Canonical Duality Method for Solving Kantorovich Mass Transfer Problem

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Abstract This paper addresses analytical solution to the Kantorovich mass transfer problem. Through an ingenious approximation mechanism, the Kantorovich problem is first reformulated as a variational form, which is equivalent to a nonlinear differential equation with Dirichlet boundary. The existence and uniqueness of the solution can be demonstrated by applying the canonical duality theory. Then, using the canonical dual transformation, a perfect dual maximization problem is obtained, which leads to an analytical solution to the primal problem. Its global extremality for both primal and dual problems can be identified by a triality theory. In addition, numerical maximizers for the Kantorovich problem are provided under different circumstances. Finally, the theoretical results are verified by applications to Monge's problem. Although the problem is addressed in one-dimensional space, the theory and method can be generalized to solve high-dimensional problems.

1 Introduction

The Monge–Kantorovich mass transfer model is widely used in modern economic activities, medical science, and mechanical processes. In these respects, some typical examples include the logistics of transport for industrial products, purification of blood in the kidneys and livers, shape optimization, etc. Interesting readers can refer to [1, 2, 9, 23, 24, 28, 29] for more details.

The original transfer problem, which was proposed by Monge [28], investigated how to move one mass distribution to another one with the least amount of work. In this paper, we consider the Monge–Kantorovich problem in the 1-D case. Let

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$\Omega = [a, b]$ and $\Omega^* = [c, d]$, $a, b, c, d \in \mathbb{R}$ and denote $U := \Omega \cup \Omega^* = [a, b] \cup [c, d]$. Here we focus on the closed case, and other bounded cases can be discussed similarly. Moreover, f^+ and f^- are two nonnegative density functions in Ω and Ω^*, respectively, and satisfy the normalized balance condition

$$\int_\Omega f^+ dx = \int_{\Omega^*} f^- dx = 1.$$

Let $c : \Omega \times \Omega^* \to [0, +\infty)$ be a cost function, which indicates the work required to move a unit mass from the position x to a new position y. There are many types of cost functions while dealing with different problems [2, 5, 9, 27]. In Monge's problem, the cost function is proportional to the distance $|x - y|$,

$$c(x, y) = |x - y|.$$

The Monge's problem consists in finding an optimal mass transfer mapping $\mathbf{s}^* : \Omega \to \Omega^*$ to minimize the cost functional $I(\mathbf{s})$:

$$I(\mathbf{s}^*) = \min_{\mathbf{s} \in \mathcal{N}} \left\{ I[\mathbf{s}] := \int_\Omega |x - \mathbf{s}(x)| f^+(x) dx \right\}, \tag{1}$$

where $\mathbf{s} : \Omega \to \Omega^*$ belongs to the class $\mathcal{N}$ of measurable mappings driving $f^+(x)$ to $f^-(y)$,

$$\mathbf{s}_\# f^+ = f^-,$$

which means, for $\forall x \in \Omega$,

$$f^+(x) = f^-(\mathbf{s}(x)) |\det(J(\mathbf{s}(x)))|,$$

where $J(\mathbf{s}(x))$ is the Jacobian matrix of the mapping $\mathbf{s}$.

In the 1940s, Kantorovich [23, 24] relaxed Monge's transfer problem (1) and proposed the task of finding a Kantorovich potential $u^* \in \mathcal{L}$ solving

$$K[u^*] = \max_{w \in \mathcal{L}} \left\{ K[w] := \int_U w f dz = \int_U w(f^+ - f^-) dz \right\}, \tag{2}$$

where $\mathcal{L}$ is the class of functionals $w : U \to \mathbb{R}$ satisfying

$$\mathrm{Lip}[w] := \sup_{x \neq y} \frac{|w(x) - w(y)|}{|x - y|} \leq 1.$$

As a matter of fact, the Kantorovich's problem (2) is not a perfect maximization dual of Monge's minimization problem (1). Following the procedure of [5, 9], one can prove the *dual criteria for optimality* in the bounded case.

Lemma 1.1. *Let* $\mathbf{s}^* \in \mathscr{N}$ *and* $u^* \in \mathscr{L}$. *If the following identity holds,*

$$u^*(x) - u^*(\mathbf{s}^*(x)) = |x - \mathbf{s}^*(x)|,$$

then

- $\mathbf{s}^*$ *is an optimal mass transfer mapping in Monge's problem (1);*
- u^* *is a Kantorovich potential maximizing Kantorovich's problem (2);*
- *The minimum* $I[\mathbf{s}^*]$ *in (1) is equal to the maximum* $K[u^*]$ *in (2);*
- *Every optimal mass transfer mapping* $\mathbf{s}^*$ *and Kantorovich potential* u^* *satisfy the above identity.*

Due to the implicitness of u^*, L.C. Evans, W. Gangbo, and J. Moser [6, 8, 9] provided an ODE recipe to build $\mathbf{s}^*$ by solving a flow problem involving Du. This method is indeed useful but very complicated. In 2001, L.A. Caffarelli, M. Feldman, R.J. McCann, N.S. Trudinger, and X.J. Wang showed a much simpler approach to construct optimal mappings by decomposition of transfer sets and measure theory [5, 30]. Once an analytical Kantorovich potential u^* is found, by checking the above identity, one can immediately judge whether it is possible to construct a suitable optimal mapping $\mathbf{s}^*$ by virtue of u^*. However, due to the nonuniform convexity of the cost function $c(x, y)$, it is difficult to find optimal mass allocation. In order to gain some insight into this problem, many approximating mechanisms were introduced. For example, L.A. Caffarelli, W. Gangbo, R.J. McCann and X.J. Wang [4, 13, 14, 30], etc. utilized an approximation of strictly convex cost functions

$$c_\varepsilon(x, y) = |x - y|^{1+\varepsilon} \quad \varepsilon > 0.$$

The existence and uniqueness of the optimal mapping $\mathbf{s}^*_\varepsilon$ can be proved by convex analysis. Then let ε tends to 0, and one can construct an optimal mapping $\mathbf{s}^*$ using transfer rays and transfer sets invoked by L.C. Evans and W. Gangbo [8]. In addition, N.S. Trudinger and X.J. Wang used the approximation

$$c_\varepsilon(x, y) = \sqrt{\varepsilon^2 + |x - y|^2}$$

in the discussion of regularity [27, 30]. All the above-mentioned approximations concentrate upon the cost function $c(x, y)$. In this paper, we are eager to explore whether the approximation of Kantorovich's problem can bring more useful information.

