Chapter 7
The Geometric Spectrum of a Graph and Associated Curvatures

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Abstract We approach the problem of defining curvature on a graph by attempting to attach a 'best-fit polytope' to each vertex, or more precisely what we refer to as a configured star. How this should be done depends upon the global structure of the graph which is reflected in its geometric spectrum. Various curvatures naturally arise from local liftings of the graph into a suitable Euclidean space.

7.1 Introduction

One of the challenges of graph theory is to define notions of curvature purely in terms of combinatorial structure without recourse to a predefined metric structure. One would like to see geometry and curvature *emerge* from combinatorial structure rather than being imposed upon it. Different approaches go back to classical work of Descartes in the context of 3-dimensional polyhedra. In this volume, Chap. 6, M. Keller discusses a notion of combinatorial curvature that arises when there exists an embedding of a graph into a surface. Defined at each vertex, this doesn't depend on any metric structure, but only on the number of vertices (the face degree) of incident faces. The Ricci curvature of Ollivier, studied in Chap. 1 of this volume, is defined in terms of optimal transport of local measures. Its relation to the clustering coefficient, a measure of local connectedness, has been explored by Jost and Liu [19], see also Chap. 1. Common themes occur in this chapter: on the one hand local embeddings of a graph arise from the geometric spectrum leading to notions of curvature; on the other hand connectedness appears to have a crucial influence on the nature of the geometric spectrum, and so on the local geometry that arises.

Our approach is to appeal directly to the way sense data correlates with our brains to infer 3-dimensionality, just as we visualize a 3D-cube in the planar graph illustrated in Fig. 7.1.

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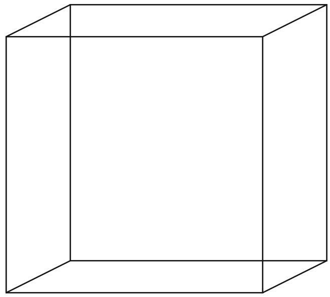


Fig. 7.1 A planar graph can produce the mental image of a 3D-object

Table 7.1 The geometric spectral values associated to the convex regular polyhedra

Polyhedron	γ
Tetrahedron	$3/4$
Cube	0
Octahedron	$1/2$
Icosahedron	$\frac{2-\sqrt{5}}{3-\sqrt{5}} < 0$
Dodecahedron	$\frac{3(1-\sqrt{5})}{2(3-\sqrt{5})} < 0$

More specifically, we would like metric structure and curvature to emerge from combinatorial structure, but in a way that derives from potential geometric realizations. In order to achieve this end, we introduce what we refer to as the *geometric spectrum* of a graph. An element of this spectrum is a real-valued function defined on the vertices which need not be constant, and occurs as the parameter γ in the quadratic difference equation:

$$\gamma(\Delta\phi)^2 = (\nabla\phi)^2, \tag{7.1}$$

where ϕ is a complex-valued function on the vertices (for definitions, see below), thought of as an orthogonal projection to the complex plane of a (local) realization of the graph in a Euclidean space. A requirement is that γ should remain invariant under any similarity transformation of this realization. In this way, the geometry that arises is implicit in the graph, rather than being imposed by the way it is embedded into Euclidean space. As an example, the convex regular polytopes in Euclidean 3-space satisfy (7.1) with γ constant as given in Table 7.1.

To a solution ϕ, we cannot in general hope for a global realization of the graph, just a local one of each vertex and its neighbors. However, this is sufficient to enable one to develop notions of (relative) edge length and curvature. More precisely, a non-trivial solution allows one to fit a *configured star* to each vertex and its neighbors, giving an invariant lifting into Euclidean space. A configured star generalizes the vertex figure of a regular polytope, so an intuitive picture to have in mind, is that of attempting to attach a 'best-fit' regular polytope to each vertex, just as in smooth geometry we can approach curvature via 'best-fit' circles, or spheres. How we do this depends on the global structure of the graph as reflected in its geometric spectrum. Such lifts to a configured star in Euclidean 3-space are essentially unique up to a sign ambiguity, however, if we wish to extend the lifting beyond the immediate neighbors of a vertex, then in general one needs to relax this condition and allow more general invariant stars.

An embedded graph is one that can be embedded in a surface. This allows one to associate faces to the graph, which in turn determines a combinatorial quantity called *face degree*, defined to be the number of boundary edges (or vertices) of each face. In this volume, Chap. 6, M. Keller discusses a natural combinatorial curvature associated to an embedded graph, and in particular, its relation to the so-called physical Laplacian (rather than the harmonic, or Tutte Laplacian that appears in (7.1) above). In view of Theorem 7.1 below, which relates our combinatorial construction to smooth submanifolds, there should be strong connections between these two approaches.

We begin in Sect. 7.2 by defining the geometric spectrum and in Sect. 7.3 we derive the lifting properties discussed above. For an arbitrary graph, the geometric spectrum may be quite complicated and difficult to compute. We indicate how this may be done using Gröbner bases in Appendix 1 and give some examples. Also, in this appendix we introduce a polynomial graph invariant, the γ-polynomial, whose roots determine the constant elements of the spectrum. At present we know of no two non-isomorphic graphs with the same γ-polynomial. Connections with vertex colorings of a graph are discussed in respect of a particular example.

The most immediate consequence of the lift to a configured star, explored in Sect. 7.4, is the designation of a Gauss map at each vertex given by the axis of the star. If the lift is into $\mathbb{R}^3$, then this defines a point on the 2-sphere, which we can connect to other points according to whether or not they are connected by an edge in the original graph. The geodesic distance between two such points now gives an *edge* curvature on the graph. Another consequence of the lift is to endow each edge with a length, which in turn allows us to define the distance between two points, for which we establish a triangle inequality. The graph now has the structure of a path metric space and we can explore notions of curvature in the sense of Alexandrov, by comparing triangles with those in a 2-dimensional space form (see also Chap. 2 in this book). To establish curvature bounds in the general case requires further investigation, but we indicate how this may be done with an example.

Our main goal is to define curvature in a way that depends only on the element γ of the geometric spectrum and not on the solution ϕ to (7.1). For Gaussian curvature, defined in terms of angular deficit at a vertex, this is possible provided we accept an approximation to well-known classical theorems for polytopes (Sect. 7.5). In Sect. 7.6, we show how to define sectional curvature and Ricci curvature in terms of the edge curvature. Finally, in Appendix 2, we explore methods to construct solutions on new graphs from given solutions. An interesting model occurs in the theory of random graphs which suggests how geometry could naturally emerge in scale-free networks, so prevalent in biology and social networks.

7.2 The Geometric Spectrum

We use the notation $G = (V, E)$ to denote a graph with vertex set V and edge set E. Graphs will be assumed simple (no multiple edges or loops) and undirected. We will suppose also that G is locally finite, that is the degree of each vertex is finite. For $x \in V$ define the *tangent space to G at x* to be the set of oriented edges with base point x together with the zero vector: $T_xG = \{\mathbf{xy} : y \sim x\} \cup \{\mathbf{0}\}$. Define the *tangent bundle to G* to be the union: $TG = \cup_{x \in V} T_xG$. Inclusion of the zero vector is useful when we come to discuss holomorphic mappings between graphs in Appendix 2.

A 1-*form* on G is a map $\omega : TG \to \mathbb{C}$ such that $\omega(\mathbf{xy}) = -\omega(\mathbf{yx})$ and $\omega(\mathbf{0}) = 0$. To a function $\phi : V \to \mathbb{C}$, we can naturally associate a 1-form, the *derivate* $\nabla\phi$, by setting $\nabla\phi(\mathbf{xy}) = \phi(y) - \phi(x)$ and $\nabla\phi(\mathbf{0}) = 0$. For differential calculus on a simplex, see the book by Romon [24], where the notion of 1-form coincides with that given above (Sect. IV, Section 1.1 of [24]).

For two 1-forms ω, η, define their *pointwise symmetric product at $x \in V$* by

$$\langle \omega, \eta \rangle_x = \frac{1}{d_x} \sum_{y \sim x} \omega(\mathbf{xy})\eta(\mathbf{xy}) \,,$$

where d_x is the degree of vertex x. The above definition is the complex symmetric analogue of standard L^2 products that arises in functional analytic theory on a graph; in the latter situation it is replaced by a Hermitian product rather than a symmetric product, see Chap. 1 in this volume. Write $(\nabla\phi)^2(x)$ for the *symmetric square of the derivative* of the function $\phi : V \to \mathbb{C}$:

$$(\nabla\phi)^2(x) := \frac{1}{d_x} \sum_{y \sim x} \big(\phi(y) - \phi(x)\big)^2 \,.$$

The Tutte Laplacian (or harmonic Laplacian) on G is defined by

$$\Delta\phi(x) = \frac{1}{d_x} \sum_{y \sim x} \big(\phi(y) - \phi(x)\big) \,.$$

The choice of conventions means that the spectrum is negative and lies in the interval $[-2, 0]$.

Given a graph $G = (V, E)$ together with a real-valued function $\gamma : V \to \mathbb{R}$, we are interested in the equation:

$$\frac{\gamma(x)}{d_x} \Big(\sum_{y \sim x} \big(\phi(y) - \phi(x)\big) \Big)^2 = \sum_{y \sim x} \big(\phi(y) - \phi(x)\big)^2 \,, \tag{7.2}$$

at each vertex x, where $\phi : V \to \mathbb{C}$ is a complex-valued function. In the notation above, this has the convenient form of (7.1) of the Introduction. Solutions with

$\gamma \equiv 0$ have been called *holomorphic functions*[1] and have been used to give a description of massless fields in a combinatorial setting [4], see also Appendix 2. Note that the equations are invariant under the transformations

$$\phi \mapsto \lambda\phi + \mu \quad (\lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{C}), \quad \text{and} \quad \phi \mapsto \overline{\phi}. \tag{7.3}$$

We shall consider two solutions related in this way as *equivalent*. Equation (7.2) only depends on the derivative $\nabla\phi$ and more generally can be defined for an arbitrary 1-form ω on replacing $\phi(y) - \phi(x)$ with $\omega(\mathbf{xy})$.

If in (7.2) we label the neighbors of x by $y_1, \ldots, y_d$ and write $z_\ell = \phi(y_\ell) - \phi(x)$, then if the z_ℓ are all real, an application of the Cauchy-Schwarz inequality gives

$$\sum_{\ell=1}^{d} {z_\ell}^2 = \frac{\gamma}{d}\left(\sum_{\ell=1}^{d} z_\ell\right)^2 \leq \gamma \sum_{\ell=1}^{d} {z_\ell}^2, \tag{7.4}$$

so that if the z_ℓ are not all zero, necessarily $1 \leq \gamma$. Otherwise said, if $\gamma < 1$ at least one of the z_ℓ must be complex. When defining the geometric spectrum below, we impose the condition $\gamma(x) \leq 1$ for all $x \in V$ with the inequality strict if $d_x \geq 3$. This is necessary for our invariance requirement that we discuss later.

For a given graph, we would like to know what are the admissible functions $\gamma : V \to \mathbb{R}$ for which (7.2) has a solution. Define the *geometric spectrum* of G to be the collection of equivalence classes of such functions:

$$\Sigma = \{\gamma : V \to [-\infty, 1] \subset \mathbb{R} : \exists \text{ non} - \text{const } \phi : V \to \mathbb{C} \text{ satisfying (7.2)} \\ \textit{with } \gamma(x) < 1 \text{ if } d_x \geq 3\},$$

where two functions are identified when they determine a common solution ϕ and agree on the compliment of the set $\{x \in V : \Delta\phi(x) = (\nabla\phi)^2(x) = 0\}$. The function γ may take on the value $-\infty$ at points where the Laplacian vanishes.

By a *framework* in Euclidean space, we mean a graph that is realized as a subset of Euclidean space with edges straight line segments joining the vertices. We say that it is *immersed* if all vertices are distinct and *embedded* if it is immersed and edges only intersect at end points. The framework is called *invariant* if for a particular γ, it satisfies (7.2) with ϕ the restriction to the vertices of some orthogonal projection to the complex plane *independently of any similarity transformation of the framework*.

[1] A notion of *holomorphic function* somewhat similar to this has been introduced by S. Barré and A. Zeghib [7]; however, in addition to (7.2) with $\gamma \equiv 0$, Barré and Zeghib require that ϕ be harmonic. An alternative notion of *discrete holomorphic function* in the special case of quad-graphs is given by Bobenko et al. [8].

Questions that arise are:

- For a given graph G, what is its geometric spectrum?
- Does a solution to (7.2) arise from an embedding of the graph as an invariant framework in Euclidean space?
- Even if the answer to the last question is no, can we still derive geometric quantities such as edge length and curvature from a solution?
- To what extent do such quantities depend only on γ rather than on the choice of solution ϕ?

For an arbitrary graph, the geometric spectrum is determined by a fairly complicated set of algebraic equations. For graphs of sufficiently small order, these can be solved with computer software, for example MAPLE; see Appendix 1. One can check that for the complete graph on $N+1$ vertices, with $2 \leq N \leq 5$, the geometric spectrum consists of the single value $N/(N+1)$, with corresponding invariant realization as the 1-skeleton of a regular simplex in $\mathbb{R}^N$. It is not known if this remains so for $N > 5$. At the other extreme, for a cyclic or linear graph (connected graphs with the least internal connections), the geometric spectrum arises from realizations in the plane of corresponding frameworks with all edges of equal length. Now, the geometric spectrum has continuous components with complicated branching behavior. These properties are discussed in Appendix 1.

In general, after taking into account the freedom (7.3), an equation count shows that for a graph on N vertices, (7.2) is equivalent to $2N$ real equations in $3N-4$ real parameters, together with the N inequalities: $\gamma < 1$ at each vertex. Empirical evidence indicates that the more connected the graph, the more restricted its geometric spectrum. Another question that now arises is:

- Does the geometric spectrum have any relation to the Laplace spectrum?

The spectrum of the Tutte Laplacian is reviewed in Sect. 1.6.1 of Chap. 1 of this volume, and for the physical Laplacian, see Chap. 6. Any connection is not immediately obvious, since both have different structures: for a graph on N vertices, its (Tutte) Laplacian has precisely N eigenvalues, whereas, as noted above, the geometric spectrum can range from a single element to continuous components. The structure of the geometric spectrum seems to be related to rigidity in the context of bar-and-joint frameworks, although this connection needs to be explored further. In recent work, Zelazo et al. introduce a rigidity matrix associated to a framework which they show is similar (in the mathematical sense) to a weighted graph Laplacian; the eigenvalues of the rigidity matrix are related to the rigidity of the bar-and-joint framework [29]. However, the rigidity that occurs in our situation is not quite the same. Although arising from embeddings in Euclidean space (see Sect. 7.3 below), edge length can change through different realizations. Examples occur with both the dodecahedron and icosahedron, which are one of a family of realizations of graphs consisting of their respective 1-skeletons with vertices given by the columns of the matrices (7.25) and (7.26) below.

In order to interpret γ, it is instructive to consider Eq. (7.1) for the case of a smooth hypersurface in Euclidean space, where we find an interesting connection with mean-curvature.

Theorem 7.1 ([2]) *Let M^n be a smooth hypersurface in $\mathbb{R}^{n+1}$ $(n \geq 1)$ and let g denote the metric on M^n induced from the standard metric on $\mathbb{R}^n$. Let $\phi : (M^n, g) \to \mathbb{C}$ be any orthogonal projection; then*

$$(\Delta\phi)^2 = -H^2(\nabla\phi)^2 \,, \tag{7.5}$$

where H is the mean curvature of M^n, and where in local coordinates, $\Delta\phi = g^{ij}(\phi_{ij} - \Gamma_{ij}^k\phi_k)$ and $(\nabla\phi)^2 = g^{ij}\phi_i\phi_j$ (summing over repeated indices).

In the case when $n = 1$, the theorem confirms the identity

$$c''(s) = \kappa(s)ic'(s) \,,$$

for a regular curve $c : I \subset \mathbb{R} \to \mathbb{C}$ parametrized with respect to arc length. It is necessary that M^n be a *hypersurface* in order to satisfy the smooth version of (7.1). For example, consider the surface in $\mathbb{R}^4$ parametrized in the form:

$$(x^1, x^2) \mapsto (x^1, x^2, x^1x^2, x^1 + x^2) \,.$$

Let $\phi : \mathbb{R}^4 \to \mathbb{C}$ be the projection $\phi(x^1, x^2, x^3, x^4) = x^1 + x^2 i$. Then it is readily checked that the function γ defined by (7.1) is not even real.

