Chapter 5
Riemann's Higher-Dimensional Geometry

5.1 The Legacy of Riemann

In mathematics we sometimes see striking examples of brilliant contributions or completely new ideas that change the ways mathematics develops in a significant fashion. A prime example of this is the work of Descartes [55], which completely changed how mathematicians looked at geometric problems. But it is rare that a single mathematician makes as many singular advances in his lifetime as did Bernhard Riemann in the middle of the nineteenth century. In this section we will discuss in some detail his fundamental creation of the theory of higher-dimensional manifolds and the additional creation of what is now called Riemannian geometry. In Part III we will review his contributions to complex analysis and complex geometry.

However, it is worth noting that he only published nine papers in his short lifetime (he lived to be only 40 years old); and several other important works, including those that concern us in this section, were published posthumously from the writings he left behind. His collected works (including in particular these posthumously published papers) were edited and published in 1876 and are still in print today [200].

In Figs. 5.1 and 5.2 we have reproduced the table of contents of Riemann's collected works [200]. Looking through the titles one is struck by the wide diversity as well as the originality. Let us give a few examples here. In Paper I (his dissertation) he formulated and proved the Riemann mapping theorem and dramatically moved the theory of functions of one complex variables in new directions. In Paper VI, in order to study Abelian functions, he formulated what became known as Riemann surfaces and this led to the general theory of complex manifolds in the twentieth century. In Paper VII he introduced the Riemann zeta function as a tool for studying the Prime Number Theorem and formulated the Riemann hypothesis, which is surely the outstanding mathematical problem in the world today. In Paper XII he formulated the first rigorous definition of a definite integral (the Riemann integral) and applied it to trigonometric series, setting the stage for Lebesgue and others in the early twentieth century to develop many consequences of the powerful theory of Fourier analysis. In Papers XIII and XXII he formulated the theory of higher-dimensional manifolds,

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Inhalt.

Erste Abtheilung.

Abhandlungen, die von Riemann selbst veröffentlicht sind.

Fig. 5.1 Table of contents, p. VI, Riemann's collected works [200]

including the important concepts of Riemannian metric, normal coordinates and the Riemann curvature tensor, which we will visit very soon in the sections below. Paper XVI contains correspondence with Enrico Betti leading to the first higher-dimensional topological invariants beyond those Riemann had earlier developed for two-dimensional manifolds.

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Fig. 5.2 Table of contents, p. II, Riemann's collected works [200]

This will suffice. The reader can glance at the other titles to see their further diversity. His contributions to the theory of partial differential equations and various problems in mathematical physics were also quite significant.

5.2 Higher-Dimensional Manifolds and a Quadratic Line Element

Riemann's paper "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen"[1] [201] (Paper XIII above) is a posthumously published version of a public lecture Riemann gave as his *Habilitationsvortrag* in 1854. This was part of the process for obtaining his *Habilitation*, a German advanced degree beyond the doctorate necessary to qualify for a professorship in Germany at the time (such requirements are still in place at most German universities today as well as in other European countries, e.g., France and Russia; it is similar to the research requirements in the US to be qualified for tenure). This paper, being a public lecture, has very few formulas, is at times quite philosophical and is amazing in its depth of vision and clarity. On the other hand, it is quite a difficult paper to understand in detail, as we shall see.

Before this paper was written, manifolds were all one- or two-dimensional curves and surfaces in $\mathbf{R}^3$, including their extension to points at infinity, as discussed in Chap. 3. In fact, some mathematicians who had to study systems parametrized by more than three variables declined to call the parametrization space a manifold or give such a parametrization a geometric significance. In addition, these one- and two-dimensional manifolds always had a differential geometric structure which was induced by the ambient Euclidean space (this was true for Gauss, as well).

In Riemann's paper [201] he discusses the distinction between discrete and continuous manifolds, where one can make comparisons of quantities by either counting or by measurement, and gives a hint, on p. 256, of concepts from set theory, which was only developed later in a single-handed effort by Cantor. Riemann begins his discussion of manifolds by moving a one-dimensional manifold, which he intuitively describes, in a transverse direction (moving in some type of undescribed ambient "space") to obtain a surface, and inductively, generating an n-dimensional manifold by moving an $n-1$-manifold transversally in the same manner. Conversely, he discusses having a nonconstant function on an n-dimensional manifold, and the set of points where the function is constant is (generically) a lower-dimensional manifold; and by varying the constant, one obtains a one-dimensional family of $n-1$-manifolds (similar to his construction above).[2]

Riemann formulates local coordinate systems $(x^1, x^2, \ldots, x^n)$ on a manifold of n dimensions near some given point, taken here to be the origin. He formulates

[1] "On the hypotheses, which are the basis for geometry".

