Multifractal Properties of Convex Hulls of Typical Continuous Functions

Zoltán Buczolich

Abstract We study the singularity (multifractal) spectrum of the convex hull of the typical/generic continuous functions defined on $[0,1]^d$. We denote by $\mathbf{E}_\phi^h$ the set of points at which $\phi : [0,1]^d \to \mathbb{R}$ has a pointwise Hölder exponent equal to h. Let H_f be the convex hull of the graph of f, the concave function on the top of H_f is denoted by $\phi_{1,f}(\mathbf{x}) = \max\{y : (\mathbf{x}, y) \in H_f\}$ and $\phi_{2,f}(\mathbf{x}) = \min\{y : (\mathbf{x}, y) \in H_f\}$ denotes the convex function on the bottom of H_f. We show that there is a dense G_δ subset $\mathscr{G} \subset C[0,1]^d$ such that for $f \in \mathscr{G}$ the following properties are satisfied. For $i = 1, 2$ the functions $\phi_{i,f}$ and f coincide only on a set of zero Hausdorff dimension, the functions $\phi_{i,f}$ are continuously differentiable on $(0,1)^d$, $\mathbf{E}_{\phi_{i,f}}^0$ equals the boundary of $[0,1]^d$, $\dim_H \mathbf{E}_{\phi_{i,f}}^1 = d - 1$, $\dim_H \mathbf{E}_{\phi_{i,f}}^{+\infty} = d$ and $\mathbf{E}_{\phi_{i,f}}^h = \emptyset$ if $h \in (0, +\infty) \setminus \{1\}$.

1 Introduction

We started with J. Nagy to study multifractal properties of typical/generic functions in [2] where multifractal properties of generic monotone functions on $[0,1]$ were treated. The higher dimensional version of this question was considered in [4] where with S. Seuret we investigated the Hölder spectrum of functions monotone in several variables. In [3] we also showed that typical Borel measures on $[0,1]^d$ satisfy a multifractal formalism. Multifractal properties of typical convex continuous functions defined on $[0,1]^d$ are discussed in [5].
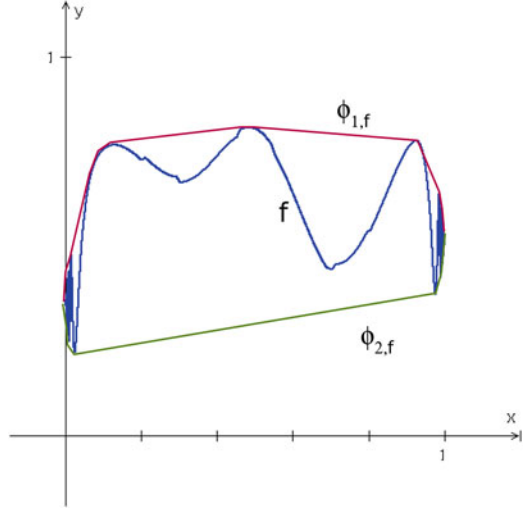
In [1] the convex hull of typical continuous functions $f \in C[0,1]$ is considered by A.M. Bruckner and J. Haussermann. In this case the boundary of this convex hull decomposes into two functions (in our notation) $\phi_{1,f}$ and $\phi_{2,f}$ see Fig. 1. The upper one $\phi_{1,f}$ is concave, the lower one $\phi_{2,f}$ is convex. It is shown that for the typical f these functions are continuously differentiable on $(0,1)$ and at the endpoints they have infinite derivatives. The aim of our paper is to describe the multifractal spectrum of these functions in the multidimensional setting, that is, generic/typical

Z. Buczolich (✉)

Department of Analysis, ELTE Eötvös Loránd University, Pázmány Péter Sétány 1/c,
1117 Budapest, Hungary
e-mail: buczo@cs.elte.hu

Fig. 1 f, $\phi_{1,f}$ and $\phi_{2,f}$ in 1D

continuous functions f in $C[0, 1]^d$ in the sense of Baire category. The topology on $C[0, 1]^d$ is the supremum metric. We also prove the multidimensional version of the above-mentioned results of A.M. Bruckner and J. Haussermann.

The convex hull of the graph of $f \in C[0, 1]^d$ is denoted by H_f, that is, $H_f =$ convex hull of $\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in [0, 1]^d\}$.

We are interested in two functions: $\phi_{1,f}(\mathbf{x}) = \max\{y : (\mathbf{x}, y) \in H_f\}$ is the function on the top of H_f, and $\phi_{2,f}(\mathbf{x}) = \min\{y : (\mathbf{x}, y) \in H_f\}$ is the function on the bottom of H_f. In Fig. 1 these functions are illustrated in dimension one.

The points where f and $\phi_{i,f}$ coincide are denoted by $E_{i,f} = \{\mathbf{x} : \phi_{i,f}(\mathbf{x}) = f(\mathbf{x})\}$, $i = 1, 2$.

The Hölder exponent and singularity spectrum for a locally bounded function is defined as follows.

Definition 1.1. Let $f \in L^\infty([0, 1]^d)$. For $h \geq 0$ and $\mathbf{x} \in [0, 1]^d$, the function f belongs to $C_{\mathbf{x}}^h$ if there are a polynomial P of degree less than $[h]$ and a constant C such that, for $\mathbf{x}'$ close to $\mathbf{x}$,

$$|f(\mathbf{x}') - P(\mathbf{x}' - \mathbf{x})| \leq C|\mathbf{x}' - \mathbf{x}|^h. \tag{1}$$

The pointwise Hölder exponent of f at $\mathbf{x}$ is $h_f(\mathbf{x}) = \sup\{h \geq 0 : f \in C_{\mathbf{x}}^h\}$.

Definition 1.2. The singularity spectrum of f is defined by

$$d_f(h) = \dim_H \mathbf{E}_f^h, \quad \text{where } \mathbf{E}_f^h = \{\mathbf{x} : h_f(\mathbf{x}) = h\}.$$

Here $\dim_H$ denotes the Hausdorff dimension (see, for example, [7] or [8]), and $\dim \emptyset = -\infty$ by convention.

We will also use the sets

$$\mathbf{E}_f^{h,\leq} = \{\mathbf{x} : h_f(\mathbf{x}) \leq h\} \supset \mathbf{E}_f^h \text{ and } \mathbf{E}_f^{h,<} = \{\mathbf{x} : h_f(\mathbf{x}) < h\}. \tag{2}$$

The faces of $[0, 1]^d$ are

$$F_{0,j} = \{(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) \in [0, 1]^d\}$$

and

$$F_{1,j} = \{(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_d) \in [0, 1]^d\}.$$

Then $\partial([0, 1]^d) = \cup_{i=1}^2 \cup_{j=1}^d F_{i,j}$.

Since our functions are defined on $[0, 1]^d$ at points of $\partial([0, 1]^d)$ we can consider only one-sided partial derivatives. For example, for a point $\mathbf{x} \in F_{0,1}$ we denote by $\partial_{1,+} f(\mathbf{x})$ the one-sided partial derivative in the first variable pointing in the direction of the interior of $[0, 1]^d$, while for points $\mathbf{x} \in F_{1,1}$ we need to use $\partial_{1,-} f(\mathbf{x})$.

