(S)PDE on Fractals and Gaussian Noise

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Abstract In the first part of this paper we give a survey on results from Hinz and Zähle (Potential Anal 36:483–515, 2012) and Issoglio and Zähle (Stoch PDE Anal Comput 3:372–389, 2015) for nonlinear parabolic (S)PDE on certain metric measure spaces of spectral dimensions less than 4 with applications to fractals. We consider existence, uniqueness, and fractional regularity properties of mild function solutions in the pathwise sense. In the second part we apply this to the special case of fractal Laplace operators as generators and Gaussian random noises.

Furthermore, we show that random space-time fields $Y(t,x)$ like fractional Brownian sheets with Hurst exponents H in time and K in space on general Ahlfors regular compact metric measure spaces X possess a modification whose sample paths are elements of $C^{\alpha}([0,t_0],C^{\beta}(X))$ for all $\alpha < H$ and $\beta < K$. This is used in the above special case of SPDE on fractals.

1 Introduction

Deterministic elliptic and parabolic PDE without noises on classes of fractals and more general metric measure spaces have been studied, e.g., in [1, 6–8, 11, 25].

In the present paper we give a survey on some of the results obtained together with Hinz [16] and Issoglio [19] and further complementary material concerning the following parabolic nonlinear Cauchy problem on a *locally compact separable metric measure space* (X,d,μ), where μ is a *Radon measure*.

$$\frac{\partial u}{\partial t} = -Au + F(u) + G(u)\cdot \dot{z}, \quad t \in (0,t_0], \quad u(0) = f\,. \tag{1}$$

Here $t_0 > 0$ is arbitrary, $-A$ is the generator of a *Markovian strongly continuous symmetric semigroup* $\{T(t), t \geq 0\}$ on $L_2(\mu)$ admitting the *heat kernel estimate* HKE(β), and F and G are sufficiently regular functions. The noise term $\dot{z}$ denotes a fractional space-time perturbation which will be made more precise later on. In the

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Euclidean case it can be interpreted as a formal time derivative of a time dependent spatial distribution, where the latter may be defined by a distributional gradient of a non-differentiable function via Fourier analysis, cf. [15]. Solutions to (1) are considered in the mild form, formally given by

$$u(t) = T(t)f + \int_0^t T(t-s)F(u(s))ds + \int_0^t T(t-s)G(u(s))dz(s). \tag{2}$$

In [16] the second integral is determined by means of fractional time derivatives and pointwise products of functions and "distributions". The spaces used to describe the regularity of the solution to (2) are fractional Sobolev spaces defined on metric measure spaces using the associated semigroup. The spatial distributions $z(s)$ are introduced as elements of appropriate dual spaces. In [19] the time regularity of the solution is also expressed in terms of Hölder exponents.

These notions and results are summarized in Sects. 2 and 3. For the proofs we refer to [16] and [19].

In Sect. 4 this is applied pathwise to SPDE with Gaussian noise, i.e.,

$$z(t) = (A + \mathrm{Id})^{\sigma} Y(t, \cdot)\,,$$

for certain exponents $\sigma > 0$, where $Y(t,x)$ is a real valued centered Gaussian random field in time and space with certain covariance structure. In order to check the conditions on z from the previous sections the existence of strong Hölder continuous modifications for such Gaussian random fields and an embedding result for the corresponding function spaces are used.

References to related literature for the Euclidean case may be found in [16] and [19].

Then we consider the special case of semigroups determined by local regular Dirichlet forms associated with Neumann Laplace operators $\Delta = -A$ on compact fractal spaces. Here such Gaussian fields are constructed by means of series expansion with the usual methods of spectral analysis. Examples are p.c.f. self-similar sets, generalized Sierpinski carpets, or certain products of fractals.

The whole approach is low dimensional and works only for spectral dimension less than 4.

As an auxiliary tool of independent interest we show in Sect. 5 the following extension of a classical result. Let $Y(t,x)$ be a centered Gaussian random space-time field on a general Ahlfors regular compact metric measure space (X, d, μ). If its mean quadratic increments satisfy upper estimates like for Euclidean fractional Brownian space-time sheets with Hurst exponents H in time and K in space, then Y has a modification $\widetilde{Y}$ such that a.s. $\widetilde{Y} \in C^{\sigma}([0,t_0], C^{\tau}(X))$ for all $\sigma < H$ and $\tau < K$.

Here and in the sequel the letter c *denotes a general finite constant* which might change from step to step.

2 Semigroups, Fractional Sobolev Spaces, and Their Dual Spaces

We now recall some background from the literature and related results which are used or shown in [16] and [19].

In the case of metric measure spaces the analogues of the classical *fractional Sobolev (or Bessel potential) spaces* in the literature are introduced by means of the given semigroup $\{T(t), t \geq 0\}$, i.e., of its generator $-A$:

The *generalized Bessel potential operator* on $L_2(\mu)$ is defined for $\sigma \geq 0$ as

$$J^\sigma(\mu) := (A + \mathrm{Id})^{-\sigma/2}.$$

To each operator there corresponds a *potential space* defined as

$$H^\sigma(\mu) := J^\sigma(L_2(\mu))$$

and equipped with the norm $\|u\|_{H^\sigma(\mu)} := \|u\|_{L_2(\mu)} + \|A^{\sigma/2}u\|_{L_2(\mu)}$, which is equivalent to $\|(A + \mathrm{Id})^{\sigma/2}u\|_{L_2(\mu)}$. In fact these spaces correspond to the domains of fractional powers of A, i.e., $D((A + \mathrm{Id})^{\sigma/2}) = D(A^{\sigma/2}) = H^\sigma(\mu)$. In particular, for any $\alpha \geq 0$ the operator J^α acts as an isomorphisms between $H^\sigma(\mu)$ and $H^{\alpha+\sigma}(\mu)$. Analogously one can define the potential spaces corresponding to the generators $-A_p, 1 < p < \infty$, of Markovian semigroups on $L_p(\mu)$. They are denoted by $H_p^\sigma(\mu)$ and clearly $H_2^\sigma(\mu) = H^\sigma(\mu)$. We also consider the spaces

$$H_\infty^\sigma(\mu) := H^\sigma(\mu) \cap L_\infty(\mu)$$

normed by $\|\cdot\|_{H^\sigma(\mu)} + \|\cdot\|_\infty$, with slight abuse of notation. Here the norm $\|\cdot\|_\infty$ in $L_\infty(\mu)$ is given by the essential supremum.