Let $\mathscr{L}_0$ be a subset of $\mathscr{L}$,

$$\mathscr{L}_0 := \left\{ \phi \in W_0^{2,\infty}(U) \cap C(U) \middle| \, |\phi_x| \le 1, \phi = 0 \text{ on } \Omega \cap \Omega^* \right\},$$

where $W_0^{2,\infty}(U)$ is a Sobolev space. Here, when $\Omega \cap \Omega^* = \emptyset$, $C(U)$ represents $C(\Omega)$ and $C(\Omega^*)$. We restrict our discussion of Kantorovich's problem (2) in $\mathscr{L}_0$, namely,

$$K[u] = \max_{w \in \mathscr{L}_0} \Big\{ K[w] := \int_U w f dz = \int_U w(f^+ - f^-) dz \Big\}. \tag{3}$$

In the survey paper [10], L.C. Evans proposed a sequence of approximated dual problems of (3). Now we explain the mechanism. We consider a sequence of approximated primal problems

$$(\mathscr{P}^{(k)}) : \min_{w_k \in \mathscr{L}_0} \Big\{ J^{(k)}[w_k] := \int_U \Big(H^{(k)}(w_{k,x}) - w_k f \Big) dx \Big\}, \tag{4}$$

where $w_{k,x}$ is the derivative of w_k with respect to x, $H^{(k)} : \mathbb{R} \to \mathbb{R}^+$ is defined as

$$H^{(k)}(\gamma) := \frac{1}{k} \mathrm{e}^{\frac{k}{2}(\gamma^2 - 1)},$$

and $J^{(k)}$ is called the potential energy functional. Notice that when $|\gamma| \le 1$, then $\lim_{k \to \infty} H^{(k)}(\gamma) = 0$ uniformly. From [10], it is clear that

$$- \lim_{k \to \infty} \min_{w_k \in \mathscr{L}_0} J^{(k)}[w_k] = \max_{w \in \mathscr{L}_0} K[w].$$

Consequently, once a sequence of functions $\{u_k^*\}_k$ satisfying $J^{(k)}[u_k^*] = \min_{w_k \in \mathscr{L}_0} J^{(k)}[w_k]$ globally is obtained, then it will help us find a Kantorovich potential $u = \lim_{k \to \infty} u_k^*$ which solves (3).

In this paper, we investigate analytic solutions to the Kantorovich potential u^* of problem (3) using *canonical duality theory*. This theory was developed from Gao and Strang's original work on nonconvex/nonsmooth variational problems [21]. During the last few years, considerable effort has been taken to illustrate these nonconvex problems from the theoretical point of view [16, 17]. Interesting readers can refer to [18–20, 22].

Before we state the main results, we introduce some useful notations.

- θ_k is the corresponding Gâteaux derivative of $H^{(k)}$ with respect to $w_{k,x}$ given by

$$\theta_k(x) = \mathrm{e}^{\frac{k}{2}(w_{k,x}^2 - 1)} w_{k,x}.$$

- $\Phi^{(k)} : \mathscr{L}_0 \to L^\infty(U)$ is a nonlinear geometric mapping given by

$$\Phi^{(k)}(w_k) := \frac{k}{2}(w_{k,x}^2 - 1).$$

For convenience's sake, denote

$$\xi_k := \Phi^{(k)}(w_k).$$

It is evident that ξ_k belongs to the function space $\mathscr{U}$ given by

$$\mathscr{U} := \left\{\phi \in L^\infty(U) \middle| \phi \leq 0\right\}.$$

- $\Psi^{(k)} : \mathscr{U} \to L^\infty(U)$ is a canonical energy defined as

$$\Psi^{(k)}(\xi_k) := \frac{1}{k}\mathrm{e}^{\xi_k},$$

which is a convex function with respect to ξ_k.
- ζ_k is the corresponding Gâteaux derivative of $\Psi^{(k)}$ with respect to ξ_k given by

$$\zeta_k = \frac{1}{k}\mathrm{e}^{\xi_k},$$

which is invertible with respect to ξ_k and belongs to the function space $\mathscr{V}^{(k)}$,

$$\mathscr{V}^{(k)} := \left\{\phi \in L^\infty(U) \middle| 0 \leq \phi \leq \frac{1}{k}\right\}.$$

- λ_k is defined as

$$\lambda_k := k\zeta_k,$$

and belongs to the function space $\mathscr{V}$,

$$\mathscr{V} := \left\{\phi \in L^\infty(U) \middle| 0 \leq \phi \leq 1\right\}.$$

Now we are ready to introduce the main theorems.

Theorem 1.2. *For positive density functions $f^+ \in C(\Omega)$, $f^- \in C(\Omega^*)$, we can always find a sequence of analytical functions $\{u_k^* \in \mathscr{L}_0\}_k$ minimizing the approximated problems (4) globally.*

By canonical duality method, we are able to find an analytical Kantorovich potential for (3).

Theorem 1.3. *For positive density functions $f^+ \in C(\Omega)$, $f^- \in C(\Omega^*)$, we can always find an analytical global maximizer $u \in \mathscr{L}_0$ for Kantorovich's mass transfer problem (3).*

Remark 1.4. *Generally speaking, there are plenty of approximating schemes, for example, one can also let*

$$H^{(k)}(\gamma) := \frac{1}{k}(\gamma^2 - 1)^2.$$

Then by following the procedure in dealing with double-well potentials in [19, 21], we could definitely find an analytical Kantorovich potential.

Remark 1.5. *Through applying the canonical duality method, we have devised a systematic procedure in finding an analytical minimizer. In fact, for other types of cost functions, for instance,* $c(x, y) = |x - y|^p, \quad p \in [1, \infty)$*, we can also use this method to construct an analytical Kantorovich potential* u^**. Compared with former results [8, 9], we obtain an explicit representation of Kantorovich potential, which helps us construct an optimal mapping* $\mathbf{s}^*$ *according to Lemma 1.1. This question will be discussed in detail in the application part.*

The rest of the paper is organized as follows. In Sect. 2, first we apply the canonical dual transformation to establish a sequence of perfect dual problems and a pure complementary energy principle. Next we explain the canonical duality theory and triality theory. In particular, the triality theory provides global extremum conditions for the problem (4). Afterward, we construct a sequence of analytical functions minimizing $J^{(k)}$ globally and a Kantorovich potential maximizing $K[w]$ of (3). In the final analysis, we use a product allocation model in 1-D to illustrate our theoretical results.

2 Proof of the Main Results: Technique of Canonical Duality Method

2.1 Proof of Lemma 1.1 in the Bounded Case:

Proof. Similar as [5], for any $\mathbf{s} \in \mathscr{N}$ and $w \in \mathscr{L}_0$, we compute

$$\begin{aligned} I[\mathbf{s}] &= \int_\Omega |x - \mathbf{s}(x)| f^+(x) dx \\ &= \int_U |x - \mathbf{s}(x)| f^+(x) dx \\ &\geq \int_U (w(x) - w(\mathbf{s}(x))) f^+(x) dx \\ &= \int_U w(x) f^+(x) dx - \int_U w(y) f^-(y) dy \\ &= K[w]. \end{aligned}$$

Taking into account the given identity, we complete the proof.

