Chapter 3
A Certified Reduced Basis Approach for Parametrized Optimal Control Problems with Two-Sided Control Constraints

Eduard Bader, Martin A. Grepl, and Karen Veroy

Abstract In this paper, we employ the reduced basis method for the efficient and reliable solution of parametrized optimal control problems governed by elliptic partial differential equations. We consider the standard linear-quadratic problem setting with distributed control *and* two-sided control constraints, which play an important role in many industrial and economical applications. For this problem class, we propose two different reduced basis approximations and associated error estimation procedures. In our first approach, we directly consider the resulting optimality system, introduce suitable reduced basis approximations for the state, adjoint, control, and Lagrange multipliers, and use a projection approach to bound the error in the reduced optimal control. For our second approach, we first reformulate the optimal control problem using two slack variables, we then develop a reduced basis approximation for both slack problems by suitably restricting the solution space, and derive error bounds for the slack based optimal control. We discuss benefits and drawbacks of both approaches and substantiate the comparison by presenting numerical results for a model problem.

3.1 Introduction

Optimal control problems governed by partial differential equations (PDEs) appear in a wide range of applications in science and engineering, such as heat phenomena, crystal growth, and fluid flow (see, e.g., [4, 8]). Their solution using classical discretization techniques such as finite elements (FE) or finite volumes can be

E. Bader (✉) • K. Veroy
Aachen Institute for Advanced Study in Computational Engineering Science (AICES), RWTH Aachen University, Schinkelstraße 2, 52062 Aachen, Germany
e-mail: bader@aices.rwth-aachen.de; veroy@aices.rwth-aachen.de

M.A. Grepl
Numerical Mathematics, RWTH Aachen University, Templergraben 55, 52056 Aachen, Germany
e-mail: grepl@igpm.rwth-aachen.de

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computationally expensive and time-consuming. Often, additional parameters enter the problem, e.g., material or geometry parameters in a design exercise.

Previous work on reduced order methods for optimal control problems considered distributed but unconstrained controls or constrained but scalar controls. Elliptic optimal control problems with distributed control have been considered recently by Negri et al. [10]. The proposed error bound is based on the Banach-Nečas-Babuška (BNB) theory applied to the first order optimality system. The approach thus provides a combined bound for the error in the state, adjoint, and control variable, but it is only applicable to problems *without* control constraints. Since the bound requires the very costly computation of a lower bound to the inf-sup constant, Negri et al. [11] compute error estimates using a heuristic interpolant surrogate of that constant.

Based on the ideas in Tröltzsch and Volkwein [13], Kärcher and Grepl [6] proposed rigorous *and* online-efficient control error bounds for reduced basis (RB) approximations of scalar elliptic optimal control problems. These ideas are extended and improved in [7] to distributed control problems.

In a recent paper [1], we employed the RB method as a surrogate model for the solution of distributed *and* one-sided constrained optimal control problems governed by parametrized elliptic partial differential equations. In this paper we extend this work to two-sided control constraints. After stating the problem in Sect. 3.2 we present the following contributions:

- In Sect. 3.3 we extend previous work on reduced basis methods for variational inequalities in [1, 3] to the optimal control setting with two-sided control constraints. While we can derive an offline-online decomposable RB optimality system, we are only able to derive a *partially* offline-online decomposable control error bound that depends on the FE dimension of the control.
- In Sect. 3.4 we build on the recent work in [1, 14] and propose an RB slack approach for optimal control. We introduce two slack formulations for the optimal control problem, which we obtain by shifting the optimal control by each constraint. We are thus able to derive an offline-online decomposable RB optimality systems *and* control error bound. The evaluation of this bound is independent of the FE dimension of the problem, but requires the solution of three RB systems.

In Sect. 3.5 we propose a greedy sampling procedure to construct the RB spaces and in Sect. 3.6 we assess the properties of our methods by presenting numerical results for a Graetz flow problem.

3.2 General Problem Statement and Finite Element Discretization

In this section we introduce the parametrized linear-quadratic optimal control problem with elliptic PDE constraint and a constrained distributed control. We introduce a finite element (FE) discretization for the continuous problem and recall the first-order necessary (and in the convex setting sufficient) optimality conditions.

3.2.1 Preliminaries

Let Y_e with $H_0^1(\Omega) \subset Y_e \subset H^1(\Omega)$ be a Hilbert space over the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}$, with boundary Γ.[1] The inner product and induced norm associated with Y_e are given by $(\cdot,\cdot)_Y$ and $\|\cdot\|_Y = \sqrt{(\cdot,\cdot)_Y}$. We assume that the norm $\|\cdot\|_Y$ is equivalent to the $H^1(\Omega)$-norm and denote the dual space of Y_e by Y_e'. We also introduce the control Hilbert space $U_e = L^2(\Omega)$, together with its inner product $(\cdot,\cdot)_U$, induced norm $\|\cdot\|_U = \sqrt{(\cdot,\cdot)_U}$, and associated dual space U_e'.[2] Furthermore, let $\mathscr{D} \subset \mathbb{R}^P$ be a prescribed P-dimensional compact parameter set in which the P-tuple (input) parameter $\mu = (\mu_1, \ldots, \mu_P)$ resides.

We directly consider a FE "truth" approximation for the exact infinite-dimensional optimal control problem. To this end, we define two conforming FE spaces $Y \subset Y_e$ and $U \subset U_e$ and denote their dimensions by $\mathscr{N}_Y = \dim(Y)$ and $\mathscr{N}_U = \dim(U)$. We assume that the truth spaces Y and U are sufficiently rich such that the FE solutions guarantee a desired accuracy over $\mathscr{D}$.

We next introduce the μ-dependent bilinear form $a(\cdot,\cdot;\mu) : Y \times Y \to \mathbb{R}$, and shall assume that $a(\cdot,\cdot;\mu)$ is (1) continuous for all $\mu \in \mathscr{D}$ with continuity constant $\gamma_a(\mu) < \infty$ and (2) coercive for all $\mu \in \mathscr{D}$ with coercivity constant $\alpha_a(\mu) > 0$. Furthermore, we introduce the μ-dependent continuous linear functional $f(\cdot;\mu) : Y \to \mathbb{R}$ and the bilinear form $b(\cdot,\cdot;\mu) : U \times Y \to \mathbb{R}$ with continuity constant $\gamma_b(\mu) < \infty$.

In anticipation of the optimal control problem, we introduce the parametrized control constraints $u_a(\mu), u_b(\mu) \in U$ and a desired state $y_d \in D$. Here, $D \subset L^2(\Omega_D)$ is a suitable FE space for the observation subdomain $\Omega_D \subset \Omega$. Furthermore, we note that the semi-norm $|\cdot|_D$ for $y \in L^2(\Omega)$ is defined by $|\cdot|_D = \|\cdot\|_{L^2(\Omega_D)}$.

The involved bilinear and linear forms as well as the control constraint are assumed to depend affinely on the parameter. For example we require for all $w, v \in Y$ and all parameters $\mu \in \mathscr{D}$ that $a(w, v;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, a^q(w, v)$ and $u_a(x;\mu) = \sum_{q=1}^{Q_{ua}} \Theta_{ua}^q(\mu)\, u_a^q(x)$ for some (preferably) small integers Q_a and Q_{ua}. Here, the coefficient functions $\Theta_\bullet^q(\cdot) : \mathscr{D} \to \mathbb{R}$ are continuous and depend on μ, whereas the continuous bilinear and linear forms, e.g., $a^q(\cdot,\cdot)$ and $u_a^q \in U$ do *not* depend on μ. Although we choose $y_d(x)$ to be parameter-independent, our approach directly extends to an affinely parameter-dependent $y_d(x;\mu)$ (see Kärcher et al. [7]).

[1] The subscript "e" denotes the "exact" infinite-dimensional continuous problem setting.

[2] The framework of this work directly extends to Neumann boundary controls $U_e = L^2(\partial\Omega)$ or finite dimensional controls $U_e = \mathbb{R}^m$. Also distributed controls on a subdomain $\Omega_U \subset \Omega$ or Neumann boundary controls on a boundary segment $\Gamma_U \subset \partial\Omega$ are possible.

