Chapter 4
Mathematical Modelling

If I were again beginning my studies, I would follow the advice of Plato and start with mathematics.

Galileo Galilei

4.1 Introductory Remarks

When talking about mathematical models, most readers normally associate this concept with a very complex notion. In this chapter, the GPS mathematical models are presented in very simple terms that will allow the reader to understand how errors are managed and positions finally extracted from the observations discussed in Sect. 3.3. In addition, it will be shown how errors can be eliminated or avoided through the design of a GNSS survey. The intended outcomes of this chapter are to assist the reader to:

(i) Know and understand the nature of code- and phase-pseudorange observation equations (derived from Sect. 3.3) and their uses.
(ii) Know how various GPS observational errors can be modelled through simple observation techniques.
(iii) Understand the basic concepts of static, kinematic, Differential GPS (DGPS), and relative positioning modes and their uses.

More detailed discussions can be found, e.g., in [1, pp. 107–116], [2, pp. 181–201], and [3, pp. 170–187, 250–261]. Here, the focus is on the observations, models, and the subsequent configurations that enable the solution of positions and other parameters required for environmental monitoring.

Before looking at the mathematical formulations that underpin the GPS observations, let us briefly examine how these satellites orbit in space (i.e., space segments discussed in Sect. 3.2.1). In general, the motions of planets in space obey the three Johannes Kepler's laws:

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(i) The orbital path of a planet takes the shape of an ellipse, with the sun located at one of its focal points.
(ii) The (imaginary) line connecting the sun to any of the planet sweeps equal areas of the orbital ellipse over equal time intervals.
(iii) The ratio of the square of the planets orbital period and the cube of the mean distance from the sun is constant.

These Keplerian laws apply not only to the planets, but also to artificial satellites (e.g., GNSS) orbiting the Earth. The orbital path takes the shape of an ellipse, with the Earth located at one of its focal points. Satellite orbits are characterized by their altitudes, i.e., Low Earth Orbiting (LEO) satellites (up to 2000 km), e.g., GRACE (Gravity Recovery And Climate Experiment) and COSMIC (Constellation Observing System for Meteorology, Ionosphere, and Climate), which together with GNSS satellites provide useful tools for environmental monitoring at continental scales as we shall see in part II of the book. Besides LEO satellites, we have Medium Earth Orbiting (MEO) satellites (5000–20,000 km) in which GNSS satellites fall, and Geostationary Earth Orbiting (GEO) satellites, orbiting at about 36,000 km, and which comprise communication satellites.

In a perfect situation, a satellite orbit should ideally follow the ideal *Keplerian laws* in which all forces except the Earth's gravitational force are neglected, the Earth's gravitational field is radially symmetric, and there exists no atmospheric drag. In reality, however, the *central* and *non-central* gravitational forces act on the satellites. In addition, other forces acting on the satellites include *gravitational attraction* of the Sun, Moon and planets; solar radiation pressure; and magnetic forces. All the other forces other than the central gravitational force are normally grouped under *perturbing forces*.

Modelling of GPS observations makes use of satellite positions broadcast through navigation messages (see e.g., Sect. 3.3.1). This is of particular importance in instantaneous positioning (i.e., in real-time positioning). Understanding how to model the orbital errors of GNSS satellites is therefore a good starting point towards achieving more accurate positions and other environmental monitoring parameters. For other applications that require post-processing of GNSS data, and which the results may not be immediately required, such as those of long term monitoring of environmental changes (e.g., sea level), *precise ephemeris* (see Sect. 3.4.1) should be used.

4.2 Observation Equations

For *code ranging*, let us consider that a signal is send by a satellite at time t^s and received by a ground-based receiver at time t_r (see e.g., Sect. 3.3.2). The time taken by the signal to travel from the satellite to the receiver would therefore be

$$\Delta t = t_r - t^s. \tag{4.1}$$

Due to clock errors (see e.g., Sect. 3.4.2) in the transmitting satellite (i.e., δ^s) and the receiving receiver (i.e., δ_r), Eq. (4.1) becomes

$$\Delta t = \{t_r + \delta_r\} - \{t^s + \delta^s\} = \Delta t + \Delta\delta, \tag{4.2}$$

which when multiplied by the speed of light c (e.g., Eq. 3.1) gives the distance between the satellite and the receiver. Since *time difference* is used, the derived distance between the satellite and the receiver, i.e., code-pseudorange (e.g., Sect. 3.3.2) is obtained from (4.2) by

$$\boxed{R(t) = c\Delta t = c(\Delta t + \Delta\delta) = \varrho(t) + c\Delta\delta(t)}, \tag{4.3}$$

where $R(t)$ is the measured pseudorange, $\varrho(t)$ the geometric (true) unknown distance between the satellite and the receiver, and $c\Delta\delta(t)$ is the range bias.

