

Small Union with Large Set of Centers

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Abstract Let $T \subset \mathbb{R}^n$ be a fixed set. By a scaled copy of T around $x \in \mathbb{R}^n$ we mean a set of the form $x + rT$ for some $r > 0$. In this survey paper we study results about the following type of problems: How small can a set be if it contains a scaled copy of T around every point of a set of given size? We will consider the cases when T is circle or sphere centered at the origin, Cantor set in \mathbb{R} , the boundary of a square centered at the origin, or more generally the k -skeleton ($0 \leq k < n$) of an n -dimensional cube centered at the origin or the k -skeleton of a more general polytope of \mathbb{R}^n . We also study the case when we allow not only scaled copies but also scaled and rotated copies and also the case when we allow only rotated copies.

1 Introduction

In this survey paper we study the following type of problems. How small can a set be if it contains a scaled copy of a given set around a large set of points of \mathbb{R}^n ? More precisely:

Problem 1.1 Let $T \subset \mathbb{R}^n$ be a fixed set and let $S, B \subset \mathbb{R}^n$ be sets such that for every $x \in S$ there exists an $r > 0$ such that $x + rT \subset B$, in other words B contains a scaled copy of T around every point of S . How small can B be if we know the size of S ?

In Sect. 2 we study the case when T is a circle or sphere centered at the origin and we present the classical deep results of Stein [19], Bourgain [1], Marstrand [17], Mitsis [18], and Wolff [23, 24]. The strongest results of this section (Theorem 2.5 [18, 24] and its corollary Theorem 2.9) state that if the Hausdorff dimension of S is $\dim_{\text{H}} S > 1$ then B must have positive Lebesgue measure and if $\dim_{\text{H}} S \leq 1$ then $\dim_{\text{H}} B \geq \dim_{\text{H}} S + n - 1$.

In Sect. 3 we study the case when $n = 1$ and T is a Cantor set with $0 \notin T$. In this case there are four results: Łaba and Pramanik [13] constructed Cantor sets $T \subset [1, 2]$ for which the Lebesgue measure of B must be positive whenever S has

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positive Lebesgue measure; Máthé [16] constructed Cantor sets T for which this is false; Hochman [9] proved that if $\dim_{\text{H}} S > 0$ then $\dim_{\text{H}} B > \dim_{\text{H}} T + \delta$ for any porous Cantor set T , where $\delta > 0$ depends only on $\dim_{\text{H}} T$ and $\dim_{\text{H}} S$; and Máthé noticed that it is a consequence of a recent projection theorem of Bourgain [2] that $\dim_{\text{H}} T > 0$ implies $\dim_{\text{H}} B \geq \frac{\dim_{\text{H}} S}{2}$.

In Sect. 4 we present the results of Nagy, Shmerkin, and the author [11] about the case when $n = 2$ and T is the boundary or the set of vertices of a fixed axis-parallel square centered at the origin. It turns out that in these cases B can be much smaller than S . If $S = \mathbb{R}^2$ and T is the boundary of the square, then the minimal Hausdorff dimension of B is 1 (Proposition 4.3), the minimal upper box, lower box, and packing dimensions of B are all $7/4$ (Theorem 4.5). If $S = \mathbb{R}^2$ and T is the set of vertices of the square, then the minimal Hausdorff dimension of B is also 1 and the minimal upper box, lower box, and packing dimensions are $3/2$ (Theorems 4.9 and 4.10). For general S when the dimension s of S is given, where dimension can be Hausdorff, lower box, upper box, and packing dimension, the smallest possible dimension of B is determined as a function of s both in the boundary and in the vertices cases, see Table 1. It is remarkable that we get three different functions for different dimensions in the boundary case.

In Sect. 5 we study the case when T is the k -skeleton ($0 \leq k < n$) of a fixed axis-parallel n -dimensional cube centered at the origin or the k -skeleton of a more general polytope. For the k -skeleton of axis-parallel cubes Thornton [22] generalized the above-mentioned two-dimensional results for packing and box dimensions (Theorem 5.2), found the estimate $\dim_{\text{H}} B \geq \max(k, \dim_{\text{H}} S - 1)$ for Hausdorff dimension, and posed the conjecture that this estimate is sharp. This conjecture was proved by Chang, Csörnyei, Héra, and the author [3] not only for cubes but also for more general polytopes (Theorem 5.6). To obtain this result first the smallest Hausdorff dimension of B was determined for any fixed compact S , in other words, instead of $\dim_{\text{H}} S$ we fixed S itself (Theorem 5.8). This was done by showing that for any fixed compact set S if we take a suitable B in a (Baire category sense) typical way then $\dim_{\text{H}} B$ is minimal, see the explanation after Theorem 5.8 for more details.

So far we allowed only scaled copies of T . In Sect. 6 first we study the modification of Problem 1.1 when scaled and rotated copies of T are also allowed. Obviously, allowing rotations can only decrease the minimal dimension of the set B . When T is the k -skeleton of a cube, it is hard to imagine that rotated copies of T can overlap “more” than non-rotated axis-parallel ones. Therefore one might think that allowing rotations of T cannot yield a smaller set B . However, this turns out to be false. In fact, for any $0 \leq k < n$ there exists a closed set B of Hausdorff dimension k that contains the k -skeleton of a (rotated) cube centered at every point of \mathbb{R}^n (Theorem 6.1). We also study the modification of Problem 1.1 when instead of scaled copies we use only rotated copies of T (of fixed size). It turns out (Theorem 6.2) that even by using only rotated copies we can get smaller union than using only scaling: for any $0 \leq k < n$ there exists a Borel set B of Hausdorff dimension $k+1$ that contains the k -skeleton of a (rotated) *unit* cube centered at every point of \mathbb{R}^n . It turns out that this $k+1$ is sharp, B must have Hausdorff dimension at

least $k + 1$ (Corollary 6.4). Most of the results of this section are very recent results of Chang, Csörnyei, Héra, and the author [3], the last mentioned lower estimate for $\dim_{\mathbb{H}} B$ is a very recent result of Héra, Máthé, and the author [8].

2 Classical Results About Circle and Sphere Packing

How small can a set be if it contains a scaled copy of a given set around a large set of points of \mathbb{R}^n ? The first result of this type seems to be the following result of E. Stein.