Given the above theorem, we expect an invariant framework that closely coincides with a smooth hypersurface to have γ approximately equal to $-1/H^2$ modulo a scaling factor (Eq. (7.5) is not scale invariant; in order to make it so, a volume term should be added).

The study of constant elements of the geometric spectrum is particularly interesting and leads to the association of a polynomial invariant to a finite graph, which we refer to as the γ-polynomial. Its definition is given in Appendix 1, where examples are given of its construction. Further work needs to be done to understand if this invariant is related to other polynomial invariants and to what extent it distinguishes isomorphism classes of graphs.

7.3 Invariant Stars and the Lifting Problem

A star graph, or bipartite graph $K_{1,d}$, has one internal vertex connected to d external vertices; there are no other connections. A star framework in $\mathbb{R}^N$ with internal vertex located at the origin can be specified by a $(N \times d)$-matrix W whose columns are the components of the external vertices. We will refer to W as the *star matrix*. Provided the center of mass of the external vertices does not coincide with the origin, then it defines a line through the origin which we refer to as the *axis of the star*. We are interested in a particular class of star frameworks whose external vertices form what we call a configuration in a plane orthogonal to the axis of the star.

7.3.1 Configured Stars

For $N \geq 2$, a collection of points $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ in $\mathbb{R}^{N-1}$ forms a *configuration* if the $((N-1) \times d)$-matrix $U = (\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_d)$ whose columns have as components the coordinates $v_{\ell j}$ of $\mathbf{v}_\ell$ $(j = 1, \ldots, N-1;\ \ell = 1, \ldots, d)$, satisfies:

$$UU^t = \rho I_{N-1}\,, \qquad \sum_{\ell=1}^{d} \mathbf{v}_\ell = \mathbf{0}\,, \tag{7.6}$$

for some non-zero constant ρ (necessarily positive), where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{N-1}$ and U^t denotes the transpose of U. Necessarily $\mathrm{rank}(U) = N-1$ so that $d \geq N$. A star in $\mathbb{R}^N$ whose external vertices form a configuration in a plane not passing through the origin, is referred to as a *configured star*. If all edges are of identical length, we call the star *regular*. An *invariant* of a configured star is a quantity that is invariant by orthogonal transformation. The following lemma characterizes configured stars [3].

Lemma 7.1 *Consider a configured star in $\mathbb{R}^N$ $(N \geq 2)$ with internal vertex the origin connected to d external vertices $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ $(d \geq N)$. Let $W = (\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_d)$ be the $(N \times d)$-matrix whose columns are the components $x_{\ell j}$ of $\mathbf{x}_\ell$ $(j = 1, \ldots, N;\ \ell = 1, \ldots, d)$. Then*

$$WW^t = \rho I_N + \sigma \mathbf{u}\mathbf{u}^t, \qquad \sum_{\ell=1}^{d} \mathbf{x}_\ell = \sqrt{d(\sigma+\rho)}\,\mathbf{u}\,, \tag{7.7}$$

where the unit vector $\mathbf{u} \in \mathbb{R}^N$ is the axis of the star, $\rho > 0$ and $\rho + \sigma > 0$. The quantities d, ρ, σ are all invariants of the star; the vector $\mathbf{u}$ is normal to the affine plane containing $\mathbf{x}_1, \ldots, \mathbf{x}_d$.

Conversely, any matrix $W = (\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_d)$ satisfying (7.7) determines a configured star with internal vertex the origin and external vertices $\mathbf{x}_1, \ldots, \mathbf{x}_d$.

Proof Consider a configured star in standard position given by (7.6). Set

$$V = \left(\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_d \\ c & c & \cdots & c \end{array} \right)$$

and let $A : \mathbb{R}^N \to \mathbb{R}^N$ be an orthogonal transformation; set $\mathbf{x}_d = A \begin{pmatrix} \mathbf{v}_d \\ c \end{pmatrix}$. Then $W = (\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_d) = AV$ and

$$WW^t = AVV^tA^t = \rho I_N + \sigma (A\mathbf{e}_N)(A\mathbf{e}_N)^t\,,$$

where

$$\sigma = dc^2 - \rho\,. \tag{7.8}$$

Furthermore $\sum_{\ell=1}^{d} \mathbf{x}_\ell = dcA\mathbf{e}_N$, which gives the form (7.7) with $\mathbf{u} = A\mathbf{e}_N$. The independence of the quantities d, ρ, σ under the orthogonal transformation A is clear.

Conversely, suppose we are given an $(N \times d)$-matrix $W = (\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_d)$ satisfying (7.7). Let A be an orthogonal transformation such that $A\mathbf{u} = \mathbf{e}_N$ and let $V = AW$. Write

$$V = \left(\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_d \\ y_{1N} & y_{2N} & \cdots & y_{dN} \end{array} \right).$$

Then

$$VV^t = \rho I_N + \sigma \mathbf{e}_N {\mathbf{e}_N}^t \qquad \text{and} \qquad \sum_\ell \begin{pmatrix} \mathbf{v}_\ell \\ y_{\ell N} \end{pmatrix} = \sqrt{d(\sigma+\rho)}\,\mathbf{e}_N\,, \tag{7.9}$$

so that $\sum_\ell \mathbf{v}_\ell = 0$ and $\sum_\ell y_{\ell N} = \sqrt{d(\sigma+\rho)}$. Furthermore, (7.9) implies that $\sum_\ell {y_{\ell N}}^2 = \rho + \sigma$. In particular

$$d \sum_\ell^d {y_{\ell N}}^2 = \left(\sum_\ell^d y_{\ell N} \right)^2 .$$

But then equality in the inequality (7.4) implies that $y_{1N} = y_{2N} = \cdots = y_{dN} = \sqrt{(\sigma+\rho)/d}$. □

Corollary 7.1 *Let* $W = (\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_d)$ *define a configured star and let* $\phi : \mathbb{R}^N \to \mathbb{C}$ *be the orthogonal projection* $\phi(y_1, y_2, \ldots, y_N) = y_1 + \mathrm{i}y_2$. *Then if* $z_\ell = \phi(\mathbf{x}_\ell) = x_{\ell 1} + \mathrm{i}x_{\ell 2}$, *we have*

$$\frac{\sigma}{d(\sigma+\rho)} \left(\sum_{\ell=1}^d z_\ell \right)^2 = \sum_{\ell=1}^d {z_\ell}^2 ,$$

where ρ *and* σ *are given by* (7.7). *In particular, with reference to Eq.* (7.2), $\gamma = \sigma/(\sigma+\rho)$ *is real and depends only on the star invariants.*

Proof Let $\mathbf{u} = (u_1, \ldots, u_N)$ be the unit normal to the plane of the star. Then for each $j = 1, \ldots, N$, we have

$$\sum_{\ell=1}^d x_{\ell j} = \sqrt{d(\sigma+\rho)}\, u_j .$$

Thus

$$\begin{aligned} \left(\sum_{\ell=1}^d z_\ell \right)^2 &= \sum_{k,\ell=1}^d (x_{k1}x_{\ell 1} - x_{k2}x_{\ell 2} + 2\mathrm{i}x_{k1}x_{\ell 2}) \\ &= d(\sigma+\rho)({u_1}^2 - {u_2}^2 + 2\mathrm{i}u_1u_2) = d(\sigma+\rho)(u_1 + \mathrm{i}u_2)^2 , \end{aligned}$$

whereas

$$\sum_{\ell=1}^{d} z_\ell{}^2 = \sum_{\ell=1}^{d} (x_{\ell 1}{}^2 - x_{\ell 2}{}^2 + 2\mathrm{i} x_{\ell 1} x_{\ell 2}) = (WW^t)_{11} - (WW^t)_{22} + 2\mathrm{i}(WW^t)_{12} = \sigma(u_1 + \mathrm{i} u_2)^2 \,.$$

The formula now follows. □

7.3.2 Invariant Stars

To test whether a framework in Euclidean space is invariant, it suffices to see whether the star about each of its vertices is invariant at the internal vertex. A consequence of Corollary 7.1 is that any *configured* star is invariant at its internal vertex. The star framework about the vertex of any regular polytope is configured, so that the underlying framework of a regular polytope is invariant [3]. On the other hand, not all invariant stars are configured. For example, the star in $\mathbb{R}^3$ with $2r$ external vertices represented by the columns of the $(3 \times (2r))$-matrix

$$W = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_1 & x_2 & \cdots & x_r \\ s_1 & s_2 & \cdots & s_r & -s_1 & -s_2 & \cdots & -s_r \\ t_1 & t_2 & \cdots & t_r & -t_1 & -t_2 & \cdots & -t_r \end{pmatrix} ,$$

where the vectors $\mathbf{s} = (s_1, \ldots, s_r)$ and $\mathbf{t} = (t_1, \ldots, t_r)$ are orthogonal and of the same length, is invariant, but it is only configured when $x_1 = x_2 = \cdots = x_r$. As a specific case, take W to be the star on four vertices represented by the following matrix.

$$W = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{7.10}$$

The external vertices lie on the corners of a cube as indicated in Fig. 7.2.

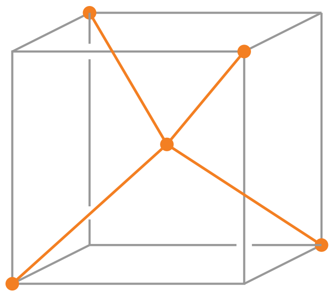


Fig. 7.2 An invariant star which is not configured

Contrary to a configured star, the external vertices don't lie in any half-space whose boundary passes through the internal vertex. When we come to discuss Gaussian curvature in Sect. 7.5, such a star will have negative curvature associated to the internal vertex. On the other hand, a configured star will always have positive curvature at the internal vertex.

We can give a characterization of all invariant stars in $\mathbb{R}^3$ with the following.

Proposition 7.1 *Consider a general star in* $\mathbb{R}^3$ *with matrix*

$$W = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & x_{22} & \cdots & x_{2d} \\ x_{31} & x_{32} & \cdots & x_{3d} \end{pmatrix},$$

where we assume that $d \geq 2$*, that the columns of* W *are non-zero and that there is at least one* $i = 1, 2, 3$ *such that* $\sum_{\ell=1}^{d} x_{i\ell}$ *is non-zero. Then* W *is invariant if and only if there exists a real number* γ *such that*

$$\frac{\gamma}{d}\left[\left(\sum_{\ell=1}^{d} x_{i\ell}\right)^2 - \left(\sum_{\ell=1}^{d} x_{j\ell}\right)^2\right] = \sum_{\ell=1}^{d} {x_{i\ell}}^2 - \sum_{\ell=1}^{d} {x_{j\ell}}^2 \tag{7.11}$$

$$\frac{\gamma}{d}\left(\sum_{\ell=1}^{d} x_{i\ell}\right)\left(\sum_{\ell=1}^{d} x_{j\ell}\right) = \sum_{\ell=1}^{d} x_{i\ell}x_{j\ell} \tag{7.12}$$

for all $i, j = 1, 2, 3,\ i \neq j$*.*

Proof Let $A = (a_{ij})$ be an arbitrary orthogonal transformation. Then

$$AW = \begin{pmatrix} a_{1j}x_{j1} & a_{1j}x_{j2} & \cdots & a_{1j}x_{jd} \\ a_{2j}x_{j1} & a_{2j}x_{j2} & \cdots & a_{2j}x_{jd} \\ a_{3j}x_{j1} & a_{3j}x_{j2} & \cdots & a_{3j}x_{jd} \end{pmatrix}$$

where, in the matrix, we sum over repeated indices. Now project to the complex plane via $(y_1, y_2, y_3) \mapsto y_i + \mathrm{i}y_2$, so that the end points of the star are projected to the complex numbers

$$z_\ell = \sum_{j=1}^{3}(a_{1j} + \mathrm{i}a_{2j})x_{j\ell}, \quad \ell = 1, \ldots, d.$$

In order that W represents an invariant star we require that

$$\frac{\gamma}{d}\left(\sum_{\ell=1}^{d} z_\ell\right)^2 = \sum_{\ell=1}^{d} {z_\ell}^2, \tag{7.13}$$

for some real number γ independently of A. Since A is an orthogonal matrix, we have

$$\sum_{j=1}^{3}(a_{1j}+\mathrm{i}a_{2j})^2=0 \quad \Rightarrow \quad (a_{13}+\mathrm{i}a_{23})^2=-(a_{11}+\mathrm{i}a_{21})^2-(a_{12}+\mathrm{i}a_{22})^2\,,$$

and (7.13) is equivalent to

$$\begin{aligned}
&\frac{\gamma}{d}\Bigg\{(a_{11}+\mathrm{i}a_{21})^2\left(\left(\sum_{\ell=1}^{d}x_{1\ell}\right)^2-\left(\sum_{\ell=1}^{d}x_{3\ell}\right)^2\right)\\
&\quad+(a_{12}+\mathrm{i}a_{22})^2\left(\left(\sum_{\ell=1}^{d}x_{2\ell}\right)^2-\left(\sum_{\ell=1}^{d}x_{3\ell}\right)^2\right)\\
&\quad+2\sum_{j<k}(a_{1j}+\mathrm{i}a_{2j})(a_{1k}+\mathrm{i}a_{2k})\left(\sum_{\ell=1}^{d}x_{j\ell}\right)\left(\sum_{\ell=1}^{d}x_{k\ell}\right)\Bigg\}\\
&=(a_{11}+\mathrm{i}a_{21})^2\sum_{\ell=1}^{d}({x_{1\ell}}^2-{x_{3\ell}}^2)+(a_{12}+\mathrm{i}a_{22})^2\sum_{\ell=1}^{d}({x_{2\ell}}^2-{x_{3\ell}}^2)\\
&\quad+2\sum_{j<k}(a_{1j}+\mathrm{i}a_{2j})(a_{1k}+\mathrm{i}a_{2k})\sum_{\ell=1}^{d}x_{j\ell}x_{k\ell}
\end{aligned}$$

and the sufficiency of Eqs. (7.11) and (7.12) follow.

To see that the equations are necessary, it suffices to set

$$A=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$$

for θ an arbitrary parameter and to compare coefficients of $\cos^2\theta, \sin^2\theta$ and $\cos\theta\sin\theta$. □

If $\sum_{\ell=1}^{d}x_{i\ell}=0$ for all $i=1,2,3$, then for γ finite, whatever the projection to the plane, the left-hand side of (7.13) vanishes. In this case, in certain circumstances, it is desirables to allow γ to take on the value $-\infty$ and to interpret (7.13) by continuity arguments, even with a non-zero right-hand side. This can occur for example, when the degree $d=2$; see Section "First Cases" in Appendix 1 and below.

In the statement of the above proposition, we do not exclude the possibility that some columns of W may coincide. In this case the star will not be embedded. Indeed, the only invariant star satisfying the hypotheses of the proposition with $d=2$ must have identical external vertices. To see this, we perform a similarity transformation so that the matrix W has the form

$$W = \begin{pmatrix} 1 & x_{12} \\ 0 & 0 \\ 0 & x_{32} \end{pmatrix} .$$

Then the projection of the star to the plane is real and provided $x_{12} \neq -1$, by the Cauchy-Schwarz inequality (7.4), we must have $\gamma = 1$. The same conclusion arises if $x_{32} \neq 0$ and $x_{12} = -1$; to see this we permute the 1st and 3rd lines of W prior to projection. Equations (7.11) and (7.12) now imply that necessarily $x_{32} = 0$ and $x_{12} = 1$. This leaves the case when $x_{12} = -1$ and $x_{32} = 0$, i.e. column two is minus column one. As discussed in the preceding paragraph, it makes sense to interpret this as an invariant star with $\gamma = -\infty$.

Any star with just one external vertex is automatically invariant with $\gamma = 1$. On the other hand, for $d = 3$ we have the following.

