Chapter 4
A Reduced Basis Method with an Exact Solution Certificate and Spatio-Parameter Adaptivity: Application to Linear Elasticity

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Abstract We present a reduced basis method for parametrized linear elasticity equations with two objectives: providing an error bound with respect to the exact weak solution of the PDE, as opposed to the typical finite-element "truth", in the online stage; providing automatic adaptivity in both physical and parameter spaces in the offline stage. Our error bound builds on two ingredients: a minimum-residual mixed formulation with a built-in bound for the dual norm of the residual with respect to an infinite-dimensional function space; a combination of a minimum eigenvalue bound technique and the successive constraint method which provides a lower bound of the stability constant with respect to the infinite-dimensional function space. The automatic adaptivity combines spatial mesh adaptation and greedy parameter sampling for reduced bases and successive constraint method to yield a reliable online system in an efficient manner. We demonstrate the effectiveness of the approach for a parametrized linear elasticity problem with geometry transformations and parameter-dependent singularities induced by cracks.

4.1 Introduction

Reduced basis (RB) methods provide rapid and reliable solution of parametrized partial differential equations (PDEs), including linear elasticity equations, in real-time and many-query applications; see, e.g., a review paper [13] and early applications to linear elasticity in [4, 7, 10, 14]. However, until recently, RB methods have focused on approximating the high-fidelity "truth" solution—typically a finite element (FE) solution on a prescribed mesh—and not the exact solution of the PDE, which is of actual interest. Classical RB methods *assume* that the "truth" model is sufficiently accurate to serve as an surrogate for the exact PDE. However, in practice, satisfying the assumption requires a careful mesh construction especially

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in the presence of sharp corners and cracks (as done in [4]), and in any event the assumption is never rigorously verified for all parameter values. In this work, we present a RB method which provides a certificate with respect to the *exact* solution of the parametrized PDE and automatically produces a reduced model that meets the desired tolerance through automatic adaptivity, eliminating the issue of the "truth".

Specifically, we present a RB method for linear elasticity problems that provides

1. error bounds with respect to the exact solution in energy norm or for functional outputs for any parameter value in the online stage;
2. automatic adaptivity in physical space and parameter space to control the error with respect to the exact solution;
3. a strict offline-online computational decomposition such that the online computational cost is independent of the offline FE solves.

Item 3 provides rapidness, as in the case for the standard RB method. Items 1 and 2, which provide certification and adaptivity with respect to the exact solution, distinguish our method from the standard RB method.

Recently, a number of RB methods has been proposed to provide error bounds with respect to the exact solution. Ali et al. [1] consider a RB method based on snapshots generated by an adaptive wavelet method. Ohlberger and Schindler [8] considers a RB method for multiscale problems with an error bound with respect to the exact solution. We have also introduced RB methods which provide error bounds with respect to the exact solution using the complementary variational principle [15] and using a minimum-residual mixed formulation [16, 17]. This work shares a common goal with the above recent works in the RB community.

The error certification and adaptation approach that we present in this paper is an extension of the method we introduced in [17] for scalar equations to linear elasticity equations with piecewise-affine geometry transformations. We provide a solution approximation and an upper bound of the residual dual norm using a minimum-residual mixed formulation. We provide a lower bound of the stability constant using a version of the successive constraint method (SCM) [5], which has been extended to provide bounds relative to an appropriate infinite-dimensional function space by appealing to Weinstein's method and a residual-based bounding technique. In extending the approach to linear elasticity, special attention is paid to the treatment of rigid-body rotation modes and the construction of the dual space in the presence of geometry transformations.

The paper is organized as follows. Section 4.2 defines the problem of interest. Section 4.3 presents our residual bound procedure. Section 4.4 presents our stability-constant bound procedure. Section 4.5 presents the error bound. Section 4.6 presents spatio-parameter adaptive algorithms. Section 4.7 presents numerical results.

4.2 Preliminaries

4.2.1 Problem Statement

Notations In order to describe tensor operations that appear in linear elasticity, we now fix the notations. Given a order-2 tensor $\underline{w}$, we "reshape" it as a vector $w \in \mathbb{R}^{d^2}$ with entries $(w)_{i \cdot d+j} = \underline{w}_{ij}$. Similarly, given a order-4 tensor $\underline{A}$, we "reshape" it as a matrix $A \in \mathbb{R}^{d^2 \times d^2}$ with entries $(A)_{i \cdot d+j, k \cdot d+l} = \underline{A}_{ijkl}$. These reshaped notations allow us to precisely express operations on order-2 and -4 tensors using the standard linear algebra notations without introducing explicit indices.

Using the convection, the derivative of a vector field $v : \Omega \to \mathbb{R}^d$ evaluated at x is expressed as a vector $v(x) \in \mathbb{R}^{d^2}$ with entries $(\nabla v(x))_{i \cdot d+j} = \frac{\partial v_i}{\partial x_j}$. Similarly, the divergence of a order-2 tensor field $q : \Omega \to \mathbb{R}^{d^2}$ evaluated at x is expressed as a vector $\nabla q(x) \in \mathbb{R}^d$ with entries $(\nabla q(x))_i = \sum_{j=1}^d \frac{\partial q_{ij}}{\partial x_j}$; the evaluation of q at x in the direction of $n \in \mathbb{R}^d$ is expressed as a vector $n \cdot q(x) \in \mathbb{R}^d$ with entries $(n \cdot q(x))_i = \sum_{j=1}^d q_{ij}(x) n_j$.

Problem Description over a Parametrized Domain We first introduce a P-dimensional parameter domain $\mathcal{D} \subset \mathbb{R}^P$. We next introduce a d-dimensional parametrized physical domain $\tilde{\Omega}(\mu) \subset \mathbb{R}^d$ with a Lipchitz boundary $\partial\tilde{\Omega}(\mu)$. For each component $i = 1, \ldots, d$, the boundary $\partial\tilde{\Omega}(\mu)$ is decomposed into a Dirichlet part $\tilde{\Gamma}_{D,i}(\mu)$ and a Neumann part $\tilde{\Gamma}_{N,i}(\mu)$ such that $\partial\overline{\tilde{\Omega}}(\mu) = \overline{\tilde{\Gamma}}_{D,i}(\mu) \cup \overline{\tilde{\Gamma}}_{N,i}(\mu)$. We then introduce a Sobolev space $\mathcal{V}(\tilde{\Omega}) = \{\tilde{v} \in (H^1(\tilde{\Omega}))^d \mid \tilde{v}_i|_{\tilde{\Gamma}_{D,i}} = 0,\ i = 1, \ldots, d\}$, where $H^1(\tilde{\Omega})$ is the standard H^1 Sobolev space over $\tilde{\Omega}$. (See, e.g., [2].)

We now introduce order-4 tensors, unwrapped as $d^2 \times d^2$ matrices, associated with our linear elasticity problem. We first introduce the strain tensor operator $E \in \mathbb{R}^{d^2 \times d^2}$ such that $E \nabla \tilde{v}(\tilde{x}) \in \mathbb{R}^{d^2}$ is the reshaped strain tensor. We next introduce a parametrized stiffness tensor field $\tilde{K} : \mathcal{D} \times \tilde{\Omega} \to \mathbb{R}^{d^2 \times d^2}$; by definition the stiffness tensor is symmetric positive definite for all $\mu \in \mathcal{D}$ and $\tilde{x} \in \tilde{\Omega}$. We also introduce the associated parametrized compliance tensor field $\tilde{C} : \mathcal{D} \times \tilde{\Omega} \to \mathbb{R}^{d^2 \times d^2}$. The stiffness and compliance tensor are related by $\tilde{K}(\mu; \tilde{x})\tilde{C}(\mu; \tilde{x}) = I_{d^2}$, where I_{d^2} denotes the $d^2 \times d^2$ identity matrix.

We now consider the following weak formulation of linear elasticity: given $\mu \in \mathcal{D}$, find $\tilde{u}(\mu) \in \mathcal{V}(\tilde{\Omega}(\mu))$ such that

$$a_{\tilde{\Omega}(\mu)}(\tilde{u}(\mu), \tilde{v}; \mu) = \ell_{\tilde{\Omega}(\mu)}(\tilde{v}; \mu) \quad \forall \tilde{v} \in \mathcal{V}(\tilde{\Omega}) \tag{4.1}$$

where

$$a_{\tilde{\Omega}(\mu)}(\tilde{w}, \tilde{v}; \mu) = \int_{\tilde{\Omega}(\mu)} \tilde{\nabla}\tilde{v}^T E^T \tilde{K}(\mu) E \tilde{\nabla}\tilde{w} d\tilde{x}, \tag{4.2}$$

$$\ell_{\tilde{\Omega}(\mu)}(\tilde{v}; \mu) = \int_{\tilde{\Omega}(\mu)} \tilde{v}^T \tilde{f}(\mu) d\tilde{x} + \int_{\tilde{\Gamma}_N(\mu)} \tilde{v}^T \tilde{g}(\mu) d\tilde{s}.$$

Here, $\tilde{f}(\mu)$ is the body force on the solid, $\tilde{g}(\mu)$ is the traction force on the Neumann boundaries, and the subscript $\tilde{\Omega}(\mu)$ on the forms emphasizes the problem is defined over a parameterized physical domain.