For precise GPS measurements, *carrier-phase measurements* are often used to obtain ranges. Consider the phase $\varphi^s(t)$ (cycles) with a frequency f^s (cycles per second) to have been sent by a satellite. At the receiver, a phase $\varphi_r(t)$ with a frequency f_r is generated. Taking the start time of the signal as t_0, the time passed in the GPS system from when the signal was sent from the satellite to the receiver will be t_{sr}. The relationship between frequency and phase is such that the frequency is the derivative of phase, see e.g., [2, p. 72]

$$f = \frac{d\varphi}{dt}. \tag{4.4}$$

From (4.4), phase can be obtained by integrating the frequency from the initial time t_0 to t as

$$\varphi = \int_{t_0}^{t} f dt, \tag{4.5}$$

which leads to

$$\varphi = f(t - t_0). \tag{4.6}$$

For the initial time $t_0 = 0$, (4.6) yields $\varphi(t_0) = 0$. When the satellite signal reaches the receiver, it is

- reconstructed with phase $\varphi^s = f^s(t - t_{sr})$, and
- compared with the receiver generated signal with phase $\varphi_r = f_r(t)$

Using Eq. (3.1) relating speed, distance and time, t_{sr} is given as ϱ/c, which when substituted for t_{sr} in $\varphi^s = f^s(t - t_{sr})$ leads to

$$\varphi^s = f^s(t - \frac{\varrho}{c}). \tag{4.7}$$

The reconstructed signal is then written as

$$\varphi^s(t) = f^s t - f^s \frac{\varrho}{c} - \varphi_{0s}, \tag{4.8}$$

while for the receiver generated signal, (4.7) becomes

$$\varphi_r(t) = f_r t - \varphi_{0r}. \tag{4.9}$$

In (4.8) and (4.9), φ_{0s} and φ_{0r} arise from the satellite and receiver clock errors. Considering the clock errors to be δ^s and δ_r for the satellite and receiver, respectively, from (4.2) and using $\varphi_{0s} = -f^s\delta^s$ and $\varphi_{0r} = -f_r\delta_r$, the phase measured at a time t is given by

$$\varphi_r^s(t) = \varphi^s(t) - \varphi_r(t) = -f^s \frac{\varrho}{c} + f^s\delta^s - f_r\delta_r + (f^s - f_r)t. \tag{4.10}$$

Expressing the receiver-satellite clock bias term as $\Delta\delta = \delta_r - \delta^s$ and using the frequency-wavelength relation $f = c/\lambda$, assuming the satellite frequency f^s and receiver frequency f_r to be equal to the nominal frequency f, (4.10) is rewritten as

$$\Delta\varphi(t) = -\frac{\varrho}{\lambda}(t) - c\frac{\Delta\delta}{\lambda}(t), \tag{4.11}$$

where $\Delta\varphi$ is the measured phase expressed in terms of the measured pseudorange ϱ, wavelength λ, and the speed of light c.

Besides expressing the phase in terms of pseudorange as in (4.11), it can also be expressed in terms of the initial instantaneous fractional measurement immediately when the receiver is switched on at time t_0. At this stage, the receiver still does not know the integer number N of cycles between it and the satellite, i.e., the integer ambiguity (cf. Eq. 3.2). If a satellite is tracked by the receiver without loosing lock, this integer number remains the same (see e.g., Fig. 4.1). The phase at time t can therefore be expressed in-terms of the *measured value* at time t plus the unknown *integer number of cycles* N since the initial time t_0 as [2, p. 89]

$$\Delta\varphi(t) = \Delta\varphi_0(t) + N, \tag{4.12}$$

Comparing Eqs. (4.11) and (4.12), one obtains

$$\boxed{\lambda\Delta\varphi_m(t) = \varrho(t) + c\Delta\delta(t) + \lambda N}, \tag{4.13}$$

where $\Delta\varphi_m(t) = -\Delta\varphi_0(t)$ is the measurable phase at epoch t. Note that (4.13) is similar to (4.3), only that it has the ambiguity term N added. Hence, the difference between code-pseudorange and phase-pseudorange in positioning with GPS satellites is that the former is only concerned with the position X, Y, Z of a receiver while the later is concerned with determining the receiver's position plus the unknown

