McShane-Whitney Pairs

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Abstract. We present a constructive version of the classical McShane-Whitney theorem on the extendability of real-valued Lipschitz functions defined on a subset of a metric space. Through the introduced notion of a McShane-Whitney pair we study some abstract properties of this extension theorem showing how the behavior of a Lipschitz function defined on the subspace of the pair affect its McShane-Whitney extensions on the space of the pair. As a consequence, a Lipschitz version of the theory around the Hahn-Banach theorem is formed. We work within Bishop's informal system of constructive mathematics BISH.

1 Introduction

According to the classical extension theorem of McShane and Whitney that first appeared in [12,19], a real-valued Lipschitz function defined on *any* subset A of a metric space X is extended to a Lipschitz function defined on X. To determine metric spaces X and Y such that a similar extension theorem for Y-valued Lipschitz functions defined on a subset A of X holds is a non-trivial problem under active current study (see [1,4,14]). Although the McShane-Whitney theorem has a highly ineffective proof similar to the proof of the Hahn-Banach theorem (see [17], pp.16–17), it also admits a proof based on an explicit definition of two such extension functions. This definition, which involves the notions of infimum and supremum of a bounded subset of reals, can be carried out constructively only if we restrict to certain subsets A of a metric space X.

We define a McShane-Whitney subset A of a metric space X in order to constructively realize the McShane-Whitney explicit definition of the extension functions. A pair (X, A), where X is a metric space and A is a subset of X on which the McShane-Whitney explicit definition is carried out constructively is called here a McShane-Whitney pair. The importance of the McShane-Whitney extension lies in the possibility to relate properties of a given Lipschitz function on A to properties of its extension functions on X in such a way that a Lipschitz-version of the theory around the Hahn-Banach theorem is formed. We present here the first basic results in this direction. We work within Bishop's informal system of constructive mathematics BISH (see [2,3,6]). The constructive reconstruction of the general theory of Lipschitz functions is quite underdeveloped. Some first results on constructive Lipschitz analysis are found in [8,10,13]. All proofs that are not included here due to space restrictions are left to the reader.

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2 Basic Notions and Facts

Definition 1. *Let* $A \subseteq \mathbb{R}$ *and* $b, l, \lambda, m, \mu \in \mathbb{R}$. *If* A *is bounded above, we define* $b \geq A :\leftrightarrow \forall_{a \in A}(b \geq a)$, $[A) := \{b \in \mathbb{R} \mid b \geq A\}$, $l = \sup A :\leftrightarrow l \geq A \wedge \forall_{\epsilon > 0} \exists_{a \in A}(a > l - \epsilon)$, *and* $\lambda = \mathrm{lub} A :\leftrightarrow \lambda \geq A \wedge \forall_{b \in [A)}(b \geq \lambda)$. *If* A *is bounded below,* $b \leq A$, $(A]$, $m = \inf A$, *and* $\mu = \mathrm{glb} A$ *are defined in a dual way.*

In [11], pp. 24–25, Mandelkern gave a necessary and sufficient condition for the existence of $\mathrm{lub}A$ and $\mathrm{glb}A$ and proved the following remark: If $A \subseteq \mathbb{R}$ bounded and $\mathrm{glb}A$ exists, then $\sup(A]$ exists and $\sup(A] = \mathrm{glb}A$, while if $\mathrm{lub}A$ exists, then $\inf[A)$ exists and $\inf[A) = \mathrm{lub}A$.

Definition 2. *We denote by* $\mathbb{F}(X, Y)$ *the set of functions of type* $X \to Y$ *and by* $\mathbb{F}(X)$ *the set of functions of type* $X \to \mathbb{R}$. *If* $a \in \mathbb{R}$, *then* $\overline{a}_X$ *denotes the constant map in* $\mathbb{F}(X)$ *with value* a, *and* $\mathrm{Const}(X)$ *is the set of constant maps. If* $(X, d), (Y, \rho)$ *are metric spaces, then* $\mathrm{C_u(X, Y)}$ *denotes the uniformly continuous functions from* X *to* Y, *and* $\mathrm{C_u(X)}$ *denotes the uniformly continuous functions from* X *to* $\mathbb{R}$, *where* $\mathbb{R}$ *is equipped with its standard metric. The metric* d_{x_0} *at the point* $x_0 \in X$ *is defined by* $d_{x_0}(x) := d(x_0, x)$, *for every* $x \in X$, *and* $U_0(X) := \{d_{x_0} \mid x_0 \in X\}$. *The set* X_0 *of the* d*-distinct pairs of* X *is defined by* $X_0 := \{(x, y) \in X \times X \mid d(x, y) > 0\}$.

Definition 3. *If* A *is a subset of a metric space* X, $x \in X$, *and* $\Delta(x, A) := \{d(x, a) \mid a \in A\}$, A *is located if* $d(x, A) := \inf \Delta(x, A)$ *exists, for every* $x \in X$, *and* A *is colocated, if* $\delta(x, A) := \sup \Delta(x, A)$ *exists, for every* $x \in X$.

If A is inhabited and colocated, then A is bounded; if a_0 inhabits A, then $d(a, b) \leq d(a, a_0) + d(a_0, b) \leq 2\delta(a_0, A)$, for every $a, b \in A$. Unless otherwise stated, for the rest X and Y are equipped with metrics d and ρ, respectively.