Furthermore, the dual spaces of $H_p^\sigma(\mu)$ are used in the sequel: for $1 < p < \infty$, $\sigma \geq 0$ they are denoted by

$$H_q^{-\sigma}(\mu) := \left(H_p^\sigma(\mu)\right)^*,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In case $p = 2$ we do not write p explicitly.

Remark 2.1 The operators $(A + \mathrm{Id})^{\sigma/2}$ can be extended to dual spaces as follows. For $0 < \rho < \sigma < \infty$ the bijective linear mappings

$$(A + \mathrm{Id})^{\sigma/2} : H_q^\rho(\mu) \to H_q^{\rho-\sigma}(\mu) \tag{3}$$

are well defined by the dual pairing

$$\left\langle g, (A + \mathrm{Id})^{\sigma/2} f\right\rangle := \left\langle (A + \mathrm{Id})^{(\sigma-\rho)/2} g, (A + \mathrm{Id})^{\rho/2} f\right\rangle,$$

where $g \in H_p^{\sigma-\rho}(\mu)$ and $f \in H_q^{\rho}(\mu)$ with p, q as before. Then there are unique extensions of the above operators acting as in (3) such that

$$(A + \mathrm{Id})^{\sigma} \circ (A + \mathrm{Id})^{\tau} = (A + \mathrm{Id})^{\sigma+\tau}$$

for all $\rho, \sigma, \tau \in \mathbb{R}$ and $q > 1$.
For the regularity in time of the solution to (2) we consider the following spaces frequently used in the literature: Let $0 < \eta < 1$ and $(B, \|\cdot\|_B)$ be a normed space. Then $W^{\eta}([0, t_0], Y)$ denotes the space of mappings $v : [0, t_0] \to B$ such that $\|v\|_{\eta,B} < \infty$, where

$$\|v\|_{\eta,B} := \sup_{0 \le t \le t_0} \left(\|v(t)\|_B + \int_0^t \frac{\|v(t) - v(s)\|_B}{(t-s)^{\eta+1}} ds \right)$$

is the norm in $W^{\eta}([0, t_0], B)$.

Furthermore, $C^{\eta}([0, t_0], B)$ is the corresponding space of *Hölder continuous mappings* of order η with the usual norm. In the sequel B is a fractional Sobolev space or a Hölder space on X.

Throughout the paper we use the *short notations* for the following norms:

$$\|\cdot\|_{\alpha,\infty} := \|\cdot\|_{H^{\alpha}_{\infty}(\mu)} \quad \text{and} \quad \|\cdot\|_{\alpha} := \|\cdot\|_{H^{\alpha}(\mu)} \quad \text{for each } \alpha \in \mathbb{R}.$$

Then we recall that for $\nu \ge 0$ and $t > 0$ the operators $T(t)$ and A^{ν} commute on $D(A^{\nu})$ and satisfy the following well-known estimates (see, e.g., [22]) for $u, v \in D(A^{\nu})$:

$$\|T(t)v\|_{\nu} \le c t^{-\frac{\nu}{2}} \|v\|_0 , \tag{4}$$

and for $0 < \nu < 1$,

$$\|T(t)u - u\|_0 \le c t^{\nu} \|u\|_{2\nu} , \tag{5}$$

where $0 < t \le t_0$.

The symmetry of the semigroup $\{T(t), t \ge 0\}$ has been used in order to extend it to elements from the dual spaces. If $w \in H^{-\beta}(\mu)$, then $T(t)w$ is the element of $L_2(\mu)$ determined by the duality relation

$$\langle v, T(t)w \rangle := \langle T(t)v, w \rangle \ , \quad v \in L_2(\mu) .$$

Then we get

$$|\langle v, T(t)w \rangle| = |\langle T(t)v, w \rangle| \le \|T(t)v\|_{\beta} \|w\|_{-\beta}$$

and hence,

$$\|T(t)w\|_0 \le c t^{-\frac{\beta}{2}} \|w\|_{-\beta}$$

in view of (4). Applying the latter again and using $T(t) = T(\frac{t}{2}) \circ T(\frac{t}{2})$ we infer

$$\|T(t)w\|_{\delta} \leq ct^{-\frac{\delta}{2}-\frac{\beta}{2}}\|w\|_{-\beta} \tag{6}$$

for any $\delta, \beta > 0$.

Similarly one obtains from (5)

$$\|T(t)w - w\|_{-\beta-2\nu} \leq ct^{\nu}\|w\|_{-\beta} , \tag{7}$$

for any $\beta > 0$, $w \in H^{-\beta}(\mu)$ and $0 < \nu < 1$.

Note that the constants in the estimates depend on the associated parameters ν, β, and δ.

The main results in [16] and [19] are derived under the following additional condition:

Assumption (HKE(β)) The transition kernel $P_t(x, dy)$ associated with the semigroup $T(t), t \geq 0$ admits a transition density $P_t(x, dy) = p(t, x, y)\mu(dy)$ which satisfies for almost all $x, y \in X$ the following *heat kernel estimate*:

$$t^{-\frac{d_H}{w}}\Phi_1(t^{-\frac{1}{w}}d(x, y)) \leq p(t, x, y) \leq t^{-\frac{d_H}{w}}\Phi_2(t^{-\frac{1}{w}}d(x, y))$$

if $0 < t < R_0$ for some constants $R_0 > 0$, $w \geq 2$ and nonnegative bounded decreasing functions Φ_i on $[0, \infty)$, where d_H is the *Hausdorff dimension* of (X, d) and w is called the *walk dimension* of the semigroup. Moreover, for a given $\beta > 0$,[1]

$$\int_0^{\infty} s^{d_H+\beta w/2-1}\Phi_2(s)ds < \infty .$$

For $t \geq R_0$, $p(t, x, y) \leq p_t$ and p_t decreases in t.