Reference-Domain Formulation Following the standard approach to treat parametrized geometric variations in the RB method (see, e.g., [13, 14]), we recast the problem over the parametrized domain $\tilde{\Omega}(\mu)$ to a parameter-independent reference domain Ω. Specifically, we consider each point $\tilde{x} \in \tilde{\Omega}(\mu)$ to be associated with a unique point $x \in \Omega$ by a piecewise affine map. We denote the Jacobian of the parametrized map by $J(\mu) \in \mathbb{R}^{d\times d}$ and the associated determinant by $|J(\mu)|$. Similarly, we denote the Jacobian associated with the mapping of a boundary segment by $|\partial J(\mu)|$. We also introduce a block matrix $Y = I_d \otimes J(\mu) \in \mathbb{R}^{d^2\times d^2}$ that facilitates transformation of tensors; here $\otimes$ is the Kronecker product.

We now introduce a Sobolev space over Ω,

$$\mathcal{V} \equiv \mathcal{V}(\Omega) \equiv \{v \in (H^1(\Omega))^d \mid v_i|_{\Gamma_{D_i}} = 0,\ i = 1,\dots,d\}$$

endowed with an inner product

$$(w,v)_{\mathcal{V}} \equiv \int_\Omega \nabla v^T \nabla w dx + \int_\Omega v^T w dx + \int_{\Gamma_N} v^T w ds \tag{4.3}$$

and the associated induced norm $\|v\|_{\mathcal{V}} \equiv \sqrt{(v,v)_{\mathcal{V}}}$. We then introduce a weak formulation that is equivalent to (4.1) but is associated with the reference domain: given $\mu \in \mathcal{D}$, find $u(\mu) \in \mathcal{V}$ such that

$$a(u(\mu), v; \mu) = \ell(v;\mu) \quad \forall v \in \mathcal{V}, \tag{4.4}$$

where

$$a(w,v;\mu) = \int_\Omega \nabla v^T Y(\mu)^{-1} EK(\mu) EY(\mu)^{-T} \nabla w |J(\mu)| dx$$
$$\ell(v;\mu) = \int_\Omega v^T f(\mu)|J(\mu)|dx + \int_{\Gamma_N} v^T g(\mu)|\partial J(\mu)|ds.$$

Here the tensor fields in the physical and reference domains are related by $\tilde{v}(\tilde{x}) = v(x)$, $\tilde{K}(\mu;\tilde{x}) = K(\mu;x)$, $\tilde{f}(\mu;\tilde{x}) = f(\mu;x)$, and $\tilde{g}(\mu;\tilde{x}) = g(\mu;x)$. We readily verify that $a(\cdot,\cdot;\mu)$ is symmetric and bounded in $\mathcal{V}$. We also note that $a(\cdot,\cdot;\mu)$ is coercive in $\mathcal{V}$ due to the Korn inequality and the trace theorem [2]; we denote the associated energy norm by $|||\cdot|||_\mu \equiv \sqrt{a(\cdot,\cdot;\mu)}$.

Remark 1 In the standard RB formulation [13], we simply treat the elasticity equation as a vector-valued equation with the stiffness matrix $\hat{K}(\mu) \equiv |J(\mu)|Y(\mu)^{-1}EK(\mu)EY(\mu)^{-T}$. Unfortunately, our exact error-bound formulation does not permit this simple treatment; our formulation [17] requires the inverse of the stiffness matrix, while the matrix $\hat{K}(\mu)$ is singular because $EK(\mu)E$ is rank-deficient. We will keep the explicit representation of the stiffness matrix to clearly show how our bound formulation for linear elasticity circumvents the issue.

Assumptions We clarify the set of assumptions for our RB formulation. First, we assume that the stiffness tensor $K(\mu)$, the compliance tensor $C(\mu)$, the body force $f(\mu)$, and the boundary traction force $g(\mu)$ each admit a decomposition that is affine in functions of parameter: $K(\mu) = \sum_{q=1}^{Q_K} \Theta_q^K(\mu) K_q$, $C(\mu) = \sum_{q=1}^{Q_C} \Theta_q^C(\mu) C_q$, $f(\mu) = \sum_{q=1}^{Q_f} \Theta_q^f(\mu) f_q$, and $g(\mu) = \sum_{q=1}^{Q_g} \Theta_q^g(\mu) g_q$, where $K_q : \Omega \to \mathbb{R}^{d^2 \times d^2}$, $C_q : \Omega \to \mathbb{R}^{d^2 \times d^2}$, $f_q : \Omega \to \mathbb{R}^d$, and $g_q : \Omega \to \mathbb{R}^d$ are parameter-independent fields, and $\Theta_q^K : \mathcal{D} \to \mathbb{R}$, $\Theta_q^C : \mathcal{D} \to \mathbb{R}$, $\Theta_q^f : \mathcal{D} \to \mathbb{R}$, and $\Theta_q^g : \mathcal{D} \to \mathbb{R}$ are parameter-dependent functions. Second, we assume that the mapping from the reference domain Ω to the physical domain $\tilde{\Omega}(\mu)$ is piecewise affine such that both the Jacobian $J(\mu)$ and the inverse Jacobian $J(\mu)^{-1}$ admit a decomposition that are affine in functions of parameter: $J(\mu) = \sum_{q=1}^{Q_J} \Theta_q^J(\mu) J_q$ and $J(\mu)^{-1} = \sum_{q=1}^{Q_{J^{\text{inv}}}} \Theta_q^{J^{\text{inv}}}(\mu) J_q^{\text{inv}}$. Finally, we assume that the fields $K(\mu)$, $C(\mu)$, $f(\mu)$, and $g(\mu)$ are piecewise polynomials such that we can integrate the fields exactly using standard quadrature rules.

4.2.2 Abstract Error Bounds: Energy Norm and Compliance Output

To simplify the presentation of our formulation, we introduce a parametrized inner product

$$(w, v)_{\mathcal{W}(\mu;\delta)} = a(w, v) + \delta (w, v)_{\mathcal{V}}$$

and the associated induced norm $\|w\|_{\mathcal{W}(\mu;\delta)} \equiv \sqrt{(w, w)_{\mathcal{W}(\mu;\delta)}}$ for a parameter $\mu \in \mathcal{D}$ and a weight $\delta \in \mathbb{R}_{>0}$. Here $a(\cdot, \cdot; \mu)$ is the bilinear form (4.2), and $(\cdot, \cdot)_{\mathcal{V}}$ is the inner product (4.3). The parametrized norm is related to the energy norm by $\|v\|^2_{\mathcal{W}(\mu;\delta)} = |||v|||^2_\mu + \delta \|v\|^2_{\mathcal{V}}$. For any $\delta \in \mathbb{R}_{>0}$, the norm $\|\cdot\|_{\mathcal{W}(\mu;\delta)}$ is equivalent to the energy norm $|||\cdot|||_\mu$, which in turn is equivalent to $\|\cdot\|_{H^1(\Omega)}$. The role of δ in our formulation is discussed in Sect. 4.5.

In order to bound the error, we now introduce the residual form

$$r(v; w; \mu) \equiv \ell(v; \mu) - a(w, v; \mu) \quad \forall w, v \in \mathcal{V} \tag{4.5}$$

and the associated dual norm $\|r(\cdot; w; \mu)\|_{\mathcal{W}'(\mu;\delta)} \equiv \sup_{v \in \mathcal{V}} \frac{r(v;w;\mu)}{\|v\|_{\mathcal{W}(\mu;\delta)}}$. We also introduce the stability constant

$$\alpha(\mu; \delta) \equiv \inf_{v \in \mathcal{V}} \frac{|||v|||^2_\mu}{\|v\|^2_{\mathcal{W}(\mu;\delta)}}. \tag{4.6}$$

The following proposition bounds the energy norm of the error.

Proposition 2 *Given $\mu \in \mathcal{D}$ and an approximation $w \in \mathcal{V}$, the error is bounded by*

$$|||u(\mu) - w|||_\mu \le \frac{1}{(\alpha(\mu;\delta))^{1/2}} \|r(\cdot; w; \mu)\|_{\mathcal{W}'(\mu;\delta)},$$

where $r(\cdot,\cdot;\cdot)$ is the residual form (4.5), and $\alpha(\cdot,\cdot)$ is the stability constant (4.6).

Proof See, e.g., Rozza et al. [13].

We can also construct an error bound for the compliance output $s(\mu) \equiv \ell(u(\mu);\mu)$.

Proposition 3 *Let the compliance output associated with an approximation $w \in \mathcal{V}$ be $\hat{s}(\mu) \equiv \ell(w;\mu) + r(w;w;\mu)$, where $r(\cdot;\cdot;\cdot)$ is the residual form (4.5). Then, the error in the compliance output is bounded by*

$$|s(\mu) - \hat{s}(\mu)| \le \frac{1}{\alpha(\mu;\delta)} \|r(\cdot; w; \mu)\|^2_{\mathcal{W}'(\mu;\delta)}.$$

Proof We suppress μ for brevity. It follows $s(\mu) - \hat{s}(\mu) = \ell(u) - (\ell(w) + r(w;w)) = \ell(u) - \ell(w) - \ell(w) + a(w,w) = \ell(u-w) - a(u-w,w) = a(u-w,u-w) = |||u-w|||^2_\mu$. Proposition 2 then yields the desired result.

