invertibility of the matrix $\mathscr{E}$ in (14) yields the result.

Corollary 1 *If $\mathbb{Z}$ does not generate entanglement, neither does $\mathbb{L}$ in (5).*

Proof Given the hypothesis, the previous proposition makes diagonal the matrix $N = [N_{ij}]$ in (29),

$$N = \begin{pmatrix} \mathscr{E}A^T\mathscr{E} & 0 \\ 0 & C \end{pmatrix} .$$

Since $\mathbb{L}$ generates a semigroup of completely positive maps, then, by Gorini-Kossakowski-Sudarshan-Lindblad theorem (see (3)), the Kossakowski matrix K must be positive semi-definite, whence $\mathscr{E}A^T\mathscr{E}$ and C are both positive semi-definite matrices. Then, by the same theorem, the partially transposed generator $\mathbb{N}$ in (27) also generates a semigroup of completely positive maps $\mathscr{N}_t = \exp(t\mathbb{N})$ (see Remark 2), so that $\mathscr{N}_t[P_{\psi^*} \otimes P_\phi]$ is always positive semi-definite for all $t \geq 0$ and $\Delta(t)$ in (31) cannot become negative.

2.3 Case 2: $\mathbb{Z}$ Generates Entanglement

Without restricting to the specific generator $\mathbb{L}$ in the previous section, we now cast the master Eq. (1) as an equation for the dynamical maps Λ_t, $\partial_t\Lambda_t = \mathbb{L} \circ \Lambda_t$, and introduce the Laplace transform of the solution Λ_t,

$$\widetilde{\Lambda}_s := \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{-st}\, \Lambda_t \qquad s \geq 0 \,. \tag{32}$$

Then, the master equation translates into

$$\begin{aligned}\widetilde{\Lambda}_s &= \frac{1}{s-\mathbb{L}} = \frac{1}{s-\mathbb{Z}-\mathbb{B}} = \frac{1}{s-\mathbb{Z}} + \frac{1}{s-\mathbb{Z}}\Big(s-\mathbb{Z}-(s-\mathbb{Z}-\mathbb{B})\Big)\frac{1}{s-\mathbb{Z}-\mathbb{B}} \\ &= \frac{1}{s-\mathbb{Z}}\sum_{k=0}^{+\infty}\left(\mathbb{B}\frac{1}{s-\mathbb{Z}}\right)^k \,. \end{aligned}\tag{33}$$

One thus sees that if Λ_t, $t \geq 0$, is positive, respectively completely positive, such is also the Laplace transform $\widetilde{\Lambda}_s$, $s \geq 0$, since the latter is an integral of positive, respectively completely positive maps weighted by positive factors. Moreover, the same is true of

$$(-)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \widetilde{\Lambda}_s = \int_0^{+\infty} \mathrm{d}t\, t^k\, \Lambda_t \qquad \forall\, k \geq 0\,. \tag{34}$$

A theorem of Bernstein [15] asserts that the latter ones are not only necessary, but also sufficient conditions for Λ_t to be positive, respectively completely positive.

Remark 3 The Laplace transform of the dynamics has been thoroughly used in dealing with the complete positivity of dynamical maps outside the Markovian regime when they are generated by time-dependent master equations of the form

$$\partial_t \varrho_t = \int_0^t \mathrm{d}\tau\, \mathbb{K}_{t-\tau}[\varrho_\tau]\,, \qquad \Lambda_{t=0} = \mathrm{id}\,, \tag{35}$$

where $\mathbb{K}_t$ is a suitable kernel. In this case, not so many results are available regarding the form it ought to have in order to generate a complete positive dynamics. Postulating a kernel of the form $\mathbb{K}_t := \mathbb{Z}_t + \mathbb{B}_t$ satisfying trace preservation (see footnote 1), the use of (33) allowed the construction of *legitimate pairs* $(\mathbb{Z}_t\,,\, \mathbb{B}_t)$, such that the generated dynamics is completely positive [15]. In this approach, it clearly emerges the pivotal role played by the the $\mathbb{Z}_t$ term with respect to $\mathbb{B}_t$ in ensuring the complete positivity of the generated time-evolution.

Since we are dealing with two qubits, the linear maps $\mathbb{Z}$ and $\mathbb{B}$ have finite norms $\|\mathbb{Z}\|$ and $\|\mathbb{B}\|$ on $M_4(\mathbb{C})$. Then, from (33) one can estimate, for $s > \|\mathbb{Z}\| + \|\mathbb{B}\|$,

$$\left\| \widetilde{\Lambda}_s - \frac{1}{s - \mathbb{Z}} \right\| \leq \sum_{k=1}^{\infty} \|\mathbb{B}\|^k \left\| \frac{1}{s - \mathbb{Z}} \right\|^{k+1} \leq \frac{\|\mathbb{B}\|}{(s - \|\mathbb{B}\|)^2} \frac{1}{1 - \frac{\|\mathbb{B}\|}{s - \|\mathbb{Z}\|}}\,,$$

whence, for large $s \geq 0$,

$$\widetilde{\Lambda}_s = \frac{1}{s - \mathbb{Z}} + o\left(s^{-1}\right)\,, \qquad (-)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \widetilde{\Lambda}_s = \frac{k!}{(s - \mathbb{Z})^{k+1}} + o\left(s^{-(k+1)}\right)\,. \tag{36}$$

Applying these considerations, we can prove the following result.