Theorem 2.1 (Stein [19]) *Let $S \subset \mathbb{R}^n$ ($n \geq 3$) be a set of positive Lebesgue measure. If $B \subset \mathbb{R}^n$ contains a sphere centered at every point of S , then B has positive Lebesgue measure.*

In fact, what Stein proved is a stronger result about the spherical maximal operator. Let

$$M_{S^{n-1}}f(x) = \sup_{r>0} \int_{S^{n-1}} |f(x + ry)| d\sigma^{n-1}y \quad (x \in \mathbb{R}^n), \tag{1}$$

where σ^{n-1} is the normalized surface measure on S^{n-1} . In other words $M_{S^{n-1}}f(x)$ is the maximal spherical average of $|f|$ around x .

Theorem 2.2 (Stein’s Spherical Maximal Theorem [19]) *The spherical maximal operator (1) is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if $n \geq 3$ and $p > n/(n - 1)$.*

Theorem 2.1 follows easily from Theorem 2.2: we claim that if $B \subset \mathbb{R}^n$ contains a sphere centered at every point of S and B has Lebesgue measure zero then so has S . Indeed, if f is the indicator function of B , then by Theorem 2.2, $\|M_{S^{n-1}}f\|_p = 0$. Since, by definition, $M_{S^{n-1}}f = 1$ on S we obtain that S has measure zero.

Note that this argument also shows that it is enough to assume that around each point of S there is a sphere that intersects B in a set of positive $(n - 1)$ -dimensional measure.

For about 10 years it was open if the above theorems hold for $n = 2$. They were settled by Marstrand and Bourgain, independently.

Theorem 2.3 (Bourgain [1], Marstrand [15]) *Let $S \subset \mathbb{R}^2$ be a set of positive Lebesgue measure. If $B \subset \mathbb{R}^2$ contains a circle centered at every point of S , then B has positive Lebesgue measure.*

Theorem 2.4 (Bourgain’s Circular Maximal Theorem [1]) *The circular maximal operator M_{S^1} defined by (1) is bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for $p > 2$.*

In fact, Bourgain proved his results not only for circles but also for any smooth curve with non-vanishing curvature.

In Theorems 2.1 and 2.3 the set B contains an n -dimensional family of $n - 1$ -dimensional surfaces (without big intersection), so it is not surprising that B has positive Lebesgue measure. We can also expect that smaller set of centers should

also guarantee that B has positive measure, perhaps a set of centers with Hausdorff dimension larger than 1 is large enough. This is indeed the case as it was shown by T. Mitsis and T. Wolff:

Theorem 2.5 (Mitsis [18] for $n \geq 3$, Wolff [24] for $n = 2$) *Let $S \subset \mathbb{R}^n$ ($n \geq 2$) be a set of Hausdorff dimension > 1 . If $B \subset \mathbb{R}^n$ contains a sphere centered at every point of S , then B has positive Lebesgue measure.*

In fact, Wolff proved the following more general statement.

Theorem 2.6 (Wolff [24]) *Let $D \subset \mathbb{R}^2 \times (0, \infty)$ be a set of Hausdorff dimension larger than 1 and suppose that $B \subset \mathbb{R}^2$ is a set such that for any $(x, r) \in D$ a positive measure part of the circle $C(x, r)$ is contained in B . Then B must have positive Lebesgue measure.*

The following example shows that the above results are sharp.

Theorem 2.7 (Talagrand [21]) *There exists a planar set of Lebesgue measure zero that contains a circle centered at every point of a straight line.*

If the Hausdorff dimension of the set of centers is at most 1, then we can estimate the Hausdorff dimension of the union of the circles:

Theorem 2.8 (Wolff [23]) *If S is a Borel set in the plane with $\dim S = s \leq 1$ and B contains a circle centered at every point of S , then $\dim B \geq s + 1$.*

In fact, Theorem 2.5 implies Theorem 2.8 and its following higher dimensional generalization. The author heard this simple argument from András Máthé but it is quite likely that Wolff, Mitsis, and others were also aware of this.

Theorem 2.9 *If $S \subset \mathbb{R}^n$ ($n \geq 2$) with $\dim S = s \leq 1$ and B contains a sphere centered at every point of S , then $\dim B \geq s + n - 1$.*

Proof Fix $\varepsilon \in (0, 1)$. First we construct a compact set $D \subset \mathbb{R}^n$ with $\dim_{\text{H}} D = \dim_{\text{B}} D = 1 - s + \varepsilon$ such that $\dim_{\text{H}}(S + D) = 1 + \varepsilon$. Let $D_1 \subset \mathbb{R}^n$ be any compact set (for example, a self-similar set) with $\dim_{\text{H}} D_1 = \dim_{\text{B}} D_1 = 1 - s + \varepsilon$. Then $\dim_{\text{H}}(S \times D_1) = 1 + \varepsilon$, so its orthogonal projection to almost every n -dimensional subspace of \mathbb{R}^{2n} also has Hausdorff dimension $1 + \varepsilon$. This implies that a suitable affine copy of D_1 has all the properties we wanted for D .

Now $B + D$ contains a sphere centered at every point of $S + D$ and $\dim_{\text{H}}(S + D) = 1 + \varepsilon > 1$, so by Theorem 2.5, $B + D$ has positive Lebesgue measure, so $\dim_{\text{H}}(B + D) = n$. On the other hand, $\dim_{\text{H}}(B + D) \leq \dim_{\text{H}} B + \dim_{\text{B}} D = \dim_{\text{H}} B + 1 - s + \varepsilon$, which implies $\dim_{\text{H}} B \geq s + n - 1 - \varepsilon$ for any $\varepsilon \in (0, 1)$. \square

Note that all of the above results are in harmony with the general principle that an a -dimensional collection of b -dimensional sets in \mathbb{R}^n must have positive measure if $a + b > n$ and Hausdorff dimension $a + b$ if $a + b \leq n$ unless the sets have large intersections.

3 Cantor Sets

In this section we study the case when we want the scaled copy of a Cantor set around a given set of points. (By Cantor set we mean nowhere dense nonempty compact set without isolated points.)

For any finite Borel measure μ on \mathbb{R}^n one can generalize the maximal operator (1) as

$$M_\mu f(x) = \sup_{r>0} \int_{\mathbb{R}^n} |f(x + ry)| d\mu(y) \quad (x \in \mathbb{R}^n). \tag{2}$$

I. Łaba and M. Pramanik [13] studied this maximal operator in the case when μ is the natural probability measure on a Cantor set $C \subset [1, 2]$. They constructed Cantor sets of Hausdorff dimension s for any $s \in (\frac{2}{3}, 1]$ such that M_μ of (2) is bounded on $L^p(\mathbb{R})$ for any $p > \frac{2-s}{s}$. Similarly as Theorem 2.2 easily implies Theorem 2.1, this result gives the following.