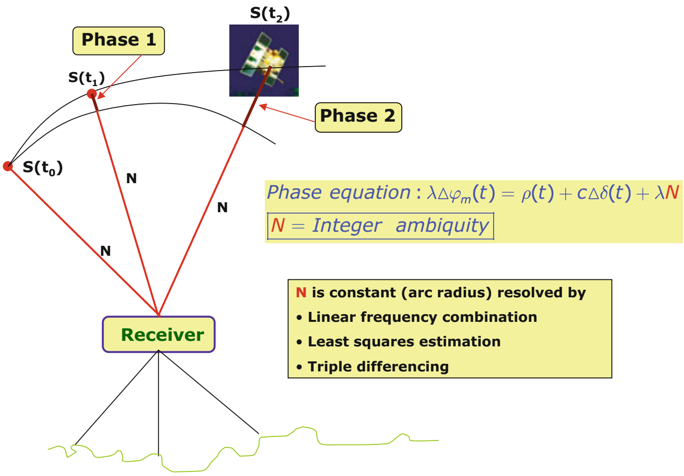

Fig. 4.1 How does the integer ambiguity arises? When a receiver is switched on, the number of complete cycles, known as integer ambiguity N, is unknown. The receiver measures the phase, i.e., fraction of a cycle shown in the figure. As subsequent measurements are taken, it is possible to determine the N whose value remains constant as shown in the figure

ambiguity term N. Note that the phase observation in (4.13) has been multiplied by the wavelength λ to convert it into a distance measurement. In Sect. 6.2.2.2 on p. 99, the ambiguity solution is discussed in more detail.

4.3 Models

The word "*model*" has about 8 meanings,[1] ranging from a small object that is built to represent in detail another, often larger, object, to an animal whose appearance is copied by a mimic in Zoology. Model is a term that we often meet in daily life. For example, it is common for architects to work with models as a representation of the real world, or for people to display clothing as we see on the TV. It is also often common to hear people speak of role models, i.e., people they would like to emulate.

Mathematical models can also be seen as a representation of ideas using formulae as objects. An encyclopedic definition of a mathematical model is given as follows[2]:

> *A mathematical model is an abstract model that uses mathematical language to describe the behaviour of a system. Mathematical models are used particularly in the natural sciences and engineering disciplines (such as physics, biology, and electrical engineering) but also in the social sciences (such as economics, sociology and political science); physicists, engineers, computer scientists, and economists use mathematical models most extensively.*

[1] http://www.thefreedictionary.com/model.

[2] http://encyclopedia.thefreedictionary.com/Mathematical+model.

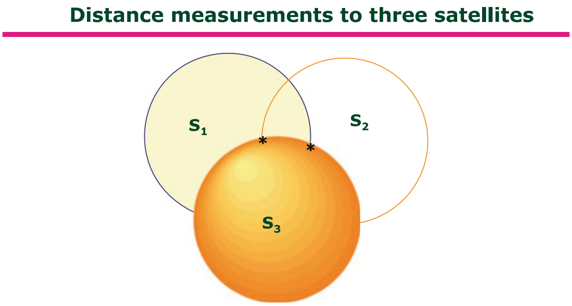

Fig. 4.2 GPS positioning geometry. Normally two positions of the receiver's position indicated by the *star* (*) in the figure will be given, one of which will be on the Earth's surface while the other in space (outside the Earth's surface)

An abstract model (or conceptual model) in the definition above is a theoretical construction that represents physical, biological or social processes, with a set of variables and a set of logical and quantitative relationships between them. In what follows, mathematical models are used to relate the measured GPS pseudoranges to their geometrical equivalent, and to model the observational errors discussed in Sect. 3.4.

A simple way to conceptualize a mathematical model in GPS operations involves distance measurements from a ground-based receiver to GPS satellites. In this case, the distance, which is a measurable quantity is geometrically represented by an equation that relates it to the position of the ground receiver and those of the satellites. The geometrical (true) distance S between the satellite and the receiver is represented by the model

$$S = \sqrt{(X^j - X_r)^2 + (Y^j - Y_r)^2 + (Z^j - Z_r)^2}, \tag{4.14}$$

where $\{X^j, Y^j, Z^j\}$ is the position of a satellite j and $\{X_r, Y_r, Z_r\}$ is the receiver's position. Since the distance from the satellite to the receiver S is measured, and the satellite's position $\{X^j, Y^j, Z^j\}$ is known from the transmitted ephemeris (e.g., Sect. 3.3.1), only the receiver's position $\{X_r, Y_r, Z_r\}$ needs to be determined from Eq. (4.14). Geometrically, to determine the three coordinates $\{X_r, Y_r, Z_r\}$ of the receiver's position, an intersection of three spherical cones, each representing a distance $S_i | i = 1, 2, 3$ is performed (see e.g., Fig. 4.2). Distance measurements to only one satellite puts the user's position anywhere within the sphere defined by distance S_1. Measurements to two satellites narrows the solution to the intersection of the two spheres S_1 and S_2. A third satellite is therefore required to definitely fix the user's position. This is achieved by the intersection of the third sphere S_3 with the other two (Fig. 4.2).