Remark 2.2 In this case the semigroup is *ultracontractive*, i.e.,

$$\|T(t)\|_{L_\infty(\mu)} \leq p_t\|f\|_{L_2(\mu)} ,$$

where $p_t := c\,t^{-d_S/4}$, if $t < R_0$, and the value $d_S = \frac{2d_H}{w}$ agrees with its *spectral dimension*. (For this the integrability condition is not needed.)

Heat kernels of this type have been studied in Grigor'yan et al. [11], Grigor'yan and Kumagai [10], and related references therein. Further relationships are presented in the recent survey [12] of Grigor'yan, Hu, and Lau. In particular, the heat kernel estimates HKE(β) imply that the measure μ is Ahlfors regular of order d_H, i.e.,

$$c^{-1}r^{d_H} \leq \mu(B(x, r)) \leq cr^{d_H} \tag{8}$$

for all $x \in X$ and $0 < r < R_0$, where $B(x, r) := \{y \in X : d(x, y) \leq r\}$ (cf. [11, 18]).

[1] In [16] the w is missing in the exponent.

For smooth domains in Euclidean spaces and various classes of fractal spaces (X, d, μ) the following special case has been investigated: The semigroup $\{T(t), t \geq 0\}$ is generated by a (fractal) Laplacian Δ associated with a local regular Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on X, i.e., $-A = \Delta$. Moreover, for those classes the corresponding heat kernels exist and satisfy Assumption (HKE(β)) for all $\beta > 0$ (see, e.g., Barlow and Bass [2] and [3], Barlow et al. [4], Fitzsimmons et al. [9], Hambly and Kumagai [13], Kigami [21], Barlow et al. [5], and the references therein).

In order to make the integral in (2) precise we need *pointwise products* of functions and dual elements from the potential spaces. In [16] the following is proved which also extends related results for the Euclidean case.

Proposition 2.3 *[16, Corollary 4.1] Suppose* (HKE(β)) *for* $0 < \beta < \delta < \min(\frac{d_S}{2}, 1)$. *Then for* $q = \frac{d_S}{\delta}$ *the product gh of* $g \in H^{\delta}(\mu)$ *and* $h \in H_q^{-\beta}(\mu)$ *is well defined in* $H^{-\beta}(\mu)$ *by the duality relation* $\langle f, gh\rangle := \langle fg, h\rangle$, $f \in H^{\delta}(\mu)$, *and the following estimate holds true:*

$$\|gh\|_{-\beta} \leq c\|g\|_{\delta}\|h\|_{H_q^{-\beta}(\mu)} .$$

Furthermore, for applications to the random case the following embedding relationship is useful. Recall that the semigroup $T(t)$ is called *conservative* if $T(t)\mathbf{1} = \mathbf{1}$ for any t.

Proposition 2.4 *Suppose that the semigroup is conservative and the underlying metric measure space* (X, d, μ) *is compact. Then for* $0 < \tau' < \tau$ *the Hölder space* $C^{\tau}(X)$ *is embedded into* $H_q^{\tau'2/w}(\mu)$ *for any q, if the semigroup satisfies the upper estimates in* (HKE(β)) *for* $\beta = \tau'2/w$.

This can be seen, e.g., from the arguments around Proposition 5.6 in [18] concerning the upper estimate of the inverse of the Bessel potential operator. (See also the proof of [16, Proposition 4.1].)

3 The Integral Equation and Mild Solution

A rigorous definition for the second integral and a contraction principle for the solution to Eq. (2) are given in [16] by means of fractional calculus in Banach spaces, in particular under the following additional conditions.

Assumption (FG) The nonlinear functions F and G are such that $F \in C^1(\mathbb{R}^n)$, $F(0) = 0$ and F has bounded Lipschitz derivative F' and $G \in C^2(\mathbb{R}^n)$, $G(0) = 0$ and G has bounded Lipschitz second derivative G''.

For the parameters we here consider the case II from [16].

Assumption (P) $0 < \alpha < \gamma, 0 < \beta < \delta < \min(\frac{d_S}{2}, 1), \gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$, where β and d_S are from (HKE(β)), and $q = \frac{d_S}{\delta}$.

We now will briefly summarize the construction.

If $u \in W^\gamma([0,t_0], H^\delta_\infty(\mu))$ the operator $U(t;s) : H_q^{-\beta}(\mu) \to H^\delta_\infty(\mu)$ is defined as

$$U(t;s)w := T(t-s)\,(G(u(s))w) \tag{9}$$

for $w \in H_q^{-\beta}(\mu)$. Here Proposition 2.3 is used for the products $G(u(s))w$. Then under the above assumptions on the function G and the parameters (P) for any $0 < \eta < \gamma$ the *left-sided Weyl–Marchaud fractional derivative* of order η is determined by

$$D^\eta_{0+}U(t;s) := \frac{\mathbf{1}_{(0,t)}(s)}{\Gamma(1-\eta)}\left(\frac{U(t;s)}{s^\eta} + \eta\int_0^s \frac{U(t;s)-U(t;\tau)}{(s-\tau)^{\eta+1}}d\tau\right)$$

as an element of $L_1([0,t], L(H_q^{-\beta}(\mu), H^\delta(\mu))$ (in the sense of Bochner integration). This is shown in [16, Lemma 5.2, (ii)].[2]

Let us now consider the regulated version of $z \in C^{1-\alpha}([0,t_0], H_q^{-\beta})$ on $[0,t]$ given by $z_t(s) := \mathbb{1}_{(0,t)}(s)(z(s)-z(t))$. If additionally $1-\eta < 1-\alpha$, which is always possible in view of (P), one can define the *right-sided Weyl–Marchaud fractional derivative* of z_t of order $1-\eta$ by

$$D^{1-\eta}_{t-}z_t(s) := \frac{(-1)^{1-\eta}\mathbf{1}_{(0,t)}(s)}{\Gamma(\eta)}\left(\frac{z(s)-z(t)}{(t-s)^{1-\eta}} + (1-\eta)\int_s^t \frac{z(s)-z(\tau)}{(\tau-s)^{(1-\eta)+1}}d\tau\right)$$

as an element of $L_\infty([0,t], H_q^{-\beta}(\mu))$.

