Improved Canonical Dual Finite Element Method and Algorithm for Post-Buckling Analysis of Nonlinear Gao Beam

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Abstract This paper deals a study on post-buckling problem of a large deformed elastic beam by using a canonical dual mixed finite element method (CD-FEM). The nonconvex total potential energy of this beam can be used to model post-buckling problems. To verify the triality theory, different types of dual stress interpolations are used. Applications are illustrated with different boundary conditions and different external loads using semi-definite programming (SDP) algorithm. The results show that the global minimizer of the total potential energy is stable buckled configuration, the local maximizer solution leads to the unbuckled state, and both of these two solutions are numerically stable. While the local minimizer is unstable buckled configuration and very sensitive.

1 Introduction

Nonconvex variational problems have always presented serious challenges not only in numerical analysis, but also in computational mechanics and engineering sciences. By numerical discretization techniques, nonconvex variational problems are linked with certain nonconvex global optimization minimization problems. Due to the lack of global optimality condition, conventional numerical methods and direct approaches cannot solve these problems deterministically. The popular primal–dual interior point methods suffer from uncertain error bounds in nonconvex analysis because of the intrinsic duality gaps produced by traditional duality theories. Therefore, most nonconvex minimization problems are considered as **NP-hard** in global

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optimization and computer sciences. Unfortunately, this fundamental difficulty is not fully recognized in computational mathematics and mechanics due to the significant gap between these fields.

Canonical duality theory is a newly developed, potentially powerful methodological theory which can transfer general multi-scale nonconvex problems in R^n to a unified convex dual problem in continuous space R^m with $m \leq n$ and without duality gap. The associated *triality theory* provides extremality criteria for both global and local optimal solutions, which can be used to develop powerful algorithms for solving general nonconvex variational problems. This talk will present a canonical dual finite element method (CD-FEM) for solving general nonconvex variational problems. Using Gao–Strang's complementary–dual principle and mixed finite element discretization, the general nonconvex variational problem can be reformulated as a min–max optimization problem of a saddle function. Based on the triality theory and the SDP method, a canonical primal–dual algorithm is proposed. Detailed application will be illustrated by post-buckling problem of a large elastic deformations of beam, which is governed by a fourth-order nonlinear differential equation. The total potential energy of this beam is a double-welled nonconvex functional with two local minimizers, representing the two buckled states, and one local maximizer representing the unbuckled state.

The purpose of the present work is to verify the triality theory to find all solutions of the post-buckling problem of a large deformation nonlinear beam. Mixed finite element method with mixed meshes of different dual stress interpolations are used to get a closed dimensions between the discretized displacement and discretized stress. Numerical results show that the our algorithm can produce a stable solutions for the global minimizer and local maximizer. However, the local minimizer is very sensitive to numerical discretization and external loads.

2 Nonconvex Problem and Canonical Dual–Complementary Principle

Let us consider an elastic beam subjected to a vertical distributed lateral load $q(x)$ and compressive external axial force F at the right end as shown in Fig. 1. It was

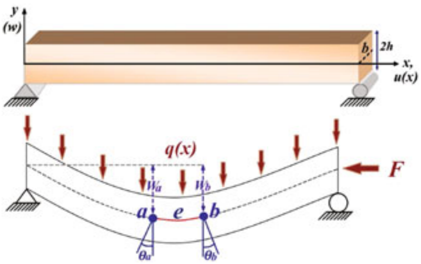

Fig. 1 Simply supported beam model

discovered by Gao in 1996 that the well-known von Karman nonlinear plate model in one dimension is actually equivalent to a linear differential equation and therefore, it cannot be used for studying post-buckling phenomena [4]. The main reason for this “paradox” is due to the fact that the stress in lateral direction of large deformed plate was ignored by von Karman. Therefore, von Karman equation works only for thin plate and cannot be used as a beam model. For a relatively thick beam such that $h/L \sim w(x) \in O(1)$, the deformation in the lateral direction can not be ignored. Based on the finite deformation theory for Hooke' material and EulerBernoulli hypothesis (i.e., straight lines normal to the mid-surface remain straight and normal to the mid-surface after deformation), a nonlinear beam model was proposed by Gao [4]:

$$EI.w_{,xxxx} - \alpha E w_{,x}^2 w_{,xx} + E\lambda w_{,xx} - f(x) = 0, \ \forall x \in [0 \ L], \tag{1}$$

where E is the elastic modulus of material, $I = 2h^3/3$ is the second moment of area of the beam's cross section, w is the transverse displacement field of the beam, $\alpha = 3h(1 - \nu^2) > 0$ with ν as the Poisson's ratio, $\lambda = (1 + \nu)(1 - \nu^2)F/E > 0$ is an integral constant, $f(x) = (1 - \nu^2)q(x)$ depends mainly on the distributed lateral load $q(x)$; $2h$ and L represent to the height and length of the beam, respectively. The axial displacement $u(x)$ is governed by the following differential equation [4]:

