



Fractional Differential Calculus and Continuum Mechanics

24

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Contents

Introduction	854
Basic Properties of Fractional Calculus	858
The Geometry of Fractional Differential	860
Differentiation	863
Linear Combinations	863
Linear Maps	864
Bilinear Maps	864
Cartesian Products	864
Compositions	864
The Fractional Arc Length	865
The Fractional Tangent Space	866
Fractional Curvature of Curves	867
The Fractional Radius of Curvature of a Curve	867
The Serret-Frenet Equations	868
Applications	870
The Fractional Geometry of a Parabola	870
The Tangent and Curvature Center of the Weierstrass Function	872
Bending of Fractional Beams	872
The Fractional Tangent Plane of a Surface	873
Fundamental Differential Forms on Fractional Differential Manifolds	875
The First Fractional Fundamental Form	875

The present essay is dedicated to Pepi Lazopoulou M.D., adorable wife and mother of the authors for her devotion in our family

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The Second Fractional Fundamental Form	876
The Fractional Normal Curvature	877
Fractional Vector Operators	878
Fractional Vector Field Theorems	879
Fractional Green's Formula	879
Fractional Stoke's Formula	880
Fractional Gauss' Formula	881
Fractional Deformation Geometry	881
Polar Decomposition of the Deformation Gradient	883
Deformation of Volume and Surface	884
Examples of Deformations	885
Homogeneous Deformations	885
The Nonhomogeneous Deformations	887
The Infinitesimal Deformations	888
Fractional Stresses	888
The Balance Principles	889
Material Derivatives of Volume, Surface, and Line Integrals	889
The Balance of Mass	891
Balance of Linear Momentum Principle	892
Balance of Rotational Momentum Principle	892
Fractional Zener Viscoelastic Model	893
The Integer Viscoelastic Model	893
The Fractional Order Derivative Viscoelastic Models	894
Proposed Fractional Viscoelastic Zener Model	895
Comparison of the Three Viscoelastic Models	898
Conclusion: Further Research	900
References	902

Abstract

The present essay is an attempt to present a meaningful continuum mechanics formulation into the context of fractional calculus. The task is not easy, since people working on various fields using fractional calculus take for granted that a fractional physical problem is set up by simple substitution of the conventional derivatives to any kind of the plethora of fractional derivatives. However, that procedure is meaningless, although popular, since laws in science are derived through differentials and not through derivatives. One source of that mistake is that the fractional derivative of a variable with respect to itself is different from one. The other source of the same mistake is that the well-known derivatives are not able to form differentials. This leads to erroneous and meaningless quantities like fractional velocity and fractional strain. In reality those quantities, that nobody understands what physically represent, alter the dimensions of the physical quantities. In fact the dimension of the fractional velocity is L/T^α , contrary to the conventional L/T . Likewise, the dimension of the fractional strain is $L^{-\alpha}$, contrary to the conventional L^0 . That fact cannot be justified. Imagine that even in relativity theory, where everything is changed, like time, lengths, velocities, momentums, etc., the dimensions remain constant. Fractional calculus is allowed up to now to change the dimensions and to accept derivatives

that are not able to form differentials, according to differential topology laws. Those handicaps have been pointed out in two recent conferences dedicated to fractional calculus by the authors, (K.A. Lazopoulos, in *Fractional Vector Calculus and Fractional Continuum Mechanics, Conference "Mechanics though Mathematical Modelling", celebrating the 70th birthday of Prof. T. Atanackovic*, Novi Sad, 6–11 Sept, Abstract, p. 40, 2015; K.A. Lazopoulos, A.K. Lazopoulos, Fractional vector calculus and fractional continuum mechanics. *Prog. Fract. Diff. Appl.* **2**(1), 67–86, 2016a) and were accepted by the fractional calculus community. The authors in their lectures (K.A. Lazopoulos, in *Fractional Vector Calculus and Fractional Continuum Mechanics, Conference "Mechanics though Mathematical Modelling", celebrating the 70th birthday of Prof. T. Atanackovic*, Novi Sad, 6–11 Sept, Abstract, p. 40, 2015; K.A. Lazopoulos, in *Fractional Differential Geometry of Curves and Surfaces, International Conference on Fractional Differentiation and Its Applications (ICFDA 2016)*, Novi Sad, 2016b; A.K. Lazopoulos, On *Fractional Peridynamic Deformations, International Conference on Fractional Differentiation and Its Applications, Proceedings ICFDA 2016*, Novi Sad, 2016c) and in the two recently published papers concerning fractional differential geometry of curves and surfaces (K.A. Lazopoulos, A.K. Lazopoulos, On the fractional differential geometry of curves and surfaces. *Prog. Fract. Diff. Appl.*, No **2**(3), 169–186, 2016b) and fractional continuum mechanics (K.A. Lazopoulos, A.K. Lazopoulos, Fractional vector calculus and fractional continuum mechanics. *Prog. Fract. Diff. Appl.* **2**(1), 67–86, 2016a) added in the plethora of fractional derivatives one more, that called Leibniz L-fractional derivative. That derivative is able to yield differential and formulate fractional differential geometry. Using that derivative the dimensions of the various quantities remain constant and are equal to the dimensions of the conventional quantities. Since the establishment of fractional differential geometry is necessary for dealing with continuum mechanics, fractional differential geometry of curves and surfaces with the fractional field theory will be discussed first. Then the quantities and principles concerning fractional continuum mechanics will be derived. Finally, fractional viscoelasticity Zener model will be presented as application of the proposed theory, since it is of first priority for the fractional calculus people. Hence the present essay will be divided into two major chapters, the chapter of fractional differential geometry, and the chapter of the fractional continuum mechanics. It is pointed out that the well-known historical events concerning the evolution of the fractional calculus will be circumvented, since the goal of the authors is the presentation of the fractional analysis with derivatives able to form differentials, formulating not only fractional differential geometry but also establishing the fractional continuum mechanics principles. For instance, following the concepts of fractional differential and Leibniz's L-fractional derivatives, proposed by the author (K.A. Lazopoulos, A.K. Lazopoulos, Fractional vector calculus and fractional continuum mechanics. *Prog. Fract. Diff. Appl.* **2**(1), 67–86, 2016a), the L-fractional chain rule is introduced. Furthermore, the theory of curves and surfaces is revisited, into the context of fractional calculus. The fractional tangents, normals, curvature vectors, and radii of curvature of curves

are defined. Further, the Serret-Frenet equations are revisited, into the context of fractional calculus. The proposed theory is implemented into a parabola and the curve configured by the Weierstrass function as well. The fractional bending problem of an inhomogeneous beam is also presented, as implementation of the proposed theory. In addition, the theory is extended on manifolds, defining the fractional first differential (tangent) spaces, along with the revisiting first and second fundamental forms for the surfaces. Yet, revisited operators like fractional gradient, divergence, and rotation are introduced, outlining revision of the vector field theorems. Finally, the viscoelastic mechanical Zener system is modelled with the help of Leibniz fractional derivative. The compliance and relaxation behavior of the viscoelastic systems is revisited and comparison with the conventional systems and the existing fractional viscoelastic systems are presented.

Keywords

Fractional Derivative · Fractional Differential · Fractional Stress · Fractional Strain · Fractional Principles in Mechanics · Fractional Continuum Mechanics

Introduction

Fractional Calculus, originated by Leibnitz (1849), Liouville (1832), and Riemann (1876) has recently applied to modern advances in physics and engineering. Fractional derivative models account for long-range (nonlocal) dependence of phenomena, resulting in better description of their behavior. Various material models, based upon fractional time derivatives, have been presented, describing their viscoelastic interaction (Atanackovic 2002; Mainardi 2010). Lazopoulos (2006) has proposed an elastic uniaxial model, based upon fractional derivatives for lifting Noll's axiom of local-action (Carpinteri et al. 2011) have also proposed a fractional approach to nonlocal mechanics. Applications in various physical areas may also be found in various books (Kilbas et al. 2006; Samko et al. 1993; Poldubny 1999; Oldham and Spanier 1974).

Since the need for Fractional Differential Geometry has extensively been discussed in various places, researchers have presented different aspects, concerning fractional geometry of manifolds (Tarasov 2010; Calcani 2012) with applications in fields of mechanics, quantum mechanics, relativity, finance, probabilities, etc. Nevertheless, researchers are raising doubtfulness about the existence of fractional differential geometry and their argument is not easily rejected.

Basically, the classical differential $df(x) = f'(x)dx$ has been substituted by the fractional one introduced by Adda (2001, 1998) in the form:

$$d^a f = g(x)(dx)^a$$

Nevertheless that definition of the differential is valid in the case of positive increments dx , whereas in the case of negative increments, the differential $d^a f(x)$

may be complex. That is exactly the reason why many researchers reasonably reject the existence of fractional differential geometry. However the variable x accepts its own fractional differential:

$$d^\alpha x = \sigma(x)(dx)^\alpha$$

with $\sigma(x) \neq 1$, differently of the conventional case when $a = 1$, where $\sigma(x)$ is always one. Relating both equations, it appears that:

$$d^a f = \frac{g(x)}{\sigma(x)} d^a x$$

In this case $d^\alpha x$ is always a real quantity accepting positive or negative incremental real values alike. On these bases, the development of fractional differential geometry may be established.

Further, fractal functions exhibiting self-similarity are nondifferentiable functions, but they exhibit fractional differentiability of order $0 < \alpha < 1$ (see Yao et al. 2005; Carpinteri et al. 2009; Goldmankhaneh et al. 2013; Liang and Su 2007). Goldmankhaneh et al. (2013) introduced the generalized fractional Riemann-Liouville and Caputo-like derivatives for functions defined on fractal sets.

Fractional Calculus in mechanics has been suggested by many researchers, Tarasov (2010, 2008), Drapaca and Sivaloganathan (2012), Sumelka (2014), and Lazopoulos and Lazopoulos (2016a), in problems of continuum mechanics with microstructure where nonlocal elasticity is necessary. Fractional continuum mechanics has been applied to various problems in hydrodynamics (Tarasov 2010; Balankin and Elizarrataz 2012). Recently fractional calculus has been introduced by the author (Lazopoulos 2016a) for the description of peridynamic theory (Silling 2003, 2010). Yet, fractional calculus has been considered as the best frame for describing viscoelastic problems (Atanackovic 2002; Beyer and Kempfle 1995). In addition fractional differential geometry affects rigid body dynamics, in holonomic and nonholonomic systems (Riewe 1996, 1997; Baleanu et al. 2013). Recent applications in quantum mechanics, physics, and relativity demand differential geometry revisited by fractional calculus (Golmankhaneh Ali et al. 2015; Baleanu et al. 2009).

In the present work, the fractional differential established in Lazopoulos and Lazopoulos (2016a) will be recalled along with the introduced Leibniz's L-fractional derivatives. Those differentials are always real and proper for establishing the fractional differential geometry. Correcting the picture of fractional differential of a function, the fractional tangent space of a manifold was defined, introducing also Leibniz's L-fractional derivative that is the only one having physical meaning. Moreover, the present work reviews description of fractional geometry of curves, describing their tangent (first differential) spaces, their normals, the curvature vectors, and the corresponding radii of curvature. In addition, the Serret-Frenet equations will be revisited into the fractional calculus context. The theory is implemented to a parabola, to the Weierstrass function and the beam bending (Lazopoulos et al. 2015), considered as applications of the curves' theory to the solid

mechanics. Yet, the theory is extended on manifolds, just to describe the fractional differential geometry of surfaces. Finally outline of fractional vector field theory is included, along with the revisited fractional vector field theorems.

The mechanics researchers have been motivated by the mechanical behavior of disordered (nonhomogeneous) materials with microstructure. Porous materials (Vardoulakis et al. 1998, Ma et al. 2002), colloidal aggregates (Wyss et al. 2005), ceramics, etc., are materials with microstructure that exert strong influence in their deformation. Major factors in determining the material deformation are microcracks, voids, material phases, etc. The nonhomogeneity of the heterogeneous materials has been tackled by various homogenization theories (Bakhalov and Panasenko 1989). Nevertheless, these materials require the lifting of the basic local action axiom of continuum mechanics (Truesdell 1977; Truesdell and Noll 1965). As defined by Noll (1958, 1959) simple materials satisfy the three fundamental axioms:

- (a) The principle of determinism
- (b) The principle of local action
- (c) The principle of material frame-indifference.

Truesdell and Noll (1965) points out in his classic continuum mechanics book: “The motion of body-points at a finite distance from a point x in some shape may be disregarded in calculating the stress at x .” Material microstructure, inhomogeneities, microcracks, etc., are some of the various important factors that affect the material deformation with nonlocal action. These factors are not considered in the simple materials formulation.

Various theories have been proposed just to introduce a long distance action in the deformation of the materials. One direction considers Taylor’s expansion of the strain tensor in the neighborhood of a point, taking in consideration one or two most important terms. Hence gradient strain theories have been appeared in nonlinear form (Toupin 1965), and in linear deformation (Mindlin 1965). Eringen (2002) has also proposed a theory dealing with micropolar elasticity. Mindlin introduced a simpler version of linear gradient theories and an even simpler model has been presented by Aifantis (1999) with his GRADELA model. In these theories, the authors introduced intrinsic material lengths that accompany the higher order derivatives of the strain. Many problems have been solved employing those theories concerning size effects, lifting of various singularities, porous materials (Aifantis 1999, 2003, 2011; Askes and Aifantis 2011), mechanics of microbeams, microplates, and microsheets (Lazopoulos 2004; Lazopoulos et al. 2010).

However various nonlocal elastic theories have been introduced, that are more reliable in taking care of nonsmooth deformations, since integrals are friendlier than derivatives to take care of various nonsmooth phenomena.

Lazopoulos (2006) introduced fractional derivatives of the strain in the strain energy density function in an attempt to introduce nonlocality in the elastic response of materials. Fractional calculus was used by many researchers, not only in the field of Mechanics but mainly in Physics and especially in Quantum Mechanics, to develop the idea of introducing nonlocality. In fact, the history of fractional calculus

is dated since seventeenth century. Particle physics, electromagnetics, mechanics of materials, hydrodynamics, fluid flow, rheology, viscoelasticity, optics, electrochemistry and corrosion, and chemical physics are some fields where fractional calculus has been introduced.

Fractional calculus in material deformations has been adopted in solving various types of problems. First we may consider the deformation problems with nonsmooth strain field. Second heterogeneous material deformations may also be studied. Furthermore, time fractional derivative is proved to be more suitable in viscoelastic deformations, since viscoelastic deformations with retarded memory materials may also be discussed. The nonlocal strain effects of deformation problems are concerned by the last type of those problems. There are many studies considering fractional elasticity theory, introducing fractional strain (Drapaca and Sivaloganathan 2012; Carpinteri et al. 2001, 2011; Di Paola et al. 2009; Atanackovic et al. 2008; Agrawal 2008). Recently Sumelka (2014) has presented applications of Fractional Calculus in the nonlocal elastic deformation of Kirchhoff-Love plates and in the rate-independent plasticity. Nevertheless a different definition of fractional strain is yielded in the present work. Jumarie (2012) has proposed modified Riemann-Liouville derivative of fractional order with an approach to differential geometry of fractional order. In addition Meerschaert et al. (2006) have presented fractional order vector calculus for fractional advection-dispersion. Recent applications of fractional calculus have been appeared in peridynamic theory (Silling 2000; Silling et al. 2003; Evangelatos 2011; Evangelatos and Spanos 2012). Tarasov (2010) has also presented a book including fractional mathematics and its applications to various physics areas. In addition Tarasov (2008) has presented a fractional vector fields theory combining fractals (Feder 1988), and fractional calculus.