If direct distance measurements to the satellites were possible, Eq. (4.14) would suffice to provide the user's position. However, as already stated earlier, distance measurements to satellites are not *direct* owing to the satellites' and receivers' clock errors, errors in the satellites' positions, atmospheric delays, and receiver related errors such as phase centering and multipath discussed in Sect. 3.4. The distance Eq. (4.14), therefore, converts to the *pseudorange* equation

$$\boxed{\varrho_i^j = \sqrt{(X^j - X_r)^2 + (Y^j - Y_r)^2 + (Z^j - Z_r)^2}}, \tag{4.15}$$

where ϱ_i^j has now replaced S_i and contains the error uncertainties (cf. Eq. 6.7 on p. 109). Since the satellites' clock errors can be modeled while the receiver's clock errors must be determined as unknowns, the code pseudorange Eq. (4.3) then becomes

$$\boxed{R(t) + c\delta^j(t) = \varrho(t) + c\delta_i(t)}, \tag{4.16}$$

where at time t, $R(t)$ is the measured range, $\delta^j(t)$ and $\delta_i(t)$ are the satellite and receiver clock errors, $\varrho(t)$ the geometrical pseudorange from Eq. (4.15) and c is the velocity of propagation (i.e., speed of light). In Eq. 4.16, the addition of $c\delta_i(t)$ as an unknown to the receiver's position $\{X_r, Y_r, Z_r\}$ leads to four unknowns to be determined. This means, therefore, that instead of three satellites required to determine the receiver's position (e.g., Fig. 4.2), four satellites are required (Fig. 4.4). The fourth satellite is used to determine the receiver clock bias $c\delta_i(t)$.

When four satellites are observed as required for positioning, four pseudorange equations are formed from (4.16) and (4.15) as

$$\begin{aligned} R_1 &= [(X^1 - X_r)^2 + (Y^1 - Y_r)^2 + (Z^1 - Z_r)^2]^{1/2} + c\delta \\ R_2 &= [(X^2 - X_r)^2 + (Y^2 - Y_r)^2 + (Z^2 - Z_r)^2]^{1/2} + c\delta \\ R_3 &= [(X^3 - X_r)^2 + (Y^3 - Y_r)^2 + (Z^3 - Z_r)^2]^{1/2} + c\delta \\ R_4 &= [(X^4 - X_r)^2 + (Y^4 - Y_r)^2 + (Z^4 - Z_r)^2]^{1/2} + c\delta \end{aligned}, \tag{4.17}$$

with the four unknowns being the receiver's 3D position (X_r, Y_r, Z_r) and the receiver clock bias $c\delta$. With the observations of more satellites, Eq. (4.17) retains the four unknown with increased number of equations thereby necessitating the use of least squares solutions discussed in Sect. 6.3. Examples of algebraic methods for solving Eq. (4.17) are presented in Awange et al. [1, 4, 5]. It should be mention here that the accuracy of the obtained receiver's position from Eq. (4.17) depends upon the accuracies of *range measurements*, the accuracy of each satellite's position, the accuracy by which the atmospheric parameters are modeled, the accuracy upon which the receiver measures time (i.e., clock synchronization, signals processing, signal noise, etc.), the geometry of the satellite and the length of time taken to observe [6]. For practical use, the determined receiver's geocentric coordinates (X_r, Y_r, Z_r) in the GPS's WGS-84 system can then be transformed to the user's local reference datum (see e.g., Sect. 5.6.1). This is often done automatically by the processing software.

For the *phase-pseudorange*, in addition to the receiver's position and receiver clock uncertainties, the integer ambiguity N which corresponds to the unknown number of complete cycles the signal has travelled from the satellite to the receiver must be added as an unknown. Equation (4.13) is then modeled by

$$\boxed{\lambda \Delta\varphi_m(t) + c\delta^j(t) = \varrho(t) + c\delta_i(t) + \lambda N}. \tag{4.18}$$

Similarly to Eqs. (4.3) and (4.13), comparison between Eqs. (4.16) and (4.18) reveal similarities, with (4.18) having the additional term of λN on the right-hand side. The solution of the right-hand side of (4.16), when inserted into the right-hand side of (4.18), leads to the solution of N. For this reason, code-pseudorange finds use in the linear combination of code and phase approaches for solving the integer ambiguity unknown N (e.g., Sect. 6.2.2.2).

The models (4.16) and (4.18) only become valid once other errors associated with the ionospheric effect I, tropospheric effect T and multipath m have been modeled.

4.3.1 Static and Kinematic Positioning

Static positioning is where a receiver is set up on top of a tripod or pillar at some point of interest and measurements are taken, see e.g., Fig. 4.3. The stationary receiver takes observations from a minimum of four satellites (e.g., Fig. 4.4) for example every 10 s (or as set by the user) for up to one hour or more to achieve a more precise position than is possible by a stand-alone instantaneous reading. During this period, it is hoped that the number of satellites either remains at the minimum of 4, or more, with an adequate PDOP (see Sect. 3.4.5). This would potentially lead to a better solution for the unknown ambiguity N, and better solutions of position X_r, Y_r, Z_r. It should be pointed out, however, that the occupation time being recommended by some receiver manufactures, e.g., Sokkia, is currently less than 30 min. This decrease in occupation time is attributed to improved receiver technology and modernization of the satellites as discussed in the preceding chapters.