$$u_x = -\frac{1}{2}(1 + \nu)w_{,x}^2 - \frac{\lambda}{2h(1 + \nu)} \ , \tag{2}$$

which shows that $u(x) \sim w_{,x}(x) \in O(\epsilon)$, $u_{,x}(x) \sim w_{,xx}(x) \in O(\epsilon^2)$. The total potential energy attendant of this problem is the function $\Pi(w) : \mathscr{U}_a \to R$ define by

$$\Pi(w) = \int_0^L \left(\frac{1}{2} EI w_{,xx}^2 + \frac{1}{12} E\alpha w_{,x}^4 - \frac{1}{2} E\lambda w_{,x}^2 - f(x)\, w \right) dx = 0, \tag{3}$$

where $\mathscr{U}_a$ is the kinematically admissible space, in which certain necessary boundary conditions are given. Thus, for the given external loads $f(x)$ and λ, the primal variational problem is to find $\bar{w} \in \mathscr{U}_a$ such that

$$(\mathscr{P}) : \quad \Pi(\bar{w}) = \inf \{\Pi(w) | w \in \mathscr{U}_a\}. \tag{4}$$

It is easy to prove that the stationary condition $\delta\Pi(w) = 0$ leads to the governing equation (1). From the classic beam theory, the Euler buckling load can be determined by

$$\lambda_{cr} = \inf_{w \in \mathscr{U}_a} \frac{\int_0^L EI w_{,xx}^2 dx}{\int_0^L E w_{,x}^2 dx} \ . \tag{5}$$

Clearly, before the axial load λ reaches to the Euler buckling load λ_{cr}, the total potential energy $\Pi(w)$ is convex on $\mathscr{U}_a$ and the nonlinear differential equation (1) has only one solution. When $\lambda > \lambda_{cr}$, the beam is in a post-buckling state. In this case, the total potential energy Π is nonconvex and Eq. (1) may have at most three (strong) solutions [6] at each material point $x \in [0\ L]$: two minimizers corresponding to the two possible buckled states, one maximizer corresponding to the possible unbuckled state. Clearly, these solutions are sensitive to both the axial load λ and the distributed lateral force field $f(x)$. By Eq. (2) we know that the axial deformation could be relatively larger, the Gao beam model can be used for studying both pre- and post-buckling problems in engineering and sciences [2, 11]. Mathematically, due to the fact that traditional numerical methods and convex optimization techniques cannot identify the global minimizer at each numerical iteration, most of nonconvex optimization problems are considered to be NP-hard in global optimization and computer science [7]. the Gao–Strang total complementary energy $\Xi : \mathscr{U}_a \times \mathscr{S}_a \to R$ [8] in nonlinear elasticity can be defined as

$$\Xi(w,\sigma) = \int_0^L \left(\frac{1}{2}EIw_{,xx}^2 + \frac{1}{2}\sigma w_{,x}^2 - \frac{3}{4E\alpha}(\sigma + E\lambda)^2 - f(x)w\right)dx$$
$$= G(w,\sigma) - \int_0^L [V^*(\sigma) - f(x)w]dx, \tag{6}$$

where $\mathscr{S}_a = \{\sigma \in C[0,L] |\ \sigma(x) \geq -\lambda E \ \ \forall x \in [0,L]\}$ and

$$G(w,\sigma) = \int_0^L \left(\frac{1}{2}EIw_{,xx}^2 + \frac{1}{2}\sigma w_{,x}^2\right)dx$$

is the generalized Gao–Strang complementary gap function [8].

3 Mixed Finite Element Method and Triality Theory

In order to apply FEM, the domain of the beam is discretized into m elements $[0,L] = \bigcup_{e=1}^m \Omega^e$. In each element $\Omega^e = [x_a, x_b]$, the deflection, rotating angular, and dual stress for the node x_a are marked as w_a, θ_a, and σ_a, respectively, and similar for the node x_b. Then, we have the nodal displacement vector $w_e^T = [w_a\ \ \theta_a\ \ w_b\ \ \theta_b]$ of the e-th element and the nodal dual stress element $\sigma_e^T = [\sigma_a\ \ \sigma_b]$. In each element, we use mixed finite element interpolations for both $w(x)$ and $\sigma(x)$, i.e.,

$$w_e^h(x) = N_w^T(x)w_e \quad , \qquad \sigma_e^h(x) = N_\sigma^T(x)\boldsymbol{\sigma}_e \quad \forall x \in \Omega^e.$$