The main result of our paper is the following theorem:

Theorem 1.3. *There exists a dense G_δ set $\mathscr{G} \subset C[0, 1]^d$ such that for every $f \in \mathscr{G}$ for $i = 1, 2$*

- *$\phi_{i,f}$ is continuously differentiable on $(0, 1)^d$;*
- *if $\mathbf{x} \in \partial([0, 1]^d)$, then*

$$h_{\phi_{i,f}}(\mathbf{x}) = 0 \tag{3}$$

  hence $d_{\phi_{i,f}}(0) = d - 1$ and $\mathbf{E}^0_{\phi_{i,f}} \cap (0, 1)^d = \emptyset$;
- *$d_{\phi_{i,f}}(1) = d - 1$;*
- *$d_{\phi_{i,f}}(+\infty) = d$;*
- *$d_{\phi_{i,f}}(h) = -\infty$, that is, $\mathbf{E}^h_f = \emptyset$ for $h \in (0, +\infty) \setminus \{1\}$;*
- *for $j = 1, \ldots, d$ if $\mathbf{x} \in F_{0,j}$, then*

$$\partial_{j,+} \phi_{i,f}(\mathbf{x}) = (-1)^{i+1}(+\infty) \tag{4}$$

  if $\mathbf{x} \in F_{1,j}$, then

$$\partial_{j,-} \phi_{i,f}(\mathbf{x}) = (-1)^i(+\infty); \tag{5}$$

- *$\dim_H E_{i,f} = 0$.*

2   Notation and Preliminary Results

The open ball with center $\mathbf{x}$ and of radius $r > 0$ is denoted by $B(\mathbf{x}, r)$. We use similar notation for open neighborhoods of subsets, for example, if $A \subset \mathbb{R}^d$, then $B(A, r) = \{\mathbf{x} \in \mathbb{R}^d : \text{dist}(\mathbf{x}, A) < r\}$.

For subsets $A \subset \mathbb{R}^d$ we denote the diameter of the set A by $|A|$ while ∂A denotes its boundary.

The j'th basis vector in $\mathbb{R}^d$ is denoted by $\mathbf{e}_j = (0, \ldots, 0, \underset{\underset{j}{\uparrow}}{1}, 0, \ldots, 0)$.

In our proofs we use a standard fixed countable dense set $\{f_n\}_{n=1}^{\infty}$ in $C[0, 1]^d$. We assume that the functions f_n are in $C^{\infty}[0, 1]^d$. (Taking an arbitrary countable dense set $\{\widetilde{f}_n\}$ in $C[0, 1]^d$ by using a mollifier function it is easy to obtain such a dense set of C^{∞} functions.)

Definition 2.1. We say that $f : [0, 1]^d \to \mathbb{R}$ is *piecewise linear* if there is a partition $Z_j, j = 1, \ldots, n_{\zeta}$ of $[0, 1]^d$ into simplices such that for each j the set $\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in Z_j\}$ is the subset of a hyperplane in $\mathbb{R}^{d+1}$.

We say that f is *independent piecewise linear* if it is piecewise linear and if V denotes the collection of the vertices of the simplices $Z_j, j = 1, \ldots, n_{\zeta}$ for a suitable partition, then

- from $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$, $\mathbf{x}_i \neq \mathbf{x}_j$, $i \neq j$, the points $(\mathbf{x}_j, f(\mathbf{x}_j))$, $j = 1, \ldots, k$ are on the same d-dimensional hyperplane in $\mathbb{R}^{d+1}$ it follows that $k \leq d + 1$,
- from $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V \cap (0, 1)^d$, $\mathbf{x}_i \neq \mathbf{x}_j$, $i \neq j$, the points $\mathbf{x}_j, j = 1, \ldots, k$ are on the same $(d - 1)$-dimensional hyperplane in $\mathbb{R}^d$ it follows that $k \leq d$.

The second assumption in the above definition is void in the one dimensional case since the zero dimensional hyperplanes are just points.

Lemma 2.2. *Suppose* $f \in C[0, 1]^d$. *For any* $\mathbf{x}_0 \in [0, 1]^d$ *there exist* $\mathbf{x}_i \in E_{1,f}$, $i = 1, \ldots, d + 1$ *and* $p_i \geq 0$, $\sum_{i=1}^{d+1} p_i = 1$, *such that* $\mathbf{x}_0 = \sum_{i=1}^{d+1} p_i \mathbf{x}_i$ *and*

$$\phi_{1,f}(\mathbf{x}_0) = \sum_{i=1}^{d+1} p_i \phi_{1,f}(\mathbf{x}_i) = \sum_{i=1}^{d+1} p_i f(\mathbf{x}_i).$$

Remark 2.3. We remark that in the above lemma some p_i's can equal zero, or some $\mathbf{x}_i$'s coincide.

Proof. By Carathéodory's theorem (see, for example, [6]) from $(\mathbf{x}_0, \phi_{1,f}(\mathbf{x}_0)) \in H_f$ it follows that one can find (not necessarily different) $\mathbf{x}_i \in [0, 1]^d$, $p_i \geq 0$, $i = 1, \ldots, d + 1$, $\sum_{i=1}^{d+1} p_i = 1$ such that $\sum_{i=1}^{d+1} p_i \mathbf{x}_i = \mathbf{x}_0$, $\sum_{i=1}^{d+1} p_i f(\mathbf{x}_i) = \phi_{1,f}(\mathbf{x}_0)$. If for an i' we had $f(\mathbf{x}_{i'}) < \phi_{1,f}(\mathbf{x}_{i'})$, then letting

$$y' = \left(\sum_{i=1, \, i \neq i'}^{d+1} p_i f(\mathbf{x}_i) \right) + p_{i'} \phi_{1,f}(\mathbf{x}_{i'})$$

we would obtain $(\mathbf{x}_0, y') \in H_f$ contradicting that $\phi_{1,f}(\mathbf{x}_0) = \max\{y : (\mathbf{x}_0, y) \in H_f\}$.

Lemma 2.4. *There exists a dense* G_{δ} *set* $\mathscr{G}_{0,d} \subset C[0, 1]^d$ *such that for every* $f \in \mathscr{G}_{0,d}$ *and for every* $\mathbf{x}_0 \in F_{0,d}$

$$\partial_{d,+} \phi_{1,f}(\mathbf{x}_0) = +\infty, \ \partial_{d,+} \phi_{2,f}(\mathbf{x}_0) = -\infty \ and \tag{6}$$

$$h_{\phi_i,f}(\mathbf{x}_0) = 0 \ for \ i = 1, 2. \tag{7}$$

Remark 2.5. For $j = 1, \ldots, d$ a version of Lemma 2.4 can provide dense G_δ sets $\mathcal{G}_{0,j} \subset C[0, 1]^d$ such that (6) and (7) hold with d replaced by j. If we use the faces $F_{1,j}$ instead of $F_{0,j}$ in analogous versions of Lemma 2.4 we can obtain dense G_δ sets $\mathcal{G}_{1,j} \subset C[0, 1]^d$.