The energy-norm and compliance-output error bound both require the same ingredients: an upper bound of the dual norm of the residual and a lower bound of the stability constant. In the next two sections, we develop offline-online efficient computational procedures for both of these quantities.

Remark 4 The output bound framework may be extended to any linear functional output by introducing the adjoint equation; see, e.g., Rozza et al. [13].

4.3 Upper Bound of the Dual Norm of the Residual

4.3.1 Bound Form

Our bound formulation is based on a mixed formulation and requires a dual field [16, 17]. Our dual space over a physical domain is the $H(\mathrm{div})$-conforming space

$$\mathcal{Q}(\tilde{\Omega}(\mu)) \equiv \{\tilde{q} \in (L^2(\tilde{\Omega}(\mu)))^{d^2} \mid \tilde{\nabla} \cdot \tilde{q} \in (L^2(\tilde{\Omega}(\mu)))^d\}.$$

The dual space over the reference domain is given by

$$\mathcal{Q} \equiv \{q \in (L^2(\Omega))^{d^2} \mid \nabla \cdot q \in (L^2(\Omega))^d\}.$$

We relate a field in a physical domain $\tilde{q} \in \mathcal{Q}(\tilde{\Omega}(\mu))$ and a field in the reference domain $q \in \mathcal{Q}$ by the Piola transformation, $\tilde{q}(\tilde{x}) = |J(\mu)|^{-1} Yq(x)$. The Piola transformation has an important property that it preserves $H(\text{div})$-conformity.

The following proposition introduces a version of the bound form introduced in [17] extended to linear elasticity equations with geometry transformations.

Proposition 5 *For any $w \in \mathcal{V}$, $q \in \mathcal{Q}$, $\mu \in \mathcal{D}$, and $\delta \in \mathbb{R}_{>0}$,*

$$\|r(\cdot; w; \mu)\|_{\mathcal{W}'(\mu;\delta)} \leq (F(w, q; \mu; \delta))^{1/2},$$

where the bound form is given by

$$\begin{aligned} F(w, q; \mu; \delta) = {} & \||J(\mu)|^{-1/2} C(\mu)^{1/2} Y(\mu) q - |J(\mu)|^{1/2} K(\mu)^{1/2} E Y(\mu)^{-T} \nabla w\|^2_{L^2(\Omega)} \\ & + \delta^{-1} \|Y(\mu)^{-1}(I - E) Y(\mu) q\|^2_{L^2(\Omega)} + \delta^{-1} \|\nabla \cdot q + f(\mu)|J(\mu)|\|^2_{L^2(\Omega)} \\ & + \delta^{-1} \|g(\mu)|\partial J(\mu)| - n \cdot q\|^2_{L^2(\Gamma_N)} \end{aligned} \tag{4.7}$$

Proof For notational simplicity, we suppress μ from parameter-dependent operators and forms in the proof. For all $v \in \mathcal{V}$, $w \in \mathcal{V}$, $q \in \mathcal{Q}$, and $\delta \in \mathbb{R}_{>0}$,

$$\begin{aligned} & r(v; w; \mu; \delta) \\ & = \int_\Omega v^T f |J| dx + \int_{\Gamma_N} v^T g |\partial J| ds - \int_\Omega \nabla v^T Y^{-1} E^T K E Y^{-T} \nabla w |J| dx \\ & \quad + \int_\Omega v^T \nabla \cdot q dx + \int_\Omega \nabla v^T q dx - \int_{\Gamma_N} v^T n \cdot q ds \\ & = \int_\Omega \nabla v^T Y^{-1} E^T K |J| (|J|^{-1} C Y q - E Y^{-T} \nabla w) dx + \int_\Omega \nabla v^T Y^{-1} (I - E) Y q dx \\ & \quad + \int_\Omega v^T (\nabla \cdot q + f|J|) dx + \int_{\Gamma_N} v^T (g|\partial J| - n \cdot q) ds \\ & \leq (\||J|^{1/2} K^{1/2} E Y^{-1} \nabla v\|^2_{L^2(\Omega)} + \delta \|\nabla v\|^2_{L^2(\Omega)} + \delta \|v\|^2_{L^2(\Omega)} + \delta \|v\|^2_{L^2(\Gamma_N)})^{1/2} \\ & \quad (\||J|^{-1/2} C^{1/2} Y q - |J|^{1/2} K^{1/2} E Y^{-T} \nabla w\|^2_{L^2(\Omega)} + \delta^{-1} \|Y^{-1}(I - E) Y q\|^2_{L^2(\Omega)} \\ & \qquad + \delta^{-1} \|\nabla \cdot q + f|J|\|^2_{L^2(\Omega)} + \delta^{-1} \|g|\partial J| - n \cdot q\|^2_{L^2(\Gamma_N)})^{1/2} \\ & = \|v\|_{\mathcal{W}(\mu;\delta)} (F(w, q; \mu; \delta))^{1/2}. \end{aligned}$$

Note, the second line of the first equality vanishes by the Green's theorem. Hence, $\|r(\cdot; w; \mu; \delta)\|_{\mathcal{W}'(\mu;\delta)} = \sup_{v \in \mathcal{V}} r(v; w; \mu; \delta) / \|v\|_{\mathcal{W}(\mu;\delta)} \leq (F(w, q; \mu; \delta))^{1/2}$, $\forall q \in \mathcal{Q}$, which is the desired inequality.

The bound form (4.7) for linear elasticity is similar to the bound form for scalar equations introduced in [17]. However, the bound form differs in that it includes the "asymmetric penalty" term $\|Y(\mu)^{-1}(I - E)Y(\mu)q\|^2_{L^2(\Omega)}$; this term penalizes asymmetry in the dual tensor field in the *physical domain*, $\tilde{q} \in \tilde{Q}(\mu)$. In our bounding procedure, this term arises because the linear elasticity equation has zero energy with respect to not only translation but also rotation. In fact, the presence of this term is closely related to the complementary variational principle for elasticity equations requiring a symmetric dual field [9], as discussed in detail in Sect. 4.5.

The form (4.7) admits a decomposition into a quadratic, linear, and constant forms:

$$F(w, p; \mu; \delta) = G((w, p), (w, p); \mu; \delta) - 2L((w, p); \mu; \delta) + H(\mu; \delta).$$

We here omit the explicit expressions for brevity and refer to a similar decomposition *without the "asymmetric penalty" term* in [17]. The forms G, L, and H inherit the affine decomposition of the parametrized operators $K(\mu)$, $C(\mu)$, $f(\mu)$, $g(\mu)$, $J(\mu)$ and $J(\mu)^{-1}$, which makes the bound form F amenable to offline-online computational decomposition. In addition, the form $G(\cdot, \cdot; \mu; \delta)$ is coercive and bounded in $\mathcal{V} \times \mathcal{Q}$; the proof relies on Korn's inequality and is omitted here for brevity.

4.3.2 Minimum-Bound Solutions and Approximations

Exact Solution We consider the following minimum bound problem: given $\mu \in \mathcal{D}$ and $\delta \in \mathbb{R}_{>0}$, find $(u(\mu), p(\mu)) \in \mathcal{V} \times \mathcal{Q}$ such that

$$(u(\mu), p(\mu)) = \underset{w \in \mathcal{V},\ q \in \mathcal{Q}}{\arg\inf}\ F(w, q; \mu; \delta).$$

The associated Euler-Lagrange equation is the following: given $\mu \in \mathcal{D}$, find $(u(\mu), p(\mu)) \in \mathcal{V} \times \mathcal{Q}$ such that

$$G((u(\mu), p(\mu)), (v, q); \mu; \delta) = L((v, q); \mu; \delta) \quad \forall v \in \mathcal{V},\ \forall q \in \mathcal{Q}.$$

The problem is wellposed due to the coercivity and boundedness of G in $\mathcal{V} \times \mathcal{Q}$.

We can readily show that the primal solution $u(\mu)$ is the weak solution of the original problem (4.4), and the dual solution $p(\mu)$ in the reference domain is related to the primal solution by $|J(\mu)|^{-1}Y(\mu)p(\mu) = K(\mu)EY^{-T}(\mu)\nabla u(\mu)$. The associated residual bound is 0 as expected. Equivalently, the dual solution and the primal solution are related in the physical domain by $\tilde{p}(\mu) = \tilde{K}(\mu)E\tilde{\nabla}\tilde{u}(\mu)$; the dual solution in the physical domain is the stress field. The tensor associated with the dual field $\tilde{p}(\mu)$ is symmetric in the physical domain, which is consistent with the constitutive relation, but is not symmetric in the reference domain.