Lazopoulos and Lazopoulos (2016a, b) have clarified the geometry of the fractional differential resulting in fractional tangent spaces of the manifolds quite different from the conventional ones. Hence the fractional differential geometry has been established, indispensable for the development of fractional mechanics. It is evident that the definition of the stress and the strain is greatly affected by the tangent spaces. Hence the fractional stress tensors and the fractional strain tensors are quite different from the conventional ones. The linear strain tensors are also revisited. Those basic concepts are important for establishing fractional continuum mechanics.

In the present work, fractional vector calculus is revisited, since the fractional differential of a function is not linearly dependent upon the conventional differential of the variables. Furthermore, the fractional derivative of a variable with respect to itself is different from one. The fractional vector calculus is revisited along with the basic field theorems of Green, Stokes, and Gauss. Applications of the fractional vector calculus to continuum mechanics are presented. The revision in the right and left Cauchy-Green deformation tensors and Green (Lagrange) and Euler-Almansi strain tensors are exhibited. The change of volume and the surface due to deformation (change of configuration) of a deformable body is also discussed. Further the revisited fractional continuum mechanics principles yielding the fractional continuity and motion equations are also derived.

Moreover, linearization of the strain tensors is performed. The change of fractional volume and fractional surface due to deformation (change of configuration) of a deformable body is also discussed. In addition, the revisited fractional continuum mechanics principles, such as mass conservation (fractional continuity equation) and motion equations (conservation of fractional linear and rotational momentum), are also discussed.

In addition, Bagley et al. (1986) introduced Fractional Calculus in viscoelasticity, and Atanackovic et al. (2002, 2002a) pursue the idea in many applications and in fractional variational problems (Atanackovic et al. 2008) with fractional derivatives. Mainardi (2010) has also discussed the application of fractional calculus in linear viscoelasticity. In Sabatier et al. (2007), there exists a section concerning viscoelastic disordered media. As an application to the present theory, the viscoelastic behavior of Zener model will be revisited using Leibniz fractional time derivatives. Comparison of the proposed model to the existing fractional ones will be discussed. Further the behavior of the proposed model concerning its compliance and relaxation is discussed and compared to the existing fractional ones and the conventional as well.

It is pointed out that the present essay will be divided into two major chapters, the chapter of fractional differential geometry and the chapter of the fractional continuum mechanics. The viscoelasticity Zener model will be discussed into the context of the proposed theory.

Basic Properties of Fractional Calculus

Fractional Calculus has recently become a branch of pure mathematics, with many applications in Physics and Engineering, (Tarasov 2008, 2010). Many definitions of fractional derivatives exist. In fact, fractional calculus originated by Leibniz, looking for the possibility of defining the derivative $\frac{d^n g}{dx^n}$ when $n = \frac{1}{2}$. The various types of the fractional derivatives exhibit some advantages over the others. Nevertheless they are almost all nonlocal, contrary to the conventional ones.

The detailed properties of fractional derivatives may be found in Kilbas et al. (2006), Podlubny (1999), and Samko et al. (1993). Starting from Cauchy formula for the n -fold integral of a primitive function $f(x)$

$$I^n f(x) = \int_a^x f(s) (ds)^n = \int_a^x dx_n \int_a^{x_n} dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \dots \int_a^{x_2} f(x_1) dx_1 \quad (1)$$

expressed by:

$${}_a I_x^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s) ds, \quad x > 0, n \in N \quad (2)$$

and

$${}_x I_b^n f(x) = \frac{1}{(n-1)!} \int_x^b (s-x)^{n-1} f(s) ds, \quad x > 0, n \in \mathbb{N} \tag{3}$$

the left and right fractional integral of f are defined as:

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x-s)^{1-\alpha}} ds \tag{4}$$

$${}_x I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(s)}{(s-x)^{1-\alpha}} ds \tag{5}$$

In Eqs. 4 and 5 we assume that α is the order of fractional integrals with $0 < \alpha \leq 1$, considering $\Gamma(x) = (x-1)!$ with $\Gamma(\alpha)$ Euler’s Gamma function.

Thus the left and right Riemann-Liouville (R-L) derivatives are defined by:

$${}_a D_x^\alpha f(x) = \frac{d}{dx} ({}_a I_x^{1-\alpha} f(x)) \tag{6}$$

and

$${}_x D_b^\alpha f(x) = -\frac{d}{dx} ({}_b I_x^{1-\alpha} f(x)) \tag{7}$$

Pointing out that the R-L derivatives of a constant c are nonzero, Caputo’s derivative has been introduced, yielding zero for any constant. Thus, it is considered as more suitable in the description of physical systems.

In fact Caputo’s derivative is defined by:

$${}_a^c D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(s)}{(x-s)^\alpha} ds \tag{8}$$

$$\text{and } {}_x^c D_b^\alpha f(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{f'(s)}{(s-x)^\alpha} ds \tag{9}$$

Evaluating Caputo’s derivatives for functions of the type:

$f(x) = (x-a)^n$ or $f(x) = (b-x)^n$ we get:

$${}_a^c D_x^\alpha (x-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} (x-a)^{\nu-\alpha}, \tag{10}$$

and for the corresponding right Caputo's derivative:

$${}_x D_b^{\alpha} (b-x)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} (b-x)^{\nu-\alpha}$$

Likewise, Caputo's derivatives are zero for constant functions:

$$f(x) = c. \tag{11}$$

Finally, Jumarie's derivatives are defined by,

$${}_a^J D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(s) - f(a)}{(x-s)^{\alpha}} ds$$

and

$${}_x^J D_b^{\alpha} f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(s) - f(b)}{(s-x)^{\alpha}} ds$$

Although those derivatives are accompanied by some derivation rules that are not valid, the derivatives themselves are valid and according to our opinion are better than Caputo's, since they accept functions less smooth than the ones for Caputo. Also Jumarie's derivative is zero for constant functions, basic property, advantage, of Caputo derivative. Nevertheless, Caputo's derivative will be employed in the present work, having in mind that Jumarie's derivative may serve better our purpose.

The Geometry of Fractional Differential

It is reminded, the n -fold integral of the primitive function $f(x)$, Eq. 1, is

$$I^n f(x) = \int_a^x f(s) (ds)^n \tag{12}$$

which is real for any positive or negative increment ds . Passing to the fractional integral

$$I^{\alpha} (f(x)) = \int_a^x f(s) (ds)^{\alpha} \tag{13}$$

the integer n is simply substituted by the fractional number α . Nevertheless, that substitution is not at all straightforward. The major difference between passing from

Eqs. 11 to 12 is that although $(ds)^n$ is real for negative values of ds , $(ds)^\alpha$ is complex. Therefore, the fractional integral, Eq. 13, is not compact for any increment ds . Hence the integral of Eq. 13 is misleading. In other words, the differential, necessary for the existence of the fractional integral, Eq. 13, is wrong. Hence, a new fractional differential, real and valid for positive and negative values of the increment ds , should be established.

It is reminded that the α -Fractional differential of a function $f(x)$ is defined by, Adda (1998):

$$d^\alpha f(x) = {}_a^c D_x^\alpha f(x)(dx)^\alpha \quad (14)$$

It is evident that the fractional differential, defined by Eq. 14, is valid for positive incremental dx , whereas for negative ones, that differential might be complex. Hence considering for the moment that the increment dx is positive, and recalling that ${}_a^c D_x^\alpha x \neq 1$, the α -fractional differential of the variable x is:

$$d^\alpha x = {}_a^c D_x^\alpha x(dx)^\alpha \quad (15)$$

Hence

$$d^\alpha f(x) = \frac{{}_a^c D_x^\alpha f(x)}{{}_a^c D_x^\alpha x} d^\alpha x \quad (16)$$

It is evident that $d^\alpha f(x)$ is a nonlinear function of dx , although it is a linear function of $d^\alpha x$. That fact suggests the consideration of the fractional tangent space that we propose. Now the definition of fractional differential, Eq. 16, is imposed either for positive or negative variable differentials $d^\alpha x$. In addition the proposed L-fractional (in honor of Leibniz) derivative ${}_a^L D_x^\alpha f(x)$ is defined by,

$$d^\alpha f(x) = {}_a^L D_x^\alpha f(x) d^\alpha x \quad (17)$$

with the Leibniz L-fractional derivative,

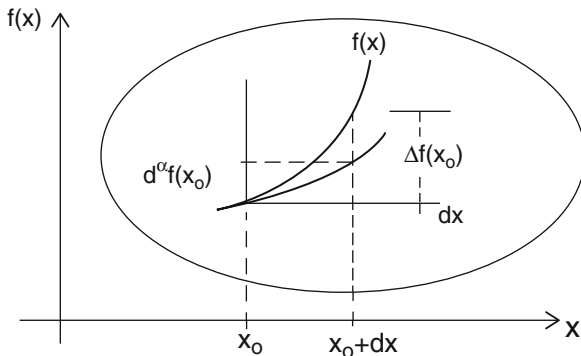
$${}_a^L D_x^\alpha f(x) = \frac{{}_a^c D_x^\alpha f(x)}{{}_a^c D_x^\alpha x} \quad (18)$$

Hence only Leibniz's derivative has any geometrical or physical meaning.

In addition, Eq. 3, is deceiving and the correct form of Eq. 3 should be substituted by,

$$f(x) - f(a) = {}_a^L I_x^\alpha \left({}_a^L D_x^\alpha f(x) \right) = \frac{1}{\Gamma(\alpha) \Gamma(2-\alpha)} \int_a^x \frac{(s-a)^{1-\alpha}}{(x-s)^{1-\alpha}} {}_a^L D_x^\alpha f(s) ds \quad (19)$$

Fig. 1 The nonlinear differential of $f(x)$



It should be pointed out that the correct forms are defined for the fractional differential by Eq. 17, the Leibniz derivative, Eq. 18 and the fractional integral by Eq. 19. All the other forms are misleading.

Configuring the fractional differential, along with the first fractional differential space (fractional tangent space), the function $y = f(x)$ has been drawn in Fig. 1, with the corresponding first differential space at a point x , according to Adda’s definition, Eq. 14.

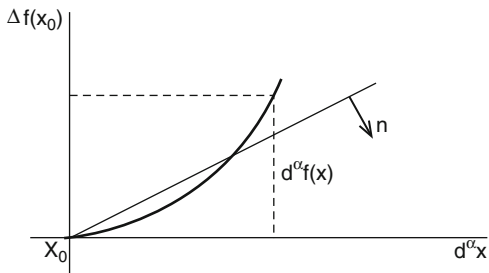
The tangent space, according to Adda’s (1998) definition, Eq. 14, is configured by the nonlinear curve $d^\alpha f(x)$ versus dx . Nevertheless, there are some questions concerning the correct picture of the configuration, (Fig. 1), concerning the fractional differential presented by Adda (1998). Indeed,

- (a) The tangent space should be linear. There is not conceivable reason for the nonlinear tangent spaces.
- (b) The differential should be configured for positive and negative increments dx . However, the tangent spaces, in the present case, do not exist for negative increments dx .
- (c) The axis $d^\alpha f(x)$, in Fig. 1, presents the fractional differential of the function $f(x)$, however dx denotes the conventional differential of the variable x . It is evident that both axes along x and $f(x)$ should correspond to differentials of the same order.

Therefore, the tangent space (first differential space), should be configured in the coordinate system with axes $(d^\alpha x, d^\alpha f(x))$. Hence, the fractional differential, defined by Eq. 17, is configured in the plane $(d^\alpha x, d^\alpha f(x))$ by a line, as it is shown in Fig. 2.

It is evident that the differential space is not tangent (in the conventional sense) to the function at x_0 , but intersects the figure $y = f(x)$ at least at one point x_0 . This space, we introduce, is the tangent space. Likewise, the normal is perpendicular to the line of the fractional tangent. Hence we are able to establish fractional differential geometry of curves and surfaces with the fractional field theory. Consequently when $\alpha = 1$, the tangent spaces, we propose, coincide with

Fig. 2 The virtual tangent space of the $f(x)$ at the point $x = x_0$



the conventional tangent spaces. As a last comment concerning the proposed L-fractional derivative, the physical dimensions of the various quantities remain unaltered from the conventional to any order Fractional Calculus.

Differentiation

The following chapter is a summary from the paragraphs 2.4–2.6 of the book, Differential topology with a view to applications (Chillingworth 1976).

Let E and F be two normed linear spaces with respective norms $\|\cdot\|_E$ and $\|\cdot\|_F$, suppose $f: E \rightarrow F$ is a given continuous (not necessarily linear) map, and let x be a particular point of E .

Definition

The map f is differentiable if there exists a linear map $L: E \rightarrow F$ which approximates f at x in the sense that for all h in E we have:

$$f(x + h) - f(x) = L(h) + \|h\|_E \eta(h) \tag{20}$$

where $h(x)$ is an element of F with,

$$\|\eta(h)\|_F \rightarrow 0 \text{ as } \|h\|_E \rightarrow 0 \tag{21}$$

In this definition, E and F are two normed linear spaces with respective norms $\|\cdot\|_E$ and $\|\cdot\|_F$, and $f: E \rightarrow F$ is a given continuous (not necessarily linear) map.

Furthermore, let x be a particular point of E and U an open subset of E . The properties and uses of the derivative are:

Linear Combinations

If $f, g: U \rightarrow F$ are differentiable then so is the map $\alpha f + \beta g$ for any constants α, β and $D(\alpha f + \beta g) = \alpha Df + \beta Dg$ as maps $U \rightarrow L(E, F)$. At this point we must point out that constant maps themselves have derivative zero.

Linear Maps

If f is the restriction to U of a continuous linear map: $E \rightarrow F$, then $Df(x) = L$ for every $x \in U$, i.e. L is already its own linear approximation.

In the special case $E = F = R$ we have $f(x) = \zeta x$ for some number ζ , and $Df(x) = \zeta$ regarded as the linear map $R \rightarrow R: x \rightarrow \zeta x$.

Bilinear Maps

If $E = E_1 \times E_2$ and f is the restriction to U of a continuous bilinear map $B: E_1 \times E_2 \rightarrow F$ (i.e. B is linear in each factor separately) then $Df(x)h = B(x_1, h_2) + B(h_1, x_2)$ where $x = (x_1, x_2)$, $h = (h_1, h_2)$ with x_1, h_1 in E_1 and x_2, h_2 in E_2 .

Cartesian Products

If $f_1: U_1 \rightarrow F_1$ and $f_2: U_2 \rightarrow F_2$ are differentiable then so is $f = f_1 \times f_2: U_1 \times U_2 \rightarrow F_1 \times F_2$ where $(f_1 \times f_2)(x_1, x_2)$ means $(f_1(x_1), f_2(x_2))$, and we have:

$$Df(x_1, x_2)(h_1 \times h_2) = (Df_1(x_1)h_1, Df_2(x_2)h_2) \quad (22)$$

That is, derivatives operate coordinate-wise.

Compositions

Chain Rule

Suppose E, F , and G are three normed linear spaces, and U, V are open sets in E, F , respectively. Let $f: U \rightarrow F$ and $g: V \rightarrow G$ be continuous maps, and suppose the image of f lies in V so that the composition $g \circ f: U \rightarrow G$ exists.

If f is differentiable at x and g is differentiable at $f(x)$ then $g \circ f$ is differentiable

$$D(g \circ f)(x) = Dg(f(x)) \bullet Df(x) \quad (23)$$

In other words, the derivative of a composition is the composition of the derivatives.