Theorem 3.1 (Łaba–Pramanik [13]) *For any $s \in (\frac{2}{3}, 1]$ there exists a Cantor set $C \subset [1, 2]$ of Hausdorff dimension s with the following property. If B and S are subsets of \mathbb{R} , S has positive Lebesgue measure and for every $x \in S$ there exists an $r > 0$ such that $rC + x \subset B$, then B must also have positive Lebesgue measure.*

It is natural to ask if the same is true for every Cantor set. Very recently A. Máthé showed that the answer is negative:

Theorem 3.2 (Máthé [16]) *For any $s \in (0, 1)$ there exists a Cantor set C of Hausdorff dimension s and a Borel set B of Hausdorff dimension $\frac{s+1}{2}$ such that for every $x \in \mathbb{R}$ there exists an $r > 0$ such that $rC + x \subset B$.*

The idea of the proof of the above result of Máthé comes from a discrete grid construction of G. Elekes [5] that shows the sharpness of the Szemerédi–Trotter theorem [20].

By the above results, for some Cantor sets C there is a set $B \subset \mathbb{R}$ of measure zero which contains scaled copies of C around every $x \in \mathbb{R}$ and for some other sets C there is no such B . For given Cantor sets C with $0 \notin C$ it seems to be hard to decide whether such a construction is possible or not. For example, we do not know this if C is the classical middle- $\frac{1}{3}$ Cantor set translated so that it is symmetric around the origin.

One can also ask if at least we can guarantee that the union of scaled copies of a Cantor set C around every point of \mathbb{R} has Hausdorff dimension strictly larger than the Hausdorff dimension of C . Very recently M. Hochman [9] gave an affirmative answer to this question for porous Cantor sets. (Here a set is called *porous* if there exist $c > 0$ and $r_0 > 0$ such that every interval of length $r < r_0$ contains an interval of length cr disjoint to the set.)

Theorem 3.3 (Hochman [9]) *Let $S \subset \mathbb{R}$ be a compact set with $\dim_{\text{H}} S > 0$, $C \subset \mathbb{R}$ be a porous Cantor set. If $B \subset \mathbb{R}$ contains a scaled copy $rC + x$ of C for every $x \in S$, then $\dim_{\text{H}} B > \dim_{\text{H}} C + \delta$, where $\delta > 0$ depends only on $\dim_{\text{H}} S$ and $\dim_{\text{H}} C$.*

Surprisingly, $\dim_{\mathbb{H}} B \geq \frac{\dim_{\mathbb{H}} S}{2}$ is guaranteed just by assuming $\overline{\dim}_{\mathbb{H}} C > 0$. This follows from a recent theorem of Bourgain, as was pointed out by A. Máthé.

Theorem 3.4 (Bourgain [2]) *For every $0 < \alpha < 2$ and $\kappa > 0$ there exists $\eta > \alpha/2$ such that if $A \subset \mathbb{R}^2$ is a set with $\dim_{\mathbb{H}} A > \alpha$, then $\dim_{\mathbb{H}} \text{proj}_{\theta}(A) \geq \eta$ for every $\theta \in S^1$ except in an exceptional set $E \subset S^1$ satisfying $\dim_{\mathbb{H}} E \leq \kappa$.*

Corollary 3.5 *Let S and C be compact subsets of \mathbb{R} with $\dim_{\mathbb{H}} C > 0$. If B is a Borel subset of \mathbb{R} and for every $x \in S$ there exists an $r(x) \in \mathbb{R}$ such that $r(x)C + x \subset B$, then $\dim_{\mathbb{H}} B \geq \frac{\dim_{\mathbb{H}} S}{2}$.*

(For a short direct proof of the corollary, see Máthé [16].)

Proof We can clearly suppose that $\dim_{\mathbb{H}} S > 0$. Let $0 < \varepsilon < \frac{\dim_{\mathbb{H}} S}{2}$, $0 < \kappa < \dim_{\mathbb{H}} C$ and $\alpha = \dim_{\mathbb{H}} S - 2\varepsilon$ and let $\eta > \alpha/2 = \frac{\dim_{\mathbb{H}} S}{2} - \varepsilon$ be the number guaranteed by Theorem 3.4. Let A be the graph of the function $r(x)$; that is, $A = \{(x, r(x)) : x \in S\}$. Then $\dim_{\mathbb{H}} A \geq \dim_{\mathbb{H}} S > \alpha$. Let $p_c(x, y) = x + cy$. Note that for every $c \in C$, $p_c(A) \subset B$ and $p_c(A)$ is a scaled copy of a projection of A . By the theorem, $\dim_{\mathbb{H}} p_c(A) \geq \eta$ for every $c \in \mathbb{R}$ except in an exceptional set E satisfying $\dim_{\mathbb{H}} E \leq \kappa$. Since $\dim_{\mathbb{H}} E \leq \kappa < \dim_{\mathbb{H}} C$ we can choose $c \in C \setminus E$. Thus $\dim_{\mathbb{H}}(B) \geq \dim_{\mathbb{H}}(p_c(A)) \geq \eta > \frac{\dim_{\mathbb{H}} S}{2} - \varepsilon$. \square

We remark that Theorem 3.3 and Corollary 3.5 are easier for upper box dimension. In case of Theorem 3.3 see the explanation in [9] before the proof of the theorem. In case of Corollary 3.5, as Máthé pointed out, if we consider upper box dimension instead of Hausdorff dimension, then Corollary 3.5 would follow very easily even if we assume only that C has at least two points. Indeed, if B contains a scaled copy $r\{p, q\} + x$ of a set $\{p, q\}$ ($p \neq q$) for every $x \in S$ then, as one can easily check, we have $\frac{p}{p-q}B - \frac{q}{p-q}B \supset S$, which implies that $\overline{\dim}_{\mathbb{B}} B \geq \frac{\overline{\dim}_{\mathbb{B}} S}{2}$.

4 Squares

The results and arguments of this section are due to D.T. Nagy, P. Shmerkin, and the author [11].

First we consider Problem 1.1 in the case when $n = 2$ and T is the boundary of a fixed axis-parallel square centered at the origin, say T is the boundary of $[-1, 1] \times [-1, 1]$. That is, now we study the following problem.

Question 4.1 (The Question for Square Boundaries) *Let T be the boundary of the square $[-1, 1] \times [-1, 1]$. Suppose that we have sets $B, S \subset \mathbb{R}^2$ such that B contains the boundary of an axis-parallel square centered at every point of S ; that is,*

$$(*) (\forall x \in S) (\exists r > 0) x + rT \subset B.$$

How big must B be (or how small can it be) if the size of S is given?