Taking

$$\mathcal{G}_0 = \bigcap_{j=1}^{d} \mathcal{G}_{0,j} \cap \mathcal{G}_{1,j} \tag{8}$$

for any $f \in \mathcal{G}_0$ we have (3)–(5) satisfied.

Proof. Without limiting generality we prove the statement for $\phi_{1,f}$.

Suppose $\mathbf{x}_0 = (x_{0,1}, \ldots, x_{0,d-1}, 0) \in F_{0,d}$. By Lemma 2.2 there exist $\mathbf{x}_i \in E_{1,f} \subset [0, 1]^d$, and $p_i \geq 0$, $i = 1, \ldots, d + 1$, such that $\sum_{i=1}^{d+1} p_i = 1$,

$$\sum_{i=1}^{d+1} p_i \mathbf{x}_i = \mathbf{x}_0 \text{ and } \sum_{i=1}^{d+1} p_i f(\mathbf{x}_i) = \sum_{i=1}^{d+1} p_i \phi_{1,f}(\mathbf{x}_i) = \phi_{1,f}(\mathbf{x}_0). \tag{9}$$

This and $\mathbf{x}_0 \in F_{0,d}$ imply

$$\mathbf{x}_i \in F_{0,d}, \ i = 1, \ldots, d + 1. \tag{10}$$

Put $M_n = ||f_n'||_\infty \geq |\partial_d f_n(\mathbf{x})|$ for all $\mathbf{x} \in [0, 1]^d$. We also let

$$f_{n,m}(\mathbf{x}) = f_n(\mathbf{x}) + \frac{1}{n+m}(\text{dist}(\mathbf{x}, F_{0,d}))^{1/m} \text{ and } \delta_{n,m} = \frac{1}{(n+m)2^{(n+m)}}. \tag{11}$$

It is clear that $G_m = \cup_{n=1}^{\infty} B(f_{n,m}, \delta_{n,m})$ is dense and open in $C[0, 1]^d$ and $\mathcal{G}_{0,d} \overset{\text{def}}{=} \cap_{m=1}^{\infty} G_m$ is dense G_δ. Suppose $f \in \mathcal{G}_{0,d}$. Then there exists a sequence n_m, $m = 1, \ldots$ such that $f \in B(f_{n_m,m}, \delta_{n_m,m})$. Since $\mathbf{x}_i \in F_{0,d}$ by (11) we have

$$f(\mathbf{x}_i) \leq f_{n_m,m}(\mathbf{x}_i) + \delta_{n_m,m} = f_{n_m}(\mathbf{x}_i) + \delta_{n_m,m}, \ i = 1, \ldots, d + 1.$$

Therefore, using (9)

$$\phi_{1,f}(\mathbf{x}_0) \leq \left(\sum_{i=1}^{d+1} p_i f_{n_m}(\mathbf{x}_i)\right) + \delta_{n_m,m}.$$

On the other hand, since $\phi_{1,f}$ is concave

$$\phi_{1,f}(\mathbf{x}_0 + t\mathbf{e}_d) \geq \left(\sum_{i=1}^{d+1} p_i \phi_{1,f}(\mathbf{x}_i + t\mathbf{e}_d)\right) \geq$$

$$\sum_{i=1}^{d+1} p_i f(\mathbf{x}_i + t\mathbf{e}_d) \geq \left(\sum_{i=1}^{d+1} p_i f_{n_m,m}(\mathbf{x}_i + t\mathbf{e}_d)\right) - \delta_{n_m,m} \geq$$

[using (10) and (11)]

$$\geq \left(\sum_{i=1}^{d+1} p_i f_{n_m}(\mathbf{x}_i + t\mathbf{e}_d)\right) + \frac{1}{n_m + m} t^{1/m} - \delta_{n_m,m}.$$

Taking difference

$$\phi_{1,f}(\mathbf{x}_0 + t\mathbf{e}_d) - \phi_{1,f}(\mathbf{x}_0) \geq$$

$$\left(\sum_{i=1}^{d+1} p_i(f_{n_m}(\mathbf{x}_i + t\mathbf{e}_d) - f_{n_m}(\mathbf{x}_i))\right) - 2\delta_{n_m,m} + \frac{1}{n_m + m} t^{1/m} \geq \qquad (12)$$

(by the Mean value Theorem and the choice of M_n)

$$\geq \frac{1}{n_m + m} t^{1/m} - 2\delta_{n_m,m} - M_n t.$$

Choosing $t_m = 2^{-(n_m+m)}$ by (11) we obtain

$$\delta_{n_m,m} = t_m \frac{1}{n_m + m}.$$

Hence,

$$\limsup_{m \to \infty} \frac{\phi_{1,f}(\mathbf{x}_0 + t_m\mathbf{e}_d) - \phi_{1,f}(\mathbf{x}_0)}{t_m^\alpha} = +\infty \text{ for any } \alpha > 0.$$

This implies (7).

Taking $\alpha = 1$ and using concavity of $\phi_{1,f}$ we also obtain $\partial_{d,+}\phi_{1,f}(\mathbf{x}_0) = +\infty$. This implies (6).

Lemma 2.6. *There exists a dense G_δ set $\mathscr{G}_1 \subset C[0,1]^d$ such that for every $f \in \mathscr{G}_1$ the functions $\phi_{1,f}$ and $\phi_{2,f}$ are both continuously differentiable on $(0,1)^d$.*

Remark 2.7. This also implies that $h_{\phi_{i,f}}(\mathbf{x}) \geq 1$ for any $\mathbf{x} \in (0,1)^d$, that is, $\mathbf{E}_{\phi_{i,f}}^{1,<} \cap (0,1)^d = \emptyset$ for $f \in \mathscr{G}_1$ and $i = 1, 2$.

Proof. Again we start with $f_n \in C^\infty[0,1]^d$, $n = 1, \ldots$ a countable dense set in $C[0,1]^d$. This time we select

$$M_n \geq 1 \text{ such that } |\partial_j^2 f_n| \leq M_n, \ j = 1, \ldots, d.$$

We also put

$$\delta_{n,m} = \left(\frac{1}{m \cdot M_n}\right)^2. \qquad (13)$$

Recall that $\mathscr{G}_0$ was defined in Remark 2.5. We put

$$\mathscr{G}_1 = \mathscr{G}_0 \cap \bigcap_m \bigcup_n B(f_n, \delta_{n,m})$$

and select $f \in \mathscr{G}_1$. Then we can choose n_m such that $f \in B(f_{n_m}, \delta_{n_m,m})$.

Suppose $\mathbf{x} \in (0,1)^d$. We need to verify that $\partial_j \phi_{i,f}(\mathbf{x})$ exists and continuous for any $j = 1, \ldots, d$ and $i = 1, 2$.

Since the other cases are similar we can suppose that $i = 1, j = 1$.