For the development of a posteriori error bounds we also require additional ingredients. We assume that we are given a positive lower bound $\alpha_a^{\mathrm{LB}}(\mu) : \mathscr{D} \to \mathbb{R}_+$ for the coercivity constant $\alpha_a(\mu)$ of $a(\cdot,\cdot;\mu)$ such that $\alpha_a^{\mathrm{LB}}(\mu) \le \alpha_a(\mu)\ \forall \mu \in \mathscr{D}$. Furthermore, we assume that we have upper bounds available for the constant $C_D^{\mathrm{UB}} \ge C_D = \sup_{w \in Y\setminus\{0\}} \frac{|w|_D}{\|w\|_Y} \ge 0\ \forall \mu \in \mathscr{D}$, and the continuity constant of the bilinear form $b(\cdot,\cdot;\mu)$: $\gamma_b^{\mathrm{UB}}(\mu) \ge \gamma_b(\mu)\ \forall \mu \in \mathscr{D}$. Here, the constant C_D depends on the parameter, since later we use $|\cdot|_D = \|\cdot\|_{L^2(\Omega_D(\mu))}$ (see Sect. 3.6). In our setting, it is possible to compute these constants (or their bounds) efficiently using an offline-online procedure (see [7, 12]).

3.2.2 Abstract Formulation of Linear-Quadratic Optimal Control Problems and the First-Order Optimality Conditions

We consider the following FE optimal control problem in weak form with $u_a(\mu) < u_b(\mu)$

$$\min_{\hat{y},\hat{u}} J(\hat{y},\hat{u}) = \frac{1}{2}|\hat{y} - y_d|_D^2 + \frac{\lambda}{2}\|\hat{u}\|_U^2, \qquad \lambda > 0 \tag{P}$$

$$\text{s.t.} \quad (\hat{y},\hat{u}) \in Y \times U \quad \text{solves} \quad a(\hat{y},\phi;\mu) = b(\hat{u},\phi;\mu) + f(\phi;\mu) \quad \forall \phi \in Y,$$

$$(u_a(\mu),\rho)_U \le (\hat{u},\rho)_U \le (u_b(\mu),\rho)_U \quad \forall \rho \in U^+,$$

where $U^+ := \{\rho \in U; \rho \ge 0 \text{ almost everywhere}\}$ and we dropped the μ-dependence of the state and control $(\hat{y},\hat{u})$ for the sake of readability. We note that the last line of (**P**) is equivalent to $\hat{u}$ being in the convex admissible set

$$U_{\mathrm{ad}} = \{\psi \in U; (u_a(\mu),\rho)_U \le (\psi,\rho)_U \le (u_b(\mu),\rho)_U\ \forall \rho \in U^+\}. \tag{3.1}$$

In the following we call problem (**P**) the "primal" problem, for which the existence and uniqueness of the solution is standard (see, e.g., [4]). The derivation of the necessary and sufficient first-order optimality system is straightforward: Given $\mu \in \mathscr{D}$, the optimal solution $(y,p,u,\sigma,\sigma_b) \in Y \times Y \times U \times U \times U$ satisfies

$$a(y,\phi;\mu) = b(u,\phi;\mu) + f(\phi;\mu) \qquad \forall \phi \in Y, \tag{3.2a}$$

$$a(\varphi,p;\mu) = (y_d - y,\varphi)_D \qquad \forall \varphi \in Y, \tag{3.2b}$$

$$(\lambda u,\psi)_U - b(\psi,p;\mu) = (\sigma,\psi)_U - (\sigma_b,\psi)_U \qquad \forall \psi \in U, \tag{3.2c}$$

$$(u_a(\mu) - u,\rho)_U \le 0 \quad \forall \rho \in U^+, \qquad (u_a(\mu) - u,\sigma)_U = 0, \quad \sigma \ge 0, \tag{3.2d}$$

$$(u_b(\mu) - u,\rho)_U \ge 0 \quad \forall \rho \in U^+, \qquad (u_b(\mu) - u,\sigma_b)_U = 0, \quad \sigma_b \ge 0. \tag{3.2e}$$

Note that we follow a first-discretize-then-optimize approach here, for a more detailed discussion see [4, Sect. 3.2.4]).

In the following we comment on the FE-setting in this paper. We assume that the state variable is discretized by P_1, i.e., continuous and piecewise linear, and the control variable by P_0, i.e., piecewise constant finite elements. Next, we introduce two bases for the FE spaces Y and U, such that

$$Y = \mathrm{span}\{\, \phi_i^y, i = 1, \ldots, \mathcal{N}_Y\} \quad \text{and} \quad U = \mathrm{span}\{\, \phi_i^u, i = 1, \ldots, \mathcal{N}_U\},$$

where $\phi_i^y \geq 0,\ i = 1, \ldots, \mathcal{N}_Y$, and $\phi_i^u \geq 0,\ i = 1, \ldots, \mathcal{N}_U$, are the usual hat and bar basis functions. Using these basis functions we can express the functions $y, p \in Y$ and $u, \sigma, \sigma_b \in U$ as, e.g., $y = \sum_{i=1}^{\mathcal{N}_Y} y_i \phi_i^y$. The corresponding FE coefficient vectors are given by, e.g., $\underline{y} = (y_1, \ldots, y_{\mathcal{N}_Y})^T \in \mathbb{R}^{\mathcal{N}_Y}$. Note that by definition of U^+ and since $\phi_i^u \geq 0$, the condition $\rho \in U^+$ in (3.2d) and (3.2e) translates into the condition $\underline{\rho} \geq 0$ for the corresponding coefficient vector. Further, we also introduce the control mass matrix M_U with entries $(M_U)_{ij} = (\phi_i^u, \phi_j^u)_U$, which is for a P_0 control discretization a positive diagonal matrix. Hence the point-wise and the 'weak' (averaged) constraint formulations are equivalent $u(x) \geq u_a(x; \mu) \Leftrightarrow (u, \rho)_U \geq (u_a(\mu), \rho)_U \ \forall \rho \in U^+$. However, this is in general not true for other control discretizations, e.g., P_1.

Based on the truth FE primal problem (**P**) we derive an RB primal problem (**P$_\mathbf{N}$**) and a rigorous a posteriori error bound for the error between the truth and RB control approximation in Theorem 1.

3.3 Reduced Basis Method for the Primal Problem

3.3.1 Reduced Basis Approximation

To begin, we define the RB spaces $Y_N \subset Y$, U_N, Σ_N, $\Sigma_{b,N} \subset U$ as well as the convex cones $\Sigma_N^+ \subset U^+$, $\Sigma_{b,N}^+ \subset U^+$ as follows: given N parameter samples $\mu^1, \ldots, \mu^N$, we set

$$Y_N = \mathrm{span}\{\, \zeta_1^y, \ldots, \zeta_{N_Y}^y\} = \mathrm{span}\{\, y(\mu^1), p(\mu^1), \ldots, y(\mu^N), p(\mu^N)\,\}, \tag{3.3a}$$

$$U_N = \mathrm{span}\{\zeta_1^u, \ldots, \zeta_{N_U}^u\} = \mathrm{span}\{u(\mu^1), \sigma(\mu^1), \sigma_b(\mu^1), \ldots, u(\mu^N), \sigma(\mu^N), \sigma_b(\mu^N)\}, \tag{3.3b}$$

$$\Sigma_N = \mathrm{span}\{\, \zeta_1^\sigma, \ldots, \zeta_{N_\sigma}^\sigma\} = \mathrm{span}\{\, \sigma(\mu^1), \ldots, \sigma(\mu^N)\,\}, \tag{3.3c}$$

$$\Sigma_{b,N} = \mathrm{span}\{\zeta_1^{\sigma_b}, \ldots, \zeta_{N_{\sigma_b}}^{\sigma_b}\} = \mathrm{span}\{\sigma_b(\mu^1), \ldots, \sigma_b(\mu^N)\}, \tag{3.3d}$$

$$\Sigma_N^+ = \mathrm{span}_+\{\, \zeta_1^\sigma, \ldots, \zeta_{N_\sigma}^\sigma\} \quad \text{and} \quad \Sigma_{b,N}^+ = \mathrm{span}_+\{\zeta_1^{\sigma_b}, \ldots, \zeta_{N_{\sigma_b}}^{\sigma_b}\}, \tag{3.3e}$$

where we assume that the basis functions, $\zeta_1^\bullet, \dots \zeta_{N_\bullet}^\bullet$, are linearly independent and $\mathrm{span}_+\{\cdot\}$ indicates the cone spanned by non-negative combinations of the elements, i.e.