Consider, for example, the code-pseudorange model (4.16). For a stationary receiver, 3 unknown receiver position $\{X_r, Y_r, Z_r\}$ and 1 unknown receiver clock bias have to be determined. For each observational epoch, we have an additional receiver clock bias as an unknown. If n is the number of satellites that can be observed in static mode and m the number of epochs, the following relation can be written [2, p. 185]

$$nm \geq 3 + m, \tag{4.19}$$

which gives

$$m \geq \frac{3}{n-1}. \tag{4.20}$$

Therefore, for static point positioning, for $n = 4$, $m \geq 1$, gives the minimum number of satellites and epoch needed to solve the three receiver coordinates and the receiver clock bias (see, e.g., Fig. 4.4). Exact solutions of these 4 equations to obtain point positions with GPS have been presented e.g., in [1, pp. 107–116], [7–10] and [11].

For *kinematic positioning*, the antenna is moving onboard a pole, boat or vehicle. In this mode of point positioning, there exist more unknown points since the receiver is in motion. The unknown coordinates now change from the static mode of 3 to $3m$, i.e., for each epoch of measurement while in motion. The addition of the receiver

Fig. 4.3 A static receiver stationed on a tripod at point G2 of Curtin University (Bentley Campus), Australia

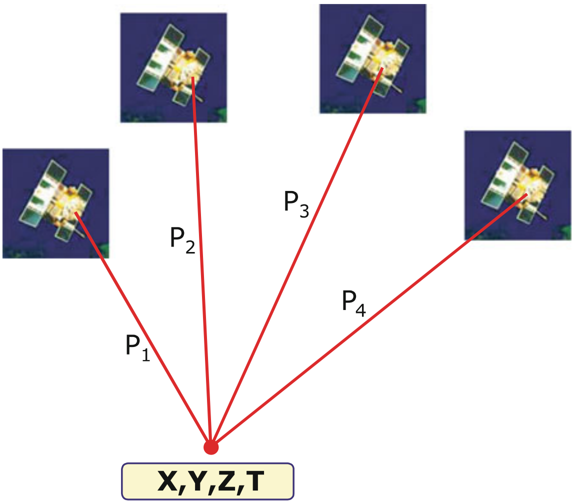

Fig. 4.4 Absolute point positioning using GPS satellites (see, e.g., Fig. 4.3)

clock bias m gives a total of $4m$ unknowns leading to

$$nm \geq 4m, \tag{4.21}$$

which leads to

$$n \geq 4. \tag{4.22}$$

The position of the receiver in motion can therefore be determined at any epoch t as long as a minimum of 4 satellites are visible.

4.3.2 *Differential GPS (DGPS)*

Absolute point in Sect. 4.3.1 (i.e., Fig. 4.4) is not suitable for applications requiring high accuracies such as the provision of controls for monitoring deformation, e.g., of dams. The reason for this is largely due to the errors discussed in Sect. 3.4. In order to minimize these errors and obtain higher accuracies, GPS can be used in relative or differential positioning mode [6]. The *differential positioning method*, also known as Differential GPS (DGPS) is one of the techniques most commonly used to model positioning errors in order to improve the accuracy of the final solutions. In this approach, code-pseudoranges are simultaneously measured by two receivers with one receiver occupying a *known reference station* (see Figs. 4.5 and 5.5). The reference station calculates the geometric pseudorange by making use of the satellite's position through $\rho_g^j = \|\mathbf{X}_r - \mathbf{X}^s\|$, where $\mathbf{X}_r$ are the coordinates of the reference station and $\mathbf{X}^s$ those of the satellite (from the received ephemeris). Since the positions of the satellite $\mathbf{X}^s = \{X^j(t), Y^j(t), Z^j(t)\}$ and that of the receiver $\mathbf{X}_r = \{X_A, Y_A, Z_A\}$ at a reference station are both known, the geometrical distance ϱ_g^j is computed from (4.15) as

$$\varrho_g^j = \|\mathbf{X}_r - \mathbf{X}^s\| = \sqrt{(X^j(t) - X_A)^2 + (Y^j(t) - Y_A)^2 + (Z^j(t) - Z_A)^2}. \tag{4.23}$$

If ϱ_{ma}^j is the measured pseudorange at the reference station A, then the pseudorange corrections (PRC) $\Delta\varrho_A$ for point A are given by

$$PRC = \Delta\varrho_A = \varrho_g^j - \varrho_{ma}^j, \tag{4.24}$$

where the correction term $PRC = \Delta\varrho_A$ comprises the range, satellite, and receiver bias terms corresponding to station A and the range rate correction. PRC are due to errors discussed in Sect. 3.4, which are assumed to be similar at both stations (A and B) due to their close proximity. $PRCs$ are transmitted to the roving receiver at location B in real-time using communication link (e.g., radio, satellite, or cell phone