Since $\phi_{1,f}(\mathbf{x} + t\mathbf{e}_1)$ is a concave function in t it is sufficient to verify that its derivative exists at $t = 0$ for any choice of $\mathbf{x} \in (0,1)^d$. This will imply that $\partial_1 \phi_{1,f}(\mathbf{x} + t\mathbf{e}_1)$ is monotone decreasing in t, without any jump discontinuities, hence for a fixed $\mathbf{x}$ it is continuous as a function of one variable. In the end of this proof we will provide a standard argument showing that from the concavity and continuity of $\phi_{1,f}$ one can deduce that $\partial_1 \phi_{1,f}$ is continuous on $(0,1)^d$.

From now on $\mathbf{x} \in (0,1)^d$ is fixed.

By Carathéodory's theorem we can select $\mathbf{x}_i \in E_{1,f}, p_i \geq 0, i = 1, \ldots, d+1$ such that $\sum_{i=1}^{d+1} p_i = 1, \sum_{i=1}^{d+1} p_i \mathbf{x}_i = \mathbf{x}$ and

$$\phi_{1,f}(\mathbf{x}) = \sum_{i=1}^{d+1} p_i \phi_{1,f}(\mathbf{x}_i) = \sum_{i=1}^{d+1} p_i f(\mathbf{x}_i).$$

By the assumption that $f \in \mathscr{G}_0$ for $\mathbf{x} \in (0,1)^d$ the points $\mathbf{x}_i$ are in $(0,1)^d$. Suppose

$$h_m = \frac{1}{m \cdot M_{n_m}}. \tag{14}$$

By the one dimensional Taylor's formula one can find $c_{n_m,i,\pm}$ such that $|c_{n_m,i,\pm}| < h_m$ and

$$f_{n_m}(\mathbf{x}_i \pm h_m \mathbf{e}_1) = f_{n_m}(\mathbf{x}_i) \pm \partial_1 f_{n_m}(\mathbf{x}_i) h_m + \frac{\partial_1^2 f_{n_m}(\mathbf{x}_i + c_{n_m,i,\pm} \mathbf{e}_1)}{2!} h_m^2 \geq$$

$$f_{n_m}(\mathbf{x}_i) \pm \partial_1 f_{n_m}(\mathbf{x}_i) h_m - \frac{M_{n_m}}{2} h_m^2. \tag{15}$$

This implies

$$\phi_{1,f}(\mathbf{x} \pm h_m \mathbf{e}_1) \geq \sum_{i=1}^{d+1} p_i \phi_{1,f}(\mathbf{x}_i \pm h_m \mathbf{e}_1) \geq$$

$$\sum_{i=1}^{d+1} p_i f(\mathbf{x}_i \pm h_m \mathbf{e}_1) \geq \left(\sum_{i=1}^{d+1} p_i f_{n_m}(\mathbf{x}_i \pm h_m \mathbf{e}_1) \right) - \delta_{n_m,m} \geq \tag{16}$$

[using (15)]

$$\geq \left(\sum_{i=1}^{d+1} p_i f_{n_m}(\mathbf{x}_i) \right) \pm \left(\sum_{i=1}^{d+1} p_i \partial_1 f_{n_m}(\mathbf{x}_i) \right) h_m - \frac{M_{n_m}}{2} h_m^2 - \delta_{n_m,m} \geq \circledast$$

using $|f - f_{n_m}| < \delta_{n_m,m}$, $f(\mathbf{x}_i) = \phi_{1,f}(\mathbf{x}_i)$ and $\sum_{i=1}^{d+1} p_i f(\mathbf{x}_i) = \phi_{1,f}(\mathbf{x})$ we can continue by

$$\circledast \geq \phi_{1,f}(\mathbf{x}) - \delta_{n_m,m} \pm \left(\sum_{i=1}^{d+1} \partial_1 f_{n_m}(\mathbf{x}_i) \right) h_m - \frac{M_{n_m}}{2} h_m^2 - \delta_{n_m,m}.$$

By (13) and (14) we obtain that

$$\frac{\phi_{1,f}(\mathbf{x}) - \phi_{1,f}(\mathbf{x} - h_m \mathbf{e}_1)}{h_m} \leq \left(\sum_{i=1}^{d+1} \partial_1 f_{n_m}(\mathbf{x}_i) \right) + \left(\frac{M_{n_m}}{2} + 2 \right) h_m$$

and similarly

$$\frac{\phi_{1,f}(\mathbf{x} + h_m \mathbf{e}_1) - \phi_{1,f}(\mathbf{x})}{h_m} \geq \left(\sum_{i=1}^{d+1} \partial_1 f_{n_m}(\mathbf{x}_i) \right) - \left(\frac{M_{n_m}}{2} + 2 \right) h_m$$

and hence,

$$\frac{\phi_{1,f}(\mathbf{x}) - \phi_{1,f}(\mathbf{x} - h_m \mathbf{e}_1)}{h_m} - \frac{\phi_{1,f}(\mathbf{x} + h_m \mathbf{e}_1) - \phi_{1,f}(\mathbf{x})}{h_m} \leq$$

$$(M_{n_m} + 4) h_m = \left(\frac{M_{n_m} + 4}{M_{n_m}} \right) \cdot \frac{1}{m} \leq \frac{5}{m}.$$

Since $\phi_{1,f}$ is concave this implies that $\partial_1 \phi_{1,f}(\mathbf{x})$ exists.

Next we verify that $\partial_1 \phi_{1,f}$ is continuous on $(0,1)^d$. We have seen that $\partial_1 \phi_{1,f}(\mathbf{x} + t\mathbf{e}_1)$ is a monotone decreasing continuous function in t for a fixed $\mathbf{x} \in (0,1)^d$. We need to show that $\partial_1 \phi_{1,f}$ is continuous as a function of several variables at any $\mathbf{x} \in (0,1)^d$. This is quite standard. Suppose $\mathbf{x} \in (0,1)^d$ and $\varepsilon > 0$ are fixed. Choose $t_0 > 0$ such that

$$\mathbf{x} \pm 2t_0 \mathbf{e}_1 \in (0,1)^d, \qquad |\partial_1 \phi_{1,f}(\mathbf{x} \pm 2t_0 \mathbf{e}_1) - \partial_1 \phi_{1,f}(\mathbf{x})| < \frac{\varepsilon}{2}.$$

The function $\phi_{1,f}$ is continuous as the "top part" of the convex hull H_f of the continuous function f. By uniform continuity of $\phi_{1,f}$ choose $\delta_1 > 0$ such that

$$|\phi_{1,f}(\mathbf{w}) - \phi_{1,f}(\mathbf{w}')| < \frac{\varepsilon t_0}{4} \text{ if } ||\mathbf{w} - \mathbf{w}'|| < \delta_1 \text{ and } \mathbf{w}, \mathbf{w}' \in (0,1)^d. \tag{17}$$

Now suppose that

$$\mathbf{z} = (z_1, \ldots, z_d) \text{ is a vector with } z_1 = 0 \text{ and } ||\mathbf{z}|| < \delta_1. \tag{18}$$