Definition 4. *The set of Lipschitz functions* $\mathrm{Lip}(X, Y)$ *from* X *to* Y *is*

$$\mathrm{Lip}(X, Y) := \bigcup_{\sigma \geq 0} \mathrm{Lip}(X, Y, \sigma),$$

$$\mathrm{Lip}(X, Y, \sigma) := \{f \in \mathbb{F}(X, Y) \mid \forall_{x, y \in X}(\rho(f(x), f(y)) \leq \sigma d(x, y))\}.$$

If $Y = \mathbb{R}$, *we use the notations* $\mathrm{Lip}(X)$ *and* $\mathrm{Lip}(X, \sigma)$, *respectively.*

Clearly, $\mathrm{Lip}(X, Y) \subseteq \mathrm{C_u(X, Y)}$. If $A \subseteq X$ and $f \in \mathrm{Lip}(X, \sigma)$, for some $\sigma \geq 0$, then $f_{|A} \in \mathrm{Lip}(A, \sigma)$. An element of $\mathrm{Lip}(X, Y)$ sends a bounded subset of X to a bounded subset of Y, which is not generally the case for elements of $\mathrm{C_u(X, Y)}$; the identity function $\mathrm{id} : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is equipped with the discrete metric, is in $\mathrm{C_u}(\mathbb{N}) \setminus \mathrm{Lip}(\mathbb{N})$ and $\mathrm{id}(\mathbb{N}) = \mathbb{N}$ is unbounded in $\mathbb{R}$. In [13], p. 370, it is shown constructively that if X is totally bounded, then $\mathrm{Lip}(X)$ is uniformly dense in $\mathrm{C_u(X)}$.

Proposition 1. *The set* $\mathrm{Lip}(X)$ *includes the sets* $U_0(X)$, $\mathrm{Const}(X)$, *and it is closed under addition and multiplication by reals. If every element of* $\mathrm{C_u(X)}$ *is a bounded function, then* $\mathrm{Lip}(X)$ *is closed under multiplication.*

Definition 5. *If $f \in \mathbb{F}(X, Y)$, we define the following sets:*

$$\Lambda(f) := \{\sigma \geq 0 \mid \forall_{x,y \in X}(\rho(f(x), f(y)) \leq \sigma d(x, y))\},$$

$$\Xi(f) := \{\sigma \geq 0 \mid \forall_{x,y \in X}(\rho(f(x), f(y)) \geq \sigma d(x, y))\},$$

$$M_0(f) := \{\sigma_{x,y}(f) \mid (x, y) \in X_0\},$$

$$\sigma_{x,y}(f) := \frac{\rho(f(x), f(y))}{d(x, y)}.$$

Proposition 2. *If $f \in \mathbb{F}(X, Y)$, then $\Lambda(f) = [M_0(f))$ and $\Xi(f) = (M_0(f)]$.*

Classically one can prove that if $f \in \mathrm{Lip}(X, Y)$ such that $\inf \Lambda(f)$ exists, then $\sup M_0(f)$ exists and $\sup M_0(f) = \inf \Lambda(f)$. The classical argument in that proof is avoided, if $\sup M_0(f)$ exists.

Proposition 3. *Let $f \in \mathrm{Lip}(X, Y)$.*

(i) If $\sup M_0(f)$ exists, then $\inf \Lambda(f)$ exists and $\inf \Lambda(f) = \min \Lambda(f) = \sup M_0(f)$.
(ii) If $\inf \Lambda(f)$ exists, then $\mathrm{lub} M_0(f)$ exists and $\mathrm{lub} M_0(f) = \inf \Lambda(f)$.
(iii) If $\mathrm{lub} M_0(f)$ exists, then $\inf \Lambda(f)$ exists and $\inf \Lambda(f) = \mathrm{lub} M_0(f)$.

In constructive analysis one usually works with the stronger notions of infima or suprema of sets and not with greatest lower bounds or least upper bounds of sets. An important exception is found in the work of Mandelkern (see his comment in [11], p. 24). Here we also find useful to keep both notions at work.

Definition 6. *Let $f \in \mathrm{Lip}(X, Y)$. We call f L-pseudo-normable, if $\sup M_0(f)$ exists, and we write $L(f) := \sup M_0(f)$. We call f weakly L-pseudo-normable, or L^*-pseudo-normable, if $\mathrm{lub} M_0(f) = \inf \Lambda(f)$ exists, and $L^*(f) := \mathrm{lub} M_0(f)$.*

In general a Lipschitz function need not be L-pseudo-normable. Note that in the case of a linear function f between normed spaces X and Y the boundedness condition is equivalent to the Lipschitz condition and the existence of its norm $||f||$ is equivalent to the existence of $L(f)$. If f is L-pseudo-normable, and since $L(f) \geq M_0(f)$, by Proposition 2 we get $\forall_{x,y \in X}(\rho(f(x), f(y)) \leq L(f) d(x, y))$, or $f \in \mathrm{Lip}(X, Y, L(f))$. If f is L^*-pseudo-normable, we work similarly.

Proposition 4. *Let $A \subseteq X$ and $f \in \mathrm{Lip}(A, Y)$ such that*

$$\exists_{g \in \mathbb{F}(X,Y)}(g_{|A} = f \ \wedge \ \forall_{\sigma \geq 0}(f \in \mathrm{Lip}(A, Y, \sigma) \rightarrow g \in \mathrm{Lip}(X, Y, \sigma))).$$

(i) If f is L-pseudo-normable, g is L-pseudo-normable and $L(g) = L(f)$.
(ii) If f is L^*-pseudo-normable, g is L^*-pseudo-normable and $L^*(g) = L^*(f)$.

Note that if $f \in \mathrm{Lip}(A, Y), g \in \mathrm{Lip}(X, Y)$ such that $L(f), L(g)$ exist and $L(f) = L(g)$, then it is immediate that $\forall_{\sigma \geq 0}(f \in \mathrm{Lip}(A, Y, \sigma) \rightarrow g \in \mathrm{Lip}(X, Y, \sigma))$. Next follows the Lipschitz-version of the extendability of a uniformly continuous function defined on a dense subset of a metric space with values in a complete metric space.

Proposition 5. *Let $D \subseteq X$ be dense in X, Y complete, and $f \in \mathrm{Lip}(D, Y)$.*

(i) $\exists_{!g \in \mathbb{F}(X,Y)}(g_{|A} = f \ \wedge \ \forall_{\sigma \geq 0}(f \in \mathrm{Lip}(D, Y, \sigma) \rightarrow g \in \mathrm{Lip}(X, Y, \sigma)))$.
(ii) If f is L-pseudo-normable, then g is L-pseudo-normable and $L(g) = L(f)$.
(iii) If f is L^*-pseudo-normable, g is L^*-pseudo-normable and $L^*(g) = L^*(f)$.