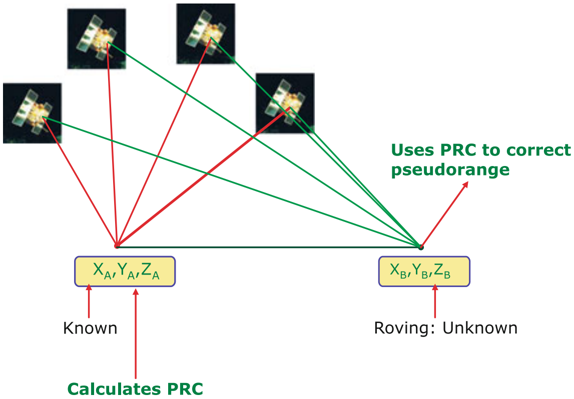

Fig. 4.5 DGPS positioning. One receiver is placed at a known reference station (e.g., A) and the other at a location B whose position is desired. The measured pseudoranges at station A are compared to the actual values computed from the satellites' and receivers' positions using Eq. 4.23. The difference between the measured and the computed values give the pseudoranges corrections PRC, which are send from receiver A to receiver B via a communication link to correct the measured pseudoranges of station B, assuming that the stations are close enough such that the same errors affecting A also affect B. If the stations are close the atmospheric conditions are largely the same for both stations. DGPS positions are obtained in real-time at decimeter level accuracy

links) to correct its measured pseudoranges, which are then used to derive its position. For example, the measured pseudorange ϱ^j_{mb} at station B would be corrected by

$$\varrho^j_{Bcorr} = \varrho^j_{mb} + \Delta\varrho_A. \tag{4.25}$$

As an example (see, e.g., [6]), consider that the computed range, i.e., the distance between the satellite antenna and the receiver antenna ϱ^j_g for station A is 20,000,000 m and the measured code-pseudorange ϱ^j_{ma} is 19,999,990. Then, $\Delta\varrho_A = 20,000,000 - 19,999,990$ gives 10 as the pseudorange correction, which is transmitted to correct the pseudorange measured by the satellite at B using (4.25) as $\varrho^j_{Bcorr} = \varrho^j_{mb} + 10$. Pseudorange measurements to each satellite at location B is corrected in the same manner and the resulting corrected pseudoranges used to determine the position of B using Eq. (4.17). The coordinates of B will be relative to those of A, hence they belong to the same datum. If more than one reference station is used to obtain the pseudorange corrections, then the corrections may further be refined using the network of reference stations. A network of stations transmitting differential GPS corrections are termed "*augmented GPS*", which are discussed in Sect. 5.4.4.2.

Improvements to the positioning accuracy depend on the accuracy of the known station's location, the accuracy of the satellite positions, and the mode of operation (i.e., whether code or phase). For phase pseudorange, (4.25) becomes

$$\lambda\varphi^j_{Bcorr} = \varrho^j_{MB} + \Delta\varrho_A + \lambda N^j_{AB}. \tag{4.26}$$

Several countries have service providers who operate DGPS. The user only needs to have one receiver and use nearby DGPS station to obtain the corrections at some cost. In Australia, for example, DGPS services are provided by the Australian Maritime Safety Authority (AMSA), as well as private companies, e.g., FUGRO. DGPS with carrier-phase ranges is applicable for real-time kinematic operations where the ambiguity is resolved using a technique known as On-The-Fly (OTF) as discussed in Sect. 5.4.6.

4.3.3 Relative Positioning

In the DGPS technique discussed above, pseudorange corrections and range-rate changes computed from a known station are transmitted via communication link and used to correct those of the unknown station. In relative positioning, instead, the baseline vector is computed and added to the coordinates of the reference station to obtain those of the unknown station (see, e.g., Fig. 4.6). Also, in contrast to DGPS, which is performed in real-time, relative positioning requires post-processing of data, where simultaneous *carrier-phase observations* from both reference and rover stations are processed.

If one considers the positions of the known reference station A (Fig. 4.6) as

$$\mathbf{X}_A = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{4.27}$$

and the computed baseline vector as

$$\Delta\mathbf{X}_{AB} = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}, \tag{4.28}$$

then the position $\mathbf{X}_B$ of the rover station B is given by

$$\mathbf{X}_B = \mathbf{X}_A + \Delta\mathbf{X}_{AB}. \tag{4.29}$$

Relative positioning can be performed through *single*-, *double*-, and *triple* differencing in both static and kinematic modes (see Sect. 4.3.1).