If $E = R^n, F = R^m, G = R^p$, and $y = f(x), z = z(y)$ the chain rule states that:

$$\frac{\partial (z_1, z_2, \dots, z_p)}{\partial (x_1, x_2, \dots, x_n)} = \frac{\partial (z_1, z_2, \dots, z_p)}{\partial (y_1, y_2, \dots, y_m)} \cdot \frac{\partial (y_1, y_2, \dots, y_m)}{\partial (x_1, x_2, \dots, x_n)} \quad (24)$$

Product Rule

For the same maps f, g holds:

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f) \quad (25)$$

Let's assume derivative of a product $f(t) = A(t) \cdot b(t)$ where $A(t)$ is an $m \times n$ matrix and $b(t)$ is an n vector. Then:

$$Df(t) = A(t) \cdot Db(t) + DA(t) \cdot b(t) \quad (26)$$

Now the question arises whether L-fractional derivative satisfies all the conditions, Eqs. 22, 23, 24, 25, and 26 required by the demand of differentiation according to differential topology rules. The answer is no. There are some rules that the L-fractional derivative satisfies, like the linearity condition, Eq. 17 that cannot be satisfied by the common fractional derivatives like Caputo etc., since the nonlinear Eq. 14 holds for them. However, there are other conditions, such as the chain rule that are not valid. Now we have to make a choice. Either to define differential that is necessary for establishing fractional differential geometry that is necessary for dealing with problems in Physics, or to forget fractional calculus. On that dilemma we make the choice to *impose* the necessary rules, like fractional chain rule, just for forming fractional differential geometry. In that case the physical problem leaves its print or trace on mathematics, that in conventional calculus is not necessary, since all differential rules are satisfied by themselves, in the conventional case.

The Fractional Arc Length

Let $y = f(x)$ be a function, which may be non-differentiable but has a fractional derivative of order α , $0 < \alpha < 1$. The fractional differential of $y = f(x)$ in the differential space is defined by:

$$d^a y = \frac{{}_a D_x^a f(x)}{{}_a D_x^a x} d^a x = {}_a^L D_x^a f(x) d^a x \quad (27)$$

Therefore the arc length is expressed by:

$$s_1(x, a) = {}_a I_x^a \left[(d^a y)^2 + (d^a x)^2 \right]^{1/2} = {}_a I_x^a \left[\left(\frac{{}_a D_x^a f(x)}{{}_a D_x^a x} \right)^2 + 1 \right]^{1/2} d^a x \quad (28)$$

Furthermore, for parametric curves of the type:

$$y = f(t), \quad x = g(t) \quad (29)$$

The fractional α -differentials are defined by:

$$\begin{aligned} d^a x &= \frac{{}_a D_t^\alpha g(t)}{{}_a D_t^\alpha t} d^a t \\ d^a y &= \frac{{}_a D_t^\alpha f(t)}{{}_a D_t^\alpha t} d^a t \end{aligned} \tag{30}$$

and the fractional differential of the arc length is expressed by:

$$d^a s = \sqrt{(d^a y)^2 + (d^a x)^2} = \left[\left(\frac{{}_a D_t^\alpha f(t)}{{}_a D_t^\alpha t} \right)^2 + \left(\frac{{}_a D_t^\alpha g(t)}{{}_a D_t^\alpha t} \right)^2 \right]^{\frac{1}{2}} d^a t \tag{31}$$

and

$$\begin{aligned} s &= {}^L I_x^a d^a s = {}^L I_t^a \left[\left(\frac{{}_a D_t^\alpha f(t)}{{}_a D_t^\alpha t} \right)^2 + \left(\frac{{}_a D_t^\alpha g(t)}{{}_a D_t^\alpha t} \right)^2 \right]^{\frac{1}{2}} d^{\alpha} t \\ &= {}^L I_t^a \left[\left({}^L D_t^\alpha f(t) \right)^2 + \left({}^L D_t^\alpha g(t) \right)^2 \right]^{\frac{1}{2}} d^{\alpha} t \end{aligned} \tag{32}$$

The Fractional Tangent Space

Let $\mathbf{r} = \mathbf{r}(s)$ be a natural representation of a curve C , where s is the α -fractional length of the curve. Since the velocity of a moving material point on the curve $r(s)$ defines the tangent space, the fractional tangent space of the curve $\mathbf{r} = \mathbf{r}(s)$ is defined by the first derivative:

$$\mathbf{r}_1 = \frac{d^a \mathbf{r}}{d^a s} = \frac{{}_a D_s^a \mathbf{r}}{{}_a D_s^a s} = {}^L D_s^a \mathbf{r} \tag{33}$$

Recalling

$$d^a |\mathbf{r}| = d^a s \tag{34}$$

the length $|\mathbf{r}_1|$ of the fractional tangent vector is unity.

The tangent space line of the curve $\mathbf{r} = \mathbf{r}(s)$ at the point $\mathbf{r}_0 = \mathbf{r}(s_0)$ is defined by:

$$\mathbf{r} = \mathbf{r}_0 + k \mathbf{t}_0 \quad 0 < k < \infty \tag{35}$$

where $\mathbf{t}_0 = \mathbf{t}(s_0)$ is the unit tangent vector at \mathbf{r} .

The plane through \mathbf{r}_0 , orthogonal to the tangent line at \mathbf{r}_0 , is called the normal plane to the curve C at s_0 . The points \mathbf{y} of that orthogonal plane are defined by:

$$(\mathbf{y} - \mathbf{r}_0) \cdot \mathbf{t}(s_0) = (\mathbf{y} - \mathbf{r}_0) \cdot \mathbf{r}_1(s_0) = 0 \tag{36}$$

Fractional Curvature of Curves

Considering the fractional tangent vector:

$$\mathbf{t} = \mathbf{r}_1(s) = \frac{{}_a D_s^a \mathbf{r}}{{}_a D_s^a s} = {}^L D_s^a \mathbf{r} \quad (37)$$

its fractional derivative may be considered:

$$\mathbf{r}_2(s) = \frac{d^a \mathbf{t}}{d^a s} = \frac{{}_a D_s^a \mathbf{t}}{{}_a D_s^a s} = {}^L D_s^a \mathbf{t} = \mathbf{t}_1(s) \quad (38)$$

The vector $\mathbf{t}_1(s)$ is called the fractional curvature vector on C at the point $\mathbf{r}(s)$ and is denoted by $\kappa = \kappa(s) = \mathbf{t}_1(s)$.

Since \mathbf{t} is a unit vector

$$\mathbf{t} \cdot \mathbf{t} = 1 \quad (39)$$

Restricted to fractional derivatives that yield zero for a constant function, such as Caputo's derivatives, the curvature vector $\mathbf{t}_1(s)$ on C is orthogonal to \mathbf{t} and parallel to the normal plane. The magnitude of the fractional curvature vector:

$$\kappa = |\kappa(s)| \quad (40)$$

is called the fractional curvature of C at $\mathbf{r}(s)$. The reciprocal of the curvature κ is the fractional radius of curvature at $\mathbf{r}(s)$:

$$\rho = \frac{1}{\kappa} = \frac{1}{|\kappa(s)|} \quad (41)$$

The Fractional Radius of Curvature of a Curve

Following Porteous (1994), for the fractional curvature of a plane curve \mathbf{r} , we study at each point $\mathbf{r}(t)$ of the curve, how closely the curve approximates there to a parameterized circle. Now in the tangent or first differential space at a point $\mathbf{r}(t_0)$, the circle, with center \mathbf{c} and radius ρ , consists of all $\mathbf{r}(t)$ in the differential space such that:

$$(\mathbf{r} - \mathbf{c}) \cdot (\mathbf{r} - \mathbf{c}) = \rho^2 \quad (42)$$

Further Eq. 42 yields:

$$\mathbf{c} \cdot \mathbf{r} - \frac{1}{2} \mathbf{r} \cdot \mathbf{r} = \frac{1}{2} (\mathbf{c} \cdot \mathbf{c} - \rho^2) \quad (43)$$

with the right hand side been constant.

Therefore the derivation of the function

$$\mathbf{V}(\mathbf{c}) : t \rightarrow \mathbf{c} \cdot \mathbf{r}(t) - \frac{1}{2} \mathbf{r}(t) \cdot \mathbf{r}(t) \quad (44)$$

Hence:

$$V(\mathbf{c})_1 = (\mathbf{c} - \mathbf{r}(t)) \cdot \mathbf{r}_1(t) = 0 \quad (45)$$

$$V(\mathbf{c})_2 = (\mathbf{c} - \mathbf{r}(t)) \cdot \mathbf{r}_2(t) - \mathbf{r}_1(t) \cdot \mathbf{r}_1(t) = 0 \quad (46)$$

Suppose that \mathbf{r} is a parametric curve with $\mathbf{r}(t)$ in the virtual tangent space. Then $V(\mathbf{c})_1(t) = 0$ when the vector $\mathbf{c} - \mathbf{r}(t)$ in the tangent space is orthogonal to the tangent vector $\mathbf{r}_1(t)$. Indeed when the point \mathbf{c} , in the tangent space, lies on the normal to $\mathbf{r}_1(t)$ at t , the line through $\mathbf{r}(t)$ is orthogonal to the tangent line.

When $\mathbf{r}_2(t)$ is not linearly dependent upon $\mathbf{r}_1(t)$, there will be a unique point $\mathbf{c} \neq \mathbf{r}(t)$, on the normal line, such that also $V(\mathbf{c})_2(t) = 0$.

The Serret-Frenet Equations

Let \mathbf{r} be a curve with unit speed, where the fractional velocity vector, (Porteous 1994),

$$\mathbf{t}(s) = \mathbf{r}_1(s) = \frac{{}_a^c D_s^a \mathbf{r}(s)}{{}_a^c D_s^a s} = {}_a^L D_s^a \mathbf{r}(s) \quad (47)$$

is of unit length.

Let $\mathbf{r}(s)$ be such a curve. The vector

$$\mathbf{t}_1(s) = \mathbf{r}_2(s) = \frac{{}_a^c D_s^a \mathbf{r}_1(s)}{{}_a^c D_s^a s} = \frac{{}_a^c D_s^a}{{}_a^c D_s^a s} \left(\frac{{}_a^c D_s^a \mathbf{r}(s)}{{}_a^c D_s^a s} \right) = {}_a^L D_s^a \left({}_a^L D_s^a \mathbf{r}(s) \right) \quad (48)$$

is normal to the curve $\mathbf{r} = \mathbf{r}(s)$ since $\mathbf{t}(s) \cdot \mathbf{r}(s) = 1$ and

$$\mathbf{t}_1(s) \cdot \mathbf{t}(s) = 0 \quad (49)$$

since for Caputo's derivative ${}_a^c D_s^a c = 0$ for any constant c .

Consider $\mathbf{t}_1(s) \kappa(s) \mathbf{n}(s)$, where $\mathbf{n}(s)$ is the unit principal normal to \mathbf{r} at s , provided that $\kappa(s) \neq 0$ where $\kappa(s)$ is the curvature of \mathbf{r} at s .

Hence the equations for the focal line are defined by

$$\begin{aligned}(\mathbf{c} - \mathbf{r}(s)) \cdot \mathbf{r}_1(s) &= 0 \\(\mathbf{c} - \mathbf{r}(s)) \cdot \kappa(s)\mathbf{n}(s) &= 1\end{aligned}\tag{50}$$

Thus, the principal center of curvature \mathbf{c} at s is the point $\mathbf{r}(s) + \rho(s)\mathbf{n}(s)$, where $\rho(s) = \frac{1}{\kappa(s)}$. Furthermore, the principal normal vector $\mathbf{n}(s)$ orthogonal to the tangent line is pointing towards the focal line (locus of the curvature centers). Likewise, the (unit) binormal $\mathbf{b}(s)$ is defined to be the vector $\mathbf{t}(s) \times \mathbf{n}(s)$, the triad of unit vectors $\mathbf{t}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$ forming a right-handed orthonormal basis for the tangent vector space to the curvature $\mathbf{r}(s)$. Each of the derivative vectors $\mathbf{t}_1(s)$, $\mathbf{n}_1(s)$, $\mathbf{b}_1(s)$ linearly depends on $\mathbf{t}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$. Considering the equations: $\mathbf{t}_1 \mathbf{t} = 0$ and $\mathbf{t}_1 \mathbf{n} = 0$ with $\mathbf{t}_1 \mathbf{n} + \mathbf{n}_1 \cdot \mathbf{t} = 0$, we get the fractional Serret-Frenet equations:

$$\begin{aligned}\mathbf{t}_1 &= \kappa \mathbf{n} \\ \mathbf{n}_1 &= -\kappa \mathbf{t} + \tau \mathbf{b} \\ \mathbf{b}_1 &= -\tau \mathbf{n}\end{aligned}\tag{51}$$

The coefficient τ is defined to be the torsion of the curve \mathbf{r} . These equations are the fractional equations for the fractional Serret-Frenet system. Considering plane curves,

$$\mathbf{r}(x) = x\mathbf{i} + y(x)\mathbf{j}\tag{52}$$

Equations 45 and 46 defining the fractional centers of curvature $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$ become,

$$\begin{aligned}(c_1 - x) + (c_2 - y(x)) {}^L D_x^\alpha y(x) &= 0 \\ (c_2 - y(x)) {}^L D_x^\alpha ({}^L D_x^\alpha y(x)) - \left(1 + {}^L D_x^\alpha y(x)^2\right) &= 0\end{aligned}\tag{53}$$

Since the fractional radius of curvature is defined by

$$\boldsymbol{\rho}^\alpha = \rho_1^\alpha \mathbf{i} + \rho_2^\alpha \mathbf{j} = (c_1 - x) \mathbf{i} + (c_2 - y(x)) \mathbf{j}\tag{54}$$

the components of the fractional curvature are given by

$$\begin{aligned}\rho_1^\alpha &= -\frac{1 + {}^L D_s^\alpha y(x)^2}{{}^L D_s^\alpha ({}^L D_s^\alpha y(x))} {}^L D_s^\alpha y(x) \\ \rho_2^\alpha &= \frac{1 + {}^L D_s^\alpha y(x)^2}{{}^L D_s^\alpha ({}^L D_s^\alpha y(x))}\end{aligned}\tag{55}$$

Further, for the case of $|y(x)| \ll 1$, that we consider in linear bending, Eq. 28 yields,

$$d^a s = d^a x + o(d^a x)^2 \tag{56}$$

with

$${}^L D_s^\alpha () = {}^L D_x^\alpha () \tag{57}$$

and

$$\rho^\alpha = |\mathbf{r}^\alpha| \approx \frac{1}{{}^L D_x^\alpha ({}^L D_x^\alpha y(x))} \tag{58}$$

Let us consider a fractional beam with its source point $(x,y,z) = (0,0,0)$. That means, the fractional u Caputo's derivatives of any function, concerning the beam, are defined by

$${}_0^c D_u^a f(u) = \frac{1}{\Gamma(1-\alpha)} \int_0^u \frac{f'(s)}{(u-s)^\alpha} ds$$

where, u might be one of the variables (x, y, z) .

Applications

The Fractional Geometry of a Parabola

Let \mathbf{r} be a parabola $t \rightarrow (t, t^2)$. Then we have

$$\mathbf{r}(t) = t\mathbf{e}_1 + t^2\mathbf{e}_2 \tag{59}$$

Hence:

$$\mathbf{r}_1(t) = \mathbf{e}_1 + \frac{{}^c D_t^a (t^2)}{{}^c D_t^a t} \mathbf{e}_2 = \mathbf{e}_1 + \frac{2t}{2-a} \mathbf{e}_2 \tag{60}$$

and

$$\mathbf{r}_2(t) = \frac{2}{2-a} \mathbf{e}_2$$

Then the centers of curvature of the parabola describe a curve:

$$\mathbf{c}(t) = c_1(t)\mathbf{e}_1 + c_2(t)\mathbf{e}_2 \tag{61}$$

Satisfying Eqs. 45 and 46 with

$$c_1 + \frac{2t}{2-a}c_2 = t + \frac{2t^3}{2-a} \tag{62}$$

$$\frac{2}{2-a}c_2 = \frac{2t^2}{2-a} + 1 + \frac{4t^2}{(2-a)^2} \tag{63}$$

Solving the system of Eqs. 62 and 63 we get:

$$c_1 = -\frac{4t^3}{(-2+a)^2} \tag{64}$$

$$c_2 = -\frac{4+a^2+8t^2-2a(2+t^2)}{(4-2a)} \tag{65}$$

Figure 3 shows the tangent space of the parabola at the point $t = 1.5$ for various values of the fractional dimension $\alpha = (1, 0.7, 0.3)$.