Corollary 1. *Let $D \subseteq X$ be dense in X, let Y be complete, and $g \in \mathrm{Lip}(X, Y)$. If g is L^*-pseudo-normable, $f = g_{|D}$ is L^*-pseudo-normable and $L^*(f) = L^*(g)$.*

3 McShane-Whitney Subsets and Pairs

We formulate a property on the subsets of a metric space so that the McShane-Whitney extension can be carried out on them constructively.

Definition 7. *If $A \subseteq X$ is inhabited, $x \in X$, $\lambda \in \mathbb{R}$, and $g \in \mathrm{Lip}(A)$, the set $\mathrm{MW}_g(A, \lambda, x)$ is defined by*

$$\mathrm{MW}_g(A, \lambda, x) := \{g(a) + \lambda d(x, a) \mid a \in A\}.$$

The set A is called a McShane-Whitney subset of X, if for every $\sigma > 0$, $g \in \mathrm{Lip}(A)$ and $x \in X$ the $\inf \mathrm{MW}_g(A, \sigma, x)$ exists.

A McShane-Whitney subset A of X is located, colocated and bounded. Since $\{d(x, a) \mid a \in A\} = \mathrm{MW}_{\overline{0}_X}(A, 1, x)$, A is located. Since $\mathrm{MW}_{-2d_x}(A, 1, x) = \{-2d(x, a) + d(x, a) \mid a \in A\} = \{-d(x, a) \mid a \in A\} = -\Delta(x, A)$, we get[1] $\delta(x, A) = \sup[-(-\Delta(x, A))] = -\inf(-\Delta(x, A)) = -\inf \mathrm{MW}_{-2d_x}(A, 1, x)$ i.e., A is colocated, and since A is inhabited, A is also bounded.

Proposition 6. *A is a McShane-Whitney subset of X if and only if for every $\sigma > 0$, $g \in \mathrm{Lip}(A)$ and $x \in X$ the $\sup \mathrm{MW}_g(A, -\sigma, x)$ exists.*

Proof. If $\sigma > 0$, $g \in \mathrm{Lip}(A)$ and $x \in X$, then $\mathrm{MW}_g(A, -\sigma, x) = \{g(a) - \sigma d(x, a) \mid a \in A\} = \{-(-g(a) + \sigma d(x, a)) \mid a \in A\} = -\{(-g)(a) + \sigma d(x, a) \mid a \in A\} = -\mathrm{MW}_{-g}(A, \sigma, x)$. Since $-g \in \mathrm{Lip}(A)$, we get $\sup \mathrm{MW}_g(A, -\sigma, x) = \sup(-\mathrm{MW}_{-g}(A, \sigma, x)) = -\inf MW_{-g}(A, \sigma, x)$. For the converse implication we use the equality $\inf(-B) = -\sup B$, where $B \subseteq \mathbb{R}$ such that $\sup B$ exists, and the similarly shown equality $\mathrm{MW}_g(A, \sigma, x) = -\mathrm{MW}_{-g}(A, -\sigma, x)$. Hence $\inf \mathrm{MW}_g(A, \sigma, x) = \inf(-\mathrm{MW}_{-g}(A, -\sigma, x)) = -\sup \mathrm{MW}_{-g}(A, -\sigma, x)$.

[1] If $B \subseteq \mathbb{R}$ is bounded and $\inf B$ exists, then $\sup(-B)$ exists and $\sup(-B) = -\inf B$; if $m = \inf B$, then by definition m is a lower bound of B and $\forall_{\epsilon>0}\exists_{b \in B}(b < m + \epsilon)$, therefore $-m$ is an upper bound of $-B$ and $\forall_{\epsilon>0}\exists_{-b \in -B}(-b > -m-\epsilon)$. The following constructively provable properties are used in this paper: if $A, B \subseteq \mathbb{R}$ are inhabited and bounded such that $\sup A, \inf A, \sup B, \inf B$ exist, then $\sup(A + B)$ exists and $\sup(A + B) = \sup A + \sup B$, $\inf(A + B)$ exists and $\inf(A + B) = \inf A + \inf B$, if $\lambda > 0$, then $\sup(\lambda A), \inf(\lambda A)$ exist and $\sup(\lambda A) = \lambda \sup A$, $\inf(\lambda A) = \lambda \inf A$, if $\lambda < 0$, then $\sup(\lambda A), \inf(\lambda A)$ exist and $\sup(\lambda A) = \lambda \inf A$, and $\inf(\lambda A) = \lambda \sup A$.

The next proposition provides examples of McShane-Whitney subsets. A locally compact (totally bounded) metric space X is one every bounded subset of which is included in a compact (totally bounded) subset of X (see [5], p. 46).

Proposition 7. *Let $A \subseteq X$ be inhabited.*

(i) If A is totally bounded, then A is a McShane-Whitney subset of X.
(ii) If X is totally bounded and A is located, then A is a McShane-Whitney subset of X.
(iii) If X is locally compact (totally bounded), then A is a McShane-Whitney subset of X if and only if A is bounded and located.
(iv) If $X = \mathbb{R}^n$, then A is a McShane-Whitney subset of $\mathbb{R}^n$ if and only if A is totally bounded.

Proof

(i) If $\sigma > 0$, $g \in \mathrm{Lip}(A)$ and $x \in X$, then $g + \sigma d_x \in \mathrm{Lip}(A) \subseteq \mathrm{C_u(A)}$, and $\inf \mathrm{MW}_g(\sigma, x)$ exists, since A is totally bounded (see [3], Corollary 4.3, p. 94).
(ii) A located subset of X is also totally bounded (see [3], p. 95), and we use (i).
(iii) If A is bounded and located, there is compact (totally bounded) $K \subseteq X$ such that $A \subseteq K$. Since A is located in X, it is a located in K, hence A is totally bounded, and we use (i). For the converse see our remark after Definition 7.
(iv) A is totally bounded if and only if it is located and bounded (see [3], p. 95), and $\mathbb{R}^n$ is locally compact as a finite product of $\mathbb{R}$ (see [3], p. 111). The required equivalence follows from (iii).