It is not hard to see that we can suppose that B is of the form $B = (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$. In this case it is easy to see that condition (*) about $B \subset \mathbb{R}^2$ is equivalent to the following condition about $A \subset \mathbb{R}$:

$$(**) \quad (\forall (x, y) \in S) (\exists r \neq 0) x - r, x + r, y - r, y + r \in A.$$

Therefore we obtained the following one-dimensional problem:

Question 4.2 (1-Dimensional Problem) *How small can A be with the following property if the size of S is given?*

$$(**) \quad (\forall (x, y) \in S) (\exists r \neq 0) x - r, x + r, y - r, y + r \in A.$$

It is very easy to check that (**) is equivalent to

$$(***) \quad (A - x) \cap (x - A) \cap (A - y) \cap (y - A) \not\subset \{0\} \quad \text{for all } (x, y) \in S.$$

Note that by the Baire category theorem (***) holds even for $S = \mathbb{R}^2$ for any dense G_δ set $A \subset \mathbb{R}$. Since there exist dense G_δ sets of Hausdorff dimension zero, this means that a set A that satisfies (***) [or equivalently (**)] can even have Hausdorff dimension zero even if $S = \mathbb{R}^2$. Thus going back to Question 4.1 we get the following.

Proposition 4.3 ([11]) *There exists a set $B \subset \mathbb{R}^2$ that contains a square centered at every point of \mathbb{R}^2 and has Hausdorff dimension 1 (same as a single square!).*

Proof Let $A \subset \mathbb{R}$ be a dense G_δ set of Hausdorff dimension 0. Then $B = (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$ is good (by our previous observations). □

Remark 4.4 A construction of Davies, Marstrand, and Taylor [4] shows that A (and therefore B) can be taken to be compact for bounded S .

The set B in Proposition 4.3 is “small” from the point of view of Hausdorff dimension, but it is “large” from the point of view of topology and other fractal dimensions.

The following results show that Question 4.1 becomes more interesting when one considers other notions of fractal dimension and/or discrete analogs. It turns out that, in some sense, Hausdorff dimension is the “wrong” dimension for this problem.

First we consider the $S = \mathbb{R}^2$ case.

Theorem 4.5 ([11])

- (i) *If $B \subset \mathbb{R}^2$ is a set that contains an axis-parallel square boundary centered at every point of $[0, 1]^2$, then its lower box, upper box, and packing dimension is at least $7/4$.*
- (ii) *There exists a compact set B that contains a square boundary centered at every point of $[0, 1]^2$ such that B has lower box, upper box, and packing dimension $7/4$.*

If we let S be a compact subset of \mathbb{R}^2 of given dimension, then even box and packing dimensions give completely different results:

Theorem 4.6 ([11])

(i) Let $B \subset \mathbb{R}^2$ be a set that contains an axis-parallel square boundary centered at every point of $S \subset \mathbb{R}^2$. Then:

(a) If $\dim = \overline{\dim}_B$ or $\underline{\dim}_B$, then $\dim B \geq \max\left(1, \frac{7}{8} \dim S\right)$.

(b) $\dim_P B \geq 1 + \frac{3}{8} \dim_P S$.

(ii) The above results are sharp: for each $s \in [0, 2]$

(a') there exist compact sets S, B as above such that

$$\dim_B S = s \quad \text{and} \quad \dim_B B = \max\left(1, \frac{7}{8}s\right).$$

(b') there exist compact sets S, B as above such that

$$\dim_P S = s \quad \text{and} \quad \dim_P B = 1 + \frac{3}{8}s.$$

To get the box and packing dimension estimates the analogous discrete results were needed. To get the analogous discrete results the following additive number theory lemma was needed.

Lemma 4.7 (1-Dimensional Main Lemma [11]) If $A \subset \mathbb{R}$ and $S \subset \mathbb{R}^2$ are finite sets such that $(\forall (x, y) \in S) (\exists r > 0) x - r, x + r, y - r, y + r \in A$, then

$$|A| \geq \frac{1}{\sqrt{2}} |S|^{\frac{3}{8}}.$$

After seeing the connection between the 2-dimensional problem Question 4.1 and the 1-dimensional problem Question 4.2, it is not surprising that we need this 1-dimensional lemma to get the 2-dimensional discrete result. The remarkable fact is that to get the 1-dimensional lemma it helps to go back to 2-dimension and to prove the following.

Lemma 4.8 (2-Dimensional Main Lemma [11]) If $B, S \subset \mathbb{R}^2$ are finite sets such that $(\forall (x, y) \in S) (\exists r > 0) (x \pm r, y \pm r) \in B$, then

$$|B| \geq \left(\frac{|S|}{2}\right)^{\frac{3}{4}}.$$

All discrete, box dimension, and packing dimension estimates in [11] are based on this combinatorial geometry lemma. Its proof is a rather short double counting argument but it is highly nontrivial.

Although this lemma was motivated by the square boundary problem Question 4.1, as a spinoff it can be directly applied to the problems when we want not the boundary but only the 4 vertices of axis-parallel squares. This leads to the following result:

Theorem 4.9 ([11]) *If $B, S \subset \mathbb{R}^2$ are such that B contains the four vertices of an axis-parallel square centered at every point of S , then*

$$\dim B \geq \frac{3}{4} \dim S.$$

if \dim is one of $\dim_{\mathbb{P}}$, $\underline{\dim}_{\mathbb{B}}$, or $\overline{\dim}_{\mathbb{B}}$.

This is sharp:

Theorem 4.10 ([11]) *For each $s \in [0, 2]$ there are compact sets S and B as above such that $\dim_{\mathbb{P}} S = \dim_{\mathbb{B}} S = s$ and $\dim_{\mathbb{P}} B = \dim_{\mathbb{B}} B = \frac{3}{4}s$. When $s = 2$ we can choose $S = [0, 1]^2$.*

All the above-mentioned results that show the sharpness of the discrete, box dimension or packing dimension estimates are based on a simple discrete construction. This construction was found independently by four students of the Eötvös Loránd University: B. Bodor, A. Mészáros, D.T. Nagy, and I. Tomon at the Miklós Schweitzer Mathematical Competition in 2012, where P. Shmerkin and the author posed the following simple but still highly nontrivial discrete version of Question 4.2.