[2] He alludes to some manifolds that cannot be described by a finite number of parameters; for instance, the manifold of all functions on a given domain, or all deformations of a spatial figure. Infinite-dimensional manifolds, such as these, were studied in great detail a century later.

a curve in the manifold as being simply n functions $(x^1(t), x^2(t), \ldots, x^n(t))$ of a single variable t. The concepts of set theory and topological space were developed only later in the nineteenth century, and so the global nature of manifolds is not really touched on by Riemann (except in his later work on Riemann surfaces and his correspondence with Betti, mentioned above). It seems clear on reading his paper that he thought of n-dimensional manifolds as being extended beyond Euclidean space in some manner, but the language for this was not yet available.

At the beginning of this paper Riemann acknowledges the difficulty he faces in formulating his new results. Here is a quote from the second page of his paper (p. 255) in [200]:

> Indem ich nun von diesen Aufgaben zunächst die erste, die Entwicklung des Begriffs mehrfach ausgedehnter Grössen, zu lösen versuche, glaube ich um so mehr auf eine nachsichtige Beurtheilung Anspruch machen zu dürfen; da ich in dergleichen Arbeiten philosophischer Natur, wo die Schwierigkeiten mehr in den Begriffen, als in der Construction liegen, wenig geübt bin und ich ausser einigen ganz kurzen Andeutungen, welche Herr Geheimer Hofrath Gauss in der zweiten Abhandlung über die biquadratischen Reste in den Göttingenschen gelehrten Anzeigen und in seiner Jubiläumsschrift darüber gegeben hat, und einigen philosophischen Untersuchungen Herbart's, durchaus keine Vorarbeiten benutzen konnte.[3]

The paper of Gauss that he cites here [78] refers to Gauss's dealing with the philosophical issue of understanding the complex number plane after some thirty years of experience with its development. We will mention this paper more explicitly in Sect. 6.3. Hebart was a philosopher whose metaphysical investigations influenced Riemann's thinking. Riemann was very aware of the speculative nature of his theory, and he used this philosophical point of view, as the technical language he needed (set theory and topological spaces) was not yet available. This was very similar to Gauss's struggle with the complex plane, as we shall see later.

As mentioned earlier, measurement of the length of curves goes back to the Archimedean study of the length of a circle. The basic idea there and up to the work of Gauss was to approximate a given curve by straight line segments and take a limit. The *length* of each straight line segment was determined by the Euclidean ambient space, and the formula, using calculus for the limiting process, became, in the plane for instance,

$$\int_{\Gamma} ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt,$$

[3] "In that my first task is to try to develop the concept of a multiply spread-out quantity [he uses the word 'Mannigfaltikeit' (manifold) later], I believe even more in being allowed an indulgent evaluation, as in such works of a philosophical nature, where the difficulties are more in the concepts than in the construction, wherein I have little experience, and except for the paper by Mr. Privy Councilor Gauss in his second commentary on biquadratic residues in the Göttingen gelehrten Anzeigen [1831] and in his Jubiläumsschrift and some investigations by Hebart, I have no precedents I could use."

where $ds^2 = dx^2 + dy^2$ is the line element of arc length in $\mathbf{R}^2$. As we saw in Chap. 4, Gauss formulated in [81] on a two-dimensional manifold with coordinates (p, q) the line element

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2, \tag{5.1}$$

where E, F, and G are induced from the ambient space. He didn't consider any examples of such a line element (5.1) that weren't induced from an ambient Euclidean space, but his remarks (see Gauss's quote in Sect. 4.2) clearly indicate that this could be a ripe area for study, and this could well include allowing coefficients of the line element (5.1) to be more general than induced from an ambient space.

