Lagrangian Intermittency Based on an Ensemble of Gaussian Velocity Time Series

Laura J. Lukassen and Michael Wilczek

Abstract We show that Lagrangian intermittency in fully developed turbulence can be captured in terms of an ensemble of Gaussian velocity time series. This is achieved by letting the individual ensemble members vary with respect to their correlation function. We briefly discuss how this can be analytically captured in terms of a suitably defined characteristic functional. Moreover, we present a numerical implementation of the ensemble showing a continuous change from Gaussian to non-Gaussian increment distributions for a decreasing time lag. In an outlook we show first results on the application to data from direct numerical simulation.

1 Introduction

Studying Lagrangian tracer particles in homogeneous isotropic turbulence gives fundamental insight into the nature of turbulent flows as tracer particles naturally sample the spatio-temporal complexity of fully developed turbulence. Among the simplest statistical measures to quantify this complexity are probability density functions (PDFs) of velocity increments. These PDFs display intermittency, i.e. they transition from Gaussian behavior at large time lags to non-Gaussian behavior with decreasing time lags. It is a central goal of statistical turbulence theory to explain this phenomenon. To fully capture the multiscale statistics of turbulence, however, more complex statistical quantities are needed. This, for example, includes joint statistics of Lagrangian increment statistics on various time scales and joint statistics of acceleration and velocity increments.

It is well known that dimensional analysis in the spirit of Kolmogorov 1941 phenomenology [1, 2] fails to capture Lagrangian intermittency. Under the assumption that the statistics depend only on the average rate of kinetic energy dissipation in the

L.J. Lukassen (✉) · M. Wilczek
Max Planck Institute for Dynamics and Self-Organization, Am Fassberg 17, 37077 Göttingen, Germany
e-mail: laura.lukassen@ds.mpg.de

M. Wilczek
e-mail: michael.wilczek@ds.mpg.de

R. Örlü et al. (eds.), *Progress in Turbulence VII*, Springer Proceedings in Physics 196, DOI 10.1007/978-3-319-57934-4_4

inertial range, the velocity increment PDFs turn out to be self-similar in scale. This self-similarity can be broken by replacing the mean dissipation rate in the theory with a fluctuating dissipation rate, which has been proposed by Kolmogorov and Oboukhov [3, 4] in terms of the refined similarity hypothesis. Originally this theory, nowadays known as K62, was derived for Eulerian statistics, but can also be applied to Lagrangian statistics, cf. [5]. Another successful approach in capturing Lagrangian intermittency is the multifractal framework [6, 7]. In its probabilistic interpretation, anomalous scaling of structure functions is obtained by time-lag dependent superposition of scaling laws.

In this paper, we present a general approach based on the characteristic functional which contains all statistical information along a trajectory and in this sense provides any joint statistics of interest. Since Gaussian characteristic functionals can be treated analytically, we generate non-Gaussian ensemble statistics by superimposing Gaussian characteristic functionals. From a conceptual point of view, our model may be regarded as a generalization of the above-mentioned K62 and multifractal approach as well as the work by Castaing et al. [8] who superimposed Gaussian PDFs with varying variance. In Sect. 2, an introduction to our model is given. In Sect. 3, we show through a numerical evaluation of an ensemble of Gaussian trajectories that this model is capable of producing intermittency. This proof of concept is the essential part of this paper. As a next step, the model will be applied to turbulence data from direct numerical simulation (DNS), preliminary results are presented in Sect. 4.

2 An Ensemble of Gaussian Characteristic Functionals

The complete statistical information along a Lagrangian trajectory can be described in terms of the characteristic functional

$$\varphi[\alpha] = \left\langle \exp\left(\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}t\, \alpha(t) u(t) \right) \right\rangle, \tag{1}$$

where the angular brackets denote an ensemble average. Equation (1) represents the inverse Fourier transform of the probability density functional of $u(t)$. Here, $u(t)$ is the velocity time series along a Lagrangian trajectory, and $\alpha(t)$ denotes the corresponding Fourier transform variable, which also is a function of time. While it is not possible to compute the average in (1) in the general case analytically, there exists an analytical expression for a Gaussian $u(t)$ [9]. Since the Lagrangian increment statistics are highly non-Gaussian for small time lags, Gaussian characteristic functionals cannot be used directly. Therefore, we consider an ensemble of Gaussian trajectories which differ with respect to the dissipation rate ε. The underlying assumption is that the velocity statistics of particles traversing regions of varying dissipation rates in the flow field can be modeled by the velocity statistics of an ensemble of particles where each particle travels through regions with a prescribed dissipation rate and simple Gaussian statistics. The characteristic functional of an ensemble of Gaussian $u(t)$ takes the form

$$\varphi[\alpha] = \int_0^\infty \mathrm{d}\varepsilon\, P(\varepsilon)\varphi_\varepsilon^G[\alpha] = \int_0^\infty \mathrm{d}\varepsilon\, P(\varepsilon) \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \alpha(t)\langle u(t)u(t')\rangle_\varepsilon \alpha(t')\right). \quad (2)$$