$$\mathrm{span}_+\{\zeta_1, \dots, \zeta_N\} = \left\{ \sum_{i=1}^{N} \alpha_i \zeta_i \,|\alpha_i \geq 0 \right\} .$$

Note that we employ integrated spaces for the state and adjoint as well as for the control (see Remarks 1 and 2). For the spaces Y_N and U_N we additionally assume that the basis functions are orthogonal, i.e., $(\zeta_i^y, \zeta_j^y)_Y = \delta_{ij}$ and $(\zeta_i^u, \zeta_j^u)_U = \delta_{ij}$, where δ_{ij} is the Kronecker delta. This orthogonality is favorable to keep the condition of the RB algebraic linear systems small [12]. In addition, we do not orthogonalize the basis ζ_i^σ, $\zeta_i^{\sigma_b}$ of the cones Σ_N^+, $\Sigma_{b,N}^+ \subset U^+$, because this non-negativity is used in the definition of the reduced problem (**$\mathbf{P_N}$**) of (**P**).[3] Although the conditions $\zeta_i^\sigma \in \Sigma_N^+$ and $\zeta_i^{\sigma_b} \in \Sigma_{b,N}^+$ appear to be much more restrictive than ζ_i^σ, $\zeta_i^{\sigma_b} \in U^+$, we observe in numerical tests (not shown) that the RB approximations converge to the FE solutions with a similar rate as the control approximations. In addition, the RB approximations are comparable to the best possible approximations derived by projecting σ to Σ_N or Σ_N^+, analogously for σ_b. We describe the greedy sampling approach to construct the RB spaces in Sect. 3.5. Next, given the RB spaces in (3.3) we derive the RB primal problem

$$\min_{\hat{y}_N, \hat{u}_N} J(\hat{y}_N, \hat{u}_N) = \frac{1}{2}|\hat{y}_N - y_d|_D^2 + \frac{\lambda}{2}\|\hat{u}_N\|_U^2 \qquad (\mathbf{P_N})$$

$$\text{s.t.}\;\; (\hat{y}_N, \hat{u}_N) \in Y_N \times U_N \;\text{ solves }\; a(\hat{y}_N, \phi; \mu) = b(\hat{u}_N, \phi; \mu) + f(\phi; \mu) \;\forall \phi \in Y_N,$$

$$(u_a(\mu), \rho)_U \leq (\hat{u}_N, \rho)_U \;\forall \rho \in \Sigma_N^+,\; (u_b(\mu), \rho)_U \geq (\hat{u}_N, \rho)_U \;\forall \rho \in \Sigma_{b,N}^+.$$

The last line of (**$\mathbf{P_N}$**) defines the admissible set for u_N: $U_{\mathrm{ad},N} = \{\psi \in U_N; (u_a(\mu), \rho)_U \leq (\psi, \rho)_U \;\forall \rho \in \Sigma_N^+,\; (u_b(\mu), \rho)_U \geq (\psi, \rho)_U \;\forall \rho \in \Sigma_{b,N}^+\}$, which is in general *not* a subset of U_{ad} in (3.1). Analogously to the primal problem (**P**) we obtain the RB optimality system: Given $\mu \in \mathscr{D}$, the optimal solution $(y_N, p_N, u_N, \sigma_N, \sigma_{b,N}) \in Y_N \times Y_N \times U_N \times \Sigma_N \times \Sigma_{b,N}$ satisfies

$$a(y_N, \phi; \mu) = b(u_N, \phi; \mu) + f(\phi; \mu) \qquad \forall \phi \in Y_N, \tag{3.4a}$$

$$a(\varphi, p_N; \mu) = (y_d - y_N, \varphi)_D \qquad \forall \varphi \in Y_N, \tag{3.4b}$$

$$(\lambda u_N, \psi)_U - b(\psi, p_N; \mu) = (\sigma_N, \psi)_U - (\sigma_{b,N}, \psi)_U \qquad \forall \psi \in U_N, \tag{3.4c}$$

$$(u_a(\mu) - u_N, \rho)_U \leq 0 \;\forall \rho \in \Sigma_N^+,\; (u_a(\mu) - u_N, \sigma_N)_U = 0, \qquad \sigma_N \in \Sigma_N^+, \tag{3.4d}$$

$$(u_b(\mu) - u_N, \rho)_U \geq 0 \;\forall \rho \in \Sigma_{b,N}^+,\; (u_b(\mu) - u_N, \sigma_{b,N})_U = 0, \qquad \sigma_{b,N} \in \Sigma_{b,N}^+. \tag{3.4e}$$

[3] Alternative methods to deal with the non-negativity can be found in [2].

Remark 1 (Existence, Uniqueness, Integrated Space Y_N) Since ($\mathbf{P_N}$) is a linear-quadratic optimal control problem over the closed convex admissible set $U_{\text{ad},N}$, the existence and uniqueness of the RB optimal control u_N follows from standard arguments (see, e.g., [4, Theorem 1.43]). Also note that we use a single "integrated" reduced basis trial and test space Y_N for the state and adjoint equations as one ingredient to ensure stability of the system (3.4), see e.g. Kärcher [5].

Remark 2 (Stability, Integrated Space U_N) For the stability of the RB solutions we need to show that the RB inf-sup constants

$$\beta_N := \inf_{\psi_\sigma \in \Sigma_N} \sup_{\psi_u \in U_N} \frac{(\psi_\sigma, \psi_u)_U}{\|\psi_\sigma\|_U \|\psi_u\|_U}, \quad \beta_{b,N} := \inf_{\psi_{\sigma_b} \in \Sigma_{b,N}} \sup_{\psi_u \in U_N} \frac{(\psi_{\sigma_b}, \psi_u)_U}{\|\psi_{\sigma_b}\|_U \|\psi_u\|_U}$$

are bounded away from zero. We guarantee that $\beta_N, \beta_{b,N} \geq \beta > 0$ by enriching the RB control space with suitable supremizers [9]. Here, these supremizers are just the multiplier snapshots $\sigma(\mu^n), \sigma_b(\mu^n),\ 1 \leq n \leq N$; we thus have $\beta_N = \beta_{b,N} = \beta = 1$.

3.3.2 Primal Error Bound

We next propose an a posteriori error bound for the optimal control. The bound is based on [1], which uses an (1) RB approach for variational inequalities of the first kind [3], and (2) an RB approach for optimal control problems with a PDE constraint [7]. Before stating the main result, we define the following approximation errors (omitting μ-dependencies) of the RB primal system (3.4)

$$e_y = y - y_N, \quad e_p = p - p_N, \quad e_u = u - u_N, \quad e_\sigma = \sigma - \sigma_N, \quad e_{\sigma_b} = \sigma_b - \sigma_{b,N},$$

as well as the residuals in the next definition.

Definition 1 (Residuals) The residuals of the state equation, the adjoint equation w.r.t. (3.2a)–(3.2c) are defined for all $\mu \in \mathscr{D}$ by

$$r_y(\phi;\mu) = b(u_N,\phi;\mu) + f(\phi;\mu) - a(y_N,\phi;\mu) \qquad \forall \phi \in Y, \tag{3.5a}$$

$$r_p(\varphi;\mu) = (y_d - y_N, \varphi)_D - a(\varphi, p_N;\mu) \qquad \forall \varphi \in Y, \tag{3.5b}$$

$$r_u(\psi;\mu) = -\lambda(u_N,\psi)_U + b(\psi, p_N;\mu) + (\sigma_N,\psi)_U - (\sigma_{b,N},\psi)_U \qquad \forall \psi \in U. \tag{3.5c}$$

Theorem 1 (Primal Error Bound) *Let u and u_N be the optimal controls of the FE primal problem* ($\mathbf{P}$) *and of the RB primal problem* ($\mathbf{P_N}$)*, respectively. Then the error* *in the optimal control satisfies for any given parameter* $\mu \in \mathscr{D}$

$$\|e_u\|_U \leq \Delta_N^{\mathrm{pr}}(\mu),$$

where $\Delta_N^{\mathrm{pr}}(\mu) := c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)}$ *with nonnegative coefficients*

$$c_1(\mu) = \frac{1}{2\lambda}\left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}}\|r_p\|_{Y'} + \lambda(\delta_1 + \delta_{1b})\right), \tag{3.6a}$$

$$\begin{aligned} c_2(\mu) = \frac{1}{\lambda}\Bigg[& \frac{2}{\alpha_a^{\mathrm{LB}}}\|r_y\|_{Y'}\|r_p\|_{Y'} + \frac{1}{4}\left(\frac{C_D^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}}\left(\|r_y\|_{Y'} + \gamma_b^{\mathrm{UB}}(\delta_1 + \delta_{1b})\right)\right)^2 \\ & + \left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}}\|r_p\|_{Y'} + \sqrt{2(\sigma_N, \sigma_{b,N})_U}\right)(\delta_1 + \delta_{1b}) + \delta_2 + \delta_{2b}\Bigg], \end{aligned} \tag{3.6b}$$

and $\delta_1 = \|[u_a - u_N]_+\|_U$, $\delta_2 = ([u_a - u_N]_+, \sigma_N)_U$, $\delta_{1b} = \|[u_N - u_b]_+\|_U$, $\delta_{2b} = ([u_N - u_b]_+, \sigma_{b,N})_U$.
Here $[\cdot]_+ = \max(\cdot, 0)$ denotes the positive part (a.e.). Note that we sometimes use $r_\bullet$ instead of $r_\bullet(\cdot;\mu)$ and omit the μ-dependencies on the r.h.s. of (3.6) for a better readability.