Since Riemann formulated an abstract n-dimensional manifold (with a local coordinate system) with no ambient space, and since he wanted to be able to measure the length of a curve on his manifold, he formulated, or rather postulated, an independent measuring system which mimics Gauss's formula (5.1). Namely, he prescribes for a given local coordinate system a metric (line element) of the form

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j, \tag{5.2}$$

where $g_{ij}(x)$ is, for each x, a symmetric positive-definite matrix, and he postulates, by the usual change of variables formulas,

$$ds^2 = \sum_{i,j=1}^{n} \tilde{g}_{ij}(\tilde{x})d\tilde{x}^i d\tilde{x}^j, \tag{5.3}$$

where $\tilde{g}_{ij}(\tilde{x})$ is the transformed positive-definite matrix in the new coordinate system $(\tilde{x}_1, \ldots, \tilde{x}^n)$. This has the form

$$g_{kl}(x) = \sum_{ij} \tilde{g}_{ij}(\tilde{x}(x))\frac{\partial \tilde{x}_i}{\partial x_k}\frac{\partial \tilde{x}_j}{\partial x_l}. \tag{5.4}$$

Using the line element (5.2), the length of a curve is defined by

$$l(\Gamma) := \int_a^b \sqrt{\sum_{i,j=1}^{n} g_{ij}(x(t))\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t)dt}.$$

The line element (5.2) is what is called a *Riemannian metric* today, and the two-form ds^2 is considered as a positive-definite bilinear form giving an inner product on the tangent space $T_p(M)$ for p a point on the manifold M. This has become the basis for almost all of modern differential geometry (with the extension to Lorentzian type spaces where $g_{ij}(x)$ is not positive-definite *à la* Minkoswki space). Riemann merely says on p. 260 of his paper (no notation here at all),

> ich beschränke mich daher auf die Mannigfaltigkeiten, wo das Linienelement durch die Quadratwurzel aus einem Differentialausdruck zweiten Grades ausgedrückt wird.[4]

Earlier he had remarked that a line element should be homogeneous of degree 1 and one could also consider the fourth root of a differential expression of fourth degree, for instance. Hence his restriction in the quote above.

5.3 Geodesic Normal Coordinates and a Definition of Curvature

The next step in Riemann's paper is his formulation of curvature. This occurs on a single page (p. 261 of [201], which we reproduce here in Fig. 5.3). It is extremely dense and not at all easy to understand. In the published collected works of Riemann one finds an addendum to Riemann's paper which analyzes this one page in seven pages of computations written by Julius Wilhelm Richard Dedekind (1831–1916). This is an unpublished manuscript that appeared only in these collected works of Riemann, pp. 384–391. In Volume 2 of Spivak's three-volume comprehensive introduction to and history of differential geometry [218], we find a detailed analysis of Riemann's paper (as well as Gauss's papers that we discussed earlier and later important works of the nineteenth century in differential geometry, including translations into English of the most important papers).

We want to summarize what Riemann says on p. 261 (again, see Fig. 5.3). He starts by introducing near a given point p on his manifold M *geodesic normal coordinates*, that is, coordinates which are geodesics emanating from the given point and whose tangent vectors at p are an orthonormal basis for T_p (this orthogonality and the geodesics use, of course, the given Riemannian metric). In this coordinate system $(x^1, \ldots, x^n)$, the metric ds^2 has a Taylor expansion through second-order terms of the form

$$ds^2 = \sum_{i=1}^{n} dx^i dx^i + \frac{1}{2} \sum_{ijkl}^{n} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(0) x^k x^l dx^i dx^j. \tag{5.5}$$

The first-order terms in this expansion involve terms of the form $\frac{\partial g_{ij}}{\partial x^k}(0)$, all of which vanish, which follows from the geodesic coordinates condition. Letting now

$$c_{ijkl} := \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(0),$$

[4] "I restrict myself therefore to manifolds where the line element is expressed by the square root of a differential expression of second degree."

sehen zu können, ist es nöthig, die von der Darstellungsweise herrührenden zu beseitigen, was durch Wahl der veränderlichen Grössen nach einem bestimmten Princip erreicht wird.

2.