It is clear that the tangent spaces for $\alpha = 0.7$ and $\alpha = 0.3$ intersect the parabola at the point $t = 1.5$, although the conventional tangent space with fractional dimension $\alpha = 1.0$ touches the parabola at $t = 1.5$.

Furthermore the centers of curvature for various values of the fractional dimension α the point $t = 1,5$ are (for the conventional case):

$\alpha = 1.0$	$c_1 = -13.5$	and $c_2 = 7.25$
$\alpha = 0.7$	$c_1 = -7.98$	and $c_2 = 6.36$
$\alpha = 0.3$	$c_1 = -4.67$	and $c_2 = 5.75$

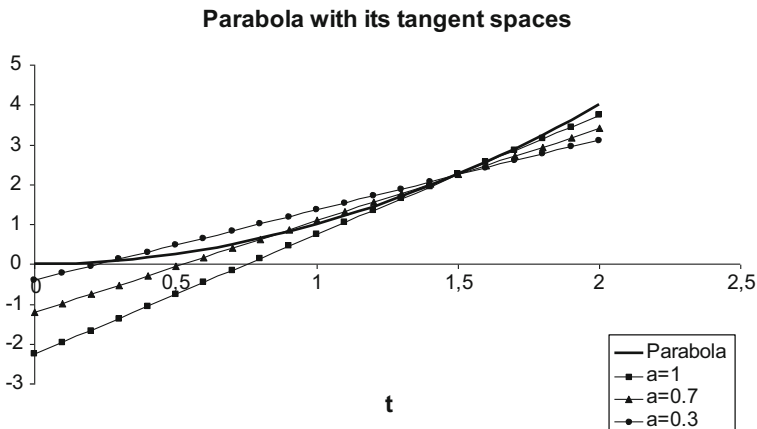


Fig. 3 The parabola with its tangent spaces at $t = 1.5$ for $\alpha = 1, \alpha = 0.7, \alpha = 0.3$

The Tangent and Curvature Center of the Weierstrass Function

Let us consider the function,

$$W(t) = \sum_{n=1}^{\infty} \lambda^{-\alpha n} \left\{ \sin\left(\frac{\lambda^n t}{2}\right) - \sin(2\lambda^n t) \right\} \tag{66}$$

the well-known Weierstrass function, continuous with discontinuous conventional derivatives at any point, (Liang and Su 2007). The parameter α has been proved to be related to the fractional dimension of the function $W(t)$. Restricting the function to $w(t)$ with

$$w(t) = \sum_{n=1}^6 \lambda^{-\alpha n} \left\{ \sin\left(\frac{\lambda^n t}{2}\right) - \sin(2\lambda^n t) \right\} \tag{67}$$

and for $\alpha = 0.5$ and $\lambda = 2$, the fractional tangent to the curve at the point $t = 1.0$ has been drawn, (Fig. 4), with the help of the Mathematica computerized pack.

Bending of Fractional Beams

Considering the pure fractional bending problem of a beam with microcracks, microvoids, various other defects, we get the fractional strain:

$$\varepsilon_{xx}^{\alpha} = -\frac{y}{\rho^{\alpha}} \tag{68}$$

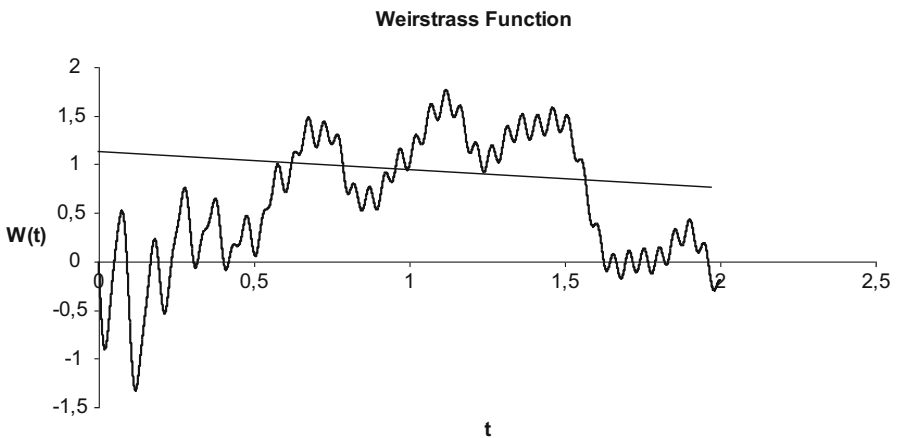


Fig. 4 The function $w(t)$ with its fractional ($\alpha = 0.5$) tangent at $t = 1.0$

where the fractional curvature is defined by,

$$\frac{1}{\rho^\alpha} = D_2 w(x) = \frac{1}{{}_a D_x^\alpha} {}_a D_x^\alpha \left(\frac{{}_a D_x^\alpha w(x)}{{}_a D_x^\alpha x} \right), \tag{69}$$

with $w(x)$ denoting the elastic line of the beam.

Likewise, the fractional bending moment is expressed by:

$$M = -2 \int_0^{h/2} \sigma_{xx}^\alpha y \, d^\alpha y = \frac{EI^\alpha}{\rho^\alpha} \tag{70}$$

with the fractional stress, (see Lazopoulos et al. 2015),

$$\sigma_{xx}^\alpha = -\frac{M}{I^\alpha} y \tag{71}$$

Hence the fractional bending of beams formula is revisited and expressed by:

$$M = EI^\alpha D_2 w(x) = \frac{EI^\alpha}{{}_0^c D_x^\alpha} D_x^\alpha \left(\frac{{}_0^c D_x^\alpha w(x)}{{}_0^c D_x^\alpha x} \right) \tag{72}$$

Therefore, the deflection curve $w(x)$ is defined by:

$$w(x) = \int_0^x \left(\int_0^s \frac{M(t)}{EI^\alpha} d^\alpha t \right) d^\alpha s + c_1 x + c_2 \tag{73}$$

In conventional integration, the deflection curve is defined by:

$$w(x) = \int_0^x \frac{s^{1-\alpha}}{\Gamma(2-\alpha)} \left(\int_0^s \frac{M(t)}{EI^\alpha} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{1}{\Gamma(\alpha)(s-t)^{1-\alpha}} dt \right) \frac{ds}{(x-s)^{1-\alpha}} + c_1 x + c_2 \tag{74}$$

The Fractional Tangent Plane of a Surface

Let us consider a manifold, with points $M(u,v)$, defined by the vectors

$$M(u, v) = \mathbf{x}(u, v) \tag{75}$$

with,

$$x_i = x_i(u, v), u_1 \leq u \leq u_2, v_1 \leq v \leq v_2, i = 1, 2, 3 \tag{76}$$

The infinitesimal distance between two points P and Q on the manifold M is defined by,

$$d^\alpha \mathbf{x} = \frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} d^\alpha u + \frac{{}^c D_v^\alpha \mathbf{x}}{{}^c D_v^\alpha v} d^\alpha v \tag{77}$$

In fact for the surface

$$z = u^2 v^2 \tag{78}$$

see, Fig. 2, the tangent space according to Eq. 77 is expressed by:

$$d^\alpha \mathbf{r} = d^\alpha x \mathbf{i} + d^\alpha y \mathbf{j} + \frac{2xy}{(2-\alpha)} (y d^\alpha x + x d^\alpha y) \mathbf{k} \tag{79}$$

Figures 5 and 6 shows the surface defined by Eq. 78 with its fractional tangent

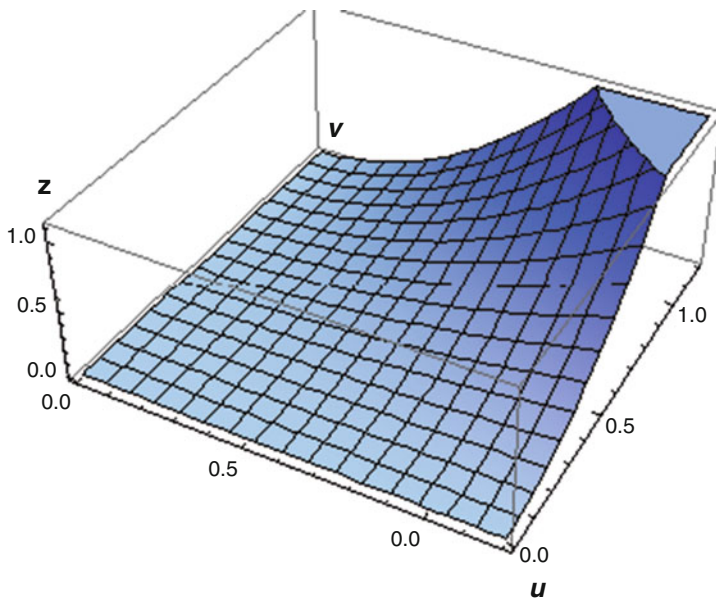


Fig. 5 The surface $z = u^2 v^2$ plane (space) at the point $(u,v) = (0.5, 0.5)$ for two fractional dimensions, $\alpha = 1$ (the conventional case) and $\alpha = 0.3$. It is clear that the fractional tangent plane is different from the conventional one ($\alpha = 1$).

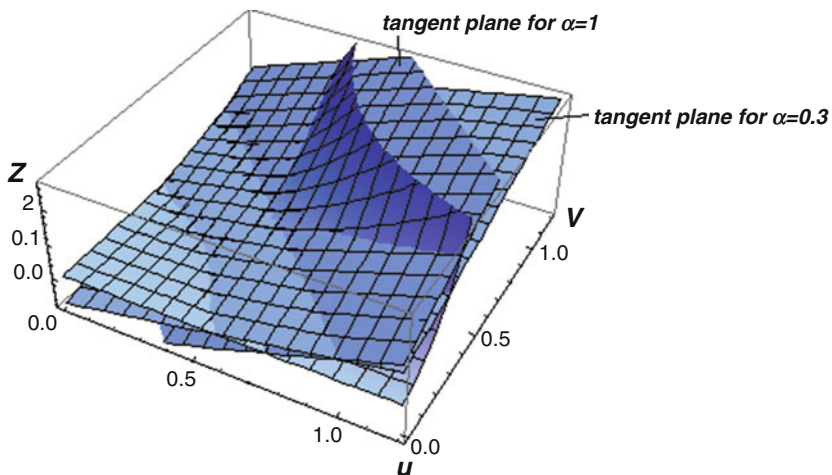


Fig. 6 The tangent planes for various values of the fractional dimension α

Fundamental Differential Forms on Fractional Differential Manifolds

The First Fractional Fundamental Form

Following formal procedure (Guggenheimer 1977), the quantity

$$\begin{aligned}
 I^\alpha &= d^\alpha \mathbf{x} \cdot d^\alpha \mathbf{x} = \left(\frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} d^\alpha u + \frac{{}^c D_v^\alpha \mathbf{x}}{{}^c D_v^\alpha v} d^\alpha v \right) \cdot \left(\frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} d^\alpha u + \frac{{}^c D_v^\alpha \mathbf{x}}{{}^c D_v^\alpha v} d^\alpha v \right) \\
 &= E d^\alpha u^2 + 2F d^\alpha u d^\alpha v + G d^\alpha v^2
 \end{aligned} \tag{80}$$

defined upon the tangent space of the manifold, as it has been clarified earlier, the I^α stands for the first fractional differential form, with the dot meaning the inner product.

$$\begin{aligned}
 E &= \frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} \cdot \frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} \\
 F &= \frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} \cdot \frac{{}^c D_v^\alpha \mathbf{x}}{{}^c D_v^\alpha v} \\
 G &= \frac{{}^c D_v^\alpha \mathbf{x}}{{}^c D_v^\alpha v} \cdot \frac{{}^c D_v^\alpha \mathbf{x}}{{}^c D_v^\alpha v}
 \end{aligned} \tag{81}$$

corresponding to

$$I^\alpha = E du^2 + 2F du dv + G dv^2$$

Furthermore the first fundamental form is positive definite, i.e., $0 \leq I^\alpha$ with $I^\alpha = 0$ if and only if $d^\alpha u$ and $d^\alpha v$ are equal to zero. Hence,

$$EG - F^2 > 0.$$

The Second Fractional Fundamental Form

Consider the manifold $M(u, v) = \mathbf{x}(u, v)$. Then, at each point of the manifold, there is a fractional unit normal \mathbf{N} to the fractional tangent plane

$$\mathbf{N} = \frac{\frac{{}_a D_u^\alpha \mathbf{x}}{{}_a D_u^\alpha u} \times \frac{{}_a D_v^\alpha \mathbf{x}}{{}_a D_v^\alpha v}}{\left| \frac{{}_a D_u^\alpha \mathbf{x}}{{}_a D_u^\alpha u} \times \frac{{}_a D_v^\alpha \mathbf{x}}{{}_a D_v^\alpha v} \right|}} \quad (82)$$

that is a function of u and v with the fractional differential

$$d^\alpha \mathbf{N} = \frac{{}_a D_u^\alpha \mathbf{N}}{{}_a D_u^\alpha u} d^\alpha u + \frac{{}_a D_v^\alpha \mathbf{N}}{{}_a D_v^\alpha v} d^\alpha v \quad (83)$$

Restricting only to Caputo's fractional derivatives with the property of zero fractional derivative of any constant, and taking into consideration that $\mathbf{N} \cdot \mathbf{N} = 1$, we get,

$$d^\alpha \mathbf{N} \cdot \mathbf{N} = 0 \quad (84)$$

where the vector $d^\alpha \mathbf{N}$ is parallel to the fractional tangent space.

The second fractional fundamental form is defined by Guggenheimer (1977),

$$\begin{aligned} \Pi^\alpha &= -d^\alpha \mathbf{x} \cdot d^\alpha \mathbf{N} = - \left(\frac{{}_a D_u^\alpha \mathbf{x}}{{}_a D_u^\alpha u} d^\alpha u + \frac{{}_a D_v^\alpha \mathbf{x}}{{}_a D_v^\alpha v} d^\alpha v \right) \cdot \left(\frac{{}_a D_u^\alpha \mathbf{N}}{{}_a D_u^\alpha u} d^\alpha u + \frac{{}_a D_v^\alpha \mathbf{N}}{{}_a D_v^\alpha v} d^\alpha v \right) \\ &= L d^\alpha u^2 + 2M d^\alpha u d^\alpha v + N d^\alpha v^2 \end{aligned} \quad (85)$$

with,

$$\begin{aligned} L &= - \frac{{}_a D_u^\alpha \mathbf{x}}{{}_a D_u^\alpha u} \cdot \frac{{}_a D_u^\alpha \mathbf{N}}{{}_a D_u^\alpha u} \\ M &= - \frac{1}{2} \left(\frac{{}_a D_u^\alpha \mathbf{x}}{{}_a D_u^\alpha u} \cdot \frac{{}_a D_v^\alpha \mathbf{N}}{{}_a D_v^\alpha v} + \frac{{}_a D_u^\alpha \mathbf{N}}{{}_a D_u^\alpha u} \cdot \frac{{}_a D_v^\alpha \mathbf{x}}{{}_a D_v^\alpha v} \right) \\ N &= - \frac{{}_a D_v^\alpha \mathbf{x}}{{}_a D_v^\alpha v} \cdot \frac{{}_a D_v^\alpha \mathbf{N}}{{}_a D_v^\alpha v} \end{aligned} \quad (86)$$

It is pointed again that the geometric procedures, that use quantities not defined upon the correct tangent spaces, are questionable. Even if analytically may yield the same results, geometrically are confusing.