Definition 8. *Let $A \subseteq X$. We call (X, A) a McShane-Whitney pair, if for all $\sigma > 0$ and $g \in \mathrm{Lip}(A, \sigma)$ the functions $g^*, {}^*g : X \to \mathbb{R}$, the smallest and largest McShane-Whitney extension of g, defined by*

$$g^*(x) = \inf M_g(A, \sigma, x), \ {}^*g(x) = \sup M_g(A, -\sigma, x),$$

for every $x \in X$, are well-defined and satisfy the following properties:

(i) $g^*, {}^*g \in \mathrm{Lip}(X, \sigma)$.
(ii) ${g^*}_{|A} = ({}^*g)_{|A} = g$.
(iii) $\forall_{f \in \mathrm{Lip}(A,\sigma)}(f_{|A} = g \to g^* \leq f \leq {}^*g)$.

The extensions g^*, ${}^*g$ of g are unique. Let $h^*, {}^*h$ satisfy conditions (i)-(iii) of Definition 8. Since ${h^*}_{|A} = ({}^*h)_{|A} = g$, $g^* \leq h^* \leq {}^*g$ and $g^* \leq {}^*h \leq {}^*g$. Since ${g^*}_{|A} = ({}^*g)_{|A} = g$, $h^* \leq g^* \leq {}^*h$ and $h^* \leq {}^*g \leq {}^*h$, hence $h^* = g^*$ and ${}^*h = {}^*g$.

Theorem 1 (McShane-Whitney). *If A is a McShane-Whitney subset of X, then (X, A) is a McShane-Whitney pair.*

Proof. By Proposition 6, the functions ${}^*g, g^*$ are well-defined. First we show that ${}^*g$ extends g. If $a_0 \in A$, then ${}^*g(a_0) = \inf\{g(a) + \sigma d(a_0, a) \mid a \in A\} \leq g(a_0)+\sigma d(a_0, a_0) = g(a_0)$. If $a \in A$, then $g(a_0)-g(a) \leq |g(a_0)-g(a)| \leq \sigma d(a_0, a)$, hence $g(a) + \sigma d(a_0, a) \geq g(a_0)$. Since a is arbitrary, ${}^*g(a_0) \geq g(a_0)$. To show ${}^*g \in \mathrm{Lip}(X, \sigma)$ let $x_1, x_2 \in X$ and $a \in A$. Then $d(x_1, a) \leq d(x_2, a) + d(x_2, x_1)$ and $\sigma d(x_1, a) \leq \sigma d(x_2, a) + \sigma d(x_1, x_2)$, therefore

$$\begin{aligned}
&g(a) + \sigma d(x_1, a) \leq (g(a) + \sigma d(x_2, a)) + \sigma d(x_1, x_2) \rightarrow \\
&{}^*g(x_1) \leq (g(a) + \sigma d(x_2, a)) + \sigma d(x_1, x_2) \rightarrow \\
&{}^*g(x_1) - \sigma d(x_1, x_2) \leq g(a) + \sigma d(x_2, a) \rightarrow \\
&{}^*g(x_1) - \sigma d(x_1, x_2) \leq {}^*g(x_2) \rightarrow \\
&{}^*g(x_1) - {}^*g(x_2) \leq \sigma d(x_1, x_2).
\end{aligned}$$

If we start from the inequality $d(x_2, a) \leq d(x_1, a) + d(x_2, x_1)$ and work as above, we get ${}^*g(x_2) - {}^*g(x_1) \leq \sigma d(x_1, x_2)$, therefore $|{}^*g(x_1) - {}^*g(x_2)| \leq \sigma d(x_1, x_2)$. Working similarly we get that g^* is an extension of g which is in $\mathrm{Lip}(X, \sigma)$. If $f \in \mathrm{Lip}(X, \sigma)$ such that $f_{|A} = g$, $x \in X$ and $a \in A$ we have that

$$\begin{aligned}
&f(x) - g(a) = f(x) - f(a) \leq |f(x) - f(a)| \leq \sigma d(x, a) \rightarrow \\
&f(x) \leq g(a) + \sigma d(a, x) \rightarrow \\
&f(x) \leq {}^*g(x), \\
&g(a) - f(x) = f(a) - f(x) \leq |f(a) - f(x)| \leq \sigma d(x, a) \rightarrow \\
&g(a) - \sigma d(a, x) \leq f(x) \rightarrow \\
&g^*(x) \leq f(x).
\end{aligned}$$

Proposition 8. *Let (X, A) be a McShane-Whitney pair and $g \in \mathrm{Lip}(A, \sigma)$.*

(i) The set A is located.
(ii) If $\inf g$ and $\sup g$ exist, then $\inf {}^*g$, $\sup g^*$ exist and

$$\inf_{x \in X} {}^*g = \inf_{a \in A} g, \qquad \sup_{x \in X} g^* = \sup_{a \in A} g.$$

Proof

(i) Let $r \in \mathbb{R}$ and $\sigma > 0$. Since $\overline{r}_A \in \mathrm{Lip}(A, \sigma)$, by hypothesis ${}^*\overline{r}_A$ is well-defined, where ${}^*\overline{r}_A(x) = \inf\{r + \sigma d(x, a) \mid a \in A\}$, for every $x \in X$. If $x \in X$ and $a \in A$, then $d(x, a) = \frac{1}{\sigma}(r + \sigma d(x, a) - r)$, and $\Delta(x, A) = \{\frac{1}{\sigma}(r + \sigma d(x, a) - r) \mid a \in A\}$. Hence $d(x, A) = \inf\{\frac{1}{\sigma}(r + \sigma d(x, a) - r) \mid a \in A\} = \frac{1}{\sigma}(\inf\{r + \sigma d(x, a) \mid a \in A\} - r) = \frac{1}{\sigma}({}^*\overline{r}_A(x) - r)$.