Question 4.11 (Miklós Schweitzer Mathematical Competition 2012) *Call $A \subset \mathbb{Z}_n (= \mathbb{Z}/n\mathbb{Z})$ rich, if for every $x, y \in \mathbb{Z}_n$ there exists $r \in \mathbb{Z}_n$ for which $x - r, x + r, y - r, y + r \in A$. What are those α values for which there exist constants $C_\alpha > 0$ such that for any odd n every rich set $A \subset \mathbb{Z}_n$ has cardinality at least $C_\alpha n^\alpha$?*

The answer is $\alpha \leq 3/4$ and the part that these α values are good is essentially a special case of the 1-dimensional main lemma (Lemma 4.7) and it can be proved by a much simpler but still nontrivial double counting argument. The part that no $\alpha > 3/4$ can be good follows from the following simple construction, on which the above-mentioned other constructions are also based on.

Construction 4.12 (Bodor–Mészáros–Nagy–Tomon, 2012) *Fix a positive integer k , let $n = k^4$ and let $A \subset \mathbb{Z}_n$ consist of those four digit numbers in base k that have at least one zero digit. Then clearly A has cardinality $O(k^3) = O(n^{3/4})$ and it is easy to check that A is “rich”: by choosing the last digit of r we can guarantee that the last digit of $x - r$ is zero, and so on, each digit of r guarantees one zero digit in $x - r, x + r, y - r, y + r$.*

The problem when we want only the four vertices of an axis-parallel square centered at every point of a set of given size is also interesting for Hausdorff dimension. So now we study the following:

Question 4.13 (Square Vertices and Hausdorff Dimension) *Let $B \subset \mathbb{R}^2$ be a set that contains the four vertices of an axis-parallel square centered at every point of $S \subset \mathbb{R}^2$. How small can the Hausdorff dimension of B be if the Hausdorff dimension of S is given?*

By projecting to the $x = y$ line we get $\text{proj}B \supset \text{proj}S$, hence clearly we have

$$\dim_{\text{H}} B \geq \dim_{\text{H}}(\text{proj}B) \geq \dim_{\text{H}}(\text{proj}S) \geq \max(\dim_{\text{H}} S - 1, 0).$$

Somewhat surprisingly this is sharp:

Theorem 4.14 ([11]) *For any $s \in [0, 2]$ there are compact sets S and B as above such that $\dim_{\text{H}} S = s$ and $\dim_{\text{H}} B = \max(s - 1, 0)$.*

The proof of this result in [11] is rather involved. In the next section we describe a more general result that can be proved by a simpler method.

To finish this section we summarize the results about squares:

5 The k -Skeletons of Cubes and Other Polytopes of \mathbb{R}^n

Note that the boundary and the set of vertices of the square can be considered as the 1-dimensional skeleton and the 0-dimensional skeleton of the 2-dimensional cube. So the results of Table 1 can be considered as the answers to the $n = 2$ special case of the following more general problem.

Question 5.1 (k -Skeleton of n -Cubes) *Let $0 \leq k < n$ and $B \subset \mathbb{R}^n$ be a set that contains the k -dimensional skeleton of an axis-parallel n -dimensional cube centered at every point of $S \subset \mathbb{R}^n$. How small can the Hausdorff/packing/box dimension of B be if the appropriate dimension of S is given?*

For box and packing dimension R. Thornton [22] answered this question:

Theorem 5.2 (Thornton [22]) *(i) For any $0 \leq k < n$ and any sets $B, S \subset \mathbb{R}^n$ such that B contains the k -skeleton of an axis-parallel n -cube centered at every point in S we have*

Table 1 ([11]) If $B \subset \mathbb{R}^2$ contains the vertices/boundary of axis-parallel squares centered at every point of $S \subset \mathbb{R}^2$ of dimension s (for some dimension) then the best lower bound for the dimension (for the same dimension) of B is

Dimension	Vertices	Boundary
\dim_{P}	$\frac{3}{4}s$	$1 + \frac{3}{8}s$
$\overline{\dim}_{\text{B}}$	$\frac{3}{4}s$	$\max\left(1, \frac{7}{8}s\right)$
$\underline{\dim}_{\text{B}}$	$\frac{3}{4}s$	$\max\left(1, \frac{7}{8}s\right)$
\dim_{H}	$\max(s - 1, 0)$	1

$$(a) \dim_{\mathbb{P}} B \geq k + \frac{(n-k)(2n-1)}{2n^2} \dim_{\mathbb{P}} S,$$

and if \dim denotes upper or lower box dimension, then

$$(b) \dim B \geq \max \left\{ k, \left(1 - \frac{n-k}{2n^2} \right) \dim S \right\}.$$

(ii) Given any $0 \leq k < n$, $s \in [0, n]$, there are compact sets $B, S, B', S' \subset \mathbb{R}^n$ where $\dim_{\mathbb{P}}(S) = \dim_{\mathbb{B}}(S') = s$, B and B' contain the k -skeleton of an axis-parallel n -cube centered at every point in S and S' , respectively, and

$$(a') \dim_{\mathbb{P}} B = k + \frac{(n-k)(2n-1)}{2n^2} s \text{ and}$$

$$(b') \dim_{\mathbb{B}} B' = \max \left\{ k, \left(1 - \frac{n-k}{2n^2} \right) s \right\}.$$

The proof of part (ii) is again based on Construction 4.12. The proof of part (i) is based on the following generalization of the 1-dimensional Main Lemma (Lemma 4.7) and the 2-dimensional Main Lemma (Lemma 4.8).

Lemma 5.3 (n, l -Dimensional Main Lemma, Thornton [22]) *If $l \leq n$ are positive integers, $A \subset \mathbb{R}^l$, $S \subset \mathbb{R}^n$ and*

$$\forall x \in S \exists r > 0 \forall 1 \leq i_1 < \dots < i_l \leq n : (x_{i_1} \pm r, \dots, x_{i_l} \pm r) \in A$$

then

$$|A| \geq C_{n,l} |S|^{l(2n-1)/(2n^2)}$$

for some $C_{n,l} > 0$ that depends only on n and l .