Corollary 7.2 *Any invariant star in $\mathbb{R}^3$ with three distinct external vertices must be configured.*

Proof By a similarity transformation of $\mathbb{R}^3$, we may suppose that W has the form

$$W = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ c & c & c \end{pmatrix} ,$$

for some constant c, that is, all the external vertices lie in the plane $x_3 = c$ and further, by a rotation about the x_3 axis and a dilation, the first external vertex has the form indicated. Write $\mathbf{r}_j$ for the jth line of W and $\sum \mathbf{r}_j$ for the sum of its components. Thus (7.11) and (7.12) can be written

$$\frac{\gamma}{3}\left((\sum \mathbf{r}_j)^2 - (\sum \mathbf{r}_k)^2\right) = \mathbf{r}_j \cdot \mathbf{r}_j - \mathbf{r}_k \cdot \mathbf{r}_k \quad \text{and} \quad \frac{\gamma}{3}\left(\sum \mathbf{r}_j\right)\left(\sum \mathbf{r}_k\right) = \mathbf{r}_j \cdot \mathbf{r}_k$$

for all $j \neq k$.

If $c = 0$, then we deduce that $\frac{\gamma}{3}\left(\sum \mathbf{r}_j\right)^2 = \mathbf{r}_j \cdot \mathbf{r}_j$ for $j = 1, 2$ and from the Cauchy-Schwarz inequality, we must have $\gamma \geq 1$ which contradicts our hypothesis that $\gamma < 1$. Thus $c \neq 0$ and from the second equation, noting that $\sum \mathbf{r}_3 = 3c$, we deduce that

$$\gamma c \sum \mathbf{r}_j = c \sum \mathbf{r}_j .$$

for $j = 1, 2$. Then if $\sum \mathbf{r}_j \neq 0$ $(j = 1, 2)$ we would have $\gamma = 1$, a contradiction, so that $\sum \mathbf{r}_j = 0$ for $j = 1, 2$, in particular $1 + x_{12} + x_{13} = 0$ and $x_{22} + x_{23} = 0$.

We exclude the case $x_{22} = x_{23} = 0$ which again gives a contradiction. But now the necessary conditions

$$1 + {x_{12}}^2 + {x_{13}}^2 = {x_{22}}^2 + {x_{23}}^2 \quad \text{and} \quad x_{12}x_{22} + x_{13}x_{23} = 0\,,$$

have only $x_{12} = x_{13} = -1/2$ and $x_{22} = -x_{23} = \pm\sqrt{3}/2$ as possible solutions. Thus the external vertices are symmetrically placed as the third roots of unity in the plane $x_3 = c$ and the star is configured. □

Note that, up to dilation and rotation, the only configuration of three points in the plane is given by the third roots of unity. In particular, any star on this configuration is regular.

Examples of invariant frameworks in $\mathbb{R}^3$ with vertices of degree three are the 1-skeleta of the tetrahedron, the cube and the dodecahedron. However, the necessity that invariant stars about a vertex of degree three be configured, means that the piecing together of such stars to form an extensive invariant framework is bound to be restrictive. On the other hand, if the degree is four, then one has more flexibility. For example, if one fixes two of the external vertices then there is a certain freedom in the choice of the other two. This may make it possible to construct an invariant mesh which approximates arbitrarily closely a smooth surface. Such problems are for future investigation.

7.3.3 The Lifting Problem

Given a solution ϕ to (7.2), at each vertex x, our aim is to construct a configured star in some Euclidean space $\mathbb{R}^N$ whose external vertices project to the points $\phi(y) - \phi(x)$ $(y \sim x)$ of the complex plane. To do this, we establish a converse to Corollary 7.1. We shall refer to the problem of constructing such a star as *the lifting problem*. At a vertex of degree three with ϕ holomorphic, this is the Theorem of Axonometry of Gauss [15].

Fix a vertex x of degree d and label its neighbors $y_1, \dots, y_d$. Set $z_\ell = \phi(y_\ell) - \phi(x)$ $(\ell = 1, \dots, d)$, which we suppose not all zero. From (7.2):

$$\frac{\gamma}{d}\left(\sum_{\ell=1}^{d} z_\ell\right)^2 = \sum_{\ell=1}^{d} {z_\ell}^2 \quad (\gamma \in \mathbb{R})\,. \tag{7.14}$$

For a given N with $2 \leq N \leq d$, we wish to construct a configured star $W = (\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_d)$ in $\mathbb{R}^N$ with z_ℓ the orthogonal projection of $\mathbf{x}_\ell$. For convenience, write $z_\ell = x_{\ell 1} + \mathrm{i}x_{\ell 2} = \alpha_\ell + \mathrm{i}\beta_\ell$, so that

$$W = \left(\begin{array}{c|c|c|c} \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ \beta_1 & \beta_2 & \cdots & \beta_d \\ x_{13} & x_{23} & \cdots & x_{d3} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1N} & x_{2N} & \cdots & x_{dN} \end{array}\right).$$

For the case $N = 2$, see Appendix 1. For $N \geq 3$, we are required to solve the system:

$$WW^t = \rho I_N + \sigma \mathbf{u}\mathbf{u}^t, \qquad \sum_{\ell=1}^{d} \mathbf{x}_\ell = \sqrt{d(\sigma + \rho)}\,\mathbf{u}\,, \tag{7.15}$$

for $x_{\ell j}$ ($\ell = 1, \ldots, d$; $j = 3, \ldots, N$), $\rho > 0$, σ such that $\rho + \sigma > 0$ and $\mathbf{u} \in \mathbb{R}^N$ unit, with $\gamma = \sigma/(\sigma + \rho)$. This is a matter of linear algebra which we now detail for the case $N = 3$. The general case is dealt with in [3].

Let $\{z_1, \ldots, z_d; \gamma\}$ be a non-trivial solution to (7.14) satisfying $\gamma < 1$. Set

$$\rho = \frac{1}{2}\sum_\ell z_\ell \bar{z}_\ell - \frac{\gamma}{2d}\Big(\sum_\ell z_\ell\Big)\Big(\sum_\ell \bar{z}_\ell\Big) > 0\,, \tag{7.16}$$

and

$$\sigma = \frac{\gamma\rho}{1-\gamma} \quad (\Rightarrow \sigma + \rho = \rho/(1-\gamma) > 0)\,. \tag{7.17}$$

Define

$$u_1 = \frac{1}{\sqrt{d(\sigma + \rho)}}\sum_{\ell=1}^{d} \alpha_\ell, \quad u_2 = \frac{1}{\sqrt{d(\sigma + \rho)}}\sum_{\ell=1}^{d} \beta_\ell\,; \tag{7.18}$$

and let $u_3 = \sqrt{1 - {u_1}^2 - {u_2}^2}$. Set

$$A := \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ \beta_1 & \beta_2 & \cdots & \beta_d \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \qquad X := \begin{pmatrix} x_{13} \\ x_{23} \\ \vdots \\ x_{d3} \end{pmatrix}. \tag{7.19}$$

Then (7.15) is equivalent to solving

$$AX = B := u_3 \begin{pmatrix} \sigma u_1 \\ \sigma u_2 \\ \sqrt{d(\sigma + \rho)} \end{pmatrix} \tag{7.20}$$

subject to the constraint:

$$X^t X = \rho I_{N-2} + \sigma {u_3}^2. \tag{7.21}$$

It is important to note the sign ambiguity: the equations are invariant under the simultaneous replacement of u_3 by $-u_3$ and of X by $-X$. This ambiguity represents two choices for the configured star.

When A has maximal rank 3, the system (7.20) and (7.21) has the unique solution $X = A^+B$, where $A^+ = A^t(AA^t)^{-1}$ (together with the sign ambiguity discussed above). If the rows of A are dependent then AA^t is no longer invertible and $u_1{}^2 + u_2{}^2 = 1 \Rightarrow u_3 = 0$, so we are required to solve the system $AX = 0$ with the constraint $X^tX = \rho$. There is now a 1-parameter family of solutions. This case occurs if and only if the complex numbers z_ℓ in (7.14) satisfy (see [3]):

$$d\sum_{\ell=1}^{d}|z_\ell|^2 + (\gamma - 2)\Big|\sum_{\ell=1}^{d} z_\ell\Big|^2 = 0\,.$$

Thus, apart from special cases, we have a lift about each vertex of a solution to (7.2) into $\mathbb{R}^3$ which is unique modulo translation along the axis of projection $\mathbb{R}^3 \to \mathbb{R}^2$ and up to the twofold ambiguity corresponding to the sign of X. This already enables certain geometric quantities to be defined in an unambiguous way, for example edge length. Furthermore, the twofold ambiguity may sometimes be removed by a requirement of global consistency, as is the case with the cube: a choice at one vertex imposes a choice of lift at neighboring vertices.

The problem of when a global lifting of a given graph exists remains relatively unexplored. An obvious geometric obstruction occurs when the lifts at neighboring vertices defines a *different* length to the connecting edge. This is particularly relevant when we try to lift into $\mathbb{R}^3$ since then, in general, edge length of each lift is unique (see, for example [3], Example 4.4). However, in general there is a smooth family of lifts into $\mathbb{R}^N$ when $N > 3$ subject to the constraint that $N \leq d$ (d = degree of the vertex), so that it may still be possible to find a global lift into a higher dimensional Euclidean space. A next step in order to extend the lifting beyond the immediate neighborhood of a vertex, will be to relax the condition that a lifted star be configured while maintaining invariance.

Examples of invariant frameworks other than the regular polytopes are given in [2, 3]. A particularly interesting example is given by a double cone on a convex planar regular polygon as illustrated in Fig. 7.3, sometimes referred to as an *n-gonal bipyramid*. In this case, there is a unique height (distance from the plane of the polygon to the apex) for which it becomes invariant. The invariant stars about the lateral vertices are only configured when the polygon is a square, which corresponds to the octahedron.

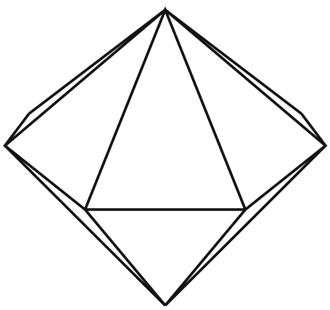


Fig. 7.3 The double cone on a regular convex polygon has a unique height for which it becomes invariant

7.4 Edge Length and the Gauss Map

Let $G = (V, E)$ be a graph and let $\gamma : V \to [-\infty, 1]$ be an element of its geometric spectrum. We are interested in quantities related to distance and curvature, defined in terms of a lift of a vertex and its neighbors to a configured star, that depend only on the geometric spectrum rather than the representative solution ϕ to (7.2). Sometimes this will be possible, sometimes not. To a given solution ϕ, suppose that the matrix A of (7.19) is of maximal rank 3. This property is clearly independent of the equivalence (7.3). The most obvious object that first arises is the axis $\mathbf{u}$ of a configured star, whose components are given by (7.18), with $u_3 = \sqrt{1 - {u_1}^2 - {u_2}^2}$. In fact, we may write

$$u_1 + \mathrm{i}u_2 = \sqrt{\frac{1-\gamma}{d\rho}} \sum_{\ell=1}^{d} z_\ell \,,$$

which, from the expression (7.16) for ρ, is scale invariant under $z_\ell \mapsto r z_\ell$ ($\forall \ell \,, r \in \mathbb{R}^*$), but not invariant under complex conjugation $z_\ell \mapsto \overline{z_\ell}$ ($\forall \ell$), which changes the sign of u_2. This ambiguity represents a global freedom in the solution of the form $\phi \mapsto \overline{\phi}$, which corresponds to the isometry $(y_1, y_2, y_3) \mapsto (y_1, -y_2, y_3)$ in the ambient Euclidean space, which will not affect subsequent quantities such as edge curvature and edge length. Note also that the axis is well-defined as $\gamma \to -\infty$.

However, there is a further sign ambiguity in the choice of u_3 which we can trace back to the two choices of lifted star. What additional information is required in order to make a unique choice for the lifted star? One way to do this is to define *orientation* on a graph.

7.4.1 Graph Orientation

In [4], a notion of orientation was considered on a regular graph of degree d, say, whereby the graph is endowed with an edge coloring of the d colors $\{1, 2, \ldots, d\}$. Thus each edge is colored in such a way that no two edges of the same color are incident at a vertex. This enables one to uniquely label the edges at each vertex to give an ordering. One could then attempt to apply a right-hand rule say, in order to make a choice of lift. However, although this can be done with the solution corresponding to the framework of the tetrahedron in a way consistent with its canonical embedding, it turns out to be impossible for the cube and the dodecahedron. In the latter examples, any edge coloring with three colors leads to at least one of the two choices of lifted stars being directed in the opposite way required. We therefore proceed to define an orientation in terms of an edge coloring together with an n-form at each vertex of degree d.

Definition 7.1 Let $G = (V, E)$ be a graph with largest vertex degree equal to M. Then an *edge coloring* of G is an association of one of the colors $\{1, 2, \ldots, M, M+1\}$ to each edge so that no two same colors are incident at any vertex. By a theorem of Vizing, any graph can be colored with either M or $M+1$ colors (see [12]). We make the convention to choose the minimum M colors when possible. Given an edge coloring of G, at each vertex, a *volume form* is an alternating mapping θ of the edges which takes on the value $+1$ or -1. Thus if x is a vertex with d incident edges $e_1, \ldots, e_d$ arranged so that $color(e_j) < color(e_k)$ for $j < k$, then $\theta_x(e_1, \ldots, e_d) = \pm 1$ with $\theta_x(e_{\sigma(1)}, \ldots e_{\sigma(d)}) = \text{sign}\,(\sigma)\theta(e_1, \ldots, e_d)$ for any permutation σ of $\{1, \ldots, d\}$. An *orientation* of G is given by an edge coloring together with a volume form at each vertex.

Suppose that $N = 3$ and that the matrix A of Eq. (7.19) is of maximal rank 3. There is now either a unique lifted configured star in the case when $u_3 = 0$, or two choices if $u_3 \neq 0$ depending on the sign chosen for $u_3 = \pm\sqrt{{u_1}^2 + {u_2}^2}$. The star matrix is now given by

$$W = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ \beta_1 & \beta_2 & \cdots & \beta_d \\ x_{13} & x_{23} & \cdots & x_{d3} \end{pmatrix},$$

where the last row is only defined up to sign. Suppose that the vertex in question has an orientation according to Definition 7.1. Without loss of generality, we can suppose that the edges are colored with the colors $\{1, 2, \ldots, d\}$, in such a way that the external vertex $\mathbf{x}_\ell$ is joined to the internal vertex by the edge with color ℓ. Suppose that the volume form satisfies $\theta(e_1, \ldots, e_d) = \varepsilon$, where $\varepsilon \in \{+1, -1\}$. Then provided the determinant of the 3×3-minor given by the first three columns of W is non-zero, we choose the sign of the third row so that

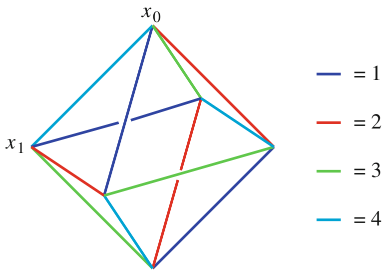
$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ x_{13} & x_{23} & x_{33} \end{vmatrix} = \varepsilon\delta,$$

where $\delta > 0$. If on the other hand this determinant vanishes, then we proceed in a lexicographic ordering, to choose next the minor formed from columns 1, 2 and 4 and so on, until we encounter a non-zero determinant and apply the above rule.

Example 7.1 Consider the framework of a regular octahedron with vertices placed at the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. Then this can be edge-colored as indicated in Fig. 7.4 and there is a volume form which gives the lifts that correspond to the standard embedding in $\mathbb{R}^3$. However, in order to do this at the lateral vertices, we have to impose an additional condition that the star be regular. This is because at these vertices $u_3 = 0$ and we do not satisfy the conditions of the discussion above. The corresponding solution to (7.2) has $\gamma = 1/2$ and $\rho = \sigma = 2$.

Consider the vertex x_0 and define the volume form θ_{x_0} by $\theta_{x_0}(1234) = -1$ (for convenience, we write $\theta(1234)$ rather than $\theta(e_1, e_2, e_3, e_4)$). Then, with this

Fig. 7.4 An edge coloring for the octahedron



edge-coloring, at this vertex $z_1 = 1$, $z_2 = \mathrm{i}$, $z_3 = -1$ and $z_4 = -\mathrm{i}$. Thus $u_1 = u_2 = 0$ and $u_3 = \pm 1$. The solution to (7.20) is given by $(x_{13}, \ldots, x_{43}) = \pm(1, 1, 1, 1)$. In order to be consistent with the orientation, we must take the negative sign, to give the lifted star:

$$W = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

whose sign of the determinant of the 3×3-minor given by the first three columns is negative, which coincides with the sign of $\theta_{x_0}(1234)$.