(ii) We show that $m := \inf\{g(a) \mid a \in A\}$ satisfies the properties of $\inf\{{}^*g(x) \mid x \in X\}$. It suffices to show that $m \leq {}^*g(X)$, since the other definitional condition of inf follows immediately; if $\epsilon > 0$, then there exists $a \in A \subseteq X$ such that $g(a) = {}^*g(a) < m + \epsilon$. If $x \in A$, then $m \leq g(x) = {}^*g(x)$, since $m = \inf g$. Since A is located, the set $-A := \{x \in X \mid d(x, A) > 0\}$ is well-defined. If $x \in -A$, then $d(x, a) \geq d(x, A) > 0$, for every $a \in A$. Hence

$$g(a) + \sigma d(x, a) > g(a) \geq \inf_{a \in A} g \rightarrow \inf_{a \in A} (g(a) + \sigma d(x, a)) \geq \inf_{a \in A} g \leftrightarrow {}^*g(x) \geq m.$$

Since A is located, the set $A \cup (-A)$ is dense in X (see [3], p.88). If $x \in X$, there is some sequence $(d_n)_{n\in\mathbb{N}} \subseteq A \cup (-A)$ such that $d_n \xrightarrow{n} x$. By the continuity of ${}^*g$ we have that ${}^*g(d_n) \xrightarrow{n} {}^*g(x)$. Suppose that ${}^*g(x) < m$. Since ${}^*g(d_n) \geq m$, for every $n \in \mathbb{N}$, if $\epsilon := (m - {}^*g(x)) > 0$, there is some n_0 such that for every $n \geq n_0$ we have that $|{}^*g(d_n) - {}^*g(x)| = {}^*g(d_n) - {}^*g(x) < m - {}^*g(x) \leftrightarrow {}^*g(d_n) < m$, which is a contradiction. Hence ${}^*g(x) \geq m$. For the existence of $\sup g^*$ and the equality $\sup_{x\in X} g^* = \sup_{a\in A} g$ we work similarly.

If $g = \overline{r}_A$ and $\sigma > 0$, then $\overline{r}_A^* = \overline{r}_A - \sigma d_A$ and ${}^*\overline{r}_A = \overline{r}_A + \sigma d_A$. If $g \in \mathrm{Lip}(A, 0)$, it is immediate that $g = \overline{r}_A$, for some $r \in \mathbb{R}$, and ${}^*g = g^* = \overline{r}_X$. If D is dense in X and (X, D) is a McShane-Whitney pair, then by Proposition 5 there is a unique σ-Lipschitz extension on X of some $g \in \mathrm{Lip}(D)$, hence ${}^*g = g^*$, a fact which is also shown by the definition of ${}^*g$ and g^*. A weaker property on A that suffices for the McShane-Whitney extension is that for every $\sigma > 0, g \in \mathrm{Lip}(A, \sigma)$ and $x \in X$ the $\inf \mathrm{MW}_g(A, \sigma, x)$ exists, but since all our examples of McShane-Whitney subsets satisfy the stronger property of Definition 7, we avoid it. The next proposition expresses a "step-invariance" of the McShane-Whitney extension. If $A \subseteq B \subseteq X$ such that $(X, A), (X, B)$, (B, A) are McShane-Whitney pairs and $g \in \mathrm{Lip}(A)$, then g^{*_X} is the $(A-X)$ extension of g, $g^{*_B *_X}$ is the $(B-X)$ extension of the $(A-B)$ extension g^{*_B} of g, and similarly for ${}^{*_X}g$ and ${}^{*_X *_B}g$.

Proposition 9. *If $A \subseteq B \subseteq X$ such that $(X, A), (X, B)$, (B, A) are McShane-Whitney pairs and $g \in \mathrm{Lip}(A, \sigma)$, for some $\sigma > 0$, then*

$$g^{*_X} = g^{*_B *_X}, \quad {}^{*_X}g = {}^{*_X *_B}g.$$

Proof. We show only the first equality and for the second we work similarly. By definition $g^{*_B} : B \to \mathbb{R} \in \mathrm{Lip}(B, \sigma)$ and $g^{*_B}(b) = \sup\{g(a) - \sigma d(b, a) \mid a \in A\}$, for every $b \in B$. Moreover, $g^{*_B *_X} : X \to \mathbb{R} \in \mathrm{Lip}(X, \sigma)$ and $g^{*_B *_X}(x) = \sup\{g^{*_B}(b) - \sigma d(x, b) \mid b \in B\}$, for every $x \in X$. For the $(A-X)$ extension of g we have that $g^{*_X} : X \to \mathbb{R} \in \mathrm{Lip}(X, \sigma)$ and $g^{*_X}(x) = \sup\{g(a) - \sigma d(x, a) \mid a \in A\}$, for every $x \in X$. Since $(g^{*_B *_X})_{|B} = g^{*_B}$, we have that $(g^{*_B *_X})_{|A} = (g^{*_B})_{|A} = g$. Therefore $g^{*_X} \leq g^{*_B *_X} \leq {}^{*_X}g$, and $(g^{*_X})_{|B} \leq (g^{*_B *_X})_{|B} = g^{*_B} \leq ({}^{*_X}g)_{|B}$. Since $(g^{*_X})_{|A} = g$, we get that $((g^{*_X})_{|B})_{|A} = g$, therefore $g^{*_B} \leq ((g^{*_X})_{|B}) \leq {}^{*_B}g$. Since $(g^{*_X})_{|B} \leq g^{*_B}$ and $g^{*_B} \leq (g^{*_X})_{|B}$, we get $(g^{*_X})_{|B} = g^{*_B}$. Hence $g^{*_B *_X} \leq g^{*_X} \leq {}^{*_X *_B}g$ i.e., we have shown both $g^{*_X} \leq g^{*_B *_X}$ and $g^{*_B *_X} \leq g^{*_X}$.

Proposition 10. *Let (X, A) be a McShane-Whitney pair and $g \in \mathrm{Lip}(A)$ such that $L(g)$ exists.*

(i) $g \in \mathrm{Lip}(A, L(g))$.
(ii) If f is an $L(g)$-Lipschitz extension of g, then $L(f)$ exists and $L(f) = L(g)$.
(iii) $L({}^*g), L(g^*)$ exist and $L({}^*g) = L(g) = L(g^*)$.

Proof

(i) Since $L(g) = \sup M_0(g)$, we have that $L(g) \geq M_0(g)$ and by Proposition 2 we get $L(g) \in \Lambda(g)$, therefore $g \in \mathrm{Lip}(A, L(g))$.