Thus, the spaces $\mathscr{U}_a$ and $\mathscr{S}_a$ can be numerically discretized to the finite-dimensional spaces $\mathscr{U}_a^h$ and $\mathscr{S}_a^h$, respectively. The shape function for $w(x)$ is based on piecewise-cubic polynomial, i.e.,

$$N_w = \begin{bmatrix} \frac{1}{4}\,(1-\xi)^2\,(2+\xi) \\ \frac{L_e}{8}(1-\xi)^2\,(1+\xi) \\ \frac{1}{4}\,(1+\xi)^2\,(2-\xi) \\ \frac{L_e}{8}(1+\xi)^2\,(\xi-1) \end{bmatrix},$$

where $\xi = 2x/L_e - 1$ and L_e is the length of e-th beam element. The shape function for σ is based on different dual stress interpolations; piecewise-linear stresses (PLS, $\delta = 1$), piecewise-quadratic stresses (PQS, $\delta = 2$), and piecewise-cubic stresses (PCS, $\delta = 3$) as follows:

$$N_\sigma|_{\delta=1} = \frac{1}{2}\begin{bmatrix} 1-\xi \\ 1+\xi \end{bmatrix}, \qquad N_\sigma|_{\delta=2} = \frac{1}{2}\begin{bmatrix} \xi^2-\xi \\ 1-\xi^2 \\ \xi^2+\xi \end{bmatrix},$$

and

$$N_\sigma|_{\delta=3} = \frac{1}{16}\begin{bmatrix} -1+\xi+9\xi^2-9\xi^3 \\ 9-27\xi-9\xi^2+27\xi^3 \\ 9+27\xi-9\xi^2-27\xi^3 \\ -1-\xi+9\xi^2+9\xi^3 \end{bmatrix},$$

where δ refers to the number of straight lines inside the element e as shown in Fig. 2.

Thus, on the discretized feasible deformation space $\mathscr{U}_a^h$, the Gao–Strang total complementary energy can be expressed in the following discretized form:

$$\begin{aligned} \Xi^h(\mathbf{w}, \boldsymbol{\sigma}) &= \sum_{e=1}^{m} \left(\frac{1}{2} w_e^T\, G^e(\sigma_e)\, w_e - \frac{1}{2}\sigma_e^T\, K_e\, \sigma_e - \lambda_e^T\, \sigma_e - f_e^T\, w_e - c_e \right) \\ &= \frac{1}{2}\mathbf{w}^T\, \mathbf{G}(\boldsymbol{\sigma})\, \mathbf{w} - \frac{1}{2}\boldsymbol{\sigma}^T\, \mathbf{K}\, \boldsymbol{\sigma} - \boldsymbol{\lambda}^T\, \boldsymbol{\sigma} - \mathbf{f}^T\, \mathbf{w} - c, \end{aligned} \tag{7}$$

where $\mathbf{w} \in \mathscr{U}_a^h \subset R^{2(m+1)}$ and $\boldsymbol{\sigma} \in \mathscr{S}_a^h \subset R^{\delta m+1}$ are nodal deflection and dual stress vectors, respectively. We let

$$\mathscr{S}_a^h = \{\boldsymbol{\sigma} \in R^{\delta m+1} | \quad \det \mathbf{G}(\boldsymbol{\sigma}) \neq 0\}. \tag{8}$$

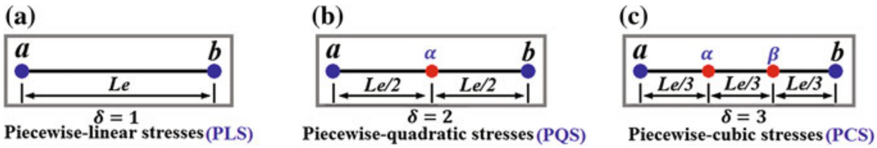

Fig. 2 Dual stress nodes in an element

The Hessian matrix of the gap function $\mathbf{G}(\boldsymbol{\sigma}) \in R^{2(m+1)} \times R^{2(m+1)}$ is obtained by assembling the following symmetric matrices $G^e(\sigma_e)$:

$$G^e(\sigma_e) = \int_{\Omega_e} \left(EI \; N''_w \; (N''_w)^T + (N_\sigma)^T \; \sigma_e \; N'_w \; (N'_w)^T \right) dx. \tag{9}$$

The matrix $\mathbf{K} \in R^{\delta m+1} \times R^{\delta m+1}$ is obtained by assembling the following positive-definite matrices K_e

$$K_e = \int_{\Omega_e} \left(\frac{3}{2E\alpha} N_\sigma \; N_\sigma^T \right) dx.$$

Also, $\boldsymbol{\lambda} = \{\lambda_e\} \in R^{\delta m+1}$, $\mathbf{f} = \{f_e\} \in R^{2(m+1)}$ are defined by assembling the corresponding element components $\lambda_e = \int_{\Omega_e} \left(\frac{3}{2\alpha} \lambda N_\sigma \right) dx$, $f_e = \int_{\Omega_e} f\big(x\big) \; N_w \; dx$, and $c = \sum_{e=1}^{m} c_e \in R$, where $c_e = \int_{\Omega_e} \left(\frac{3E}{4\alpha} \lambda^2 \right) dx = \frac{3}{4\alpha} E L_e \lambda^2$.