2.2 Proof of Theorem 1.2:

Here we apply the variational method to discuss problem (4). Now we show an important lemma in this respect.

Lemma 2.1. *The Euler–Lagrange equation for* $(\mathscr{P}^{(k)})$ *takes the following form,*

$$\theta_{k,x} + f = (\mathrm{e}^{\frac{k}{2}(u_{k,x}^2-1)} u_{k,x})_x + f = 0, \quad \textit{in } U. \tag{5}$$

Remark 2.2. *The term* $\mathrm{e}^{\frac{k}{2}(u_{k,x}^2-1)}$ *is called the transport density. Clearly, like* $p-$*Laplace operator,* $\mathrm{e}^{\frac{k}{2}(u_{k,x}^2-1)}$ *is a highly nonlinear and nonlocal function of* $u_k \in \mathscr{L}_0$*. With the hidden boundary value* $u_k = 0$ *on* ∂U*, we are able to prove the existence and uniqueness of the solution of (5). This important fact will be explained later.*

Proof. Indeed, the Gâteaux derivative of $J^{(k)}$ with respect to u_k belongs to $L^1(U)$. For any given $\mu > 0$ and any test function $\phi \in \mathscr{L}_0$, by integrating by parts, we have

$$\lim_{\mu\to 0^+} \frac{J^{(k)}[u_k + \mu\phi] - J^{(k)}[u_k]}{\mu}$$

$$= \lim_{\mu\to 0^+} \int_U \left\{ \frac{\frac{1}{k}\mathrm{e}^{\frac{k}{2}((u_k+\mu\phi)_x^2-1)} - \frac{1}{k}\mathrm{e}^{\frac{k}{2}(u_{k,x}^2-1)}}{\mu} - \phi f \right\} dx$$

$$= \int_U \left\{ \frac{1}{k}\mathrm{e}^{\frac{k}{2}(u_{k,x}^2-1)} \lim_{\mu\to 0^+} \frac{\mathrm{e}^{\frac{k}{2}(\mu^2\phi_x^2+2\mu u_{k,x}\phi_x)} - 1}{\frac{k}{2}(\mu^2\phi_x^2 + 2\mu u_{k,x}\phi_x)} \cdot \frac{\frac{k}{2}(\mu^2\phi_x^2 + 2\mu u_{k,x}\phi_x)}{\mu} - \phi f \right\} dx$$

$$= \int_U \left\{ \frac{1}{k}\mathrm{e}^{\frac{k}{2}(u_{k,x}^2-1)} k u_{k,x}\phi_x - \phi f \right\} dx$$

$$= -\int_U \left\{ (\mathrm{e}^{\frac{k}{2}(u_{k,x}^2-1)} u_{k,x})_x \phi + \phi f \right\} dx.$$

Actually, since u_k and ϕ are both in $\mathscr{L}_0$, then for any given $\mu < 0$, when $\mu \to 0^-$, the above calculation still holds.

Now we are going to apply the canonical duality method invoked by David Y. Gao [17]. By Legendre transformation, we define a *Gao–Strang total complementary energy functional.*

Definition 2.3. With the notations in Sect. 1, we define a Gao–Strang total complementary energy $\Xi^{(k)}$ in the form

$$\Xi^{(k)}(u_k, \zeta_k) := \int_U \left\{ \Phi^{(k)}(u_k)\zeta_k - \Psi_*^{(k)}(\zeta_k) - f u_k \right\} dx, \tag{6}$$

where the function $\Psi_*^{(k)} : \mathscr{V}^{(k)} \to L^\infty(U)$ is defined as

$$\Psi_*^{(k)}(\zeta_k) := \xi_k\zeta_k - \Psi^{(k)}(\xi_k) = \zeta_k(\ln(k\zeta_k) - 1). \tag{7}$$

Next we introduce an important *criticality criterium* for the Gao–Strang total complementary energy functional.

Definition 2.4. $(\bar{u}_k, \bar{\zeta}_k) \in \mathscr{L}_0 \times \mathscr{V}^{(k)}$ is called a critical pair of $\Xi^{(k)}$ if and only if

$$D_{u_k} \Xi^{(k)}(\bar{u}_k, \bar{\zeta}_k) = 0, \tag{8}$$

and

$$D_{\zeta_k} \Xi^{(k)}(\bar{u}_k, \bar{\zeta}_k) = 0, \tag{9}$$

where D_{u_k}, D_{ζ_k} denote the partial Gâteaux derivatives of $\Xi^{(k)}$, respectively.

By variational method, we explore the criticality criterium (8) and (9). Indeed, on the one hand, we have the following observation from (8).

Lemma 2.5. *For a fixed $\zeta_k \in \mathscr{V}^{(k)}$, (8) leads to the equilibrium equation*

$$(\lambda_k \bar{u}_{k,x})_x + f = 0, \quad in\ U. \tag{10}$$

Remark 2.6. *It is easy to check the equilibrium equation (10) is consistent with (5) except that the transport density is replaced by $\lambda_k = k\zeta_k$. We will use this fact to construct a sequence of analytical solutions later.*

Proof. Indeed, the partial Gâteaux derivative of $\Xi^{(k)}$ with respect to u_k belongs to $L^1(U)$. For $\forall \mu > 0$ and any test function $\phi \in \mathscr{L}_0$, by integrating by parts, we have

$$\begin{aligned}
&\lim_{\mu \to 0^+} \frac{\Xi^{(k)}(\bar{u}_k + \mu\phi, \zeta_k) - \Xi^{(k)}(\bar{u}_k, \zeta_k)}{\mu} \\
&= \lim_{\mu \to 0^+} \int_U \frac{\Lambda^{(k)}(\bar{u}_k + \mu\phi) - \Lambda^{(k)}(\bar{u}_k)}{\mu} \zeta_k dx - \int_U f\phi dx \\
&= \lim_{\mu \to 0^+} \int_U \frac{k(\bar{u}_{k,x} + \mu\phi_x)^2 - \bar{u}_{k,x}^2)}{2\mu} \zeta_k dx - \int_U f\phi dx \\
&= \int_U k\bar{u}_{k,x}\phi_x \zeta_k dx - \int_U f\phi dx \\
&= -\int_U \Big\{ (k\zeta_k \bar{u}_{k,x})_x \phi dx + f\phi \Big\} dx.
\end{aligned}$$

Since u_k and ϕ are both in $\mathscr{L}_0$, then for $\forall \mu < 0$, when $\mu \to 0^-$, the above calculation still holds.

On the other hand, from (9), we have the following observation.