The angular brackets with the subscript ε indicate an average over the respective ensemble members (i.e. trajectories) with identical dissipation rates. It is assumed that the mean velocity of the respective ensemble members is zero, i.e. $\langle u(t)\rangle_\varepsilon = 0$. The characteristic functional in (2) is determined by the distribution for the dissipation rate $P(\varepsilon)$ and the velocity correlation function $\langle u(t)u(t')\rangle_\varepsilon$.

In order to ensure that the ensemble one-time velocity statistics preserve Gaussianity, we construct the ensemble such that each ensemble member has the same second moment of velocity $\langle u(t)^2\rangle$. By an appropriate choice of α, the characteristic functional (2) can be projected to yield the desired joint statistics or multipoint statistics. For example, one-time velocity statistics are derived by the choice $\alpha(t) = \alpha_1\delta(t - t_1)$ and evaluating the integrals in (2). This turns the characteristic functional into a characteristic function $\varphi(\alpha_1)$. Since the second moment of velocity is by construction independent of ε, the ensemble velocity statistics are Gaussian:

$$\varphi(\alpha_1) = \int_0^\infty \mathrm{d}\varepsilon\, P(\varepsilon) \exp\left(-\frac{1}{2}\langle u(t_1)^2\rangle_\varepsilon\, \alpha_1^2\right) = \exp\left(-\frac{1}{2}\langle u(t_1)^2\rangle\, \alpha_1^2\right). \quad (3)$$

In contrast, the second-order moments of acceleration and increment statistics explicitly depend on ε. With the choices $\alpha(t) = -\beta_1 \frac{\mathrm{d}}{\mathrm{d}t}\delta(t - t_1)$ for acceleration statistics and $\alpha(t) = \mu_1\delta(t - t_2) - \mu_1\delta(t - t_1)$ for increment statistics, the characteristic functional (2) becomes the characteristic function for the acceleration and the velocity increment, respectively:

$$\varphi(\beta_1) = \int_0^\infty \mathrm{d}\varepsilon\, P(\varepsilon) \exp\left(-\frac{1}{2}\left\langle\left(\frac{\mathrm{d}}{\mathrm{d}t_1}u(t_1)\right)^2\right\rangle_\varepsilon \beta_1^2\right), \quad (4)$$

$$\varphi(\mu_1) = \int_0^\infty \mathrm{d}\varepsilon\, P(\varepsilon) \exp\left(-\frac{1}{2}\langle v^2\rangle_\varepsilon(\tau)\, \mu_1^2\right), \quad (5)$$

with the velocity increment $v(\tau) = u(t_1 + \tau) - u(t_1)$ and the time lag $\tau = t_2 - t_1$. Even though the respective ensemble members show Gaussian acceleration and increment statistics, the ensemble statistics can become non-Gaussian by the superposition of the Gaussian ensemble members with varying dissipation rates.

All time-lag dependence in (5) is contained in the second moment of the velocity increment, while $P(\varepsilon)$ does not depend on the time lag. This marks a difference to K62 and the multifractal approach where the distribution for the dissipation rate averaged over the time lag (K62), or the distribution for the scaling exponents (multifractal approach), explicitly depends on the time lag. More detailed information on the analytical background of superimposing Gaussian characteristic functionals can be found in [10].

3 Numerical Implementation of an Ensemble of Gaussian Trajectories

As we have seen throughout the previous section, intermittent increment statistics can be constructed by an ensemble of Gaussian characteristic functionals with a dissipation rate assigned to each ensemble member and subsequent averaging over a distribution of dissipation rates. Here, we numerically create an ensemble of Gaussian velocity trajectories $u_\varepsilon(t)$ and evaluate velocity increment PDFs directly from the trajectories as a proof of concept. By a superposition of Fourier modes with random phases, we generate approximately Gaussian trajectories, cf. [11]. Since the velocity correlation $\langle u(t)u(t')\rangle_\varepsilon$ is given by the inverse Fourier transform of the energy spectrum, we directly model the energy spectrum to determine the amplitude of the Fourier coefficients. For each ensemble member, a dissipation rate ε is drawn from a distribution $P(\varepsilon)$. In the context of the K62 theory, a log-normal distribution for the averaged dissipation rate has been proposed. Also for our model a log-normal distribution is a plausible choice for the distribution for the kinetic energy dissipation rate. For the present numerical evaluation, we chose a distribution for $\ln(\varepsilon)$ with the mean $\mu_{\ln(\varepsilon)} = -0.25$ and the standard deviation $\sigma_{\ln(\varepsilon)} = 1.3$. Consequently, the coefficients of the log-normal distribution do not depend on the time lag here. The range for ε in the numerical implementation is the interval [0.05, 50] which covers 98.2% of the probability distribution. The ensemble consists of 200 members, the number of Fourier modes is $n = 500{,}000$. Throughout the numerical implementation, all variables and parameters are treated in a non-dimensionalized form. The periodic domain has a size of $T = 1\,000\pi$. Then, the velocity for a trajectory in the ensemble is given by