FE For a FE approximation of the minimum bound problem, we first introduce a primal FE space $\mathcal{V}^{\mathcal{N}}$ of H^1-conforming Lagrange elements and a dual FE space $\mathcal{Q}^{\mathcal{N}}$ of $H(\text{div})$-conforming Raviart-Thomas elements [11]. We then consider the minimum-bound FE approximation: given $\mu \in \mathcal{D}$ and $\delta \in \mathbb{R}_{>0}$, find $(u^{\mathcal{N}}(\mu), p^{\mathcal{N}}(\mu)) \in \mathcal{V}^{\mathcal{N}} \times \mathcal{Q}^{\mathcal{N}}$ such that

$$G((u^{\mathcal{N}}(\mu), p^{\mathcal{N}}(\mu)), (v, q); \mu; \delta) = L((v, q); \mu; \delta) \quad \forall v \in \mathcal{V}^{\mathcal{N}}, \ \forall q \in \mathcal{Q}^{\mathcal{N}}. \tag{4.8}$$

The problem is wellposed due to the coercivity and boundedness of G and L. The dual norm of the residual is bounded by $\|r(\cdot; u^{\mathcal{N}}(\mu); \mu)\|_{\mathcal{W}'(\mu;\delta)} \leq F(u^{\mathcal{N}}(\mu), p^{\mathcal{N}}(\mu); \mu; \delta)^{1/2}$.

RB For a RB approximation of the minimum bound problem, we first introduce primal and dual RB spaces $\mathcal{V}_N = \text{span}\{\xi_i\}_{i=1}^N \subset \mathcal{V}$ and $\mathcal{Q}_N = \text{span}\{\eta_i\}_{i=1}^N \subset \mathcal{Q}$. We then introduce a minimum-bound RB approximation: given $\mu \in \mathcal{D}$ and $\delta \in \mathbb{R}_{>0}$, find $(u_N(\mu), p_N(\mu)) \in \mathcal{V}_N \times \mathcal{Q}_N$ such that

$$G((u_N(\mu), p_N(\mu)), (v, q); \mu; \delta) = L((v, q); \mu; \delta) \quad \forall v \in \mathcal{V}_N, \ \forall q \in \mathcal{Q}_N.$$

The problem is again wellposed due to the coercivity and boundedness of G and L. The dual norm of the residual is bounded by $\|r(\cdot; u_N(\mu); \mu)\|_{\mathcal{W}'(\mu;\delta)} \leq F(u_N(\mu), p_N(\mu); \mu; \delta)^{1/2}$.

4.4 Stability Constant

4.4.1 Transformation of the Stability Constant

We recall that a lower bound of the stability constant $\alpha(\mu; \delta)$ is needed to bound the energy norm of the error. In our approach, we do not compute a lower bound of $\alpha(\mu; \delta)$ directly but rather consider a related problem associated with another quantity $\tau(\mu)$. The following proposition relates the two quantities.

Proposition 6 *For any $\mu \in \mathcal{D}$ and $\delta \in \mathbb{R}_{>0}$, the stability constant $\alpha(\mu; \delta)$ is bounded from the below by*

$$\alpha(\mu; \delta) \equiv \inf_{v \in \mathcal{V}} \frac{|||v|||_\mu^2}{\|v\|_{\mathcal{W}(\mu;\delta)}^2} \geq \left(1 + \frac{\delta}{\tau_{\text{LB}}(\mu)}\right)^{-1} \equiv \alpha_{\text{LB}}(\mu; \delta),$$

where $\tau_{\text{LB}}(\mu)$ satisfies $\tau_{\text{LB}}(\mu) \leq \tau(\mu) \equiv \inf_{v \in \mathcal{V}} |||v|||_\mu^2 / \|v\|_{\mathcal{V}}^2$.

Proof We note that

$$\frac{1}{\alpha(\mu;\delta)} = \sup_{v\in\mathcal{V}} \frac{\|v\|^2_{\mathcal{W}(\mu;\delta)}}{|||v|||^2_\mu} = \sup_{v\in\mathcal{V}} \frac{|||v|||^2_\mu + \delta\|v\|^2_{\mathcal{V}}}{|||v|||^2_\mu} = 1 + \delta \sup_{v\in\mathcal{V}} \frac{\|v\|^2_{\mathcal{V}}}{|||v|||^2_\mu} = 1 + \frac{\delta}{\tau(\mu)}.$$

Appealing to $\tau_{\mathrm{LB}}(\mu) \leq \tau(\mu)$ provides the desired inequality.
We make a few observations. First, if we can provide a lower bound of $\tau(\mu)$, then we can provide a lower bound of $\alpha(\mu;\delta)$. Second, the stability constant is close to unity if we choose $\delta \ll \tau_{\mathrm{LB}}(\mu)$; in particular, the effectivity of $\alpha_{\mathrm{LB}}(\mu;\delta)$ is desensitized from the effectivity of $\tau_{\mathrm{LB}}(\mu)$ as long as $\delta \ll \tau_{\mathrm{LB}}(\mu)$. Third, in the limit of $\delta \to 0$, the stability constant is unity; this is closely related to the complementary variational principle, as discussed in detail in Sect. 4.5. Fourth, the fraction that appears in the definition of $\tau(\mu)$ admits an affine decomposition because $|||v|||^2_\mu \equiv a(v, v;\mu)$ admits an affine decomposition and $\|v\|^2_{\mathcal{V}}$ is parameter independent.

4.4.2 A Residual-Based Lower Bound of the Minimum Eigenvalue

By the Rayleigh quotient, the constant $\tau(\mu)$ is related to the minimum eigenvalue of the following eigenproblem: given $\mu \in \mathcal{D}$, find $(z_i(\mu), \lambda_i(\mu)) \in \mathcal{V} \times \mathbb{R}$ such that

$$a(z_i(\mu), v;\mu) = \lambda_i(\mu)(z_i(\mu), v)_{\mathcal{V}} \quad \forall v \in \mathcal{V} \quad \text{and} \quad \|z_i(\mu)\|_{\mathcal{V}} = 1; \tag{4.9}$$

here the subscript i denotes the index of the eigenpair. We order the eigenpairs in the ascending order of eigenvalues; hence $\tau(\mu) = \min_i \lambda_i(\mu) = \lambda_1(\mu)$.

To compute a lower bound of the minimum eigenvalue, we appeal to Weinstein's method. Towards this end, we introduce the eigenproblem residual associated with any approximate eigenpair $(w, \chi) \in \mathcal{V} \times \mathbb{R}$,

$$r_{\mathrm{eig}}(v; w, \chi;\mu) = a(w, v;\mu) - \chi(w, v)_{\mathcal{V}},$$

and the associated dual norm $\|r_{\mathrm{eig}}(\cdot; w, \chi;\mu)\|_{\mathcal{V}'} \equiv \sup_{v\in\mathcal{V}} \frac{r_{\mathrm{eig}}(v;w,\chi;\mu)}{\|v\|_{\mathcal{V}}}$. The eigenproblem residual is sometimes called the "defect" in the literature. We then introduce the following proposition by Weinstein. (See [3, Chap. 6].)

Proposition 7 *For any $\mu \in \mathcal{D}$ and a pair $(w, \chi) \in \mathcal{V} \times \mathbb{R}$ such that $\|w\|_{\mathcal{V}} = 1$, the distance between χ and the closest eigenvalue is bounded by*

$$\min_i |\lambda_i(\mu) - \chi| \leq \|r_{\mathrm{eig}}(\cdot; w, \chi;\mu)\|_{\mathcal{V}'}.$$

Proof See [3, Chap. 6] for a general case or [17] for the specific case.

Corollary 8 *Consider any* $\mu \in \mathcal{D}$ *and a pair* $(w, \chi) \in \mathcal{V} \times \mathbb{R}$ *such that* $\|w\|_{\mathcal{V}} = 1$. *If* $|\lambda_1(\mu) - \chi| < |\lambda_2(\mu) - \chi|$, *then* $\lambda_1(\mu) \geq \chi - \|r_{\text{eig}}(\cdot; w, \chi; \mu)\|_{\mathcal{V}'}$.

In order to provide a lower bound of the minimum eigenvalue, the corollary requires that the eigenvalue of the approximate eigenpair $(\chi, w) \in \mathcal{V} \times \mathbb{R}$ is closer to $\lambda_1(\mu)$ than to $\lambda_2(\mu)$. *Assuming* this condition is satisfied, we can provide a lower bound of the minimum eigenproblem by bounding the dual norm of the eigenproblem residual, as shown in the following proposition.

Proposition 9 *For any* $w \in \mathcal{V}$, $\chi \in \mathbb{R}$, $q \in \mathcal{Q}$, *and* $\mu \in \mathcal{D}$,

$$\|r_{\text{eig}}(\cdot; w, \chi; \mu)\|_{\mathcal{V}'} \leq (F_{\text{eig}}(w, \chi, q; \mu))^{1/2} \quad \forall q \in \mathcal{Q},$$

where the bound form is given by

$$F_{\text{eig}}(w, \chi, q; \mu) \equiv \chi^2(\|\chi^{-1}|J(\mu)|Y(\mu)^{-1}EK(\mu)EY(\mu)^{-T}\nabla w - \nabla w - q\|^2_{L^2(\Omega)} + \|w + \nabla \cdot q\|^2_{L^2(\Omega)} + \|w - n \cdot q\|^2_{L^2(\Gamma_N)}). \tag{4.10}$$

Proof The proof is omitted here for brevity. We refer to [17] for a complete proof; unlike the proof of Proposition 5, rigid-body rotation modes do not introduce additional difficulties relative to the scalar case in [17].