(ii) Since $f \in \mathrm{Lip}(X, L(g))$, we get $L(g) \in \Lambda(f)$ and $L(g) \geq M_0(f)$. Let $\epsilon > 0$. Since $L(g) = \sup M_0(g)$, there exists $(a,b) \in A_0 \subseteq X_0$ such that $\sigma_{a,b}(g) > L(g) - \epsilon$. Since f extends g, $\sigma_{a,b}(g) = \sigma_{a,b}(f)$.

(iii) By definition ${}^*g, g^* \in \mathrm{Lip}(X, L(g))$ and they extend g. Hence we use (ii).

Definition 9. *Let $(X, ||.||)$ be a normed space. A subset C of X is called convex, if $\forall_{x,y\in C}\forall_{\lambda\in(0,1)}(\lambda x + (1-\lambda)y \in C)$. If $C \subseteq X$ is convex, a function $g : C \to \mathbb{R}$ is called convex, if $\forall_{x,y\in C}\forall_{\lambda\in(0,1)}(g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y))$, and g is called concave, if $\forall_{x,y\in C}\forall_{\lambda\in(0,1)}(g(\lambda x + (1-\lambda)y) \geq \lambda g(x) + (1-\lambda)g(y))$. A function $f : X \to \mathbb{R}$ is called sublinear if it is subadditive and positive homogeneous i.e., if $f(x+y) \leq f(x) + f(y)$, and $f(\lambda x) = \lambda f(x)$, for every $x, y \in X$ and $\lambda > 0$, respectively. Similarly, f is called superlinear, if it is superadditive i.e., if $f(x+y) \geq f(x) + f(y)$, for every $x, y \in X$, and positive homogeneous.*

Proposition 11. *Let $(X, ||.||)$ be a normed space, $C \subseteq X$ convex and inhabited, (X, C) a McShane-Whitney pair, and $g \in \mathrm{Lip}(C, \sigma)$, for some $\sigma > 0$.*

(i) If g is convex, then ${}^*g$ is convex.
(ii) If g is concave, then g^* is concave.

Proof. We show only (i), and for (ii) we work similarly. Let $x, y \in X$, and $\lambda \in (0,1)$. If we consider the sets $C_x = \{\lambda(g(c) + \sigma||x-c||) \mid c \in C\}$ and $C_y = \{(1-\lambda)(g(c) + \sigma||y-c||) \mid c \in C\}$, an element of $C_x + C_y$ has the form $\lambda(g(c) + \sigma||x-c||) + (1-\lambda)(g(d) + \sigma||y-d||)$, for some $c, d \in C$. We show that ${}^*g(\lambda x + (1-\lambda)y) \leq \lambda(g(c)+\sigma||x-c||) + (1-\lambda)(g(d)+\sigma||y-d||)$, where $c, d \in C$. Since C is convex, $c' := \lambda c + (1-\lambda)d \in C$, and by the convexity of g we get

$$\begin{aligned}{}^*g(\lambda x + (1-\lambda)y) &\leq g(c') + \sigma||\lambda x + (1-\lambda)y - c'|| \\ &\leq \lambda g(c) + (1-\lambda)g(d) + \lambda\sigma||x-c|| + (1-\lambda)\sigma||y-d|| \\ &= \lambda(g(c) + \sigma||x-c||) + (1-\lambda)(g(d) + \sigma||y-d||).\end{aligned}$$

Since the element of $C_x + C_y$ considered is arbitrary, we get that ${}^*g(\lambda x + (1-\lambda)y) \leq \inf(C_x + C_y) = \inf C_x + \inf C_y = \lambda {}^*g(x) + (1-\lambda){}^*g(y)$.

Proposition 12. *Let (X, A) be McShane-Whitney pair, $g_1 \in \mathrm{Lip}(A, \sigma_1), g_2 \in \mathrm{Lip}(A, \sigma_2)$ and $g \in \mathrm{Lip}(A, \sigma)$, for some $\sigma_1, \sigma_2, \sigma > 0$.*

(i) $(g_1 + g_2)^* \leq g_1^* + g_2^*$ and ${}^*(g_1 + g_2) \geq {}^*g_1 + {}^*g_2$.
(ii) If $\lambda > 0$, then $(\lambda g)^* = \lambda g^*$ and ${}^*(\lambda g) = \lambda {}^*g$.
(iii) If $\lambda < 0$, then $(\lambda g)^* = \lambda {}^*g$ and ${}^*(\lambda g) = \lambda g^*$.

Proof. In each case we show only one of the two facts.

(i) We have that $g_1 + g_2 \in \mathrm{Lip}(A, \sigma_1 + \sigma_2)$ and

$$\begin{aligned}(g_1 + g_2)^*(x) &= \sup\{g_1(a) + g_2(a) - (\sigma_1 + \sigma_2)d(x,a) \mid a \in A\} \\ &= \sup\{(g_1(a) - \sigma_1 d(x,a)) + (g_2(a) - \sigma_2 d(x,a)) \mid a \in A\} \\ &\leq \sup\{g_1(a) - \sigma_1 d(x,a) \mid a \in A\} + \sup\{g_2(a) - \sigma_2 d(x,a) \mid a \in A\} \\ &= g_1^*(x) + g_2^*(x).\end{aligned}$$

(ii) If $\lambda \in \mathbb{R}$, then $\lambda g \in \mathrm{Lip}(A, |\lambda|\sigma)$ and if $\lambda > 0$, then

$$\begin{aligned}(\lambda g)^*(x) &= \sup\{\lambda g(a) - |\lambda|\sigma d(x,a) \mid a \in A\}\\ &= \lambda \sup\{g(a) - \sigma d(x,a) \mid a \in A\}\\ &= \lambda g^*(x).\end{aligned}$$

(iii) If $\lambda < 0$, then

$$\begin{aligned}(\lambda g)^*(x) &= \sup\{\lambda g(a) - |\lambda|\sigma d(x,a) \mid a \in A\}\\ &= \sup\{\lambda g(a) - (-\lambda)\sigma d(x,a) \mid a \in A\}\\ &= \sup\{\lambda(g(a) + \sigma d(x,a)) \mid a \in A\}\\ &= \lambda \inf\{g(a) + \sigma d(x,a) \mid a \in A\}\\ &= \lambda {}^*g(x).\end{aligned}$$