By the critical condition $\delta \Xi^h(\mathbf{w}, \boldsymbol{\sigma}) = 0$, we obtain the two equations $\mathbf{G}(\boldsymbol{\sigma}) \; \mathbf{w} - \mathbf{f} = 0$, and $\frac{1}{2} \mathbf{w}^T \; \mathbf{G}_{,\boldsymbol{\sigma}}(\boldsymbol{\sigma}) \; \mathbf{w} - \mathbf{K} \; \boldsymbol{\sigma} - \boldsymbol{\lambda} = 0$, where $\mathbf{G}_{,\boldsymbol{\sigma}}(\boldsymbol{\sigma})$ is gradient of $\mathbf{G}$ respect to $\boldsymbol{\sigma}$. The discretized pure complementary energy $\Pi_d^h : \mathscr{S}_a^h \rightarrow R$ can be obtained by the following canonical dual transformation:

$$\Pi_d^h(\boldsymbol{\sigma}) = -\frac{1}{2} \mathbf{f}^T \; \mathbf{G}^{-1}(\boldsymbol{\sigma}) \; \mathbf{f} - \frac{1}{2} \boldsymbol{\sigma}^T \; \mathbf{K} \; \boldsymbol{\sigma} - \boldsymbol{\lambda}^T \; \boldsymbol{\sigma} - c \tag{10}$$

Suppose $(\bar{\mathbf{w}}, \bar{\boldsymbol{\sigma}})$ is a stationary point of $\Xi^h(\mathbf{w}, \boldsymbol{\sigma})$, and let $\mathscr{S}_a^+ = \{\boldsymbol{\sigma} \in \mathscr{S}_a^h \mid \mathbf{G}(\boldsymbol{\sigma}) \succ 0\}$, and $\mathscr{S}_a^- = \{\boldsymbol{\sigma} \in \mathscr{S}_a^h \mid \mathbf{G}(\boldsymbol{\sigma}) \prec 0\}$. Then, by "*Complementary–duality Principle theorem*" [5], we have the following theorem.

Theorem 1. *Suppose* $(\bar{\mathbf{w}}, \bar{\boldsymbol{\sigma}})$ *is a stationary point of* $\Xi^h(\mathbf{w}, \boldsymbol{\sigma})$*, then* $\Pi_p^h(\bar{\mathbf{w}}) = \Xi^h(\bar{\mathbf{w}}, \bar{\boldsymbol{\sigma}}) = \Pi_d^h(\bar{\boldsymbol{\sigma}})$*. Moreover, if* $\bar{\boldsymbol{\sigma}} \in \mathscr{S}_a^+$*, then we have*

Canonical Min–Max Duality: *The stationary point* $\bar{\mathbf{w}}$ *is a global minimizer of* $\Pi_p^h(\mathbf{w})$ *on* $\mathscr{U}_a^h$ *if and only if* $\bar{\boldsymbol{\sigma}}$ *is a global maximizer of* $\Pi_d^h(\boldsymbol{\sigma})$ *on* $\mathscr{S}_a^+$*, i.e.,*

$$\Pi_p^h(\bar{\mathbf{w}}) = \min_{\mathbf{w} \in \mathscr{U}_a^h} \Pi_p^h(\mathbf{w}) \quad \Leftrightarrow \quad \max_{\boldsymbol{\sigma} \in \mathscr{S}_a^+} \Pi_d^h(\boldsymbol{\sigma}) = \Pi_d^h(\bar{\boldsymbol{\sigma}}). \tag{11}$$

If $\bar{\boldsymbol{\sigma}} \in \mathscr{S}_a^-$*, then on a neighborhood* $\mathscr{U}_o \times \mathscr{S}_o \subset \mathscr{U}_a^h \times \mathscr{S}_a^-$ *of* $(\bar{\mathbf{w}}, \bar{\boldsymbol{\sigma}})$ *we have*

Canonical Double-max Duality: *The stationary point* $\bar{\mathbf{w}}$ *is a local maximizer of* $\Pi_p^h(\mathbf{w})$ *on* $\mathscr{U}_o$ *if and only if the stationary point* $\bar{\boldsymbol{\sigma}}$ *is a local maximizer of* $\Pi_d^h(\boldsymbol{\sigma})$ *on* $\mathscr{S}_o$*, i.e.,*

$$\Pi_p^h(\bar{\mathbf{w}}) = \max_{\mathbf{w} \in \mathscr{U}_o} \Pi_p^h(\mathbf{w}) \quad \Leftrightarrow \quad \max_{\boldsymbol{\sigma} \in \mathscr{S}_o} \Pi_d^h(\boldsymbol{\sigma}) = \Pi_d^h(\bar{\boldsymbol{\sigma}}) \tag{12}$$

Canonical Double-min Duality: *The stationary point* $\bar{\mathbf{w}}$ *is a local minimizer of* $\Pi_p^h(\mathbf{w})$ *on* $\mathscr{U}_o$ *if and only if the stationary point* $\bar{\boldsymbol{\sigma}}$ *is a local minimizer of* $\Pi_d^h(\boldsymbol{\sigma})$ *on* $\mathscr{S}_o$*, i.e.,*

$$\Pi_p^h(\bar{\mathbf{w}}) = \min_{\mathbf{w}\in\mathscr{U}_o} \Pi_p^h(\mathbf{w}) \quad \Leftrightarrow \quad \min_{\boldsymbol{\sigma}\in\mathscr{S}_o} \Pi_d^h(\boldsymbol{\sigma}) = \Pi_d^h(\bar{\boldsymbol{\sigma}}). \tag{13}$$