We can readily show that for an eigenpair $(z_1(\mu), \lambda_1(\mu)) \in \mathcal{V} \times \mathbb{R}$ of (4.9), $\inf_{q \in \mathcal{Q}} F_{\text{eig}}(z_1(\mu), \lambda_1(\mu), q; \mu) = 0$. Hence, given the exact eigenvalue $\lambda_1(\mu)$, there exists $(w, q) \in \mathcal{V} \times \mathcal{Q}$ such that the lower bound collapses to the exact eigenvalue.

4.4.3 FE Approximation of Bounds of $\tau(\mu)$

Upper Bound An upper bound of $\tau(\mu)$ is readily given by a FE approximation of the eigenproblem (4.9): given $\mu \in \mathcal{D}$, find $(z_1^{\mathcal{N}}(\mu), \lambda_1^{\mathcal{N}}(\mu)) \in \mathcal{V}^{\mathcal{N}} \times \mathbb{R}$ such that

$$a(z_1^{\mathcal{N}}(\mu), v; \mu) = \lambda_1^{\mathcal{N}}(\mu)(z_1^{\mathcal{N}}(\mu), v)_{\mathcal{V}} \quad \forall v \in \mathcal{V} \quad \text{and} \quad \|z_i^{\mathcal{N}}(\mu)\|_{\mathcal{V}} = 1. \tag{4.11}$$

Because $\lambda_1(\mu) \equiv \inf_{v \in \mathcal{V}} |||v|||^2_{\mu}/\|v\|^2_{\mathcal{V}} \leq \inf_{v \in \mathcal{V}^{\mathcal{N}}} |||v|||^2_{\mu}/\|v\|^2_{\mathcal{V}} \equiv \lambda_1^{\mathcal{N}}(\mu)$, we conclude $\tau(\mu) \equiv \lambda_1(\mu) \leq \lambda_1^{\mathcal{N}}(\mu) \equiv \tau^{\mathcal{N}}_{\text{UB}}(\mu)$. We hence set $\tau^{\mathcal{N}}_{\text{UB}}(\mu) \equiv \lambda_1^{\mathcal{N}}(\mu)$.

Lower Bound To compute a lower bound of $\tau(\mu)$ using a FE approximation, we first solve the Galerkin FE problem (4.11) to obtain an approximate eigenpair $(z_1^{\mathcal{N}}(\mu), \lambda_1^{\mathcal{N}}(\mu)) \in \mathcal{V}^{\mathcal{N}} \times \mathbb{R}$. We then solve the minimum bound problem associated with (4.10) for the dual field: given $\mu \in \mathcal{D}$, find $y^{\mathcal{N}}(\mu) \in \mathcal{Q}^{\mathcal{N}}$ such that

$$y^{\mathcal{N}}(\mu) = \arg\inf_{q \in \mathcal{Q}^{\mathcal{N}}} F_{\text{eig}}(z_1^{\mathcal{N}}(\mu), \lambda_1^{\mathcal{N}}(\mu), q; \mu).$$

We then *assume* that $|\lambda_1(\mu) - \lambda_1^{\mathcal{N}}(\mu)| < |\lambda_2(\mu) - \lambda_1^{\mathcal{N}}(\mu)|$ and set

$$\tau_{\rm LB}^{\mathcal{N}}(\mu) \equiv \lambda_1^{\mathcal{N}}(\mu) - (F_{\rm eig}(z_1^{\mathcal{N}}(\mu), \lambda_1^{\mathcal{N}}(\mu), y^{\mathcal{N}}(\mu); \mu))^{1/2} \leq \tau(\mu). \tag{4.12}$$

We unfortunately have no means to verify whether the assumption $|\lambda_1(\mu) - \lambda_1^{\mathcal{N}}(\mu)| < |\lambda_2(\mu) - \lambda_1^{\mathcal{N}}(\mu)|$ is satisfied. However, in practice, we have found that smaller eigenvalues of (4.9) are well separated, and the associated eigenfunctions are well approximated even on very coarse meshes. Hence, $\tau_{\rm LB}^{\mathcal{N}}(\mu)$ defined by (4.12) provides a lower bound of the stability constant $\tau(\mu)$.

4.4.4 Offline-Online Efficient SCM and RB Bounds of $\tau(\mu)$

Lower Bound While the approach described in Sect. 4.4.3 provides a lower bound of the stability constant $\tau(\mu)$ under a plausible assumption, the approach requires FE approximations and is not suited for rapid online evaluation. To overcome the difficultly, we appeal to a version of the successive constraint method (SCM) of Huynh et al. [5] that has been extended to compute a lower bound of the stability constant with respect to an infinite-dimensional function spaces [17]. We refer to [5, 17] for detailed discussion of the algorithm; we here simply present the mechanics for completeness.

For notational simplicity, we first define an operator associated with the bilinear form $a(w, v; \mu)$, $A(\mu) \equiv |J(\mu)|Y(\mu)^{-1}EK(\mu)EY(\mu)^{-T}$. Because $K(\mu)$ and $Y(\mu)^{-1} = I_d \otimes J(\mu)^{-1}$ admit affine decompositions, $A(\mu)$ also admits an affine decomposition, which we denote by $A(\mu) = \sum_{q=1}^{Q_A} \Theta_q^A(\mu) A_q$. The number of terms in the affine expansion Q_A is at most $Q_J Q_{J\rm inv}^2 Q_K$.

The SCM computes the lower bound as follows. We first introduce a bounding box $B_{Q_A} \equiv \prod_{q=1}^{Q_A}[\hat{\gamma}_q^-, \hat{\gamma}_q^+] \subset \mathbb{R}^{Q_A}$, where $\hat{\gamma}_q^\pm \equiv \|\lambda_{\max}(A_q)\|_{L^\infty(\Omega)}$; we can readily evaluate $\|\lambda_{\max}(A_q)\|_{L^\infty(\Omega)}$ since A_q are known. We then define $\mathcal{Y}_{{\rm LB},M} \equiv \left\{ y \in B_{Q_A} \mid \sum_{q=1}^{Q_A} \Theta_q^A(\mu') \geq \tau_{\rm LB}^{\mathcal{N}}(\mu'),\ \forall \mu' \in \Xi_{\rm con} \right\}$; here $\Xi_{\rm con} \subset \mathcal{D}$ is a set of judiciously chosen "SCM constraint points" (e.g., by a greedy algorithm) of cardinality M, and $\tau_{\rm LB}^{\mathcal{N}}(\mu')$, $\mu' \in \Xi_{\rm con}$, are the FE approximations of lower bound of eigenvalues in (4.12). The SCM lower bound of $\tau(\mu)$ is then given by

$$\tau_{{\rm LB},M}(\mu) = \inf_{y \in \mathcal{Y}_{{\rm LB},M}} \sum_{q=1}^{Q_A} \Theta_q^A(\mu) y_q. \tag{4.13}$$

We can readily show $\tau_{{\rm LB},M}(\mu) \leq \tau(\mu)$; we refer to [5] or [17] for a proof.

The SCM algorithm is online-offline efficient: in the offline stage, we evaluate the constants $\{\gamma_q^\pm\}$ by taking the L^∞-norm of A_q and $\{\tau_{\rm LB}^{\mathcal{N}}(\mu')\}_{\mu' \in \Xi_{\rm con}}$ by solving $M \equiv |\Xi_{\rm con}|$ FE problems (4.12); in the online stage, we solve a linear programming problem (4.13) with Q_A variables and M inequality constraints.

Upper Bound While bounding the error in the online stage requires only the lower bound $\tau_{\mathrm{LB},M}(\mu)$, our offline training algorithm also requires a rapidly computable upper bound of $\tau(\mu)$ to select Ξ_{con}. Towards this end, we appeal to a Galerkin RB approximation of $\tau(\mu)$ (c.f. [12]). We introduce a RB space spanned by the eigenfunctions associated with M parameter values: $\mathcal{V}_M^{\mathrm{eig}} = \mathrm{span}\{z_1^{\mathcal{N}}(\mu')\}_{\mu' \in \Xi_{\mathrm{con}}}$. We then solve a RB eigenproblem: given $\mu \in \mathcal{D}$, find $(z_{M,1}(\mu), \lambda_{M,1}(\mu)) \in \mathcal{V}_M^{\mathrm{eig}} \times \mathbb{R}$ such that $\|z_{M,1}(\mu)\|_{\mathcal{V}} = 1$ and

$$a(z_{M,1}(\mu), v; \mu) = \lambda_{M,1}(\mu)(z_{M,1}(\mu), v)_{\mathcal{V}} \quad \forall v \in \mathcal{V}_M^{\mathrm{eig}}. \tag{4.14}$$

Because $\lambda_1(\mu) \equiv \inf_{v \in \mathcal{V}} |||v|||_\mu^2 / \|v\|_{\mathcal{V}}^2 \leq \inf_{v \in \mathcal{V}_M^{\mathrm{eig}}} |||v|||_\mu^2 / \|v\|_{\mathcal{V}}^2 \equiv \lambda_{1,M}(\mu)$, we conclude $\tau(\mu) \equiv \lambda_1(\mu) \leq \lambda_1^{\mathcal{N}}(\mu) \equiv \tau_{\mathrm{UB},M}(\mu)$. We hence set $\tau_{\mathrm{UB},M}(\mu) \equiv \lambda_{M,1}(\mu)$. The RB eigenproblem (4.14) is amenable to offline-online computational decomposition because the form $a(\cdot, \cdot; \mu)$ admits an affine decomposition. In addition, the basis $\mathcal{V}_M^{\mathrm{eig}}$ is generated as a biproduct of computing $\{\tau_{\mathrm{LB}}^{\mathcal{N}}(\mu')\}_{\mu' \in \Xi_{\mathrm{con}}}$ by FE eigenproblem (4.11) in the offline stage.