Zu diesem Ende denke man sich von einem beliebigen Punkte aus das System der von ihm ausgehenden kürzesten Linien construirt; die Lage eines unbestimmten Punktes wird dann bestimmt werden können durch die Anfangsrichtung der kürzesten Linie, in welcher er liegt, und durch seine Entfernung in derselben vom Anfangspunkte und kann daher durch die Verhältnisse der Grössen dx^0, d. h. der Grössen dx im Anfang dieser kürzesten Linie und durch die Länge s dieser Linie ausgedrückt werden. Man führe nun statt dx^0 solche aus ihnen gebildete lineäre Ausdrücke $d\alpha$ ein, dass der Anfangswerth des Quadrats des Linienelements gleich der Summe der Quadrate dieser Ausdrücke wird, so dass die unabhängigen Variabeln sind: die Grösse s und die Verhältnisse der Grössen $d\alpha$; und setze schliesslich statt $d\alpha$ solche ihnen proportionale Grössen $x_1, x_2, \ldots, x_n$, dass die Quadratsumme $= s^2$ wird. Führt man diese Grössen ein, so wird für unendlich kleine Werthe von x das Quadrat des Linienelements $= \Sigma dx^2$, das Glied der nächsten Ordnung in demselben aber gleich einem homogenen Ausdruck zweiten Grades der $n\frac{n-1}{2}$ Grössen $(x_1\,dx_2 - x_2\,dx_1)$, $(x_1\,dx_3 - x_3\,dx_1)$, $\ldots$, also eine unendlich kleine Grösse von der vierten Dimension, so dass man eine endliche Grösse erhält, wenn man sie durch das Quadrat des unendlich kleinen Dreiecks dividirt, in dessen Eckpunkten die Werthe der Veränderlichen sind $(0, 0, 0, \ldots)$, $(x_1, x_2, x_3, \ldots)$, $(dx_1, dx_2, dx_3, \ldots)$. Diese Grösse behält denselben Werth, so lange die Grössen x und dx in denselben binären Linearformen enthalten sind, oder so lange die beiden kürzesten Linien von den Werthen 0 bis zu den Werthen x und von den Werthen 0 bis zu den Werthen dx in demselben Flächenelement bleiben, und hängt also nur von Ort und Richtung desselben ab. Sie wird offenbar $= 0$, wenn die dargestellte Mannigfaltigkeit eben, d. h. das Quadrat des Linienelements auf Σdx^2 reducirbar ist, und kann daher als das Mass der in diesem Punkte in dieser Flächenrichtung stattfindenden Abweichung der Mannigfaltigkeit von der Ebenheit angesehen werden. Multiplicirt mit $-\frac{3}{4}$ wird sie der Grösse gleich, welche Herr Geheimer Hofrath Gauss das Krümmungsmass einer Fläche genannt hat. Zur Bestimmung der Massverhältnisse einer $n$fach ausgedehnten in der vorausgesetzten Form darstellbaren Mannigfaltigkeit wurden vorhin $n\frac{n-1}{2}$ Functionen des Orts nöthig gefunden;

Fig. 5.3 Page 261 of Riemann's foundational paper on differential geometry [201] we have the natural symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{jilk},$$

due to the symmetry of the indices in g_{ij} and in the commutation of the second-order partial derivatives. Moreover, and this is *not* easy to verify, the coefficients *also* satisfy

$$\begin{aligned} c_{ijkl} &= c_{klij}, \\ c_{lijk} + c_{ljki} + c_{lkij} &= 0. \end{aligned} \tag{5.6}$$

This is proved in six pages of computation in Spivak's Vol. 2 (pp. 172–178 of [218]), and we quote from the top of p. 174: "We now proceed to the hardest part of the computation, a hairy computation indeed." These symmetry conditions use the fact that the coordinates are specifically linked to the metric (our geodesic coordinates). For instance, on p. 175 Spivak points out that

$$x^i = \sum_{j=1}^{n} g_{ij} x^j,$$

illustrating vividly the relation between the coordinates and the metric.

Let now

$$Q(x, dx) := \sum_{ijkl} c_{ijkl} x^k x^l dx^i dx^j \tag{5.7}$$

be the biquadratic form defined by the second-order terms in (5.5), then Riemann asserts on p. 61 of [201] that $Q(x, dx)$ can be expressed in terms of the $n(\frac{n-1}{2})$ expressions $\{(x^1dx^2 - x^2dx^1), (x^1dx^3 - x^3dx^1), \ldots\}$ (see Fig. 5.3), that is,

$$Q(x, dx) = \sum_{ijkl}^{n} C_{ijkl}(x^i dx^j - x^j dx^i)(x^k dx^l - d^l dx^k). \tag{5.8}$$

Spivak proves that the conditions (5.6) are necessary and sufficient for $Q(x, dx)$ to be expressed in the form (5.8), and he shows, moreover, that

$$C_{ijkl} = \frac{1}{3} c_{ijkl}.$$

Riemann simply asserts that this is the case, which is, of course, indeed true!

The expression $Q(x, dx)$ defined by (5.8) is Riemann's definition of *curvature* for the manifold at the point 0 defined by the metric (5.2) using geodesic normal coordinates. This has become known as *Riemannian curvature* ever since.