Lemma 2.7. *For a fixed $u_k \in \mathscr{L}_0$, (9) is in fact the constructive law*

$$\Phi^{(k)}(u_k) = D_{\zeta_k}\Psi_*^{(k)}(\bar{\zeta}_k). \tag{11}$$

Remark 2.8. *It is worth noticing that (11) is consistent with the notations in Sect. 1.*

Proof. Indeed, the partial Gâteaux derivative of $\Xi^{(k)}$ with respect to ζ_k belongs to $L^1(U)$. For $\forall\mu > 0$ and any test function $\phi \in \mathscr{L}_0$, by integrating by parts, we have

$$\begin{aligned}
&\lim_{\mu\to 0^+} \frac{\Xi^{(k)}(u_k, \bar{\zeta}_k + \mu\phi) - \Xi^{(k)}(u_k, \bar{\zeta}_k)}{\mu} \\
&= \lim_{\mu\to 0^+} \int_U \left\{\Phi^{(k)}(u_k)\phi - \frac{\Psi_*^{(k)}(\bar{\zeta}_k + \mu\phi) - \Psi_*^{(k)}(\bar{\zeta}_k)}{\mu}\right\}dx \\
&= \int_U \left\{\Phi^{(k)}(u_k) - D_{\zeta_k}\Psi_*^{(k)}(\bar{\zeta}_k)\right\}\phi dx.
\end{aligned}$$

Since u_k and ϕ are both in $\mathscr{L}_0$, then for $\forall\mu < 0$, when $\mu \to 0^-$, the above calculation still holds.

Lemmas 2.5 and 2.7 indicate that $\bar{u}_k$ from the critical pair $(\bar{u}_k, \bar{\zeta}_k)$ solves (5). Now we introduce the canonical duality theory. For our purpose, we define the following *Gao–Strang pure complementary energy functional*.

Definition 2.9. From Definition 2.3, we define a Gao–Strang pure complementary energy $J_d^{(k)}$ in the form

$$J_d^{(k)}[\zeta_k] := \Xi^{(k)}(\bar{u}_k, \zeta_k), \tag{12}$$

where $\bar{u}_k$ solves (5).

For the sake of convenience, we give another representation of $J_d^{(k)}$ by the following lemma.

Lemma 2.10. *The pure complementary energy functional $J_d^{(k)}$ can be rewritten as*

$$J_d^{(k)}[\zeta_k] = -\frac{1}{2}\int_U \left\{\frac{\theta_k^2}{k\zeta_k} + k\zeta_k + 2\zeta_k(\ln(k\zeta_k) - 1)\right\}dx. \tag{13}$$

Remark 2.11. *$\bar{u}_k$ is included in this representation in an implicit manner, which will simplify our further discussion considerably.*

Proof. With Definition 2.3, by integrating by parts, we have

$$
\begin{aligned}
\Xi^{(k)}(\bar{u}_k,\zeta_k) &= \int_U \Big\{\Big(\frac{k}{2}(\bar{u}_{k,x}^2-1\Big)\zeta_k - \Psi_*^{(k)}(\zeta_k) - f\bar{u}_k\Big\}dx \\
&= \int_U \Big\{k\zeta_k\bar{u}_{k,x}^2 - f\bar{u}_k\Big\}dx \\
&\quad - \textstyle\int_U \Big\{\frac{k}{2}(\bar{u}_{k,x}^2-1)\zeta_k + k\zeta_k + \zeta_k(\ln(k\zeta_k)-1)\Big\}dx \\
&= \underbrace{-\int_U \Big\{(k\zeta_k\bar{u}_{k,x})_x + f\Big\}\bar{u}_k dx}_{(I)} \\
&\quad \underbrace{-\frac{1}{2}\int_U \Big\{k\zeta_k\bar{u}_{k,x}^2 + k\zeta_k + 2\zeta_k(\ln(k\zeta_k)-1)\Big\}dx.}_{(II)}
\end{aligned}
\tag{14}
$$

Since $\bar{u}_k$ solves (5), then the first part (I) disappears. Keeping in mind the definition of θ_k, we reach the conclusion immediately.

With the above discussion, next we establish the dual variational problem of (4).

$$
(\mathscr{P}_d^{(k)}):\ \max_{\zeta_k\in\mathscr{V}^{(k)}}\Big\{J_d^{(k)}[\zeta_k] = -\frac{1}{2}\int_U\Big\{\frac{\theta_k^2}{k\zeta_k} + k\zeta_k + 2\zeta_k(\ln(k\zeta_k)-1)\Big\}dx\Big\}. \tag{15}
$$

By variational calculus, we have the following lemma.

Lemma 2.12. *The variation of $J_d^{(k)}$ with respect to ζ_k leads to the dual algebraic equation(DAE), namely,*

$$
\theta_k^2 = k\bar{\zeta}_k^2(2\ln(k\bar{\zeta}_k)+k), \tag{16}
$$

where $\bar{\zeta}_k$ is from the critical pair $(\bar{u}_k,\bar{\zeta}_k)$.

Proof. Indeed, by calculating the Gâteaux derivative of $J_d^{(k)}$ with respect to ζ_k, we can prove the lemma immediately.

Remark 2.13. *Taking into account the notation of λ_k, we can rewrite (16) as*

$$
\theta_k^2 = \lambda_k^2\ln(\mathrm{e}\lambda_k^{\frac{2}{k}}). \tag{17}
$$

From (17), we know that θ_k^2 is monotonously increasing with respect to $\lambda_k > \mathrm{e}^{-\frac{k}{2}}$.

As a matter of fact, we have the following asymptotic expansion of θ_k^2.

Lemma 2.14. *When $k \geq 3$, θ_k^2 has the expansion of the form*

$$\theta_k^2 = (1 - \frac{2}{k})\lambda_k^2 + \frac{2}{k}\lambda_k^3 + R_k(\lambda_k),$$

where the remainder term $|R_k(\lambda_k)| \leq \frac{1}{k}$ uniformly for any $\lambda_k \in [e^{-\frac{k}{2}}, 1]$. In particular, for a fixed $x \in U$, if $\lim_{k\to\infty} \lambda_k = \lambda$ and $\lim_{k\to\infty} \theta_k = \theta$ in L^∞, then we have the limit version of (17),

$$\theta^2 = \lambda^2. \tag{18}$$

Remark 2.15. *In the 1-D case, later on we will demonstrate how to find the limit θ of the sequence $\{\theta_k\}_k$ as k tends to infinity.*

Proof. Since $\lambda_k \in [e^{-\frac{k}{2}}, 1]$, we can rewrite (17) using Taylor's expansion formula for $\ln \lambda_k$ at the point 1,

$$\theta_k^2 = \lambda_k^2\Big(1 + \frac{2}{k}(\lambda_k - 1) - \frac{1}{k\eta_k^2}(\lambda_k - 1)^2\Big) = (1 - \frac{2}{k})\lambda_k^2 + \frac{2}{k}\lambda_k^3 - \frac{1}{k}\frac{\lambda_k^2}{\eta_k^2}(\lambda_k - 1)^2,$$

where $\eta_k \in (\lambda_k, 1)$. It is evident that

$$|\frac{1}{k}\frac{\lambda_k^2}{\eta_k^2}(\lambda_k - 1)^2| \leq |\frac{1}{k}\frac{\lambda_k^2}{\lambda_k^2}(\lambda_k - 1)^2| \leq \frac{1}{k}.$$

This concludes our proof.