The Fractional Normal Curvature

Let P be a point on a surface $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x}(t) = \mathbf{x}(u(t), v(t))$ a regular curve C at P . The fractional curvature of curves has been discussed in ► Chap. 6, “Uniqueness of Elastoplastic Properties Measured by Instrumented Indentation”. The normal curvature \mathbf{k}_n^α vector of C at P is the vector projection of the curvature vector \mathbf{k}^α onto the normal vector \mathbf{N} at P . The component of \mathbf{k}^α in the direction of the normal \mathbf{N} is called the normal fractional curvature of C at P and is denoted by k_n^α . Therefore,

$$k_n^\alpha = \mathbf{k}^\alpha \cdot \mathbf{N} \quad (87)$$

Since the unit tangent to C at P is the vector

$$\mathbf{t} = \frac{d^\alpha \mathbf{x}}{d^\alpha s} = \frac{d^\alpha \mathbf{x}}{d^\alpha t} / \left| \frac{d^\alpha \mathbf{x}}{d^\alpha t} \right| \quad (88)$$

where s denotes the fractional arc length of the curve and \mathbf{t} is the unit perpendicular to the normal \mathbf{N} along the curve, we get:

$$0 = \frac{d^\alpha (\mathbf{t} \cdot \mathbf{N})}{d^\alpha t} = \frac{d^\alpha \mathbf{t}}{d^\alpha t} \cdot \mathbf{N} + \mathbf{t} \cdot \frac{d^\alpha \mathbf{N}}{d^\alpha t} \quad (89)$$

Therefore, the normal curvature of a curve is equal to:

$$\begin{aligned} k_n^\alpha &= \mathbf{k} \cdot \mathbf{N} = \frac{d^\alpha \mathbf{t}}{d^\alpha t} \cdot \mathbf{N} / \left| \frac{d^\alpha \mathbf{x}}{d^\alpha t} \right| = -\mathbf{t} \cdot \frac{d^\alpha \mathbf{N}}{d^\alpha t} / \left| \frac{d^\alpha \mathbf{x}}{d^\alpha t} \right| \\ &= -\frac{d^\alpha \mathbf{x}}{d^\alpha t} \cdot \frac{d^\alpha \mathbf{N}}{d^\alpha t} / \left| \frac{d^\alpha \mathbf{x}}{d^\alpha t} \right|^2 \\ &= -\frac{\left(\frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} \frac{d^\alpha u}{d^\alpha t} + \frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha v} \frac{d^\alpha v}{d^\alpha t} \right) \cdot \left(\frac{{}^c D_u^\alpha \mathbf{N}}{{}^c D_u^\alpha u} \frac{d^\alpha u}{d^\alpha t} + \frac{{}^c D_u^\alpha \mathbf{N}}{{}^c D_u^\alpha v} \frac{d^\alpha v}{d^\alpha t} \right)}{\left(\frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} \frac{d^\alpha u}{d^\alpha t} + \frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha v} \frac{d^\alpha v}{d^\alpha t} \right) \cdot \left(\frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha u} \frac{d^\alpha u}{d^\alpha t} + \frac{{}^c D_u^\alpha \mathbf{x}}{{}^c D_u^\alpha v} \frac{d^\alpha v}{d^\alpha t} \right)} \\ &= \frac{L(d^\alpha u/d^\alpha t)^2 + 2M(d^\alpha u/d^\alpha t)(d^\alpha v/d^\alpha t) + N(d^\alpha v/d^\alpha t)^2}{E(d^\alpha u/d^\alpha t)^2 + 2F(d^\alpha u/d^\alpha t)(d^\alpha v/d^\alpha t) + G(d^\alpha v/d^\alpha t)^2} \quad (90) \end{aligned}$$

Recalling Eqs. 87 and 90, the normal curvature is defined by:

$$k_n^\alpha = \frac{\Pi^\alpha}{I^\alpha} \tag{91}$$

Fractional Vector Operators

In the present section the fractional tangent spaces along with their fractional normal vectors should be reminded, as they were defined in the preceding sections “The Geometry of Fractional Differential,” “The Fractional Arc Length,” and “The Fractional Tangent Space.”

For Cartesian coordinates, fractional generalizations of the divergence or gradient operators are defined by:

$$\nabla^{(a)} f(\mathbf{x}) = \text{grad}^{(a)} f(\mathbf{x}) = \nabla_i^{(a)} f(x) \mathbf{e}_i = \frac{{}_\omega^c D_i^a f(\mathbf{x})}{{}_\omega^c D_i^a x_i} + \mathbf{e}_i = {}_\omega^L D_i^a f(x) \mathbf{e}_i \tag{92}$$

where ${}_\omega^c D_i^a$ are Caputo’s fractional derivatives of order α and the subline meaning no contraction. Further, ${}_\omega^L D_i^a f(\mathbf{x})$ is Leibniz’s derivative, Eq. 18. Hence, the gradient of the vector \mathbf{x} is

$$\nabla^{(a)} \mathbf{x} = \mathbf{I} \tag{93}$$

with \mathbf{I} denoting the identity matrix. Consequently for a vector field

$$\mathbf{F}(x_1, x_2, x_3) = \mathbf{e}_1 F_1(x_1, x_2, x_3) + \mathbf{e}_2 F_2(x_1, x_2, x_3) + \mathbf{e}_3 F_3(x_1, x_2, x_3) \tag{94}$$

where $F_i(x_1, x_2, x_3)$ are absolutely integrable, the circulation is defined by:

$$C_L^{(a)}(\mathbf{F}) = \left({}_\omega I_L^{(a)}, \mathbf{F} \right) = \int_L (d\mathbf{L}, \mathbf{F}) = {}_\omega I^{(a)}{}_L (F_1 d^\alpha x_1) + {}_\omega I^{(a)}{}_L (F_2 d^\alpha x_2) + {}_\omega I^{(a)}{}_L (F_3 d^\alpha x_3) \tag{95}$$

Furthermore, the divergence of a vector $\mathbf{F}(\mathbf{x})$ is defined by:

$$\nabla^{(a)} \cdot \mathbf{F}(x) = \text{div}^{(a)} \mathbf{F}(x) = \frac{{}_\omega^c D_k^a F_k(\mathbf{x})}{{}_\omega^c D_k^a x_k} = {}_\omega^L D_k^a F_k(\mathbf{x}) \tag{96}$$

where the subline denotes no contraction.

Moreover, the fractional curl \mathbf{F} ($\text{curl}^{(a)}\mathbf{F}(\mathbf{x})$) of a vector \mathbf{F} is defined by:

$$\text{curl}^{(a)}\mathbf{F} = \mathbf{e}_l \varepsilon_{lmn} \frac{\omega^c D_m^a F_n}{\omega^c D_m^a x_m} = \mathbf{e}_l \varepsilon_{lmn\omega} {}^L D_m^a F_n \quad (97)$$

A fractional flux of the vector \mathbf{F} expressed in Cartesian coordinates across surface S is a fractional surface integral of the field with:

$$\Phi_s^\alpha(\mathbf{F}) = ({}_\omega I_s^\alpha, \mathbf{F}) = \iint_{\omega S}^{(\alpha)} (F_1 d^\alpha x_2 d^\alpha x_3 + F_2 d^\alpha x_3 d^\alpha x_1 + F_3 d^\alpha x_2 d^\alpha x_3) \quad (98)$$

A fractional volume integral of a triple fractional integral of a scalar field $f = f(x_1, x_2, x_3)$ is defined by:

$${}_\omega V_\Omega^{(a)}[f] = {}_\omega I_\Omega^{(a)}[x_1, x_2, x_3] f(x_1, x_2, x_3) = \iiint_{\omega \Omega}^{(\alpha)} f(x_1, x_2, x_3) d^\alpha x_1 d^\alpha x_2 d^\alpha x_3 \quad (99)$$

It should be pointed out that the triple fractional integral is not a volume integral, since the fractional derivative of a variable with respect to itself is different from one. So there is a clear distinction between the simple, double, or triple integrals and the line, surface, and volume integrals respectively.

Fractional Vector Field Theorems

Fractional Green's Formula

Green's theorem relates a line integral around a simple closed curve ∂B and a double integral over the plane region B with boundary ∂B . With positively oriented boundary ∂B , the conventional Greens theorem for a vector field $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2$ is expressed by:

$$\int_{\partial B} (F_1 dx_1 + F_2 dx_2) = \int \int_B \left(\frac{\partial (F_1)}{\partial x_2} - \frac{\partial (F_2)}{\partial x_1} \right) dx_1 dx_2 \quad (100)$$

Recalling that:

$$d^a \mathbf{x} = (d^a x_1, d^a x_2) = ({}_\omega^c D_{x_1}^a [x_1] d^\alpha x_1, {}_\omega^c D_{x_2}^a [x_2] d^\alpha x_2) \quad (100a)$$

and substituting into conventional Green's theorem Eq. 100 we get:

$$\int_{\omega \partial W}^{(\alpha)} (F_1 d^\alpha x_1 + F_2 d^\alpha x_2) = \int_{\omega}^{(\alpha)} \int_W \left(\frac{c}{\omega} D_{x_2}^a (F_1) - \frac{c}{\omega} D_{x_1}^a (F_2) \right) d^\alpha x_1 d^\alpha x_2 \quad (101)$$

Fractional Stoke's Formula

Restricting in the consideration of a simple surface W , if we denote its boundary by ∂W and if \mathbf{F} is a vector field defined on W , then the conventional Stokes' Theorem asserts that:

$$\oint_W \mathbf{F} \cdot d\mathbf{L} = \iint_W \text{curl } \mathbf{F} \cdot d\mathbf{S} \quad (102)$$

In Cartesian coordinates it yields:

$$\begin{aligned} \int_{\partial W} (F_1 dx_1 + F_2 dx_2 + F_3 dx_3) &= \iint_W \left(\frac{\partial(F_3)}{\partial x_2} - \frac{\partial(F_2)}{\partial x_3} \right) dx_2 dx_3 \\ &+ \left(\frac{\partial(F_1)}{\partial x_3} - \frac{\partial(F_3)}{\partial x_1} \right) dx_3 dx_1 + \left(\frac{\partial(F_2)}{\partial x_1} - \frac{\partial(F_1)}{\partial x_2} \right) dx_1 dx_2 \end{aligned} \quad (103)$$

where $\mathbf{F}(x_1, x_2, x_3) = \mathbf{e}_1 F_1(x_1, x_2, x_3) + \mathbf{e}_2 F_2(x_1, x_2, x_3) + \mathbf{e}_3 F_3(x_1, x_2, x_3)$.

In this case, the fractional curl operation is defined by:

$$\begin{aligned} \text{curl}_W^\alpha (\mathbf{F}) &= e_l \varepsilon_{lmn} \frac{\alpha}{\omega} D_{x_m}^a (F_n) / \frac{c}{\omega} D_{x_k}^a (x_i) \delta_{mk} = \mathbf{e}_1 \left(\frac{\alpha}{\omega} D_{x_2}^a F_3 - \frac{\alpha}{\omega} D_{x_3}^a F_2 \right) \\ &+ \mathbf{e}_2 \left(\frac{\alpha}{\omega} D_{x_3}^a F_1 - \frac{\alpha}{\omega} D_{x_1}^a F_3 \right) + \mathbf{e}_3 \left(\frac{\alpha}{\omega} D_{x_1}^a F_2 - \frac{\alpha}{\omega} D_{x_2}^a F_1 \right) \end{aligned} \quad (104)$$

Therefore transforming the conventional Stokes' theorem into the fractional form we get:

$$\begin{aligned} \int_{\omega}^{(\alpha)} \int_W (F_1 d^\alpha x_1 + F_2 d^\alpha x_2 + F_3 d^\alpha x_3) &= \int_{\omega}^{(\alpha)} \iint_W \left\{ \left(\frac{\alpha}{\omega} D_{x_2}^a F_3 - \frac{\alpha}{\omega} D_{x_3}^a F_2 \right) d^\alpha x_2 d^\alpha x_3 \right. \\ &+ \left(\frac{\alpha}{\omega} D_{x_3}^a F_1 - \frac{\alpha}{\omega} D_{x_1}^a F_3 \right) d^\alpha x_3 d^\alpha x_1 \\ &+ \left. \left(\frac{\alpha}{\omega} D_{x_1}^a F_2 - \frac{\alpha}{\omega} D_{x_2}^a F_1 \right) d^\alpha x_1 d^\alpha x_2 \right\} \end{aligned} \quad (105)$$

Fractional Gauss' Formula

For the conventional fields theory, let $\mathbf{F} = \mathbf{e}_1 F_1 + \mathbf{e}_2 F_2 + \mathbf{e}_3 F_3$, be a continuously differentiable real-valued function in a domain W with boundary ∂W . Then the conventional divergence Gauss' theorem is expressed by:

$$\iint_{\partial W} \mathbf{F} \cdot \mathbf{dS} = \iiint_W \text{div} \mathbf{F} dV \quad (106)$$

Since

$$\mathbf{d}^{(\alpha)} \mathbf{S} = \mathbf{e}_1 d^\alpha x_2 d^\alpha x_3 + \mathbf{e}_2 d^\alpha x_3 d^\alpha x_1 + \mathbf{e}_3 d^\alpha x_1 d^\alpha x_2 \quad (107)$$

where $d^\alpha x_i$, $i = 1, 2, 3$ is expressed by Eq. 15,

$$d^{(\alpha)} V = d^\alpha x_1 d^\alpha x_2 d^\alpha x_3 \quad (108)$$

Furthermore, see Eq. 96,

$$\text{div}^{(\alpha)} \mathbf{F}(x) = \frac{{}^c D_k^\alpha F_k(\mathbf{x})}{{}^c D_k^\alpha x_i} \delta_{km} \quad (109)$$

The Fractional Gauss divergence theorem becomes:

$$\iint_{\partial W} \mathbf{F} \cdot d^{(\alpha)} \mathbf{S} = \iiint_W \text{div}^{(\alpha)} \mathbf{F} d^{(\alpha)} V \quad (110)$$

Remember that the differential $d^\alpha S = \mathbf{n}^\alpha d^\alpha S$, where \mathbf{n}^α is the unit normal of the fractional tangent space as it has been defined in section “[The Geometry of Fractional Differential](#).”

Fractional Deformation Geometry

Outlining the Fractional Deformation Geometry presented in Lazopoulos and Lazopoulos (2016a), we assume the description in the Euclidean space, considering the reference configuration B with the boundary ∂B of a body displaced to its current configuration b with the boundary ∂b (see Truesdell 1977). The points in the reference placement B , defining the material points, are described by \mathbf{X} , whereas the set of the displaced points \mathbf{y} describe the current configuration b with the boundary ∂b . The coordinate system in B is denoted by $\{X_A\}$, while the corresponding to the current configuration b is reflected to $\{y_i\}$. In the present description, both systems have the same axial directions with base vectors $\{\mathbf{e}_A\}, \{\mathbf{e}_i\}$ whether the

reference concerns the current or the initial (unstressed) configuration. The motion of a reference point \mathbf{X} is described by the function

$$\mathbf{y} = \Psi(\mathbf{X}, t) \tag{111}$$

the conventional gradient of the deformation is defined by:

$$\mathbf{F}(\Psi, t) = \frac{\partial \Psi(\mathbf{X}, t)}{\partial \mathbf{X}} \text{ or } F_{iA} = \frac{\partial \Psi_i}{\partial X_A} \tag{112}$$

with

$$\mathbf{F}^{(\alpha)} = F_{ij}^{(\alpha)} = \nabla_{\mathbf{X}}^{(\alpha)} \mathbf{y}_i = {}^c D_{X_j}^\alpha \mathbf{y}_i = \frac{{}^c D_{X_j}^\alpha (\mathbf{X} + \mathbf{u})}{{}^c D_{X_j}^\alpha \mathbf{X}} = \nabla_{\mathbf{X}}^{(\alpha)} \mathbf{X} + \nabla_{\mathbf{X}}^{(\alpha)} \mathbf{u} \tag{113}$$

Furthermore, the right Cauchy-Green fractional deformation tensor,

$$\mathbf{C}^{(\alpha)} = \mathbf{F}^{(\alpha)T} \mathbf{F}^{(\alpha)} \tag{114}$$

and

$$\mathbf{B}^{(\alpha)} = \mathbf{F}^{(\alpha)} \mathbf{F}^{(\alpha)T} \tag{115}$$

is the left Cauchy-Green fractional deformation tensor.