Proof This proof follows the proof of the primal error bound from [1]. Since the FE optimal solution $(y, p, u, \sigma, \sigma_b)$ satisfies the optimality conditions (3.2), we obtain the following error-residual equations:

$$a(e_y, \phi; \mu) - b(e_u, \phi; \mu) = r_y(\phi; \mu) \quad \forall \phi \in Y, \tag{3.7a}$$

$$a(\varphi, e_p; \mu) + (e_y, \varphi)_D = r_p(\varphi; \mu) \quad \forall \varphi \in Y, \tag{3.7b}$$

$$\lambda(e_u, \psi)_U - b(\psi, e_p; \mu) - (e_\sigma, \psi)_U + (e_{\sigma_b}, \psi)_U = r_u(\psi; \mu) \quad \forall \psi \in U. \tag{3.7c}$$

From (3.7a) with $\phi = e_y$, (3.7b) with $\varphi = e_p$, and $\alpha_a^{\mathrm{LB}}(\mu) \leq \alpha_a(\mu)$ we infer that

$$\|e_y\|_Y \leq \frac{1}{\alpha_a^{\mathrm{LB}}}\left(\|r_y\|_{Y'} + \gamma_b^{\mathrm{UB}}\|e_u\|_U\right), \quad \|e_p\|_Y \leq \frac{1}{\alpha_a^{\mathrm{LB}}}\left(\|r_p\|_{Y'} + C_D^{\mathrm{UB}}|e_y|_D\right). \tag{3.8}$$

Choosing $\phi = e_p$, $\varphi = e_y$, $\psi = e_u$ in (3.7), adding (3.7b) and (3.7c), and subtracting (3.7a) results in

$$\lambda\|e_u\|_U^2 + |e_y|_D^2 \leq \|r_y\|_{Y'}\|e_p\|_Y + \|r_p\|_{Y'}\|e_y\|_Y + \|r_u\|_{U'}\|e_u\|_U + (e_\sigma - e_{\sigma_b}, e_u)_U. \tag{3.9}$$

Next, we bound $(e_\sigma, e_u)_U$ and $-(e_{\sigma_b}, e_u)_U$. We first consider $(e_\sigma, e_u)_U$ and note that

$$(e_\sigma, e_u)_U = (\sigma - \sigma_N, u - u_N)_U = (\sigma, u - u_N)_U + (\sigma_N, u_N - u)_U$$
$$= (\sigma, u - u_a(\mu))_U + (\sigma, u_a(\mu) - u_N)_U + (\sigma_N, u_N - u_a(\mu))_U + (\sigma_N, u_a(\mu) - u)_U,$$

where, except for the second term, all terms are nonpositive, see (3.2d) and (3.4d). Hence

$$(e_\sigma, e_u)_U \le (\sigma, u_a(\mu) - u_N)_U \le (\sigma, [u_a(\mu) - u_N]_+)_U$$
$$= (\sigma - \sigma_N, [u_a(\mu) - u_N]_+)_U + (\sigma_N, [u_a(\mu) - u_N]_+)_U \le \|e_\sigma\|_U\ \delta_1 + \delta_2. \tag{3.10}$$

Analogously, we bound $-(e_{\sigma_b}, e_u)_U \le \|e_{\sigma_b}\|_U\ \ \delta_{1b} + \delta_{2b}$. Most significantly, it remains to bound the terms $\|e_\sigma\|_U$ and $\|e_{\sigma_b}\|_U$, which we achieve in two steps: First, we relate $\|e_\sigma\|_U$ and $\|e_{\sigma_b}\|_U$ with $\|e_\sigma - e_{\sigma_b}\|_U$ by

$$\|e_\sigma\|_U^2 + \|e_{\sigma_b}\|_U^2 = \|e_\sigma - e_{\sigma_b}\|_U^2 + 2\big((\sigma, \sigma_b)_U - (\sigma_N, \sigma_b)_U + (\sigma_N, \sigma_{b,N})_U - (\sigma, \sigma_{b,N})_U\big)$$

If we employ $(\sigma, \sigma_b)_U = 0$, $(\sigma_N, \sigma_b)_U \ge 0$, and $(\sigma, \sigma_{b,N})_U \ge 0$, we obtain

$$\|e_\sigma\|_U, \|e_{\sigma_b}\|_U \le \|e_\sigma - e_{\sigma_b}\|_U + \sqrt{2(\sigma_N, \sigma_{b,N})_U}. \tag{3.11}$$

Second, we focus on the optimality residual (3.7c), use the inf-sup stability of $(\cdot,\cdot)_U$ and (3.8) to derive $\|e_\sigma - e_{\sigma_b}\|_U \le \|r_u\|_{U'} + \lambda\|e_u\|_U + \frac{\gamma_b^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}}\left(\|r_p\|_{Y'} + C_D^{\mathrm{UB}}|e_y|_D\right)$. Next, we employ the inequalities (3.8) and (3.11) in (3.9) to obtain

$$\lambda\|e_u\|_U^2 + |e_y|_D^2 \le \|e_u\|_U\left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}}\|r_p\|_{Y'} + \lambda(\delta_1 + \delta_{1b})\right) \tag{3.12}$$
$$+ \frac{2}{\alpha_a^{\mathrm{LB}}}\|r_y\|_{Y'}\|r_p\|_{Y'} + |e_y|_D\frac{C_D^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}}\left(\|r_y\|_{Y'} + \gamma_b^{\mathrm{UB}}(\delta_1 + \delta_{1b})\right)$$
$$+ \left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}}\|r_p\|_{Y'} + \sqrt{2(\sigma_N, \sigma_{b,N})_U}\right)(\delta_1 + \delta_{1b}) + \delta_2 + \delta_{2b}.$$

It thus follows from applying Young's inequality to the $|e_y|_D$-terms in (3.12) that

$$\|e_u\|_U^2 - 2c_1(\mu)\|e_u\|_U - c_2(\mu) \le 0,$$

where $c_1(\mu)$ and $c_2(\mu)$ are given in (3.6). Solving the last inequality for the larger root yields $\|e_u\|_U \le c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)} = \Delta_N^{\mathrm{pr}}(\mu)$.

We note that most of the ingredients of the primal error bound $\Delta_N^{\mathrm{pr}}(\mu)$ introduced in Theorem 1 are standard, i.e., the dual norms of state, adjoint, and control residuals, as well as coercivity and continuity constants or rather their lower and upper bounds [7, 12]. The only non-standard terms are $\delta_1, \delta_2, \delta_{1b}$ and δ_{2b}, which measure the constraint-violation of the RB optimal control u_N. As a result, the online computational cost to evaluate $\delta_\bullet$—and hence the error bound $\Delta_N^{\mathrm{pr}}(\mu)$—depends on the FE control dimension $\mathscr{N}_U$, requiring $\mathscr{O}((N_U + N_\sigma + N_{\sigma_b})\mathscr{N}_U)$ operations.