The proof of this theorem follows from the general results in global optimization [3, 9, 10]. The canonical min–max duality can be used to find global minimizer of the nonconvex problem by the canonical dual problem $\max\{\Pi_d^h(\boldsymbol{\sigma})|\ \boldsymbol{\sigma} \in \mathscr{S}_a^+\}$, which is a concave maximization problem and can be solved easily by well-developed convex analysis and optimization techniques. The canonical double-max and double-min duality statements can be used to find the biggest local maximizer and a local minimizer of the nonconvex primal problem, respectively. It was proved in [3, 9, 10] that both the canonical min–max and double-max duality statements hold strongly regardless the dimensions of $\mathscr{U}_a^h$ and $\mathscr{S}_a^h$, while the canonical double-min duality statement (13) holds weakly for $\dim \mathscr{U}_a^h \neq \dim \mathscr{S}_a^h$, but it holds strongly if $\dim \mathscr{U}_a^h = \dim \mathscr{S}_a^h$. This case is within our reach in the following applications.

4 Semi-definite Programming Algorithm

According to Schur complement lemma [12], the global optimization problem $\min_{\mathbf{w}\in\mathscr{U}_a^h} \Pi_p^h(\mathbf{w}) = \min_{\mathbf{w}\in\mathscr{U}_a^h} \max_{\boldsymbol{\sigma}\in\mathscr{S}_a^h} \Xi(\mathbf{w}, \boldsymbol{\sigma})\ \ s.t.\ \ \mathbf{G}(\boldsymbol{\sigma}) \succeq 0,$ can be relaxed to the following SDP problem [1]:

$$\begin{aligned} &\max_{\boldsymbol{\sigma},t}\ t \\ &s.t.\ \mathbf{G}(\boldsymbol{\sigma}) \succeq 0, \quad \begin{bmatrix} 2\mathbf{K}^{-1} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma}^T & \frac{1}{2}\mathbf{w}^T\mathbf{G}(\boldsymbol{\sigma})\mathbf{w} - \boldsymbol{\lambda}^T\boldsymbol{\sigma} - \mathbf{f}^T\mathbf{w} - c - t \end{bmatrix} \succeq 0, \end{aligned} \tag{14}$$

where $\mathbf{w} = \mathbf{w}(\boldsymbol{\sigma}) = \mathbf{G}^{-1}(\boldsymbol{\sigma})\mathbf{f}$. By the fact that $\mathbf{K} \succeq 0$, the second inequality constraint implies to; $t(\mathbf{w}, \boldsymbol{\sigma}) \leq \frac{1}{2}\mathbf{w}^T\mathbf{G}(\boldsymbol{\sigma})\mathbf{w} - \boldsymbol{\lambda}^T\boldsymbol{\sigma} - \mathbf{f}^T\mathbf{w} - c$.

By the same way, the SDP relaxation for the canonical double-max duality statement, $\max_{\mathbf{w}\in\mathscr{U}_a^h} \Pi_p^h(\mathbf{w}) = \max_{\mathbf{w},\boldsymbol{\sigma}} \Xi(\mathbf{w}, \boldsymbol{\sigma}) = \max \Pi_d^h(\boldsymbol{\sigma})$ s.t. $\boldsymbol{\sigma} \in \mathscr{S}_a^-$ should be equivalent to [1]:

$$\begin{aligned} &\max_{\boldsymbol{\sigma},t}\ t \\ &s.t.\ -\mathbf{G}(\boldsymbol{\sigma}) \succeq 0, \quad \begin{bmatrix} 2\mathbf{K}^{-1} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma}^T & \frac{1}{2}\mathbf{w}^T\mathbf{G}(\boldsymbol{\sigma})\mathbf{w} - \boldsymbol{\lambda}^T\boldsymbol{\sigma} - \mathbf{f}^T\mathbf{w} - c - t \end{bmatrix} \succeq 0. \end{aligned} \tag{15}$$

which leads to a local maximum solution to the post-buckling problem.