For more details on these fractional derivatives we refer the reader to Samko et al. [23]. The corresponding generalized Lebesgue–Stieltjes integral was introduced in [26, 27], see [14] for the Banach space version. (It coincides with the Young integral and other forward integrals on the joint domains of definition.) This type of integral is used for the term concerning the noise in the integral equation (2) for the mild solution:

Proposition 3.1 *[16, Lemma 5.1] Suppose* (HKE(β)), (FG)*, the parameter conditions* (P) *and* $z \in C^{1-\alpha}([0,t_0], H_q^{-\beta}(\mu))$*. Then for the operator* $U(t;s) = T(t-s)(G(u(s)\cdot)$ *as in (9) with* $t \in [0,t_0]$ *and* $u \in W^\gamma([0,t_0], H^\delta_\infty(\mu))$ *the integral* $\int_0^t U(t;s)dz(s)$ *is well defined by*

$$\int_0^t U(t;s)dz(s) := (-1)^\eta \int_0^t D^\eta_{0+}U(t;s)D^{1-\eta}_{t-}z_t(s)ds, \tag{10}$$

independently of the choice of η *with* $\eta < \gamma$ *and* $1-\eta < 1-\alpha$*. (In particular, the integrand on the right side is a Lebesgue integrable real function.)*

[2] We remark that there is a typo in [16, Lemma 5.2], namely in (ii) and (iii) the right-hand side of the main condition on the parameters should read $2-2\eta-\beta$ instead of $2-2\eta-(\beta \vee \frac{d_S}{2})$.

(Note that the complex fractional powers of -1 are cancelled by those of the right sided fractional derivatives. They guarantee an integration-by-part rule including the marginal case $\eta = 1$ in the limit.)

We now can formulate the main results from [16] and [19] concerning *existence, uniqueness, and Hölder regularity of the mild solution* to (1).

Theorem 3.2 *Suppose* (HKE(β)), (FG), *and* (P), *i.e.,* $0 < \alpha < \gamma$, $0 < \beta < \delta < \min(\frac{d_S}{2}, 1)$, $\gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$, *and* $q = \frac{d_S}{\delta}$. *If* $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ *and the initial condition* f *is an element of* $H_\infty^{\delta+2\gamma+\varepsilon}(\mu)$ *for some* $\varepsilon > 0$, *then we have the following.*

(a) [16, Theorem 1.2] There exists a unique solution u *to Eq. (2) for the definition (10) of the integral such that* $u \in W^\gamma([0, t_0], H_\infty^\delta(\mu))$.
(b) [19, Theorem 1.2] The unique solution $u \in W^\gamma([0, t_0], H_\infty^\delta(\mu))$ *is also an element of* $C^\gamma([0, t_0], H^\delta(\mu))$.

With slightly more restrictive assumptions on the noise it follows that the unique solution u belongs to the spaces $W^\gamma([0, t_0], H_\infty^\delta(\mu))$, and thus to $C^\gamma([0, t_0], H^\delta(\mu))$, for all (γ, δ) such that $\alpha < \gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$ and $\beta < \delta < \min(\frac{d_S}{2}, 1)$.

Corollary 3.3 ([19, Corollary 3.6]) *Suppose* (HKE(β)), (FG) *and that the given parameters* α, β *satisfy* $0 < \alpha < \frac{1}{2}$ *and* $0 < \beta < \min(\frac{d_S}{2}, 1 - 2\alpha, 2(1-\alpha) - \frac{d_S}{2})$. *Suppose that* $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ *for any* $1 < q < \frac{d_S}{\beta}$ *and* $f \in H_\infty^{2(1-\alpha)-\beta}(\mu)$. *Then for any* $\beta < \delta < \min(\frac{d_S}{2}, 1)$ *and* $\alpha < \gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$ *Eq. (2) has a unique solution in the space* $W^\gamma([0, t_0], H_\infty^\delta(\mu))$ *and hence, it has a unique solution belonging to these spaces for all such* γ, δ.

Moreover, this solution is an element of $C^\gamma([0, t_0], H^\delta(\mu))$ *for all* γ *and* δ *as before.*

This will be applied in the next section to the random case with Gaussian noise. Some extensions are discussed in [19].

4 Application to an SPDE with Gaussian Noise

The above approach can be applied stochastic models where the noise $\dot{z}$ is random and satisfies the previous conditions with probability 1. Then the integral equation for the mild solution is understood in the pathwise sense.

As an example we consider the following. Suppose additionally that the metric space (X, d) is compact and let $Y(t, x)$ be a real valued Gaussian random field on $[0, t_0] \times X$ with mean zero such that

$$\mathbb{E}(Y(s,x) - Y(t,x))^2 \le c\,|s-t|^{2H} \tag{11}$$

$$\mathbb{E}(Y(t,x) - Y(t,y))^2 \le c\,d(x,y)^{2K} \tag{12}$$

$$\mathbb{E}\big(Y(s,x) - Y(t,y) - (Y(s,x) - Y(t,y))\big)^2 \leq c\,|s-s|^{2H} d(x,y)^{2K} \tag{13}$$

for all $s, t \in [0, t_0]$ and $x, y \in X$.

(In the Euclidean case the equalities hold for the corresponding fractional Brownian space-time sheet.) Note that (12) is a special case of (13) if $Y(0,x) = 0$ a.s..