4.5 Error Bounds

Bounds Having devised offline-online efficient approach for computing an upper bound of the dual norm of the residual and a lower bound of the stability constant, we appeal to Proposition 2 to obtain a computable bound of an energy norm of the error:

$$|||u(\mu) - u_N(\mu)|||_\mu \leq \Delta_N(\mu) \equiv \frac{1}{(\alpha_{\mathrm{LB},M}(\mu;\delta))^{1/2}} (F(u_N(\mu), p_N(\mu); \mu; \delta))^{1/2}.$$

Similarly, we appeal to Proposition 3 to define an approximate compliance output $s_N(\mu) = \ell(u_N(\mu)) + r(u_N(\mu), u_N(\mu); \mu)$ and to provide an error bound

$$|s(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu) \equiv \frac{1}{\alpha_{\mathrm{LB},M}(\mu;\delta)} F(u_N(\mu), p_N(\mu); \mu; \delta).$$

We note that the term $r(u_N(\mu), u_N(\mu); \mu)$ is nonzero because our approximation $u_N(\mu)$ is based on the minimum-bound formulation and not a Galerkin projection.

Complementary Variational Principle There exists a close relationship between our error bound formulation and finite-element error bounds based on the complementary variational principle in, e.g., [6, 9]. If we consider the limit of $\delta \to 0$ for our norm $\|\cdot\|_{\mathcal{W}(\mu;\delta)}$, our bound form (4.7) expressed in the *physical domain* $\tilde{\Omega}(\mu)$

becomes

$$F(w,q;\mu;\delta) = \begin{cases} \|\tilde{C}(\mu)^{1/2}\tilde{q} - \tilde{K}(\mu)^{1/2}\tilde{\nabla}\tilde{w}\|^2_{L^2(\tilde{\Omega})}, & q \in \tilde{\mathcal{Q}}^{\star}(\mu), \\ \infty, \quad q \notin \mathcal{Q}^{\star}(\mu), \end{cases}$$

where

$$\tilde{\mathcal{Q}}^{\star}(\mu) = \{\tilde{q} \in \tilde{\mathcal{Q}}(\mu) \mid -\tilde{\nabla}\cdot\tilde{q} = \tilde{f}(\mu),\ \tilde{n}\cdot\tilde{q} = g(\mu),\ \tilde{q}\text{-tensor is symmetric}\} \tag{4.15}$$

The associated stability constant for $\delta \to 0$ is $\lim_{\delta\to 0}\alpha(\mu;\delta) = 1$.

The conditions that define $\tilde{\mathcal{Q}}(\mu)$ in (4.15) are the dual-feasibility conditions associated with the complementary variational principle. The symmetry of the dual field is a required condition for linear elasticity [9], which is not present for scalar equations. In addition, for $\tilde{q} \in \tilde{Q}(\mu)$, the complementary variational principle yields $|||\tilde{w}|||^2_{\mu} \leq \|\tilde{C}(\mu)^{1/2}\tilde{q} - \tilde{K}(\mu)^{1/2}\tilde{\nabla}\tilde{w}\|^2_{L^2(\tilde{\Omega})}$, which implies that the stability constant is unity. Hence, our bound formulation in the limit $\delta \to 0$ is equivalent to the complementary variational principle.

For $\delta > 0$, our approach is a "relaxation" of the complementary variational principle in the sense that it does not require the dual field to lie in the dual-feasible space (4.15). This relaxation facilitates offline-online decomposition, as the construction of the parameter-dependent dual-feasible space $\tilde{\mathcal{Q}}^{\star}(\mu)$ in an online-efficient manner seems only possible for rather limited cases [15]. However, as a consequence, our stability constant $\alpha(\mu;\delta)$ is not unity, and we require an explicit computation of a lower bound of the stability constant.

4.6 Spatio-Parameter Adaptation

Our spatio-parameter adaptation algorithm for SCM and RB offline training are presented in [17]; we here reproduce the algorithms for completeness.

SCM The SCM training algorithm is shown as Algorithm 1. The algorithm leverages the offline-online efficient upper and lower bounds of τ introduced in Sect. 4.4. In short, the algorithm computes the relative bound gap for each $\mu \in \Xi_{\text{train}}$, identifies μ with the largest bound gap, computes $\tau^{\mathcal{N}}_{\text{UB}}$ and $\tau^{\mathcal{N}}_{\text{UB}}$ to prescribed accuracy $\epsilon_{\text{SCM,FE}}$ using the adaptive FE eigensolver, and updates the SCM constraint set and reduced basis for the eigenproblem. The process is repeated until the bound gap meets ϵ_{SCM} for all $\mu \in \Xi_{\text{train}}$. The two threshold parameters must satisfy $\epsilon_{\text{SCM,FE}} \leq \epsilon_{\text{SCM}} < 1$; in practice we set $\epsilon_{\text{SCM}} \approx 0.8$ and $\epsilon_{\text{SCM,FE}} \leq \epsilon_{\text{SCM,FE}}/2$.

Algorithm 1 Spatio-parameter adaptive SCM training

input : $\Xi_{\text{train}} \subset \mathcal{D}$: SCM training set
$\epsilon_{\text{SCM}}, \epsilon_{\text{SCM,FE}}$: greedy and finite-element bound-gap tolerances
output : $\{\tau_{\text{LB}}^{\mathcal{N}}(\mu')\}_{\mu' \in \Xi_{\text{con}}}$: SCM constraints
$\mathcal{V}_M^{\text{eig}} = \{z_1^{\mathcal{N}}(\mu')\}_{\mu' \in \Xi_{\text{con}}}$: RB eigenproblem space

1 **for** $M = 1, 2, \ldots$ **do**
2 Identify the maximum relative $\tau(\mu)$ gap parameter $\mu^{(M)} = \arg\sup_{\mu \in \Xi_{\text{train}}} (\tau_{\text{UB},M-1}(\mu) - \tau_{\text{LB},M-1}(\mu))/\tau_{\text{UB},M-1}(\mu)$.
3 If $\sup_{\mu \in \Xi_{\text{train}}} (\tau_{\text{UB},M}(\mu) - \tau_{\text{LB},M}(\mu))/\tau_{\text{UB},M}(\mu) < \epsilon_{\text{SCM}}$, terminate.
4 Solve (4.11) and (4.12) to obtain eigenpair $(z_1^{\mathcal{N}}(\mu^{(M)}), \lambda_1^{\mathcal{N}}(\mu^{(M)}) \equiv \tau_{\text{UB}}^{\mathcal{N}}(\mu^{(M)}))$ and a lower bound $\tau_{1,\text{LB}}^{\mathcal{N}}(\mu)$; invoke mesh adaptivity as necessary such that $(\tau_{\text{UB}}^{\mathcal{N}}(\mu_M) - \tau_{\text{LB}}^{\mathcal{N}}(\mu_M))/\tau_{\text{UB}}^{\mathcal{N}}(\mu) < \epsilon_{\text{SCM,FE}}$.
5 Augment the SCM constraint set, $\Xi_{\text{con}} \leftarrow \Xi_{\text{con}} \cup \mu^{(M)}$, and update $\{\tau_{\text{LB}}^{\mathcal{N}}(\mu')\}_{\mu' \in \Xi_{\text{con}}}$ and $\mathcal{V}_M^{\text{eig}} = \{z_1^{\mathcal{N}}(\mu')\}_{\mu' \in \Xi_{\text{con}}}$ accordingly.
6 **end**

Algorithm 2 Spatio-parameter adaptive RB training

input : Ξ_{train}: RB training set
$\epsilon_{\text{RB}}, \epsilon_{\text{RB,FE}}$: greedy and finite-element error tolerance
output : $\mathcal{V}_N$, $\mathcal{Q}_N$: RB spaces

1 **for** $N = 1, 2, \ldots$ **do**
2 Identify the maximum bound parameter $\mu^{(N)} = \arg\sup_{\mu \in \Xi_{\text{train}}} \Delta_{N-1}(\mu)$.
3 If $\sup_{\mu \in \Xi_{\text{train}}} \Delta_{N-1}(\mu) \le \epsilon_{\text{RB}}$, terminate.
4 Solve (4.8) to obtain FE approximations $u^{\mathcal{N}}(\mu^{(N)})$ and $p^{\mathcal{N}}(\mu^{(N)})$; invoke mesh adaptivity as necessary such that $\Delta^{\mathcal{N}}(\mu) \le \epsilon_{\text{RB,FE}}$.
5 Update RB spaces: $\mathcal{V}_N = \text{span}\{\mathcal{V}_{N-1}, u^{\mathcal{N}}(\mu^{(N)})\}$ and $\mathcal{Q}_N = \text{span}\{\mathcal{Q}_{N-1}, p^{\mathcal{N}}(\mu^{(N)})\}$.
6 **end**

RB The RB training algorithm is shown as Algorithm 2. The algorithm leverages the offline-online efficient error bound Δ_N. In short, the algorithm computes the error bound for each $\mu \in \Xi_{\text{train}}$, identifies μ with the largest error bound, approximate the solution to prescribed accuracy using the adaptive mixed FE solver, and updates the reduced basis. The process is repeated until the error bound meets ϵ_{RB} for all $\mu \in \Xi_{\text{train}}$. The two threshold parameters must satisfy $\epsilon_{\text{RB,FE}} \le \epsilon_{RB}$; in practice we set $\epsilon_{\text{RB,FE}} \le \epsilon_{\text{RB}}/2$. We set $\delta \equiv \min_{\mu \in \Xi_{\text{train}}} \tau_{\text{LB},M}(\mu)/10$ throughout the training (and in online evaluation); the choice ensures that the stability constant satisfies $10/11 \le \alpha_{\text{LB},M}(\mu) \le 1$ and in particular is close to unity.