To find the local minimum for the beam post-buckling problem, it is appropriate to use the following new formula of pure complementary energy [1]:

$$\widehat{\Pi}^d(\boldsymbol{\sigma}, \mathbf{w}) = -\frac{1}{2}\mathbf{f}^T\ \mathbf{G}^{-1}(\boldsymbol{\sigma})\ \mathbf{f} - \frac{1}{2}\mathbf{w}^T\ \mathbf{M}(\boldsymbol{\sigma})\ \mathbf{w} - \frac{1}{2}\boldsymbol{\lambda}^T\ \boldsymbol{\sigma} - c. \tag{16}$$

The SDP relaxation for the canonical double-min duality statement $\min_{\mathbf{w}} \Pi_p^h(\mathbf{w}) = \min_{\mathbf{w},\boldsymbol{\sigma}} \Xi(\mathbf{w}, \boldsymbol{\sigma}) = \min_{\mathbf{w},\boldsymbol{\sigma}} \widehat{\Pi}^d(\boldsymbol{\sigma}, \mathbf{w})$ s.t. $\boldsymbol{\sigma} \in \mathscr{S}_a^-$ and for $\mathbf{w} = \mathbf{w}(\boldsymbol{\sigma})$ should be equivalent to

$$\min_{\boldsymbol{\sigma},t} \; t$$
$$s.t. \; -\mathbf{G}(\boldsymbol{\sigma}) \succ 0, \quad \begin{bmatrix} -2\mathbf{G}(\boldsymbol{\sigma}) & \mathbf{f} \\ \mathbf{f}^T & \frac{1}{2}\mathbf{w}^T\mathbf{M}(\boldsymbol{\sigma})\mathbf{w} + \frac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{\sigma} + c + t \end{bmatrix} \succeq 0. \tag{17}$$

Where $\mathbf{M}(\boldsymbol{\sigma})$ is obtained by assembling the following symmetric matrices $M^e(\sigma_e)$:

$$M^e(\sigma_e) = \int_{\Omega_e} \frac{1}{2}\left((N_\sigma)^T \; \sigma_e \; N'_w \; (N'_w)^T \right) dx \tag{18}$$

The post-buckling configurations of a large deformed nonlinear beam can be found by the following steps:

1. With an initial point $\mathbf{w}^{(k=1)}$, the next steps are repeated as $\mathbf{w}^{(k+1)}$ converges to the solution.
2. Find $\boldsymbol{\sigma}^{(k+1)}$ by applying SDP algorithm for global maximizer and local minimizer problems in (15) and (17), respectively.
3. Compute $\mathbf{w}^{(k+1)} = \mathbf{G}^{-1}(\boldsymbol{\sigma}^{(k+1)})\mathbf{f}$.
4. Check convergence; if $\|\mathbf{w}^{(k+1)} - \mathbf{w}^{(k)}\| / \|\mathbf{w}^{(k)}\| \le \epsilon$, stop with optimal solution $\mathbf{w}^* = \mathbf{w}^{(k+1)}$, where ϵ is a small positive real number. Otherwise, put $k = k + 1$ and return to step 2.

For applying SDP algorithm, a software package named SeDuMi [13] is used to solve the problems (15) and (17) via the interior point method.

5 Numerical Solutions with Different Dual Stress Interpolations

According to the triality theory, the canonical double-min duality statement (13) holds strongly if $\dim \mathscr{U}_a^h = \dim \mathscr{S}_a^h$. So, the piecewise-quadratic stress ($\delta = 2$) is the most convenient to verify this theory to obtain closed dimensions between the discretized displacement $\mathbf{w} \in R^{2(m+1)}$ and discretized stress $\boldsymbol{\sigma} \in R^{\delta m+1}$. But these two dimensions are still not equal. However, it is possible to make these dimensions equal if we use mixed different dual stress interpolations on the elements of the same beam. So, many mixed meshes of dual stress interpolations are used in this paper beside to the "PLS mesh" and "PQS mesh" in order to improve the local unstable buckled configuration solution of a large deformed beam.

We present four different types of beams which are controlled by different boundary conditions. Some geometrical data are kept fixed for all computations; $E = 1000\,\text{Pa}$, $\nu = 0.3$, $L = 1\,\text{m}$, $h = 0.05\,\text{m}$ with an odd number of beam elements

$m = 51$. Different loading conditions, including both axial and transverse arrangements, are considered in our applications.

5.1 Simply Supported Beam

A simply supported beam model is fixed in both directions at $x = 0$ and fixed only in the y-direction at $x = L$ as shown in Fig. 3-a. By applying the boundary conditions, $w(0) = w''(0) = w(L) = w''(L) = 0$, two elements of discretized displacement $\mathbf{w} = \{w_e\} \in R^{2(m+1)}$ should be zero. Then, the remaining nonzero elements of the vector w is $(2m)$. We used three types of dual stress interpolations to construct a mixed mesh of dual stress fields in order to obtain $\dim \mathcal{U}_a^h = \dim \mathcal{S}_a^h$. The PQS is applied on $(m - 3)$ elements and the PCS is used for only one element that is on the central of the beam. While the PLS is applied on two beam elements which surround the central element as shown in "*Mesh-1*" in Fig. 4. So, we have $dim(\boldsymbol{\sigma}) = dim(\mathbf{w}) = 2m$, and this dimension equals 102 for $m = 51$. The critical load of the simply supported beam is $\lambda_{cr} = 0.00097m^2$, see Eq. (5). The approximate deflections with $\lambda > \lambda_{cr}$ under both of uniformly distributed load and concentrated force are shown in Figs. 5 and 6, respectively.