Theorem 4.1 *Suppose that the metric space (X, d) is compact and the semigroup $T(t)$ is conservative. If the Gaussian space-time field Y fulfills the above conditions, then it has a modification $\widetilde{Y}$ such that a.s.*

$$\widetilde{Y} \in C^{\sigma}([0, t_0], H_q^{\tau\, 2/w}(\mu))$$

for all $q > 1$, $0 < \sigma < H$ and $0 < \tau < K$ provided that the corresponding upper heat kernel estimates (HKE($\tau\, 2/w$)) *are satisfied. Furthermore, for fixed $\beta^* > 0$ set*

$$z(t) := (A + \mathrm{Id})^{(\beta^* - K2/w)/2}\widetilde{Y}(t, \cdot)\,. \tag{14}$$

Then we get a.s. that

$$z \in C^{\sigma}([0, t_0], H_q^{-\beta}(\mu))$$

for all $\sigma < H$, $\beta > \beta^*$ and $q > 1$.
In this way we can obtain a random noise z satisfying the conditions of Corollary 3.3.

Proof Below we will show that under the conditions (11)–(13) the random field Y has a modification $\widetilde{Y}$ such that a.s.

$$\widetilde{Y} \in C^{\sigma}([0, t_0], C^{\tau}(X))$$

for all $\sigma < H$ and $\tau < K$, see Theorem 5.2. Therefore the embedding property from Proposition 2.4 implies the first statement. From this one infers the second assertion by means of Remark 2.1
In order to construct an example of such a Gaussian field Y on an appropriate space we now consider the situation mentioned in Sect. 2. Let (X, d, μ) be a compact metric measure space admitting a conservative semigroup $\{T(t), t \geq 0\}$ generated by a (fractal) *Neumann Laplacian* Δ associated with a local regular Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on X, i.e., $-A = \Delta$. Equation (1) reads then

$$\frac{\partial u}{\partial t} = \Delta u + F(u) + G(u) \cdot \dot{z}, \quad t \in (0, t_0], \quad u(0) = f\,. \tag{15}$$

Recall that for various classes of fractals the corresponding Neumann heat kernels exist and satisfy Assumption (HKE(β)) for any $\beta > 0$ (see, e.g., [2–5, 9, 13, 21] and the references therein). Moreover, for these cases there exists a complete orthonormal system $e_0, e_1, e_2, \ldots$ of eigenfunctions of $-\Delta$ in $L_2(\mu)$ with the corresponding eigenvalues λ_i, where $\lambda_0 = 0$ and $\lim_{n\to\infty} \lambda_n = \infty$. An equivalent condition in the general case is that the operator A has a compact resolvent. We additionally suppose that we have, up to an exceptional set,

$$|e_i(x) - e_i(y)| \le c\lambda_i^{a/2} d(x,y)^K \tag{16}$$

for some positive constants a and K.

If we work with the resistance distance $R(x, y)$ w.r.t. the Dirichlet form $\mathscr{E}$, then this estimate holds true for $a = K = 1$. Under some mild additional assumptions on such fractals in Euclidean spaces the resistance metric R satisfies $R(x, y) \le c|x-y|^K$ for some $K > 0$, see Hu and Wang [17]. Hence, in this case (16) is also fulfilled for the Euclidean metric.

Then a standard example for the auxiliary Gaussian field Y is the following:

Corollary 4.2 *Let the metric measure space and the semigroup be as above with eigenfunctions e_i and eigenvalues λ_i of the generator $-\Delta$ satisfying (16). $B_1^H(t), B_2^H(t), \ldots$ are i.i.d. fractional Brownian motions in $\mathbb{R}$ with Hurst exponent $\frac{1}{2} < H < 1$, $b := \max(a, d_S/2)$, and q_i are real coefficients such that*

$$\sum_{i=1}^{\infty} q_i^2 \, \lambda_i^b < \infty \,. \tag{17}$$

Then we have the following:

(i) The Gaussian random field

$$Y(t,x) := \sum_{i=1}^{\infty} B_i^H(t)\, q_i\, e_i$$

determined by convergence in the mean squared satisfies conditions (11)–(13).

The corresponding noise z in (14) is a stochastic modification of the series $\sum_{i=1}^{\infty} B_i^H(t)\,(1+\lambda_i)^{(\beta^*-K2/w)/2} q_i\, e_i$.

(ii) If $0 < \beta^* < \min\left(\frac{d_S}{2}, 2H-1, 2H-\frac{d_S}{2}\right)$, then for any $1-H < \gamma < H - \frac{\beta^*}{2} - \frac{d_S}{4}$ and $\beta^* < \delta < \min\left(\frac{d_S}{2}, 1\right)$ Eq. (15) with initial condition $f \in H_\infty^{2H-\beta^*}(\mu)$ has a pathwise unique solution in $W^\gamma([0,t_0], H_\infty^\delta(\mu))$. It is also an element of $C^\gamma([0,t_0], H^\delta(\mu))$. Consequently, the unique solution belongs to all these spaces with parameters γ and δ satisfying the above inequalities.

Proof It suffices to show (i), since (ii) follows then from Corollary 3.3.

We first consider the increments in time.

$$\mathbb{E}(Y(s,x) - Y(t,x))^2$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} q_i^2 \mathbb{E}(B_i^H(s) - B_i^H(t))^2 |e_i(x)|^2 \\
&= \sum_{i=0}^{\infty} q_i^2 |s-t|^{2H} |e_i(x)|^2 \\
&\leq \left(q_0^2 \|e_0\|_\infty + \sum_{i=1}^{\infty} q_i^2 c \lambda_i^{d_S/2} \right) |s-t|^{2H} \\
&= c\,|s-t|^{2H}
\end{aligned}
$$

which yields (11). (Recall that c denotes a varying constant.) Here we have used that the above ultracontractivity of the semigroup [cf. (HKE(β))] implies $\|e_i\|_\infty \leq c\lambda_i^{d_S/4}$ for $i \geq 1$ and then the convergence of the series (17).

Note that convergence of the series defining Y may be considered as a special case setting $s = 0$, since $B^H(0) = 0$ a.s..