To prove the above lemma, first the 2-dimensional Main Lemma (Lemma 4.8) had to be generalized to n -dimension, and then the key observation was that Lovász's following corollary of the Katona–Kruskal theorem [10, 12] can be applied:

Theorem 5.4 (Katona–Kruskal–Lovász [14]) *Let $k < n$ be positive integers, X be a finite set of n -element sets and let Y be the k element subsets of the sets of X . If $x > 0$ are chosen such that $\binom{x}{n} = |X|$, where $\binom{x}{n} = x \cdot (x-1) \cdot \dots \cdot (x-n+1)/n!$, then $|Y| \geq \binom{x}{k}$.*

The case of Hausdorff dimension is again completely different. Thornton [22] noticed that a simple projection argument again gives a bound:

If B contains a k -skeleton of an axis-parallel n -cube centered at every point of S with $\dim_{\mathbb{H}} S = s$ then, denoting the orthogonal projection to the hyperplane with normal vector $(1, \dots, 1)$ by proj , we have $\text{proj} B \supset \text{proj} S$, so again $\dim_{\mathbb{H}} B \geq \dim_{\mathbb{H}}(\text{proj} B) \geq \dim_{\mathbb{H}}(\text{proj} S) \geq \dim_{\mathbb{H}} S - 1 = s - 1$. Since clearly $\dim_{\mathbb{H}} B \geq k$, we obtained the following:

Proposition 5.5 (Thornton [22]) *If B contains a k -skeleton of an axis-parallel n -cube centered at every point of S with $\dim_{\mathbb{H}} S = s$, then $\dim_{\mathbb{H}} B \geq \max\{k, s - 1\}$.*

After proving that this is sharp in some special cases, Thornton [22] posed the conjecture that this is always sharp; that is, for any integers $0 \leq k < n$ and any $s \in [0, n]$ there exist compact sets S and B with the above property such that $\dim_{\mathbb{H}} B = \max\{k, s - 1\}$.

This conjecture was proved by A. Chang, M. Csörnyei, K. Héra, and the author [3], even for more general polytopes:

Theorem 5.6 ([3]) *Let $0 \leq k < n$ and T be the k -skeleton of an n -dimensional polytope such that 0 is not contained in any of the k -dimensional affine subspaces defined by T . Then for every $s \in [0, n]$ there exist compact sets S and B such that*

$$(\spadesuit) \quad (\forall x \in S) (\exists r > 0) x + rT \subset B,$$

$$\dim_{\mathbb{H}} S = s \text{ and } \dim_{\mathbb{H}} B = \max(s - 1, k).$$

We will see later that the previous simple projection argument works also in this more general case, so (\spadesuit) implies $\dim_{\mathbb{H}} B \geq \max(\dim_{\mathbb{H}} S - 1, k)$, therefore the above result shows that this is a sharp estimate for these more general k -skeletons. If 0 is contained in one of the k -dimensional affine subspaces defined by T , then the projection argument implies $\dim_{\mathbb{H}} B \geq \max(\dim_{\mathbb{H}} S, k)$ and it is also proved in [3] that in this case this is sharp. So the minimal Hausdorff dimension of a compact B such that (\spadesuit) holds for some compact S with $\dim_{\mathbb{H}} S = s$ is $\max\{s, k\}$ if 0 is contained in one of the k -dimensional affine subspaces defined by T and $\max\{s - 1, k\}$ otherwise.

The following table extends Table 1 with the results about the k -skeletons of an n -cube (Table 2).

To get the above-mentioned constructions of Chang, Csörnyei, Héra, and the author [3] a slightly different problem was studied first: instead of fixing the Hausdorff dimension s of S , the set S itself was fixed.

Problem 5.7 (The Modified Problem) Let T and S be fixed subsets of \mathbb{R}^n . We want to find a (compact) set $B \subset \mathbb{R}^n$ with minimal Hausdorff dimension such that B contains a scaled copy of T around every point of S ; that is,

$$(\spadesuit) \quad (\forall x \in S) (\exists r > 0) x + rT \subset B.$$

Table 2 ([3, 11, 22]) If $0 \leq k < n$ and $B \subset \mathbb{R}^n$ contains a k -dimensional skeleton of an n -dimensional axis-parallel cube centered at every point $S \subset \mathbb{R}^n$ of dimension s (for some dimension) then the best lower bound for the dimension (for the same dimension) of B is shown in the last column of the following table

Dimension	Square vertices ($n = 2, k = 0$)	Square boundary ($n = 2, k = 1$)	k -skeleton of an n -cube
$\dim_{\mathbb{P}}$	$\frac{3}{4}s$	$1 + \frac{3}{8}s$	$k + \frac{(n-k)(2n-1)}{2n^2}s$
$\overline{\dim}_{\mathbb{B}}$	$\frac{3}{4}s$	$\max\left\{1, \frac{7}{8}s\right\}$	$\max\left\{k, \left(1 - \frac{n-k}{2n^2}\right)s\right\}$
$\underline{\dim}_{\mathbb{B}}$	$\frac{3}{4}s$	$\max\left\{1, \frac{7}{8}s\right\}$	$\max\left\{k, \left(1 - \frac{n-k}{2n^2}\right)s\right\}$
$\dim_{\mathbb{H}}$	$\max\{0, s - 1\}$	1	$\max\{k, s - 1\}$

In [3] this problem was answered for a bit more general T but for simplicity here we consider only the case when T is the k -skeleton of a polytope P .

Again, we can easily get lower estimate for the Hausdorff dimension using appropriate projections:

Let F_1, \dots, F_m be the k -dimensional faces of the polytope P . Let $W_i = \text{span}F_i$, where span means the linear span (not the affine span). Then for every $u \in \text{proj}_{W_i^\perp} S$ the set B contains a similar copy of F_i in $u + W_i$, so $\dim_{\text{H}} B \geq \dim_{\text{H}}(\text{proj}_{W_i^\perp} S) + k$ for every i . Therefore we get the estimate

$$\dim_{\text{H}} B \geq \max_i \dim_{\text{H}}(\text{proj}_{W_i^\perp} S) + k \stackrel{\text{def}}{=} d_{T,S}.$$

Again, this turned out to be sharp:

Theorem 5.8 ([3]) *Let $0 \leq k < n$ and T be the k -skeleton of an n -dimensional polytope and let $S \subset \mathbb{R}^n$ be a compact set. Let F_1, \dots, F_m be the k -faces of T (so $T = \cup_i F_i$), let $W_i = \text{span}F_i$ and $d_{T,S} = \max_i \dim_{\text{H}}(\text{proj}_{W_i^\perp} S) + k$.*

- (a) *If B contains a scaled copy of T around every point of S , then $\dim_{\text{H}} B \geq d_{T,S}$.*
- (b) *There exists a compact set B with $\dim_{\text{H}} B = d_{T,S}$ that contains a scaled copy of T around every point of S .*

We saw the very simple proof of part (a) above. Part (b) was shown by proving that in some sense a typical compact B with property (♠) has Hausdorff dimension $d_{T,S}$. Now we make this more precise.

We have fixed T and S . The set B will be of the form

$$B = \bigcup_{(x,r) \in K} x + rT \stackrel{\text{def}}{=} \phi(K),$$

where $K \subset S \times [1, 2]$ is compact.

Note that $B = \phi(K)$ contains a scaled copy of T around every point of S if $\text{proj}_1 K = S$, where proj_1 denotes the projection to the first coordinate.