By comparing (5) with (10), we deduce that an analytical solution of (5) can be given as

$$\bar{u}_k(x) = \int_{x_0}^{x} \frac{\theta_k(t)}{\lambda_k(t)} dt + C, \tag{19}$$

where $x, x_0 \in U$. Together with (18), we see that

$$\lim_{k\to\infty} |\bar{u}_{k,x}| = 1,$$

which is consistent with the conclusion in [9]. Summarizing the above discussion, we have the following duality theorem.

Theorem 2.16 (Canonical Duality Theory)**.** *For positive density functions $f^+ \in C(\Omega)$, $f^- \in C(\Omega^*)$, if θ_k is a solution of the Euler–Lagrange equation (5), which is not identically equal to 0, then (17) has a unique positive root $\bar{\lambda}_k$ due to the monotonicity property. Furthermore, an analytical function given by*

$$\bar{u}_k(x) = \int_{x_0}^{x} \frac{\theta_k(t)}{\bar{\lambda}_k(t)} dt + C \tag{20}$$

is a local minimizer of (4) and satisfies the following duality identity locally,

$$J^{(k)}[\bar{u}_k] = J_d^{(k)}[\bar{\zeta}_k], \tag{21}$$

where $(\bar{u}_k, \bar{\zeta}_k)$ *is a critical pair for* $\Xi^{(k)}$.

Proof. It suffices to prove the identity (21). Indeed, this identity is obtained by direct variational calculus of $J^{(k)}[u_k]$ and $J_d^{(k)}[\zeta_k]$ in (4) and (15), respectively.

$$J^{(k)}[\bar{u}_k] = \Xi^{(k)}(\bar{u}_k, \bar{\zeta}_k) = J_d^{(k)}[\bar{\zeta}_k]. \tag{22}$$

Remark 2.17. *Theorem 2.16 demonstrates that the maximization of the pure complementary energy functional* $J_d^{(k)}$ *is perfectly dual to the minimization of the potential energy functional* $J^{(k)}$. *In effect, the identity (22) indicates there is no duality gap between them.*

Up to now, we have constructed a critical pair $(\bar{u}_k, \bar{\zeta}_k)$ satisfying (22) locally. Next we verify that $\bar{u}_k$ and $\bar{\zeta}_k$ are exactly a global minimizer for $J^{(k)}$ and a global maximizer for $J_d^{(k)}$, respectively. In the following theorem, we apply the *triality theory* to obtain the extremum conditions for the critical pair.

Theorem 2.18 (Triality Theory)**.** *For positive density functions* $f^+ \in C(\Omega)$, $f^- \in C(\Omega^*)$, *we have,* θ_k *is the unique solution of the Euler–Lagrange equation (5) with hidden Dirichlet boundary. Moreover,* $\bar{\zeta}_k$ *is a global maximizer of* $J_d^{(k)}$ *over* $\mathscr{V}^{(k)}$, *and the corresponding* $\bar{u}_k$ *in the form of (20) is a global minimizer of* $J^{(k)}$ *over* $\mathscr{L}_0$, *namely,*

$$J^{(k)}(u_k^*) = J^{(k)}(\bar{u}_k) = \min_{u_k \in \mathscr{L}_0} J^{(k)}(u_k) = \max_{\zeta_k \in \mathscr{V}^{(k)}} J_d^{(k)}(\zeta_k) = J_d^{(k)}(\bar{\zeta}_k). \tag{23}$$

Proof. We divide our proof into three parts. In the first and second parts, we discuss the uniqueness of θ_k. Extremum conditions will be illustrated in the third part.

First Part:

Without loss of generality, we consider the disjoint case $\Omega = [a, b]$ and $\Omega^* = [c, d]$, $b < c$. In Ω, we have a general solution for the nonlinear differential equation (5) in the form of

$$\theta_k(x) = -F(x) + C_k, \quad F(x) := \int_a^x f^+(x)dx, \quad x \in [a, b].$$

Since $f^+ > 0$, then $F \in C[a, b]$ is a strictly increasing function with respect to $x \in [a, b]$ and consequently is invertible. Let F^{-1} be its inverse function, which is also a strictly increasing function, then

$$F^{-1} : [0, 1] \to [a, b].$$

From Remark 2.13, we see that there exists a unique piecewise continuous function $\lambda_k(x) > \mathrm{e}^{-\frac{k}{2}}$ except for the point $x = F^{-1}(C_k)$. By paying attention to the fact that $\bar{u}_k(a) = 0$, we represent the analytical solution $\bar{u}_k$ in the following form:

$$\bar{u}_k(x) = \int_a^x \frac{-F(x) + C_k}{\lambda_k(x)} dx, \quad x \in [a, b].$$

Since

$$\lim_{x \to F^{-1}(C_k)^-} \frac{-F(x) + C_k}{\lambda_k(x)} = 0, \quad \lim_{x \to F^{-1}(C_k)^+} \frac{-F(x) + C_k}{\lambda_k(x)} = 0,$$

thus $\bar{u}_k$ is continuous at the point $x = F^{-1}(C_k)$. As a result, $\bar{u}_k \in C[a, b]$. Recall that

$$\bar{u}_k(b) = \int_a^{F^{-1}(C_k)} \frac{-F(x) + C_k}{\lambda_k(x)} dx + \int_{F^{-1}(C_k)}^b \frac{-F(x) + C_k}{\lambda_k(x)} dx = 0,$$

and we can determine the constant $C_k \in (0, 1)$ uniquely. Indeed, let

$$M(t) := \int_a^{F^{-1}(t)} \frac{-F(x) + t}{\lambda_k(x, t)} dx + \int_{F^{-1}(t)}^b \frac{-F(x) + t}{\lambda_k(x, t)} dx,$$

where $\lambda_k(x, t)$ is from (17). It is evident that λ_k depends on C_k. As a matter of fact, M is strictly increasing with respect to $t \in (0, 1)$, which leads to

$$C_k = M^{-1}(0).$$

Indeed, for $t_1 < t_2, t_1, t_2 \in (0, 1)$, by keeping in mind the identity (17), we have