At the vertex x_1, we choose $\theta_{x_1}(1234) = +1$. Then the edge-coloring dictates that $z_1 = -1 + \mathrm{i}$, $z_2 = 1 + \mathrm{i}$, $z_3 = \mathrm{i}$, $z_4 = \mathrm{i}$, so that $u_1 = 0$, $u_2 = 1$ and $u_3 = 0$. The solution to (7.20) gives a 1-parameter family of lifted stars:

$$W = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\frac{\cos t}{\sqrt{2}} & -\frac{\cos t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} + \sin t & \frac{\cos t}{\sqrt{2}} - \sin t \end{pmatrix}.$$

If we now impose the condition that the lift must be a regular star, then there are just two solutions given by $t = \pi/2$ or $t = 3\pi/2$. The choice $t = 3\pi/2$ is required in order that the determinant of the 3×3-minor consisting of the first three columns be positive, to coincide with the sign of $\theta_{x_1}(1234)$. This gives the lift that coincides with the canonical embedding of the octahedron. We proceed similarly with the other vertices, defining the appropriate volume form, with the proviso that the stars at the lateral vertices be regular.

7.4.2 *Edge Curvature and Edge Length*

For a given graph $G = (V, E)$ together with a solution ϕ to (7.2), a unique choice of lifted star at each vertex now determines a unit vector $\mathbf{u}$ which gives the associated Gauss map $\mathbf{u} : V \to S^2$. If two vertices are connected by an edge, we may connect them by the shortest geodesic arc in S^2, so realizing a copy of G in S^2. For a given edge $e = \overline{xy} \in E$, we can define the *edge-curvature* to be the length of the spherical arc joining $\mathbf{u}(x)$ to $\mathbf{u}(y)$. For example, for the standard cube, the curvature of each edge is $\arccos(1/3)$; for the octahedron, it is $\pi/2$.

Edge-length clearly depends on the choice of representative solution ϕ. A convenient way to obtain the mean length of edges incident with a particular vertex is to reverse the order of multiplication of $W = (\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_d)$ and W^t:

$$W^t W = \begin{pmatrix} ||\mathbf{x}_1||^2 & \langle \mathbf{x}_1, \mathbf{x}_2\rangle & \cdots & \langle \mathbf{x}_1, \mathbf{x}_d\rangle \\ \langle \mathbf{x}_1, \mathbf{x}_2\rangle & ||\mathbf{x}_2||^2 & \cdots & \langle \mathbf{x}_2, \mathbf{x}_d\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{x}_1, \mathbf{x}_d\rangle & \langle \mathbf{x}_2, \mathbf{x}_d\rangle & \cdots & ||\mathbf{x}_d||^2 \end{pmatrix},$$

where $\langle \mathbf{x}_j, \mathbf{x}_k\rangle$ denotes the standard Euclidean inner product of $\mathbf{x}_i$ and $\mathbf{x}_j$. But then

$$\begin{aligned} \sum_\ell ||\mathbf{x}_\ell||^2 &= \operatorname{trace} W^t W = \operatorname{trace} W W^t \\ &= N\rho + \sigma||\mathbf{u}||^2 = N\rho + \sigma\,. \end{aligned}$$

The latter quantity can be expressed in terms of γ and z_ℓ from (7.16) and the relation $\gamma = \sigma/(\sigma + \rho)$ to give the mean of the values $||\mathbf{x}_\ell||^2$:

$$\frac{1}{d}\sum_\ell ||\mathbf{x}_\ell||^2 = \frac{\big(N + (1-N)\gamma\big)}{d(1-\gamma)}\rho\,. \tag{7.22}$$

This equation expresses the mean length of the edges of a lift to a configured star in $\mathbb{R}^N$ whose external vertices $\mathbf{x}_\ell$ project to z_ℓ. This motivates our definition of edge length in a graph.

Let $G = (V, E)$ be a graph coupled to a solution $\phi : V \to \mathbb{C}$ to Eq. (7.2). For each $x \in V$, following (7.16), set

$$\rho(x) = \frac{1}{2}\left\{\sum_{y\sim x}|\phi(y) - \phi(x)|^2 - \frac{\gamma(x)}{d_x}\Big|\sum_{y\sim x}(\phi(y) - \phi(x))\Big|^2\right\},$$

where d_x is the degree of G at x.

Definition 7.2 If $x \in V$ is a vertex of degree d_x such that $\gamma(x) < 1$, then we define the *median edge length at x relative to ϕ* to be the quantity $r(x) \geq 0$ whose square is given by

$$r(x)^2 = \frac{\big[N + (1-N)\gamma(x)\big]}{d_x[1-\gamma(x)]}\,\rho(x)\,.$$

If $\overline{xy} \in E$ is an edge which joins x to y such that both $\gamma(x) < 1$ and $\gamma(y) < 1$, then we define the *length of $\overline{xy}$ relative to ϕ* to be the mean $\ell(\overline{xy})$ of the median edge lengths at x and y:

$$\ell(\overline{xy}) = \frac{r(x) + r(y)}{2} .$$

As emphasized in the above definition, the lengths so defined are *relative* to the solution ϕ of (7.2), which is only defined up to $\phi \mapsto \lambda\phi + \mu$ for $\lambda, \mu \in \mathbb{C}$. This means that the only meaningful quantities are *relative* lengths, say $\ell(e)/\ell(f)$, for two edges $e, f \in E$. In particular, if both d and γ are constant on the graph, we may take the quantity 2ρ defined by (7.16) as a measure of median edge length at each vertex:

$$r(x)^2 = 2\rho = \sum_{y \sim x} |\phi(y) - \phi(x)|^2 - \frac{\gamma}{d}\Big|\sum_{y \sim x}(\phi(y) - \phi(x))\Big|^2 .$$

If desired, we can define an absolute length by normalizing with respect to the *square L^2-norm of the derivative of ϕ*:

$$||\mathrm{d}\phi||^2 = \sum_{\overline{xy} \in E} |\mathrm{d}\phi(\overline{xy})|^2 = \frac{1}{2} \sum_{x,y \in V, x \sim y} |\phi(y) - \phi(x)|^2 ,$$

and setting

$$r_{\mathrm{abs}}(x)^2 = \frac{r(x)^2}{||\mathrm{d}\phi||^2} ,$$

where $r(x)$ is the median edge length at x relative to ϕ. If $e \in E$ is an edge joining x to y such that both $\gamma(x) < 1$ and $\gamma(y) < 1$, then we define the *absolute length of e relative to ϕ* to be the mean $\ell_{\mathrm{abs}}(e)$ of the absolute median edge lengths at x and y:

$$\ell_{\mathrm{abs}}(e) = \frac{r_{\mathrm{abs}}(x) + r_{\mathrm{abs}}(y)}{2} .$$

Then both the quantities $r_{\mathrm{abs}}(x)$ and $\ell_{\mathrm{abs}}(e)$ are independent of the freedom (7.3).

The median edge length of Definition 7.2 is defined so as to give the length of the edges of a corresponding regular star in $\mathbb{R}^N$, when such exists. In particular, if $G = (V, E)$ is the 1-skeleton of a regular polytope and $\phi : V \to \mathbb{C}$ associates to each vertex its value after an orthogonal projection, then the edge-length at each vertex coincides with the lengths of the edges of the regular polytope. More generally, we can interpret edge length at each vertex as the length of the edges of the "best fit" configured star at that vertex. The median edge length then gives the average length at two adjacent vertices.

Example 7.2 Consider the graph on five vertices sketched in Fig. 7.5, with solutions ϕ to (7.2) normalized so as to take the value 0 at the central vertex and 1 on one of

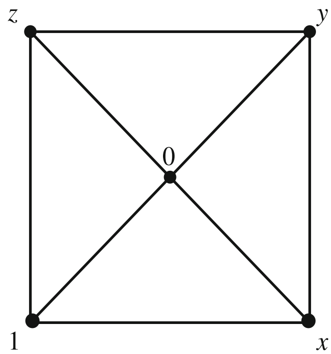


Fig. 7.5 The most general non-constant solution can be normalized to take on the value 0 at the central vertex and 1 at any one of the other vertices

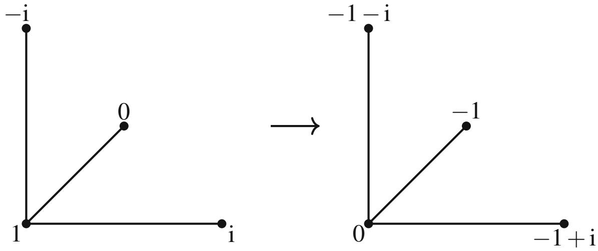


Fig. 7.6 The solution is now normalized to take on the value 0 at the internal vertex of the star

the other vertices. The symmetry of the figure means that this determines the most general non-constant solution.

There are two solutions to (7.2) with γ constant, namely:

$$\begin{array}{ll} \gamma = 1/3\,; & x = \pm\,\mathrm{i}\,,\; y = -1\,,\; z = \mp\,\mathrm{i}\,; \\ \gamma = 1\,; & x = yz\,,\; y = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\,,\; z = 3 \pm 2\sqrt{2}\,. \end{array}$$

We reject the latter solution, since the inequality $\gamma < 1$ is violated. Consider the solution with $\gamma = 1/3$. Let us construct the lift at the bottom left-hand vertex. First, we normalize so that the solution takes on the value 0 at this vertex as indicated in Fig. 7.6.

The choice $n = N = 3$ is determined and from (7.16) and (7.17); we obtain $\rho = 2$ and $\sigma = 1$. From (7.18) we find that ${u_1}^2 + {u_2}^2 = 1$ so that $u_3 = 0$. Then the 3×1-matrix B vanishes and system (7.20) has general solution $\mathbf{x}_3 := (x_{13}, x_{23}, x_{33}) = (2\lambda, -\lambda, -\lambda)$. The constraint (7.21) requires that $\langle \mathbf{x}_3, \mathbf{x}_3 \rangle = \rho = 2$, so that $\lambda = \pm 1/\sqrt{3}$. The star matrix W (whose columns give the positions of the external star vertices) is given by:

$$W = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ \pm\frac{2}{\sqrt{3}} & \mp\frac{1}{\sqrt{3}} & \mp\frac{1}{\sqrt{3}} \end{pmatrix}.$$

We can proceed similarly with the central vertex of degree 4. Now we can choose $N = 3$ or $N = 4$. In either case, $u_1 = u_2 = 0$, $\rho = 2$ and $\sigma = 1$, so that, for $N = 3$ we must have $u_3 = \pm 1$. Then

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 2\sqrt{3} \end{pmatrix}$$

and the unique (minimizing) solution is given by

$$Z = A^{+}B = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}.$$

The star matrix W is given by

$$W = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

where the last line is only defined up to sign.

The (unique) common dimension to define edge length is $N = 3$. At the central vertex the edge length is $\sqrt{7}/2$ and at any of the other vertices, it is $\sqrt{7/3}$. Thus the median edge length of the edge joining the central vertex to one of the other vertices is $\frac{(\sqrt{7}/2)+\sqrt{7/3}}{2} \sim 1.4252$, whereas the median edge length of one of the outside edges is $\sqrt{7/3} \sim 1.5275$. So, for example, the shortest path joining x to z is given by passing through the central vertex. Note that, as already remarked, the edge lengths are only defined up to a multiple and so only relative edge lengths have meaning.

7.4.3 Path Metric Space Structure and Curvature in the Sense of Alexandrov

Does the notion of edge length, either relative or absolute, that we have defined above, endow a graph with the structure of a *path metric space* (see [16])? We first of all note a triangle inequality around complete subgraphs on three vertices.

Given a function $\phi : V \to \mathbb{C}$ and a vertex $x \in V$, we say that ϕ *is constant on the star centered on* x if the restriction of ϕ to x and its neighbors $y \sim x$, is constant.

Proposition 7.2 (Local Triangle Inequality) *Let $G = (V, E)$ be a graph coupled to a solution $\phi : V \to \mathbb{C}$ to Eq. (7.2). Suppose $x, y, z \in V$ are three vertices of a complete subgraph: $x \sim y$, $y \sim z$, $z \sim x$, such that the inequality $\gamma < 1$ is satisfied at each vertex. Then the triangle inequality is satisfied:*

$$\ell(\overline{xy}) + \ell(\overline{xz}) \geq \ell(\overline{yz})\,.$$

If further ϕ is non-constant on the star centered on x, then the inequality is strict.

Proof This is an immediate consequence of the definition. Specifically,

$$\ell(\overline{xy}) + \ell(\overline{xz}) = \frac{1}{2}(r(x) + r(y)) + \frac{1}{2}(r(x) + r(z)) = \ell(\overline{yz}) + r(x) \geq \ell(\overline{yz})\,,$$

since because of the inequality $\gamma(x) < 1$, we have $r(x) \geq 0$. If further, ϕ is non-constant on the star centered on x, then $r(x) > 0$ and the inequality is strict. □

In spite of this local triangle inequality, we may encounter a difficulty in trying to endow a graph coupled to a solution ϕ to (7.2) with a metric space structure. This may arise when, for a given vertex x, the function ϕ is constant on the star centered on x, as well as on the star centered on one of its neighbors y. Then $\ell(\overline{xy}) = 0$. We can either agree to allow distinct points to have zero distance between them, and so consider rather a *pseudo-metric space structure*, or we can avoid this situation by introducing a notion of *collapsing*.

Definition 7.3 Let (G, ϕ) be a graph coupled to a solution to Eq. (7.2). Then we *collapse* G to a new graph $\widetilde{G}$ by removing all edges that connect vertices at which ϕ takes on identical values; then remove all isolated vertices.

It is clear that after collapse, if we let $\widetilde{\phi}$ denote the restriction of ϕ to $\widetilde{G}$, then $\widetilde{\phi}$ also satisfies (7.2) with $\widetilde{\gamma} = \gamma\widetilde{d}/d$ where $\widetilde{d}$ is the new degree at each vertex. Indeed, if we check at a vertex x, then if y is a neighbor at which $\phi(y) = \phi(x)$, then since only the difference $\phi(y) - \phi(x)$ occurs in (7.2), removing the edge $\overline{xy}$ only affects the degree. However, it is to be noted that collapsing may disconnect a graph.

Let (G, ϕ) be a connected graph coupled to a solution of Eq. (7.2). Suppose that for each edge $\overline{xy}$, its length $\ell(\overline{xy}) > 0$. Given a path $\overline{c} := \overline{x_0 x_1 x_2 \cdots x_n}$ joining x to y (so we have $x = x_0$, $y = x_n$ and $x_j \sim x_{j+1}$ for all $j = 0, \ldots, n-1$), we define the length $\ell(\overline{c})$ to be the sum:

$$\sum_{j=0}^{n-1} \ell(\overline{x_j x_{j+1}})\,.$$

Let X be the underlying topological space formed from the union of the vertices and the edges (where edges only intersect at a common end point). We can extend path length to include points belonging to edges in the obvious way, by identifying the segment $[0, \ell(\overline{xy})] \subset \mathbb{R}$ with $\overline{xy}$ and attributing length linearly along the interval $[0, \ell(\overline{xy})]$. Now any two points $\xi, \eta \in X$ have a well-defined distance $d(\xi, \eta)$ between them given as the length of the shortest path joining them. A geodesic segment $[\xi, \eta]$ is a path of length $d(\xi, \eta)$ joining ξ and η. Since any pair of points $\xi, \eta \in X$ can be joined by a continuous path of length $d(\xi, \eta)$, X has the structure of a geodesic space. In this setting, curvature bounds in the sense of Alexandrov arise by comparing triangles in the path metric space X with those in a 2-dimensional space form $M(K)$ of constant Gaussian curvature K; see Chap. 2 by E. Saucan in this volume. In Example 7.2, due to the simple nature of the graph, we can find an exact comparison.