3.4 Slack Problem and the Primal-Slack Error Bound

In this section we introduce a reformulation of the original primal problem by means of a slack variable. We extend the ideas presented for the one-sided control-constrained problem in [1] to the two-sided control-constrained problem. First, we reformulate the original optimization problem (**P**) by replacing the control variable with a slack variable that depends on one of the two constraints $u_a(\mu)$ or $u_b(\mu)$. Second, we use snapshots of the slack variable to construct an associated convex cone, leading to strictly feasible approximations w.r.t. either the lower or upper constraint. Third, we derive two RB slack problems by restricting the RB-slack coefficients to a convex cone. And finally, we propose an a posteriori $\mathscr{N}$-independent error bound for RB slack approximation w.r.t. either the lower or upper constraint in Theorem 2.

3.4.1 FE and RB Slack Problem

We consider the FE optimization problem (**P**) and introduce the slack variable $s \in U^+$ given by

$$s = u - u_a(\mu) \tag{3.13}$$

together with the corresponding FE coefficient vector $\underline{s} = \underline{u} - \underline{u}_a(\mu)$, where we state the slack variable w.r.t. $u_a(\mu)$. Here, we again omit the explicit dependence of u and s on the parameter μ. We note that, by construction, the feasibility of u w.r.t. $u_a(\mu)$ is equivalent to $M_U \underline{s} \geq 0$, which in turn is equivalent to $\underline{s} \geq 0$, if we are using P_0 elements.

If we substitute u by $s + u_a(\mu)$ in (**P**), we obtain the "slack" optimization problem

$$\min_{\hat{y},\hat{s}} J_s(\hat{y},\hat{s}) = \frac{1}{2}|\hat{y} - y_d|_D^2 + \frac{\lambda}{2}\|\hat{s} + u_a(\mu)\|_U^2 \tag{S}$$

$$\text{s.t.} \quad (\hat{y},\hat{s}) \in Y \times U^+ \quad \text{solves} \quad a(\hat{y},\phi;\mu) = b(\hat{s} + u_a(\mu),\phi;\mu) + f(\phi;\mu)\ \forall \phi \in Y,$$

$$(u_b(\mu),\rho)_U \geq (\hat{s} + u_a(\mu),\rho)_U \quad \forall \rho \in U^+.$$

Analogously, we define a slack variable $s_b = u_b(\mu) - u$ w.r.t. $u_b(\mu)$ and recast (**P**) w.r.t. s_b. We do not state this minimization problem explicitly since it is analogous to (**S**).

In the following we derive two RB slack problems w.r.t. $u_a(\mu)$ and $u_b(\mu)$. We start with the former and reuse the RB space Y_N, introduced in Sect. 3.3.1, for the state and adjoint variables. Furthermore, for the RB approximation of the slack variable s we simply introduce an RB slack space S_N and a convex cone S_N^+ by shifting the control snapshots of (**P**) with the control constraint $u_a(\mu)$

$$S_N = \text{span}\{\,\zeta_1^s, \dots, \zeta_{N_S}^s\,\} = \text{span}\{\,u(\mu^1) - u_a(\mu^1), \dots, u(\mu^N) - u_a(\mu^N)\,\}, \tag{3.14a}$$

$$S_N^+ = \text{span}_+\{\,\zeta_1^s, \dots, \zeta_{N_S}^s\,\} \subset U^+. \tag{3.14b}$$

We assume that the snapshots $\zeta_1^s, \dots, \zeta_{N_s}^s$ are linearly independent and not orthogonalized. Further, we need to consider a Lagrange multiplier $\sigma_b^s \in U^+$ for the constraint $u_b(\mu)$ by incorporating the RB space $\Sigma_{b,N} \subset U$, as well as the convex cone $\Sigma_{b,N}^+$ from (3.3d) and (3.3e). Overall, for an RB approximation $s_N \in S_N^+ \subset U^+$ of s, we have $s_N \geq 0$. From the definition of the slack variable $s = u - u_a(\mu)$, see (3.13), we derive the control approximation $u^s := s_N + u_a(\mu)$ that satisfies $u^s \geq u_a(\mu)$. However, we can not conclude $u^s \leq u_b(\mu)$ since the slack approximation s_N is constructed—as the slack variable s in (3.13)—using information from $u_a(\mu)$ but not $u_b(\mu)$.

Overall, employing the RB spaces in (**S**) results in the RB slack problem

$$\min_{\hat{y}_N^s, \hat{s}_N} J_s(\hat{y}_N^s, \hat{s}_N) = \frac{1}{2}|\hat{y}_N^s - y_d|_D^2 + \frac{\lambda}{2}\|\hat{s}_N + u_a(\mu)\|_U^2 \tag{$\mathbf{S_N}$}$$

$$\text{s.t. } (\hat{y}_N^s, \hat{s}_N) \in Y_N \times S_N^+ \ \text{ solves } \ a(\hat{y}_N^s, \phi; \mu) = b(\hat{s}_N + u_a(\mu), \phi; \mu) + f(\phi; \mu) \ \forall \phi \in Y_N,$$

$$(u_b(\mu), \rho)_U \geq (\hat{s}_N + u_a(\mu), \rho)_U \quad \forall \rho \in \Sigma_{b,N}^+,$$

As in the RB primal problem ($\mathbf{P_N}$), the existence and uniqueness of the RB optimal control follows from the same arguments as in the Remark 1. Next, we derive the optimality conditions for s_N; however, we here follow the 'first-discretize-then-optimize' approach that will eventually lead to a feasible—w.r.t. $u_a(\mu)$—approximation of the control. We perform two steps.

First, we use the RB-representations of y_N^s, s_N with their RB-coefficient vectors $\underline{y}_N^s, \underline{s}_N$ to discretize ($\mathbf{S_N}$). Since the algebraic RB slack problem is simple to derive, we only state the main crucial condition that $\underline{s}_N \geq 0$. Next, we derive the first-order optimality conditions. We introduce a discrete Lagrange multiplier $\underline{\hat{\omega}}_N \in \mathbb{R}^{N_S}$, $\underline{\hat{\omega}}_N \geq 0$, ensuring the non-negativeness of $\underline{\hat{s}}_N$ and derive the following necessary (and here sufficient) first-order optimality system: Given $\mu \in \mathscr{D}$, the optimal RB slack solution coefficients $(\underline{y}_N^s, \underline{s}_N, \underline{p}_N^s, \underline{\sigma}_{b,N}^s, \underline{\omega}_N) \in \mathbb{R}^{N_Y} \times \mathbb{R}^{N_Y} \times \mathbb{R}^{N_S} \times$

$\mathbb{R}^{N_{\sigma_b}} \times \mathbb{R}^{N_S}$ satisfy (omitting all μ-dependencies)

$$A_N \underline{y}_N^s = F_N + B_N^s \underline{s}_N + B_{a,N}^s, \tag{3.15a}$$

$$A_N^T \underline{p}_N^s = Y_{d,N} - D_N \underline{y}_N^s, \tag{3.15b}$$

$$\lambda U_N^s \underline{s}_N + \lambda U_{a,N}^s - (B_N^s)^T \underline{p}_N^s = \underline{\omega}_N - U_N^{\sigma_b,s} \underline{\sigma}_{b,N}^s, \tag{3.15c}$$

$$\underline{s}_N{}^T \underline{\omega}_N = 0, \quad \underline{s}_N \geq 0, \quad \underline{\omega}_N \geq 0, \tag{3.15d}$$

$$(U_{b,N}^{\sigma_b} - U_{a,N}^{\sigma_b} - U_N^{\sigma_b,s} \underline{s}_N)^T \underline{\sigma}_{b,N}^s = 0, \quad U_{b,N}^{\sigma_b} - U_{a,N}^{\sigma_b} \geq U_N^{\sigma_b,s} \underline{s}_N, \quad \underline{\sigma}_{b,N}^s \geq 0. \tag{3.15e}$$

where the reduced basis matrices and vectors are given by

$$(A_N)_{ij} = a(\zeta_i^y, \zeta_j^y), \quad (F_N)_i = f(\zeta_i^y), \quad (B_N^s)_{ij} = b(\zeta_j^s, \zeta_i^y), \ (B_{a,N}^s)_i = b(u_a, \zeta_i^y),$$
$$(Y_{d,N})_i = (y_d, \zeta_i^y)_D, \quad (D_N)_{ij} = (\zeta_i^y, \zeta_j^y)_D, \quad (U_N^s)_{ij} = (\zeta_i^s, \zeta_j^s)_U, \ (U_{a,N}^s)_i = (u_a, \zeta_i^s)_U,$$
$$(U_N^{\sigma_b,s})_{ij} = (\zeta_i^{\sigma_b}, \zeta_j^s)_U, \ (U_{b,N}^{\sigma_b})_i = (u_b, \zeta_i^{\sigma_b})_U, \ (U_{a,N}^{\sigma_b})_i = (u_a, \zeta_i^{\sigma_b})_U$$

and $1 \leq i, j \leq N_\bullet$ [see (3.3) and (3.14)].