$$\begin{aligned}
M(t_1) &= \int_a^{F^{-1}(t_1)} \frac{-F(x) + t_1}{\lambda_k(x, t_1)} dx + \int_{F^{-1}(t_1)}^b \frac{-F(x) + t_1}{\lambda_k(x, t_1)} dx \\
&= \int_a^{F^{-1}(t_1)} \frac{-F(x) + t_1}{\lambda_k(x, t_1)} dx + \int_{F^{-1}(t_1)}^{F^{-1}(t_2)} \frac{-F(x) + t_1}{\lambda_k(x, t_1)} dx + \int_{F^{-1}(t_2)}^b \frac{-F(x) + t_1}{\lambda_k(x, t_1)} dx \\
&< \int_a^{F^{-1}(t_1)} \frac{-F(x) + t_2}{\lambda_k(x, t_2)} dx + \int_{F^{-1}(t_1)}^{F^{-1}(t_2)} \frac{-F(x) + t_2}{\lambda_k(x, t_2)} dx + \int_{F^{-1}(t_2)}^b \frac{-F(x) + t_2}{\lambda_k(x, t_2)} dx \\
&= M(t_2).
\end{aligned}$$

More information concerned with C_k will be explained in the proof of Theorem 1.3.

Second Part:

Applying the similar procedure, we see that

$$\theta_k(x) = G(x) - D_k, \quad G(x) := \int_c^x f^-(x)dx, \quad x \in [c, d],$$

where the constant $D_k \in (0, 1)$. Since $f^- > 0$, then $G \in C[c, d]$ is a strictly increasing function with respect to $x \in [c, d]$ and consequently is invertible. Let G^{-1} be its inverse function, which is also a strictly increasing function, then

$$G^{-1} : [0, 1] \to [c, d].$$

We can represent the analytical solution $\bar{u}_k$ in the following form:

$$\bar{u}_k(x) = \int_c^x \frac{G(x) - D_k}{\lambda_k(x)} dx, \quad x \in [c, d].$$

Since

$$\lim_{x \to G^{-1}(D_k)^-} \frac{G(x) - D_k}{\lambda_k(x)} = 0, \quad \lim_{x \to G^{-1}(D_k)^+} \frac{G(x) - D_k}{\lambda_k(x)} = 0,$$

thus $\bar{u}_k$ is continuous at the point $x = G^{-1}(D_k)$. As a result, $\bar{u}_k \in C[c, d]$. Recall that

$$\bar{u}_k(d) = \int_c^{G^{-1}(D_k)} \frac{G(x) - D_k}{\lambda_k(x)} dx + \int_{G^{-1}(D_k)}^d \frac{G(x) - D_k}{\lambda_k(x)} dx = 0,$$

and we can determine the constant $D_k \in (0, 1)$ uniquely. Indeed, let

$$N(t) := \int_c^{G^{-1}(t)} \frac{G(x) - t}{\lambda_k(x, t)} dx + \int_{G^{-1}(t)}^d \frac{G(x) - t}{\lambda_k(x, t)} dx,$$

where $\lambda_k(x, t)$ is from (17). As a matter of fact, N is strictly decreasing with respect to $t \in (0, 1)$, which leads to

$$D_k = N^{-1}(0).$$

Indeed, for $t_1 < t_2, t_1, t_2 \in (0, 1)$, by keeping in mind the identity (17), we have

$$
\begin{aligned}
N(t_1) &= \int_c^{G^{-1}(t_1)} \frac{G(x)-t_1}{\lambda_k(x,t_1)}dx + \int_{G^{-1}(t_1)}^d \frac{G(x)-t_1}{\lambda_k(x,t_1)}dx \\
&= \int_c^{G^{-1}(t_1)} \frac{G(x)-t_1}{\lambda_k(x,t_1)}dx + \int_{G^{-1}(t_1)}^{G^{-1}(t_2)} \frac{G(x)-t_1}{\lambda_k(x,t_1)}dx + \int_{G^{-1}(t_2)}^d \frac{G(x)-t_1}{\lambda_k(x,t_1)}dx \\
&> \int_c^{G^{-1}(t_1)} \frac{G(x)-t_2}{\lambda_k(x,t_2)}dx + \int_{G^{-1}(t_1)}^{G^{-1}(t_2)} \frac{G(x)-t_2}{\lambda_k(x,t_2)}dx + \int_{G^{-1}(t_2)}^d \frac{G(x)-t_2}{\lambda_k(x,t_2)}dx \\
&= N(t_2).
\end{aligned}
$$

Furthermore, the other cases, such as $b = c$ and $b > c$, can also be discussed similarly due to the fact that $\bar{u}_k = 0$ on $\Omega \cap \Omega^*$. Therefore, θ_k is uniquely determined in U and the analytic solution $\bar{u}_k \in C(U)$.

Third Part:

In order to prove the extremum of the critical pair, we recall the second variational formula for both $J^{(k)}$ and $J_d^{(k)}$.

On the one hand, for any test function $\phi \in \mathscr{L}_0$ satisfying $\phi_x \neq 0$ a.e. in U, the second variational form $\delta^2_\phi J^{(k)}$ with respect to ϕ is equal to

$$
\int_U \frac{d^2}{dt^2}\Big\{H^{(k)}((\bar{u}_k + t\phi)_x)\Big\}\Big|_{t=0} dx = \int_U \mathrm{e}^{\frac{k}{2}(\bar{u}_{k,x}^2-1)}\Big\{k(\bar{u}_{k,x}\phi_x)^2 + \phi_x^2\Big\}dx. \quad (24)
$$

On the other hand, for any test function $\psi \in \mathscr{V}^{(k)}$ satisfying $\psi \neq 0$ a.e. in U, the second variational form $\delta^2_\psi J_d^{(k)}$ with respect to ψ is equal to

$$
\begin{aligned}
&-\frac{1}{2}\int_U \frac{d^2}{dt^2}\Big\{\frac{\theta_k^2}{k(\zeta_k + t\psi)} + 2(\zeta_k + t\psi)\Big(\ln(k(\zeta_k + t\psi)) - 1\Big)\Big\}\Big|_{t=0} dx \\
&= -\int_U \Big\{\frac{\theta_k^2\psi^2}{k\zeta_k^3} + \frac{\psi^2}{\zeta_k}\Big\}dx.
\end{aligned}
$$

From (24) and (25), we know immediately that

$$
\delta^2_\phi J^{(k)}(\bar{u}_k) > 0, \quad \delta^2_\psi J_d^{(k)}(\bar{\zeta}_k) < 0. \quad (25)
$$

Then Theorem 2.16 and the uniqueness of θ_k discussed in the first and second parts complete our proof.

Consequently, we reach the conclusion of Theorem 1.2.