Proposition 3 *Consider a semigroup of completely positive maps $\Lambda_t = \exp(t\mathbb{L})$, $t \geq 0$, on the state space of two qubits, with $\mathbb{L} = \mathbb{Z} + \mathbb{B}$ as in (3). Then, if $\mathbb{Z}$ generates entanglement, so does $\mathbb{L}$.*

Proof Since $\mathbb{Z}$ generates entanglement, the dynamical maps $\gamma_t := \exp(t\mathbb{N}_Z)$, with $\mathbb{N}_Z := (\mathbb{T} \otimes \mathrm{id}) \circ \mathbb{Z} \circ (\mathbb{T} \otimes \mathrm{id})$, cannot be positive. Indeed, there must exist an initial separable projection $P_\psi \otimes P_\phi$ such that $\gamma_t[P_\psi \otimes P_\phi]$ becomes entangled at some

$t > 0$. Thus, Theorem 2 together with the fact that

$$(\mathbb{T} \otimes \mathrm{id}) \circ \mathrm{e}^{t\mathbb{Z}} \circ (\mathbb{T} \otimes \mathrm{id}) = \mathrm{e}^{t\mathbb{N}_Z} ,$$

and $((\mathbb{T} \otimes \mathrm{id}))^2 = \mathrm{id}$ implies that

$$(\mathbb{T} \otimes \mathrm{id}) \circ \mathrm{e}^{t\mathbb{Z}}[P_\psi \otimes P_\phi] = \gamma_t[P_{\psi^*} \otimes P_\phi]$$

is no longer positive semi-definite. Then, going to the Laplace trasform $\widetilde{\gamma}_t$, Bernstein theorem (see (34)) implies that there must exist an integer $k \geq 0$ such that

$$(-)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \widetilde{\gamma}_s = \frac{k!}{(s - \mathbb{N}_Z)^{k+1}}$$

is not a positive map. Let us now consider the full generator $\mathbb{L} = \mathbb{Z} + \mathbb{B}$ and its partially transposed partner

$$\mathbb{N} = (\mathbb{T} \otimes \mathrm{id}) \circ \mathbb{L} \circ (\mathbb{T} \otimes \mathrm{id}) = \mathbb{N}_Z + \mathbb{N}_B , \qquad \mathbb{N}_B := (\mathbb{T} \otimes \mathrm{id}) \circ \mathbb{B} \circ (\mathbb{T} \otimes \mathrm{id}) .$$

Then, regarding the Laplace transform $\widetilde{\mathscr{N}}_s$ of the maps $\mathscr{N}_t = \mathrm{e}^{t\mathbb{N}}$, from the asymptotic behaviour (36) for large $s \geq 0$, one can conclude that also

$$(-)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \widetilde{\mathscr{N}}_s = \frac{k!}{(s - \mathbb{N}_Z)^{k+1}} + o\left(s^{-(k+1)}\right)$$

cannot be a positive map for sufficiently large $s \geq 0$. Therefore, again by Bernstein theorem, $\mathscr{N}_t = (\mathbb{T} \otimes \mathrm{id}) \circ \Lambda_t$ cannot be positive and thus Λ_t must be entanglement generating for some $t \geq 0$.

3 Conclusions

Two qubits have been considered in weak interaction with a common environment that makes them evolve according to a dissipative semigroup of completely positive maps $\Lambda_t = \exp(t\mathbb{L})$ that do not provide mediated interaction between the two open quantum systems, but only statistically mix them. The paper provides necessary and sufficient conditions on the generator $\mathbb{L} = \mathbb{Z} + \mathbb{B}$ for the dynamical maps Λ_t to be able to entangle initial separable states. As is the case with master equations generating semigroups of completely positive maps, the generator $\mathbb{L}$ consists of two terms: a pseudo-commutative term $\mathbb{Z}$ responsible for a damped time-evolution transforming vector states into non-normalized vector states and a noise term $\mathbb{B}$ transforming vector states into mixtures of projections. A Laplace transform technique has been used that reduced the problem to the discussion of the properties of the pseudo-commutator $\mathbb{Z}$ showing that if the latter alone generates entanglement so does $\mathbb{L}$ and vice versa. As shortly mentioned in Remark 3, the Laplace transform had been used in [15] to study the complete positivity of the maps generated by a master equation with a time-dependent memory kernel $\mathbb{K}_t = \mathbb{Z}_t + \mathbb{B}_t$. There, it became apparent the prominence of the role of the pseudo-commutator $\mathbb{Z}_t$ in fixing the properties of the generator $\mathbb{K}_t$. The result in Theorem 1 confirms such an evidence in a memory-less context and in relation to entanglement generation.

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