Proposition 13. *Let $(X, ||.||)$ be a normed space, A a non-trivial subspace of X such that (X, A) is a McShane-Whitney pair, and let $g \in \mathrm{Lip}(A, \sigma)$, for some $\sigma > 0$, be linear. Moreover, let $x_1, x_2, x \in X$ and $\lambda \in \mathbb{R}$.*

(i) $g^*(x_1 + x_2) \geq g^*(x_1) + g^*(x_2)$ *and* ${}^*g(x_1 + x_2) \leq {}^*g(x_1) + {}^*g(x_2)$.
(ii) If $\lambda > 0$, *then* $g^*(\lambda x) = \lambda g^*(x)$ *and* ${}^*g(\lambda x) = \lambda {}^*g(x)$.
(iii) If $\lambda < 0$, *then* $g^*(\lambda x) = \lambda {}^*g(x)$ *and* ${}^*g(\lambda x) = \lambda g^*(x)$.

Proof. In each case we show only one of the two facts.

(i) If $a_1, a_2 \in A$, then

$$\begin{aligned}{}^*g(x_1 + x_2) &= \inf\{g(a) + \sigma||(x_1 + x_2) - a|| \mid a \in A\}\\ &\leq g(a_1 + a_2) + \sigma||x_1 + x_2 - (a_1 + a_2)||\\ &\leq g(a_1) + \sigma||x_1 - a_1|| + g(a_2) + \sigma||x_2 - a_2||,\end{aligned}$$

therefore

$$\begin{aligned}{}^*g(x_1 + x_2) &\leq \inf\{g(a_1) + \sigma||x_1 - a_1|| + g(a_2) + \sigma||x_2 - a_2|| \mid a_1, a_2 \in A\}\\ &= \inf\{g(a_1) + \sigma||x_1 - a_1|| \mid a_1 \in A\} + \inf\{g(a_2) + \sigma||x_2 - a_2|| \mid a_2 \in A\}\\ &= {}^*g(x_1) + {}^*g(x_2).\end{aligned}$$

(ii) First we show that ${}^*g(\lambda x) \leq \lambda {}^*g(x)$. If $a \in A$, then

$$\begin{aligned}{}^*g(\lambda x) &= \inf\{g(a) + \sigma||\lambda x - a|| \mid a \in A\}\\ &\leq g(\lambda a) + \sigma||\lambda x - \lambda a||\\ &= \lambda g(a) + |\lambda|\sigma||x - a||\\ &= \lambda(g(a) + \sigma||x - a||),\end{aligned}$$

therefore

$$\begin{aligned}{}^*g(\lambda x) &\leq \inf\{\lambda(g(a) + \sigma||x - a||) \mid a \in A\}\\ &= \lambda \inf\{g(a) + \sigma||x - a|| \mid a \in A\}\\ &= \lambda {}^*g(x).\end{aligned}$$

For the inclusion ${}^*g(\lambda x) \geq \lambda^* g(x)$ we work as follows.

$$\begin{aligned}\lambda^* g(x) &= \lambda \inf\{g(a) + \sigma||x - a|| \mid a \in A\} \\ &\leq \lambda(g(\frac{1}{\lambda}a) + \sigma||x - \frac{1}{\lambda}a||) \\ &= g(a) + \sigma|\lambda|||x - \frac{1}{\lambda}a|| \\ &= g(a) + \sigma||\lambda(x - \frac{1}{\lambda}a)|| \\ &= g(a) + \sigma||\lambda x - a||,\end{aligned}$$

therefore

$$\lambda^* g(x) \leq \inf\{g(a) + \sigma||\lambda x - a|| \mid a \in A\} = {}^*g(\lambda x).$$

(iii) First we show that ${}^*g(\lambda x) \leq \lambda g^*(x)$. If $a \in A$, then

$$\begin{aligned}{}^*g(\lambda x) &= \inf\{g(a) + \sigma||\lambda x - a|| \mid a \in A\} \\ &\leq g(\lambda a) + \sigma||\lambda x - \lambda a|| \\ &= \lambda g(a) + |\lambda|\sigma||x - a|| \\ &= \lambda(g(a) - \sigma||x - a||),\end{aligned}$$

therefore

$$\begin{aligned}{}^*g(\lambda x) &\leq \inf\{\lambda(g(a) - \sigma||x - a||) \mid a \in A\} \\ &= \lambda \sup\{g(a) - \sigma||x - a|| \mid a \in A\} \\ &= \lambda g^*(x).\end{aligned}$$

For the inclusion ${}^*g(\lambda x) \geq \lambda g^*(x)$ we work as follows. Since

$$\begin{aligned}g^*(x) &= \sup\{g(a) - \sigma||x - a|| \mid a \in A\} \\ &\geq g(\frac{1}{\lambda}a) - \sigma||x - \frac{1}{\lambda}a||\end{aligned}$$

and $\lambda < 0$, we get

$$\begin{aligned}\lambda g^*(x) &\leq \lambda(g(\frac{1}{\lambda}a) - \sigma||x - \frac{1}{\lambda}a||) \\ &= g(a) - \lambda\sigma||x - \frac{1}{\lambda}a|| \\ &= g(a) + \sigma|\lambda|||x - \frac{1}{\lambda}a|| \\ &= g(a) + \sigma||\lambda(x - \frac{1}{\lambda}a)|| \\ &= g(a) + \sigma||\lambda x - a||,\end{aligned}$$

therefore

$$\lambda g^*(x) \leq \inf\{g(a) + \sigma||\lambda x - a|| \mid a \in A\} = {}^*g(\lambda x).$$

Proposition 13 says that ${}^*g$ is sublinear and g^* is superlinear. If X is a normed space and $x_0 \in X$, then it is not generally the case that $\mathbb{R}x_0 := \{\lambda x_0 \mid \lambda \in \mathbb{R}\}$ is a located subset of X. If $X = \mathbb{R}$, this is equivalent to LPO, the limited principle of omniscience[2] (see [3], p. 122). Things change, if $||x_0|| > 0$. In this case $\mathbb{R}x_0$ is a 1-dimensional subspace of X i.e., a closed and located linear subset of X of dimension one (see [3], p. 307). Of course, $\mathbb{R}x_0$ is a convex subset of X. A standard corollary of the classical Hahn-Banach theorem is that if $x_0 \neq 0$, there is a bounded linear functional u on X such that $||u|| = 1$ and $u(x_0) = ||x_0||$. Its proof is based on the extension of the obvious linear map on $\mathbb{R}x_0$ to X through the Hahn-Banach theorem. Next follows a first approach to the translation of this corollary in Lipschitz analysis. First we need a simple lemma.