2.3 Proof of Theorem 1.3:

Proof. Without loss of generality, we still consider the disjoint case, $b < c$. First we show an important lemma which describes the asymptotic behavior of C_k and D_k when k tends to infinity.

Lemma 2.19. *When $b < c$, the sequences of $\{C_k\}_k$ and $\{D_k\}_k$ are given in the proof of Theorem 2.18, then we have*

$$\lim_{k\to\infty} C_k = F(\frac{a+b}{2}), \tag{26}$$

$$\lim_{k\to\infty} D_k = G(\frac{c+d}{2}). \tag{27}$$

Proof. Recall the identity

$$\bar{u}_k(b) = \int_a^{F^{-1}(C_k)} \frac{-F(x)+C_k}{\lambda_k(x)}dx + \int_{F^{-1}(C_k)}^b \frac{-F(x)+C_k}{\lambda_k(x)}dx = 0. \tag{28}$$

Since

$$\lim_{k\to\infty} \frac{-F(x)+C_k}{\lambda_k(x)} = 1, \quad x \in [a, F^{-1}(C_k)),$$

$$\lim_{k\to\infty} \frac{-F(x)+C_k}{\lambda_k(x)} = -1, \quad x \in (F^{-1}(C_k), b],$$

then for $\forall \varepsilon > 0$, there exists an $N \in \mathbb{N}^+$, such that for $\forall k > N$, the following inequalities hold:

$$(1-\varepsilon)(F^{-1}(C_k)-a) \le \int_a^{F^{-1}(C_k)} \frac{-F(x)+C_k}{\lambda_k(x)}dx \le (1+\varepsilon)(F^{-1}(C_k)-a), \tag{29}$$

$$(-1-\varepsilon)(b-F^{-1}(C_k)) \le \int_{F^{-1}(C_k)}^b \frac{-F(x)+C_k}{\lambda_k(x)}dx \le (-1+\varepsilon)(b-F^{-1}(C_k)). \tag{30}$$

Combining (29)–(31) together, we have

$$\frac{a+b}{2} - \frac{b-a}{2}\varepsilon \le F^{-1}(C_k) \le \frac{a+b}{2} + \frac{b-a}{2}\varepsilon. \tag{31}$$

Then (27) follows immediately. It is obvious that we can prove (28) in a similar manner.

As a result, we define the limit of θ_k in L^∞ as

$$\theta(x) := \begin{cases} \lim\limits_{k\to\infty}(-F(x)+C_k) = F_{ab}(x), \quad F_{ab}(x) = -F(x) + F(\dfrac{a+b}{2}),\ x\in[a,b], \\ \lim\limits_{k\to\infty}(G(x)-D_k) = G_{cd}(x), \quad G_{cd}(x) = G(x) - G(\dfrac{c+d}{2}), \quad x\in[c,d]. \end{cases}$$

Next, according to (18), we define the limit of λ_k in L^∞ as

$$\lambda(x) := \begin{cases} |F_{ab}(x)|,\ x\in[a,b], \\ |G_{cd}(x)|,\ x\in[c,d]. \end{cases}$$

Finally, we calculate the limit of $\bar{u}_k$ in $\mathscr{L}_0$ as follows:

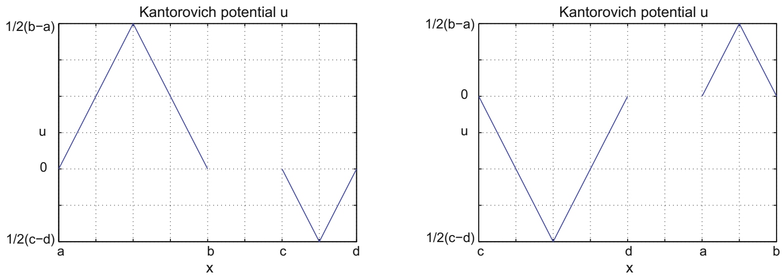
$$u(x) := \begin{cases} \displaystyle\int_a^x \frac{F_{ab}(x)}{|F_{ab}(x)|}dx = x - a, & x\in[a, \frac{a+b}{2}], \\ \displaystyle\int_a^{\frac{a+b}{2}} \frac{F_{ab}(x)}{|F_{ab}(x))|}dx + \int_{\frac{a+b}{2}}^x \frac{F_{ab}(x)}{|F_{ab}(x)|}dx = -x + b, & x\in(\frac{a+b}{2}, b], \\ \displaystyle\int_c^x \frac{G_{cd}(x)}{|G_{cd}(x)|}dx = -x + c, & x\in[c, \frac{c+d}{2}], \\ \displaystyle\int_c^{\frac{c+d}{2}} \frac{G_{cd}(x)}{|G_{cd}(x)|}dx + \int_{\frac{c+d}{2}}^x \frac{G_{cd}(x)}{|G_{cd}(x)|}dx = x - d, & x\in(\frac{c+d}{2}, d]. \end{cases} \tag{32}$$

This solution is illustrated in Fig. 1. Several other cases can be proved similarly, and the corresponding Kantorovich potentials are depicted in Figs. 1, 2, 3 and 4. As a result, we have constructed a global maximizer for Kantorovich's mass transfer problem (3) in 1-D.

2.4 Application to Monge's Problem

During the past few decades, Monge's and Kantorovich's problems have been the subject of active inquiry, since it covers the domains of optimization, probability theory, partial differential equations, allocation mechanism in economics and membrane filtration in biology, etc. In this application part, we apply the main theorems to solve a product allocation model in 1-D.

We want to transport some products from $[a, b]$ to $[c, d]$. Assume that the products are distributed uniformly in $[a, b]$, that means, the density function f^+ satisfies



(a) $a<b<c<d,\ b-a\geq d-c$ (b) $c<d<a<b,\ b-a\leq d-c$

Fig. 1 The unique continuous Kantorovich potential of Problem (3) while Ω and Ω^* are disjoint in 1-D

$$f^+(x) = \frac{1}{b-a}, \quad x \in [a,b].$$

Figure 1: When $f^- > 0$ in $[c,d]$, according to Theorem 1.3, one can check that the unique Kantorovich potential does not satisfy the dual criteria for optimality. Therefore, in this case, the Kantorovich problem (3) is not a perfect dual problem of Monge's problem (1). We know, the optimal mapping should be $\mathbf{s}^*(x) = c$, in which case, the density f^- is a δ-function in the form of

$$f^- = \begin{cases} \infty & \text{if } x = c, \\ 0 & \text{if } x \in (c,d], \end{cases}$$

satisfying

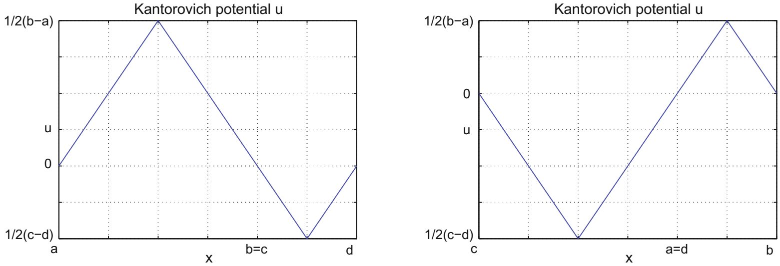
$$\int_c^d f^-(x)dx = 1.$$

Figure 2: When $f^- > 0$ in $[c,d]$, according to Theorem 1.3, one can check that the unique Kantorovich potential satisfies the dual criteria for optimality only for $x \in [\frac{a+b}{2}, b],\ \ y \in [c, \frac{c+d}{2}]$. Therefore, in this case, the Kantorovich problem (3) is not a perfect dual problem of Monge's problem (1). We know, the optimal mapping should be $\mathbf{s}^*(x) = c$.