5.2 Doubly/Clamped Beam

Doubly/clamped beam is fixed at both ends (see Fig. 3-c). The boundary conditions, $w(0) = w'(0) = w(L) = w'(L) = 0$, force the first two and the last two elements

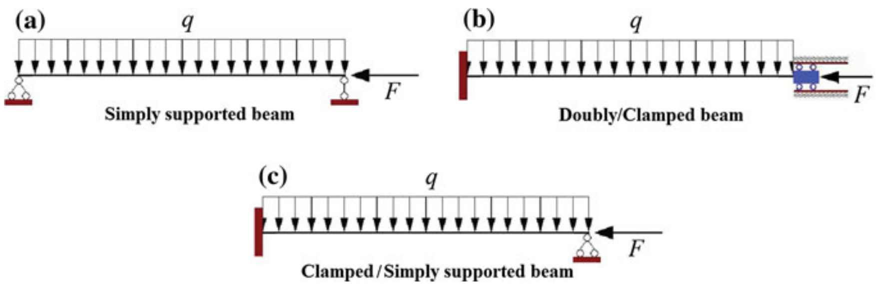

Fig. 3 Different types of beams

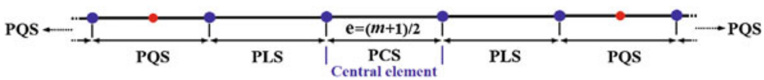

Fig. 4 **Mesh-1**: Mixed dual stress interpolations of beam elements

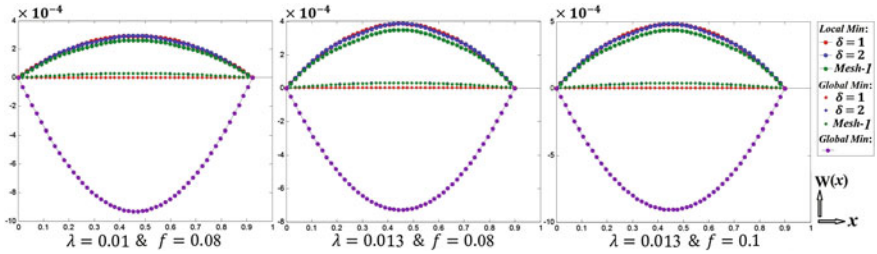

Fig. 5 Post-buckling solutions of simply supported beam under uniformly distributed load

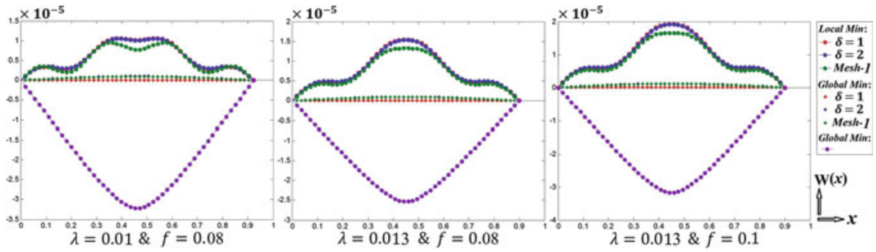

Fig. 6 Post-buckling solutions of simply supported beam under a concentrated force

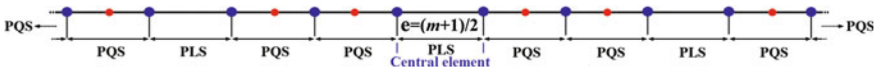

Fig. 7 **Mesh-3**: Mixed dual stress interpolations of beam elements

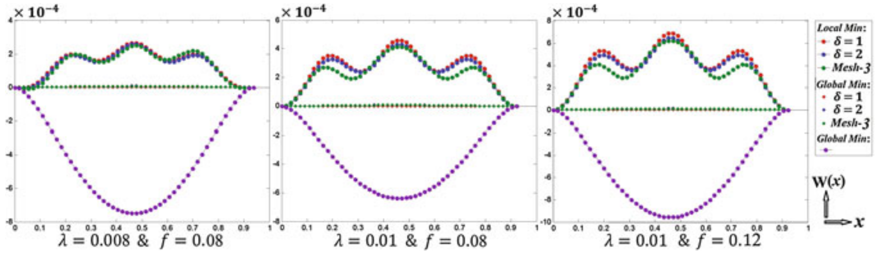

Fig. 8 Post-buckling configurations of clamped beam under uniformly distributed load

of discretized displacement w to be zero. Thus, the remaining nonzero element of displacement vector is $(2m - 2)$. The selected mixed mesh for dual stress field is "*Mesh-3*" which contains $(m - 3)$ of PQS, while PLS is used for three beam elements (see Fig. 7). For $m = 51$, the $dim(\boldsymbol{\sigma}) = dim(\mathbf{w}) = 100$. The approximate deflections for $\lambda > \lambda_{cr}$ with $\lambda_{cr} = 0.0041m^2$ under uniformly distributed load and concentrated force are shown in Figs. 8 and 9, respectively.