Second, by solving (3.15) we have $\underline{s}_N \geq 0$ and through the definition of s we obtain a feasible—w.r.t. $u_a(\mu)$—approximation for the control by $u^s = s_N + u_a(\mu)$. In order to derive an error bound for $\|u - u^s\|_U$ we need, however, to analogously repeat the RB reduction for the second RB slack problem with $s_b = u_b(\mu) - u$. There we likewise introduce the RB space $S_{b,N}$, as well as its convex cone $S_{b,N}^+$ and follow the previous steps to obtain $s_{b,N} \geq 0$. Using this, we obtain a control approximation $u^{s_b} = u_b(\mu) - s_{b,N}$ that is feasible w.r.t. the constraint $u_b(\mu)$.

3.4.2 Primal-Slack Error Bound

In the following we will focus on the primal-slack error bound for $\|u - u^s\|_U$ w.r.t. $u_a(\mu)$. Similarly to the primal error bound in Theorem 1 we use residuals and properties of the bilinear and linear forms to derive a quadratic inequality in $\|u - u^s\|_U$. We consider the following RB primal-slack approximation $(y_N^s, p_N^s, u^s, \sigma_N, \sigma_{b,N}) \in Y_N \times Y_N \times U_{\text{ad}} \times \Sigma_N^+ \times \Sigma_{b,N}^+$, which depends on the solutions of the RB primal and slack problem. We define the corresponding errors $e_y^s = y - y_N^s$, $e_p^s = p - p_N^s$, $e_u^s = u - u^s$. Further, we revisit Definition 1 and insert on the r.h.s. of (3.5) the approximation $(y_N^s, p_N^s, u^s, \sigma_N, \sigma_{b,N})$ to obtain on the l.h.s. the residuals r_y^s, r_p^s, r_u^s.

We state the main result in the following theorem.

Theorem 2 (Primal-Slack Error Bound) *Let u, s_N, and $s_{b,N}$ be the optimal solutions of the FE primal problem (**P**) and the RB slack problems (**S_N**) and its equivalent w.r.t. $u_b(\mu)$, respectively. Then the error in the optimal control satisfies* *for all parameters* $\mu \in \mathscr{D}$

$$\|e_u^s\|_U \le \Delta_N^{\text{pr-sl}}(\mu),$$

where $\Delta_N^{\text{pr-sl}}(\mu) := c_1^s(\mu) + \sqrt{c_1^s(\mu)^2 + c_2^s(\mu)}$ *with nonnegative coefficients*

$$c_1^s(\mu) = \frac{1}{2\lambda}\left(\|r_u^s\|_{U'} + \frac{\gamma_b^{\text{UB}}}{\alpha_a^{\text{LB}}}\|r_p^s\|_{Y'} + \lambda\|u^s - u^{s_b}\|_U\right), \tag{3.16a}$$

$$c_2^s(\mu) = \frac{1}{\lambda}\left[\frac{2}{\alpha_a^{\text{LB}}}\|r_y^s\|_{Y'}\|r_p^s\|_{Y'} + \frac{1}{4}\left(\frac{C_D^{\text{UB}}}{\alpha_a^{\text{LB}}}(\|r_y^s\|_{Y'} + \gamma_b^{\text{UB}}\|u^s - u^{s_b}\|_U)\right)^2 + (\sigma_N, s_N)_U\right.$$
$$\left. + \|u^s - u^{s_b}\|_U\left(\|r_u^s\|_{U'} + \frac{\gamma_b^{\text{UB}}}{\alpha_a^{\text{LB}}}\|r_p^s\|_{Y'} + \sqrt{2(\sigma_N, \sigma_{b,N})_U}\right) + (\sigma_{b,N}, s_{b,N})_U\right] \tag{3.16b}$$

Proof Let the FE primal solution $(y, p, u, \sigma, \sigma_b)$ satisfy the optimality conditions (3.2). We follow the proof of Theorem 1, and derive analogously to (3.9) the inequality

$$\lambda\|e_u^s\|_U^2 + |e_y^s|_D^2 \le \|r_y^s\|_{Y'}\|e_p^s\|_Y + \|r_p^s\|_{Y'}\|e_y^s\|_Y + \|r_u^s\|_{U'}\|e_u^s\|_U + (e_\sigma - e_{\sigma_b}, e_u^s)_U. \tag{3.17}$$

We first focus on $(e_\sigma, e_u^s)_U$ and exploit the feasibility of u^s w.r.t. $u_a(\mu)$. Again we have $(e_\sigma, e_u^s)_U = -(\sigma, u_a(\mu) - u)_U - (\sigma, s_N)_U + (\sigma_N, u_a(\mu) - u)_U + (\sigma_N, s_N)_U$, where the first three terms are non-positive and hence $(e_\sigma, e_u^s)_U \le (\sigma_N, s_N)_U$. In order to bound $-(e_{\sigma_b}, e_u^s)_U$, we need to solve the second RB slack problem for u^{s_b} and derive

$$-(e_{\sigma_b}, e_u^s)_U = (e_{\sigma_b}, u^s - u^{s_b} + u^{s_b} - u)_U \le \|e_{\sigma_b}\|_U\|u^s - u^{s_b}\|_U + (\sigma_{b,N}, s_{b,N})_U.$$

We restate that $\|e_{\sigma_b}\|_U \le \|e_\sigma - e_{\sigma_b}\|_U + \sqrt{2(\sigma_N, \sigma_{b,N})_U}$ and $\|e_\sigma - e_{\sigma_b}\|_U$ is bounded by $\|e_\sigma - e_{\sigma_b}\|_U \le \|r_u^s\|_{U'} + \lambda\|e_u^s\|_U + \frac{\gamma_b^{\text{UB}}}{\alpha_a^{\text{LB}}}\left(\|r_p^s\|_{Y'} + C_D^{\text{UB}}|e_y^s|_D\right)$. Using the bounds for $(e_\sigma, e_u^s)_U - (e_{\sigma_b}, e_u^s)_U$ and the inequalities (3.8) in (3.17) we obtain

$$\lambda\|e_u^s\|_U^2 + |e_y^s|_D^2 \le \|e_u^s\|_U\left(\|r_u^s\|_{U'} + \frac{\gamma_b^{\text{UB}}}{\alpha_a^{\text{LB}}}\|r_p^s\|_{Y'} + \lambda\|u^s - u^{s_b}\|_U\right) + (\sigma_N, s_N)_U$$
$$+ \frac{2}{\alpha_a^{\text{LB}}}\|r_y^s\|_{Y'}\|r_p^s\|_{Y'} + |e_y^s|_D\frac{C_D^{\text{UB}}}{\alpha_a^{\text{LB}}}\left(\|r_y^s\|_{Y'} + \gamma_b^{\text{UB}}\|u^s - u^{s_b}\|_U\right)$$
$$+ \left(\|r_u^s\|_{U'} + \frac{\gamma_b^{\text{UB}}}{\alpha_a^{\text{LB}}}\|r_p^s\|_{Y'} + \sqrt{2(\sigma_N, \sigma_{b,N})_U}\right)\|u^s - u^{s_b}\|_U + (\sigma_{b,N}, s_{b,N})_U.$$

It thus follows from employing Young's inequality to the $|e_y^s|_D$-term that $\|e_u^s\|_U^2 - 2c_1^s(\mu)\|e_u^s\|_U - c_2^s(\mu) \leq 0$, where $c_1^s(\mu)$ and $c_2^s(\mu)$ are given in (3.16). Solving the last inequality for the larger root yields $\|e_u^s\|_U \leq c_1^s(\mu) + \sqrt{c_1^s(\mu)^2 + c_2^s(\mu)} = \Delta_N^{\mathrm{pr-sl}}(\mu)$.