Then by (17)

$$\left| \frac{\phi_{1,f}(\mathbf{x} + \mathbf{z} + 2t_0\mathbf{e}_1) - \phi_{1,f}(\mathbf{x} + \mathbf{z} + t_0\mathbf{e}_1)}{t_0} - \right. \tag{19}$$

$$\left. \frac{\phi_{1,f}(\mathbf{x} + 2t_0\mathbf{e}_1) - \phi_{1,f}(\mathbf{x} + t_0\mathbf{e}_1)}{t_0} \right| < \frac{\varepsilon}{2}.$$

By the Mean Value Theorem there exist $c_\mathbf{x}$ and $c_\mathbf{z}$ in $(t_0, 2t_0)$ such that

$$\frac{\phi_{1,f}(\mathbf{x} + \mathbf{z} + 2t_0\mathbf{e}_1) - \phi_{1,f}(\mathbf{x} + \mathbf{z} + t_0\mathbf{e}_1)}{t_0} = \partial_1\phi_{1,f}(\mathbf{x} + \mathbf{z} + c_\mathbf{z}\mathbf{e}_1) \tag{20}$$

and

$$\frac{\phi_{1,f}(\mathbf{x} + 2t_0\mathbf{e}_1) - \phi_{1,f}(\mathbf{x} + t_0\mathbf{e}_1)}{t_0} = \partial_1\phi_{1,f}(\mathbf{x} + c_\mathbf{x}\mathbf{e}_1).$$

By monotonicity of $\partial_1\phi_{1,f}$ we have

$$|\partial_1\phi_{1,f}(\mathbf{x} + c_\mathbf{x}\mathbf{e}_1) - \partial_1\phi_{1,f}(\mathbf{x})| < |\partial_1\phi_{1,f}(\mathbf{x} + 2t_0\mathbf{e}_1) - \partial_1\phi_{1,f}(\mathbf{x})| < \frac{\varepsilon}{2}. \tag{21}$$

From (19)–(21) it follows that

$$|\partial_1\phi_{1,f}(\mathbf{x}) - \partial_1\phi_{1,f}(\mathbf{x} + \mathbf{z} + c_\mathbf{z}\mathbf{e}_1)| < \varepsilon \text{ where } t_0 < c_\mathbf{z}. \tag{22}$$

A similar argument can show that there is $c_\mathbf{z}' \in (t_0, 2t_0)$ such that

$$\frac{\phi_{1,f}(\mathbf{x} + \mathbf{z} - t_0\mathbf{e}_1) - \phi_{1,f}(\mathbf{x} + \mathbf{z} - 2t_0\mathbf{e}_1)}{t_0} = \partial_1\phi_{1,f}(\mathbf{x} + \mathbf{z} - c_\mathbf{z}'\mathbf{e}_1)$$

and

$$|\partial_1\phi_{1,f}(\mathbf{x}) - \partial_1\phi_{1,f}(\mathbf{x} + \mathbf{z} - c_\mathbf{z}'\mathbf{e}_1)| < \varepsilon \text{ where } t_0 < c_\mathbf{z}'. \tag{23}$$

By monotonicity of $\partial_1\phi_{1,f}(\mathbf{x} + \mathbf{z} + t\mathbf{e}_1)$ this implies that for $-t_0 \leq t \leq t_0$ we have

$$|\partial_1\phi_{1,f}(\mathbf{x} + \mathbf{z} + t\mathbf{e}_1) - \partial_1\phi_{1,f}(\mathbf{x})| < \varepsilon.$$

Since this holds for any $\mathbf{z}$ satisfying (18) we obtain that $\partial_1\phi_{1,f}$ is continuous on $(0, 1)^d$.

Lemma 2.8. *There exists a dense G_δ set $\mathscr{G}_{2,1}$ in $C[0,1]^d$ such that for every $f \in \mathscr{G}_{2,1}$*

$$\dim_H(E_{1,f}) = 0, \ \dim_H(\mathbf{E}_{\phi_{1,f}}^{\leq 1} \cap (0,1)^d) = d - 1, \tag{24}$$

$$\mathbf{E}_{\phi_{1,f}}^h \cap (0,1)^d = \varnothing \ for \ 1 < h < +\infty, \ and \ \dim_H(\mathbf{E}_{\phi_{1,f}}^{+\infty}) = d.$$

Proof. We choose again a countable dense set $f_n \in C[0,1]^d$. The functions f_n are uniformly continuous and for $\frac{1}{16(n+m)}$ we choose $\eta_{n,m} > 0$ such that

$$|f_n(\mathbf{x}) - f_n(\mathbf{y})| < \frac{1}{16(n+m)} \ \text{if} \ ||\mathbf{x} - \mathbf{y}|| < \eta_{n,m}, \ \mathbf{x}, \mathbf{y} \in [0,1]^d. \tag{25}$$

We partition $[0,1]^d$ into non-overlapping simplices $Z_j, j = 1, \ldots, \zeta_{n,m}$ such that the diameter of each simplex is less than $\eta_{n,m}$. We assume that $V(n,m)$ is the set of vertices of these simplices. We can also assume that these vertices are sufficiently independent, that is, from $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V(n,m) \cap (0,1)^d$, $\mathbf{x}_i \neq \mathbf{x}_j$, $i \neq j$, the points $\mathbf{x}_j, j = 1, \ldots, k$ are on the same $d-1$-dimensional hyperplane in $\mathbb{R}^d$ it follows that $k \leq d$. This means that the second assumption in Definition 2.1 is satisfied.

We denote by $\widetilde{f}_{n,V(n,m)}$ the function which is defined on $V(n,m)$ and for any $\mathbf{x} \in V(n,m), \widetilde{f}_{n,V(n,m)}(\mathbf{x}) = f_n(\mathbf{x})$. In Fig. 2 we illustrate the procedure of selecting $f_{n,m}$. On the left half of this figure there is f_n with the little circles on its graph. We suppose that on $[0,1]^1$ we used the "simplices," which are equally spaced line segments of length 0.2. The function $\widetilde{f}_{n,V(n,m)}$ is defined on these points and its graph is represented by the little circles on the graph of f_n.
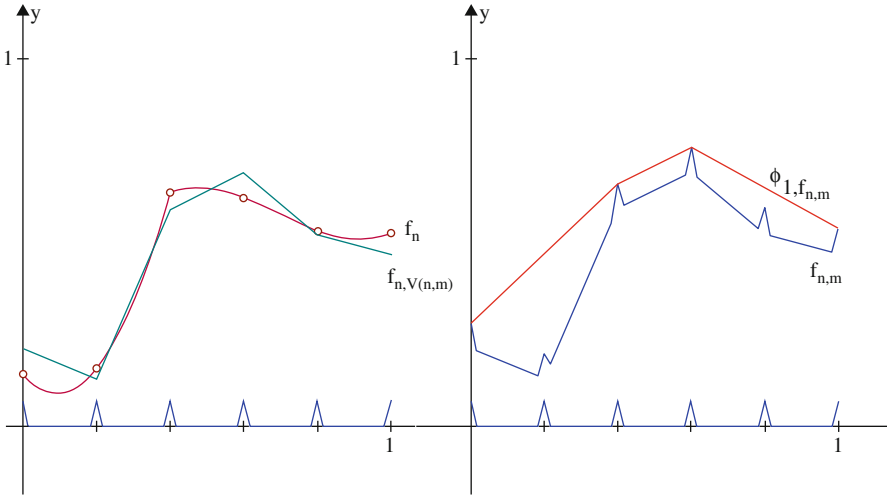
Fig. 2 The functions $f_n, f_{n,V(n,m)}, f_{n,m}$, and $\phi_{1,f_{n,m}}$