Likewise, the fractional nonlinear fractional Green-Lagrange strain tensor $\mathbf{E}^{(\alpha)}$ may be defined by:

$$E_N^{(\alpha)} = \mathbf{N} \cdot \overset{\alpha}{\mathbf{E}} \mathbf{N} \tag{116}$$

where \mathbf{N} is the unit vector of the considered fiber in the reference placement, with

$$\mathbf{E}^{(\alpha)} = \frac{1}{2} (\mathbf{C}^{(\alpha)} - \mathbf{I}) \tag{117}$$

Recalling that the current placement $\mathbf{y} = \mathbf{X} + \mathbf{u}$, where \mathbf{u} denotes the displacement vector, the fractional Green-Lagrange deformation tensor becomes:

$$\mathbf{E}^{(\alpha)} = \frac{1}{2} \left[\left(\nabla_{\mathbf{X}}^{(\alpha)} \mathbf{u} \right)^T + \nabla_{\mathbf{X}}^{(\alpha)} \mathbf{u} + \left(\nabla_{\mathbf{X}}^{(\alpha)} \mathbf{u} \right)^T \nabla_{\mathbf{X}}^{(\alpha)} \mathbf{u} \right] \tag{118}$$

Proceeding to define the fractional strain tensor referred to the current placement, i.e., Euler-Almansi strain tensor:

$$\mathbf{A}^{(\alpha)} = \frac{1}{2} \left(\mathbf{I} - \left(\mathbf{B}^{(\alpha)} \right)^{-1} \right) \tag{119}$$

with the strain

$$\varepsilon_{\mathbf{n}} = \mathbf{n}^T \cdot \mathbf{A}^{(\alpha)} \mathbf{n} \quad (120)$$

where \mathbf{n} is the unit vector along the deformed fiber, corresponding to the \mathbf{N} unit vector along the reference placement fiber. It is evident that in the conventional case with $\alpha = 1$ the fractional Euler Almansi strain tensor $\mathbf{A}^{(\alpha)}$ reduces to the conventional strain tensor \mathbf{A} . It should be pointed out that the fractional stretches $\lambda^{(\alpha)}$, adopting the right Cauchy-Green strain tensor $\mathbf{C}^{(\alpha)}$, are defined by:

$$\lambda^{(\alpha)} = \frac{d^{(\alpha)}s}{d^{(\alpha)}S} \quad (121)$$

as the ratio of the measures of the final infinitesimal length over the corresponding length of the fractional differential $d^{(\alpha)}\mathbf{X}$ vector with $d^{(\alpha)}S = (d^{(\alpha)}\mathbf{X} \cdot d^{(\alpha)}\mathbf{X})^{\frac{1}{2}}$. In fact the stretches $\lambda^{(\alpha)}$ are defined by:

$$\left(\lambda^{(\alpha)}\right)^2 = \mathbf{N}^T \cdot \mathbf{C}^{(\alpha)} \mathbf{N} \quad (122)$$

where $\mathbf{N} = N_A e_A$ is the unit vector directed along the material reference fiber.

Furthermore for

$$\frac{1}{\left(\lambda^{(\alpha)}\right)^2} = \left(\frac{d^{(\alpha)}S}{d^{(\alpha)}s}\right)^2 = \mathbf{n}^T \cdot \left(\mathbf{B}^{(\alpha)}\right)^{-1} \mathbf{n} \quad (123)$$

where \mathbf{n} the unit vector along the deformed fiber.

Polar Decomposition of the Deformation Gradient

It is well known that every nonsingular matrix may be decomposed into a product of an orthogonal and a symmetric positive tensor. Applying the property to the deformation gradient we get:

$$\mathbf{F}^{(\alpha)} = \mathbf{R}^{(\alpha)} \mathbf{U}^{(\alpha)} = \mathbf{V}^{(\alpha)} \mathbf{R}^{(\alpha)} \quad (124)$$

where \mathbf{R} is orthogonal $\mathbf{R}=\mathbf{R}^{-T}$ and \mathbf{U} and \mathbf{V} are symmetric positive ($\mathbf{U}=\mathbf{U}^T$ and $\mathbf{V}=\mathbf{V}^T$). Therefore:

$$\mathbf{C}^{(\alpha)} = \left(\mathbf{U}^{(\alpha)}\right)^2 \text{ and } \mathbf{B}^{(\alpha)} = \left(\mathbf{V}^{(\alpha)}\right)^2 \quad (125)$$

Moreover, the eigenvalues of $\mathbf{C}^{(\alpha)}$ and $\mathbf{V}^{(\alpha)}$ are the same, but the eigenvectors $\mathbf{u}^{(\alpha)}$ of $\mathbf{U}^{(\alpha)}$ and $\mathbf{v}^{(\alpha)}$ of $\mathbf{V}^{(\alpha)}$ are related by $\mathbf{v}^{(\alpha)} = \mathbf{R}^{(\alpha)} \mathbf{u}^{(\alpha)}$. In fact $\mathbf{v}^{(\alpha)}$ is directed

along a principal direction (eigenvector) of the strain tensor $\mathbf{V}^{(\alpha)}$ with $\mathbf{u}^{(\alpha)}$ been the eigenvector of $\mathbf{U}^{(\alpha)}$. In other words, principal directions refer to the vectors $\mathbf{u}^{(\alpha)} = u_A^{(\alpha)} e_A$ and $\mathbf{v}^{(\alpha)} = v_A^{(\alpha)} e_A$.

Deformation of Volume and Surface

Consider three noncoplanar line elements $d^\alpha \mathbf{X}^{(1)}$, $d^\alpha \mathbf{X}^{(2)}$, $d^\alpha \mathbf{X}^{(3)}$ at the point \mathbf{X} in B so that:

$$\mathbf{d}^\alpha \mathbf{y}^{(i)} = \mathbf{F}^{(\alpha)} d^\alpha \mathbf{X}^{(i)} \quad (126)$$

with $d^\alpha \mathbf{y}^i$ the corresponding fractional differential vectors in the current placement.

Further, the volume $d^\alpha V$ is derived by

$$d^\alpha V = d^\alpha \mathbf{X}^{(1)} \cdot (d^\alpha \mathbf{X}^{(2)} \wedge d^\alpha \mathbf{X}^{(3)}) \quad (127)$$

Alternatively

$$d^\alpha V = \det(d^\alpha \mathbf{X}^{(1)}, d^\alpha \mathbf{X}^{(2)}, d^\alpha \mathbf{X}^{(3)}) \quad (128)$$

in which $d^\alpha \mathbf{X}^{(i)}$ denotes a column vector ($i = 1, 2, 3$). The corresponding volume $d^\alpha v$ in the deformed configuration is

$$d^\alpha v = \det(d^\alpha \mathbf{y}^{(1)}, d^\alpha \mathbf{y}^{(2)}, d^\alpha \mathbf{y}^{(3)}) \quad (129)$$

and

$$d^\alpha v = \det(\mathbf{F}^{(\alpha)}) d^\alpha V \equiv J^{(\alpha)} d^\alpha V \quad (130)$$

where

$$J^{(\alpha)} = \det \mathbf{F}^{(\alpha)} \quad \text{and} \quad d^\alpha V = d^\alpha \mathbf{X}^{(1)} \cdot d^\alpha \mathbf{X}^{(2)} \wedge d^\alpha \mathbf{X}^{(3)} \quad (131)$$

Consider, further, an infinitesimal vector element of material surface dS in the neighborhood of the point \mathbf{X} in B with $d^\alpha \mathbf{S} = \mathbf{N}^\alpha d^\alpha S$ the fractional tangent surface vector corresponding to the normal fractional normal vector \mathbf{N}^α . Likewise, $d^\alpha \mathbf{X}$ is an arbitrary fiber cutting the edge $d^\alpha S$ such that $d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S} > 0$. The volume of the cylinder with base $d^\alpha \mathbf{S}$ and generators $d^\alpha \mathbf{X}$ has volume $d^\alpha V = d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S}$. If $d^\alpha \mathbf{x}$ and $d^\alpha \mathbf{s}$ are the deformed configurations of $d^\alpha \mathbf{X}$ and $d^\alpha \mathbf{S}$ respectively, with $d^\alpha \mathbf{s} = \mathbf{n}^\alpha d^\alpha s$, where \mathbf{n}^α is the normal vector to the deformed surface, the volume $d^\alpha V$ in the reference placement corresponds to the volume $d^\alpha v = d^\alpha \mathbf{x} \cdot d^\alpha \mathbf{s}$ in the current configuration so that:

$$d^\alpha v = d^\alpha \mathbf{y} \cdot d^\alpha \mathbf{s} = J^{(\alpha)} d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S} \quad (132)$$

Since $d^\alpha \mathbf{y} = \mathbf{F}^{(\alpha)} d^\alpha \mathbf{X}$, we obtain

$$\mathbf{F}^{(\alpha)T} d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{s} = J^{(\alpha)} d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S} \quad (133)$$

removing the arbitrary $d^\alpha \mathbf{X}$. Consequently,

$$d^\alpha \mathbf{s} = J^{(\alpha)} \left(\mathbf{F}^{(\alpha)} \right)^{-T} d^\alpha \mathbf{S} \quad (134)$$

and

$$\mathbf{n}^\alpha d^\alpha s = J^{(\alpha)} \left(\mathbf{F}^{(\alpha)} \right)^{-T} \mathbf{N}^\alpha d^\alpha S \quad (135)$$

The relation between the area elements corresponding to reference and current configurations is the well-known Nanson's formula for the fractional deformations.

Examples of Deformations

Homogeneous Deformations

The most general homogeneous deformation of the body B from its reference configuration is expressed by:

$$\mathbf{x} = \mathbf{A}\mathbf{X} \quad (136)$$

Choosing as fractional derivatives Caputo ones that are given by:

$${}_a^C D_t^\alpha (t-a)^v = \frac{\Gamma(v+1)}{\Gamma(-\alpha+v+1)} (t-a)^{v-\alpha}$$

and

$${}_a^C D_X^\alpha X = \frac{\Gamma(2)}{\Gamma(2-\alpha)} X^{1-\alpha} \quad (137)$$

and specializing the homogeneous deformations with the example of simple shear, we discuss the deformation,

$$\begin{aligned} x_1 &= X_1 + \gamma X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \quad (138)$$

Therefore the Cauchy-Green deformation tensors $\mathbf{C}^{(\alpha)}$ and $\mathbf{B}^{(\alpha)}$ become:

$$\mathbf{C}^{(\alpha)} = \left(\mathbf{F}^{(\alpha)} \right)^T \mathbf{F}^{(\alpha)} = \begin{vmatrix} 1 & \gamma & 0 \\ \gamma & \gamma^2 + 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{139}$$

and

$$\mathbf{B} = \mathbf{F}^{(\alpha)} \left(\mathbf{F}^{(\alpha)} \right)^T = \begin{vmatrix} 1 + \gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{140}$$

The Green (Langrange) Fractional strain tensor is:

$$\mathbf{E}^{(\alpha)} = \frac{1}{2} \left(\mathbf{C}^{(\alpha)} - \mathbf{I} \right) = \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & \gamma^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{141}$$

And the Euler-Almansi strain tensor is given by:

$$\mathbf{A}^{(\alpha)} = \frac{1}{2} \left(\mathbf{I} - \left(\mathbf{B}^{(\alpha)} \right)^{-1} \right) = \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & -\gamma^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{142}$$

It would seem strange that the results are exactly the same as the ones of the conventional elasticity. However, there is mathematical explanation for the homogeneous deformations. Just taking into consideration the definition of the fractional derivative and differential, the differential for a linear function of the form:

$$f(x) = Ax \tag{143}$$

The fractional differential is given by:

$$d^\alpha f(x) = A d^\alpha x \tag{144}$$

the fractional differential of the function has almost the same form as the conventional one. Nevertheless, for the nonlinear function:

$$f(x) = x^4 \tag{145}$$

the fractional differential

$$d^\alpha f(x) = {}^C D_x^\alpha x^4 \cdot \left({}^C D_x^\alpha x \right)^{-1} d^\alpha x \tag{146}$$

Is equal to

$$d^\alpha f(x) = \frac{\Gamma(5)\Gamma(2-\alpha)}{\Gamma(5-\alpha)} x^3 d^\alpha x \tag{147}$$

with coefficient depending upon the α fractional dimension. That makes the difference in nonhomogeneous deformations discussed in the next section.

The Nonhomogeneous Deformations

The nonhomogeneous deformation is defined by the equations:

$$\begin{aligned} x_1 &= X_1 + \gamma X_2^4 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \tag{148}$$

Taking into consideration Eqs. 124 and 125, the fractional Cauchy-Green deformation tensors are expressed by:

$$\mathbf{C}^{(\alpha)} = \left(\mathbf{F}^{(\alpha)} \right)^T \mathbf{F}^{(\alpha)} = \begin{vmatrix} 1 & \frac{24\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{24\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & 1 + \frac{576\gamma^2\Gamma(2-\alpha)^2X_2^6}{\Gamma(5-\alpha)^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{149}$$

and

$$\mathbf{B}^{(\alpha)} = \mathbf{F}^{(\alpha)} \left(\mathbf{F}^{(\alpha)} \right)^T = \begin{vmatrix} 1 + \frac{576\gamma^2\Gamma(2-\alpha)^2X_2^6}{\Gamma(5-\alpha)^2} & \frac{24\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{24\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{150}$$

Hence Green-Lagrange strain tensor is expressed by:

$$\mathbf{E}^{(\alpha)} = \begin{vmatrix} 0 & \frac{12\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{12\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & \frac{288\gamma^2\Gamma(2-\alpha)^2X_2^6}{\Gamma(5-\alpha)^2} & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{151}$$

and Euler-Almansi strain tensor is defined by:

$$\mathbf{A}^{(\alpha)} = \begin{vmatrix} 0 & \frac{12\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{12\gamma\Gamma(2-\alpha)X_2^3}{\Gamma(5-\alpha)} & -\frac{288\gamma^2\Gamma(2-\alpha)^2X_2^6}{\Gamma(5-\alpha)^2} & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{152}$$

It is evident that the strain tensors strongly depend upon the fractional dimension α for the present nonhomogeneous deformation. However, the present deformation also depends upon the source ω of the fractional analysis.

The Infinitesimal Deformations

Since there has been pointed out in the introduction, fractional strain tensors have been considered in the literature, mainly in infinitesimal deformations, simply by substituting the common derivatives to fractional ones. It would be wise to study whether that idea is valid or not. Unfortunately it is proven a mistake. Fractional strain with simple substitution of derivatives does not have any physical meaning.

Considering the fractional Euler-Lagrange strain tensor, Eq. 118, where \mathbf{u} is the small displacement vector with $|\mathbf{u}| \ll 1$ and $|\nabla_{\mathbf{x}}^{(\alpha)} \mathbf{u}| \ll 1$, we restrict into the linear deformation analysis, and the infinitesimal (linear) fractional Euler-Lagrange strain tensor \mathbf{E}_{lin}^α becomes:

$$\mathbf{E}_{lin}^{(\alpha)} = \frac{1}{2} \left[(\nabla_{\mathbf{x}}) \mathbf{u}^T + \nabla_{\mathbf{x}} \mathbf{u} \right] \tag{153}$$

It is recalled from Eq. 92 that:

$$\nabla^{(a)} f(\mathbf{x}) = grad^{(a)} f(\mathbf{x}) = \nabla_i^{(\alpha)} f(x) \mathbf{e}_i = \frac{{}^c D_i^a f(\mathbf{x})}{{}^c D_i^a x_i} \mathbf{e}_i, \tag{154}$$

Hence the linear fractional strain is not simply the half of the sum of the fractional derivatives of the displacement vector and its transport, but the half of the fractional gradient (as it has been defined by Eq. 92) and its transport.