$$u_\varepsilon(t_l) = \sum_{k=-n/2+1}^{n/2} \sqrt{E_\varepsilon(\omega_{|k|})\frac{2\pi}{T}}\exp(\mathrm{i}\varphi_k)\ \exp\left(\mathrm{i}\omega_k t_l\right), \tag{6}$$

with $\omega_k = \frac{2\pi k}{T}$, $t_l = l\frac{T}{n}$ and $\varphi_k = -\varphi_{-k}$ uniformly distributed in the interval $[0, 2\pi]$. In order to guarantee a zero mean velocity, φ_0 is randomly set to 0 or π. In Lagrangian turbulence it is observed that the velocity correlation function decays exponentially which corresponds to a Lorentzian spectrum, cf. [12, 13]. We model the spectrum including a viscous cut-off as

$$E_\varepsilon(\omega_k) = \frac{1}{\pi}\frac{T_L(\varepsilon)}{1+(\omega_k T_L(\varepsilon))^2}\langle u^2\rangle\ A_\varepsilon \exp\left(-\omega_k\tau_\eta(\varepsilon)\right). \tag{7}$$

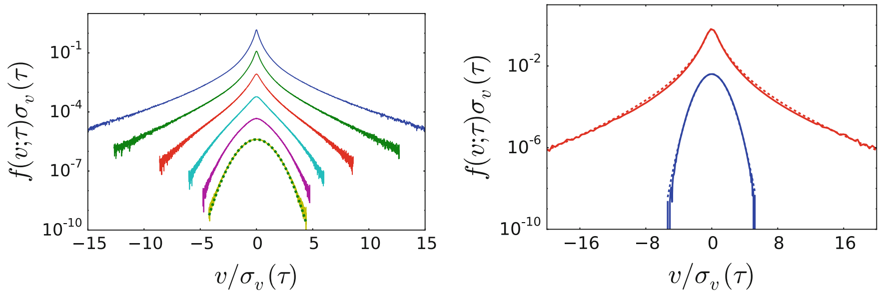

Fig. 1 *Left* Standardized PDFs for velocity increments obtained from the ensemble of Gaussian trajectories. The curves are shifted vertically for clarity. The time lags are increasing from top to bottom: {0.27, 1.07, 4.28, 17.12, 68.49, 273.96} $\tau_{\eta\langle\varepsilon\rangle}$. The Kolmogorov time scale $\tau_{\eta\langle\varepsilon\rangle}$ is determined through the expectation value of the log-normally distributed ε. The *dashed bottom curve* is a standardized Gaussian PDF as a reference which overlies the PDF for the largest time lag. *Right* Comparison of standardized velocity increment PDFs from DNS data (*solid lines*) and preliminary results of an evaluation of equation (8) (*dashed lines*) with an optimized $P(\varepsilon)$. $P(\varepsilon)$ used in this example is chosen as a log-normal distribution where the mean $\mu_{\ln(\varepsilon)}$ and the standard deviation $\sigma_{\ln(\varepsilon)}$ of $\ln(\varepsilon)$ are fitted for a small (*top curve*) and a large (*bottom curve*) time lag yielding $\mu_{\ln(\varepsilon)} = -2.8$ and $\sigma_{\ln(\varepsilon)} = 1.03$. The curves are shifted vertically for clarity. The graphs are created with matplotlib [14]

The factor A_ε is determined such that the kinetic energy $\int_0^\infty E_\varepsilon(\omega)\mathrm{d}\omega = \frac{1}{2}\langle u^2\rangle$ is kept fixed to one for each ensemble member. The choice of ε determines the integral time scale which we here define as $T_L(\varepsilon) = \frac{1}{2}\frac{\langle u^2\rangle}{\varepsilon}$ and the Kolmogorov time scale $\tau_\eta = (\nu/\varepsilon)^{1/2}$ where we choose $\nu = 0.001$.