Now we perturb $\widetilde{f}_{n,V(n,m)}$ a bit in order to obtain an "independent" function $\bar{f}_{n,V(n,m)}$ such that

$$|\bar{f}_{n,V(n,m)} - \widetilde{f}_{n,V(n,m)}| < \frac{1}{16(n+m)} \tag{26}$$

and if $\mathbf{x}_0, \ldots, \mathbf{x}_k \in V(n,m)$, $\mathbf{x}_i \neq \mathbf{x}_j$ if $i \neq j$, and $(\mathbf{x}_j, \bar{f}_{n,V(n,m)}(\mathbf{x}_j)), j = 1, \ldots, k$ are on the same hyperplane in $\mathbb{R}^{d+1}$, then $k \leq d + 1$. This implies that the first assumption of Definition 2.1 is satisfied for $\bar{f}_{n,V(n,m)}$.

If $\mathbf{x} \in [0,1]^d \setminus V(n,m)$ and $\mathbf{x}$ is in the simplex $Z_j, j \in \{1, \ldots, \zeta_{n,m}\}$ with vertices $\mathbf{z}_{j,1}, \ldots, \mathbf{z}_{j,d+1} \in V(n,m)$ we define $f_{n,V(n,m)}(\mathbf{x})$ so that $(\mathbf{x}, f_{n,V(n,m)}(\mathbf{x}))$ is on the hyperplane determined by the points

$$(\mathbf{z}_{j,1}, \bar{f}_{n,V(n,m)}(\mathbf{z}_{j,1})), \ldots, (\mathbf{z}_{j,d+1}, \bar{f}_{n,V(n,m)}(\mathbf{z}_{j,d+1})).$$

Therefore, $f_{n,V(n,m)}$ is an independent piecewise linear function (recall Definition 2.1 and see the illustration on the left half of Fig. 2).

By (25) and (26)

$$|f_{n,V(n,m)}(\mathbf{x}) - f_n(\mathbf{x})| < \frac{1}{4(n+m)}. \tag{27}$$

Now we want to perturb the functions $f_{n,V(n,m)}$ a little further. Let $\kappa(\mathbf{x}) = \max\{1 - ||\mathbf{x}||, 0\}$ and for a $\gamma > 0$ put

$$\Gamma_{V(n,m),\gamma}(\mathbf{x}) = \chi_{[0,1]^d}(\mathbf{x}) \sum_{\mathbf{v} \in V(n,m)} \kappa\left(\frac{\mathbf{x} - \mathbf{v}}{\gamma}\right).$$

Then $\lim_{\gamma \to 0+} \Gamma_{V(n,m),\gamma}(\mathbf{x}) = \chi_{V(n,m)}(\mathbf{x})$.

We denote by $\nu(n,m)$ the minimum distance among points of $V(n,m)$ and will select a sufficiently small $\gamma_{n,m} > 0$ later. We put

$$f_{n,m} = f_{n,V(n,m)} + \frac{1}{4(n+m)} \Gamma_{V(n,m),\gamma_{n,m}}.$$

On both halves of Fig. 2 one can see $\frac{1}{4(n+m)} \Gamma_{V(n,m),\gamma_{n,m}}$ which is the function with the equally spaced small peaks at the points which are multiples of 0.2. On the right half of Fig. 2 one can see $f_{n,m}$ which is obtained from $f_{n,V(n,m)}$ (pictured on the left half of Fig. 2) after we added the small peaks.

We suppose that

$$\gamma_{n,m} < \frac{\nu(n,m)}{100} \tag{28}$$

and if $\mathfrak{G}(n,m)$ denotes the maximum of the norms of the gradients of the hyperplanes in the definition of $f_{n,V(n,m)}$, then

$$\frac{1}{\gamma_{n,m}} > 100 \cdot \mathfrak{G}(n,m). \tag{29}$$

This way if we take the convex hull of $f_{n,m}$, then

$$E_{1,f_{n,m}} = \{\mathbf{x} : f_{n,m}(\mathbf{x}) = \phi_{1,f_{n,m}}(\mathbf{x})\}$$

will be a subset of $V(n,m)$. See the right half of Fig. 2. We remark that the resolution of our drawings does not make it possible to take into all the above assumptions and hence they are distorted, but we hope that they can help to understand our procedure.

Our next aim is to select a sufficiently small $\delta_{n,m} > 0$. It is clear that given $r_{n,m} > 0$ if $\delta_{n,m}$ is sufficiently small, then f can coincide with $\phi_{1,f}$ only close to some points in $V(n,m)$, that is, for any $f \in B(f_{n,m}, \delta_{n,m})$ and any $\mathbf{x} \in E_{1,f}$ there is $\mathbf{w_x} \in V(n,m)$ such that

$$||\mathbf{w_x} - \mathbf{x}|| < r_{n,m}. \tag{30}$$

On the left half of Fig. 3 one can see $f_{n,m}$ and f. We also graphed the functions $\phi_{1,f_{n,m}}$ and $\phi_{1,f}$ which will be very close to each other. The latter function is not exactly piecewise linear but a close approximation to such a function, namely to $\phi_{1,f_{n,m}}$. In the one dimensional case, like in Fig. 3 the nonlinear parts (not pictured) are very close to some elements of $V(n,m)$. The higher dimensional case is a bit more complicated and we discuss it below.

We will select a sufficiently small $r_{n,m} > 0$ later. At this point we suppose that
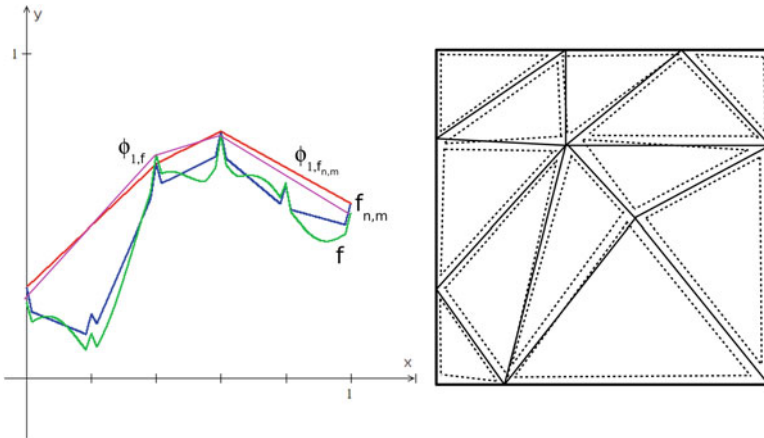
$$r_{n,m} < \frac{\nu(n,m)}{1000}. \tag{31}$$

Fig. 3 On the *left*: $f_{n,m}$ and f, on the *right*: $\mathscr{S}_\phi$ and $\Phi_{\phi,f}$ when $d = 2$

This implies that for any $\mathbf{x} \in E_{1,f}$ there is a unique $\mathbf{w_x} \in V(n,m)$. That is, f will "almost" look like a piecewise linear function. This implies that if $f \in B(f_{n,m}, \delta_{n,m})$, then

$$E_{1,f} \text{ can be covered by the set } \cup_{\mathbf{w} \in V(n,m)} B(\mathbf{w}, r_{n,m}). \tag{32}$$

We can suppose that $r_{n,m}$ is chosen so small, that

$$\#(V(n,m)) \cdot r_{n,m}^{1/m} < 1/m. \tag{33}$$

This estimate will imply that $\dim_H E_{1,f} = 0$ for the typical $f \in C[0,1]^d$, that is for $f \in \mathscr{G}_{2,1}$.