Therefore we want to find a “code set” K from

$$\mathcal{K} = \{K \subset S \times [1, 2] : K \text{ is compact, } \text{proj}_1 K = S\}$$

such that $\dim_{\text{H}}(\phi(K)) = d_{T,S}$. By considering the Hausdorff metric, \mathcal{K} is a compact metric space, so Baire category theorem can be applied. Thus we can say that (in the Baire category sense) for a typical $K \in \mathcal{K}$ the set $B = \phi(K)$ has Hausdorff dimension $d_{T,S}$ if there is a dense G_δ subset \mathcal{G} of \mathcal{K} such that for any $K \in \mathcal{G}$ we have $\dim_{\text{H}}(\phi(K)) = d_{T,S}$.

The point is that this way it is enough to consider only one k -face of T , and for one k -face it is not hard to show the claim.

To get Theorem 5.6 about the original problem in which only the Hausdorff dimension s of S is given we have to apply Theorem 5.8 for a compact set S of

Hausdorff dimension s such that the dimension drop of S is maximal for all the projections $\text{proj}_{W_i^\perp}$. The existence of such an S can be proved using sets with large intersection properties defined by K. Falconer [6].

Finally we explain what the above-mentioned maximal dimension drops are for which S has to be constructed, how we get the dimension $\max\{k, s - 1\}$ or $\max\{k, s\}$ as the dimension of B , as it was mentioned after Theorem 5.6, and why we have two cases depending on the positions of 0 and T . The point is that the linear span W_i of the k -dimensional face F_i is k -dimensional if the affine span of F_i contains 0 and $k + 1$ dimensional if not. So the maximal dimension drop of the s -dimensional S when we project to W_i^\perp is $\min\{s, k\}$ in the first case and $\min\{s, k + 1\}$ in the second case. Therefore, if we have maximal dimension drop for each i , then $d_{T,S} = \max_i \dim_{\mathbb{H}}(\text{proj}_{W_i^\perp} S) + k$ equals to $s - \min\{s, k\} + k = \max\{k, s\}$ if we are in the first case for at least one i and $s - \min\{s, k + 1\} + k = \max\{k, s - 1\}$ otherwise.

6 Rotated Cubes and Polytopes

So far we allowed only scaled copies of a cube or a more general polytope but it seems to be natural to study the case when we allow rotations as well. For example, how small can $B \subset \mathbb{R}^n$ be if it contains the k -skeleton of a (possibly rotated) n -cube centered at every point of a set S of given size?

When we did not allow rotation and used only axis-parallel cubes, we saw that B can be much smaller than S and this was possible because there was a huge overlap between the k -skeletons of different cubes. This seems to indicate that there is no point using rotated copies, since their intersection is small. This intuition turned out to be wrong, at least for Hausdorff dimension, as the following recent result of Chang, Csörnyei, Héra and, the author [3] shows.

Theorem 6.1 ([3]) *For any $0 \leq k < n$ there exists a closed set B of Hausdorff dimension k that contains the k -skeleton of a (rotated) cube centered at every point of \mathbb{R}^n .*

Recall from the previous section that without allowing rotations the dimension of the above Borel set cannot be less than $n - 1$.

We do not know if the results of the previous sections about the box and packing dimension are also changed if we allow rotations as well.

Like the constructions of the previous section for Hausdorff dimension, Theorem 6.1 was also shown by Baire category argument. Again, we make this more precise. Let T be the k -skeleton of a fixed n -dimensional cube centered at the origin. We will put a cube centered at every point of an arbitrary fix compact set $S \subset \mathbb{R}^n$.

The set B will be of the form

$$B = \bigcup_{(x,r,\alpha) \in K} x + r\alpha(T) \stackrel{\text{def}}{=} \Psi(K),$$

where $K \subset S \times [1, 2] \times SO(n)$ is compact. Now $B = \Psi(K)$ contains a scaled and rotated copy of T centered at every point of S if $\text{proj}_1 K = S$, where proj_1 still denotes the projection to the first coordinate. Therefore now we want to find a code set K from

$$\mathcal{K}' = \{K \subset S \times [1, 2] \times SO(n) : K \text{ is compact, } \text{proj}_1 K = S\}$$

such that $\dim_{\mathbb{H}}(\Psi(K)) = k$. By considering a natural compact metric on $SO(n)$, \mathcal{K}' is also a compact metric space with the Hausdorff metric, so Baire category theorem can be applied again. It is proved in [3] that for a typical $K \in \mathcal{K}'$ indeed we have $\dim_{\mathbb{H}}(\Psi(K)) = k$, which proves Theorem 6.1.

The above argument works of course not only for cubes but also for more general sets, among others for the k -skeleton of any polytope provided that 0 is not contained in any of the k -dimensional affine subspaces defined by the polytope.

In [3] we also study what happens if we allow rotation but do not allow scaling. For example, how small the Hausdorff dimension of a set can be if it contains the k -skeleton of an n -dimensional (rotated) unit cube centered at every point of \mathbb{R}^n ? This is closely related to the problems when we want a k -dimensional affine subspace at distance 1 from every point, or when we want k -skeleton of n -dimensional (rotated) cubes of every size centered at every point, or when we want k -planes at every positive distance from every point. Combining results of Chang et al. [3] and Héra et al. [8] it turns out that the smallest possible Hausdorff dimension is $k + 1$ in all of these problems.

First we present the result about the constructions.

Theorem 6.2 ([3]) *For any integers $0 \leq k < n$ there exist Borel sets B_1 and B_2 of Hausdorff dimension $k + 1$ such that for every $x \in \mathbb{R}^n$*

- (i) *the set B_1 contains a k -dimensional affine subspace at every positive distance from x ,*
- (ii) *the set B_2 contains a k -skeleton of (rotated) cubes of every size centered at x .*

Part (i) is very easy: B_1 can be chosen as the union of any countable dense collection of $k + 1$ -dimensional affine subspaces. Like the previously mentioned constructions, part (ii) is also proved by showing that a typical construction has Hausdorff dimension at most $k + 1$ and it also works for the k -skeleton of any polytope provided that 0 is not contained in any of the k -dimensional affine subspaces defined by the polytope.

Recall that, by Proposition 5.5, if instead of k -skeleton of rotated cubes of every size we are allowed to use axis-parallel cubes of arbitrary size, then $\dim_{\mathbb{H}} B \geq n - 1$. Therefore, somewhat surprisingly, Theorem 6.2 (ii) shows that in the $k < n - 2$ case we can get smaller union if we take k -skeletons of rotated cubes of every size centered at every point than in the case when we take k -skeletons of axis-parallel cubes of arbitrary size centered at every point.