Furthermore,

$$
\begin{aligned}
&\mathbb{E}\big(Y(s,x) - Y(s,y) - (Y(t,x) - Y(t,y))\big)^2 \\
&\quad = \sum_{i=0}^{\infty} q_i^2 \mathbb{E}(B_i^H(s) - B_i^H(t))^2 (e_i(x) - e_i(y))^2 \\
&\quad = \sum_{i=0}^{\infty} q_i^2 |s-t|^{2H} (e_i(x) - e_i(y))^2 \\
&\quad \leq \sum_{i=0}^{\infty} q_i^2 |s-t|^{2H} c \lambda_i^a \, d(x,y)^{2K} \\
&\quad \leq c\,|s-t|^{2H} d(x,y)^{2K}
\end{aligned}
$$

in view of (16) and (17). Hence, (13) is fulfilled. Equation (12) is here the special case where $s = 0$.
For p.c.f. fractals with regular harmonic structures we have $d_S = \frac{2d_H}{d_H+1} < 2$, see Kigami [20]. Examples with spectral dimension greater than 2 are provided by generalized Sierpinski carpets, see Barlow and Bass [3], or by certain products of fractals, see Strichartz [24, 25].

According to Corollary 4.2 function solutions to Eq. (15) which are Hölder regular in time can be found for $d_S = \frac{2d_H}{w} < 4$, $H > \frac{1}{2} + \frac{\beta^*}{2} + \frac{d_S}{8}$ and $0 < \beta^* < \frac{d_S}{2}$. (Recall that d_H denotes the Hausdorff dimension of X and w the walk dimension of the semigroup.)

5 Modifications of Gaussian Space-Time Fields on Ahlfors Regular Metric Measure Spaces

In the classical setting of Gaussian fields in Euclidean spaces there are several methods for obtaining Hölder regular modifications. In this paper we extend the approach via the Borel–Cantelli lemma going back to Kolmogorov and Chentsov to space-time fields on more general metric spaces. Recall the following notion.

Definition 5.1 For $D > 0$ a compact metric measure space (X, d, μ) is called (Ahlfors) regular of order D if

$$c^{-1}r^D < \mu(B(x,r)) < c\, r^D\ , \quad x \in X\ , \ r \leq \operatorname{diam} X\ .$$

It is well known that in this case the number D agrees with the Hausdorff dimension d_H of X. In view of (8) the compact spaces considered in the previous sections are Ahlfors regular.

We show now that in such spaces Gaussian random fields with properties like fractional Brownian space-time sheets possess Hölder continuous modifications in the following strong sense.

Theorem 5.2 *Let (X, d, μ) be an Ahlfors D-regular compact metric measure space and Y a Gaussian field on $[0, t_0] \times X$ with mean zero and*

(a) $\mathbb{E}(Y(s,x) - Y(t,x))^2 \leq c\,|s-t|^{2H}$
(b) $\mathbb{E}(Y(t,x) - Y(t,y))^2 \leq c\, d(x,y)^{2K}$
(c) $\mathbb{E}\big(Y(s,x) - Y(s,y) - (Y(t,x) - Y(t,y))\big)^2 \leq c\,|s-t|^{2H} d(x,y)^{2K}$

for some $0 < H < 1$, $K > 0$ and all $s, t \in [0, t_0]$, $x, y \in X$.

Then Y has a modification $\widetilde{Y}$ such that a.s. $\widetilde{Y} \in C^\alpha([0,t_0], C^\beta(X))$ for all $\alpha < H$ and $\beta < K$.

For brevity we write $C^{\alpha,\beta} := C^\alpha([0,t_0], C^\beta(X))$ in the sequel.

An auxiliary tool for proving this theorem is the sufficient part of the following criterion.

Proposition 5.3 *Let (X,d) be a separable metric space. Then a random space-time field Y on $[0, t_0] \times X$ admits a $C^{\alpha,\beta}$-modification if and only if the following two conditions are fulfilled:*

1. *$Y(t,x)$ is stochastically continuous w.r.t. the product metric.*
2. *Y is in $C^{\alpha,\beta}$ restricted to a countable dense subset of $[0,t_0] \times X$ with probability 1.*

This can be shown in the same way as in the classical case of Hölder continuous modifications for real valued stochastic processes: One determines the new random field a.s. by continuous extension of the primary one from the countable dense subset from Condition 2 to the whole space. Then it easily follows from stochastic continuity that this is a modification.

Proof (of Theorem 5.2) According to Proposition 5.3 it suffices to show that Conditions 1 and 2 are fulfilled in our case.

First recall that the N-th moment of a centered Gaussian random variable is equal to a multiple of the $\frac{N}{2}$-th power of the second moment. Therefore the Chebyshev inequality together with (a), (b), and (c) yields for any $\varepsilon > 0$ and $N \in \mathbb{N}$,

$$\mathbb{P}\left(\frac{|Y(s,x)-Y(t,x)|}{|s-t|^{\alpha}} > \varepsilon\right) \le c_N\, \varepsilon^{-N} |s-t|^{(H-\alpha)N} \tag{18}$$

$$\mathbb{P}\left(\frac{|Y(t,x)-Y(t,y)|}{d(x,y)^{\beta}} > \varepsilon\right) \le c_N\, \varepsilon^{-N} d(x,y)^{(K-\beta)N} \tag{19}$$

$$\mathbb{P}\left(\frac{|Y(s,x)-Y(s,y)-(Y(t,x)-Y(t,y))|}{|s-t|^{\alpha} d(x,y)^{\beta}} > \varepsilon\right) \le c_N\, \varepsilon^{-N} |s-t|^{(H-\alpha)N} d(x,y)^{(K-\beta)N} . \tag{20}$$

In particular, (20) implies that Y is stochastically continuous, i.e., Condition 1.

To construct a countable dense subset as in Condition 2 is more extensive. We do this by adapting the Euclidean set of dyadic rational numbers to our metric measure space.