If we let $a = (\sqrt{7}/2 + \sqrt{7/3})/2$ and $b = \sqrt{7/3}$, then there exists a unique radius r of the 2-sphere for which the graph can be placed as indicated sketched in Fig. 7.7, where we require two neighboring exterior vertices to be subtended by a right-angle at the central vertex and the two geodesic segments to have respective lengths a and b. A numerical calculation shows that the required radius r is approximately given by 0.9811, which yields Gauss curvature $K = 1/r^2 \sim 1.0389$, very close to 1.

For a general graph coupled to a solution ϕ of (7.2), we can only expect to obtain bounds on the curvature in the sense of Alexandrov, rather than an exact figure as in the above example. If we first normalize ϕ so as to use absolute edge length, then an interesting problem would be to express such bounds in terms of the geometric spectrum γ.

Note that the positive curvature of the above example can also be deduced by drawing a Euclidean triangle with side-lengths as indicated in Fig. 7.8.

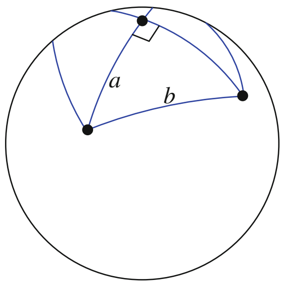


Fig. 7.7 There is a unique 2-sphere into which the graph can be embedded to realize the correct edge lengths

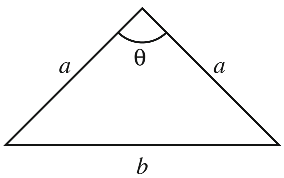


Fig. 7.8 Angle deficit at the internal vertex also implies positive curvature

The angle θ subtended by the two external vertices is calculated to be about 1.13 radians, yielding an angular deficit for the sum of the interior angles at the central vertex to be approximately $2\pi - 4 \times 1.13 \sim 2.04$ radians. Gaussian curvature in this sense, defined in terms of angular deficit is the subject of the next section.

7.5 Gaussian Curvature

We first review some classical notions of curvature associated to polytopes defined in terms of angular deficiency at a vertex, see also Sect. 6.2.1 of Chap. 6 in this volume. Even though we are mainly interested in liftings into dimension $N = 3$, we shall in the first instance consider general N, since, as we shall see, this provides a practical way to calculate γ for any regular polytope. Our approach is pragmatic in that our expression for curvature is approximate, exact formulae being difficult in general to write down.

7.5.1 *The Theorem of Descartes*

A *convex* polytope is by definition, the closed intersection of a finite number of half-spaces (whether this be in Euclidean space, or in spherical space). In the case when a polytope is regular (convex or not), its vertices all lie on a sphere called the *circumsphere* [11]. It is useful to use *absolute angle measure* when measuring solid angles (see [17, 26]). We will write $H^M(\Lambda)$ for the M-dimensional Hausdorff measure of a set Λ in these units. Then, in any dimension, the angle is measured as a fraction of the total angle subtended by a sphere centered at the point in question. Thus in two dimensions, a right-angle has value $1/4$, whereas in three dimensions, the angle subtended by the vertex figure of a cube has value $1/8$. Equivalently, $H^2(\Lambda) = 1/8$, where $\Lambda \subset S^2$ represents one eighth portion: $x, y, z > 0$ of the sphere $x^2 + y^2 + z^2 = 1$.

A family $\{F_1, \dots, F_r\}$ of $(N-1)$-dimensional convex polytopes in $\mathbb{R}^N$ form an *elementary polytope* P of dimension N if: (i) for all j, k, $F_j \cap F_k$ is either empty or a face of each of F_j and F_k; (ii) $\cup_j F_j$ is an $(N-1)$-dimensional manifold.

Given an elementary polytope $P \subset \mathbb{R}^N$, denote by $\mathscr{P}$ the *face decomposition of* P; thus $\mathscr{P}$ is the collection of all (open) faces of all dimension, consisting of the vertices, edges, ..., $(N-1)$-faces. For $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{w} \in S^{N-1}$, following Ehrenborg [14], we define the quantity $R(\mathbf{x}, \mathbf{w}) := \lim_{s \to 0^+} \mathbf{1}_P(\mathbf{x} + s \cdot \mathbf{w})$, where $\mathbf{1}_P$ is the characteristic function of P. Note that this takes on the value 0 or 1. Then given a face $F \in \mathscr{P}$, we have $R(\mathbf{x}, \mathbf{w}) = R(\mathbf{y}, \mathbf{w})$ for all $\mathbf{x}, \mathbf{y} \in F$; write this as $R(F, \mathbf{w})$ and define $\Lambda_F = \{\mathbf{w} \in S^{N-1} : R(F, \mathbf{w}) = 1\}$.

Let $S(\Lambda_F) = H^{N-2}(\partial \Lambda_F)$ and let $\sigma_N = H^N(S^N)$, so that in absolute angle measure, $\sigma_N = 1$. Let F be a face of P of dimension $\leq N-3$. We define the *deficiency at F* to be the quantity:

$$\delta(F) := \sigma_{N-2} - S(\Lambda_F)\,.$$

Note that if $Q \subset S^{N-1}$ is spherically convex (that is, it is the intersection of hemispheres), then $S(Q) = H^{N-2}(\partial Q)$ is proportional to the Haar measure of all the great circles which intersect Q. The following theorem generalizes a classical result of Descartes.

Theorem 7.2 ([14, 17, 26]) *Let P be an elementary polytope with face decomposition $\mathscr{P}$. Then*

$$\sum_{F\in\mathscr{P},\ \dim F\leq N-3} \varepsilon(F)\delta(F) = \sigma_{N-2}\,\varepsilon(P)\,,$$

where $\varepsilon(F)$ denotes the Euler characteristic of F given by $(-1)^k$ when F is of dimension k and $\varepsilon(P)$ is the Euler characteristic of P.

In the case when $N=3$ and P is a convex polyhedron (now using radians for our measure), we obtain the classical theorem of Descartes:

$$\sum_{\ell} \delta(\mathbf{v}_\ell) = 4\pi\,,$$

where the sum is taken over the vertices of P. Here, the deficiency $\delta(\mathbf{v}_\ell)$ is $2\pi-$ (the sum of the internal angles at $\mathbf{v}_\ell$ of the faces which contain $\mathbf{v}_\ell$) (see also Theorem 2.1 of Chap. 6).

7.5.2 Vertex Curvature

The definition of curvature that we give below coincides with the angular deficiency discussed in the above paragraph when the solution to (7.2) arises from the underlying framework of a regular convex polyhedron. The term *vertex-curvature* is used to distinguish it from the previously defined *edge-curvature*. In the case when $N=3$, the vertex curvature coincides with the same notion considered by M. Keller, Chap. 6, Sect. 6.1.3 in this volume for graphs embedded in a surface, provided the embedding permits a configuration about the vertex in question.

Definition 7.4 (*N*-dimensional Vertex-Curvature) Let (G,ϕ) be a pair consisting of a graph G coupled to a solution ϕ to *(7.2)* with $\gamma < 1$. Let y be a vertex of G and let d be the degree of G at y. Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_d\}$ be the configuration in $\mathbb{R}^{N-1}$

(centered on $\mathbf{0} \in \mathbb{R}^{N-1}$) of an associated lifted configured star in $\mathbb{R}^N$ with invariant ρ given by *(7.6)*. Let

$$\mathbf{x}_\ell = \frac{1}{\sqrt{\rho + dr^2(1-\gamma)}} \begin{pmatrix} \sqrt{d(1-\gamma)}\,\mathbf{v}_\ell \\ \sqrt{\rho} \end{pmatrix} \in S^{N-1} \quad (\ell = 1, \ldots, d)\,,$$

be the corresponding normalized vertices of a configured star in $\mathbb{R}^N$ centered on $\mathbf{0}$. Let Λ be the convex hull in S^{N-1} of the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\} \subset S^{N-1}$. We define the *N-dimensional vertex-curvature of* (G, ϕ) *at* y to be the deficit:

$$\delta_\gamma(y) = 1 - H^{N-2}(\partial \Lambda)\,,$$

in absolute angle measure.

Note that in the above definition, if we rescale the vertices $\mathbf{v}_\ell$ by $\mathbf{v}_\ell \mapsto \mathbf{v}_\ell^{\sim} = \lambda \mathbf{v}_\ell$, say, then $r \mapsto \widetilde{r} = \lambda r$, $\rho \mapsto \widetilde{\rho} = \lambda^2 \rho$ and both $\mathbf{x}_j$ and the curvature $\delta_\gamma(y)$ are well-defined and independent of this scaling.

Let us now consider the different dimensional curvatures that arise from Definition 7.4. In general, we would like this curvature to depend only on γ and not on the solution ϕ to (7.2). For this reason, we will consider only cases when the configuration given by $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ in the above definition, coincides with the vertices of a regular polytope in $\mathbb{R}^{N-1}$. Note that the vertices of a regular polytope always form a configuration [3].

By convention, at a vertex of degree 1, we assign the curvature $\delta = 1$. At a vertex of degree 2, the 2-curvature just measures the exterior angle in absolute angle measure. Note that in this dimension, a configured star with two external vertices corresponds to two line segments in the plane of equal length meeting at the internal vertex (see Appendix 1). If θ is the exterior angle, then from Section "First Cases" in Appendix 1,

$$\gamma = \frac{2\cos\theta}{\cos\theta - 1}\,,$$

from which one deduces the following expression for the vertex curvature.

Proposition 7.3 *Let* (G, ϕ) *be a pair consisting of a graph* G *coupled to a solution* ϕ *to* (7.2)*. Let* y *be a vertex of degree* 2 *where* $\gamma \leq 1$*. Then the* 2*-dimensional vertex-curvature of* (G, ϕ) *at* y *is given by the quantity:*

$$\delta_\gamma(y) = \frac{1}{2\pi} \arccos\left(\frac{\gamma}{\gamma - 2}\right).$$

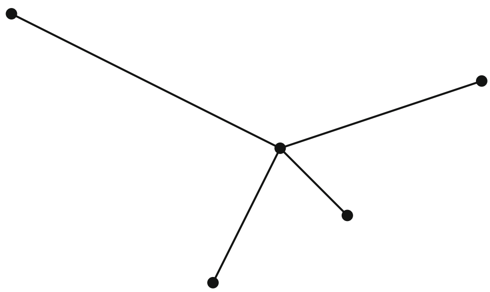
Note that $\lim_{\gamma \to 1^-} \delta_\gamma(y)$ is well-defined and equals $1/2$. For example, if G is a cyclic graph of even order $2n$ and ϕ is a function taking on alternate values at neighboring vertices, then ϕ satisfies (7.2) with $\gamma = 1$. The total curvature is then given by n. If G is a convex polygon in the plane and ϕ the corresponding position function, then the total 2-curvature is equal to 1, or in radians, to 2π, as required. We now proceed to higher dimensional curvature.

Proposition 7.4 (3-dimensional Vertex-Curvature) *Let (G, ϕ) be a pair consisting of a graph G coupled to a solution ϕ to (7.2). Let y be a vertex of G and let d be the degree of G at y. Suppose that the vertices of a lifted configured star coincide with those of a regular planar polygon. Then the 3-dimensional vertex-curvature of (G, ϕ) at y is given by the quantity:*

$$\delta_\gamma(y) = 1 - \frac{d}{2\pi} \arccos \left\{ \frac{1 + 2(1-\gamma)\cos\frac{2\pi}{d}}{3 - 2\gamma} \right\} . \tag{7.23}$$

Proof The configuration of vertices is given by $\mathbf{v}_\ell = e^{2\pi i \ell/d}$ ($\ell = 1, \ldots, d$). The boundary of the convex hull Λ of the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ in S^2 is made up of arcs of great circles of length $\alpha = \arccos(\mathbf{x}_1 \cdot \mathbf{x}_2)$. In absolute angle measure, the deficit, or 3-curvature, is given by $1 - \frac{d}{2\pi}\alpha$. Substitution of the expressions for $\mathbf{x}_\ell$ given by Definition 7.4 gives the required formula. □

Although the above proposition only applies to configured stars whose vertices coincide with a regular planar polygon, since the expression for the curvature depends only on γ, we can apply the formula to more general situations. However, we can only expect this to give an approximation of the true angular deficit. Note that $\lim_{\gamma \to 1^-} \delta_\gamma(y)$ is well-defined and equals 1 and that $\lim_{\gamma \to -\infty} \delta_\gamma(y) = 0$. As a further remark, an invariant framework may contain an invariant star which is not configured, for which one would desire a negative curvature, as illustrated:



An example of an invariant star with negative curvature at the internal vertex

The challenge in future investigations will be to characterize conditions that allow for a global lifting to an invariant framework, but without the constraint that each star be configured. Thus we allow stars such as that given by (7.10) with negative curvature at the internal vertex given as $2\pi - \sum \theta_j$ where the θ_j are the angles between the incident edges. This requires an ordering of the edges in order to decide how to calculate the angles. However, since in this example, all the angles are equal, simple trigonometry gives $2\pi - 8\arcsin(\sqrt{2/3}) \sim -1.36$ for the curvature in radian measure.

For the convex polyhedra, Proposition 7.4 gives the required result. For example, the 1-skeleton of a tetrahedron in $\mathbb{R}^3$ has $d = 3$ and $\gamma = 3/4$. The 3-curvature at each vertex is given by $\delta = 1/2$, giving a total curvature of 2. To obtain the total curvature in radian measure, we multiply by 2π to give the value 4π, which confirms the theorem of Descartes. The same proposition can also be used to calculate the values of γ for the regular convex polyhedra as given in the Introduction. For example, the dodecahedron has 20 vertices, so by the Theorem of Descartes, the deficit δ at each vertex must equal $1/10$. The degree of each vertex is $d = 3$ and substitution into (7.23) gives the required value $\gamma = \frac{3(1-\sqrt{5})}{2(3-\sqrt{5})}$.

Example 7.3 Consider the double cone on the triangle discussed at the end of Sect. 7.3. Now there is an underlying polytope and we can calculate angular deficit at each vertex in the traditional way. By the theorem of Descartes, the total angular deficit is 4π. However, let us calculate it by taking a lift to a configured star at each vertex as determined by the corresponding solution to (7.2). In the original figure, the stars at the lateral vertices are not configured, so an error will occur. We find:

$$\delta_{\text{apex}} = 2\pi - 3\arccos\frac{1}{7} \quad \text{and} \quad \delta_{\text{lat}} = 2\pi - 4\arccos\frac{5}{7},$$

to give a total curvature of

$$\delta_{\text{tot}} = 3\delta_{\text{lat}} + 2\delta_{\text{apex}} = 10\pi - 6\left(\arccos\frac{1}{7} + 2\arccos\frac{5}{7}\right) \sim 4.244 \times \pi\ .$$

The above example illustrates one of the problems in defining the curvature. The advantage of lifting to a *configured* star is that, in dimension $N = 3$, the lift is unique up to sign changes, to give a uniquely defined curvature. However, any expression of the total curvature as an invariant quantity would need to involve some approximation.

For the 4-curvature, there are some special cases to consider. We list these in the proposition below.