The fact that we cannot get sets of Hausdorff dimension less than $k + 1$ even if we want only k -skeleton of unit cubes around every point or k -dimensional affine subspace at distance 1 from every point is based on the following very recent result of Héra, Máthé, and the author.

Theorem 6.3 ([8]) *Let $0 \leq k < n$ be integers and let $A(n, k)$ denote the space of all k -dimensional affine subspaces of \mathbb{R}^n and consider any natural metric on $A(n, k)$.*

Suppose that $B \subset \mathbb{R}^n$, $E \subset A(n, k)$ and every k -dimensional affine subspace $P \in E$ intersects B in a set of positive k -dimensional Hausdorff measure. Then

$$\dim_{\text{H}} B \geq k + \min(\dim_{\text{H}} E, 1). \tag{3}$$

For the special case when $k = n - 1$ this was proved by Falconer and Mattila [7]. Note that for $0 < k < n - 1$ the right-hand side of (3) cannot be replaced by the more natural $k + \min(\dim_{\text{H}} E, n - k)$: if B is a $k + 1$ -dimensional affine subspace and E is the set of all k -planes of B , then $\dim_{\text{H}} E = k + 1 > 1$, so $\dim_{\text{H}} B = k + 1 < k + \min(\dim_{\text{H}} E, n - k)$.

Corollary 6.4 ([3]) *Let $0 \leq k < n$ be integers, $d > 0$ be fixed, and B be a subset of \mathbb{R}^n such that for every $x \in \mathbb{R}^n$ there exists a k -dimensional affine subspace P at distance d from x such that P intersects B in a set of positive k -dimensional Hausdorff measure. Then $\dim_{\text{H}} B \geq k + 1$.*

Proof (Sketch) Let E be the set of those k -dimensional affine subspaces that intersect B in a set of positive k -dimensional Hausdorff measure. By Theorem 6.3 it is enough to prove that $\dim_{\text{H}} E \geq 1$. For each $P \in E$ let $D(P)$ be the set of those k -dimensional affine subspaces that are parallel to P and are at distance d from P (in the Euclidean distance of \mathbb{R}^n), and let $C(P)$ be the union of the k -planes of $D(P)$. Note that $C(P)$ is a cylinder, and it is exactly the set of those points of \mathbb{R}^n that are at distance d from P . Thus, by assumption, $\cup_{P \in E} C(P) = \mathbb{R}^n$.

One can easily check that for any P we have $\dim_{\text{H}} D(P) = n - k - 1$. From this it is not hard to show that $\dim_{\text{H}}(\cup_{P \in E} D(P)) \leq \dim_{\text{H}} E + n - k - 1$ and then $\dim_{\text{H}}(\cup_{P \in E} C(P)) \leq \dim_{\text{H}}(\cup_{P \in E} D(P)) + k \leq \dim_{\text{H}} E + n - 1$. Since $\cup_{P \in E} C(P) = \mathbb{R}^n$ this implies that indeed $\dim_{\text{H}} E \geq 1$. □

Combining Theorem 6.2 and Corollary 6.4 we immediately get the following four statements.

Corollary 6.5 *For any integers $0 \leq k < n$ and any of the following four properties the minimal Hausdorff dimension of a Borel set $B \subset \mathbb{R}^n$ with that property is $k + 1$.*

- (i) *B contains the k -skeleton of a (rotated) unit cube centered at every point of \mathbb{R}^n .*
- (ii) *B contains the k -skeleton of (rotated) cubes of every size centered at every point of \mathbb{R}^n .*
- (iii) *B contains a k -dimensional affine subspace at distance 1 from every point of \mathbb{R}^n .*
- (iv) *B contains a k -dimensional affine subspace at every positive distance from every point of \mathbb{R}^n .*

For $k = n - 1$ in Corollaries 6.4 and 6.5 it is natural to ask if instead of full Hausdorff dimension we can also guarantee positive Lebesgue measure. The following results give negative answers in case of Corollaries 6.4 and 6.5 (i) and (iii).

Theorem 6.6 ([3]) *For any $n \geq 2$ there exist Borel sets $B_1, B_2 \subset \mathbb{R}^n$ of Lebesgue measure zero such that for every $x \in \mathbb{R}^n$*

- (i) *the set B_1 contains an $n - 1$ -dimensional hyperplane at distance 1 from x .*
- (ii) *the set B_2 contains the boundary of a (rotated) unit cube centered at x .*

These results are also proved by showing that a typical construction has Lebesgue measure zero and the proof of (ii) also works for any polytope provided that 0 is not contained in any of the $n - 1$ -dimensional affine subspaces defined by the polytope.

7 Open Problems

Here we collect some of those problems that seem to be open. Most of these were already mentioned in the previous sections.

Questions 7.1 *Let C be the classical middle-third Cantor set translated by $-\frac{1}{2}$ to move its center to 0. Let $B \subset \mathbb{R}$ be a set such that for any $x \in \mathbb{R}$ there exists $r > 0$ such that $x + rC \subset B$. How small can B be? How small can $\dim_{\mathbb{H}} B$ be? Can B have Lebesgue measure zero? What can we say about other Cantor sets C ?*

Recall from Sect. 3 that by the theorem of Łaba and Pramanik [13] there exist Cantor sets for which such a B must have positive Lebesgue measure, by the result of Máthé [16] there exist Cantor sets for which such a B can have zero measure, and by the results of Hochman [9] and Bourgain [2] for any porous Cantor set C with $\dim_{\mathbb{H}} C > 0$ such a B must have Hausdorff dimension strictly larger than $\dim_{\mathbb{H}} C$ and at least $1/2$.

Questions 7.2 *What can we say about the box and packing dimension in the problems of Sect. 6? In other words, how the box and packing dimension results about squares and cubes of Sects. 4 and 5 are changed if we allow rotated squares and cubes? For example, is there a compact subset of the plane that contains a (rotated) square boundary centered at every point of $[0, 1] \times [0, 1]$ with box or packing dimension less than $7/4$ (which is the smallest possible box and packing dimension for axis-parallel squares by Theorem 4.5)?*

Theorem 6.1 shows that in some cases we can get sets with smaller Hausdorff dimension if we allow rotation but we do not know if smaller box or smaller packing dimension can be also obtained by allowing rotated squares or cubes.

Acknowledgements The author is grateful to András Máthé for helpful discussions and for checking the paper very carefully.

This research was supported by Hungarian Scientific Foundation grant no. 104178.

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