Let D_1 be a set of centers of disjoint closed balls of radius $\frac{1}{2}$ which form an optimal packing of X, i.e., the number of such balls is maximal. Then we define inductively D_{n+1} to be a set of centers of disjoint closed balls of radius $\frac{1}{2^{n+1}}$ such that $D_n \subset D_{n+1}$ and the number of these balls is maximal. Since those numbers are maximal one infers the covering property

$$\bigcup_{x\in D_j} B\left(x, \frac{1}{2^j}\right) = X \quad \text{for any } j \in \mathbb{N} . \tag{21}$$

The classical family $D'_m := \left\{\frac{k}{2^m} t_0 : k = 0, \ldots, 2^m\right\}$, $m \in \mathbb{N}$, of sets on $[0, t_0]$ has analogous properties with respect to the Euclidean distance.

Denote $\mathscr{D}_{m,n} := D'_m \times D_n$ We will prove now that for the countable dense set

$$\mathscr{D} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathscr{D}_{m,n} \subset [0, t_0] \times X$$

Condition 2 holds true. To this aim we first show that any two points in $\bigcup_{n=1}^{\infty} D_n$ can be connected by a chain of successive neighbors. In view of (21) for fixed n, any $q \ge n$ and $x = x_q \in D_q$ there exists a sequence $(x_n, \ldots, x_{q-1})$ such that $x_j \in D_j$ and $d(x_j, x_{j+1}) \le \frac{1}{2^j}$ for $j = n, \ldots, q-1$. For $y = y_q \in D_q$ let $(y_n, \ldots, y_{q-1})$ be defined similarly. Then one obtains

$$d(x_n, y_n) \le \sum_{j=n}^{q-1} d(x_j, x_{j+1}) + d(x, y) + \sum_{j=n}^{q-1} d(y_j, y_{j+1})$$
$$\le \sum_{j=0}^{q-1} \frac{1}{2^j} + \sum_{j=n}^{p-1} \frac{1}{2^j} + d(x, y) \le 2\frac{1}{2^n} + d(x, y)$$

and hence,

$$d(x_n, y_n) \le \frac{3}{2^n}\,, \quad \text{if} \quad d(x, y) \le \frac{1}{2^n}\,. \tag{22}$$

The same procedure works on the time interval $[0, t_0]$. For fixed m, $p \ge m$ and $s, t \in D'_p$ let $s_p = s$, $t_p = t$ and $(s_m, \dots, s_{p-1})$, $(t_m, \dots, t_{p-1})$ be defined analogously.

Next we consider on the basic probability space the sets

$$A_{m,n} := \bigcup_{\left\{s \in D'_m, t \in D'_{m+1} \cap B'(s, \frac{3}{2^m})\right\}} \; \bigcup_{\left\{x \in D_n, y \in D_{n+1} \cap B(x, \frac{3}{2^n})\right\}} \left\{\omega : \left|Y(s, x) - Y(s, y) - (Y(t, x) - Y(t, y))\right| > \frac{1}{2^{m\alpha}} \frac{1}{2^{n\beta}}\right\}.$$

From (19) we get for s, t, x, y as in the union sets

$$\mathbb{P}\left(\left|Y(s, x) - Y(s, y) - (Y(t, x) - Y(t, y))\right| > \frac{1}{2^{m\alpha}} \frac{1}{2^{n\beta}}\right) \le \widetilde{c_N} \frac{1}{2^{N m(H-\alpha)}} \frac{1}{2^{Nn(K-\beta)}}$$

for some constant $\widetilde{c_N}$ and arbitrary $N \in \mathbb{N}$.

At the end of the proof we will derive from D-regularity of the compact metric measure space (X, d, μ) that the number of the sets in the union in the definition of $A_{m,n}$ is bounded by $c\, 2^m 2^{Dn}$.

Choosing N large enough so that $N(H - \alpha) > 1$ and $N(K - \beta) > D$ we infer that the series $\sum_{m,n=1}^{\infty} \mathbb{P}(A_{m,n})$ converges. Therefore the Borel–Cantelli lemma yields

$$\mathbb{P}\left(\bigcup_{m_0=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{m=m_0}^{\infty} \bigcap_{n=n_0}^{\infty} A^c_{m,n}\right) = 1\,,$$

i.e., for a.a. ω there exist $m_0(\omega) \in \mathbb{N}$ and $n_0(\omega) \in \mathbb{N}$ such that for any $m \ge m_0(\omega)$, $n \ge n_0(\omega)$, $s \in D'_m$, $t \in D'_{m+1} \cap B(s, \frac{3}{2^m})$, $x \in D_n$, $y \in D_{n+1} \cap B(x, \frac{3}{2^n})$ we have

$$\left|Y(s, x) - Y(s, y) - (Y(t, x) - Y(t, y))\right| \le \frac{1}{2^{m\alpha}} \frac{1}{2^{n\beta}}\,. \tag{23}$$

For such ω this property can be extended to arbitrary space-time points by means of the above chain construction:

Let now $(s,x),(s,y),(t,x),(t,y) \in \mathscr{D}$ be such that $\frac{t_0}{2^{m+1}} < |s-t| \le \frac{t_0}{2^m}$ for some $m \ge m_0(\omega)$ and $\frac{1}{2^{n+1}} < d(x,y) \le \frac{1}{2^n}$ for some $n \ge n_0(\omega)$.