Fractional Stresses

Pointing out that the fractional tangent space of a surface has different orientation of the conventional one, the fractional normal vector \mathbf{n}^α does not coincide with the conventional normal vector \mathbf{n} . Hence we should expect the stresses and consequently the stress tensor to differ from the conventional ones not only in the values.

If $d^\alpha \mathbf{P}$ is a contact force acting on the deformed area $d^\alpha \mathbf{a} = \mathbf{n}^\alpha d^\alpha a$, lying on the fractional tangent plane where \mathbf{n}^α is the unit outer normal to the element of area $d^\alpha \mathbf{a}$, then the α -fractional stress vector is defined by:

$$\mathbf{t}^\alpha = \lim_{d^\alpha a \geq 0} \frac{d^\alpha \mathbf{P}}{d^\alpha a} \tag{155}$$

However, the α -fractional stress vector does not have any connection with the conventional one

$$\mathbf{t} = \lim_{da \geq 0} \frac{d\mathbf{P}}{da} \quad (156)$$

since the conventional tangent plane has different orientation from the α -fractional tangent plane and the corresponding normal vectors too.

Following similar procedures as the conventional ones we may establish Cauchy's fundamental theorem (see Truesdell 1977).

If $\mathbf{t}^\alpha(\cdot, \mathbf{n}^\alpha)$ is a continuous function of the transplacement vector \mathbf{x} , there is an α -fractional Cauchy stress tensor field

$$\mathbf{T}^\alpha = \left[\sigma_{ij}^\alpha \right] \quad (157)$$

The Balance Principles

Almost all balance principles are based upon Reynold's transport theorem. Hence the modification of that theorem, just to conform to fractional analysis is presented. The conventional Reynold's transport theorem is expressed by:

$$\frac{d}{dt} \int_W A dV = \int_W \frac{dA}{dt} dV + \int_{\partial W} A \mathbf{v}_n dS \quad (158)$$

For a vector field A applied upon region W with boundary ∂W and \mathbf{v}_n is the normal velocity of the boundary ∂W .

Material Derivatives of Volume, Surface, and Line Integrals

For any scalar, vector or tensor property that may be represented by:

$$P_{ij}(t) = \int_V P_{ij}^*(x, t) dV \quad (159)$$

where V is the volume of the current placement in the conventional calculus. The material time derivative of $P_{ij}^*(x, t)$ is expressed by:

$$\frac{dP_{ij}^*}{dt} = \frac{\partial P_{ij}^*}{\partial t} + v \frac{\partial P_{ij}^*}{\partial x_j} = \frac{\partial P_{ij}^*}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} P_{ij}^* \quad (160)$$

recalling that we consider constant material points during time derivation. Similarly in fractional calculus, the material time derivative is given, for any tensor field P_{ij} by:

$$\frac{dP_{ij}^*}{dt} = \frac{\partial P_{ij}^*}{\partial t} + \frac{\partial^\alpha P_{ij}^*}{\partial x_k^a} \frac{d^a x_l}{dx_k^a} \frac{dg(x_m)}{dt} \delta_{km} \tag{161}$$

Since

$$v_k = \frac{\partial^a x_e}{\partial x_k^a} \frac{dg(x_m)}{dt} \delta_{km} \tag{162}$$

The material derivative, into the context of fractional calculus, is expressed by:

$$\frac{d(\quad)}{dt} = \frac{\partial(\quad)}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}}^\alpha (\quad) \tag{163}$$

Hence, the acceleration is defined by:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}}^\alpha \mathbf{v} \tag{164}$$

Furthermore the material time derivative of $P_{ij}(t)$ is expressed in conventional analysis by:

$$\frac{d}{dt} [P_{ij}(t)] = \frac{d}{dt} \int_V P_{ij}^*(x, t) dV \tag{165}$$

where b is the current placement of the region. It is well known that:

$(d\dot{V}) = JdV$, and the Eq. 165 yields:

$$\frac{d}{dt} \int_V P_{ij}^*(x, t) dV = \int_V \left[\frac{\partial P_{ij}^*(x, t)}{\partial t} + P_{ij}^*(x, t) \frac{\partial v_p}{\partial x_p} \right] dV \tag{166}$$

Recalling the material derivative operator Eqs. 160 and 166 yields,

$$\frac{d}{dt} \int_V P_{ij}^*(x, t) dV = \int_b \left[\frac{\partial P_{ij}^*(x, t)}{\partial t} + \frac{\partial (u_p P_{ij}^*(x, t))}{\partial x_p} \right] dV \tag{167}$$

yielding Reynold's Transport theorem:

$$\frac{d}{dt} \int_v P_{ij}^*(x, t) dV = \int_V \frac{\partial P_{ij}^*(x, t)}{\partial t} dV + \int_S v_p [P_{ij}^*(x, t)] dS_p \quad (168)$$

Expressing Reynold's Transfer Theorem in fractional form we get:

$$\frac{d}{dt} \int_{\omega V} P_{ij}^*(x, t) d^\alpha V = \int_{\omega V} \frac{\partial P_{ij}^*(x, t)}{\partial t} d^\alpha V + \int_{\omega S} u_p [P_{ij}^*(x, t)] d^\alpha S_p \quad (169)$$

where $d^\alpha V$ and $d^\alpha S$ the infinitesimal fractional volume and surface, respectively.

The volume integral of the material time derivative of $P_{ij}(t)$ may also be expressed by:

$$\frac{d}{dt} \int_{\omega V}^{(\alpha)} P_{ij}^*(\mathbf{x}, t) d^\alpha V = \int_{\omega V}^{(\alpha)} \left(\frac{\partial P_{ij}^*(x, t)}{\partial t} + div^\alpha [\mathbf{v} P_{ij}^*] \right) d^\alpha V \quad (170)$$

The Balance of Mass

The conventional balance of mass, expressing the mass preservation, is expressed by:

$$\frac{d}{dt} \int_W \rho dV = 0 \quad (171)$$

In the fractional form it is given by:

$$\frac{d}{dt} \int_{\omega W}^{(\alpha)} \rho d^\alpha V = 0 \quad (172)$$

where $\frac{d}{dt}$ is the total time derivative.

Recalling the fractional Reynold's Transport Theorem, we get:

$$\frac{d}{dt} \int_{\omega V}^{(\alpha)} \rho d^\alpha V = \int_{\omega V}^{(\alpha)} \left(\frac{\partial \rho(x, t)}{\partial t} + div^\alpha [\mathbf{v} \rho] \right) d^\alpha V \quad (173)$$

Since Eq. 173 is valid for any volume V , the continuity equation is:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}^a [\mathbf{v}\rho] = 0 \quad (174)$$

where div^a has already been defined by Eq. 96. That is the continuity equation expressed in fractional form.

Balance of Linear Momentum Principle

It is reminded that the conventional balance of linear momentum is expressed in continuum mechanics by:

$$\frac{d}{dt} \int_{\Omega} \rho \mathbf{v} dV = \int_{\partial\Omega} \mathbf{t}^{(n)} dS + \int_{\Omega} \rho \mathbf{b} dV \quad (175)$$

where \mathbf{v} is the velocity, $\mathbf{t}^{(n)}$ is the traction on the boundary, and \mathbf{b} is the body force per unit mass. Likewise that principle in fractional form is expressed by:

$$\frac{d}{dt} \int_{\omega \Omega}^{(\alpha)} \rho \mathbf{v} d^{(\alpha)} V = \int_{\omega \Omega}^{(\alpha)} [\rho \mathbf{b} + \operatorname{div}^a (\mathbf{T}^\alpha)] d^{(\alpha)} V \quad (176)$$

Hence the equation of linear motion, expressing the balance of linear momentum, is defined by:

$$\operatorname{div}^a [\mathbf{T}^\alpha] + \rho \mathbf{b} - \rho \dot{\mathbf{v}} = 0 \quad (177)$$

It should be pointed out that div^a has already been defined by Eq. 96 and is different from the common definition of the divergence.

Following similar steps as in the conventional case, the balance of rotational momentum yields the symmetry of Cauchy stress tensor.

Balance of Rotational Momentum Principle

Following similar procedure as in the conventional case, we may end up to the symmetry property of the fractional stress tensor, i.e.:

$$\mathbf{T}^\alpha = (\mathbf{T}^\alpha)^T \quad (178)$$

Fractional Zener Viscoelastic Model

The Integer Viscoelastic Model

It was proposed by Zener consists of the elastic springs with elastic constraints E_1 and E_2 and the viscous dashpot having a viscosity constant C . Then the standard linear solid model called Zener viscoelastic model is indicated in Fig. 7.

The three-parameter Zener model (Fig. 7) yields the following constitutive equation:

$$\left[1 + \frac{C}{E_1 + E_2} \frac{d}{dt} \right] \sigma(t) = \frac{E_2}{E_1 + E_2} \left[E_1 + C \frac{d}{dt} \right] \varepsilon(t) \tag{179}$$

In this case the creep and relaxation functions take the form:

$$J(t) = \frac{E_1 + E_2}{E_1 E_2} \{ 1 - \exp[-(t/\tau_\varepsilon)] \} \tag{180}$$

$$G(t) = \frac{E_1 E_2}{E_1 + E_2} \{ 1 + \exp[-(t/\tau_\sigma)] \} \tag{181}$$

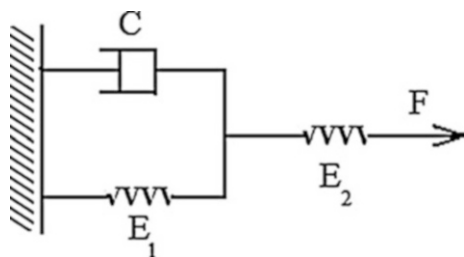
where

$$\tau_\varepsilon = \frac{C}{E_1}, \tau_\sigma = \frac{C}{E_1 + E_2}. \tag{182}$$

For the viscoelastic materials with $E_1 = 0.16 \cdot 10^9$ N/m, $E_2 = 1.5 \cdot 10^9$ N/m and $C = 16 \cdot 10^9$ Ns/m, and the initial condition $y_0 = 0$, the function of the compliance with respect to time is shown in Fig. 8.

Furthermore, the function of the relaxation modulus $G(t)$ with respect to time t is shown in Fig. 9:

Fig. 7 The three-parameter Zener model



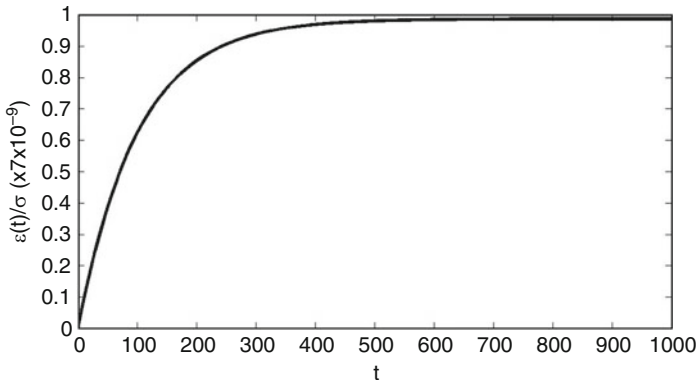


Fig. 8 The variation of the compliance with respect to time for the integer Zener viscoelastic model

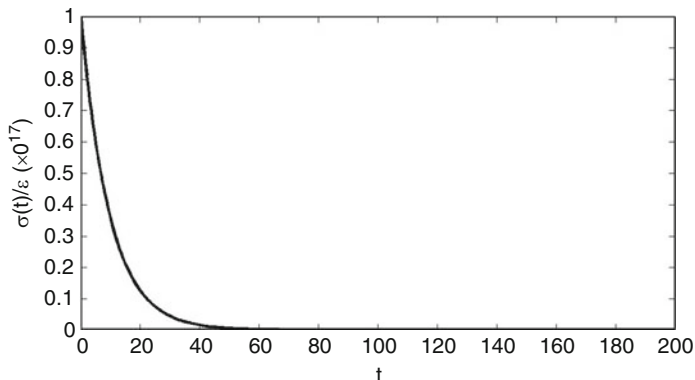


Fig. 9 Variation of the relaxation modulus with respect to time for the integer Zener viscoelastic model

The Fractional Order Derivative Viscoelastic Models

Viscoelastic models have been proposed using fractional time derivatives, just to take into account the material memory effects. The fractional Zener model introduced by Bagley and Torvik (1986) has been extensively studied in the literature (see Atanackovic 2002 and Sabatier et al. 2007). Consequently, the fractional Zener model has been proposed, using fractional time derivatives and more specifically the Caputo time derivatives. Therefore the constitutive equation for the model is:

$$\left[1 + \frac{C}{E_1 + E_2} \frac{d^\alpha}{dt^\alpha} \right] \sigma(t) = \frac{E_2}{E_1 + E_2} \left[E_1 + C \frac{d^\alpha}{dt^\alpha} \right] \varepsilon(t) \quad (183)$$

with creep compliance and relaxation modulus:

$$J(t) = \frac{1}{E_2} + \frac{1}{E_1} \{1 - E_\alpha [-(t/\tau_\epsilon)^\alpha]\} \tag{184}$$

$$G(t) = \frac{E_2}{E_1 + E_2} \{E_0 + E_1 E_\alpha [-(t/\tau_\epsilon)^\alpha]\} \tag{185}$$

where $\tau_\epsilon^\alpha = \frac{C}{E_1}$, $\tau_\sigma^\alpha = \frac{C}{E_1 + E_2}$.

For the viscoelastic materials with $E_1 = 0.16 \cdot 10^9$ N/m, $E_2 = 1.5 \cdot 10^9$ N/m, $C = 16 \cdot 10^9$ Ns/m, and the initial condition $y_0 = 0$, the function of the compliance with respect to time is shown in Fig. 10.

Furthermore, the function of the relaxation modulus $G(t)$ with respect to time t is shown in Fig. 11. It is shown that the lower the fractional order the slower the convergence to the final zero value of the relaxation.