The reduced model constructed by Algorithms 1 and 2 provides an RB approximation $u_N(\mu)$ such that the error $|||u(\mu) - u_N(\mu)|||_\mu$ with respect to the *exact* solution is guaranteed to be less than ϵ_{RB} for all $\mu \in \Xi_{\text{train}}$; for $\mu \notin \Xi_{\text{train}}$, the model may yield an approximation with an error greater than ϵ_{RB}, but the approximation is nevertheless equipped with an error bound with respect to the *exact* solution.

4.7 Numerical Results

4.7.1 Problem Description

We consider a linear elasticity problem associated with a cracked square patch of unit-length edges shown in Fig. 4.1. We will refer to the crack embedded in the domain as the "embedded crack" and crack in the center as the "primary crack." Two parameters characterize the embedded crack: the first parameter, $\mu_1 \in [0.25, 0.4]$, controls the vertical location of the crack; the second parameter, $\mu_2 \in [0.3, 0.7]$, controls the length of the crack. The patch is clamped along Γ_D, is subjected to vertical traction force along Γ_T, and is traction-free on all other boundaries. The output of interest is compliance.

4.7.2 Uniform Spatio-Parameter Refinement

We first solve the parametrized cracked patch problem using uniform refinement. The spatial meshes are obtained by uniformly refining the initial mesh shown in Fig. 4.2a. The snapshot locations are 2^2, 3^2, 4^2, and 5^2 equispaced points over $\mathcal{D} \equiv [0.25, 0.4] \times [0.3, 0.7]$. All mixed FE discretization is based on $\mathbb{P}^3$ Lagrange and $\mathbb{RT}^2$ Raviart-Thomas elements. For the purpose of assessment, the error bounds are computed on the sampling set $\Xi \subset \mathcal{D}$ consisting of $31 \times 41 = 1271$ equidistributed parameter points.

Figure 4.2b shows the result of the uniform refinement study. On the coarsest mesh with $\mathcal{N} = 1008$ degrees of freedom, the output error bound stagnates for $N \geq 9$ and is of $\mathcal{O}(1)$ independent of the number of snapshots; the error is dominated by the insufficient spatial resolution. Even on the finest mesh with $\mathcal{N} \approx 220{,}000$, the convergence of the error bound is affected by the spatial resolution for $N \geq 16$. This behavior is due to the relatively slow convergence of the FE method in the presence of spatial singularity and a rapid convergence of the RB method for the parametrically smooth problem.

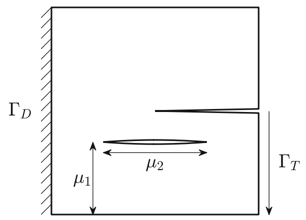

Fig. 4.1 Geometry and parametrization of the cracked patch problem

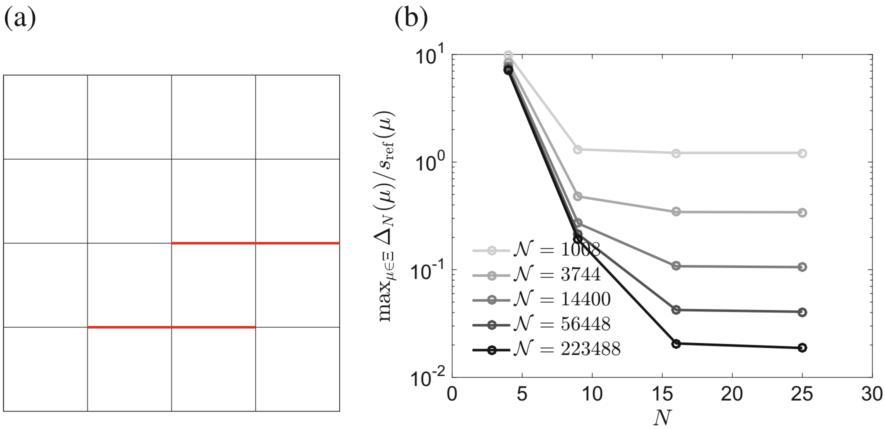

Fig. 4.2 Uniform refinement convergence study: (**a**) initial mesh with the cracks denoted in *red*; (**b**) convergence with N for several FE meshes

4.7.3 Spatio-Parameter Adaptive SCM and RB Refinement

SCM We now apply the spatio-parameter adaptive SCM training, Algorithm 1, using threshold parameters $\epsilon_{\rm SCM} = 0.8$ and $\epsilon_{\rm SCM,FE} = 0.2$. Figure 4.3 summarizes the result of the training process. Figure 4.3a shows that the dimension of the adaptive FE space varies from ≈ 3500 to ≈ 7500, depending on the configuration. Figure 4.3b shows that the target maximum relative SCM bound gap of $\epsilon_{\rm SCM} = 0.8$ is achieved using $M = 40$ constraint points for all $\mu \in \Xi \subset \mathcal{D}$. Figure 4.3c shows that, similar to the original SCM [5], the SCM lower bound of the eigenvalue is rather pessimistic away from the constraint points; as discussed earlier, we accept the pessimistic estimate for the rigor it provides, and in any event the effectivity of the stability constant $\alpha_{{\rm LB},M}$ will be desensitized from the pessimistic estimate $\tau_{{\rm LB},M}$ thanks to the transformation introduced in Sect. 4.4.1. Figure 4.3d shows that the Galerkin approximation of the upper bound—which in fact approximates very closely the true value of τ—varies smoothly over the parameter domain. The minimum $\tau_{\rm LB}$ is bounded from the below by 0.0018; we hence set $\delta = 0.00018$ to ensure that $\alpha_{{\rm LB},M}(\mu) > 0.9$.

In order to more closely analyze the adaptive FE approximation of the stability eigenproblem, we show in Fig. 4.4 the adaptation behavior for two configurations associated with the smallest and largest FE spaces. Figure 4.4a–c summarize the behavior for $\mu^{(6)}$, the configuration where the embedded crack is shortest and is far from the primary crack; the final $\mathcal{N} = 3514$ mesh exhibits strong refinement

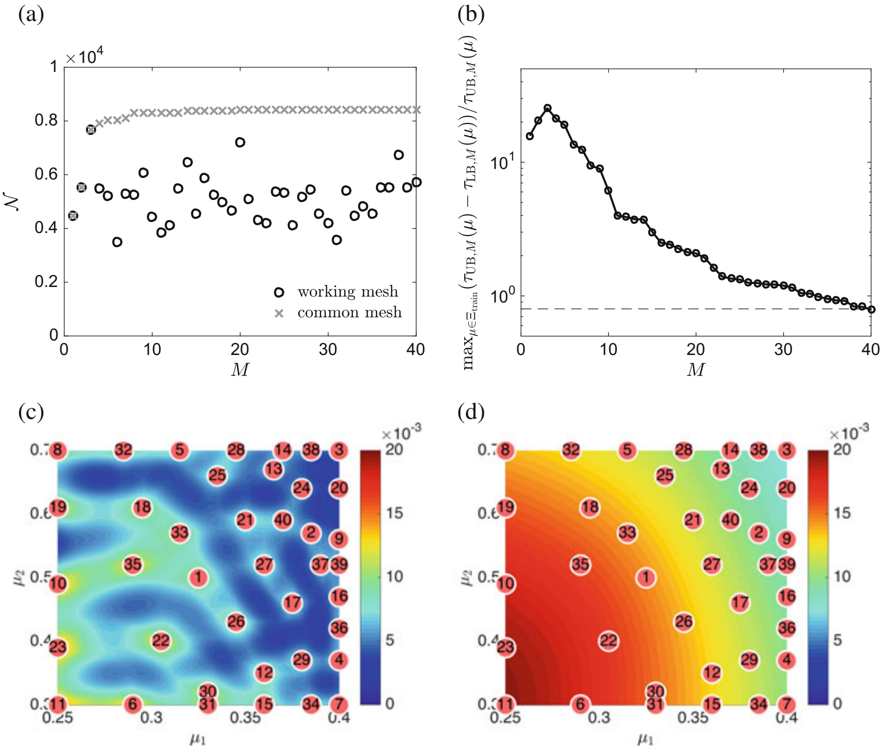

Fig. 4.3 Behavior of the spatio-parameter adaptive greedy method for SCM: (**a**) the dimension of the FE spaces; (**b**) reduction in the bound gap with number of SCM constraints; (**c**) SCM lower bound of τ over $\mathcal{D}$; (**d**) Galerkin reduced-basis upper bound of τ over $\mathcal{D}$

towards the primary crack tip, but relatively weak refinement towards the embedded crack tips. Figure 4.4d–f summarize the behavior for $\mu^{(3)}$, the configuration where the embedded crack is longest and is closest to the primary crack; the final $\mathcal{N} = 7690$ mesh exhibits much stronger refinement towards the embedded crack tips compared to the mesh for $\mu^{(6)}$. As shown in Fig. 4.4c and f, the lower bound is not as effective as the upper bound in general, but we accept the ineffectiveness for the rigor it provides.