Figure 3: When $f^- > 0$ in $[c,d]$, according to Theorem 1.3, one can check that the unique Kantorovich potential satisfies the dual criteria for optimality only when we choose

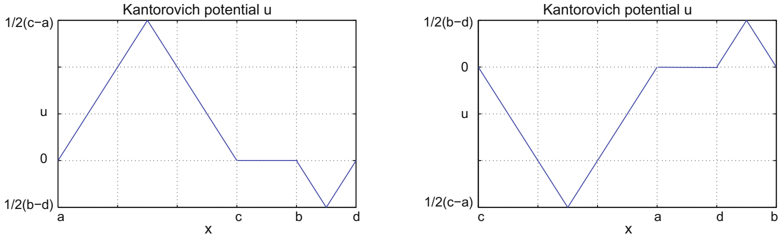
$$\mathbf{s}(x) = \begin{cases} c & \text{if } x \in [\frac{a+c}{2}, c] \ \text{ and } y = c, \\ x & \text{if } x, y \in (c,b), \\ y & \text{if } x = b \ \text{ and } y \in [b, \frac{b+d}{2}]. \end{cases}$$

Therefore, in this case, the Kantorovich problem (3) is not a perfect dual problem of Monge's problem (1). We know, the optimal mapping should be



(a) $a < b = c < d,\ b - a \geq d - c$ (b) $c < d = a < b,\ b - a \leq d - c$

Fig. 2 The unique continuous Kantorovich potential of Problem (3) while Ω and Ω^* have a unique common point in 1-D



(a) $a < c < b < d,\ b - a \geq d - c$ (b) $c < a < d < b,\ b - a \leq d - c$

Fig. 3 The unique continuous Kantorovich potential of Problem (3) while $\Omega \cap \Omega^*$ have more than one common point and $\Omega \nsubseteq \Omega^*$ or $\Omega^* \nsubseteq \Omega$ in 1-D

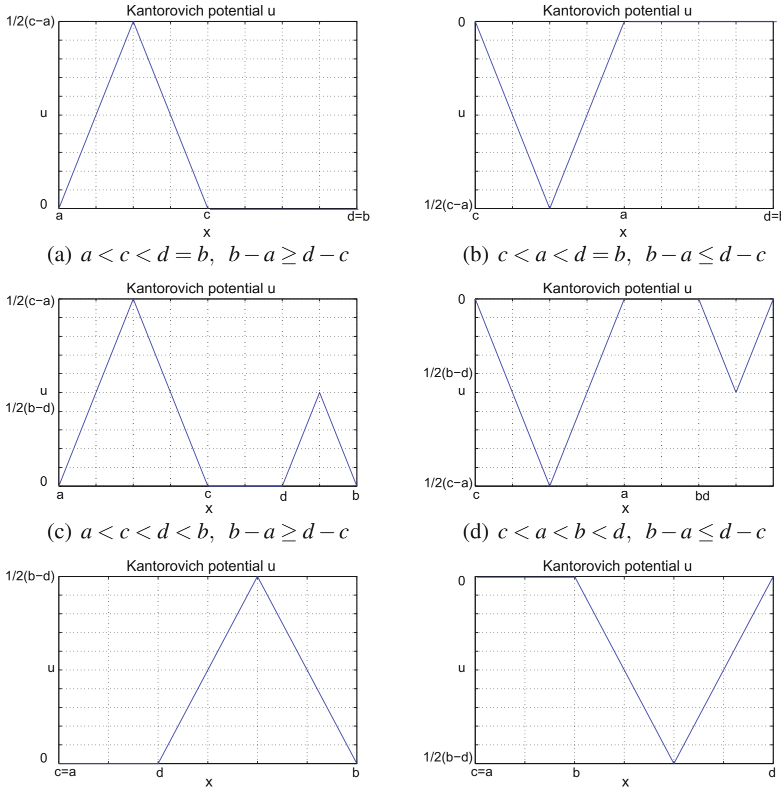
$$\mathbf{s}^*(x) = \begin{cases} c & \text{if } x \in [a, c], \\ x & \text{if } x \in (c, b]. \end{cases}$$

Figure 4a, c, e: When $f^- > 0$ in $[c, d]$, according to Theorem 1.3, one can check that the unique Kantorovich potential satisfies the dual criteria for optimality only when we choose

$$\mathbf{s}(x) = \begin{cases} c & \text{if } x \in [\frac{a+c}{2}, c] \text{ and } y = c, \\ x & \text{if } x, y \in (c, d], \\ d & \text{if } x \in (d, \frac{b+d}{2}] \text{ and } y = d. \end{cases}$$

Therefore, in this case, the Kantorovich problem (3) is not a perfect dual problem of Monge's problem (1). We know, the optimal mapping should be

$$\mathbf{s}^*(x) = \begin{cases} c & \text{if } x \in [a, c], \\ x & \text{if } x \in (c, d], \\ d & \text{if } x \in (d, b]. \end{cases}$$



(a) $a < c < d = b,\ \ b - a \geq d - c$

(b) $c < a < d = b,\ \ b - a \leq d - c$

(c) $a < c < d < b,\ \ b - a \geq d - c$

(d) $c < a < b < d,\ \ b - a \leq d - c$

(e) $c = a < d < b,,\ \ b - a \geq d - c$

(f) $c = a < b < d,,\ \ b - a \leq d - c$

Fig. 4 The unique continuous Kantorovich potential of Problem (3) while $\Omega \subseteq \Omega^*$ or $\Omega^* \subseteq \Omega$ in 1-D

Figure 4b, d, f: When $f^- > 0$ in $[a, b]$, according to Theorem 1.3, one can check that the unique Kantorovich potential does satisfy the dual criteria for optimality when we choose $\mathbf{s}^*(x) = x, \quad x \in [a, b]$. Therefore, the Kantorovich problem (3) in this case is a perfect dual problem of Monge's problem (1).

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