Proposition 7.5 (4-dimensional Vertex-Curvature) *Let (G, ϕ) be a pair consisting of a graph G coupled to a solution ϕ to (7.2). Let y be a vertex of G and let d be the degree of G at y. Suppose that $d \in \{4, 6, 12, 20\}$ and that the configuration associated to a lifted configured star coincides with the vertices of a regular 3-polytope (having d vertices). Then depending on the degree, the 4-dimensional vertex-curvature of (G, ϕ) at y is given by one of the expressions below:*

Degree	*Configuration*	*Curvature*
4	*Tetrahedron*	$2 - \frac{3}{\pi}\arccos\left(\frac{\gamma}{4-2\gamma}\right)$
6	*Octahedron*	$3 - \frac{6}{\pi}\arccos\left(\frac{1}{5-3\gamma}\right)$
12	*Icosahedron*	$6 - \frac{15}{\pi}\arccos\left(\frac{6(\sqrt{5}+1)\gamma - 11 - 7\sqrt{5}}{2[6(\sqrt{5}+3)\gamma - 23 - 7\sqrt{5}]}\right)$
20	*Dodecahedron*	$10 - \frac{15}{\pi}\arccos\left(\frac{5\gamma - 1 - 2\sqrt{5}}{2[-5\gamma + 8 - \sqrt{5}]}\right)$

Proof Given two vectors $\mathbf{u}, \mathbf{v} \in S^M(r)$ in a sphere of radius r, the arc of the great circle joining $\mathbf{u}$ to $\mathbf{v}$ is given by

$$\theta \mapsto \left(\cos\theta - \frac{(\mathbf{u}\cdot\mathbf{v})\sin\theta}{\sqrt{r^4 - (\mathbf{u}\cdot\mathbf{v})^2}}\right)\mathbf{u} + \frac{r^2 \sin\theta}{\sqrt{r^4 - (\mathbf{u}\cdot\mathbf{v})^2}}\mathbf{v}$$
$$\left(0 \leq \theta \leq \arcsin\frac{\sqrt{r^4 - (\mathbf{u}\cdot\mathbf{v})^2}}{r^2}\right).$$

When $r = 1$, this is unit speed. Furthermore, the tangent vector to this arc at $\mathbf{v}$ is given by

$$\mathbf{t} = \frac{1}{\sqrt{r^4 - (\mathbf{u}\cdot\mathbf{v})^2}}(-r^2\mathbf{u} + (\mathbf{u}\cdot\mathbf{v})\mathbf{v}). \tag{7.24}$$

The area of a spherical polygon with n sides and with interior angles θ_k ($k = 1, \ldots, n$) is given by

$$KA = \sum_k \theta_k - (n-2)\pi,$$

where K is the curvature of the sphere. We are required to calculate the spherical surface area of the boundaries of the various vertex figures in S^3. These are made up of faces lying in great 2-spheres which are either triangles, or in the case of the dodecahedron, pentagons, whose edges are arcs of great circles. In order to calculate the interior angles, we calculate the scalar product of the unit tangents to these edges at a vertex. By symmetry, any vertex will do. We calculate this for the icosahedron and the dodecahedron, the other cases being similar.

For the dodecahedron, a configuration of vertices is given by

$$\begin{pmatrix} 0 & \pm\lambda & \pm\lambda^{-1} & \pm 1 \\ \pm\lambda^{-1} & 0 & \pm\lambda & \pm 1 \\ \pm\lambda & \pm\lambda^{-1} & 0 & \pm 1 \end{pmatrix}, \tag{7.25}$$

where $\lambda = \frac{1+\sqrt{5}}{2}$ (see [11], Sect. 3.8). With the notation of Definition 7.4, three consecutive vertices around one pentagonal face are given by

$$\mathbf{x}_1 = \frac{1}{\sqrt{8+3a^2}}\begin{pmatrix} a\lambda^{-1} \\ a\lambda \\ 0 \\ \sqrt{8} \end{pmatrix}, \mathbf{x}_2 = \frac{1}{\sqrt{8+3a^2}}\begin{pmatrix} a \\ a \\ a \\ \sqrt{8} \end{pmatrix}, \mathbf{x}_3 = \frac{1}{\sqrt{8+3a^2}}\begin{pmatrix} 0 \\ a\lambda^{-1} \\ a\lambda \\ \sqrt{8} \end{pmatrix},$$

where $a = \sqrt{d(1-\gamma)}$ and $\lambda = (1+\sqrt{5})/2$. These three vertices determine a great 2-sphere in S^3 which contains the pentagonal face. With this arrangement, $\mathbf{x}_2$ is the central vertex joined to $\mathbf{x}_1$ and $\mathbf{x}_3$ by arcs of great circles. In order to calculate the interior angle at each vertex of the pentagon, we calculate the tangent to each of these arcs at $\mathbf{x}_2$. To do this we apply (7.24). For the first arc we set $\mathbf{u} = \mathbf{x}_1$ and $\mathbf{v} = \mathbf{x}_2$, to obtain the tangent vector:

$$\mathbf{t}_1 = \frac{1}{2\sqrt{3a^2+8}\sqrt{a^2+12-4\sqrt{5}}}\begin{pmatrix} \frac{a^2}{2}(3-\sqrt{5})+4(3-\sqrt{5}) \\ -\frac{a^2}{2}(3+\sqrt{5})+4(1-\sqrt{5}) \\ \sqrt{5}a^2+8 \\ -a\sqrt{8}(3-\sqrt{5}) \end{pmatrix}.$$

For the second arc, we set $\mathbf{u} = \mathbf{x}_3$ and $\mathbf{v} = \mathbf{x}_2$, to obtain:

$$\mathbf{t}_2 = \frac{1}{2\sqrt{3a^2+8}\sqrt{a^2+12-4\sqrt{5}}}\begin{pmatrix} \sqrt{5}a^2+8 \\ \frac{a^2}{2}(3-\sqrt{5})+4(3-\sqrt{5}) \\ -\frac{a^2}{2}(3+\sqrt{5})+4(1-\sqrt{5}) \\ -a\sqrt{8}(3-\sqrt{5}) \end{pmatrix}.$$

Then

$$\mathbf{t}_1 \cdot \mathbf{t}_2 = \frac{-a^2+16-8\sqrt{5}}{2(a^2+12-4\sqrt{5})},$$

which gives the cosine of the interior angle (it is indeed the interior angle, being greater than $\pi/2$). Then the area (in absolute angle measure) of each pentagonal face is given by

$$\frac{1}{4\pi}\left\{5\arccos\left(\frac{-a^2+16-8\sqrt{5}}{2(a^2+12-4\sqrt{5})}\right) - 3\pi\right\},$$

so that the surface area of the spherical dodecahedron is given by twelve times this quantity. We then obtain the angular deficiency, or 4-curvature:

$$\delta = 10 - \frac{15}{\pi}\arccos\left(\frac{-a^2+16-8\sqrt{5}}{2(a^2+12-4\sqrt{5})}\right).$$

On substituting the value of a, we obtain the required formula.

For the icosahedron, a configuration of vertices is given by

$$\begin{pmatrix} 0 & \pm 1 & \pm\lambda \\ \pm 1 & \pm\lambda & 0 \\ \pm\lambda & 0 & \pm 1 \end{pmatrix}, \tag{7.26}$$

where $\lambda = \frac{1+\sqrt{5}}{2}$. Three vertices which form one of the triangular faces are given by:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ \lambda \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} \lambda \\ 0 \\ 1 \end{pmatrix},$$

where $\lambda = (1+\sqrt{5})/2$. Then $\rho = 2+2\lambda^2 = 5+\sqrt{5}$, $n = 12$ and $r^2 = (5+\sqrt{5})/2$. This gives the corresponding vertices in S^3 as:

$$\mathbf{x}_\ell = \frac{\sqrt{2}}{\sqrt{2+d(1-\gamma)}} \begin{pmatrix} \sqrt{\frac{d(1-\gamma)}{5+\sqrt{5}}}\,\mathbf{v}_\ell \\ 1 \end{pmatrix} \quad (\ell = 1, 2, 3)\,.$$

We then proceed as above for the dodecahedron to calculate the angular deficiency. □

Example 7.4 The 600-cell is a convex 4-dimensional regular polytope made up of 600 tetrahedral 3-polytopes. It has 120 vertices and 720 edges. Its vertex figure is a regular icosahedron. If we consider an orthogonal projection onto the complex plane and let ϕ associate the corresponding value to each vertex, then ϕ satisfies (7.2) with γ constant. We can find the value of γ as follows.

Since the edges of the 600-cell all have the same length, in the notation of the above proof, we must have the distance from the origin to $\mathbf{x}_1$, that is 1, equal to the distance between two neighbors of the vertex figure: $||\mathbf{x}_1 - \mathbf{x}_2||$. One can readily calculate:

$$||\mathbf{x}_1 - \mathbf{x}_2||^2 = \frac{8d(1-\gamma)}{(5+\sqrt{5})[2+d(1-\gamma)]}\,,$$

to obtain the negative value:

$$\gamma = \frac{5(1-2\sqrt{5})}{3}\,.$$

One can now confirm the generalization of the theorem of Descartes (Theorem 7.2).

The 600-cell has 5 tetrahedra around each of its edges, each having dihedral angle $\arccos(1/3)$. Thus the angular deficiency at each edge (in absolute angle measure) is given by:

$$\delta_e = 1 - \frac{5}{2\pi}\arccos\frac{1}{3}\,.$$

Substitution of the above value of γ into the third formula of Proposition 7.5 gives the deficit, or curvature at each vertex, as

$$\delta_v = 6 - \frac{15}{\pi}\arccos\frac{1}{3}\,.$$

On then finds that $120\delta_v - 720\delta_e = 0$, as required.

We can proceed similarly to obtain explicit formulae for higher dimensional N-curvature. This is simplified by the fact that there are just three regular polytopes in dimensions $N \geq 5$, namely the N-simplex, the N-cube and the cross-polytope, with vertex figures an $(N-1)$-simplex in the first two cases and another cross-polytope in the last case. To find the N-curvature requires the calculation of the $(N-2)$-dimensional measure of $(N-2)$-simplices in great spheres of S^{N-1}. This is a standard, but non-trivial procedure using Schläfli's differential equality [25]. See also the expository article of J. Milnor for a nice account and references [21, pp. 281–295]. The article of J. Murakami provides explicit expressions in the 3-sphere [22]. We do not attempt to derive these formulae here.

7.6 Other Curvatures

Recall from Sect. 7.4, that given an edge $e = \overline{xy} \in E$, the *edge-curvature* of e is defined to be the unique angle $\theta(e) := \arccos(\langle \mathbf{u}(x), \mathbf{u}(y)\rangle_{\mathbb{R}^N}) \in [0, \pi]$ between the axes of lifted configured stars. By analogy with Riemannian geometry, various other curvatures can now be defined. If we let $\ell(e)$ denote the length of an edge $e = \overline{xy}$ as given by Definition 7.2, and $\theta(e)$ its edge-curvature, then the radius of the best-fit circle is given by $r(e) = \ell(e)/\theta(e)$ (by *best-fit circle*, we mean the circle subtending the same arc length $\ell(e)$ for the given angle $\theta(e)$). The *normal curvature* of e is then the reciprocal $1/r(e) = \theta(e)/\ell(e)$. The *mean curvature* at a vertex x can then be defined to be the mean of the normal curvatures of the edges incident with x. Since $\ell(e)$ depends on the scaling $\phi \mapsto \lambda\phi$, this quantity also depends on the scaling; the mean curvature should be thought of as the analogue of the same notion in the smooth setting, whereby we locally embed a Riemannian manifold in a Euclidean space.

From Theorem 7.1, we already have γ related to mean curvature by the formula $\gamma = -1/H^2$. However, this latter quantity will only approximate mean curvature for an invariant framework that closely approximates a smooth surface. For example, the cube has $\gamma \equiv 0$, which is far from the mean curvature of a sphere, whereas the 600-cell has negative γ, which is more realistic. We don't know if the two notions are related, or if they are related to more classical definitions of mean curvature for polytopes such as are given in [18, 23].

Ricci curvature is one of the most natural curvatures intrinsic to a Riemannian manifold. Recall that given two unit directions $\mathbf{X}$ and $\mathbf{Y}$, the sectional curvature $\mathrm{Sec}\,(\mathbf{X}, \mathbf{Y})$ can be interpreted as the Gaussian curvature of a small geodesic surface generated by the plane $\mathbf{X} \wedge \mathbf{Y}$. This in turn is the product of the principal curvatures which are the extremal values of the normal curvatures. The Ricci curvature $\mathrm{Ric}\,(\mathbf{X}, \mathbf{X})$ is then the sum: $\sum_j \mathrm{Sec}\,(\mathbf{X}, \mathbf{Y}_j)$ taken over an orthonormal frame $\{\mathbf{Y}_j\}$ with each $\mathbf{Y}_j$ orthogonal to $\mathbf{X}$. This motivates the following definition.

Definition 7.5 Given a vertex $x \in V$ and two edges $e_1 = \overline{xy_1}$ and $e_2 = \overline{xy_2}$ with endpoint x, we define the *sectional curvature* $\mathrm{Sec}\,(e_1, e_2)$ *determined by* e_1 *and* e_2 to be the product:

$$\mathrm{Sec}_x\,(e_1, e_2) = \frac{\theta(e_1)}{\ell(e_1)}\frac{\theta(e_2)}{\ell(e_2)}\,,$$

where $\theta(e_j)$ $(j = 1, 2)$ are the edge-curvatures. For an edge $e = \overline{xy}$, the *Ricci curvature* $\mathrm{Ric}\,(e, e)$ is the sum

$$\mathrm{Ric}_x\,(e, e) = \ell(e)^2 \sum_{z\sim x, z\neq y} \mathrm{Sec}_x(e, \overline{xz}) = \ell(e)^2 \sum_{z\sim x, z\neq y} \frac{\theta(\overline{xy})}{\ell(\overline{xy})}\frac{\theta(\overline{xz})}{\ell(\overline{xz})}\,,$$

and the *scalar curvature at* x is given by

$$\mathrm{Scal}_x = \sum_{y\sim x} \mathrm{Ric}\,(\overline{xy}, \overline{xy})/\ell(\overline{xy})^2.$$

As in Riemannian geometry, the Ricci curvature is scale invariant. In fact, it makes complete sense to introduce the factor $\ell(e)^2$, since the Ricci curvature is bilinear in its arguments. On the other hand, both the sectional curvature and the scalar curvature will decrease as we dilate a solution to (7.2) by some scaling $\phi \mapsto \lambda\phi$ (λ constant), since that will modify edge length by the same factor λ. In Riemannian geometry, one usually applies the polarization identity to define $\mathrm{Ric}\,(\mathbf{X}, \mathbf{Y})$, however, there would seem to be no reasonable interpretation for the sum of two edges in our setting.

In [20], Romon, provides numerical calculations of various curvatures associated to the convex regular polyhedra, notably the Ricci curvature of Ollivier. In Table 7.2, we produce similar calculations for the above curvatures, together with Ollivier's Ricci curvature for comparison. Since the sectional and scalar curvatures are scale dependent, we have taken edge length equal to 1 in their definitions. Rational values are exact.

Table 7.2 Table of the various curvatures associated to the convex polyhedra with Ollivier's Ricci curvature included for comparison

Curvature	Tetrahedron	Cube	Octahedron	Icosahedron	Dodecahedron
Vertex (Gauss)	1/2 = 0.50	1/4 = 0.25	1/3 = 0.33	1/6 = 0.17	1/10 = 0.10
Edge	0.30	1/4 = 0.25	1/4 = 0.25	0.18	0.11
Sectional	0.09	1/16 = 0.63	1/16 = 0.63	0.03	0.01
Scalar	0.54	3/8 = 0.36	3/4 = 0.75	0.60	0.07
Ricci	0.18	1/8 = 0.13	3/16 = 0.19	0.12	0.02
Ollivier's Ricci	4/3 = 1.33	2/3 = 0.67	1 = 1.00	2/5 = 0.40	0 = 0.00

Appendix 1: The Geometric Spectrum, Gröbner Bases and the γ-Polynomial

To compute the geometric spectrum, even for simple graphs, is quite challenging. We consider some fundamental cases and then make a simplifying assumption in order to apply the technique of Gröbner bases. This motivates the construction of a new polynomial invariant associated to a graph.

First Cases

Consider a vertex v_0 of degree 2. Suppose that a solution ϕ to (7.2) is non-constant on a neighborhood of this vertex, thus ϕ takes a different value on at least one of its two neighbors v_1, v_2, say v_1. By the normalization (7.3), we may suppose that $\phi(v_0) = 0$ and that $\phi(v_1) = 1$. If we let $\phi(v_2) = z$ as illustrated in Fig. 7.9, then at vertex v_0, Eq. (7.2) takes the form:

$$\frac{\gamma}{2}(1+z)^2 = 1 + z^2 \,.$$

Suppose that $z \neq -1$. Then the requirement that γ be real is equivalent to

$$\text{either} \quad \Im(z) = 0 \quad \text{or} \quad |z| = 1 .$$

If z is real and $z \neq \pm 1$, then $\gamma > 1$, so we reject this case since γ will not lie in the geometric spectrum. Suppose that z is not real and write it in polar coordinates: $z = \mathrm{e}^{\mathrm{i}\alpha}$. Then

$$\gamma = \frac{2\cos\alpha}{1+\cos\alpha} = \frac{2\cos\theta}{\cos\theta - 1} \,,$$

where $\theta = \pi - \alpha$ is the external angle. The two limiting cases $\alpha = 0$ and $\alpha = \pi$ correspond to $\gamma = 1$ and $\gamma = -\infty$, respectively. This justifies the admissibility of the value 1 in the geometric spectrum at a vertex of degree 2 (see Sect. 7.3.2).