Lemma 1. *If $(X, ||.||)$ is a normed space and $x_0 \in X$ such that $||x_0|| > 0$, then $\mathbb{I}x_0 := \{\lambda x_0 \mid \lambda \in [-1, 1]\}$ is a compact subset of X.*

Proposition 14. *If $(X, ||.||)$ is a normed space and $x_0 \in X$ such that $||x_0|| > 0$, there exists $f \in \mathrm{Lip}(X)$ such that $f(x_0) = ||x_0||$ and $L(f) = 1$.*

Proof. The function $g : \mathbb{I}x_0 \to \mathbb{R}$, defined by $g(\lambda x_0) = \lambda||x_0||$, for every $\lambda \in [-1, 1]$, is in $\mathrm{Lip}(\mathbb{I}x_0)$ and $L(g) = 1$; if $\lambda, \mu \in [-1, 1]$, then $|g(\lambda x_0) - g(\mu x_0)| = |\lambda||x_0|| - \mu||x_0||| = |\lambda - \mu|||x_0|| = ||(\lambda - \mu)x_0|| = ||\lambda x_0 - \mu x_0||$, and since

$$M_0(g) = \{\sigma_{\lambda x_0, \mu x_0}(g) = \frac{|g(\lambda x_0) - g(\mu x_0)|}{||\lambda x_0 - \mu x_0||} = 1 \mid (\lambda, \mu) \in [-1, 1]_0\},$$

we get that $L(g) = \sup M_0(g) = 1$. Since $\mathbb{I}x_0$ is inhabited and totally bounded, since by Lemma 1 it is compact, by Proposition 7(i) and Theorem 1 the extension ${}^*g$ of g is in $\mathrm{Lip}(X)$, while by Proposition 10 we have that $L({}^*g) = L(g) = 1$.

Theorem 2. *Let $(X, ||.||)$ be a normed space and $x_0 \in X$ such that $||x_0|| > 0$. If $(X, \mathbb{R}x_0)$ is a McShane-Whitney pair, there exist a sublinear Lipschitz function f on X such that $f(x_0) = ||x_0||$ and $L(f) = 1$, and a superlinear Lipschitz function h on X such that $h(x_0) = ||x_0||$ and $L(h) = 1$.*

Proof. As in the proof of Proposition 14 the function $g : \mathbb{R}x_0 \to \mathbb{R}$, defined by $g(\lambda x_0) = \lambda||x_0||$, for every $\lambda \in \mathbb{R}$, is in $\mathrm{Lip}(\mathbb{R}x_0)$ and $L(g) = 1$. Since $(X, \mathbb{R}x_0)$ is a McShane-Whitney pair, the extension ${}^*g$ of g is a Lipschitz function, and by Proposition 10 $L({}^*g) = L(g) = 1$. Since g is linear, by Proposition 13 we get that ${}^*g$ is sublinear. Similarly, the extension g^* of g is a Lipschitz function, and by Proposition 10 $L(g^*) = L(g) = 1$. Since g is linear, by Proposition 13 we get that g^* is superlinear.

[2] From this we can explain why it is not constructively acceptable that any pair (X, A) is McShane-Whitney. If $x_0 \in \mathbb{R}$ and $(\mathbb{R}, \mathbb{R}x_0)$ is a McShane-Whitney pair, then by Proposition 8(i) we have that $\mathbb{R}x_0$ is located, which implies LPO.

4 Concluding Remarks

Similarly to Theorem 1, one can prove an extension theorem for Hölder continuous functions, or for functions which are continuous with respect to a given modulus of continuity λ i.e., a function of type $[0, +\infty) \rightarrow [0, +\infty)$, which is subadditive, strictly increasing, uniformly continuous on every bounded subset of $[0, +\infty)$, and $\lambda(0) = 0$ (see also [3], p. 102). Note that one could have defined a McShane-Whitney pair such that the functions g^* and $^*g$ are given by $g^*(x) = \text{glb} M_g(A, \sigma, x)$ and $^*g(x) = \text{lub} M_g(A, -\sigma, x)$, for every $x \in X$, respectively, since only the properties of glb and lub are used in the proof of Theorem 1.

Some open problems related to the material presented here are the following:

a. To find necessary and sufficient conditions on X, Y and $f \in \text{Lip}(X, Y)$ for the L-pseudo-normability of f.
b. To find conditions on $(X, ||.||)$ under which one can show constructively that $(X, \mathbb{R}x_0)$ is a McShane-Whitney pair, if $||x_0|| > 0$. A similar attitude is taken by Ishihara in his constructive proof of the Hahn-Banach theorem, where the property of Gâteaux differentiability of the norm is added (see [5,7], p.126).
c. To elaborate the Lipschitz version of the theory of the Hahn-Banach theorem.
d. If $(\mathbb{R}^n, A)$ is a McShane-Whitney pair and $g \in \text{Lip}(A, \mathbb{R}^m, \sigma)$, then by Theorem 1 there are extensions g^* and $^*g$ of g in $\text{Lip}(\mathbb{R}^n, \mathbb{R}^m, \sqrt{m}\sigma)$. According to the classical Kirszbraun theorem there is an extension of g in $\text{Lip}(\mathbb{R}^n, \mathbb{R}^m, \sigma)$ (see [9,15,16,18]). The constructive study of the Kirszbraun theorem is a non-trivial enterprise.

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