RB We now train the RB model using the spatio-parameter adaptive method, Algorithm 2, for threshold parameters $\epsilon_{\rm RB} = 0.01$ and $\epsilon_{\rm RB,FE} = 0.005$. Figure 4.5 summarizes the result of the greedy training. Figure 4.5a shows that the number

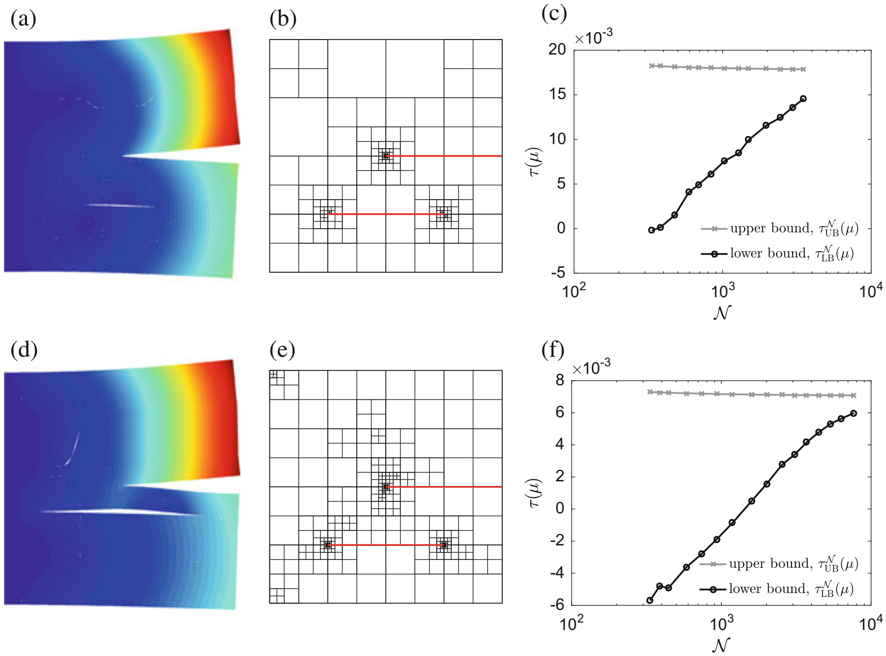

Fig. 4.4 Adaptive FE eigenproblem approximation for (**a**)–(**c**) $\mu^{(6)} = (0.29, 0.3)$ and (**d**)–(**f**) $\mu^{(3)} = (0.4, 0.7)$

of degrees of freedom varies from $\approx$13,000 to $\approx$21,000. Figure 4.5b shows the exponential convergence of the compliance output error with the dimension of the RB space; this is contrary to the behavior for uniform meshes for which the convergence with respect to the parameter dimension is limited by the insufficient spatial resolution. Figure 4.5c shows that reduced model produces an error less than $\epsilon_{\text{RB}} = 10^{-2}$ for any parameter value in $\mathcal{D}$ (or more precisely at least Ξ). Figure 4.5d shows that the final common mesh which reflects refinement required for all configurations over $\mathcal{D}$ exhibits strong refinement towards the crack tips and some corners.

As we have done for the eigenproblem, we show in Fig. 4.6 the adaptive FE solution for two configurations associated with the smallest and largest FE spaces. Figure 4.6a–c summarize the behavior for $\mu^{(17)}$, the configuration where the embedded crack is shortest and far from the primary crack; the final $\mathcal{N} = 13{,}270$ mesh shows relatively weak refinement towards the embedded crack tips. Figure 4.6d–f summarize the behavior for $\mu^{(2)}$, the configuration where the embedded crack

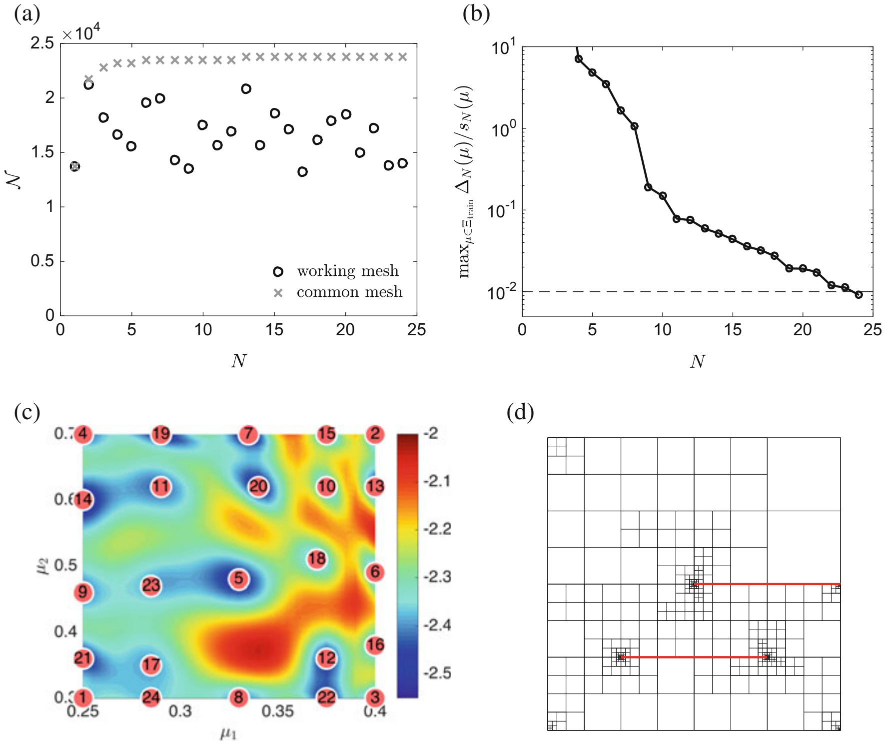

Fig. 4.5 Behavior of the spatio-parameter adaptive RB generation: (**a**) the dimension of the FE spaces; (**b**) reduction in the error bound with the dimension of RB space; (**c**) output error bound over $\mathcal{D}$; (**d**) final common mesh

is longest and closest to the primary crack; we observe much stronger refinement towards all crack tips. For both cases, the effectivity of the compliance output error bound is less than 10, which is acceptable given that this is (rigorous) bounds of the error in the outputs. For assessment purpose, the reference output is computed using an adaptive FE method with an error tolerance that is ten times tighter than the target tolerance.

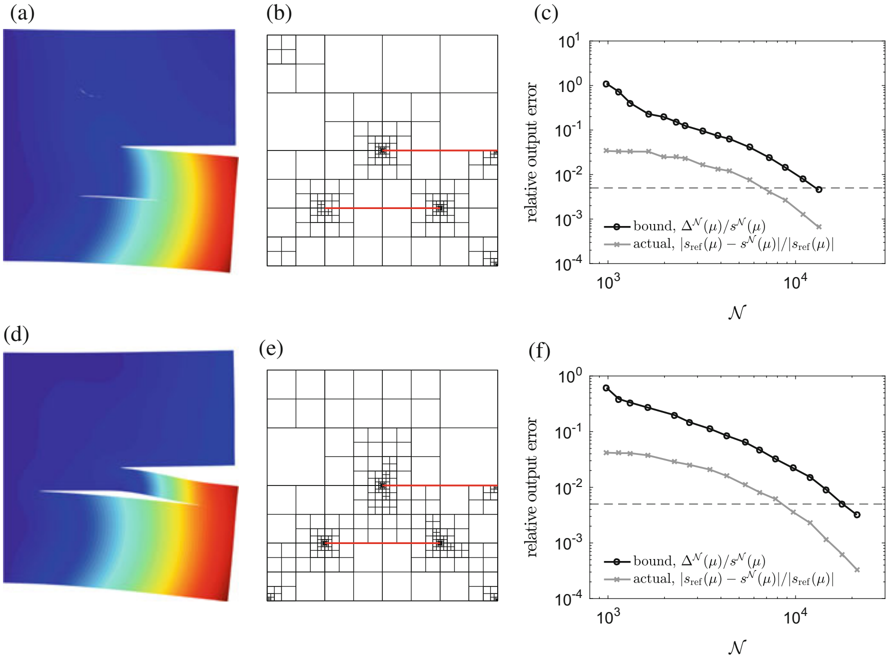

Fig. 4.6 Adaptive FE approximation for (**a**)–(**c**) $\mu^{(17)} = (0.285, 0.35)$ and for (**d**)–(**f**) $\mu^{(2)} = (0.4, 0.7)$

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