First note that, $x, y \in D_q$ for some $q \ge n$. Choosing the chains $(x_n,\dots,x_q)$ and $(y_n,\dots,y_q)$ with $x_q = x$ and $y_q = y$ we use for any $t \in [0,t_0]$ the decomposition

$$Y(t,x) - Y(t,y) = \sum_{j=n}^{q-1} \big(Y(t,x_j) - Y(t,x_{j+1})\big) - \sum_{j=n}^{q-1} \big(Y(t,y_j) - Y(t,y_{j+1})\big) - (Y(t,x_n) - Y(t,y_n))\,.$$

Secondly, $s,t \in D'_p$ for some $p \ge m$. Choosing the chains $(s_m,\dots,s_p)$ and $(t_m,\dots,t_p)$ with $s_p = s$ and $t_p = t$ as above we can further decompose

$$\begin{aligned}
&Y(s,x) - Y(s,y) - (Y(t,x) - Y(t,y)) \\
&= \sum_{i=m}^{p-1}\sum_{j=n}^{q-1} \big(Y(s_i,x_j) - Y(s_i,x_{j+1}) - (Y(s_{i+1},x_j) - Y(s_{i+1},x_{j+1}))\big) \\
&\quad - \sum_{i=m}^{p-1}\sum_{j=n}^{q-1} \big(Y(s_i,y_j) - Y(s_i,y_{j+1}) - (Y(s_{i+1},y_j) - Y(s_{i+1},y_{j+1}))\big) \\
&\quad - \sum_{i=m}^{p-1}\sum_{j=n}^{q-1} \big(Y(t_i,x_j) - Y(t_i,x_{j+1}) - (Y(t_{i+1},x_j) - Y(t_{i+1},x_{j+1}))\big) \\
&\quad + \sum_{i=m}^{p-1}\sum_{j=n}^{q-1} \big(Y(t_i,y_j) - Y(t_i,y_{j+1}) - (Y(t_{i+1},y_j) - Y(t_{i+1},y_{j+1}))\big) \\
&\quad - \sum_{i=m}^{p-1} \big(Y(s_i,x_n) - Y(s_i,y_n) - (Y(s_{i+1},x_n) - Y(s_{i+1},y_n))\big) \\
&\quad + \sum_{i=m}^{p-1} \big(Y(t_i,x_n) - Y(t_i,y_n) - (Y(t_{i+1},x_n) - Y(t_{i+1},y_n))\big) \\
&\quad - (Y(s_m,x_n) - Y(t_m,y_n))\,.
\end{aligned}$$

For the absolute values of all summands on the right-hand side the inequality (23) holds true for i and j instead of m and n. Therefore we conclude

$$|Y(s,x) - Y(s,y) - (Y(t,x) - Y(t,y))|$$

$$\leq 4 \sum_{i=m}^{p-1} \sum_{j=n}^{q-1} \frac{1}{2^{i\alpha}} \frac{1}{2^{j\beta}} + 2 \sum_{i=m}^{p-1} \frac{1}{2^{i\alpha}} \frac{1}{2^{n\beta}} + \frac{1}{2^{m\alpha}} \frac{1}{2^{n\beta}} \leq c \, \frac{1}{2^{m\alpha}} \frac{1}{2^{n\beta}} \, .$$

Recall that $(s,x),(s,y),(t,x),(t,y) \in \mathscr{D}$ are such that $\frac{t_0}{2^{m+1}} < |s-t| \leq \frac{t_0}{2^m}$ and $\frac{1}{2^{n+1}} < d(x,y) \leq \frac{1}{2^n}$, where $m \geq m_0(\omega)$ and $n \geq n_0(\omega)$. Hence,

$$|Y(s,x) - Y(s,y) - (Y(t,x) - Y(t,y))| \leq c|s-t|^\alpha d(x,y)^\beta$$

for a.a. ω and (s,x), (t,y) from $\mathscr{D}$ with the property $|s-t| < \delta_1(\omega)$, $d(x,y) < \delta_2(\omega)$. Using that the space $[0,t_0] \times X$ is compact we conclude that

$$|Y(s,x) - Y(s,y) - (Y(t,x) - Y(t,y))| \leq c(\omega)|s-t|^\alpha d(x,y)^\beta$$

for all $(s,x),(s,y),(t,x),(t,y) \in \mathscr{D}$.

The proofs that for a.a. ω,

$$|Y(s,x) - Y(t,x)| \leq c(\omega)|s-t|^\alpha$$

$$|Y(t,x) - Y(t,y)| \leq c(\omega) d(x,y)^\beta \, ,$$

for any $(s,x),(s,y),(t,x),(t,y) \in \mathscr{D}$ are similar, but simpler. Therefore Condition 2 is fulfilled.

In order to complete the proof it remains to show that the numbers of elements of the sets $\left\{(s,t) : s \in D'_m, t \in D'_{m+1} \cap B'(s, \frac{3}{2^m})\right\}$ and $\left\{(x,y) : x \in D_n, y \in D_{n+1} \cap B(x, \frac{3}{2^n})\right\}$ are bounded by $c2^m$ and $c2^{Dn}$, respectively. (Cf. the definition of $A_{m,n}$.) We derive only the second bound, since the first one can be considered as a special case. (In fact, the dyadic construction of the sets D'_m provides here more direct arguments.)

By definition, for fixed $x \in D_n$ the balls $B(y, \frac{1}{2^{n+2}})$ for different $y \in D_{n+1} \cap B(x, \frac{3}{2^n})$ are disjoint and all are contained in $B(x, \frac{3}{2^n} + \frac{1}{2^{n+2}})$. Then we infer for the measure μ,

$$\sum_{y \in D_{n+1} \cap B(x, \frac{3}{2^n})} \mu\left(B\left(y, \frac{1}{2^{n+2}}\right)\right) \leq \mu\left(B\left(x, \frac{3}{2^n} + \frac{1}{2^{n+2}}\right)\right) .$$

The D-regularity of μ implies $\mu\left(B(y, \frac{1}{2^{n+2}})\right) \geq c_1 2^{-nD}$ and

$\mu\left(B(x, \frac{3}{2^n} + \frac{1}{2^{n+2}})\right) \leq c_2 2^{-nD}$ for some constants c_1, c_2. Therefore the number of elements of the set $D_{n+1} \cap B(x, \frac{3}{2^n})$ is uniformly bounded by a constant. Consequently, it suffices to show that the number of elements in D_n is bounded by $c2^{Dn}$ for some constant c. This follows by similar arguments, since

$$\sum_{x\in D_n} \mu\left(B\left(x, \frac{1}{2^{n+1}}\right)\right) \leq \mu(X) < \infty$$

and $\mu\big(B(x, \frac{1}{2^{n+1}})\big) \geq c_3 2^{-Dn}$.

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