1. In *single differencing*, ***two ground receivers*** occupying the reference station A and a rover station B simultaneously observe ***<u>one</u> satellite*** j (see, e.g., Fig. 4.7). Using Eq. (4.18), the two phase range equations are differenced (subtracting one from the other) as

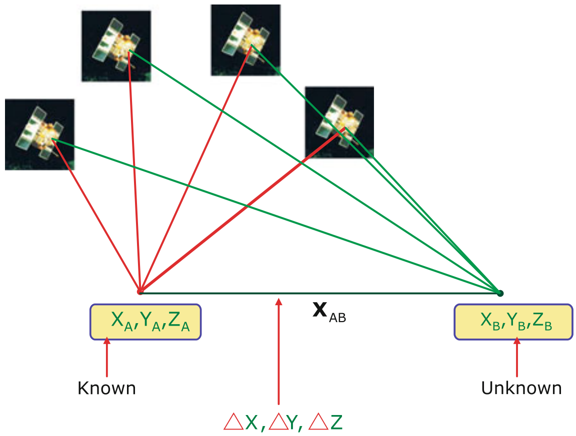

Fig. 4.6 Relative positioning. One receiver is placed at a reference station A whose position is known and the other at a rover location B whose position is sought. Baseline vectors are computed during post-processing of the data and added to the coordinates of A to obtain those of B. No radio link is required as in the case of DGPS (Fig. 4.5), and post-processing of the data is necessary hence positions are not obtained in real-time

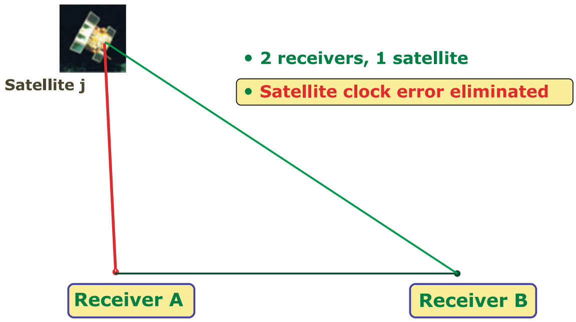

Fig. 4.7 Single differencing. Two receivers A and B simultaneously observing the same satellite j. This technique eliminates the satellite clock errors (e.g., Eq. 4.30)

$$\begin{array}{c}
\lambda \Delta\varphi_A^j(t) + c\delta^j(t) = \varrho_A^j(t) + c\delta_A(t) + \lambda N_A^j \\
\lambda \Delta\varphi_B^j(t) + c\delta^j(t) = \varrho_B^j(t) + c\delta_B(t) + \lambda N_B^j \\
\hline
\lambda\left[\Delta\varphi_B^j - \Delta\varphi_A^j\right](t) = \left[\varrho_B^j - \varrho_A^j\right](t) + c\left[\delta_B - \delta_A\right](t) + \lambda\left[N_B^j - N_A^j\right],
\end{array} \tag{4.30}$$

which simplifies to

$$\lambda \Delta\varphi_{AB}^j(t) = \varrho_{AB}^j(t) + c\delta_{AB}(t) + \lambda N_{AB}^j. \tag{4.31}$$

The significance of the results from Eq. (4.31) is that the *satellite clock bias* term $c\delta^j(t)$ is eliminated.

2. *Double differencing* refers to the case where, again, ***two ground receivers*** occupy the reference station A and rover station B, but now simultaneously observe ***two satellites, j and*** k (see, e.g., Fig. 4.8). In this case, (4.31) is written for the second satellite k as

$$\lambda \Delta\varphi_{AB}^k(t) = \varrho_{AB}^k(t) + c\delta_{AB}(t) + \lambda N_{AB}^k. \tag{4.32}$$

Differencing (4.31) and (4.32) leads to

$$\boxed{\begin{array}{c} \lambda\Delta\varphi^{j}_{AB}(t) = \varrho^{j}_{AB}(t) + c\delta_{AB}(t) + \lambda N^{j}_{AB} \\ \lambda\Delta\varphi^{k}_{AB}(t) = \varrho^{k}_{AB}(t) + c\delta_{AB}(t) + \lambda N^{k}_{AB} \\ \hline \lambda\left[\Delta\varphi^{k}_{AB} - \Delta\varphi^{j}_{AB}\right](t) = \left[\varrho^{k}_{AB} - \varrho^{j}_{AB}\right](t) + \lambda\left[N^{k}_{AB} - N^{k}_{AB}\right] \end{array}}, \quad (4.33)$$

which simplifies to

$$\lambda\Delta\varphi^{jk}_{AB}(t) = \varrho^{jk}_{AB}(t) + \lambda N^{jk}_{AB}. \quad (4.34)$$

Equation (4.34), known as the *double differencing equation*, is the most commonly used equation in processing GPS data. The importance of this equation is that both the satellite and receiver clock errors are eliminated.