As a consequence, for a cyclic graph, solutions to (7.2) correspond to realizations in the plane as a polygonal framework with sides of equal length. A cyclic graph on

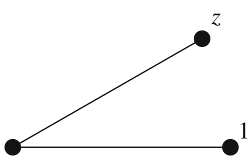


Fig. 7.9 The general invariant star at a vertex of degree 2

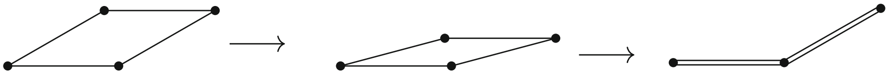


Fig. 7.10 An invariant cyclic graph of order 4 has geometric spectrum corresponding to realizations in the plane as a polygonal framework with sides of equal length

three vertices can only have one such realization (up to similarity transformation) as an equilateral triangle, so that $\gamma = 2/3$ is the only value in the geometric spectrum.

A cyclic graph on four vertices is realized as a rhombus, with the possibility of its edges collapsing onto themselves as indicated in Fig. 7.10.
In this case, the geometric spectrum has continuous components corresponding to continuous deformations of the rhombus which may be parametrized by one of the internal angles. Branching phenomena occurs as edges collapse onto themselves. As the number of vertices increases, so too does the number of variables that parametrize the geometric spectrum.

Let us turn to the other extreme, that of a complete graph. The complete graph on three vertices is cyclic and as shown above, there is just one element in its geometric spectrum, the constant value $\gamma = 2/3$. Since any invariant star in $\mathbb{R}^3$ with three external vertices is necessarily configured [3], it follows that a complete graph on four vertices which has a realization as an invariant framework in $\mathbb{R}^3$ is necessarily a tetrahedron with γ taking the constant value $3/4$. However, the results of Sect. 7.3 only guarantee a local lifting of a vertex and its neighbors in an invariant way, so in general, we don't know if such a realization exists. However, we can use a computer to solve the equations for a sufficiently small number of vertices.

Consider the complete graph on $N+1$ vertices. Label the vertices by the integers $0, 1, \ldots, N$ and set $\gamma_j = \gamma(j)$, $z_j = \phi(j)$. After normalization (7.3), we may suppose that $z_0 = 0$ and $z_1 = 1$. Then (7.2) becomes:

$$\frac{\gamma_k}{N}\left(\sum_{j=0}^{N}(z_k - z_j)\right)^2 = \sum_{j=0}^{N}(z_k - z_j)^2 \quad (k = 0, 1, 2, \ldots, N),$$

with the constraints that each γ_k be real and < 1. The software MAPLE can now solve this algebraic system at least up to $N = 5$ (the complete graph on 6 vertices), which confirms that the only element in the geometric spectrum is given by γ constant equal to $N/(N+1)$. We currently don't have a mathematical (non-computer) proof of this fact and don't know if the constancy of γ persists for $N > 5$.

The Constant Geometric Spectrum and the γ-Polynomial

From now on, to simplify matters, we consider only constant values of γ that may lie in the spectrum and attempt an algebraic geometric approach to compute this set which we refer to as the *constant geometric spectrum*.

Let $G = (V, E)$ be a connected graph. We are interested in the possible real numbers γ for which there are non-constant solutions to the equation:

$$\gamma \Delta \phi^2 = (d\phi)^2 \,. \tag{7.27}$$

Any solution is invariant by $\phi \mapsto \lambda\phi + \mu$, for complex constants $\lambda, \mu (\lambda \neq 0)$. We will no longer insist that $\gamma < 1$. Consider first how to parametrize all possible complex fields on the graph under this invariance.

Label the vertices of the graph $x_1, x_2, \dots, x_N$ and consider a non-constant complex field ϕ that assigns the value $\phi(x_k) = z_k$ to vertex x_k. Then the space of all such fields is identified with the complex space $\mathbb{C}^N \setminus \{\mu(1, 1, \dots, 1) : \mu \in \mathbb{C}\}$. Up to the equivalence $(z_1, \dots, z_N) \sim (z_1 + \mu, \dots, z_N + \mu)$, we can identify these fields with the set $\Pi \setminus \{0\}$, where Π is the linear subspace $\Pi = \{\mathbf{Z} = (z_1, \dots, z_N) \in \mathbb{C}^N : z_1 + \cdots + z_n = 0\} \subset \mathbb{C}^N$. In effect, given any non-constant field $(z_1, \dots, z_N)$, then $(z_1 + \mu, \dots, z_N + \mu)$ lies in the plane $z_1 + \cdots + z_N = 0$, when we set $\mu = -\frac{1}{N}(z_1 + \cdots + z_N)$. By non-constancy, this is non-zero. Furthermore, it is clear that any two equivalent fields correspond to the same point.

Now consider the relation $\mathbf{Z} \sim \lambda \mathbf{Z}$, for $\lambda \in \mathbb{C} \setminus \{0\}$. This defines the *moduli space* of fields up to equivalence to be $\mathscr{X} := \mathbb{C}\mathrm{P}^{N-2}$. Specifically, given a point $[z_1, \dots, z_{N-1}] \in \mathscr{X}$ in homogeneous coordinates, we define a representative field by $(z_1, \dots, z_{N-1}, z_N = -\sum_{k=1}^{N-1} z_k) \in \mathbb{C}^N$. In practice, we can set a field equal to 0 and 1 on two selected vertices x_0 and x_1, respectively, and label the other vertices arbitrarily. This is only one chart and we miss those fields which coincide at these two vertices.

If we consider γ as an arbitrary *complex* parameter, then (7.27) imposes a constraint at each vertex, so we have N equations in $N - 1$ parameters. In general these are independent so that this is an overdetermined system, which may have no solutions.

For each $\ell = 2, \dots, N$, consider the following set of N polynomials defined over the algebraically closed field $\mathbb{C}$. The variables are the values $\{z_1, \dots, z_N\}$ of a field on G with constraints $z_1 = 0$ and $z_\ell = 1$; we suppose the degree of vertex j is d_j and that $z_{jk} \in \{z_1, \dots, z_N\}$ $(k = 1, \dots, d_j)$ are the values of the field on the neighbors x_{jk} of x_j. The polynomials are then defined by

$$f_j^\ell := \frac{\gamma}{d_j}\left(\sum_{k=1}^{d_j}(z_j - z_{jk})\right)^2 - \sum_{k=1}^{d_j}(z_j - z_{jk})^2 \qquad (z_1 = 0,\ z_\ell = 1)\,,$$

in the $N - 1$ *complex* variables $\{\gamma, z_2, z_3, \dots, \widehat{z_\ell}, \dots, z_N\}$. Recall some facts and terminology from commutative algebra. We are particularly interested in the techniques of Gröbner bases, for which we refer the reader to [1, 27].

For an ideal $I = < f_1, \dots, f_N >$ in a polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_M]$, we denote by $V(I)$ the corresponding variety: $f_1 = 0, f_2 = 0, \dots, f_N = 0$. Then I is said to be *zero-dimensional* if $V(I)$ is finite. A *Gröbner basis* for I is a basis of polynomials which can be constructed from $f_1, \dots, f_N$ using a particular algorithm, called the *Buchberger algorithm*. To employ this algorithm, one is required first to choose an order on monomials. We shall only be concerned with *lexicographical order* here, which means we first choose an ordering of the variables, say $x_1 > x_2 > \cdots > x_M$ and then order monomials $x^\alpha := x_1{}^{\alpha_1} \cdots x_M{}^{\alpha_M}$, $x^\beta := x_1{}^{\beta_1} \cdots x_M{}^{\beta_M}$, by $x^\alpha < x^\beta$ if and only if the first coordinate from the left for which α_i and β_i are different, satisfies $\alpha_i < \beta_i$. With respect to the monomial order, every polynomial f in I has a leading term $\mathrm{lt}\,(f)$ which is the product $\mathrm{lt}\,(f) = \mathrm{lc}\,(f)\mathrm{lm}\,(f)$ of the leading coefficient with the leading monomial.

A set of non-zero polynomials $G = \{g_1, \dots, g_P\}$ in I is called a *Gröbner basis for I* if and only if for all $f \in I$ such that $f \neq 0$, there is a g_j in G such that $\mathrm{lm}\,(g_j)$ divides $\mathrm{lm}\,(f)$. The Gröbner basis is further called *reduced* if for all j, $\mathrm{lc}\,(g_j) = 1$ and g_j is reduced with respect to $G \setminus \{g_j\}$, that is, no non-zero term in g_j is divisible by any $\mathrm{lm}\,(g_k)$ for any $k \neq j$. A theorem of Buchberger states that every non-zero ideal has a unique reduced Gröbner basis with respect to a monomial order [9]. Gröbner bases are particularly useful for understanding the solution set of a system of polynomial equations.

Let I be an ideal in the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_M]$ and let $G = \{g_1, \dots, g_P\}$ be the unique reduced Gröbner basis with respect to the lexicographical ordering induced by the order $x_1 > x_2 > \cdots > x_M$. Then $V(I)$ is finite if and only if for each $j = 1, \dots, M$, there exists a $g_k \in G$ such that $\mathrm{lm}\, g_k = x_j{}^{n_j}$ for some natural number n_j. As a consequence, if I is a zero-dimensional ideal, it follows that we can order $g_1, \dots, g_P$ so that g_1 contains only the variable x_M, g_2 contains only x_M, x_{M-1} and so on. This is because the leading monomial of one element, g_1 say, of G must be a power of x_M and then no other term of g_1 can contain powers of any other variable (for such terms would be greater that any power of x_M with respect to the monomial order), and so on for successive elements $g_2, g_3, \dots$ of G. We note also that $V(I)$ is empty if and only if $1 \in G$.

It is also the case that, with the above hypothesis, the polynomial g_1 is the *least degree univariate polynomial in x_M which belongs to I* (any zero-dimensional ideal contains such a polynomial for every variable). For if there was another univariate polynomial $p(x_M)$ with $\deg p < \deg g_1$, then $\mathrm{lm}\, p$ would divide $\mathrm{lm}\, g_1$ in a strict sense, which would contradict the fact that G is a reduced Gröbner basis. Let us now return to the case under consideration.

For each $\ell = 2, \dots, N$, consider the ideal $I_\ell =< f_1{}^\ell, \dots, f_N{}^\ell >$. Suppose that for each $\ell = 2, \dots, N$ this admits a least degree univariate polynomial p_ℓ in γ. This can be constructed by first choosing a lexicographical ordering of the variables with γ the smallest and then applying an algorithm (say the Buchberger algorithm) to construct the unique reduced Gröbner basis for I_ℓ. The first element of this basis gives p_ℓ.

Definition 7.6 We define the *γ-polynomial $p = p_G$* of the connected finite graph $G = (V, E)$ to be the least common multiple of the least degree univariate polynomials p_ℓ ($\ell = 2, \ldots, N$) in γ associated to the Eq. (7.2) for fields $(z_1, \ldots, z_N)$ on G with $z_1 = 0$ and $z_\ell = 1$:

$$p := \mathrm{lcm}\,(p_2, \ldots, p_N)\,,$$

when each p_ℓ exists.

The γ-polynomial $p(\gamma)$ is defined up to rational multiple and has rational coefficients. This is because the initial polynomials $f_j^{\,\ell}$ used to define p all have integer coefficients and the Buchberger algorithm then generates polynomials with rational coefficients—it involves at most division by coefficients—see [1]. Clearly p depends only on the isomorphism class of a graph and in the case when the Eq. (7.27) admit no solutions for γ constant and complex, then $p \equiv 1$. In this case we shall say that *p is trivial*. The polynomials p_ℓ and so p may still be well-defined even if the solution set of the equations is infinite (that is the corresponding ideal is no longer zero-dimensional). In fact we know of no case when they are not well-defined.

The constant geometric spectrum arises as real roots of p (the problem of establishing the discreteness of the spectrum is clearly intimately related to knowing if p is well-defined in all cases). However, not all real roots may occur in the spectrum, for in general they must also solve the other equations determined by the Gröbner basis: $g_1 = 0, \ldots, g_P = 0$. Examples below illustrate this property. We know of no two non-isomorphic connected graphs with non-trivial γ-polynomial having the same γ-polynomial.

The examples of the triangle C_3 (the cyclic graph on three vertices) and the bipartite graphs K_{23} and K_{33} are instructive. We label the vertices as shown in Fig. 7.11 and consider fields ϕ taking the values $\phi(x_j) = z_j$ at each vertex x_j.

For the triangle, there are precisely two solutions to (7.27) when we normalize so that $z_1 = 0, z_2 = 1$; specifically $z_3 = \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$. Then $p = p_2 = p_3 = 3\gamma - 2$ is the γ polynomial and the only constant element (in fact the only element) of the geometric spectrum is the unique root $\gamma = 2/3$.

For K_{23}, we find $p_2 = 1$ with no solution and $p_3 = \gamma^2 - 2\gamma + 1 = (\gamma - 1)^2$ with solution $z_1 = 0$, $z_3 = 1, z_2 = 0, z_5 = \lambda$ arbitrary and $z_4 = [1 + \lambda \pm \sqrt{3}(1 - \lambda)\mathrm{i}]/2$. Then $p = \gamma^2 - 2\gamma + 1$ and the constant geometric spectrum is given by $\Sigma = \{1\}$.

For K_{33}, we find $p_2 = 9\gamma^2 - 26\gamma + 17 = (\gamma - 1)(9\gamma - 17)$ and $p_4 = 9\gamma^3 - 35\gamma^2 + 43\gamma - 17 = (\gamma - 1)p_2$, so that the γ-polynomial $p = 9\gamma^3 - 35\gamma^2 + 43\gamma - 17$.

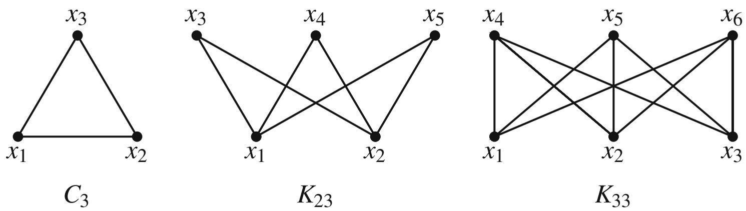


Fig. 7.11 Three graphs for which the γ-polynomial can be easily calculated by an appropriate labeling of the vertices

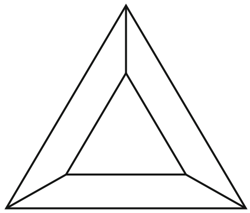


Fig. 7.12 An example of a simple graph with complicated γ-polynomial

Although this has $\gamma = 17/9$ as a root, the constant geometric spectrum $\Sigma = \{1\}$. In fact for $\gamma = 1, z_1 = 0, z_2 = 1$ we find a two complex parameter family of solutions to (7.27). The next example shows that even for simple graphs, the γ-polynomial can be quite complicated.

Consider the graph of constant degree three on six vertices whose edges form two concentric triangles as shown in Fig. 7.12.
The γ-polynomial is given by

$$\begin{aligned}5859375\gamma^{10} - 67656250\gamma^9 + 333521875\gamma^8 - 926025000\gamma^7 \\ +1603978830\gamma^6 - 1808486028\gamma^5 + 1339655598\gamma^4 \\ -639892872\gamma^3 + 186760323\gamma^2 - 29598858\gamma + 1883007\,.\end{aligned}$$

This has eight real roots, four of which are rational: $\gamma = 3/5, 21/25, 1, 3$. There remain two conjugate complex roots. Since the γ-polynomial differs from that of the bipartite graph K_{33} calculated above, we deduce that these two graphs of constant degree three on six vertices cannot be isomorphic.