3.5 Greedy Sampling Procedure

The reduced basis spaces for the two-sided control-constrained optimal control problem in Sects. 3.3.1, and 3.4.1 are constructed using the greedy sampling procedure outlined in Algorithm 1. Suppose $\Xi_{\mathrm{train}} \subset \mathscr{D}$ is a finite but suitably large parameter train sample, $\mu^1 \in \Xi_{\mathrm{train}}$ is the initial parameter value, $N_{\max}$ the maximum number of greedy iterations, $\varepsilon_{\mathrm{tol,min}} > 0$ is a prescribed desired error tolerance, and $\Delta_N^{\bullet}(\mu)/\|u_N^{\bullet}(\mu)\|_U$, $\bullet \in \{\mathrm{pr}, \mathrm{pr-sl}\}$, is the primal or primal-slack error bound from (3.6) or (3.16) with $u_N^{\bullet} \in \{u_N, u^s\}$.

We make two remarks: First, by using the bounds $\Delta_N^{\bullet}(\mu)$, $\bullet \in \{\mathrm{pr}, \mathrm{pr-sl}\}$ we only refer to the bounds derived for the primal error $\|u - u_N\|_U$ and the slack error $\|u - u^s\|_U$ w.r.t. to $u_a(\mu)$. Therefore, using the primal-slack bound $\Delta_N^{\mathrm{pr-sl}}(\mu)$ in the greedy sampling procedure, we expect not only to construct an accurate RB space S_N for s_N but also an accurate RB space $S_{b,N}$ for $s_{b,N}$. Second, we comment on two special cases: (1) if one control constraint is fully active in each greedy step, i.e. we have, e.g., $u(\mu^n) = u_a$, $n = 1, \ldots, N$, we set $S_N = \{\}$ and $s_N = 0$ (analogously for u_b we set $S_{b,N} = \{\}$ and $s_{b,N} = 0$); and (2) if the control constraint is never active, i.e., for all snapshots $\sigma(\mu^n) = \sigma_b(\mu^n) = 0$, $n = 1, \ldots, N$, we set $\Sigma_N = \Sigma_N^+ = \Sigma_{b,N} = \Sigma_{b,N}^+ = \{\}$ and $\sigma_N = \sigma_{b,N} = 0$.

Algorithm 1 Greedy sampling procedure

Choose $\Xi_{\mathrm{train}} \subset \mathscr{D}$, $\mu^1 \in \Xi_{\mathrm{train}}$ (arbitrary), $N_{\max}$, and $\varepsilon_{\mathrm{tol,min}} > 0$
Set $N \leftarrow 1$, $Y_0 \leftarrow \{0\}$, $U_0 \leftarrow \{0\}$, $S_0 \leftarrow \{0\}$, $S_{b,0} \leftarrow \{0\}$, $\Sigma_0 \leftarrow \{0\}$, $\Sigma_{b,0} \leftarrow \{0\}$
Set $\Delta_N^{\bullet}(\mu^N) \leftarrow \infty$
while $\Delta_N^{\bullet}(\mu)/\|u_N^{\bullet}(\mu^N)\|_U > \varepsilon_{\mathrm{tol,min}}$ **and** $N \leq N_{\max}$ **do**
    $Y_N \leftarrow Y_{N-1} \oplus \mathrm{span}\{\, y(\mu^N), p(\mu^N)\,\}$
    $U_N \leftarrow U_{N-1} \oplus \mathrm{span}\{\, u(\mu^N), \sigma(\mu^N), \sigma_b(\mu^N)\,\}$
    $S_N \leftarrow S_{N-1} \oplus \mathrm{span}\{\, s(\mu^N)\,\}$
    $S_{b,N} \leftarrow S_{b,N-1} \oplus \mathrm{span}\{\, s_b(\mu^N)\,\}$
    $\Sigma_N \leftarrow \Sigma_{N-1} \oplus \mathrm{span}\{\, \sigma(\mu^N)\,\}$
    $\Sigma_{b,N} \leftarrow \Sigma_{b,N-1} \oplus \mathrm{span}\{\, \sigma_b(\mu^N)\,\}$
    $\mu^{N+1} \leftarrow \arg\max_{\mu \in \Xi_{\mathrm{train}}} \Delta_N^{\bullet}(\mu)/\|u_N^{\bullet}(\mu^N)\|_U$
    $N \leftarrow N + 1$
end while

3.6 Numerical Results: Graetz Flow with Parametrized Geometry and Lower and Upper Control Constraints

We consider a Graetz flow problem, which describes a heat convection and conduction in a duct. The main goal of this example is to demonstrate the different properties of the approximations and their error bounds. The problem is parametrized by a varying Péclet number $\mu_1 \in [5, 18]$ and a geometry parameter $\mu_2 \in [0.8, 1.2]$. Hence, the parameter domain is $\mathscr{D} = [5, 18] \times [0.8, 1.2]$. The parametrized geometry is given by $\Omega(\mu) = [0, 1.5 + \mu_2] \times [0, 1]$ and is subdivided into three subdomains $\Omega_1(\mu) = [0.2\mu_2, 0.8\mu_2] \times [0.3, 0.7]$, $\Omega_2(\mu) = [\mu_2 + 0.2, \mu_2 + 1.5] \times [0.3, 0.7]$, and $\Omega_3(\mu) = \Omega(\mu) \setminus \{\Omega_1(\mu) \cup \Omega_2(\mu)\}$. A sketch of the domain is shown in Fig. 3.1. We impose boundary condition of homogeneous Neumann and of non-homogeneous Dirichlet type: $y_n = 0$ on $\Gamma_N(\mu)$, and $y = 1$ on $\Gamma_D(\mu)$. Thus the trial space is given by $Y(\mu) \subset Y_{\mathrm{e}}(\mu) = \{v \in H^1(\Omega(\mu)); v|_{\Gamma_D(\mu)} = 1\}$. The amount of heat supply in the whole domain $\Omega(\mu)$ is regulated by the distributed control $u \in U(\mu) \subset U_{\mathrm{e}}(\mu) = L^2(\Omega(\mu))$ and bounded by the lower and upper constraints $u_a = -0.5$ and $u_b = 1.25$. The observation domain is $\Omega_D(\mu) = \Omega_1(\mu) \cup \Omega_2(\mu)$ and the desired state is given by $y_d = 0.5$ on $\Omega_1(\mu)$ and $y_d = 2$ on $\Omega_2(\mu)$.

Overall, the parametrized optimal control problem is given by

$$\min_{\hat{y} \in Y(\mu), \hat{u} \in U(\mu)} J(\hat{y}, \hat{u}; \mu) = \frac{1}{2} |\hat{y} - y_d|^2_{D(\mu)} + \frac{\lambda}{2} \|\hat{u}\|^2_{L^2(\Omega(\mu))}$$

$$\text{s.t.} \quad \frac{1}{\mu_1} \int_{\Omega(\mu)} \nabla \hat{y} \cdot \nabla \phi \, \mathrm{d}x + \int_{\Omega(\mu)} \beta(x) \cdot \nabla \hat{y} \, \phi \, \mathrm{d}x = \int_{\Omega(\mu)} \hat{u} \, \phi \, \mathrm{d}x \quad \forall \phi \in Y(\mu),$$

$$(u_a, \rho)_{U(\mu)} \le (\hat{u}, \rho)_{U(\mu)} \le (u_b, \rho)_{U(\mu)} \quad \forall \rho \in U(\mu)^+,$$

for the given parabolic velocity field $\beta(x) = (x_2(1 - x_2), 0)^T$. The regularization parameter λ is fixed to 0.01.