3. Finally, double differencing can be done at two epochs, t_1 and t_2, to give a *triple difference* (see, e.g., Fig. 4.9). In this case, Eq. (4.34) will be written for both epochs and differenced to give

$$\begin{array}{c} \lambda\Delta\varphi^{jk}_{AB}(t_1) = \varrho^{jk}_{AB}(t_1) + \lambda N^{jk}_{AB} \\ \lambda\Delta\varphi^{jk}_{AB}(t_2) = \varrho^{jk}_{AB}(t_2) + \lambda N^{jk}_{AB} \\ \hline \lambda\left[\Delta\varphi^{jk}_{AB}(t_2) - \Delta\varphi^{jk}_{AB}(t_1)\right] = \varrho^{jk}_{AB}(t_2) - \varrho^{jk}_{AB}(t_1) \end{array}, \quad (4.35)$$

which simplifies to

$$\lambda\Delta\varphi^{jk}_{AB}(t_{12}) = \varrho^{jk}_{AB}(t_{12}), \quad (4.36)$$

where the *unknown ambiguity* term λN^{jk}_{AB} corresponding to ambiguities λN_A and λN_B for satellites j and k have been eliminated. Triple differencing is relevant to GPS positioning in that clock errors and the unknown integer ambiguity term N have been eliminated. Triple differencing is useful as an alternative approach for solving the unknown integer ambiguity term N and is often used to obtain the initial solutions of station coordinates in what is known as a *float solution*. These initial solutions are then used in the double differencing models to obtain a rigorous solution in what is termed the *fixed solution*. Section 6.2.3.1 presents detailed discussion on float and fixed solutions.

4.4 Concluding Remarks

In summary, this chapter has introduced the codes and phase GPS observation equations and has highlighted the similarities and differences between these equations. The differences occur in terms of the measured quantities (i.e., time and phase) and

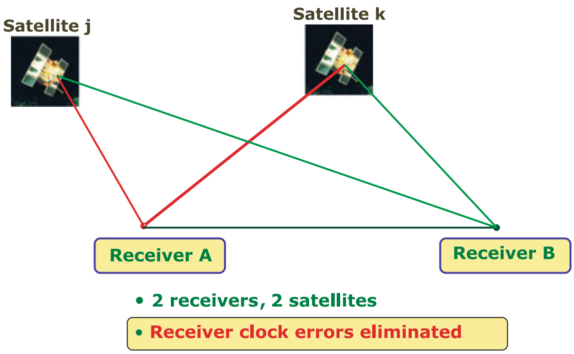

Fig. 4.8 Double differencing. Two receivers, A and B, simultaneously observe the same satellites j and k. This is the most commonly used Eq. (4.34) to process GPS data during post-processing

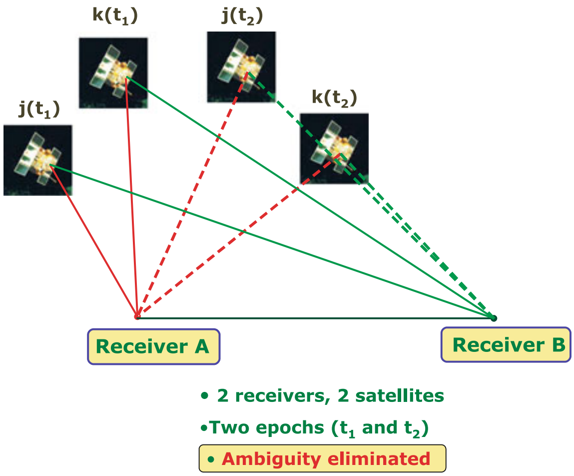

Fig. 4.9 Triple differencing. Two receivers, A and B, simultaneously observe the same satellites, j and k, at times t_1 and t_2. The advantage of this differenced solution is the capability of eliminating the unknown ambiguity term N leading to float solutions needed to obtain more accurate fixed solutions (see e.g., Sect. 6.2.3.1)

the addition of the unknown integer ambiguity term N on the phase equation. In general, the configurations confirm the well-known practise of using more than four satellites for positioning in both static and kinematic modes. Positioning accuracies can be improved by using DGPS or relative positioning techniques, which model errors associated with the atmosphere and clock errors. In particular, *single*, *double* and *triple differencing* eliminates satellite, receiver and ambiguity error terms, respectively.

It should be emphasized that the accuracies of phase observations are in the cm-mm level range, while those of code are in meters. Code-pseudorange observations are, however, relevant since they do not suffer from the unknown integer ambiguity and can thus be used to offer quick (or even instantaneous) environmental solutions that do not require higher accuracy, e.g., locating a waste damping site. They could also be used (similarly to triple differencing solutions) to provide the initial solutions required in the double differencing to resolve the integer ambiguity N in order to obtain fixed solutions. This could be of benefit in environmental monitoring tasks that require accurate observations. In the next chapter, we will provide a more in-depth discussion of the field procedures of the techniques discussed in this chapter.

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