It is interesting to consider the real roots that are ≥ 1, which can be seen to arise from surjective mappings $\phi : V \to S_N := \{z_0, \ldots, z_N\} \subset \mathbb{C}$, where $z_0, \ldots, z_N$ are the images under any orthogonal projection to the complex plane of the vertices of a regular N-simplex in $\mathbb{R}^N$. This is because of the translation-invariant relation

$$(z_0 + \cdots + z_N)^2 = (N + 1)({z_0}^2 + \cdots + {z_N}^2)$$

between these projections whenever $N \geq 2$ [13]. For example, if $N = 4$, we may take $S_4 = \{0, 1, \mathrm{i}, 1+\mathrm{i}\}$. When $N = 1$, we take $S_1 = \{0, 1\}$ and $(z_0+z_1)^2 = {z_0}^2+{z_1}^2$.

The root 3 corresponds to the solution $\phi : V \to S_1 = \{0, 1\}$ to (7.2) which takes the value 0 say, on the vertices of the inner triangle and 1 on the vertices of the outer triangle. The root 1 corresponds to the solution $\phi : V \to S_2 = \{0, 1, \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\}$, as indicated in Fig. 7.13. Thus, ϕ takes on the colors red, green and blue which are in bijective correspondence with S_2.

More generally, if we can "color" the vertices of a regular graph G of degree d with S_N in such a way that each vertex is connected by an edge to precisely one vertex of each of the other colors, then $\gamma = d/(N + 1)$ will be an element of the constant geometric spectrum. Clearly we must have $d \geq N$. For the complete graph on $N + 1$ vertices, we can take the coloring given by any bijection $V \to S_N$ to give $\gamma = N/(N + 1)$. The above example can be generalized by taking two

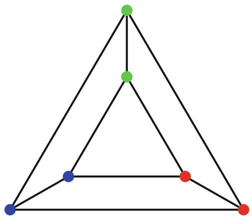


Fig. 7.13 Certain roots of the γ-polynomial correspond to graph colorings

Table 7.3 The γ-polynomial for the complete graph on $N+1$ vertices

N	$p(\gamma)$
3	$8\gamma^2 - 18\gamma + 9$
4	$5\gamma - 4$
5	$18\gamma^2 - 45\gamma + 25$
6	$7\gamma - 6$
7	$32\gamma^2 - 84\gamma + 49$

copies of the complete graph on $N+1$ vertices and connecting each vertex of one of the graphs to precisely one vertex of the other in a bijective correspondence. Now $d = N + 1$ and $\gamma = 1$ lies in the constant geometric spectrum. Relations to vertex colorings suggest potential connections between the γ-polynomial and other more well-known polynomial invariants, such as the Tutte polynomial.

Finally, let us consider the γ-polynomial of the complete graph on $N+1$ vertices. We may apply the method of Gröbner bases with an appropriate lexicographical ordering which produces the γ-polynomial as its first basis element. In fact, by the symmetry of the complete graph, it is clear that all the univariate polynomials p_ℓ ($\ell = 2, \dots, N$) in Definition 7.6 are identical. If we denote the γ-polynomial by $p(\gamma)$, then using MAPLE, we obtain the suggestive list given in Table 7.3.

It is reasonable to conjecture that for N even, the γ-polynomial is given by $p(\gamma) = (N+1)\gamma - N$ and that for $N = 2k+1$ odd, it is given by

$$\begin{aligned} p(\gamma) &= 2(k+1)^2\gamma^2 - 3(k+1)(2k+1)\gamma + (2k+1)^2 \\ &= \Big(2(k+1)\gamma - (2k+1)\Big)\Big((k+1)\gamma - (2k+1)\Big). \end{aligned}$$

Appendix 2: New Solutions from Old

Holomorphic Mappings Between Graphs

The natural class of mappings between graphs which preserve Eq. (7.2) are the so-called *holomorphic* mappings. These were introduced for simple graphs under the name *semi-conformal* mappings by Urakawa [28], as the class of maps which preserve local harmonic functions (i.e. functions which are harmonic at a vertex). The notion was later extended to non-simple graphs by Baker and Norine [5, 6], who used the term *holomorphic mapping*. In [4], it was shown that the holomorphic mappings are precisely the class of mappings which preserve local *holomorphic* functions, defined as solutions to (7.2) for which $\gamma \equiv 0$. The definition requires that we restrict to mappings of graphs that determine a well-defined mapping of the tangent space at each vertex, which also justifies our inclusion of the zero vector in the definition of tangent space (see Sect. 7.2).

Definition 7.7 Let $f : G = (V, E) \to H = (W, F)$ be a mapping between graphs. Then f is *holomorphic* if

(i) $x \sim y$ implies either $f(x) = f(y)$ or $f(x) \sim f(y)$;
(ii) there exists a function $\lambda : V \to \mathbb{N}$ such that for all $x \in V$ and for all $z' \sim z = f(x)$, we have

$$\lambda(x) = \lambda(x, z') = \sharp\{x' \in V : x' \sim x, f(x') = z'\},$$

is independent of the choice of z'; we set $\lambda(x) = 0$ if $f(x') = z$ for all $x' \sim x$. Call λ the *dilation of* f.

Proposition 7.6 *Let $f : G = (V, E) \to H = (W, F)$ be a holomorphic mapping between graphs of dilation $\lambda : V \to \mathbb{N}$. Suppose $\psi : W \to \mathbb{C}$ satisfies the equation*

$$\mu(\Delta\psi)^2 = (\mathrm{d}\psi)^2,$$

for some $\mu : W \to \mathbb{R}$. Then for each $x \in V$ such that $\lambda(x) \neq 0$, the function $\phi = \psi \circ f$ satisfies (7.2) at x with

$$\gamma(x) = \frac{d_x \mu(f(x))}{\lambda(x) d_{f(x)}}, \tag{7.28}$$

where d_z also denotes the degree of a vertex $z \in W$.

Proof Let $f : G = (V, E) \to H = (W, F)$ be a holomorphic mapping between graphs of dilation $\lambda : V \to \mathbb{N}$. Let $x \in V$ and set $z = f(x)$. Then

$$\frac{\mu(z)}{d_z}\left(\sum_{z' \sim z}(\psi(z') - \psi(z))\right)^2 = \sum_{z' \sim z}(\psi(z') - \psi(z))^2.$$

Since f is holomorphic

$$\sum_{x' \sim x}[(\psi \circ f)(x') - (\psi \circ f)(x)] = \lambda(x) \sum_{z' \sim z}(\psi(z') - \psi(z)).$$

Suppose that $\lambda(x) \neq 0$. Then

$$\begin{aligned}\sum_{x'\sim x}[(\psi\circ f)(x') - (\psi\circ f)(x)]^2 &= \lambda(x)\sum_{z'\sim z}(\psi(z') - \psi(z))^2 \\ &= \frac{\lambda(x)\mu(z)}{d_z}\left(\sum_{z'\sim z}(\psi(z') - \psi(z))\right)^2 \\ &= \frac{\mu(z)}{\lambda(x)d_z}\left(\sum_{x'\sim x}[(\psi\circ f)(x') - (\psi\circ f)(x)]\right)^2,\end{aligned}$$

from which the formula follows. If on the other hand $\lambda(x) = 0$, then $f(x') = f(x)$ for all $x' \sim x$ and both sides of (7.2) vanish. □

Given a holomorphic mapping between graphs, the above proposition shows how an element of the geometric spectrum on the co-domain determines one on the domain. A simple example of a holomorphic mapping between planar graphs is given in Fig. 7.14. In this example, the outer "wheel" of the domain graph is mapped cyclically onto the outer wheel of the image, covering it twice, while the central vertex of the domain is mapped onto the central vertex of the image.

The dilation at the central vertex is given by $\lambda = 2$, whereas it equals 1 at the other vertices. With reference to Example 7.2, we see that the constant value $\gamma \equiv 1/3$ is also an element of the geometric spectrum of the domain graph, the doubling of the degree at the central vertex being exactly compensated for by the doubling of the dilation in the formula (7.28).

Applying Normalization to Construct New Solutions

Another method to construct new solutions to (7.2) is to exploit the freedom to normalize a solution ϕ by (7.3). Whilst not exhaustive, we describe some examples.

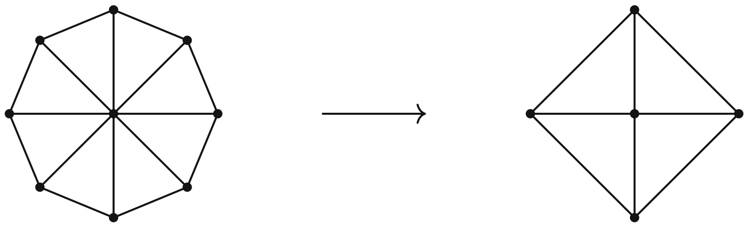


Fig. 7.14 An example of a holomorphic mapping between graphs: the outer wheel covers its image twice, while the central vertex is preserved

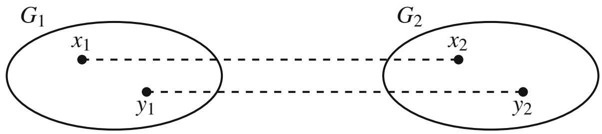


Fig. 7.15 Solutions to equation (7.2) on two graphs can be normalized at two vertices in such a way that the graphs can be connected by edges to give a new solution

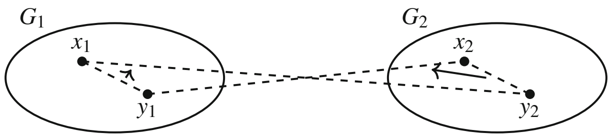


Fig. 7.16 Solutions to equation (7.2) on two graphs can be normalized at two vertices to allow edge rotations which connect the two graphs, giving a new solution

The process of collapsing was described at the end of Sect. 7.4, whereby edges that connect vertices on which a solution to (7.2) takes on the same value can be removed. This operation can be reversed as follows. Given a non-constant solution ϕ_1 to (7.2) on a graph $G_1 = (V_1, E_1)$, then for two vertices $x_1, y_1 \in V_1$ where $\phi_1(x_1) \neq \phi_1(y_1)$, we can apply the normalization (7.3) and suppose that $\phi_1(x_1) = 0$ and $\phi_1(y_1) = 1$. Similarly for a non-constant solution ϕ_2 to (7.2) on a graph $G_2 = (V_2, E_2)$, we may find two vertices x_2, y_2 and normalize so that $\phi_2(x_2) = 0$ and $\phi_2(y_2) = 1$. As illustrated in Fig. 7.15, define a new graph G whose vertex set $V = V_1 \cup V_2$ and whose edge set $E = E_1 \cup E_2 \cup \{\overline{x_1x_2}, \overline{y_1y_2}\}$.

Note that we are not obliged to add both edges, and indeed we can connect any vertices on which ϕ_1 and ϕ_2 take on the same value. We then define $\phi : V \to \mathbb{C}$ by $\phi(x) = \phi_1(x)$ if $x \in V_1$ or $\phi(x) = \phi_2(x)$ if $x \in V_2$. Clearly ϕ satisfies (7.2), but with γ modified to take into account the fact that the degrees at x_1, y_1, x_2, y_2 have increased by one.

As a variant, if both x_1 and y_1 and x_2 and y_2 are connected by edges in G_1 and G_2, respectively, then we can remove these edges and replace them by $\overline{x_1y_2}$ and $\overline{y_1x_2}$. Thus the new graph $G = (V, E)$ (illustrated in Fig. 7.16) has $V = V_1 \cup V_2$ and $E = (E_1 \setminus \{\overline{x_1y_1}\}) \cup (E_2 \setminus \{\overline{x_2y_2}\}) \cup \{\overline{x_1y_2}, \overline{y_1x_2}\}$. As before we can define a solution to (7.2) to be the restriction of ϕ_1 to V_1 and ϕ_2 to V_2, but now the degrees are preserved, so γ is unchanged.

The Preferential Attachment Model

In random graph theory, an important generative model is the preferential attachment scheme which proceeds as follows. For the current graph G, add a new vertex y and add an edge $\overline{xy}$ from y by randomly and independently choosing x in proportion to the degree of x in G. As demonstrated rigorously by Chung and Lu, as the order of the graph approaches infinity, this model generates the scale-free graphs that are so prevalent in biology and social networks [10]. Geometry may be seen to emerge from this process by exploiting our construction of invariant stars.

Consider an invariant (not necessarily configured) star in $\mathbb{R}^N$ with internal vertex located at the origin and with d external vertices. Let $\mathbf{x} \in \mathbb{R}^N$ be the center of mass of the external vertices. Suppose that $\mathbf{x} \neq \mathbf{0}$. Let $b > 0$ denote the distance of the center of mass from the origin along the axis of the star.

Lemma 7.2 ([2]) *The addition of a new external vertex at any point other than $-db$ along the axis of the star produces a new invariant star. Furthermore, if γ denotes the invariant of the original star and $x \in \mathbb{R}$ is the position along the axis of the star, then the new star invariant is given by*

$$\widetilde{\gamma} = \frac{(d+1)(x^2 + db^2\gamma)}{(x+db)^2} . \tag{7.29}$$

Proof Without loss of generality, we may suppose that the center of mass of the star lies along the y^N-axis. In particular, if $\mathbf{v}_1, \ldots, \mathbf{v}_d$ denote the external vertices, then

$$\sum_{\ell=1}^{d} \mathbf{v}_\ell = db\mathbf{e}_N .$$

We now add a new vertex at the point $x\mathbf{e}_N$, for some $x \in \mathbb{R}$. Thus the new star matrix is given by

$$(\mathbf{v}_1 | \cdots | \mathbf{v}_d | x\mathbf{e}_N) .$$

Let $A = (a_{jk})$ be an arbitrary orthogonal transformation of $\mathbb{R}^N$ and let $\phi : \mathbb{R}^N \to \mathbb{C}$ be the projection $\phi(y^1, \ldots, y^N) = y^1 + \mathrm{i}y^2$. Set $z_\ell = \phi \circ A(\mathbf{v}_\ell)$ for $\ell = 1, \ldots, d$ and $z_{d+1} = \phi \circ A(x\mathbf{e}_N)$. Then

$$z_\ell = \sum_{j=1}^{N} (a_{1j} + \mathrm{i}a_{2j}) v_\ell{}^j , \qquad z_{d+1} = x(a_{1N} + \mathrm{i}a_{2N}) .$$

Furthermore, $\sum_{\ell=1}^{d} z_\ell = db(a_{1N} + \mathrm{i}a_{2N})$, so that

$$\sum_{\ell=1}^{d} z_\ell{}^2 = \frac{\gamma}{d} \left(\sum_{\ell=1}^{d} z_\ell \right)^2 = db^2\gamma(a_{1N} + \mathrm{i}a_{2N})^2 ,$$

where γ is the invariant of the original star. We require that there is a real number $\widetilde{\gamma}$ such that

$$\widetilde{\gamma} \left(z_{d+1} + \sum_{\ell=1}^{d} z_\ell \right)^2 = (d+1) \left(z_{d+1}{}^2 + \sum_{\ell=1}^{d} z_\ell{}^2 \right) .$$

But this is uniquely given by (7.29). □

We note that as x approaches $-db$, then $|\widetilde{\gamma}|$ becomes arbitrary large. Indeed, when $x = -db$, then the Laplacian of ϕ vanishes at the internal vertex of the new star, so that $\widetilde{\gamma}$ is not well-defined in this case.

Suppose we are given a solution to (7.2) on a graph $G = (V, E)$. For a given $x \in V$, let $\{y_1, \ldots, y_d\}$ be its neighbors. Then by the lifting property described in Sect. 7.3, there is an invariant star K in $\mathbb{R}^3$ (or $\mathbb{R}^2$ if the degree of x is two) whose external vertices project to the values $\phi(y_1), \ldots, \phi(y_d)$ and whose internal vertex projects to $\phi(x)$. Introduce a new vertex y and form the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ for which $\widetilde{V} = V \cup \{y\}$ and $\widetilde{E} = E \cup \{\overline{xy}\}$. Define $\widetilde{\phi} : \widetilde{V} \to \mathbb{C}$ by $\widetilde{\phi}(u) = \phi(u)$ for $u \in V$ and $\widetilde{\phi}(y) = z$, where z is the projection of any point along the axis K. Then $\widetilde{\phi}$ satisfies (7.2) on $\widetilde{G}$ with γ modified according to the above lemma (note that any vertex of degree one always satisfies (7.2) with $\gamma = 1$). The choice of distance along the axis of the star may depend on some other parameter of the model, for example, so as to best uniformize γ.

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