Once the ensemble is numerically generated, arbitrary statistics can be obtained. As mentioned above, we are interested in velocity increment PDFs as an example. Equation (6) is used to compute velocity increments via $v_\varepsilon(\tau) = u_\varepsilon(t+\tau) - u_\varepsilon(t)$. We present the velocity increment PDFs in Fig. 1 on the left which shows a continuous transition from a Gaussian PDF for large time lags to a highly non-Gaussian PDF for smaller time lags. This demonstrates that it is in principle possible to generate intermittent increment statistics by an ensemble of Gaussian trajectories with varying dissipation rates.

4 Outlook: Validation with DNS Results

The numerical proof of concept in Sect. 3 shows that an ensemble of Gaussian trajectories captures intermittency qualitatively by model assumptions about the forms of the underlying spectra and the distribution $P(\varepsilon)$. The application to turbulence DNS data is work in progress. In contrast to the procedure presented in the previous section, we here take the Fourier transform of (5) as a starting point. Here, the velocity increment PDF is given by a superposition of Gaussian increment distributions which vary in terms of their variances:

$$f(v;\tau) = \int_0^\infty \mathrm{d}\varepsilon\, P(\varepsilon) \frac{1}{\sqrt{2\pi \langle v^2\rangle_\varepsilon(\tau)}} \exp\left(-\frac{v^2}{2\langle v^2\rangle_\varepsilon(\tau)}\right). \tag{8}$$

We choose a lognormal distribution for $P(\varepsilon)$ and obtain the mean and the standard deviation of $\ln(\varepsilon)$ via an optimization procedure. The goal of the optimization procedure is to determine the parameters of $P(\varepsilon)$ in (8) for a small and a large time lag as a best fit to the respective increment distributions from DNS data.

To generate a test data set, a standard pseudo-spectral DNS at $\mathrm{Re}_\lambda = 109$ was run in a quasi-stationary regime with a resolution of $k_M\eta \approx 2$ where k_M is the maximum wavenumber and η is the Kolmogorov length. 150,000 particles were placed in this flow. The velocity of the particles was sampled along their trajectories. Lagrangian increment statistics were then estimated from the DNS data.

Figure 1 on the right shows preliminary results for the comparison of the DNS data to the superposition of Gaussian increment distributions which are weighted according to an optimized log-normal distribution $P(\varepsilon)$. The approach is in general capable of reproducing the velocity increment PDF for a small and a large time lag. We are currently extending the approach to represent Lagrangian increment statistics across the full range of temporal scales and plan to explore the model with respect to more general statistical quantities including joint statistics of velocities and acceleration.

Acknowledgements This work was supported by the Max Planck Society. We gratefully acknowledge the DNS data provided by Cristian Lalescu.

References

1. A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. Dokl. Akad. Nauk SSSR. **30** (1941), Reprinted translation by V. Levin: Proc. R. Soc. Lond. A **434**, 9–13 (1991)
2. A.S. Monin, A.M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*. Dover Books on Physics, vol. II (Dover Publications, Mineola, 2007)
3. A.N. Kolmogorov, A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. J. Fluid Mech. **13**(01), 82–85 (1962)
4. A.M. Oboukhov, Some specific features of atmospheric turbulence. J. Fluid Mech. **13**(01), 77–81 (1962)
5. R. Benzi, L. Biferale, E. Calzavarini, D. Lohse, F. Toschi, Velocity-gradient statistics along particle trajectories in turbulent flows: the refined similarity hypothesis in the Lagrangian frame. Phys. Rev. E **80**(6), 066318 (2009)
6. U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995)

7. L. Chevillard, B. Castaing, A. Arneodo, E. Lévêque, J.-F. Pinton, S.G. Roux, A phenomenological theory of Eulerian and Lagrangian velocity fluctuations in turbulent flows. C. R. Phys. **13**(9–10), 899–928 (2012)
8. B. Castaing, Y. Gagne, E.J. Hopfinger, Velocity probability density functions of high Reynolds number turbulence. Physica D **46**(2), 177–200 (1990)
9. J.L. Lumley, *Stochastic Tools in Turbulence*, Dover Books on Engineering (Dover Publications, Mineola, 2007)
10. M. Wilczek, Non-Gaussianity and intermittency in an ensemble of Gaussian fields. New J. Phys. **18**, 125009 (2016)
11. J. Jiménez, Turbulent velocity fluctuations need not be Gaussian. J. Fluid Mech. **376**, 139–147 (1998)
12. S.B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, 2000)
13. N. Mordant, P. Metz, O. Michel, J.-F. Pinton, Measurement of Lagrangian velocity in fully developed turbulence. Phys. Rev. Lett. **87**(21), 214501 (2001)
14. J.D. Hunter, Matplotlib: a 2D graphics environment. Comput. Sci. Eng. **9**(3), 90–95 (2007)