By the choice of $\gamma_{n,m}$ if we consider $\phi_{1,f_{n,m}}$, then it is an independent piecewise linear function. There is a system of non-overlapping simplices

$$\mathscr{S}_\phi = \{S_k : k = 1, \ldots, s_\phi\}$$

such that $(\mathbf{x}, \phi_{1,f_{n,m}}(\mathbf{x}))$ for any $\mathbf{x} \in [0,1]^d$ is on a hyperplane determined by a simplex S_k containing $\mathbf{x}$. On the left half of Fig. 3 the one dimensional case is illustrated and these simplices are simply the line segments $[0, 0.4]$, $[0.4, 0.6]$, and $[0.6, 1]$. The endpoints 0.4 and 0.6 are points where this function "breaks" and these points are on two non-parallel lines ("hyperplanes"). These breakpoints/folding regions will be used to find those points where the Hölder exponent is 1. On the right half of Fig. 3 the two dimensional case $d = 2$ is illustrated. This time we have simplices (triangles) in $[0,1]^d$ bounded by solid lines on which $\phi_{1,f_{n,m}}$ is linear. On the right half of Fig. 3 only the domain of $\phi_{1,f_{n,m}}$ is shown. The system of the simplices (triangles) bounded with dashed lines will be simplices corresponding to $\phi_{1,f}$. Later we will explain this in more detail.

By the independence property of $f_{n,V(n,m)}$ the hyperplanes determined by the simplices S_k are different for different S_k.

We denote by V_ϕ the set of the vertices of the simplices S_k, $k = 1, \ldots, s_\phi$. Clearly, $V_\phi \subset V(n,m)$. The union of the faces of these simplices will be denoted by $\Phi_\phi = \cup_{k=1}^{s_\phi} \partial(S_k)$. If $\mathbf{x_0} \in \partial(S_k) \cap \partial(S_{k'})$ with $k \neq k'$, then $\{(\mathbf{x}, \phi_{1,f_{n,m}}(\mathbf{x})) : \mathbf{x} \in S_k\}$ and $\{(\mathbf{x}, \phi_{1,f_{n,m}}(\mathbf{x})) : \mathbf{x} \in S_{k'}\}$ are on different hyperplanes and hence the graph of $\phi_{1,f_{n,m}}$ "breaks" at $\mathbf{x_0}$. This implies that we can choose $\tau_{1,n,m}$ such that for any $\mathbf{x}_\phi \in \Phi_\phi \cap (0,1)^d$ and for any hyperplane $L_\mathbf{x}$ passing through $(\mathbf{x}_\phi, \phi_{1,f_{n,m}}(\mathbf{x}_\phi))$ one can choose a point $\mathbf{x}'_\phi$ such that
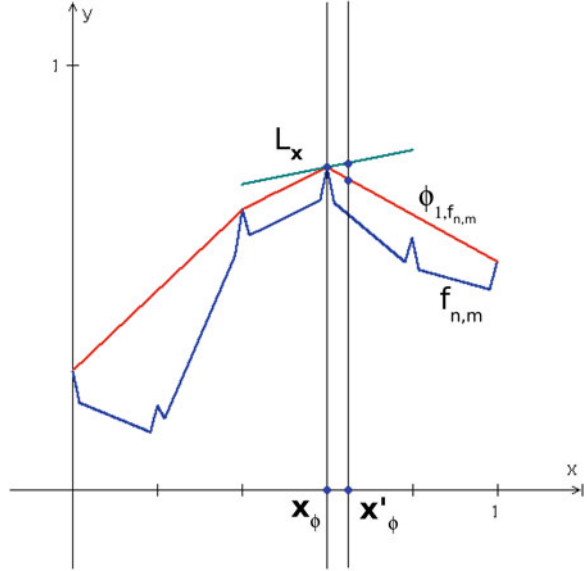
$$\tau_{1,n,m} \leq \text{dist}(\mathbf{x}'_\phi, \Phi_\phi) \leq \frac{1}{n+m}, \tag{34}$$

and

$$|L_\mathbf{x}(\mathbf{x}'_\phi) - \phi_{1,f_{n,m}}(\mathbf{x}'_\phi)| \geq ||\mathbf{x}_\phi - \mathbf{x}'_\phi||^{1+\frac{1}{m}} \geq \tau_{1,n,m}^{1+\frac{1}{m}}, \tag{35}$$

where we used that (34) implies $||\mathbf{x}'_\phi - \mathbf{x}_\phi|| > \tau_{1,n,m}$. See Fig. 4.

Fig. 4 The breaking point at $(\mathbf{x}_\phi, \phi_{1,f_{n,m}}(\mathbf{x}_\phi))$

It is also clear that if f is a good approximation of $f_{n,m}$, then one can see similar "breaking" properties on $\phi_{1,f}$. This time there are no "folding edges" like in the case of $f_{n,m}$ on $\Phi_\phi \cap (0,1)^d$ but there are regions around Φ_ϕ where we can see similar phenomena.

Using that $f_{n,m}$ and $\phi_{1,f_{n,m}}$ are both independent piecewise linear functions one can see that

$$\Delta_{1,n,m}(\delta) = \sup\{|\phi_{1,f_{n,m}}(\mathbf{x}) - \phi_{1,f}(\mathbf{x})| : \mathbf{x} \in [0,1]^d, f \in B(f_{n,m}, \delta)\} \to 0 \tag{36}$$

as $\delta \to 0+$. Apart from (31) and (33) we also assume that $r_{n,m} > 0$ is chosen so small that

$$r_{n,m} < \frac{\tau_{1,n,m}}{100} \tag{37}$$

and (using that $\dim_H \Phi_\phi = d - 1$)

$$B(\Phi_\phi, r_{n,m}) \text{ can be covered by balls } B_i \text{ such that} \tag{38}$$

$$|B_i| < \frac{1}{n+m}$$

and

$$\sum_i |B_i|^{(d-1)+\frac{1}{m}} < \frac{1}{m}. \tag{39}$$

Recall that we started to make assumptions about $\delta_{n,m}$ in the paragraph containing (30). The smaller $r_{n,m}$ we need to use the smaller $\delta_{n,m}$. Next we suppose that using (36) we chose a $\delta_{n,m}$ such that in addition to our other assumptions we have

$$\Delta_{1,n,m}(\delta_{n,m}) < \frac{1}{100}\tau_{1,n,m}^{1+\frac{1}{m}}. \tag{40}$$

Now we want to use the folding property in (34) and (35) for functions f which approximate $f_{n,m}$. This time the "folding edges" are not any more $(d-1)$-dimensional surfaces, but some neighborhoods of them. By (38) and (39) we will be able to bound the dimension of these regions.