Let's look at the special case where the manifold has two dimensions. In this case we see that there is only one coefficient of the nonzero term $(x^1dx^2 - x^2dx^1)^2$ which has the form

$$Q(x, dx) = \frac{1}{3}[c_{2211} + c_{1122} - c_{2112} - c_{1221}](x^1 dx^2 - x^2 dx^1)^2.$$

Now using Gauss's notation for the Riemannian metric (5.1), that is, $g_{11} = E$, $g_{12} = g_{21} = F$, and $g_{22} = G$, we see that

$$\begin{aligned} c_{2211} &= \frac{1}{2} G_{xx}, \\ c_{1122} &= \frac{1}{2} E_{yy}, \\ c_{2112} &= \frac{1}{2} F_{xy}. \\ c_{1221} &= \frac{1}{2} F_{xy}, \end{aligned}$$

and thus we have

$$Q(x, dx) = \frac{1}{6}[G_{xx} + E_{yy} - 2F_{xy}].$$

Looking at Gauss's formula for Gaussian curvature at the point 0 (4.10), we see that, since the first derivatives of the metric vanish at the origin, the curvature at $x = 0$ is

$$k = -\frac{1}{2}(G_{xx} G_{yy} - 2F_{xy}), \tag{5.9}$$

and hence

$$Q(x, dx) = -\frac{k}{3}(x^1 dx^2 - x^2 dx^1)^2. \tag{5.10}$$

Thus the coefficient of the single term $(x^1 dx^2 - x^2 dx^1)^2$ in the biquadratic form $Q(x, dx)$ is, up to a constant, the Gaussian curvature. As Riemann asserts it (and we paraphrase here): 'divide the expression $Q(x, dx)$ by the square of the area of the (infinitesimal) triangle formed by the three points $(0, x, dx)$, and the result of the division is $-\frac{4}{3}k$'. The factor 4 appears since the square of the area of the infinitesimal parallelogram[5] is $(x^1 dx^2 - x^2 dx^1)^2$, and thus the square of the area of the infinitesimal triangle is $\frac{1}{4}(x^1 dx^2 - x^2 dx^1)^2$. This yields the relation between Riemann's coefficient in (5.10) and Gaussian curvature (one can see this coefficient of $-\frac{3}{4}$ near the bottom of p. 261 in Fig. 5.3). Namely, except for a constant factor, Riemann's curvature expressed in normal coordinates on a two-dimensional manifold coincides with Gaussian curvature.

Riemann then considers the biquadratic form $Q(x, dx)$ in an n-dimensional manifold M and its restriction to any two-dimensional submanifold N passing through the point p, obtaining a curvature (constant multiple of the Gaussian

[5] Riemann visualizes the parallelogram formed by the points $(0, x, dx, x + dx)$ in $\mathbf{R}^2$, and the area of such a rectangle is simply given by the cross product $\|x \times dx\| = \|x^1 dx^2 - x^2 dx^2\|$, and the area of the triangle formed $(0, x, dx)$ is $\frac{1}{2}\|x^1 dx^2 - x^2 dx^1\|$.

curvature as we saw above) for the submanifold at that point. This is the sectional curvature of Riemann, introduced on this same p. 261.

In the remainder of the paper he discusses questions of flat manifolds, manifolds of positive or negative constant curvature, and numerous other questions.

The coefficients $\{c_{ijkl}\}$ in (5.7) or C_{ijkl} in (5.8) are effectively the components of the Riemannian curvature tensor for this special type of coordinate system (geodesic normal coordinates). How does one define such a curvature tensor for n-dimensional manifolds with a Riemannian metric in a general coordinate system (in the spirit of Gauss's curvature formula (4.10))? Clearly this will involve the first derivatives of the Riemannian metric as well. In a paper written in Latin for a particular mathematical prize in Paris (Paper No. XXII in Fig. 5.2), Riemann provides the first glimpse of the general Riemann curvature tensor, and this is again translated and elaborated on by Spivak [218]. The purpose of this paper was to answer a question in the Paris competition dealing with the flow of heat in a homogeneous solid body.

Riemann's ideas in these two posthumously published papers were developed and expanded considerably in the following decades in the work of Christoffel, Levi-Cevita, Ricci, Beltrami and many others. This is all discussed very elegantly in Spivak's treatise [218], and we won't elaborate on this any further at this point. The main point of our discussion has been that Riemann created on these few pages the basic idea of an n-dimensional manifold not considered as a subset of Euclidean space *and* of the independent concept of a Riemannian metric and the Riemann curvature tensor. What is missing at this point in time is the notion of a topological space on the basis of which one could formulate the contemporary concepts of a differentiable manifold or a Riemannian manifold.