Proposed Fractional Viscoelastic Zener Model

Since only Leibnitz L-fractional derivatives have physical sense, the fractional viscoelastic equations should be expressed in terms of L-fractional derivatives. Therefore the L-fractional Zener model should be expressed by:

$$\left[1 + \frac{C}{E_1 + E_2} {}^L_0 D_t^\alpha \right] \sigma(t) = \frac{E_2}{E_1 + E_2} [E_1 + C {}^L_0 D_t^\alpha] \epsilon(t) \tag{186}$$

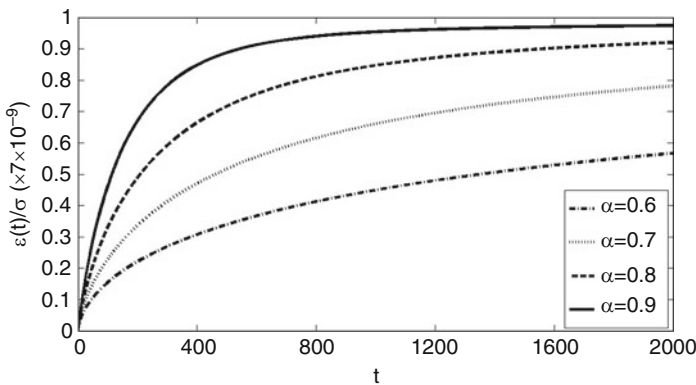


Fig. 10 The variation of the compliance with respect to time for various values of the fractional order for the existing fractional models

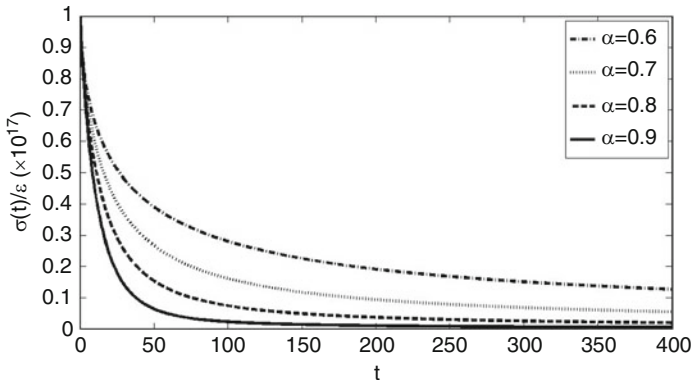


Fig. 11 The variation of the relaxation with respect to time for various values of the fractional order for the existing fractional models

Hence for constant applied stress the compliance

$$J(t) = \frac{\varepsilon(t)}{\sigma} \tag{187}$$

expressing the variation of strain with respect to time is given by the equation:

$$\left[E_1 + C_0^L D_t^\alpha \right] J(t) = \frac{E_1 + E_2}{E_2} \tag{188}$$

Therefore we have to solve the fractional linear equation:

$$C_0^L D_t^\alpha y(t) + E_1 y(t) = 1 + \frac{E_1}{E_2} \tag{189}$$

Looking now for a solution of the type:

$$y(t) = \sum_{k=0}^{\infty} y_k t^k \tag{190}$$

and substituting $y(t)$ from Eq. 190 to the governing compliance, Eq. 189 we get:

$$\sum_{k=0}^{\infty} C y_{k+1} \frac{\Gamma(2-\alpha) \Gamma(\kappa+2)}{\Gamma(\kappa+2-\alpha)} t^k + \sum_{k=0}^{\infty} E_1 y_k t^k = 1 + \frac{E_1}{E_2} \tag{191}$$

Since the algebraic Eq. 191 is valid for any t the various coefficients y_i are defined by:

$$y_1 = \frac{1}{C} \left(-E_1 y_0 + 1 + \frac{E_1}{E_2} \right) \tag{192}$$

$$y_{k+1} = -\frac{E_1}{C} \frac{\Gamma(\kappa + 2 - \alpha)}{\Gamma(\kappa + 2) \Gamma(2 - \alpha)} y_k, \forall k \geq 1 \tag{193}$$

Hence

$$y_k = \left(-\frac{E_1}{C \Gamma(2 - \alpha)} \right)^{k-1} \prod_{m=1}^{k-1} \frac{\Gamma(m + 2 - \alpha)}{\Gamma(m + 2)} y_1, \forall k \geq 2 \tag{194}$$

For the viscoelastic materials with $E_1 = 0.16 \cdot 10^9$ N/m, $E_2 = 1.5 \cdot 10^9$ N/m, $C = 16 \cdot 10^9$ Ns/m, and the initial condition $y_0 = 0$; the function of the compliance with respect to time is shown in Fig. 6.

It is clear from Fig. 6 that the fractional order has influence upon the time of convergence of the compliance modulus to the final value. The lower the value of the fractional order the slower the convergence of the compliance modulus to the final value.

Now, proceeding to the relaxation behavior of the Fractional Zener Viscoelastic model, we consider constant strain $\varepsilon(t) = \varepsilon$, then the governing Eq. 185 becomes:

$$C_0^L D_t^\alpha \left(\frac{\sigma(t)}{\varepsilon} \right) + (E_1 + E_2) \left(\frac{\sigma(t)}{\varepsilon} \right) = E_1 E_2 \tag{195}$$

For the relaxation modulus $y(t) = G(t) = \frac{\sigma(t)}{\varepsilon}$, the Eq. 197 above takes the form:

$$C_0^L D_t^\alpha y(t) + (E_1 + E_2) y(t) = E_1 E_2 \tag{196}$$

Looking for solution of the type

$$y(t) = \sum_{k=0}^{\infty} y_k t^k \tag{197}$$

and substituting in Eq. 196 we get:

$$\sum_{k=0}^{\infty} C y_{k+1} \frac{\Gamma(2 - \alpha) \Gamma(k + 2)}{\Gamma(k + 2 - \alpha)} t^k + \sum_{k=0}^{\infty} (E_1 + E_2) y_k t^k = E_1 E_2 \tag{198}$$

Since Eq. 198 is valid for any t, it is an identity. Hence:

$$y_1 = -\frac{E_1 + E_2}{C} y_0 + \frac{E_1 E_2}{C} \tag{199}$$

$$y_{k+1} = -\frac{E_1 + E_2}{C} \frac{\Gamma(m+2-\alpha)}{\Gamma(2-\alpha)\Gamma(k+2)} y_1, \forall k \geq 2 \quad (200)$$

Those relations yield:

$$y_k = \left(-\frac{E_1 + E_2}{c \Gamma(2-\alpha)} \right)^{k-1} \prod_{m=1}^{k-1} \frac{\Gamma(m+2-\alpha)}{\Gamma(m+2)} y_1, \forall k \geq 2. \quad (201)$$

For the viscoelastic materials with $E_1 = 0.16 \cdot 10^9$ N/m, $E_2 = 1.5 \cdot 10^9$ N/m, and $C = 16 \cdot 10^9$ Ns/m, the function of the relaxation modulus with respect to time is shown in Fig. 7.

It is clear from Fig. 7 that the fractional order has influence upon the time of convergence of the relaxation modulus to the zero value. The lower the value of the fractional order the slower the convergence of the relaxation modulus to the zero value.

Comparison of the Three Viscoelastic Models

For the viscoelastic materials with $E_1 = 0.16 \cdot 10^9$ N/m, $E_2 = 1.5 \cdot 10^9$ N/m, and $C = 16 \cdot 10^9$ Ns/m, and for two different values of the fractional orders $a = 0.5$, $a = 0.7$, and $a = 0.9$ the (Figs. 8, 9, and 10) show the behavior of the compliance modulus and (Figs. 11, 12, 13, 14, 15, and 16) the behavior of the relaxation modulus.

Comparing (Figs. 17, 18, and 19) the behavior of the proposed model exhibits a more mild convergence for the compliance and relaxation moduli to the final values as far as to Caputo's viscoelastic models are concerned. Further, the lower

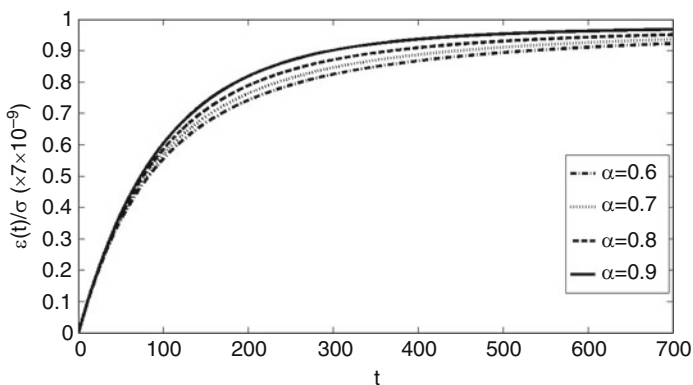


Fig. 12 The variation of the compliance with respect to time for various values of the fractional order for the existing fractional models

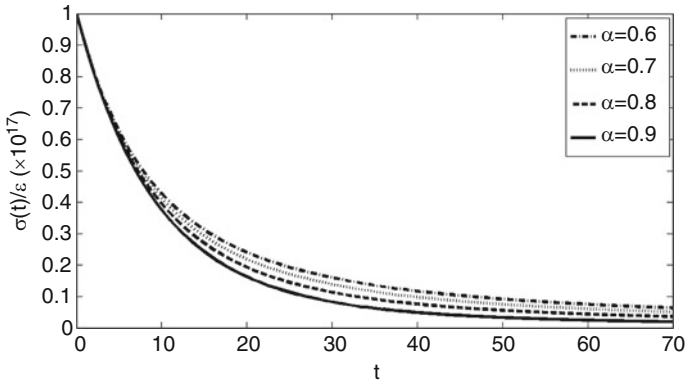


Fig. 13 Variation of the relaxation modulus with respect to time for various values of the fractional order for the existing fractional models

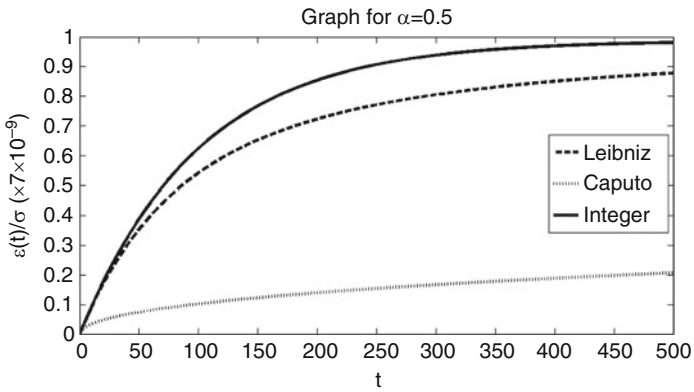


Fig. 14 The variation of the compliance with respect to time for fractional order $a = 0.5$

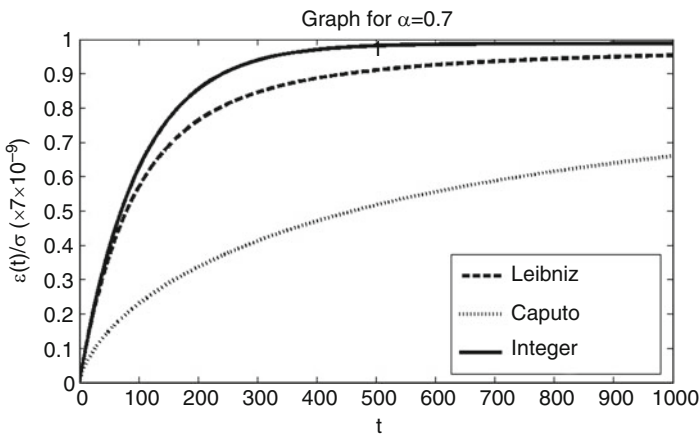


Fig. 15 The variation of the compliance with respect to time for fractional order $a = 0.7$

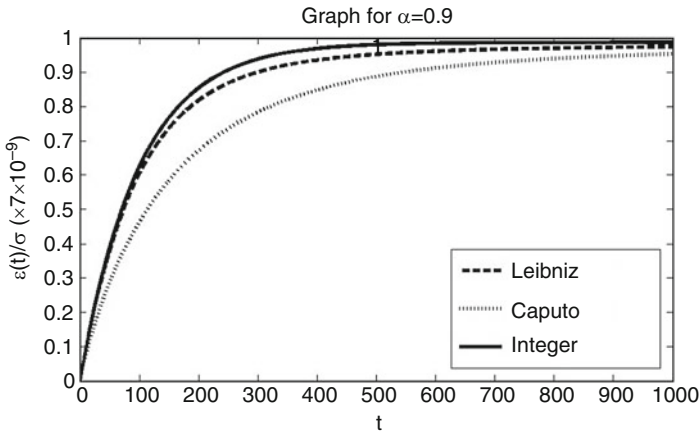


Fig. 16 The variation of the compliance with respect to time for fractional order $a = 0.9$

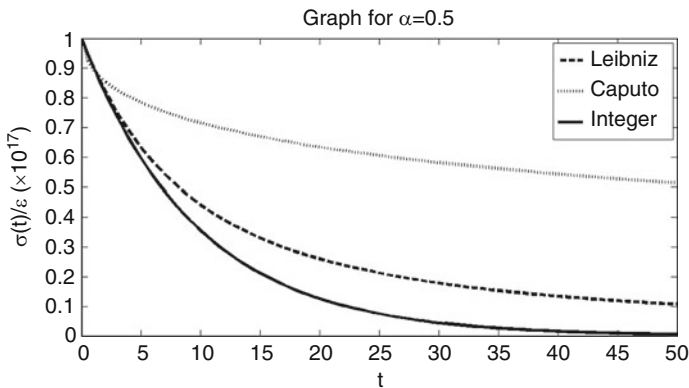


Fig. 17 Variation of the relaxation modulus with respect to time for fractional order $a = 0.5$

the fractional order of the model the slower the convergence to the final values for the same proposed model.

Conclusion: Further Research

Correcting the picture of fractional differential of a function, the fractional tangent space of a manifold was defined, introducing also Leibniz's L-fractional derivative that is the only one having physical meaning. Further, the L-fractional chain rule is imposed, that is necessary for the existence of fractional differential. After establishing the fractional differential of a function, the theory of fractional differential geometry of curves is developed. In addition, the basic forms concerning the first and second differential forms of the surfaces were defined, through the

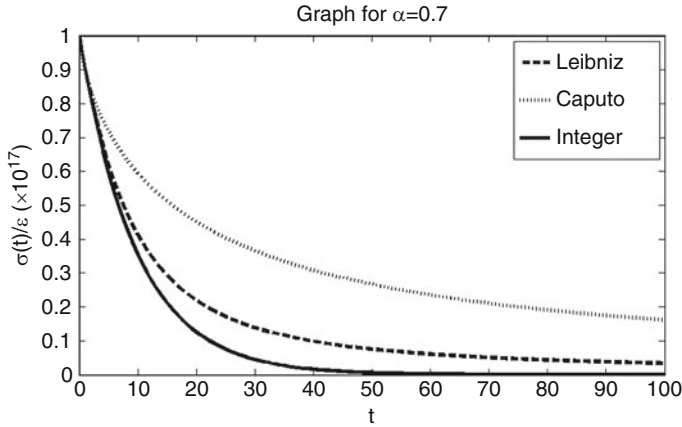


Fig. 18 Variation of the relaxation modulus with respect to time for fractional order $a = 0.7$

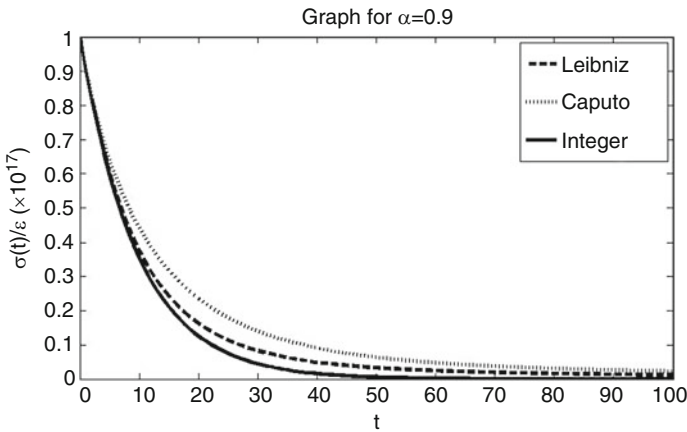


Fig. 19 Variation of the relaxation modulus with respect to time for fractional order $a = 0.9$

tangent spaces defined earlier, having mathematical meaning without any confusion, contrary to the existing procedures. Further the field theorems have been outlined in an accurate manner that may not cause confusion in their applications. The present work will help in discussing many applications concerning mechanics, quantum mechanics, and relativity, that need a clear description, based upon the fractional differential geometry. Moreover, the basic theorems of fractional vector calculus have been revised along with the basic concepts of fractional continuum mechanics. Especially the concept of fractional strain has been pointed out, because it is widely used in various places in a quite different way. The present analysis may be useful for solving updated problems in Mechanics and especially for lately proposed theories such as peridynamic theory. Finally, viscoelasticity models are studied using L-fractional derivatives. Those models exhibit milder behavior from the ones formulated through Caputo derivatives, compared to the conventional (integer)

models. In the present study, only the Zener viscoelastic model was discussed and its behavior concerning the compliance and relaxation moduli was studied.

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