Suppose $S_k \in \mathscr{S}_\phi$ with $k \in \{1,\ldots,s_\phi\}$ with vertices $\mathbf{z}_{k,1},\ldots,\mathbf{z}_{k,d+1}$. Since $f_{n,m}$ and $\phi_{1,f_{n,m}}$ are both independent piecewise linear functions there is $\delta_{\phi,k} > 0$ such that if $f \in B(f_{n,m}, \delta_{\phi,k})$, then one can choose vertices $\mathbf{z}_{k,j,f}, j = 1,\ldots,d+1$ such that

$$||\mathbf{z}_{k,j} - \mathbf{z}_{k,j,f}|| < r_{n,m} \text{ and } \mathbf{z}_{k,j,f} \in E_{1,f}, \, j = 1,\ldots,d+1, \tag{41}$$

moreover if $S_{k,f}$ denotes the simplex determined by $\{\mathbf{z}_{k,j,f} : j = 1,\ldots,d+1\}$, then $\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in S_{k,f}\}$ is on the surface of H_f inside a hyperplane determined by $\{(\mathbf{z}_{k,j,f}, f(\mathbf{z}_{k,j,f})) : k = 1,\ldots,d+1\}$, that is, $\{(\mathbf{x}, \phi_{1,f}(\mathbf{x})) : \mathbf{x} \in S_{k,f}\}$ is a "face" of $\phi_{1,f}$ approximating $\{(\mathbf{x}, \phi_{1,f_{n,m}}(\mathbf{x})) : \mathbf{x} \in S_k\}$. On the right half of Fig. 3 we have the two dimensional illustration. The simplices (triangles) $S_k \in \mathscr{S}_\phi$ are bounded by solid lines. The simplices (triangles) $S_{k,f}$ are bounded by dashed lines.

We can suppose that $\delta_{n,m} < \min\{\delta_{\phi,k} : k = 1,\ldots,s_\phi\}$ and by using independent piecewise linearity of $f_{n,m}$ and $\phi_{1,f_{n,m}}$ we obtain that the hyperplanes containing $\{(\mathbf{x}, \phi_{1,f}(\mathbf{x})) : \mathbf{x} \in S_{k,f}\}$ are different for different k's.

Hence the simplices $S_{k,f}$ are non-overlapping.

Put

$$\Phi_{\phi,f} = (0,1)^d \setminus \cup_{k=1}^{s_\phi}\text{int}(S_{k,f}). \tag{42}$$

These sets $\Phi_{\phi,f}$ will replace the folding edges $\Phi_\phi \cap (0,1)^d$. In Fig. 3 this is the region which is not covered by the interiors of the simplices (triangles) bounded by dashed lines.

From $||\mathbf{z}_{k,j} - \mathbf{z}_{k,j,f}|| < r_{n,m}$ in (41) it follows that any point $\mathbf{x}$ in S_k which is of distance no less than $r_{n,m}$ from $\partial(S_k)$ is covered by $S_{k,f}$.

Thus $\Phi_{\phi,f} \subset B(\Phi_\phi, r_{n,m})$ and hence by (38) and (39)

$$\Phi_{\phi,f} \text{ can be covered by balls } B_i \text{ such that} \tag{43}$$

$$|B_i| < \frac{1}{n+m} \text{ and } \sum_i |B_i|^{d-1+\frac{1}{m}} < \frac{1}{m}.$$

Using all the above restrictions we can select $\delta_{n,m} > 0$.
Set $\mathscr{G}_{2,1} = \cap_{m=1}^\infty \cup_{n=1}^\infty B(f_{n,m}, \delta_{n,m})$.
It is clear that $\mathscr{G}_{2,1}$ is a dense G_δ set in $C[0,1]^d$.

Suppose $f \in \mathscr{G}_{2,1}$. Then there exists a sequence n_m such that $f \in B(f_{n_m,m}, \delta_{n_m,m})$. For each m we can define the "folding region" as in (42). Since these regions depend on m we denote them by $\Phi_{\phi,f,m}$. Set $\Phi_f = \cap_{m=1}^{\infty} \Phi_{\phi,f,m}$. If $\mathbf{x} \in (0,1)^d \setminus \Phi_f$, then there exists an m such that $\mathbf{x} \in \mathrm{int}(S_{k,f,m})$ with a simplex $S_{k,f,m}$ and $\{(\mathbf{x}, \phi_{1f}(\mathbf{x})) : \mathbf{x} \in S_{k,f,m}\}$ is the subset of a hyperplane in $\mathbb{R}^{d+1}$. This implies that ϕ_{1f} is locally linear in a neighborhood of $\mathbf{x}$ and $h_{\phi_{1f}}(\mathbf{x}) = +\infty$.

Using (43) one can easily see that $\dim_H(\Phi_f) \leq d-1$. On the other hand, from (43) it also follows that if $S \subset (0,1)^d$ is a simplex such that its vertices are $\mathbf{z}_1, \ldots, \mathbf{z}_{d+1} \in E_{1,f}$, then there exists m_0 such that for $m \geq m_0$, $S \not\subset \Phi_{\phi,f,m}$. Since S is a "face" of ϕ_{1f} if $\mathbf{x} \in \partial(S)$, then $\mathbf{x}$ cannot belong to the interior of any other "face" of ϕ_{1f}. Hence $\partial(S) \subset \Phi_f$. Since $\dim_H \partial(S) = d-1$ we obtain that $\dim_H(\Phi_f) = d-1$. If $\mathbf{x} \in \Phi_f$, then (34) and (35) imply that $h_f(\mathbf{x}) \leq 1$.

The property $\dim_H E_{1,f} = 0$ follows from (32) and (33).

Proof (Proof of Theorem 1.3). We can take $\mathscr{G}_0$ from (8) in Remark 2.5 and for any $f \in \mathscr{G}_0$ we have (3)–(5) satisfied.

By Lemma 2.6 there exists a dense open set $\mathscr{G}_1 \subset C[0,1]^d$ such that for any $f \in \mathscr{G}_1$ the functions ϕ_{1f} and ϕ_{2f} are continuously differentiable on $(0,1)^d$.

There is nothing special about ϕ_{1f} in Lemma 2.8. A similar lemma can provide a dense G_δ set $\mathscr{G}_{2,2}$ such that for any $f \in \mathscr{G}_{2,2}$ we have (24) for $E_{2,f}$ and ϕ_{2f}.

If we take $\mathscr{G} = \mathscr{G}_0 \cap \mathscr{G}_1 \cap \mathscr{G}_{2,1} \cap \mathscr{G}_{2,2}$, then taking into consideration Remark 2.7 as well any $f \in \mathscr{G}$ satisfies the conclusions of Theorem 1.3. $\qquad \blacksquare$

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