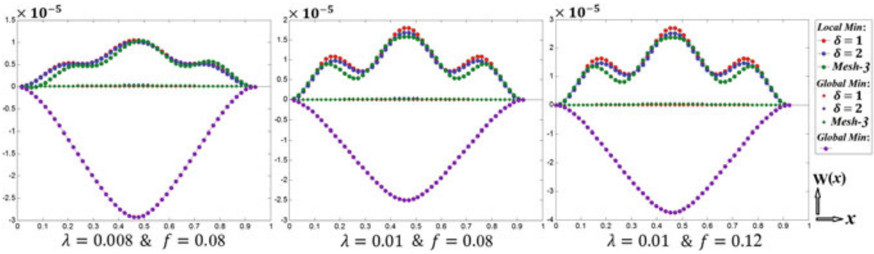

Fig. 9 Post-buckling configurations of clamped beam under a concentrated force

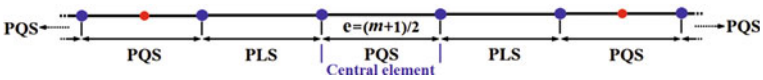

Fig. 10 **Mesh-4**: Mixed dual stress interpolations of beam elements

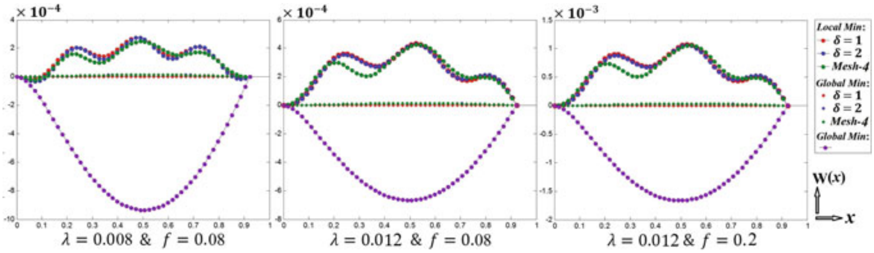

Fig. 11 Post-buckling configurations of clamped/simply supported beam under uniformly distributed load

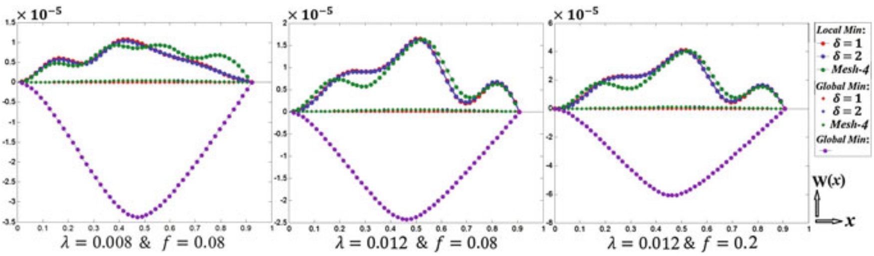

Fig. 12 Post-buckling configurations of clamped/simply supported beam under a concentrated force

5.3 Clamped/Simply Supported Beam

Clamped/simply supported beam is clamped at $x = 0$ and fixed in both directions at $x = L$ as shown in Fig. 3-d. Three elements of discretized displacement w should be zero after applying the boundary conditions; $w(0) = w'(0) = w(L) = w''(L) = 0$. The remaining nonzero element of w is $(2m - 1)$. The "*Mesh-4*" is designed by applying two different dual stress interpolations. The PQS is applied for $(m - 3)$

beam elements, while the PLS is applied on two beam elements which surround the central element as shown in Fig. 10. Thus, for $m = 51$, the $dim(\boldsymbol{\sigma}) = dim(\mathbf{w}) = 101$. The critical load of this beam is $\lambda_{cr} = 0.0034m^2$. The approximate deflections under uniformly distributed load and concentrated force are shown in Figs. 11 and 12, respectively.

6 Conclusions

This paper presents a CD-FEM for the post-buckling analysis with a large elastic deformations beam which is governed by a fourth-order nonlinear differential equation which was introduced by Gao in 1996. The generalized total complementary energy $\Xi(\mathbf{w}, \boldsymbol{\sigma})$ associated with this model is a nonconvex functional and was used to study the post-buckling problems. Combining the generalized total complementary energy and the proposed formula of pure complementary energy $\widehat{\Pi}^d(\boldsymbol{\sigma}, \mathbf{w})$ with the triality theory, a canonical duality algorithm is studied for solving post-buckling problems using SDP algorithm. According to the triality theory, the dimensions of discretized displacement and dual stress have been made equal by designing a number of mixed meshes of different dual stress interpolations. Different boundary conditions and different loading conditions, including both axial and transverse arrangements are considered in our applications. The numerical results show that the global minimizer and local maximizer of the total potential energy are stable buckled configuration for different dual stress meshes. While the local minimizer present unstable deformation states and the solutions of unstable buckled state is sensitive to both stress interpolations and external loads.

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