After recasting the problem to a reference domain $\Omega = \Omega(\mu^{\mathrm{ref}}) = [0, 2.5] \times [0, 1]$ for $\mu^{\mathrm{ref}} = (5, 1)$, and introducing suitable lifting functions that take into account the non-homogeneous Dirichlet boundary conditions, we can reformulate

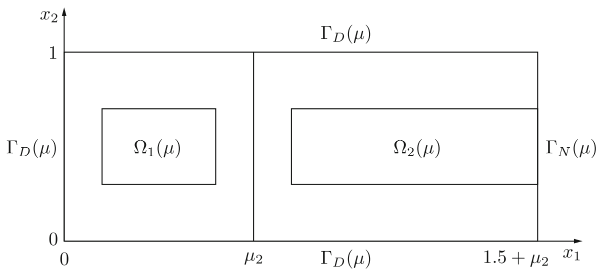

Fig. 3.1 Domain $\Omega(\mu)$ for the Graetz flow problem with distributed control

the problem in terms of the parameter-independent FE space $Y \subset Y_e = H_0^1(\Omega)$ and $U \subset U_e = L^2(\Omega)$ [12]. We then obtain the affine representation of all involved quantities with $Q_a = Q_f = 4$, $Q_b = Q_d = Q_u = Q_{yd} = 2$, and $Q_{ua} = 1$. The details of these calculations are very similar to the details presented by Rozza et al. [12] and Kärcher [5], and are thus omitted. The inner product for the state space is given by $(w, v)_Y = \frac{1}{\mu_1^{\text{ref}}} \int_\Omega \nabla w \cdot \nabla v \, dx + \frac{1}{2}(\int_\Omega \beta(x) \cdot \nabla w \, v \, dx + \int_\Omega \beta(x) \cdot \nabla v \, w \, dx)$ and we obtain a lower bound $\alpha_a^{\text{LB}}(\mu)$ for the coercivity constant by the so-called min-theta approach [12]. Note that for the control space we obtain a parameter-dependent inner product $(\cdot,\cdot)_{U(\mu)}$ from the affine geometry parametrization. Hence the control error is measured in the parameter-dependent energy norm $\|\cdot\|_{U(\mu)}$. The derivations of the primal and primal-slack error bounds remain the same in this case and they bound the control error in the energy norm.

Although the introduction of a domain parametrization seems to add an entirely new μ-dependence to the primal and the slack problems (**P**) and (**S**), the reductions and the error bound derivations can be analogously derived w.r.t. $(\cdot,\cdot)_{U(\mu)}$ instead of $(\cdot,\cdot)_{U(\mu^{\text{ref}})}$. Also the definition of the integrated space U_N in (3.3) remains, while in the inf-sup condition of Remark 2 we use $(\cdot,\cdot)_{U(\mu)}$ instead of $(\cdot,\cdot)_U$.

We choose a P_1 discretization for the state and adjoint, and a P_0 discretization for the control to obtain $\dim(Y) = \mathcal{N}_Y \approx 11{,}000$ and $\dim(U) = \mathcal{N}_U \approx 22{,}000$. The chosen discretization induces a discretization error of roughly 2%. In Fig. 3.2 we present control snapshots and associated active sets for two different parameters displayed on the reference domain $\Omega(\mu_2^{\text{ref}} = 1)$. We observe strongly varying control solutions and active sets.

We construct the RB spaces using the greedy procedure described in Algorithm 1 by employing an equidistant train sample $\Xi_{\text{train}} \subset \mathcal{D}$ of size $30 \cdot 30 = 900$ (log-scale in μ_1 and lin-scale in μ_2) and stop the greedy enrichment after 30 steps. We also introduce a test sample with $10 \cdot 5$ (log $\times$ lin) equidistant parameter points in $[5.2, 17.5] \times [0.82, 1.17] \subset \mathcal{D}$.

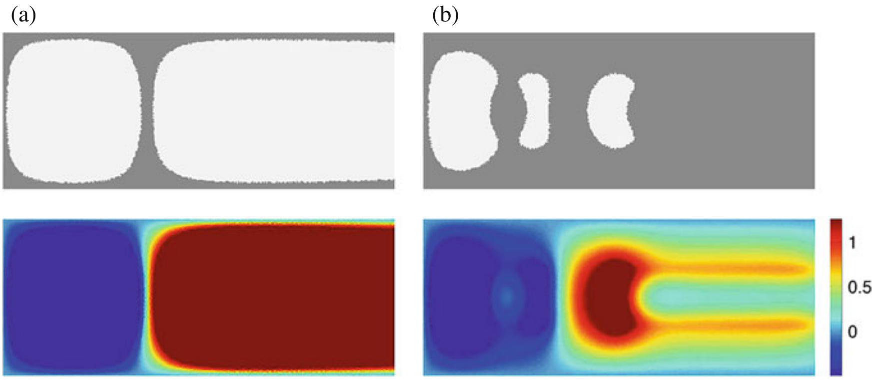

Fig. 3.2 Snapshots of active sets (*upper row*) and optimal control (*lower row*) on the reference domain. The active (inactive) sets are displayed in *light gray* (*gray*). (**a**) $\mu = (5, 0.8)$. (**b**) $\mu = (18, 1.2)$

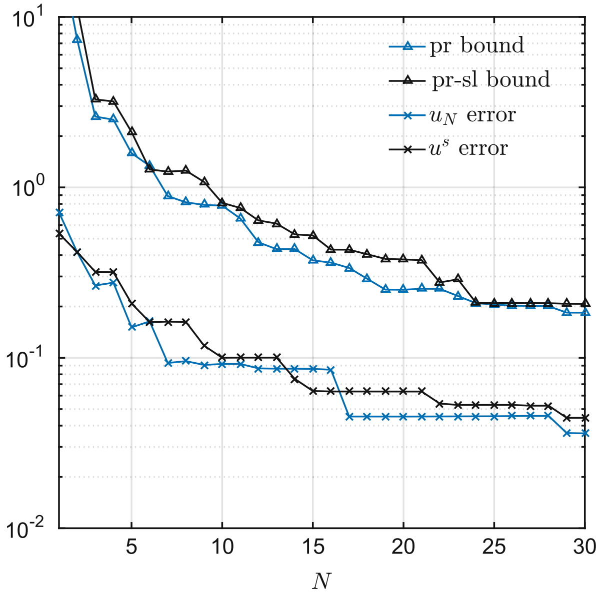

Fig. 3.3 Maximal relative control errors and bounds over the number of greedy iterations. For each N the maximal value over Ξ_{test} is displayed for both errors and both bound

In Fig. 3.3 we present, as a function of N, the resulting energy norm errors and bounds over Ξ_{test}. Here, the errors and bounds are defined as follows: the primal-slack bound is the maximum of $\Delta_N^{\text{pr-sl}}(\mu)/\|u(\mu)\|_{U(\mu)}$ over Ξ_{test}, the primal bound is the maximum of $\Delta_N^{\text{pr}}(\mu)/\|u(\mu)\|_{U(\mu)}$ over Ξ_{test}, and the u^s and u_N errors are the maxima of $\|u(\mu) - u^s(\mu)\|_{U(\mu)}/\|u(\mu)\|_{U(\mu)}$ and $\|u(\mu) - u_N(\mu)\|_{U(\mu)}/\|u(\mu)\|_{U(\mu)}$ over Ξ_{test}, respectively. We observe that both errors and both bounds decay very similarly. Quantitatively, the error bounds are comparable throughout all N, since the dominating primal-slack terms $\|r_u^s\|_{U'(\mu)}$ and $\lambda\|u^s - u^{s_b}\|_{U(\mu)}$ are comparable to the dominating primal terms $\|r_u\|_{U'(\mu)}$ and $\lambda(\delta_1+\delta_{1b})$, resulting in $\Delta_N^{\text{pr}}(\mu) \approx \Delta_N^{\text{pr-sl}}(\mu)$.

We briefly report the computational timings: the solution of the FE optimization problem takes $\approx$ 4 s (for a discretization error of 2%). The RB primal problem, for $N = 25$, is solved in $\approx$ 0.066 s and the RB slack problem is solved faster in $\approx$ 0.029 s, since $\dim(S_N) = 25$. We turn to the evaluation of error bounds: the primal bound takes 0.01 s, whereas the primal-slack bound, given σ_N, $\sigma_{b,N}$, takes 0.0065 s. From this we can conclude that for $N = 25$ the overall cost for one primal bound evaluation is roughly 0.076 s = 0.066 s + 0.01 s and for the primal-slack bound evaluation is roughly 0.13 $\approx$ 0.029 + 0.029 + 0.066 + 0.0065 s, since it relies on three RB solutions.

3.7 Conclusions

In this paper we extended the ideas from [1] to propose two certified reduced basis approaches for distributed elliptic optimal control problems with two-sided control constraints: a primal and a primal-slack approach. Albeit the reduction for the primal approach was straightforward, the primal-slack approach needed more consideration. We proposed for each constraint a corresponding RB slack problem with an additional Lagrange multiplier. The primal a posteriori error bound from [1] could be extended for the two-sided case by special properties of the Lagrange multipliers of the two-sided problem. The primal-slack error bound also relies on these properties and in addition uses three RB solutions to derive an $\mathcal{N}$-independent error bound. Both the primal and slack RB approximation can be evaluated efficiently using the standard offline-online decomposition. However, on the one hand the primal error bound depends on the FE control dimension and on the other hand the primal-slack error bound relies on three reduced order optimization problems.

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